COMP2610/6261 - Information Theory

Lecture 17: Noisy Channels

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Research School of Computer Science

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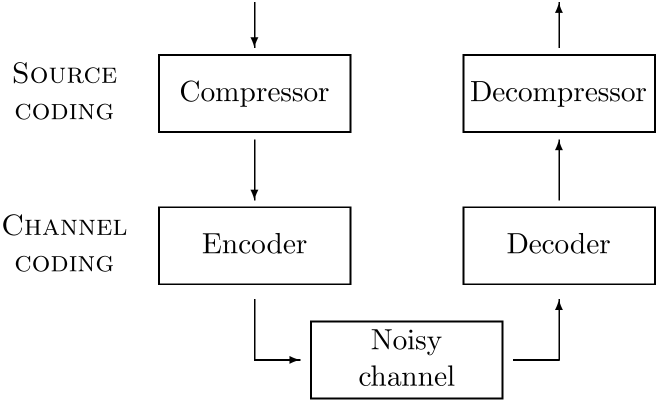
2 October, 2018

The Big Picture



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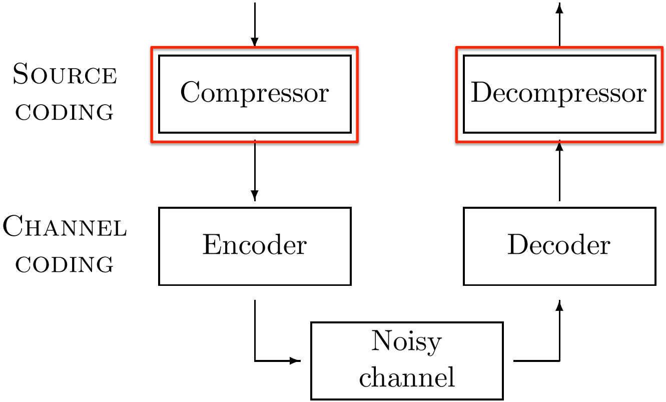

Concept: Expected code length

Theorem: Source coding theorem

Algorithms: { Huffman, Arithmetic } codes

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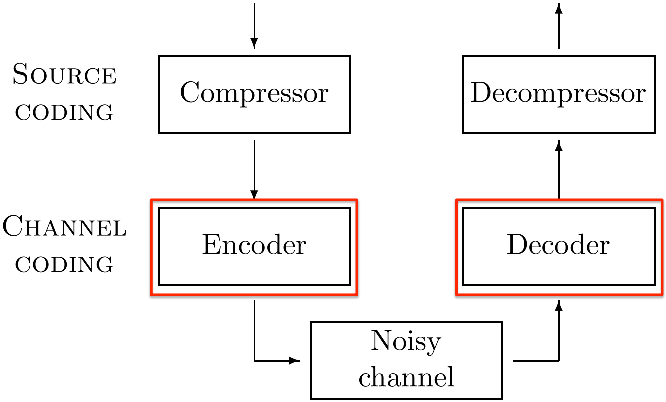


Concept: Channel capacity

Theorem: Channel coding theorem

Algorithms: Repetition codes, Hamming codes

Communication over Noisy Channels

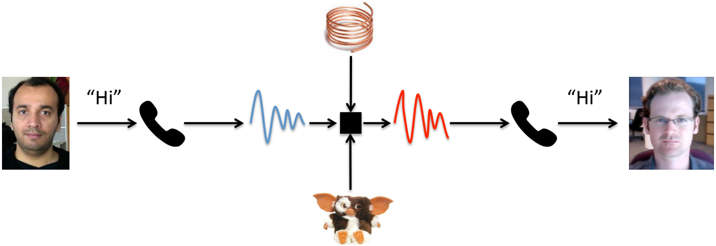
A channel is some medium for transmitting messages

A noisy channel is a channel that potentially introduces errors in the sender's message

The Problem of Communication

"The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point." (Claude Shannon, 1948)

Example: Telephone Network



Source : Aditya

Encoder : Telephone handset

Channel : Analogue telephone line

Decoder : Telephone handset

Destination : Mark

Key Questions

How do we model noisy communication abstractly?

What are the theoretical limits of noise correction?

What are the practical approaches to noise correction?

Encoder/Decoder Pairs

Suppose we have some set $\mathcal{S} = \{1, 2, \ldots, S\}$ of possible messages

- e.g. codewords from Huffman coding on some ensemble

- Sender and receiver agree on what these are

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- Sender and receiver agree on what these are

When communicating over a channel, the sender must encode messages into some input alphabet $\mathcal{X}$

The receiver then receives some (possibly corrupted) element from an output alphabet $\mathcal{Y}$

- Simple case: $\mathcal{X} = \mathcal{Y} = \{0, 1\}$
- The bit the sender transmits may not be what the receiver sees

Encoder/Decoder Pairs

Formally, the sender encodes messages via

$$\texttt{enc}\colon \mathcal{S} \to \mathcal{X}^N$$

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Isn't the compressor already "encoding" a message?

- Yes, but we might want to add something for noise tolerance

Might have $\mathcal{X} \neq \mathcal{Y}$

- e.g. if we allow a special "erased" symbol

$N > 1$ can be thought of as multiple uses of a channel

- e.g. use a bitstring of length 4 to represent messages

Channels: Informally

A **discrete channel** Q will:

- accept an **input** from $\mathcal{X}$
- produce an **output** from $\mathcal{Y}$

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A **discrete channel** Q will:

- accept an **input** from $\mathcal{X}$
- produce an **output** from $\mathcal{Y}$

There is a **probability** of observing various outputs, given an input

- This represents some inherent noise
- Noise could depend on the input

Channels: Formally

A **discrete channel** Q consists of:

- an *input alphabet* $\mathcal{X} = \{a_1, \ldots, a_I\}$
- an *output alphabet* $\mathcal{Y} = \{b_1, \ldots, b_J\}$
- *transition probabilities* $P(y|x)$.

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The channel Q can be expressed as a matrix

$$Q_{j,i} = P(y = b_j | x = a_i)$$

This represents the probability of observing b_j given that we transmit a_i

Channels: Example

Example: A channel Q with inputs $\mathcal{X} = \{a_1, a_2, a_3\}$, outputs $\mathcal{Y} = \{b_1, b_2\}$, and transition probabilities expressed by the matrix

$$Q = \begin{bmatrix} 0.8 & 0.5 & 0.2 \\ 0.2 & 0.5 & 0.8 \end{bmatrix}$$

So $P(b_1|a_1) = 0.8 = P(b_2|a_3)$ and $P(b_1|a_2) = P(b_2|a_2) = 0.5$.

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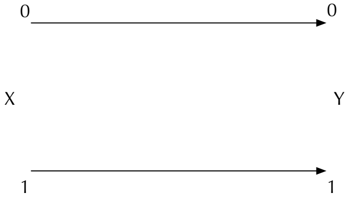
We arrange the inputs along the columns, and outputs along the rows

Actual details of alphabet are abstracted away

The Binary Noiseless Channel

One of the simplest channels is the **Binary Noiseless Channel**. The received symbol is always equal to the transmitted symbol – there is no probability of error, hence *noiseless*.



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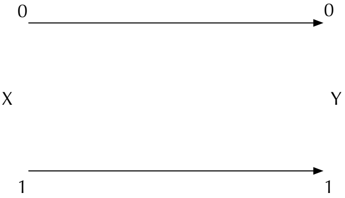

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What was transmitted over the channel if `0000 1111` was received?

$$\xrightarrow{Q} \texttt{0000 1111}$$

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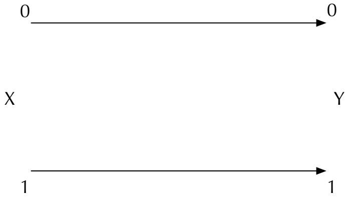

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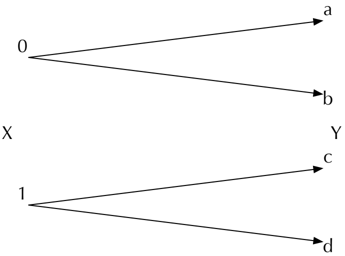

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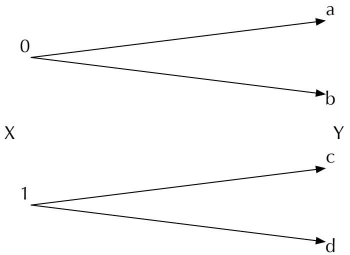

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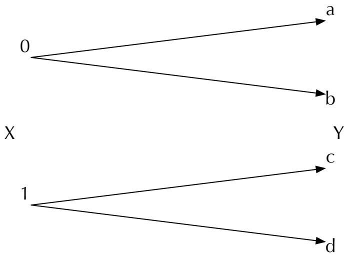

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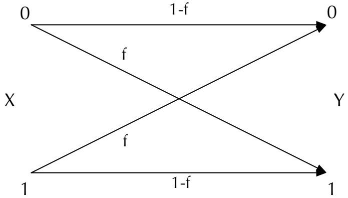
$$\color{red}{0000\,1111} \xrightarrow{Q} \mathtt{abba\,ddcd}$$

The Binary Symmetric Channel

Each symbol sent across a **binary symmetric channel** has a chance of being "flipped" to its counterpart ($\texttt{0} \rightarrow \texttt{1}$; $\texttt{1} \rightarrow \texttt{0}$)



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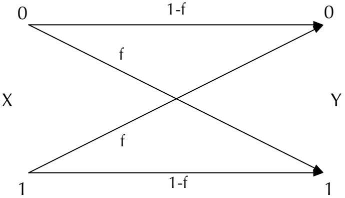

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What was most likely transmitted over the channel if $\mathtt{0010\,1001}$ was received, assuming $f = 0.1$ and $P(x = 0) = P(x = 1) = 0.5$?

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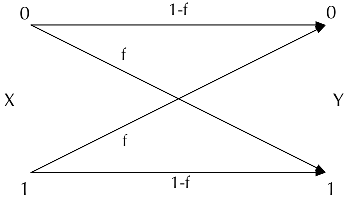

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$$0010\,1001 \xrightarrow{Q} 0010\,1001$$

Inferring the Input

Suppose we know the $P(x)$ — the probability x is transmitted.

Given a particular output $y \in \mathcal{Y}$ received over a channel Q, how likely was it that input $x \in \mathcal{X}$ was transmitted?

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$$P(x|y) = \frac{P(y|x)P(x)}{P(y)} = \frac{P(y|x)P(x)}{\sum_{x' \in \mathcal{X}} P(y|x')P(x')}$$

Inferring the Input: Example

Suppose $P(x = 0) = P(x = 1) = 0.5$. What are the probability that a $x = 0$ was transmitted over a *binary symmetric channel* Q with $f = 0.1$ given that a $y = 0$ was received?

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$$P(x = 0|y = 0) = \frac{0.9 \times 0.5}{0.9 \times 0.5 + 0.1 \times 0.5} = 0.9$$

Similarly, $P(x = 1|y = 1) = 0.9$.

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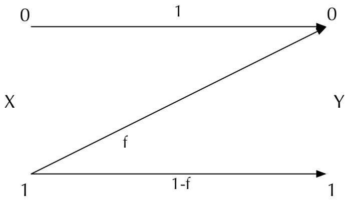
Similarly, $P(x=1|y=1) = 0.9$.

What if $P(x=0) = 0.01$?

$$P(x=0|y=0) = \frac{0.9 \times 0.01}{0.9 \times 0.01 + 0.1 \times 0.99} \approx 0.0833.$$

The Z Channel

Symbols may be corrupted over the channel asymmetrically.



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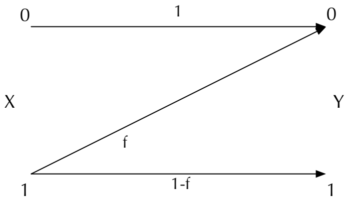

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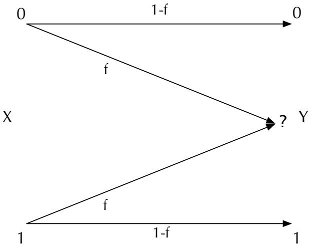
Inferring the input: Clearly $P(x = 1|y = 1) = 1$ but

$$P(x = 0|y = 0) = \frac{P(y = 0|x = 0)P(x = 0)}{\sum_{x' \in \mathcal{X}} P(y = 0|x')P(x')} = \frac{P(x = 0)}{P(x = 0) + f\,P(x = 1)}$$

So $P(x = 0|y = 0) \to 1$ as $f \to 0$, and goes to $P(x = 0)$ as $f \to 1$

The Binary Erasure Channel

We can model a channel which "erases" bits by letting one of the output symbols be the symbol '?' with associated probability f. The receiver knows which bits are erased.



Inputs $\mathcal{X} = \{0, 1\}$; Outputs $\mathcal{Y} = \{0, ?, 1\}$;
Transition probabilities

$$Q = \begin{bmatrix} 1-f & 0 \\ f & f \\ 0 & 1-f \end{bmatrix}$$

Example:

$$0000\,1111 \xrightarrow{Q} 00?0\,?11?$$

The Noisy Typewriter Channel

This channel simulates a noisy "typewriter". Inputs and outputs are 26 letters A through Z plus space. With probability $\frac{1}{3}$, each letter is either: unchanged; changed to the next letter, changed to the previous letter.

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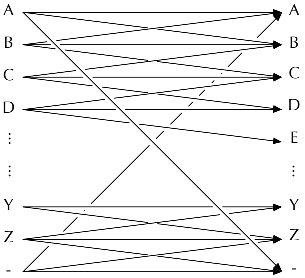

Inputs $\mathcal{X} = \{\mathtt{A}, \mathtt{B}, \dots, \mathtt{Z}, \_\}$;
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$$Q = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{3} & 0 & 0 & & \cdots & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

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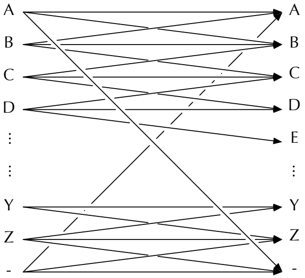

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The transition matrix for this channel has a diagonal structure: all of the probability mass is concentrated around the diagonal.

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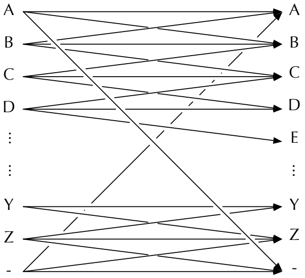

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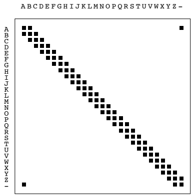

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Communicating over Noisy Channels

Suppose we know we have to communicate over some channel Q and we want build an *encoder/decoder* pair to reliably send a message $\mathbf{s}$ over Q.

$$\boxed{\mathbf{s}_{in}} \xrightarrow{\text{Encoder}} \boxed{\mathbf{x}} \xrightarrow{Q} \boxed{\mathbf{y}} \xrightarrow{\text{Decoder}} \boxed{\mathbf{s}_{out}}$$

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Reliability is measured via **probability of error** — that is, the probability of incorrectly decoding $\mathbf{s}_{out}$ given $\mathbf{s}_{in}$ as input:

$$P(\mathbf{s}_{out} \neq \mathbf{s}_{in}) = \sum_{\mathbf{s}} P(\mathbf{s}_{out} \neq \mathbf{s}_{in} | \mathbf{s}_{in} = \mathbf{s}) P(\mathbf{s}_{in} = \mathbf{s})$$

Probability of Error: Example

Let $\mathcal{S} = \{\mathtt{a}, \mathtt{b}\}$ and $\mathcal{X} = \mathcal{Y} = \{0, 1\}$

Assume *encoder*: $\mathtt{a} \to \mathtt{0}$; $\mathtt{b} \to \mathtt{1}$, *decoder*: $\mathtt{0} \to \mathtt{a}$; $\mathtt{1} \to \mathtt{b}$.

Consider binary symmetric Q,

$$Q = \begin{bmatrix} 1 - f & f \\ f & 1 - f \end{bmatrix}$$

with $f = 0.1$

Probability of Error: Example

If base probabilities of symbol transmission are $(p_{\mathtt{a}}, p_{\mathtt{b}}) = (0.5, 0.5)$,

$$P(\mathbf{s}_{in} \neq \mathbf{s}_{out}) = P(\mathbf{s}_{in} = \mathtt{a}, \mathbf{s}_{out} = \mathtt{b}) + P(\mathbf{s}_{in} = \mathtt{b}, \mathbf{s}_{out} = \mathtt{a})$$

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$$\begin{aligned}
P(\mathbf{s}_{in} \neq \mathbf{s}_{out}) &= P(\mathbf{s}_{in} = \mathtt{a}, \mathbf{s}_{out} = \mathtt{b}) + P(\mathbf{s}_{in} = \mathtt{b}, \mathbf{s}_{out} = \mathtt{a}) \\
&= P(\mathbf{s}_{out} = \mathtt{b} | \mathbf{s}_{in} = \mathtt{a}) P(\mathbf{s}_{in} = \mathtt{a}) + \\
&\quad P(\mathbf{s}_{out} = \mathtt{a} | \mathbf{s}_{in} = \mathtt{b}) P(\mathbf{s}_{in} = \mathtt{b}) \\
&= P(y = 1 | x = 0)\, p_{\mathtt{a}} + P(y = 0 | x = 1)\, p_{\mathtt{b}} \\
&= f
\end{aligned}$$

Probability of Error: Example

If base probabilities of symbol transmission are $(p_{\mathtt{a}}, p_{\mathtt{b}}) = (0.5, 0.5)$,

$$\begin{aligned}
P(\mathbf{s}_{in} \neq \mathbf{s}_{out}) &= P(\mathbf{s}_{in} = \mathtt{a}, \mathbf{s}_{out} = \mathtt{b}) + P(\mathbf{s}_{in} = \mathtt{b}, \mathbf{s}_{out} = \mathtt{a}) \\
&= P(\mathbf{s}_{out} = \mathtt{b} | \mathbf{s}_{in} = \mathtt{a}) P(\mathbf{s}_{in} = \mathtt{a}) + \\
&\quad P(\mathbf{s}_{out} = \mathtt{a} | \mathbf{s}_{in} = \mathtt{b}) P(\mathbf{s}_{in} = \mathtt{b}) \\
&= P(y = 1 | x = 0)\, p_{\mathtt{a}} + P(y = 0 | x = 1)\, p_{\mathtt{b}} \\
&= f \\
&= 0.1
\end{aligned}$$

A Simple Coding Scheme

Suppose **s** $\in \{\mathtt{a}, \mathtt{b}\}$ and we encode by $\mathtt{a} \to \mathtt{000}$ and $\mathtt{b} \to \mathtt{111}$.
To decode we count the number of `1`s and `0`s and set all bits to the majority count to determine **s**

$$\underbrace{\mathtt{000}, \mathtt{001}, \mathtt{010}, \mathtt{100}}_{A} \to \mathtt{a} \quad \text{and} \quad \underbrace{\mathtt{111}, \mathtt{110}, \mathtt{101}, \mathtt{011}}_{B} \to \mathtt{b}$$

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If the channel Q is binary symmetric with $f = 0.1$ again

$$\begin{aligned} P(\mathbf{s}_{in} \neq \mathbf{s}_{out}) &= P(\mathbf{y} \in B|000)\, p_{\mathrm{a}} + P(\mathbf{y} \in A|111)\, p_{\mathrm{b}} \\ &= [f^3 + 3f^2(1-f)]p_{\mathrm{a}} + [f^3 + 3f^2(1-f)]p_{\mathrm{b}} \\ &= f^3 + 3f^2(1-f) = 0.028 \end{aligned}$$

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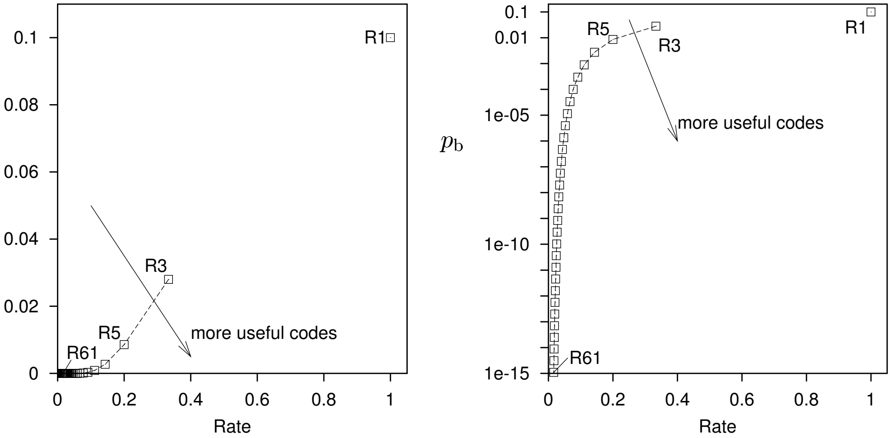
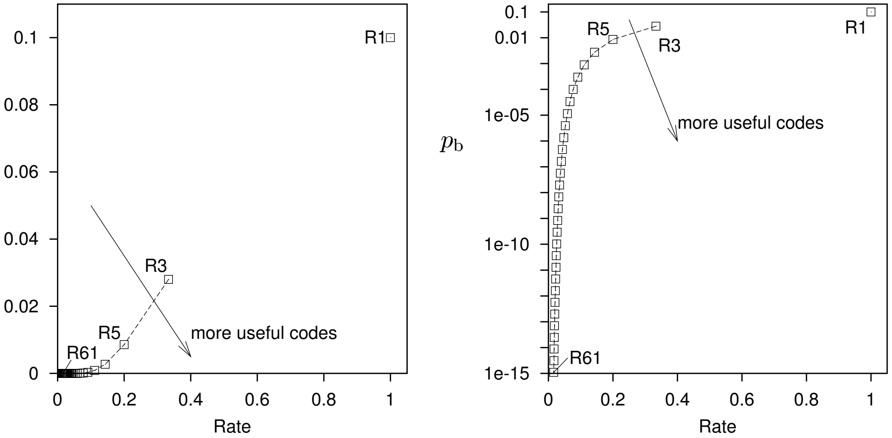
So the *error* has dropped from 0.1 to 0.028
but so has the *rate*: from 1 symbol/bit to 1/3 symbol/bit.

A Simple Coding Scheme



A Simple Coding Scheme



Can we make the error arbitrarily small without the rate going to zero?

Summary and Reading

Main Points:

- Modelling Noisy Channels
  - Noiseless, Overlap, Symmetric, Z, Erasure
- Simple Coding via Repetition
  - Probability of Error vs Transmission Rate

Reading:

- MacKay §9.1 - §9.5
- Cover & Thomas §7.1 - §7.3