COMP2610/COMP6261 - Information Theory

Tutorial 9: Stream and Noisy Channel Coding

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1. Complete arithmetic coding (Question 4, Tutorial 8) from previous tutorial if you have not completed.

2. Consider a channel with inputs $\mathcal{X} = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$, outputs $\mathcal{Y} = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\}$, and transition matrix

$$Q = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix}.$$

   (a) Assuming $p_{\mathcal{X}} = (0.25, 0.25, 0.5)$, what is the mutual information $I(X;Y)$ between the input and output of the channel?

   (b) Assuming $p_{\mathcal{X}} = (0.25, 0.25, 0.5)$, what is the average probability of error of the channel?

   (c) Calvin claims that he has constructed a block code for Q with rate 0.01 bits per transmission and maximal block error probability 1%. Is his claim possible? Justify your answer.

   (d) Hobbes claims that he has constructed a block code for Q with rate 100 bits per transmission and maximal block error probability 1%. Is his claim possible? Justify your answer.

3. *Noisy Coding (Exercise 10.12 in MacKay)*

   (a) A binary erasure channel with input $x \in \{0, 1\}$ and output $y \in \{0, ?, 1\}$ has transition matrix

$$Q_E = \begin{bmatrix} 1-q & 0 \\ q & q \\ 0 & 1-q \end{bmatrix}.$$

   Find the mutual information $I(X;Y)$ between the input and output for a general distribution $\mathbf{p}_X = (p_0, p_1)$ over inputs. Show that the capacity of this channel is $C_E = 1 - q$ bits.

   (b) A Z-channel has transition probability matrix

$$Q_Z = \begin{bmatrix} 1 & q \\ 0 & 1-q \end{bmatrix}.$$

   Show that, using a $(2, 1)$ code, that two uses of a Z-channel can be made to emulate one use of an erasure channel, and state the erasure probability of that erasure channel. Hence show the capacity of the Z-channel $C_Z \geq \frac{1}{2}(1-q) = \frac{1}{2}C_E$ bits.
   Explain why this result is an inequality rather than an equality.