COMP2610 Summary

October 30, 2018

Contents

1 Others

** Approximate to $3^{rd} - 4^{th}$ decimal number.

** "Assuming", "if", "given", "for" are key words for identify conditions.

** Be careful the $99\% = 0.99$ but $\frac{999}{1000} \neq 99\%$

** Translate and write notation/formula at first. Then find the given conditions.

** Be careful the text number conditions

** Write in log_2

2 Basic

$(log_2(x))' = \frac{1}{x\ln(2)}$ $\quad (log_2(1-x))' = -\frac{1}{(1-x)\ln(2)}$

Marginal $p(X)$

Joint $p(X, Y)$

Conditional $p(X|Y)$

Marginal Distribution

$p(X) = \sum_j p(X = x_i, Y = y_j)$(Law of Total Probability)

$p(Apple) = p(Apple, r)+p(Apple, g)+p(Apple, b) = p(Apple|r)p(r)+p(Apple|g)p(g)+p(Apple|b)p(b)$

If $X \perp Y$ (X and Y are independent random variables):

$p(X, Y) = p(X)p(Y)$

Conditional

If $X \perp Y$ (X and Y are independent random variables):

$p(X|Y) = p(X)$

$p(Coin = HH|Fair) = p(H|Fair) * p(H|Fair)$

If X is binary $\{0, 1\}$ then $p(X = 1|Y = 1) = 1 - p(X = 0|Y = 1)$

Product Rule

$p(A, B) = p(A|B)p(B) = p(B|A)p(A)$

$p(A|B)$ & $p(B|A)$ can use above formula or Bayesian Rule

Bayesian Rule

$p(X|Y) = \frac{p(Y|X)p(X)}{p(Y)}$

Posterior:$p(X|Y)$ Prior(uncertain quantity):$p(X)$ Likelihood: $p(Y|X)$

Prior $\uparrow$, then Posterior $\uparrow$, they are proportional

Use to calculate the fraction of a probability.

Prior: what we believe X is likely to be **before** looking at the data

Posterior: what we believe X is likely to be **after** looking at the data.

Expetation

$E[X] = \sum_{x \in \{X\}} p(x) * x$

$E[X|Y = y] = \sum_{x \in \{X\}} p(X = x|Y = y) * x$

$E[XY] = \sum_{x=0}^{1} \sum_{y=-1}^{1} p(X = x, Y = y) * x * y$

$Var(X) = E[X^2] - E[X]^2$

$Cov(X, Y) = E[XY] - E[X]E[Y]$ Cov is the correlation coefficient

Cox Axioms

Given B(x), B($\bar{X}$), B(X,Y), B(Y):

* Degree of belif can be ordered

*$B(X) = f[B(\bar{x})]$

*$B(X, X) = g[B(X|Y), B(Y)]$

3 Bernoulli, Binomial, Max Likelihood, MAP

Bernoulli Distribution

Variable has outputs $\{0, 1\}$

$p(X = 1|\theta) = \theta \quad p(X = 0|\theta) = 1 - \theta$

$p(X = x|\theta) = \theta^x(1 - \theta)^{1-x}$

Expected Value(Mean): $E[X|\theta] = \sum_{x \in X} p(x|\theta) * x = N\theta$

Variance(SquareStandard Deviation): $V[X|\theta] = E[(X - E[X])^2] = N\theta(1 - \theta)$

Binomial Distribution

Distributions of Bernoulli variables

Expected Value(Mean): $E[Y] = \sum_{m=0}^{N} Bin(m|N, \theta) = N\theta$

Variance(SquareStandard Deviation): $V[X|\theta] = \sum_{m=0}^{N}(m - E[m])^2 Bin(|N, \theta) = N\theta(1 - \theta)$

$p(Y = m) = Bin(m|N, \theta) = \binom{N}{m}\theta^m(1 - \theta)^{N-m}$

Num of ways get m heads out of N coin flip:$\binom{N}{m} = \frac{N!}{(N-m)!m!}$

$Binomial(x; n, P) = nCx * P^x * (1 - P)^{n-x}$ X: value; n: $\#times$; p:probability

$nCx = \frac{n!}{(n-x)!x!}$

Parameter Estimation

According to sequence D $\{0/1\}$:

$$\begin{cases} \text{if } D[i] = 1 \quad p(x_i|\theta) = (1-\theta) \\ \text{if } D[i] = 0 \quad p(x_i|\theta) = \theta \end{cases}$$

Maximum Likelihood Estimator(MLE)$\hat{\theta} = \arg max_\theta L(\theta|x)$

Likelihood Function(probability of a sequence): $L(\theta) = p(D|\theta) = \prod_{i=1}^{10} p(x_i|\theta) = \prod_{i=1}^{N} \theta^{x_i}(1-\theta)^{1-x_i}$

Maximum Likelihood(maxing likelihood): $\mathcal{L}(\theta) = log(p(D|\theta)) = \sum_{i=1}^{N} logp(x_i|\theta) = \sum_{i=1}^{N}[x_i log\theta + (1-x_i)log(1-\theta)]$

** Max likelihood uses $\sum$ and log, while likelihood uses $\prod$.

$\theta_{ML} = \frac{1}{N}\sum_{i=1}^{N} x_i$

Beta Prior

The likelihood of X given θ: Bern$(X = x|\theta) = \theta^x(1-\theta)^{1-x}$

prior (Beta Distribution):Beta$(\theta|a, b) = \frac{1}{Z(a,b)}\theta^{a-1}(1-\theta)^{b-1}$

Z(a,b) is a suitable normaliser.

posterior(By using Beta prior): $p(\theta|D, a, b) = \frac{p(D|\theta)p(\theta|a,b)}{p(D|a,b)} = Beta(\theta|m+a, l+b) \quad m = x; l = 1 - x$

If more data are coming, this posterior will become the new prior of next round.

Maximum A Posterior(MAP) uniform distribution: $\theta_{MAP} = \frac{m+a-1}{N+a+b-2}$

Maxim Likelihood: $\theta_{ML} = \frac{m}{N}$ not like max likelihood of Bernoulli model, it doesn't use prior.

If a=b=1, then $\theta_{MAP} = \theta_{ML}$

4 Entropy

how informative is a message?

information content(outcome of x):$h(x) = log_2 \frac{1}{p(x)}$ (unit in bits)

average information content of outcome: $H(X) = E_x[h(X)] \geq 0$

average information length: $\sum_{i=1}^{N} p_i(X) * N * l$ N:Num of this probability, l:Num of bit X used. (unit in bits)

$H(X) = \sum_{x \in x_1, x_2, x..} p(X) log_2(\frac{1}{p(X)}) = -\sum_{x \in x_1, x_2, x..} p(X) log_2(p(X))$

$H(X|\ Y) = \sum_{y \in y_1, y_2, y..} H(X|Y = y) \cdot p(Y = y)$; $y \in Y$ is known; $H(X|Y = 0) = H(p(x1|y = 0), p(x2|y = 0), ...) = H(p(x1, y = 0)/p(y = 0), p(x2, y = 0)p(y = 0), ...) =$ sum up all $p(x|y = 0) log_2 \frac{1}{p(x|y=0)}$

Entropy is maximised if p peaked; entropy is minimised if p uniformly distributed.

If p is **uniform** $\frac{1}{|X|}$, entropy is maximised. If p is uniform, then $H(X) = log_2(X)$ $\quad log_2|X|$ is the Num of bits needed to describe an outcome of X.

If not uniform, use shorter codes for object with higher probability, use longer codes for object with lower probability.

Entropy is the lower bound on average number of bits.

Adding outcomes of probability 0 does not affect H.

Joint Entropy

$H(X, Y) = E_{X,Y}[log \frac{1}{p(X,Y)}] = \sum_{x \in X} \sum_{y \in Y} p(x, y) log \frac{1}{p(x,y)}$

By chain rule: $H(X, Y) = -\sum_{x \in X} \sum_{y \in Y} p(x, y) log(p(x, y)) = -\sum_{x \in X} \sum_{y \in Y} p(x, y)[log(p(x)) + log(p(y|x))]$

$H(X, Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$

The joint uncertainty of X and Y is the uncertainty of X plus the uncertainty of Y given X.

If $X \perp Y$ (X and Y are independent random variables):H(X,Y)=H(X)+H(Y)

Conditional Entropy

Entropy of the probability distribution $p(X|Y=y)$: $H(X|Y=y)=\sum_{x\in X}p(x|Y=y)log\frac{1}{p(x|Y=y)}$

The average over Y of the conditional entropy X given Y=y: $H(X|Y)=\sum_{y\in Y}p(y)H(X|Y=y)=\sum_{y\in Y}p(y)\sum_{y\in Y}p(y|x)log\frac{1}{p(x|y)}=\sum_{x\in X}\sum_{y\in Y}p(x,y)log(\frac{1}{p(x|y)})=E_{x,y}[log(\frac{1}{p(X|Y)})]$

5 Kullback-Leibler(KL) Divergence

$D_{KL}(p||q)=\sum_{x\in X}p(x)(log\frac{1}{q(x)}-log\frac{1}{p(x)})=\sum_{x\in X}p(x)log(\frac{p(x)}{q(x)})=E_{p(x)}[log(\frac{p(x)}{q(x)})]$

If q uniform, $q(x)=\frac{1}{|X|}$: $D_{KL}(p||q)=\sum_{x\in X}p(x)log(\frac{p(x)}{q(x)})=\sum_{x\in X}p(x)(log(p(x))+log|X|)=-H(X)+log|X|$

If p=q(symmetry), $D_{KL}(p||q)=D_{KL}(q||p)$

6 Mutual Information

$I(X;Y)\geq 0$

Since all mutual information is symmetry, $I(X;Y)=I(Y;X)$

Self Information: I(X;X)= H(X)

$I(X;Y)=D_{KL}(p(X,Y)||p(X)p(Y))=\sum_{x\in X}\sum_{y\in Y}p(x,y)log(\frac{p(x,y)}{p(x)p(y)})$

$I(X;Y)=H(X)-H(X|Y)=H(Y)-H(Y|X)=H(X)+H(Y)-H(X,Y)$

If $X\perp Y$ (X and Y are independent random variables):I(X;Y)=0

The mutual info btw X and Y given $Z=z_k$: $I(X;Y|Z=z_k)=H(X|Z=z_k)-H(X|Y,Z=z_k)=E_{p(x,y,z)}log(\frac{p(X,Y|Z)}{p(X|Z)p(Y|Z)})$

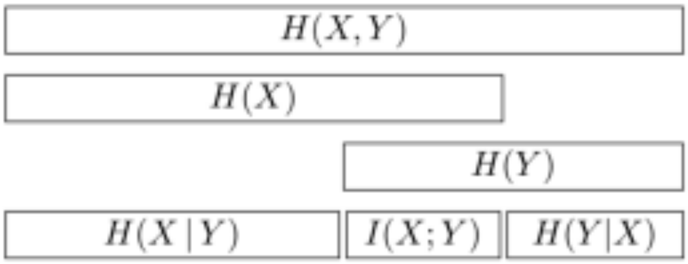


Figure 1: lec07-0

I(X;Y,Z)=$I(X;Y) + I(X;Z|Y) = I(X;Z) + I(X;Y|Z)$

$I(X,Y;Z) \neq I(X;Y,Z)$(Most of cases) but $I(X;Y,Z) = I(Y,Z;X) = I(X;Z) + I(X;Y|Z) = I(X;Y) + I(X;Z|Y)$

7 Convex Functions

$f(\lambda X_1 + (1-\lambda)X_2) \leq \lambda f(X_1) + (1-\lambda)f(X_2)$

If $\lambda = \{0, 1\}$, f is strictly convex

* concave: $\frown$ convex: $\smile$

If concave, there is a point: $\frac{df}{dx} = 0$ (Maximum point); But the derivative is not the Must condition that shows there is a max point.

8 Inequality

Jensen's Inequality

convex: $\smile$: $f(E[X]) \leq E[f(X)]$ verse versa.

Gibbs's Inequality

$D_{KL}(p||q) \geq 0$if p(x)=q(x)

If $X \perp Y$: $I(X;Y) = D_{KL}(p(X,Y)||p(X)p(Y)) \geq 0$

If $X \perp Y$: $H(X|Y) \leq H(X)$ $(I(X;Y) \geq 0)$Data don't increase uncertainty on average, info cannot hurt on average.

discret variable X: $H(X) \leq log|X|$

Large Number

Independent distribution

$E[X_i] = \mu$ and $overlinex_n = \frac{X_1+...+X_n}{n}$ $lim_{n\to\infty} p(|\overline{X}_n - \mu| > \epsilon) = 0$

$V[\overline{X}_n] = \frac{V[X_1+..+X_n]}{n^2} = \frac{n\sigma^2}{n^2}$

Markov's Inequality

Markov Chain: $p(X,Y,Z) = p(X)p(Y|X)p(Z|Y)$, it gives $X \perp Z|Y \Rightarrow p(X|Y,Z) = p(X|Y)$

(1) X and Z are conditional independent given Y;

(2)If Z=f(Y), then $X \to Y \to Z$, then $I(X;Y) \geq I(X;f(Y))$

(3)$X \to Y \to Z$ implies $Z \to Y \to X$

(4)If $X \to Y \to Z$, then $I(X;Y) \geq I(X;Z)$

Eatimate the MAX possible number of students who scored more than 80.

(1)Bound: $p(X \geq \lambda) \leq \frac{E[X]}{\lambda}$e.g. $\lambda < E[X]$ and $0 < \lambda < 1$ E[X]=n*P(X)

Only use mean of distribution

The bound from Markov's inequality A be within 1% of exact probability B: $A - B = 1\%$

Chebychev's Inequality

$p(|X - E[X]| \geq \lambda\sqrt{V[X]}) \leq \frac{V[X]}{(\lambda\sqrt{V[X]})^2} \leq \frac{1}{\lambda^2}$

9 Ensembles

Ensemble X $\in \{x, A_x, P_x\}$\{Random Value, Values, Probability\};

$H(X^N) = NH(X)$

Probability of types: $P(x) = p_1^{n_1} * p_2^{n_2} * ...$

# of sequence with n_i copies of $a_i = \frac{N!}{n_1!n_2!...}$

$N * p_x$: expectation of most likely sequence (single sequence has all same variables e.g. p(hhhh)).

Variation is small when β(clossness) is small.

# of sequences in the typical set: $T_{N\beta} \leq 2^{N(H(X)+\beta)}$; If binary low entropy: $T_{N\beta} \leq 2^{N(H(X)+\beta)} << 2^N$

$H(X) - \beta < -\frac{1}{N}log_2P(k) < H(X) + \beta$

For large X: $H(X) - \delta < \frac{1}{N}H_\delta(X^N)$; H(X) is the boundary

The most likely sequence may not belong to the typical set.

Asymptotic Equipartition Property AEP

Almost all sequences are typical

Informal: $log_2P(x_1, x_2....x_N)$ is close to $-NH(X)$ with high probability

If N large enough, it guarantees draw a sequence from a small set

If N larger, S_δ and $T_{N\beta}$ increasingly overlap. $|S_\delta| \leq T_{N\beta}$

If $< \beta$ $lim_{N\to\infty}p(...) = 1$; If $> \beta$ $lim_{N\to\infty}p(...) = 0$

Length of sequence $\uparrow$, the probability of "typical" sequence $\uparrow$ and larger than "atypical" sequence.

10 Source Coding Theorem SCT

If large sequence $\rightarrow$ average bits of each outcome = entropy of source

Average bit per outcome: $\sum l(x_i)p_i$

p_i are the weight

Lossless/Unique Decode-ability for variable-length code:

$x \neq y \rightarrow c(x) \neq c(y)$ Lossless: for outcomes; Unique Decodable: for string results

Prefix free $\Rightarrow$ Unique Decodable; Loss less $\neq$ Unique decodable unique decodable $\neq$ prefix free

Lossy Coding

Raw Bit Content: $H_0(X) = log_2|A_x|$ A_x: # of outcomes. probability of outcomes are ignored

* Reliable Probability of N outcomes: $(1 - p_{none})^N = 1 - nCx * p(X = 0)^{n-x} * p(X = 1)^x$

* Expected Bit of N outcomes: $N\frac{\sum l(x_i)p_i}{1-p_{none}} = N$

* Lossless bit = xN (x: # of non-none code)

Essential Bit Content

* Essential Bit content: $H_\delta(X) = log_2|S_\delta|$ S_δ: # of elements in set

Trade off in **uniform lossy** code: define S_δ as smallest subset of A_x; $P(x \in S_\delta) \geq 1 - \delta$

If only uniformly code elements in S_δ:

* Reliable Probability of N outcomes: $(1 - \delta)^N$

* Expected Bit of N outcomes: $Nlog_2|S_\delta|$

Set δ from 0, then to the smallest p_i, then 1 - smallest(p_i) until no more left.

For **uniform code**:

There is a N_0 such that $N \geq N_0$: $|\frac{1}{N}H_\delta(X^N) - H| < \epsilon$

* Average bit in uniform code δ portion: $\frac{1}{N}H_\delta(X^N)$

Tiny probability of error $\delta \rightarrow$ close to H

large probability of error $\delta \rightarrow$ cannot compress more than H bits per symbol.

** Raw bit content: $H_\delta(X^4) \neq 4H_\delta(X)$ But entropy $H(X^4) = 4H(X)$

Why curve is flat? $\because \delta \uparrow$ the quicker encounter sequence; It makes small and similar sized changes to $|S_\delta|$

If N $\uparrow$ $\quad log_2|S_\delta| \approx NH$

If use $< NH(X)$ bits per block, then lose information

** Bad practical to perform coding, needs huge block size N_0 and look up tables size is huge.

Kraft Inequality

If there is a prefix code C , then any binary prefix this code C, its code length: $\sum_{i=1}^{I} 2^{-l_i} \leq 1$

For Prefix Exclude Codes:

An l-bits codeword excludes $2^{k-l}*$k-bit codewords. e.g. 4*3-bit codewords:$\{000, 001, 010, ...\}$

There are only 2^{l*} possible l^*-bit codewords

Remove descendants(tree like) once its root is chosen as the prefix.

Even given code length satisfy the Kraft Inequality, doesn't mean code itself is prefix, just mean the code of length can be used to be the standard length to construct prefix.

Expected Code Length

L(C,X)=$E[l(x)] = \sum_{x \in A_x} p(x)l(x) = \sum_{i=1}^{l} p_i l_i$

Probability vector: $q_i = \frac{2^{-l_i}}{z}; \quad z = \sum_i 2^{-l_i}; \quad l_i = log_2(\frac{1}{zq_i})$

Limited of Compression: $L(C, X) = H(X) + D_{KL}(p||q) + log_2\frac{1}{z} \geq H(X)$(with equality when $l_i = log_2\frac{1}{p_i}$)

q: probability of code length; p: probability of ensemble. p=q(Gibb's Inequality)

Shannon Codes

Shannon codes are suboptimal, don't need smallest expected length

$\lceil log_2 \frac{1}{p_i} \rceil$

$H(X) \leq L(C, X) \leq H(X) + 1$

Huffman Codes

Huffman codes (prefix) are optimal. Find the smallest two to assign 0 and 1, then combine them in the next step. The non-combined will be kept in the next step.

$H(X) \leq L(C_{Huff}, X) \leq L(C_{other}, X) \leq H(X) + 1$

Best sybol codes, but notthe best code.

Shannon-Dano-Elias Codes

Worse than Huffman, prefix free; if code lossless, then F(x-1)< $\lfloor \overline{F}(x) \rfloor_{l(x)} < \lfloor \overline{F}(x) \rfloor_{l(x)} + \frac{1}{2^l} <$ F(x)(don't overlap, guarantee interval)

Cumulative Distribution: $F(x) = \sum_{i \leq x} p_i$;

$\overline{F}(x) = \sum_{i \leq x} p_i + \frac{1}{2} p(x) = F(x) - \frac{1}{2} p(x)$

Using the first bits of $\lceil log_2 \frac{1}{p(x)} \rceil + 1$

$H(X) \leq L(C_{Huff}, X) \leq H(X) + 1 \leq L(C_{SFE}, X) \leq H(X) + 2$

Arithmetic Coding

ba $p(a|b)$: first symbol is b, then a.

Mid-point = $[(U+V)/2]_2$; p(ab...)=p(a)p(b)...; The first bit: $\lfloor log_2 \frac{1}{p(ab...)} \rfloor + 1$

Dirichlet Model

Generalisation of Beta distribution $P(x = a|x_1....x_n) = \frac{\#(a_i)+m_i}{\sum[\#(a_k)+m_k]}$

Flexible, m to be the frequency of english letters. $\sum m_k$ Large $\rightarrow$ Stable;

Small $\rightarrow$ Responsive.

11 Noisy Channel

Between Encoder and Decoder

Formally: $P(y|x)$

How likely the input x transmitted when out put y is given:$P(x|y) = \frac{P(y|x)P(x)}{P(y)} = \frac{P(y|x)P(x)}{\sum_{x \in X} P(y|x')P(x')}$

(Reliability) Probability of (Block) Error $\Rightarrow$ Probability of incorrectly decoding s_{out} given S_{in}: $P(S_{out} \neq S_{in}) = \sum P(s_{out} \neq s_{in}|s_{in} = s)P(s_{in} = s) = P(S_{in} = a, S_{out} = b) + P(S_{in} = b, S_{out} = a) = P(S_{out} = b|S_{in} = a)P(S_{in} = a) + P(S_{out} = a|S_{in} = b)P(S_{in} = b)$

Maximum probability of Block error: $P_{max} = max_{s_{in}} P(s_{out} \neq s_{in}|s_{in})$ if $P_{max} \rightarrow 0; P(S_{out} \neq S_{in}) \rightarrow 0$

Achievable Rate R: guarantee small maximum probability of block error, $P_{max} < \epsilon > 0$ and $K/N \geq R$

If H(X) is small then $I(X;Y)$ is small, $I(X;Y) \leq H(X)$

Reliability Channel: noiseless > Z > Symmetric; Less I(X;Y) and less reliable

The Capacity C of a channel Q is the largest mutual information between input X and out Y for any choice of input ensemble $C = max_{px} I(X;Y)$

Capacity determine rate at which communicate across a channel with arbitrarily small error

If Q symmetric, its capacity is achieved by a uniform distribution over X. P_x:$I(X;Y) \leq 1 - H(f)$;

$H_2(q) = -qlog_2(q) - (1-q)log_2(1-q)$; $H_2(p)' = log_2 \frac{1-p}{p}$

Channel Types

Binary Noiseless Channel: no error probability(it's reliable); $I(X;Y) = H(X), I(X;Y) > H(X|Y) = 0$

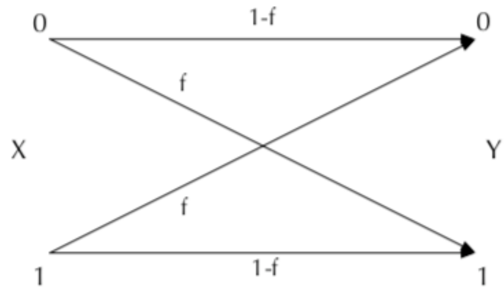
Binary symmetric Channel: P(flip)=f, most likely = transmitted code;

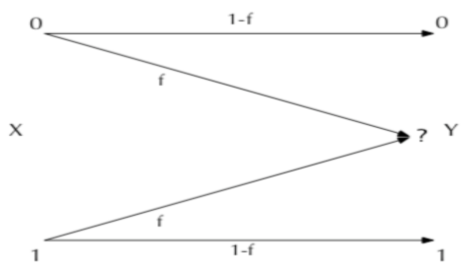
Non-overlapping Channel: uncertainty output, but certain input

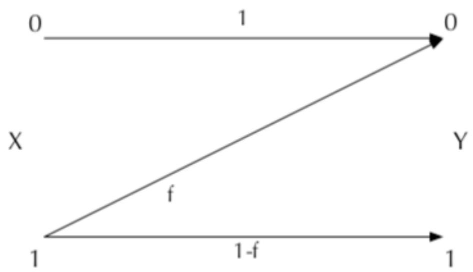
Binary Erasure: output = ?.

Z channel: asymmetrical, uncertain input when y = 0;

Noisy Typewriter Channel: all probability mass is concentrated around diagonal







Channel Capacity- Binary symmetric

**Step 1: Compute H(Y)

$q = P_y = Qp_x$:

$P(y=0) = (1-f)P(x=0) + fP(X=1) = (1-f)p_0 + fp_1$

$P(y=1) = p(Y=1|X=1)P(X=1) + p(Y=1|X=0)p(X=0) = (1-f)P(x=1) + fP(X=0) = fp_0 + (1-f)p_1$

$H(Y) = H_2(q1) = H_2(fp_0 + (1-f)p_1)$ when $q = q_1 = P(y=1)$

**Step 2: Compute $H(Y|X)$

$H(Y|X) = \sum H(Y|x)P(x) = \sum_x H_2(f)P(x) = H_2(f)\sum P(x) = H_2(f)$

$I(X;Y) = H(Y) - H(Y|X) = H_2(fp_0 + (1-f)p_1) - H_2(f)$

Channel Capacity- non symmetric

I(X;Y) concave in $p_x = (1-p, p)$, single max $\Rightarrow$ stationary point $(I(X;Y))' = 0$. If Z channel with $P(y=0|x=1) = f$:

$H(Y) = H_2(P(y=1)) = H_2(0p_0 + (1-f)p_1) = H_2((1-f)p_1)$

$H(Y|X) = p_0 H_2(P(y=1|x=0)) + p_1 H_2(P(y=0|x=1)) = p_0 H_2(0) + p_1 H_2(f)$

$I(X;Y) = H_2((1-f)p_1) - p_1 H_2(f)$

$p = \frac{1/(1-f)}{1+2^{H_2(f)/(1-f)}}$

Channel Capacity- communication

Message S has a unique block of symbol x with block length N(symbols).

(N,K) Block Code: for Q, a list of S=2^k codewords; rate $= \frac{log_2 S}{N} = \frac{K}{N}$; rate $\uparrow$, efficient communication$\uparrow$

Optimal decoder for code S, channel Q, prior P(s) maps y to output s $P(s|y)$ is maximal.$dec_{opt}(y) = maxP(s|y) = maxP(y|s)P(s)$; y is known.

12 Noisy-Channel Coding Theorem

If Q is a channel with capacity C then the rate R is achievable if and only if

13 formulas

$p(A|B,C) = \frac{p(B|A,C)p(A|C)}{p(B|C)}$

$p(A,B,C) = p(C|A,B) * p(B|A) * P(A) = p(C|A,B) * p(A,B)$

$p(A,B|C) = p(A|C,B)p(B|C)$