

COMP2610 / COMP6261 - Information Theory

Lecture 10: Typicality and Asymptotic Equipartition Property

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Last time

Markov's inequality

Chebyshev's inequality

Law of large numbers

Law of Large Numbers

Theorem

Let X_1, \dots, X_n be a sequence of iid random variables, with

$$\mathbb{E}[X_i] = \mu$$

and $\mathbb{V}[X_i] < \infty$. Define

$$\bar{X}_n = \frac{X_1 + \dots + X_n}{n}.$$

Then, for any $\beta > 0$,

$$\lim_{n \rightarrow \infty} p(|\bar{X}_n - \mu| < \beta) = 1.$$

This is also called $\bar{X}_n \rightarrow \mu$ **in probability**.

Definition: For random variables v_1, v_2, \dots , we say $v_n \rightarrow v$ **in probability** if for all $\beta > 0$ $\lim_{n \rightarrow \infty} P(|v_n - v| > \beta) = 0$.

β is fixed (not shrinking like $\frac{1}{n}$). Not max/min. Reduction in variability.

This time

- Ensembles and sequences
- Typical sets
- Asymptotic Equipartition Property (AEP)

- 1 Ensembles and sequences
 - Counting Types of Sequences
- 2 Typical sets
- 3 Asymptotic Equipartition Property (AEP)
- 4 Wrapping Up

Ensembles

Ensemble

An **ensemble** X is a triple $(x, \mathcal{A}_X, \mathcal{P}_X)$; x is a **random variable** taking **values** in $\mathcal{A}_X = \{a_1, a_2, \dots, a_I\}$ with **probabilities** $\mathcal{P}_X = \{p_1, p_2, \dots, p_I\}$.

We will call \mathcal{A}_X the **alphabet** of the ensemble

Ensembles

Example: Bent Coin



Let X be an **ensemble** with outcomes \mathbf{h} for *heads* with probability 0.9 and \mathbf{t} for *tails* with probability 0.1.

- The **outcome set** is $\mathcal{A}_X = \{\mathbf{h}, \mathbf{t}\}$
- The **probabilities** are
$$\mathcal{P}_X = \{p_{\mathbf{h}} = 0.9, p_{\mathbf{t}} = 0.1\}$$

Extended Ensembles

We can also consider **blocks** of outcomes, which will be useful to describe sequences:

Example (Coin Flips):

hhhhthhththh	→ hh hh th ht ht hh	(6 × 2 outcome blocks)
	→ hhh hth hth thh	(4 × 3 outcome blocks)
	→ hhhh thht hthh	(3 × 4 outcome blocks)

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Extended Ensemble

Let X be a single ensemble. The **extended ensemble** of blocks of size N is denoted X^N . Outcomes from X^N are denoted $\mathbf{x} = (x_1, x_2, \dots, x_N)$. The **probability** of \mathbf{x} is defined to be $P(\mathbf{x}) = P(x_1)P(x_2) \dots P(x_N)$.

Extended Ensembles

Example: Bent Coin



Let X be an ensemble with outcomes $\mathcal{A}_X = \{h, t\}$ with $p_h = 0.9$ and $p_t = 0.1$.

Consider X^4 – i.e., 4 flips of the coin.

$$\mathcal{A}_{X^4} = \{hhhh, hhht, hhth, \dots, tttt\}$$

Extended Ensembles

Example: Bent Coin



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Consider X^4 – i.e., 4 flips of the coin.

$$\mathcal{A}_{X^4} = \{hhhh, hhht, hhth, \dots, tttt\}$$

$$P(hhhh) = (0.9)^4 \approx 0.6561$$

$$P(tttt) = (0.1)^4 = 0.0001$$

$$P(hthh) = 0.9 \cdot 0.1 \cdot 0.9 \cdot 0.9 = (0.9)^3(0.1) \approx 0.0729$$

$$P(htht) = 0.9 \cdot 0.1 \cdot 0.9 \cdot 0.1 = (0.9)^2(0.1)^2 \approx 0.0081.$$

Extended Ensembles

Example: Bent Coin

Entropy of extended ensembles

We can view X^4 as comprising 4 independent random variables, based on the ensemble X

Entropy is additive for independent random variables

Thus,

$$H(X^4) = 4H(X) = 4. (-0.9 \log_2 0.9 - 0.1 \log_2 0.1) = 1.88\text{bits}.$$

More generally,

$$H(X^N) = NH(X).$$

Counting Types of Sequences

Criteria for dividing 2^N sequences into **types**

In the bent coin example,

$$\begin{aligned}(0.9)^2(0.1)^2 &= P(\text{hh}\text{tt}) \\ &= P(\text{ht}\text{ht}) \\ &= P(\text{ht}\text{th}) \\ &= P(\text{th}\text{ht}) \\ &= P(\text{th}\text{th}) \\ &= P(\text{tt}\text{hh}).\end{aligned}$$

The **order** of outcomes in the sequence is **irrelevant**

Counting Types of Sequences

Let X be an ensemble with alphabet $\mathcal{A}_X = \{a_1, \dots, a_I\}$.

Let $p(X = a_i) = p_i$.

For a sequence $\mathbf{x} = x_1, x_2, \dots, x_N$, how to compute $p(\mathbf{x})$?

let $n_i = \#$ of times symbol a_i appears in \mathbf{x} (symbol count)

Given the n_i 's, we can compute the probability of seeing \mathbf{x} :

$$\begin{aligned} P(\mathbf{x}) &= P(x_1) \cdot P(x_2) \cdot \dots \cdot P(x_N) \\ &= P(a_1)^{n_1} \cdot P(a_2)^{n_2} \cdot \dots \cdot P(a_I)^{n_I} \\ &= p_1^{n_1} \cdot p_2^{n_2} \cdot \dots \cdot p_I^{n_I} \end{aligned}$$

Sufficient statistics: $\{n_1, n_2, \dots, n_I\}$. Use it as a criteria of partitioning.

Counting Types of Sequences

Sequence Types

Each unique choice of (n_1, n_2, \dots, n_l) gives a different **type** of sequence

- 4 heads, (3 heads, 1 tail), (2 heads, 2 tails), ...
- Sequences in each type are equiprobable.

For a given **type** of sequence how many sequences are there with these symbol counts?

$$\# \text{ of sequences with } n_i \text{ copies of } a_i = \frac{N!}{n_1! n_2! \dots n_l!}$$

$$\begin{aligned} & \binom{N}{n_1} \binom{N-n_1}{n_2} \binom{N-n_1-n_2}{n_3} \dots \\ &= \frac{N!}{n_1!(N-n_1)!} \cdot \frac{(N-n_1)!}{n_2!(N-n_1-n_2)!} \cdot \frac{(N-n_1-n_2)!}{n_3!(N-n_1-n_2-n_3)!} \dots \end{aligned}$$

Counting Types of Sequences

Example

Probability of **types**

Let $\mathcal{A} = \{a, b, c\}$ with $P(a) = 0.2$, $P(b) = 0.3$, $P(c) = 0.5$.

Counting Types of Sequences

Example

Probability of **types**

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Each sequence of type $(n_a, n_b, n_c) = (2, 1, 3)$ has length 6 and probability $(0.2)^2(0.3)^1(0.5)^3 = 0.0015$.

Counting Types of Sequences

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There are $\frac{6!}{2!1!3!} = 60$ such sequences.

Counting Types of Sequences

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The probability **x** is of type $(2, 1, 3)$ is $(0.0015) \cdot 60 = 0.09$.

Study probabilities at the level of types (most likely, average/typical)

- 1 Ensembles and sequences
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Extended Ensembles

Example

With $p_h = 0.75$, what are the probabilities for X^N ?

$$N = 2$$

\mathbf{x}	$P(\mathbf{x})$
hh	0.5625
ht	0.1875
th	0.1875
tt	0.0625

Extended Ensembles

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$N = 2$

\mathbf{x}	$P(\mathbf{x})$
hh	0.5625
ht	0.1875
th	0.1875
tt	0.0625

$N = 3$

\mathbf{x}	$P(\mathbf{x})$
hhh	0.4219
hht	0.1406
hth	0.1406
thh	0.1406
htt	0.0469
tht	0.0469
tth	0.0469
ttt	0.0156

Extended Ensembles

Example

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$N = 3$

\mathbf{x}	$P(\mathbf{x})$
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hth	0.1406
thh	0.1406
htt	0.0469
tht	0.0469
tth	0.0469
ttt	0.0156

$N = 4$

\mathbf{x}	$P(\mathbf{x})$	\mathbf{x}	$P(\mathbf{x})$
hhhh	0.3164	thht	0.0352
hhht	0.1055	thth	0.0352
hhtth	0.1055	tthh	0.0352
hthhh	0.1055	httt	0.0117
thhhh	0.1055	thtt	0.0117
hthtt	0.0352	ttht	0.0117
httht	0.0352	ttth	0.0117
hhttt	0.0352	tttt	0.0039

Observations

As N increases, there is an increasing spread of probabilities

The most likely single sequence will always be the all h's

However, for $N = 4$, the most likely sequence **type** is 3 h's and 1 t

Not surprising because $3 = N \cdot p_h$, pretty much average case.

Symbol Frequency in Long Sequences

To judge if a sequence is typical/average, a natural question to ask is:

How often does each symbol appear in a sequence \mathbf{x} from X^N ?

Intuitively, in a sequence of length N , let a_i appear for n_i times.

Then **in expectation**

$$n_i \approx N \cdot P(a_i)$$

Note $p_i = P(a_i)$, and

$$P(\mathbf{x}) = P(a_1)^{n_1} P(a_2)^{n_2} \dots P(a_l)^{n_l} \approx p_1^{Np_1} p_2^{Np_2} \dots p_l^{Np_l}$$

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So the *information content* $-\log_2 P(\mathbf{x})$ of that sequence is approximately

$$-p_1 N \log_2 p_1 - \dots - p_I N \log_2 p_I = -N \sum_{i=1}^I p_i \log_2 p_i = NH(X)$$

Typical Sets

We want to consider elements \mathbf{x} that have $-\log_2 P(\mathbf{x})$ “close” to $NH(X)$

Typical Set

For “closeness” $\beta > 0$ the typical set $T_{N\beta}$ for X^N is

$$\begin{aligned} T_{N\beta} &\stackrel{\text{def}}{=} \{\mathbf{x} : |-\log_2 P(\mathbf{x}) - NH(X)| < N\beta\} \\ &= \left\{ \mathbf{x} : \left| -\frac{1}{N} \log_2 P(\mathbf{x}) - H(X) \right| < \beta \right\} \end{aligned}$$

Union of types

Typical Sets

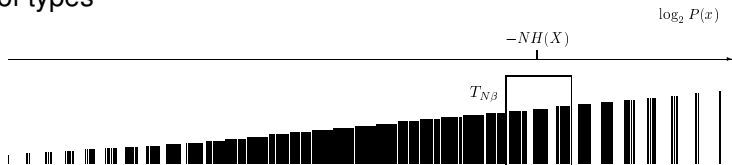
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Union of types



What when $\beta = 0$ (and replace $<$ by \leq)?

Criterion based on information content. Other criterion (KL divergence)?

Typical Sets

The name “typical” is used since $\mathbf{x} \in T_{N\beta}$ will have roughly $p_1 N$ occurrences of symbol a_1 , $p_2 N$ of a_2 , \dots , $p_K N$ of a_K .

[illegible]

Randomly drawn sequences for $P(1) = 0.1$. Note: $H(X) \approx 0.47$

Typical Sets

Properties

Typical sequences are nearly equiprobable: Every $\mathbf{x} \in T_{N\beta}$ has

$$2^{-N(H(X)+\beta)} \leq P(\mathbf{x}) \leq 2^{-N(H(X)-\beta)}.$$

Variation is small when β is small

Number of sequences in the typical set: For any N, β ,

$$|T_{N\beta}| \leq 2^{N(H(X)+\beta)}.$$

Typical Sets

Proof of Cardinality Bound

For every $\mathbf{x} \in T_{N\beta}$,

$$p(\mathbf{x}) \geq 2^{-N(H(X)-\beta)}.$$

Thus,

$$\begin{aligned} 1 &= \sum_{\mathbf{x}} p(\mathbf{x}) \\ &\geq \sum_{\mathbf{x} \in T_{N\beta}} p(\mathbf{x}) \\ &\geq \sum_{\mathbf{x} \in T_{N\beta}} 2^{-N(H(X)-\beta)} \\ &= 2^{-N(H(X)-\beta)} \cdot |T_{N\beta}|. \end{aligned}$$

Thus

$$|T_{N\beta}| \leq 2^{N(H(X)+\beta)}$$

Typical Sets

Most Likely Sequence

The most likely sequence **may not** belong to the typical set

e.g. with $p_h = 0.75$, we have

$$-\frac{1}{4} \log_2 P(\text{hhhh}) = 0.4150$$

whereas $H(X) = 0.8113$

The most likely single sequence $\rightarrow \text{hhhh}$

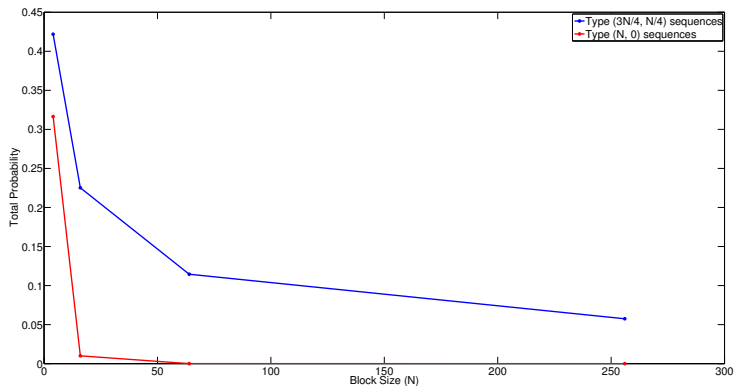
The most likely single sequence **type** $\rightarrow \{\text{hhht}, \text{hthh}, \dots\}$

Typical Sets

Most Likely Sequence

Probability of most likely sequence decays like $(p_h)^N$ ($p_h = 0.75$)

Sequences with $N \cdot p_h$ heads contain much more total probability mass



Blue curve corresponds to typical set with $\beta = 0$. What if $\beta > 0$?

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Asymptotic Equipartition Property

Eventually
Informally

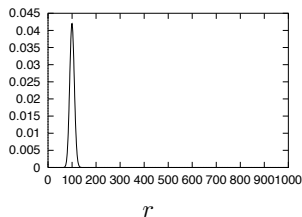
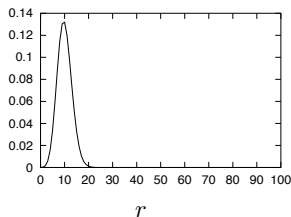
Equally Divided

Asymptotic Equipartition Property (Informal)

As $N \rightarrow \infty$, $\log_2 P(x_1, \dots, x_N)$ is close to $-NH(X)$ with high probability.

For large block sizes “almost all sequences are typical” (i.e., in $T_{N\beta}$)

$$n(r)P(\mathbf{x}) = \binom{N}{r} p_1^r (1 - p_1)^{N-r}$$



Probability sequence \mathbf{x} has r heads for $N = 100$ (left) and $N = 1000$ (right). Here $P(X = \text{head}) = 0.1$.

Asymptotic Equipartition Property

Formally

Asymptotic Equipartition Property

If x_1, x_2, \dots are i.i.d. with distribution P then, in probability,

$$-\frac{1}{N} \log_2 P(x_1, \dots, x_N) \rightarrow H(X).$$

In precise language:

$$(\forall \beta > 0) \lim_{N \rightarrow \infty} p \left(\left| -\frac{1}{N} \log_2 P(x_1, \dots, x_N) - H(X) \right| < \beta \right) = 1.$$

Exactly the probability of $\mathbf{x} \in T_{N\beta}$.

Asymptotic Equipartition Property

Formally

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If x_1, x_2, \dots are i.i.d. with distribution P then, **in probability**,

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Exactly the probability of $\mathbf{x} \in T_{N\beta}$.

Recall definition: for random variables v_1, v_2, \dots , we say $v_N \rightarrow v$ **in probability** if for all $\beta > 0$ $\lim_{N \rightarrow \infty} P(|v_N - v| > \beta) = 0$

Here v_N corresponds to $-\frac{1}{N} \log_2 P(x_1, \dots, x_N)$.

Asymptotic Equipartition Property

Comments

Why is it surprising/significant?

For an ensemble with binary outcomes, and low entropy,

$$|T_{N\beta}| \leq 2^{NH(X)+\beta} \ll 2^N$$

i.e. the typical set is a **small fraction** of all possible sequences

AEP says that for N sufficiently large, we are virtually guaranteed to draw a sequence from this small set

Significance in information theory

Asymptotic Equipartition Property

Proof

Since x_1, \dots, x_N are independent,

$$\begin{aligned} -\frac{1}{N} \log p(x_1, \dots, x_N) &= -\frac{1}{N} \log \prod_{n=1}^N p(x_n) \\ &= -\frac{1}{N} \sum_{n=1}^N \log p(x_n). \end{aligned}$$

Let $Y = -\log p(X)$ and $y_n = -\log p(x_n)$. Then, $y_n \sim Y$, and

$$\mathbb{E}[Y] = H(X).$$

But then by the law of large numbers,

$$(\forall \beta > 0) \lim_{N \rightarrow \infty} p \left(\left| \frac{1}{N} \sum_{n=1}^N y_n - H(X) \right| > \beta \right) = 0.$$

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Summary & Conclusions

- Ensembles and sequences
- Typical sets
- Asymptotic Equipartition Property (AEP)

Next: Source Coding.