COMP2610 / COMP6261 - Information Theory

Lecture 3: Probability Theory and Bayes' Rule

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Last time

- A general communication system
- Why do we need probability?
- Basics of probability theory
- Joint, marginal and conditional distributions

Review Exercise

Suppose I go through the records for $N = 1000$ students, checking their admission status, $A = \{0, 1\}$, and whether they are "brilliant" or not, $B = \{0, 1\}$

(Aside: "Brilliance" is a dodgy concept, and does not predict scientific achievement as well as persistence and combinatorial ability; see e.g. Dean Simonton, *Scientific Genius: A Psychology of Science*, Cambridge University Press, 2009; this is just a toy example!)

Say that the counts for admission and brilliance are

	$B = 0$	$B = 1$
$A = 0$	680	10
$A = 1$	220	90

Then:

$$
\begin{array}{|l|}
\hline
p(A = 1, B = 0) \\
p(B = 1) \\
p(A = 0) \\
p(B = 1 | A = 1) \\
p(A = 0 | B = 0) \\
\hline
\end{array}
$$

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Then:

$p(A = 1, B = 0)$	$220/1000$
$p(B = 1)$	
$p(A = 0)$	
$p(B = 1 \mid A = 1)$	
$p(A = 0 \mid B = 0)$	

Review Exercise

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Say that the counts for admission and brilliance are

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Then:

$p(A = 1, B = 0)$	$220/1000$
$p(B = 1)$	$100/1000$
$p(A = 0)$	
$p(B = 1 \mid A = 1)$	
$p(A = 0 \mid B = 0)$	

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Then:

$p(A = 1, B = 0)$	$220/1000$
$p(B = 1)$	$100/1000$
$p(A = 0)$	$690/1000$
$p(B = 1 \| A = 1)$	
$p(A = 0 \| B = 0)$	

Review Exercise

Suppose I go through the records for $N = 1000$ students, checking their admission status, $A = \{0, 1\}$, and whether they are “brilliant” or not, $B = \{0, 1\}$

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Then:

$p(A = 1, B = 0)$	$220/1000$
$p(B = 1)$	$100/1000$
$p(A = 0)$	$690/1000$
$p(B = 1 \mid A = 1)$	$90/310$
$p(A = 0 \mid B = 0)$	

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Say that the counts for admission and brilliance are

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Then:

$p(A = 1, B = 0)$	$220/1000$	Joint
$p(B = 1)$	$100/1000$	
$p(A = 0)$	$690/1000$	
$p(B = 1 \| A = 1)$	$90/310$	
$p(A = 0 \| B = 0)$	$680/900$	

Review Exercise

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Then:

$p(A = 1, B = 0)$	$220/1000$	Joint
$p(B = 1)$	$100/1000$	Marginal
$p(A = 0)$	$690/1000$	
$p(B = 1 \| A = 1)$	$90/310$	
$p(A = 0 \| B = 0)$	$680/900$	

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Then:

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$p(B = 1)$	$100/1000$	Marginal
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Then:

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$p(B = 1)$	$100/1000$	Marginal
$p(A = 0)$	$690/1000$	Marginal
$p(B = 1 \mid A = 1)$	$90/310$	Conditional
$p(A = 0 \mid B = 0)$	$680/900$	

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Say that the counts for admission and brilliance are

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Then:

$p(A = 1, B = 0)$	$220/1000$	Joint
$p(B = 1)$	$100/1000$	Marginal
$p(A = 0)$	$690/1000$	Marginal
$p(B = 1 \mid A = 1)$	$90/310$	Conditional
$p(A = 0 \mid B = 0)$	$680/900$	Conditional

This time

- More on joint, marginal and conditional distributions
- When can we say that X, Y do not influence each other?
- What, if anything, does $p(X = x|Y = y)$ tell us about $p(Y = y|X = x)$?

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Philosophically related to “How do we know / learn about the world?”

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Philosophically related to "How do we know / learn about the world?"

I am *not* providing a general answer; but keep it in mind!

Outline

Document Modelling Example

Suppose we have a large document of English text, represented as a sequence of characters:

$$x_1 x_2 x_3 \ldots x_N$$

- e.g. `hello␣how␣are␣you`

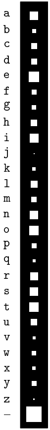
Treat each consecutive pair of characters as the outcome of "random variables" X, Y, i.e.

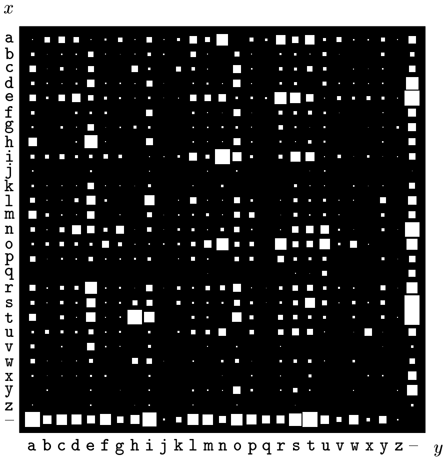
```
X = 'h', Y = 'e'

X = 'e', Y = 'l'

X = 'l', Y = 'l'
```

$$\vdots$$

Document Modelling: Marginal and Joint Distributions

i	a_i	p_i
1	a	0.0575
2	b	0.0128
3	c	0.0263
4	d	0.0285
5	e	0.0913
6	f	0.0173
7	g	0.0133
8	h	0.0313
9	i	0.0599
10	j	0.0006
11	k	0.0084
12	l	0.0335
13	m	0.0235
14	n	0.0596
15	o	0.0689
16	p	0.0192
17	q	0.0008
18	r	0.0508
19	s	0.0567
20	t	0.0706
21	u	0.0334
22	v	0.0069
23	w	0.0119
24	x	0.0073
25	y	0.0164
26	z	0.0007
27	–	0.1928



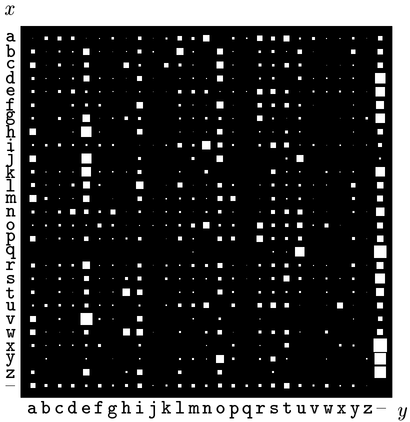


Unigram / Monogram

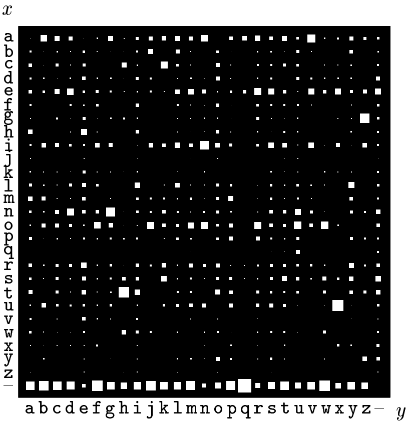
Bigram

Marginal and joint distributions for English alphabet, estimated from the "FAQ manual for Linux". Figure from Mackay (ITILA, 2003); areas of squares proportional to probability (the right way to do it!).

Document Modelling: Conditional Distributions



(a) $P(y \mid x)$



(b) $P(x \mid y)$

Conditional distributions for English alphabet, estimated from the "FAQ manual for Linux". Are these distributions "symmetric"? Figure from Mackay (ITILA, 2003)

$P(X = x|Y = y) = P(Y = y|X = x)$? $P(X = x|Y = y) = P(X = y|Y = x)$?.

Recap: Sum and Product Rules

Sum rule:

$$p(X = x_i) = \sum_j p(X = x_i, Y = y_j)$$

Product rule:

$$p(X = x_i, Y = y_j) = p(Y = y_j | X = x_i) p(X = x_i)$$

Relating the Marginal, Conditional and Joint

Suppose we knew $p(X = x, Y = y)$ for all values of x, y. Could we compute all of $p(X = x|Y = y)$, $p(X = x)$ and $p(Y = y)$?

Relating the Marginal, Conditional and Joint

Suppose we knew $p(X = x, Y = y)$ for all values of x, y. Could we compute all of $p(X = x|Y = y)$, $p(X = x)$ and $p(Y = y)$? Yes.

Relating the Marginal, Conditional and Joint

Suppose we knew $p(X = x, Y = y)$ for all values of x, y. Could we compute all of $p(X = x|Y = y)$, $p(X = x)$ and $p(Y = y)$? Yes.

Now suppose we knew $p(X = x)$ and $p(Y = y)$ for all values of x, y. Could we compute $p(X = x, Y = y)$ or $p(X = x|Y = y)$?

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The difference in answers above is of great significance

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The difference in answers above is of great significance

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	$B = 0$	$B = 1$
$A = 0$	640	50
$A = 1$	260	50

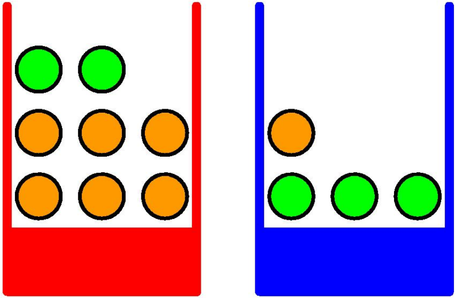
These have the same marginals, but different joint distributions

Joint as the "Master" Distribution

In general, there can be many consistent joint distributions for a given set of marginal distributions

The joint distribution is the "master" source of information about the dependence

Recall: Fruit-Box Experiment



Statistical Independence

Suppose that both boxes (red and blue) contain the same proportion of apples and oranges.

If fruit is selected uniformly at random from each box:

$$p(F = \mathtt{a}|B = \mathtt{r}) = p(F = \mathtt{a}|B = \mathtt{b}) \quad (= p(F = \mathtt{a}))$$
$$p(F = \mathtt{o}|B = \mathtt{r}) = p(F = \mathtt{o}|B = \mathtt{b}) \quad (= p(F = \mathtt{o}))$$

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The probability of selecting an apple (or an orange) is independent of the box that is chosen.

We may study the properties of F and B separately: this often simplifies analysis

Statistical Independence: Definition

Definition: Independent Variables

Two variables X and Y are statistically independent, denoted $X \perp\!\!\!\perp Y$, if and only if their joint distribution *factorizes* into the product of their marginals:

$$X \perp\!\!\!\perp Y \leftrightarrow p(X, Y) = p(X)p(Y)$$

This definition generalises to more than two variables.

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This definition generalises to more than two variables.

Are the variables in the language example statistically independent?

A Note on Notation

When we write

$$p(X, Y) = p(X)p(Y)$$

we have not specified the outcomes of X, Y explicitly

This statement is a shorthand for

$$p(X = x, Y = y) = p(X = x)p(Y = y)$$

for every possible x and y

This notation is sometimes called implied universality

Conditional independence

We may also consider random variables that are conditionally independent given some other variable

Definition: Conditionally Independent Variables

Two variables X and Y are conditionally independent given Z, denoted $X \perp\!\!\!\perp Y|Z$, if and only if

$$p(X, Y|Z) = p(X|Z)p(Y|Z)$$

Intuitively, Z is a common cause for X and Y

Example: X = whether I have a cold
Y = whether I have a headache
Z = whether I have the flu

Revisiting the Product Rule

The product rule tells us:

$$p(X, Y) = p(Y|X)p(X)$$

This can equivalently be interpreted as a *definition* of conditional probability:

$$p(Y|X) = \frac{p(X, Y)}{p(X)}$$

Can we use these to relate $p(X|Y)$ and $p(Y|X)$?

Posterior Inference:

Example 1 (Mackay, 2003)

- Dicksy Sick had a test for a rare disease
  - Only 1% people of Dicksy's background have the disease

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Example 1 (Mackay, 2003)

- Dicksy Sick had a test for a rare disease
  - Only 1% people of Dicksy's background have the disease
- The test simply classifies a person as having the disease, or not
- The test is reliable, but not infallible
  - It correctly identifies a sick individual 95% of the time
    $p(\text{identifies sick} \mid \text{sick}) = 95\%$.
  - It correctly identifies a healthy individual 96% of the time
    $p(\text{identifies healthy} \mid \text{healthy}) = 96\%$.

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  - It correctly identifies a healthy individual 96% of the time
    $p(\text{identifies healthy} \mid \text{healthy}) = 96\%$.
- Dicksy has tested positive (apparently sick)
- What is the probability of Dicksy having the disease?

Posterior Inference:

Example 1: Formalization

Let $D \in \{0, 1\}$ denote whether Dicksy has the disease, and $T \in \{0, 1\}$ the outcome of the test:

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Let $D \in \{0, 1\}$ denote whether Dicksy has the disease, and $T \in \{0, 1\}$ the outcome of the test:

$$
\begin{aligned}
p(D = 1) &= 0.01 & p(D = 0) &= 0.99 \\
p(T = 1|D = 1) &= 0.95 & p(T = 1|D = 0) &= 0.04 \\
p(T = 0|D = 1) &= 0.05 & p(T = 0|D = 0) &= 0.96
\end{aligned}
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p(T = 1|D = 1) &= 0.95 & p(T = 1|D = 0) &= 0.04 \\
p(T = 0|D = 1) &= 0.05 & p(T = 0|D = 0) &= 0.96
\end{aligned}
$$

We need to compute $p(D = 1|T = 1)$, *the probability of Dicksy having the disease given that the test has resulted positive.*

Posterior Inference:

Example 1: Solution

$$p(D = 1 | T = 1) = \frac{p(D = 1, T = 1)}{p(T = 1)}$$ Def. conditional prob.

Posterior Inference:

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$$
\begin{aligned}
p(D = 1 | T = 1) &= \frac{p(D = 1, T = 1)}{p(T = 1)} && \text{Def. conditional prob.} \\
&= \frac{p(T = 1, D = 1)}{p(T = 1)} && \text{Symmetry}
\end{aligned}
$$

Posterior Inference:

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&= \frac{p(T=1, D=1)}{p(T=1)} && \text{Symmetry} \\
&= \frac{p(T=1|D=1)p(D=1)}{p(T=1)} && \text{Product rule}
\end{aligned}
$$

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&= \frac{p(T=1|D=1)p(D=1)}{\sum_d p(T=1|D=d)p(D=d)} && \text{Sum rule}
\end{aligned}
$$

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Example 1: Solution

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&= \frac{p(T=1|D=1)p(D=1)}{\sum_d p(T=1|D=d)p(D=d)} && \text{Sum rule} \\
&= \frac{p(T=1|D=1)p(D=1)}{p(T=1|D=1)p(D=1)+p(T=1|D=0)p(D=0)}
\end{aligned}
$$

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Example 1: Solution

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&= \frac{p(T=1|D=1)p(D=1)}{\sum_d p(T=1|D=d)p(D=d)} && \text{Sum rule}\\
&= \frac{p(T=1|D=1)p(D=1)}{p(T=1|D=1)p(D=1)+p(T=1|D=0)p(D=0)}\\
&\approx 0.19.
\end{aligned}
$$

Posterior Inference:

Example 1: Solution

$$
\begin{aligned}
p(D=1|T=1) &= \frac{p(D=1,T=1)}{p(T=1)} && \text{Def. conditional prob.} \\
&= \frac{p(T=1,D=1)}{p(T=1)} && \text{Symmetry} \\
&= \frac{p(T=1|D=1)p(D=1)}{p(T=1)} && \text{Product rule} \\
&= \frac{p(T=1|D=1)p(D=1)}{\sum_d p(T=1|D=d)p(D=d)} && \text{Sum rule} \\
&= \frac{p(T=1|D=1)p(D=1)}{p(T=1|D=1)p(D=1)+p(T=1|D=0)p(D=0)} \\
&\approx 0.19.
\end{aligned}
$$

Despite testing positive and the high accuracy of the test, the probability of Dicksy having the disease is only 0.19!

Why is the Probability So Low?

A “Natural Frequency” Approach

In 100 people, only 1 is expected to have the disease ($p(D = 1) = 0.01$)

This sick person will most likely test positive ($p(T = 1|D = 1) = 0.95$)

But around 4 healthy people are expected to be wrongly flagged as sick ($p(T = 1|D = 0) = 0.04$, and $0.04 \times 99 \approx 4$)

So when the test is positive, the chance of being sick is $\approx 1/5$

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(Aside: If you can correctly perform the calculation on the previous slide, you are doing better than most medical doctors! See Gird Gigerenzer and Adrian Edwards, Simple tools for understanding risks: from innumeracy to insight, *British Medical Journal*, 327(7417), 741–744, 27 September 2003; Gird Gigerenzer, *Reckoning with risk: Learning to live with uncertainty*, Penguin, 2002.

Moral of the story — if you get sick, don’t delegate conditional probability computations to your doctor!)

Bayes' Theorem

We have implicitly used the following (at first glance remarkable) fact:

Bayes' Theorem:

$$
\begin{aligned}
p(Z|X) &= \frac{p(Z,X)}{p(X)} \\
&= \frac{p(X,Z)}{p(X)} \\
&= \frac{p(X|Z)p(Z)}{p(X)} \\
&= \frac{p(X|Z)p(Z)}{\sum_{Z'} p(X|Z')p(Z')}
\end{aligned}
$$

If we can express what knowledge of X (test) tells us about Z (disease), then we can express what knowledge of Z tells us about X

The Bayesian Inference Framework

Bayesian Inference

Bayesian inference provides a mathematical framework explaining how to change our (prior) beliefs in the light of new evidence.

$$\underbrace{p(Z|X)}_{\text{posterior}} = \frac{\overbrace{p(X|Z)}^{\text{likelihood}} \times \overbrace{p(Z)}^{\text{prior}}}{\underbrace{p(X)}_{\text{evidence}}}$$

Prior: Belief that someone is sick

Likelihood: Probability of testing positive given you are sick

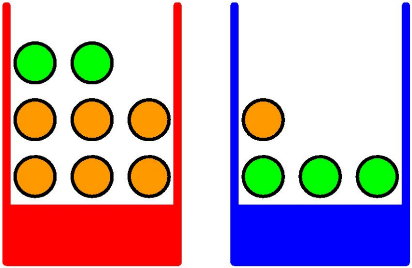
Posterior: Probability of being sick given you test positive

Posterior Inference:

Example 2 (Bishop, 2006)

Recall our fruit-box example:

- The proportion of oranges and apples are given by



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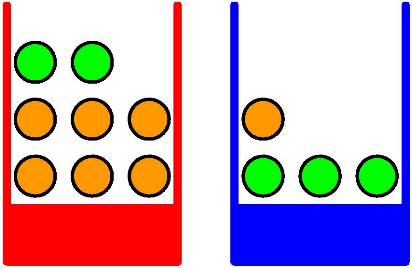

- Someone told us that in a previous experiment they ended up picking up the red box 40% of the time and the blue box 60% of the time.

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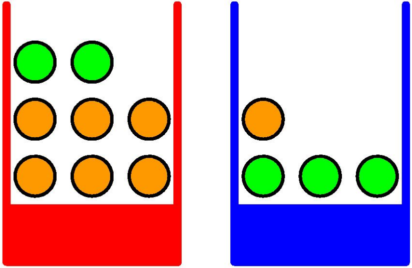

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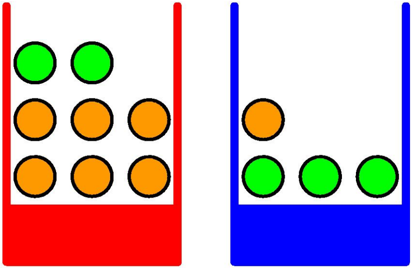

- Someone told us that in a previous experiment they ended up picking up the red box 40% of the time and the blue box 60% of the time.
- A piece of fruit has been picked up and it turned out to be an orange.
- What is the probability that it came from the red box?

Posterior Inference:

Example 2: Formalization

Let $B \in \{r, b\}$ denote the selected box and $F \in \{a, o\}$ the selected fruit.

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$$
\begin{aligned}
p(B = r) &= 4/10 & p(B = b) &= 6/10 \\
p(F = a|B = r) &= 1/4 & p(F = o|B = r) &= 3/4 \\
p(F = a|B = b) &= 3/4 & p(F = o|B = b) &= 1/4
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We need to compute $p(B = r|F = o)$, *the probability that a picked up orange came from the red box.*

Posterior Inference:

Example 2: Solution

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Posterior Inference:

Example 2: Solution

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$$
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p(B = r|F = o) &= \frac{p(F = o|B = r)p(B = r)}{p(F = o)} \\
&= \frac{p(F = o|B = r)p(B = r)}{p(F = o|B = r)p(B = r) + p(F = o|B = b)p(B = b)}
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&= \frac{2}{3}
\end{aligned}
$$

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$$\begin{aligned} p(B=r|F=o) &= \frac{p(F=o|B=r)p(B=r)}{p(F=o)} \\ &= \frac{p(F=o|B=r)p(B=r)}{p(F=o|B=r)p(B=r)+p(F=o|B=b)p(B=b)} \\ &= \frac{2}{3} \end{aligned}$$

and therefore $p(B=b|F=o)=1/3$.

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- Once we get new information that an orange has been picked, this increases the probability of the selected box being the red one
  - *Because the red box contains more oranges than the blue box*
- In fact, the proportion of oranges is so much higher in the red box that this is strong evidence that the orange came from it
  - *So after picking up the orange the red box is much more likely to have been selected than the blue one*

Summary

- Recap on joint, marginal and conditional distributions
- Interpretation of conditional probability
- Statistical Independence
- Bayes rule: combination of prior, likelihood to get a posterior
- Reading: Mackay § 2.1, § 2.2 and § 2.3

Homework Exercise

Suppose we know that random variables X, Y satisfy

$$p(X|Y) = p(Y|X)$$

What can you conclude about the relationship between X and Y?

If X and Y are independent, does that imply $p(X|Y) = p(Y|X)$?

Repeat the above questions for the statement

$$\frac{p(X|Y)}{p(Y|X)} = \frac{p(X)}{p(Y)}$$

Next time

- More examples on Bayes' theorem:
  - Eating hamburgers
  - Detecting terrorists
  - The Monty Hall problem
  - Document modelling
- Are there notions of probability beyond frequency counting?