COMP2610 / COMP6261 — Information Theory: Assignment 0

Australian National University

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Out: 30 July 2018. **Due**: 6 August 2018 by 5 PM

Instructions

This assignment is designed to test your background knowledge in elementary probability theory for the course. It does not count in the assessment of the course in any way. And it is not obligatory. It will be marked, so you get early feedback as to how well prepared you are for the course.

Answer all questions, showing all your working. Incomplete or unclear working may result in points deducted, regardless of whether the final answer is correct. That is, it will be marked in the same style as the two assignments that do contribute to your assessment.

Submission: This is an individual assignment. You should submit a printed version of your assignment to the RSCS asignment submission box. A printed submission make sit easier for the tutors to provide feedback. Make sure your uniID is clearly written on the paper.

Handwritten submissions are acceptable if the handwriting is neat and legible. If you have doubts, or you want practice typing maths, you should type it. Typing can be in LATEX (very useful skill to learn — it is the defacto format used for typing maths; there is a template provided on Wattle) or Microsoft Word - the newer versions have a form of LateX built into the equation editor.

Cheating and Plagiarism: I do hope it obvious to all that cheating on this assignment is utterly pointless since its sole purpose is to provide you with feedback on how well placed you are to do the rest of the course since its sole purpose is to provide you with feedback on how well placed you are to do the rest of the course. Cheating on this exercise will literally only cause you harm.

Furthermore, for assessable items *plagiarism is a university offence* and will be dealt with according to university procedures. Please refer to the the corresponding ANU policies: `http://academichonesty.anu.edu.au/UniPolicy.html`.

LATEX primer: `http://ctan.mackichan.com/info/lshort/english/lshort.pdf` (Chapter 3)

1. [8 points] Two fair, six-sided die are rolled. Compute the probability that the sum of the outcomes of the two rolls is:

   (a) (3 pt) Equal to 1

   (b) (3 pt) Equal to 4

   (c) (2 pt) Less than 13

2. [15 points] Suppose that X, Y are discrete random variables with joint probability

	$Y = -1$	$Y = 0$	$Y = 1$
$X = 0$	0	$\frac{1}{3}$	0
$X = 1$	$\frac{1}{3}$	0	$\frac{1}{3}$

   (a) (3 pt) Compute the conditional distributions $p(X|Y = y)$ for all possible values of y.

   (b) (2 pt) Compute the conditional distributions $p(Y|X = x)$ for all possible values of x.

   (c) (3 pt) Compute $E[X|Y = y]$ for all values of y.

   (d) (2 pt) Plot $E[X|Y = y]$ against y.

   (e) (5 pt) Are X and Y dependent? If yes, explain why. If no, how do you think X and Y are related?

3. [20 points] You have a large jar containing 999 fair coins and one two-headed coin (i.e. a coin that is guaranteed to come up heads). Suppose you pick one coin at random out of the jar and flip it 10 times.

   (a) (2 pt) What is the probability you selected a fair coin?

   (b) (2 pt) What is the probability of the first flip being heads, assuming you selected a fair coin?

   (c) (2 pt) What is the probability of the first flip being heads, assuming you selected the two-headed coin?

   (d) (3 pt) What is the overall probability of the first flip being heads?

   (e) (3 pt) What is the probability of the first *and* second flip being heads, assuming you selected a fair coin?

   (f) (8 pt) What is the probability you selected a fair coin if *all* ten flips turn up heads?

4. [32 points] A kidnapping recently occurred in the city of Twin Peaks. Dale, an FBI agent, has been assigned to catch the person who did it. He has determined that the guilty person must be one of the 1,000 members of a club called the White Lodge.

   (a) (2 pt) Leland is a member of the White Lodge. Based on only this information, what is the probability that Leland is guilty?

   (b) (6 pt) Dale has an expensive DNA test that he can use to help his investigation. Suppose the probability of a DNA match given that a person is innocent is 0.1%, and the probability of a DNA match given that a person is guilty is 99%. What is the probability of a DNA match for a randomly chosen member of the White Lodge?

(c) (9 pt) Dale has found DNA evidence at the scene of the kidnapping. An analysis of the DNA matches a sample that Leland provides. What is the probability that Leland is guilty given the positive outcome of the DNA test? Explain intuitively why this number is higher or lower than that of part (a).

(d) (12 pt) Bob is trying to sell Dale a lie detector machine that he claims is very accurate. He says: *"If my machine starts beeping, you can be 99% sure the person really is guilty"*. What must be the relative fraction of times the machine starts beeping for a guilty person, over the times it starts beeping for an innocent person?

(e) (3 pt) Given just the above information, is it possible to determine the fraction of times that Bob's machine starts beeping for a guilty person? If yes, compute the fraction. If no, explain why not.

5. [25 points] Suppose that X is a random variable with values $\{0, 1\}$, and $p(X = 1) = \theta$. In the following, express your answer in terms of θ.

(a) (3 pt) Define and calculate the expectation of X.

(b) (4 pt) Define and calculate the variance of X.

(c) (6 pt) Calculate the quantity $\phi(t) := E[e^{tX}]$ for a fixed $t > 0$.

(d) (6 pt) Obtain an expression for the derivative $\phi'(t)$. Explain the relation of $\phi'(0)$ to the result of (a).

(e) (6 pt) Obtain an expression for the second derivative $\phi''(t)$. Explain the relation of $\phi''(0)$ to the result of (b).