

COMP2610/COMP6261

Tutorial 9 Sample Solutions

Tutorial 9: Stream and Noisy Channel Coding

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1. Solution in Tutorial 8.

2. (a) We have $p(Y = \mathbf{a}) = 0.5 \cdot 0.25$, $p(Y = \mathbf{b}) = 0.5 \cdot 0.5$, $p(Y = \mathbf{c}) = 0.5 \cdot 0.75$, and $p(Y = \mathbf{d}) = 0.5 \cdot 0.5$. Thus $H(Y) = 1.9056$. Similarly, $H(Y | \mathbf{a}) = H(Y | \mathbf{b}) = H(Y | \mathbf{c}) = H(0.5) = 1$. So, $H(Y | X) = 1$. Thus, $I(X; Y) = 0.9056$ bits.

(b) We have

$$\begin{aligned} p_{\text{err}} &= p(y \neq \mathbf{a}) \cdot p(x = \mathbf{a}) + p(y \neq \mathbf{b}) \cdot p(x = \mathbf{b}) + p(y \neq \mathbf{c}) \cdot p(x = \mathbf{c}) \\ &= 0.5 \cdot 0.25 + 0.5 \cdot 0.25 + 0.5 \cdot 0.5 \\ &= 0.5. \end{aligned}$$

(c) We know that the capacity is the maximal mutual information. So, from (a), the capacity is at least 0.9056 bits. We know that we can transmit at rate below the capacity with arbitrarily small error probability. Thus, Calvin's claim is possible.

(d) We know that the capacity is at most $\min(\log_2 |\mathcal{X}|, \log_2 |\mathcal{Y}|) = 1.58$. We know that we cannot transmit at rate above $\frac{C}{1-H_2(p_b)}$ while achieving bit error p_b . Here, the maximal block error is 0.01, which bounds the average probability of error, and hence the bit error. Here $1 - H_2(0.1) = 0.9192$, and so the maximal rate is $\frac{1.58}{0.9192} = 1.7189$ bits per transmission. Thus, Hobbes' claim is not possible.

3. Using $I(X; Y) = H(X) - H(X|Y)$ it is easy to show that

$$I(X; Y) = H_2(p_0) - qH_2(p_0)$$

since the first term is $H(X)$ by definition of H_2 . The second term is $H(X|Y)$ because $P(X = 0|Y = 1) = 0$ and $P(X = 1|Y = 0) = 0$ and $P(X = 0|Y = ?) = p_0$ and $P(X = 1|Y = ?) = p_1$ and $P(Y = ?) = q$. Using $H(X|Y) = \sum_y H(X|Y = y)P(Y = y) = 0 \cdot P(Y = 0) + 0 \cdot P(Y = 1) + H_2(p_0) \cdot P(Y = ?)$ gives the above form for $H(X|Y)$.

When maximised over p_0 the above gives

$$C = \max_{p_x} I(X; Y) = \max_{p_0} H_2(p_0) - q \cdot H_2(p_0) = 1 - q$$

since $H_2(p_0)$ is maximised to 1 when $p_0 = 0.5$.

The $(2, 1)$ code required is $\{01, 10\}$. When 01 is passed through the Z -channel there is a probability of q that the 1 is flipped and so 00 is output. Similarly, if 10 is input then there is a probability q that 00 is output. By mapping 01 to 0; 10 to 1; and 00 to ? we see that this code simulates the erasure channel.

Because we can simulate the erasure channel with two uses of the Z -channel we can always convert any error-correcting code for the erasure channel into an error-correcting code for a two-use Z -channel. Therefore, a code with rate R on the erasure channel will have rate $R/2$ on the Z -channel.

Since the Noisy-Channel Coding Theorem says any rate up to the capacity C is achievable and we have shown how to convert any rate R code for the erasure channel into a $R/2$ rate code for the Z -channel the capacity of the Z -channel must be at least $C/2$. Thus, the capacity of the Z -channel with flip probability q is at least $\frac{1}{2}(1 - q)$.

The reason this is a bound and not an exact derivation of the capacity is that we have implicitly used the uniform distribution (i.e., the one that maximises the erasure mutual information). This may not be the same distribution that maximises the mutual information for the Z -channel.