COMP2610 / COMP6261 - Information Theory

Lecture 7: Relative Entropy and Mutual Information

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Last time

- Information content and entropy: definition and computation
- Entropy and average code length
- Entropy and minimum expected number of binary questions
- Joint and conditional entropies, chain rule

Information Content: Review

Let X be a random variable with outcomes in $\mathcal{X}$

Let $p(x)$ denote the probability of the outcome $x \in \mathcal{X}$

The (Shannon) information content of outcome x is

$$h(x) = \log_2 \frac{1}{p(x)}$$

As $p(x) \to 0$, $h(x) \to +\infty$ (rare outcomes are more informative)

Entropy: Review

The entropy is the average information content of all outcomes:

$$H(X) = \sum_x p(x) \log_2 \frac{1}{p(x)}$$

Entropy is minimised if **p** is peaked, and maximized if **p** is uniform:

$$0 \leq H(X) \leq \log|\mathcal{X}|$$

Entropy is related to minimal number of bits needed to describe a random variable

This time

- The decomposability property of entropy
- Relative entropy and divergences
- Mutual information

Outline

Decomposability of Entropy

Example 1 (Mackay, 2003)

Let $X \in \{1, 2, 3\}$ be a r.v. created by the following process:

1. Flip a fair coin to determine whether $X = 1$
2. If $X \neq 1$ flip another fair coin to determine whether $X = 2$ or $X = 3$

The probability distribution of X is given by:

$$\begin{aligned} p(X = 1) = &\quad \tfrac{1}{2} \\ p(X = 2) = &\quad \tfrac{1}{4} \\ p(X = 3) = &\quad \tfrac{1}{4} \end{aligned}$$

Decomposability of Entropy

Example 1 (Mackay, 2003) — Cont'd

By definition,

$$H(X) = \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4 = 1.5 \text{ bits.}$$

But imagine learning the value of X *gradually*:

1. First we learn whether $X = 1$:
   - Binary variable with $\mathbf{p}^{(1)} = (\frac{1}{2}, \frac{1}{2})$
   - Hence $H(1/2, 1/2) = \log_2 2 = 1$ bit.
2. If $X \neq 1$ we learn the value of the second coin flip:
   - Also binary variable with $\mathbf{p}^{(2)} = (\frac{1}{2}, \frac{1}{2})$
   - Therefore $H(1/2, 1/2) = 1$ bit.

However, the second revelation only happens half of the time:

$$H(X) = H(1/2, 1/2) + \frac{1}{2}H(1/2, 1/2) = 1.5 \text{ bits.}$$

Decomposability of Entropy

Generalization

For a r.v. with probability distribution $\mathbf{p} = (p_1, \ldots, p_{|\mathcal{X}|})$:

$$H(\mathbf{p}) = H(p_1, 1 - p_1) + (1 - p_1) H\left(\frac{p_2}{1 - p_1}, \ldots, \frac{p_{|\mathcal{X}|}}{1 - p_1}\right)$$

$H(p_1, 1 - p_1)$: entropy for a random variable corresponding to “Is $X = 1$?”

$1 - p_1$: probability of $X \neq 1$

$\frac{p_2}{1-p_1}, \ldots, \frac{p_{|\mathcal{X}|}}{1-p_1}$: conditional probability of $X = 2, \ldots, |\mathcal{X}|$ given $X \neq 1$.

$H\left(\frac{p_2}{1-p_1}, \ldots, \frac{p_{|\mathcal{X}|}}{1-p_1}\right)$: entropy for a random variable corresponding to outcomes when $X \neq 1$.

Decomposability of Entropy

Generalization

In general, we have that for any m between 1 and $|\mathcal{X}| - 1$:

$$
\begin{aligned}
H(\mathbf{p}) = & H\left(\sum_{i=1}^{m} p_i, \sum_{i=m+1}^{|\mathcal{X}|} p_i\right) \\
& + \left(\sum_{i=1}^{m} p_i\right) H\left(\frac{p_1}{\sum_{i=1}^{m} p_i}, \ldots, \frac{p_m}{\sum_{i=1}^{m} p_i}\right) \\
& + \left(\sum_{i=m+1}^{|\mathcal{X}|} p_i\right) H\left(\frac{p_{m+1}}{\sum_{i=m+1}^{|\mathcal{X}|} p_i}, \ldots, \frac{p_{|\mathcal{X}|}}{\sum_{i=m+1}^{|\mathcal{X}|} p_i}\right)
\end{aligned}
$$

Apply this formula with $m = 1$, $|\mathcal{X}| = 3$, $\mathbf{p} = (p_1, p_2, p_3) = (1/2, 1/4, 1/4)$

Entropy in Information Theory

If a random variable has distribution p, there exists an encoding with an average length of

$$H(p) \text{ bits}$$

and this is the “best” possible encoding

What happens if we use a “wrong” encoding?

- e.g. because we make an incorrect assumption on the probability distribution

If the true distribution is p, but we assume it is q, it turns out we will need to use

$$H(p) + D_{\mathrm{KL}}(p \| q) \text{ bits}$$

where $D_{\mathrm{KL}}(p \| q)$ is some measure of “distance” between p and q

Relative Entropy

Definition

The relative entropy or Kullback-Leibler (KL) divergence between two probability distributions $p(X)$ and $q(X)$ is defined as:

$$
\begin{aligned}
D_{\mathrm{KL}}(p \| q) &= \sum_{x \in \mathcal{X}} p(x) \left(\log \frac{1}{q(x)} - \log \frac{1}{p(x)} \right) \\
&= \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)} = \mathbb{E}_{p(X)} \left[\log \frac{p(X)}{q(X)} \right].
\end{aligned}
$$

- Note:
  - Both $p(X)$ and $q(X)$ are defined over the same alphabet $\mathcal{X}$
- Conventions on log likelihood ratio:

$$
0 \log \frac{0}{0} \stackrel{\text{def}}{=} 0 \qquad 0 \log \frac{0}{q} \stackrel{\text{def}}{=} 0 \qquad p \log \frac{p}{0} \stackrel{\text{def}}{=} \infty
$$

Relative Entropy

Properties

- $D_{\mathrm{KL}}(p\|q) \geq 0$ (proof next lecture)
- $D_{\mathrm{KL}}(p\|q) = 0 \Leftrightarrow p = q$ (proof next lecture)
- Not symmetric: $D_{\mathrm{KL}}(p\|q) \neq D_{\mathrm{KL}}(q\|p)$
- Not satisfy triangle inequality: $D_{\mathrm{KL}}(p\|q) \neq D_{\mathrm{KL}}(p\|r) + D_{\mathrm{KL}}(r\|q)$
  - Not a true distance since is not symmetric and does not satisfy the triangle inequality
  - Hence, “KL divergence” rather than “KL distance”

Relative Entropy

Uniform q

Let q correspond to a uniform distribution: $q(x) = \frac{1}{|\mathcal{X}|}$

Relative entropy between p and q:

$$\begin{aligned} D_{\mathrm{KL}}(p \| q) &= \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)} \\ &= \sum_{x \in \mathcal{X}} p(x) \cdot (\log p(x) + \log |\mathcal{X}|) \\ &= -H(X) + \sum_{x \in \mathcal{X}} p(x) \cdot \log |\mathcal{X}| \\ &= -H(X) + \log |\mathcal{X}|. \end{aligned}$$

Matches intuition as penalty on number of bits for encoding

Relative Entropy

Example (from Cover & Thomas, 2006)

Let $X \in \{0, 1\}$ and consider the distributions $p(X)$ and $q(X)$ such that:

$$p(X=1) = \theta_p \qquad p(X=0) = 1 - \theta_p$$
$$q(X=1) = \theta_q \qquad q(X=0) = 1 - \theta_q$$

What distributions are these?

Compute $D_{\mathrm{KL}}(p\|q)$ and $D_{\mathrm{KL}}(q\|p)$ with $\theta_p = \frac{1}{2}$ and $\theta_q = \frac{1}{4}$

Relative Entropy

Example (from Cover & Thomas, 2006) — Cont'd

$$
\begin{aligned}
D_{\mathrm{KL}}(p\|q) &= \theta_p \log \frac{\theta_p}{\theta_q} + (1-\theta_p) \log \frac{1-\theta_p}{1-\theta_q} \\
&= \frac{1}{2} \log \frac{\frac{1}{2}}{\frac{1}{4}} + \frac{1}{2} \log \frac{\frac{1}{2}}{\frac{3}{4}} = 1 - \frac{1}{2} \log 3 \approx 0.2075 \text{ bits}
\end{aligned}
$$

$$
\begin{aligned}
D_{\mathrm{KL}}(q\|p) &= \theta_q \log \frac{\theta_q}{\theta_p} + (1-\theta_q) \log \frac{1-\theta_q}{1-\theta_p} \\
&= \frac{1}{4} \log \frac{\frac{1}{4}}{\frac{1}{2}} + \frac{3}{4} \log \frac{\frac{3}{4}}{\frac{1}{2}} = -1 + \frac{3}{4} \log 3 \approx 0.1887 \text{ bits}
\end{aligned}
$$

Mutual Information

Definition

Let X, Y be two r.v. with joint distribution $p(X, Y)$ and marginals $p(X)$ and $p(Y)$:

Definition

The *mutual information* $I(X; Y)$ is the relative entropy between the joint distribution $p(X, Y)$ and the product distribution $p(X)p(Y)$:

$$\begin{aligned} I(X; Y) &= D_{\mathrm{KL}}\left(p(X, Y) \| p(X)p(Y)\right) \\ &= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log \frac{p(x, y)}{p(x)p(y)} \end{aligned}$$

Non-negativity: $I(X; Y) \geq 0$
Symmetry: $I(Y; X) = I(X; Y)$

Intuitively, how much information, on average, X conveys about Y.

Relationship between Entropy and Mutual Information

We can re-write the definition of mutual information as:

$$
\begin{aligned}
I(X;Y) &= \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log\frac{p(x,y)}{p(x)p(y)} \\
&= \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log\frac{p(x|y)}{p(x)} \\
&= -\sum_{x\in\mathcal{X}}\log p(x)\sum_{y\in\mathcal{Y}} p(x,y) - \left(-\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(x|y)\right) \\
&= H(X) - H(X|Y)
\end{aligned}
$$

The average reduction in uncertainty of X due to the knowledge of Y.

Self-information: $I(X;X) = H(X) - H(X|X) = H(X)$

Mutual Information:

Properties

- Mutual Information is non-negative:

$$I(X;Y) \geq 0$$

- Mutual Information is symmetric:

$$I(X;Y) = I(Y;X)$$

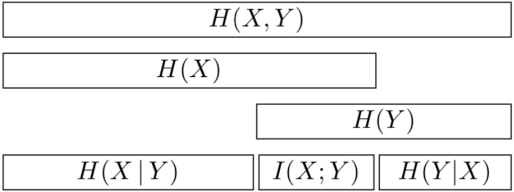
- Self-information:

$$I(X;X) = H(X)$$

- Since $H(X,Y) = H(Y) + H(X|Y)$ we have that:

$$I(X;Y) = H(X) - H(X|Y) = H(X) + H(Y) - H(X,Y)$$

Breakdown of Joint Entropy



(From Mackay, p140; see his exercise 8.8)

Mutual Information

Example 1 (from Mackay, 2003)

Let X, Y, Z be r.v. with $X, Y \in \{0, 1\}$, $X \perp\!\!\!\perp Y$ and:

$$
\begin{aligned}
p(X = 0) = p \qquad & p(X = 1) = 1 - p \\
p(Y = 0) = q \qquad & p(Y = 1) = 1 - q
\end{aligned}
$$

$$Z = (X + Y) \mod 2$$

(a) if $q = 1/2$ what is $P(Z = 0)$? $P(Z = 1)$? $I(Z; X)$?

(b) For general p and q what is $P(Z = 0)$? $P(Z = 1)$? $I(Z; X)$?

Mutual Information

Example 1 (from Mackay, 2003) — Solution (a)

(a) As $X \perp\!\!\!\perp Y$ and $q = 1/2$ the noise will flip the outcome of X with probability $q = 0.5$ regardless of the outcome of X. Therefore:

$$p(Z = 1) = 1/2 \qquad p(Z = 0) = 1/2$$

Hence:

$$I(X; Z) = H(Z) - H(Z|X) = 1 - 1 = 0$$

Indeed for $q = 1/2$ we see that $Z \perp\!\!\!\perp X$

Mutual Information

Example 1 (from Mackay, 2003) — Solution (b)

(b)

$$
\begin{aligned}
\ell \stackrel{\text{def}}{=} p(Z=0) &= p(X=0) \times p(\text{no flip}) + p(X=1) \times p(\text{flip}) \\
&= pq + (1-p)(1-q) \\
&= 1 + 2pq - q - p
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
p(Z=1) &= p(X=1) \times p(\text{no flip}) + p(X=0) \times p(\text{flip}) \\
&= (1-p)q + p(1-q) \\
&= q + p - 2pq
\end{aligned}
$$

and:

$$
\begin{aligned}
I(Z;X) &= H(Z) - H(Z|X) \\
&= H(\ell, 1-\ell) - H(q, 1-q) \quad \text{why?}
\end{aligned}
$$

Joint Mutual Information

Recall that for random variables X, Y,

$$I(X;Y) = H(X) - H(X|Y)$$

- Reduction in uncertainty in X due to knowledge of Y

More generally, for random variables $X_1, \dots, X_n, Y_1, \dots, Y_m$,

$$I(X_1, \dots, X_n; Y_1, \dots, Y_m) = H(X_1, \dots, X_n) - H(X_1, \dots, X_n | Y_1, \dots, Y_m)$$

- Reduction in uncertainty in $X_1, \dots, X_n$ due to knowledge of $Y_1, \dots, Y_m$

Symmetry also generalises:

$$I(X_1, \dots, X_n; Y_1, \dots, Y_m) = I(Y_1, \dots, Y_m; X_1, \dots, X_n)$$

Conditional Mutual Information

The conditional mutual information between X and Y given $Z = z_k$:

$$I(X; Y|Z = z_k) = H(X|Z = z_k) - H(X|Y, Z = z_k).$$

Averaging over Z we obtain:

The conditional mutual information between X and Y given Z:

$$\begin{aligned} I(X; Y|Z) &= H(X|Z) - H(X|Y, Z) \\ &= \mathbb{E}_{p(X,Y,Z)} \log \frac{p(X, Y|Z)}{p(X|Z)p(Y|Z)} \end{aligned}$$

The reduction in the uncertainty of X due to the knowledge of Y when Z is given.

Note that $I(X; Y; Z)$, $I(X|Y; Z)$ are illegal terms while e.g. $I(A, B; C, D|E, F)$ is legal.

Summary

- Decomposability of entropy
- Relative entropy
- Mutual information
- Reading: Mackay §2.5, Ch 8; Cover & Thomas §2.3 to §2.5

Next time

Mutual information chain rule

Jensen's inequality

"Information cannot hurt"

Data processing inequality