CS 294-115 Algorithmic Human-Robot Interaction **Fall 2016**

Lecture 6: Trajectory Optimization II

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Today's Outline

- Problem Statement
- Functional Gradient Descent
- CHOMP: Covariant Hamiltonian Optimization for Motion Planning
- Next Lecture: Non-Euclidean Inner Product

6.1 Problem Statement

- Trajectory (function) $\xi : [0, T] \to \mathbb{C}$
- Cost (functional) $U : \Xi \to \mathbb{R}^+$
- Optimization Problem:

$$\xi^* = \arg\min_{\xi \in \Xi} U(\xi)$$

  subject to

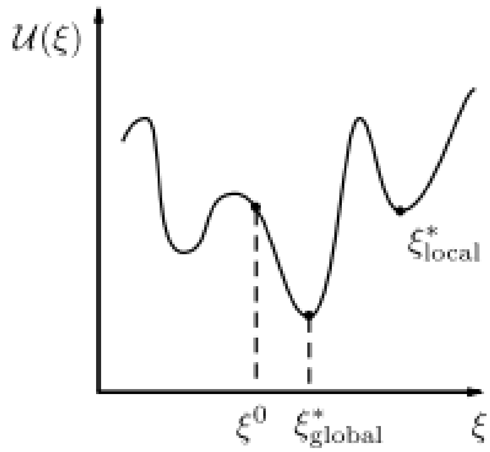
$$\xi(0) = q_s,$$

$$\xi(T) = q_g$$

- Update Equation (functional gradient descent):

$$\xi_{i+1} \leftarrow \xi_i - \frac{1}{\alpha} \nabla_{\xi} U(\xi_i)$$

  $\nabla_{\xi} U(\xi_i)$: also a function of time



- Ξ is a Hilbert space, a complete vector space with an inner product.
- Inner product:
  For this lecture, we assume a particular Hilbert space specified by the Euclidean inner product. Given two trajectories $\xi_1, \xi_2 \in \Xi$, the Euclidean inner product is

  $$< \xi_1, \xi_2 >= \int_0^T \xi_1(t)^T \xi_2(t) dt$$

  In discrete time, $\xi = [q_1, ..., q_N]^T$, thus $< \xi_1, \xi_2 >= \xi_1^T \xi_2$.
  Properties of inner products:
  Symmetry:

  $$< \xi_1, \xi_2 >=< \xi_2, \xi_1 >$$

  Positive definite:

  $$\forall \xi, < \xi, \xi > \geq 0; \ < \xi, \xi >= 0 \iff \xi = 0$$

  ($\xi = 0$ is the zero trajectory that always maps time to the zero configuration.)
  Linearity in the first argument:

  $$< \xi_1 + \xi_2, \xi_3 >=< \xi_1, \xi_3 > + < \xi_2, \xi_3 >$$

  (The same holds true for the second argument by *symmetry*.)

6.2 Functional Gradient Descent

We use calculus of variation in computing the derivatives of a functional.

- Euler-Lagrange Equation:
  If

  $$< \xi_1, \xi_2 >= \int_0^T \xi_1(t)^T \xi_2(t) dt$$

(i.e. Ξ is a Hilbert space with the Euclidean inner product)
and

$$U[\xi] = \int_0^T F(t, \xi(t), \xi'(t))dt$$

then

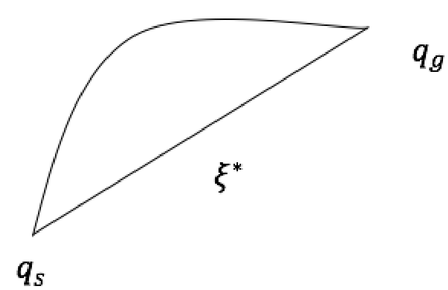
$$\nabla_\xi U(t) = \frac{\partial F}{\partial \xi(t)}(t) - \frac{d}{dt}\frac{\partial F}{\partial \xi'(t)}(t)$$

Note that $\nabla_\xi U(t) \in \Xi$

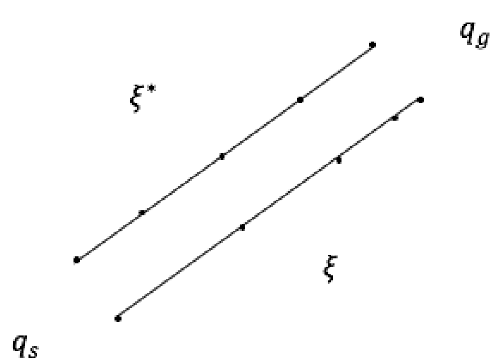
- Example:
  Consider the example where you minimize the squared norm of velocity in trajectory subject to starting at q_s and ending at q_g:

  $$U[\xi] = \frac{1}{2}\int_0^T \|\xi'(t)\|^2 dt$$

  The optimal trajectory has
  Shape: straight line. Intuitively, in the same amount time, a trajectory traversing a longer path needs a faster velocity, thus has higher cost.



Timing: constant velocity. Intuitively, in discrete time with T time steps, $U[\xi^*] < U[\xi]$



$$F(t, \xi(t), \xi'(t)) = \frac{1}{2}\|\xi'(t)\|^2$$

Apply Euler-Lagrange equation,

$$\nabla_\xi U(t) = 0 - \frac{d}{dt}\xi'(t) = -\xi''(t)$$

Since $\mathcal{U}$ is quadratic/convex, to find ξ that minimizes *cost* U, we set the gradient to 0, and solve for ξ^*, which is global minimum.

$$\xi''(t) = 0$$

$$\xi'(t) = a$$

shows that optimal trajectory has constant velocity.

$$\xi(t) = at + b$$

shows that optimal trajectory is a straight line.
Then to solve for a and b in ξ^*, we use constraints $\xi(0) = q_s$ and $\xi(T) = q_g$.

- Proof:
  First order Taylor series expansion (relates f to f')
  $f : \mathbb{R} \to \mathbb{R}$,

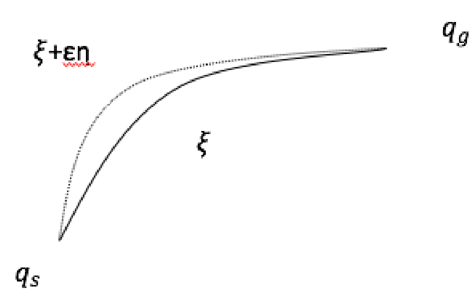
$$f(x+\epsilon) \approx f(x) + \epsilon f'(x)$$

$$f'(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

$U : \Xi \to \mathbb{R}^+$,

$$U[\xi + \epsilon\eta] \approx U[\xi] + \epsilon < \nabla_\xi U, \eta >$$

smooth disturbance $\eta \in \Xi$ s.t. $\eta(0) = \eta(T) = 0$



arbitrary small $\epsilon \in \mathbb{R}$

$$< \nabla_\xi U, \eta >= \lim_{\epsilon \to 0} \frac{U[\xi + \epsilon\eta] - U[\xi]}{\epsilon} \quad (1)$$

$$< \nabla_\xi U, \eta >= \int_0^T \nabla_\xi U(t)^T \eta(t) dt \quad (2)$$

We are going to massage equation (1) to equation (2) and term match to find $\nabla_\xi U$.
Let $\phi(\epsilon) = U[\xi + \epsilon\eta]$,

$$< \nabla_\xi U, \eta >= \lim_{\epsilon \to 0} \frac{\phi(\epsilon) - \phi(0)}{\epsilon}$$

$$= \frac{d\phi}{d\epsilon}\Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\int_0^T F[t, \xi(t) + \epsilon\eta(t), \xi'(t) + \epsilon\eta'(t)]dt\Big|_{\epsilon=0}$$

Exchange differentiation with integration,

$$= \int_0^T \frac{d}{d\epsilon} F[t, \xi(t) + \epsilon\eta(t), \xi'(t) + \epsilon\eta'(t)]dt\Big|_{\epsilon=0}$$

Change of variables, denote $x(\epsilon) = \xi(t) + \epsilon\eta(t)$ and $y(\epsilon) = \xi'(t) + \epsilon\eta'(t)$, then apply chain rule,

$$= \int_0^T (\frac{\partial F[t, x(\epsilon), y(\epsilon)]}{\partial x})^T \frac{dx}{d\epsilon} + (\frac{\partial F[t, x(\epsilon), y(\epsilon)]}{\partial y})^T \frac{dy}{d\epsilon} dt\Big|_{\epsilon=0}$$

$$= \int_0^T (\frac{\partial F[t, x(\epsilon), y(\epsilon)]}{\partial x})^T \eta(t) + (\frac{\partial F[t, x(\epsilon), y(\epsilon)]}{\partial y})^T \eta'(t) dt\Big|_{\epsilon=0}$$

$$= \int_0^T \frac{\partial F}{\partial \xi(t)}[t, \xi(t) + \epsilon\eta(t), \xi'(t) + \epsilon\eta'(t)]^T \eta(t) + \frac{\partial F}{\partial \xi'(t)}[t, \xi(t) + \epsilon\eta(t), \xi'(t) + \epsilon\eta'(t)]^T \eta'(t) dt\Big|_{\epsilon=0}$$

Evaluate at $\epsilon = 0$,

$$= \int_0^T (\frac{\partial F[t, \xi(t), \xi'(t)]}{\partial \xi(t)})^T \eta(t) + (\frac{\partial F[t, \xi(t), \xi'(t)]}{\partial \xi'(t)})^T \eta'(t) dt$$

Write in compact form,

$$= \int_0^T (\frac{\partial F}{\partial \xi(t)})^T \eta(t) + (\frac{\partial F}{\partial \xi'(t)})^T \eta'(t) dt$$

Apply integration by parts to solve $\int_0^T (\frac{\partial F}{\partial \xi'(t)})^T \eta'(t) dt$,

$$\int_0^T (\frac{\partial F}{\partial \xi'(t)})^T \eta'(t) dt$$

$$= (\frac{\partial F}{\partial \xi'(t)})^T \eta(t)\Big|_0^T - \int_0^T \frac{d}{dt}(\frac{\partial F}{\partial \xi'(t)})^T \eta(t) dt$$

By definition, $\eta(0) = \eta(T) = 0$,

$$= -\int_0^T \frac{d}{dt}(\frac{\partial F}{\partial \xi'(t)})^T \eta(t) dt$$

Thus,

$$< \nabla_\xi U, \eta >= \int_0^T (\frac{\partial F}{\partial \xi(t)} - \frac{d}{dt}\frac{\partial F}{\partial \xi'(t)})^T \eta(t) dt = \int_0^T \nabla_\xi U(t)^T \eta(t) dt \text{ for every } \eta$$

Therefore,

$$\nabla_\xi U(t) = \frac{\partial F}{\partial \xi(t)}(t) - \frac{d}{dt}\frac{\partial F}{\partial \xi'(t)}(t)$$

□

6.3 CHOMP: Covariant Hamiltonian Optimization for Motion Planning

CHOMP instantiates functional gradient descent for cost

$$U[\xi] = U_{smooth}[\xi] + \lambda U_{obs}[\xi]$$

Smoothness cost is defined as in our example

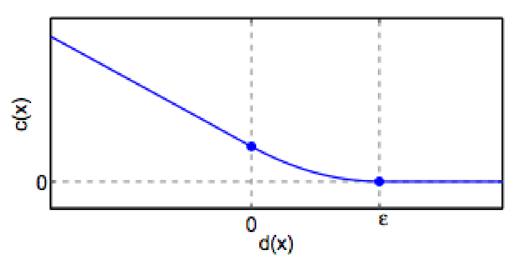
$$U_{smooth}[\xi] = \frac{1}{2}\int_0^T \|\xi'(t)\|^2 dt$$

Obstacle cost is defined as

$$U_{obs}[\xi] = \int_t \int_u c(\phi_u(\xi(t))) \cdot \|\frac{d}{dt}\phi_u(\xi(t))\| du dt$$

Understanding $U_{obs}[\xi]$:

- Define a cost function in W, $c : W \rightarrow \mathbb{R}$ that uses a signed distance field to compute distance to the closes obstacle, and returns a higher cost the closer the point is.
- Then for each time point along the trajectory (thus the integral over time), look at the configuration $\xi(t)$.
- For each body point on the robot u (thus the integral over body points), apply for forward kinematics mapping ϕ_u to get the xyz locations of the points when the robot is in configuration $\xi(t)$.
- For each body point location, compute the cost c.



The second term in the integral (the norm of the velocity) is there to create a path integral formulation.