

# A Formalization of the Ionescu-Tulcea Theorem in mathlib

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## Abstract

## 1 Introduction

Being able to talk about the joint distribution of an infinite family of random variables is crucial in probability theory. For example, one often requires a family of independent random variables. The existence of such a family relies on the existence of an infinite product measure. Indeed, given  $(\Omega_i, \mathcal{F}_i, \mu_i)_{i \in \iota}$  a family of probability spaces, the existence of the product measure  $\bigotimes_{i \in \iota} \mu_i$  yields a new probability space  $(\prod_{i \in \iota} \Omega_i, \bigotimes_{i \in \iota} \mathcal{F}_i, \bigotimes_{i \in \iota} \mu_i)$ , and the projections  $X_i : \prod_{j \in \iota} \Omega_j \rightarrow \Omega_i$  give the desired family. For another example, consider discrete-time Markov chains. Given a measurable space  $(E, \mathcal{A})$  and a Markov kernel  $\kappa : E \rightarrow E$ , one might want to build a sequence  $(X_n)_{n \in \mathbb{N}}$  of random variables with values in  $E$  such that the conditional distribution of  $X_{n+1}$  given  $X_0, \dots, X_n$  is  $\kappa(X_n, \cdot)$ . Such objects are fundamental in probability theory : families of independent variables allow to build more complicated objects, such as Brownian motion, while discrete-time Markov chains form a huge class of stochastic processes which contains random walks for instance. It so happens that those objects always exist without any restrictions on the spaces we consider. This is a direct consequence of the Ionescu-Tulcea theorem.