A Formalization of the Ionescu-Tulcea Theorem in mathlib

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Abstract

1 Introduction

Being able to talk about the joint distribution of an infinite family of random variables is crucial in probability theory. For example, one often requires a family of independent random variables. The existence of such a family relies on the existence of an infinite product measure. Indeed, given $(\Omega_i, \mathcal{F}_i, \mu_i)_{i\in\iota}$ a family of probability spaces, the existence of the product measure $\bigotimes_{i\in\iota} \mu_i$ yields a new probability space $(\prod_{i\in\iota} \Omega_i, \bigotimes_{i\in\iota} \mathcal{F}_i, \bigotimes_{i\in\iota} \mu_i)$, and the projections $X_i : \prod_{j\in\iota} \Omega_j \to \Omega_i$ give the desired family. For another example, consider discrete-time Markov chains. Given a measurable space $(E, \mathcal{A})$ and a Markov kernel $\kappa : E \to E$, one might want to build a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables with values in E such that the conditional distribution of X_{n+1} given $X_0, ..., X_n$ is $\kappa(X_n, \cdot)$. Such objects are fundamental in probability theory : families of independent variables allow to build more complicated objects, such as Brownian motion, while discrete-time Markov chains form a huge class of stochastic processes which contains random walks for instance. It so happens that those objects always exist without any restrictions on the spaces we consider. This is a direct consequence of the Ionescu-Tulcea theorem.