ERROR CREDITS

Resourceful Reasoning about Error Bounds

Alejandro Aguirre Philipp Haselwarter Markus de Medeiros
Kwing Hei Li Simon Gregersen Joseph Tassarotti Lars Birkedal
alejandro@cs.au.dk

AARHUS
UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE

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SOLIDUM PETIT IN PROFUNDIS · UNIVERSITAS ARHUSIENSIS

```
Σ : gFunctors
irisGS0 : irisGS Σ
E : coPset
Φ : val → iPropI Σ
n, m : nat
Hnm : n < m
============================
---------------------------∗
⌜m ≤ n⌝
```

Introducing... error credits

$$\vdash \;\mid\!\Rrightarrow \lightning(0)$$

$$\lightning(\varepsilon_1) * \lightning(\varepsilon_2) \dashv\vdash \lightning(\varepsilon_1 + \varepsilon_2)$$

$$\lightning(1) \vdash \bot$$

```
Σ : gFunctors
irisGS0 : irisGS Σ
E : coPset
Φ : val → iPropI Σ
n, m : nat
Hnm : n < m
============================
"err" : ↯(1)
--------------------------*
⌜m ≤ n⌝
```

```
Σ : gFunctors
irisGS0 : irisGS Σ
E : coPset
Φ : val → iPropI Σ
n, m : nat
Hnm : n < m
============================
"err" : False
---------------------------*
⌜m ≤ n⌝
```

```
No more goals.
```

Randomized algorithms

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Monte Carlo algorithms

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Las Vegas algorithms

Randomized algorithms

Monte Carlo algorithms

- Efficient runtime
- Result might be wrong
- E.g. probabilistic primality tests

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Monte Carlo algorithms

- Efficient runtime
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Las Vegas algorithms

- Probabilistic runtime
- Correct result
- E.g. randomized Quicksort

Contributions

We introduce Eris, a logic to reason about probability of errors in probabilistic programs

- Error budget represented as a separation logic resource
- $\lightning(\varepsilon)$ allows specifications to fail with probability $\leq \varepsilon$
- The resource representation of error inherits the expresiveness of HO separation logic and enables new reasoning principles
- Partial and total correctness versions of the logic
- Fully mechanized in Coq and Iris

A probabilistic sequential language

$\lambda_{\mathsf{ref}}^{\mathsf{rand}}$: sequential HeapLang plus sampling

$$e \in \mathsf{Expr} ::= v \mid \mathsf{x} \mid \mathsf{rec}\,\mathsf{f}\,\mathsf{x} = e \mid e_1(e_2) \mid \mathsf{if}\ e\ \mathsf{then}\ e_1\ \mathsf{else}\ e_2 \mid \mathsf{fst}(e) \mid \mathsf{snd}(e) \mid$$
$$\mathsf{ref}(e) \mid {!}\,e \mid e_1 \leftarrow e_2 \mid \cdots \mid \mathsf{rand}\ e$$

Probabilistic step function: $\mathrm{step} : \mathsf{Cfg} \to \mathcal{D}(\mathsf{Cfg})$.

$$\mathrm{step}((\mathsf{rec}\,\mathsf{f}\,\mathsf{x} = e)v, \sigma) = \{(e[v/\mathsf{x}][(\mathsf{rec}\,\mathsf{f}\,\mathsf{x} = e)/\mathsf{f}] \mapsto 1\}$$
$$\mathrm{step}(\mathsf{rand}\ n, \sigma) = \left\{(0,\sigma) \mapsto \frac{1}{n+1}, \ldots, (n,\sigma) \mapsto \frac{1}{n+1}\right\}$$

Partial probabilistic correctness

In Eris, WP has the following meaning:

Assume φ is a pure Coq proposition, $\varepsilon \geq 0$. If

$$\lightning(\varepsilon) \vdash \mathsf{wp}\ e\ \{v.\varphi(v)\}$$

Then, with probability at least $1 - \varepsilon$, e will either diverge or return a result v satisfying $\varphi(v)$

Generating and using error credits

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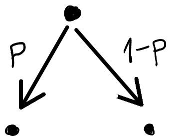

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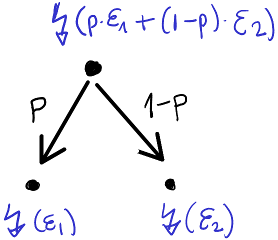

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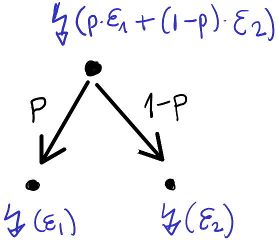
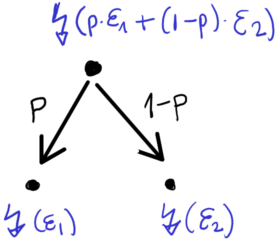
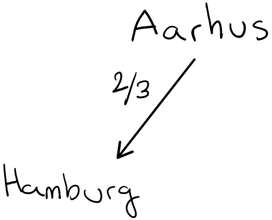


Aarhus

Generating and using error credits





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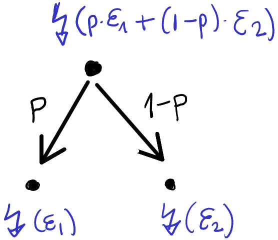

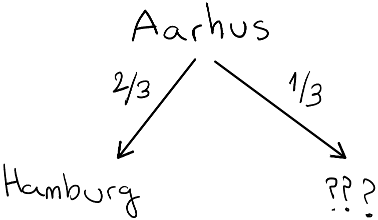

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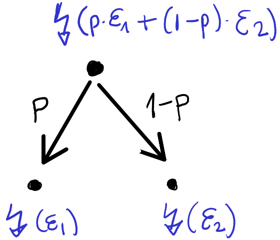

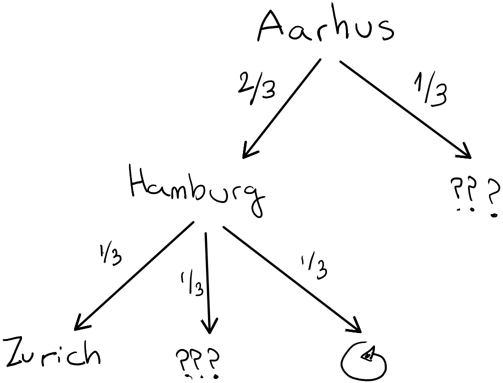

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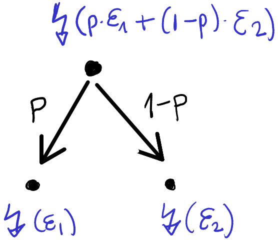

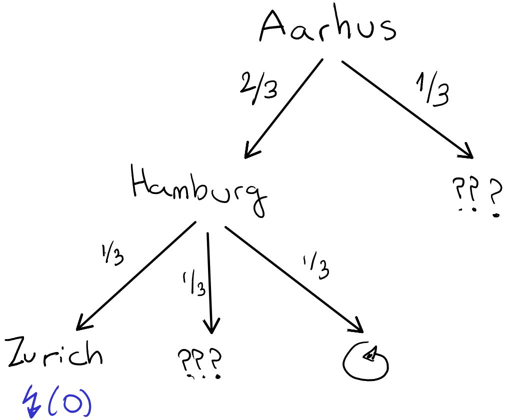

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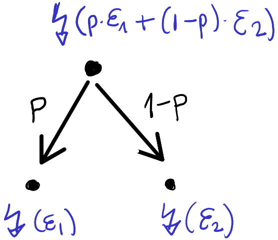

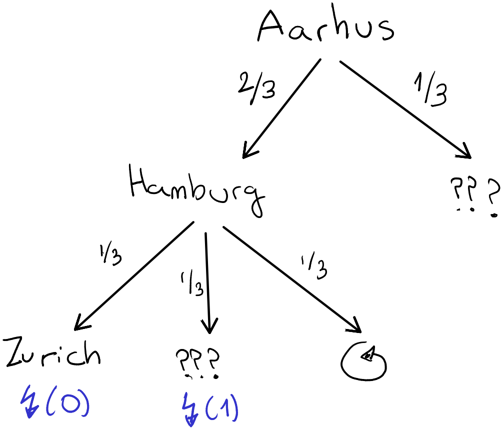

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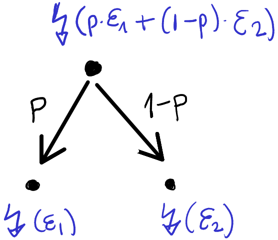

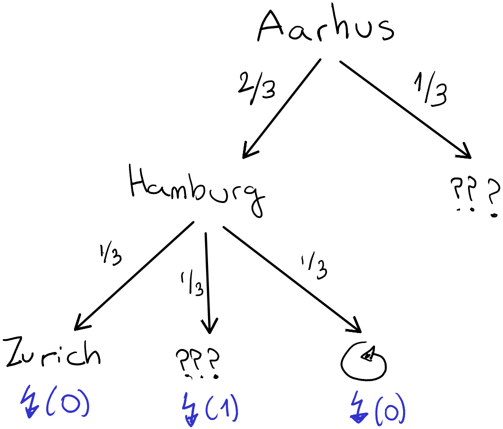

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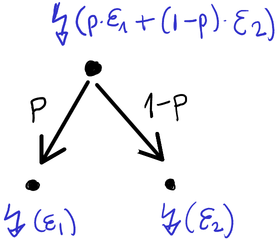

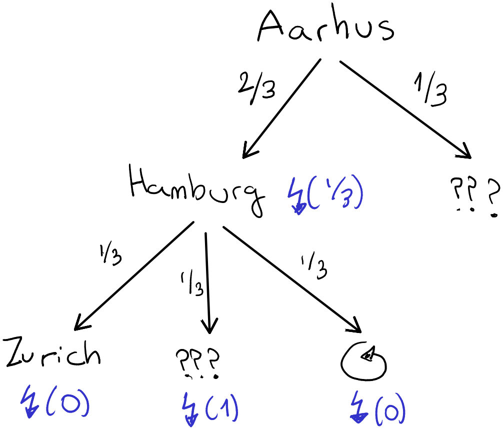

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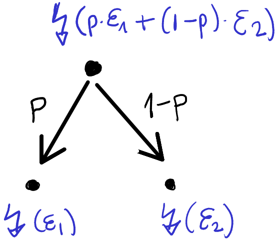

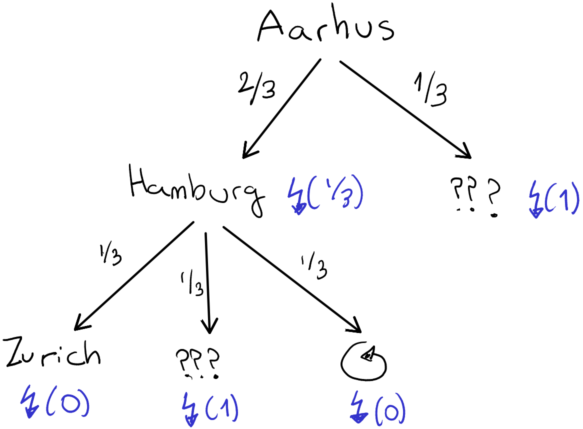

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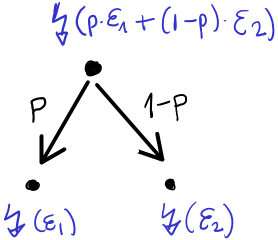
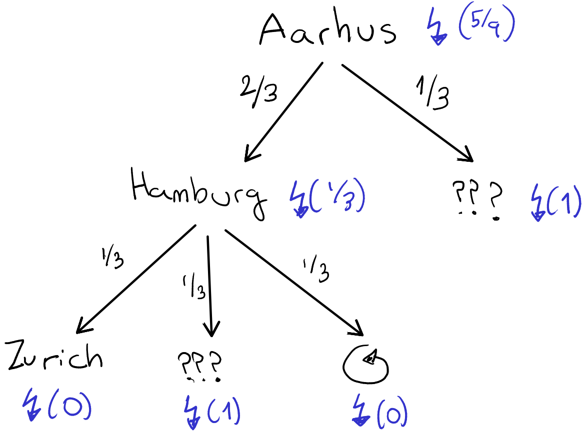
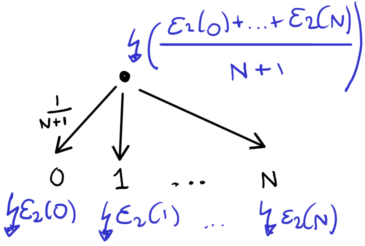




Generating and using error credits

$$\frac{\mathcal{E}_2 \colon \{0, \ldots, N\} \to \mathbb{R}^{\geq 0} \qquad \sum_{i=0}^{N} \dfrac{\mathcal{E}_2(i)}{N+1} = \varepsilon_1}{\vdash \{\lightning(\varepsilon_1)\}\ \textsf{rand}\ N\ \{n\ .\ \lightning(\mathcal{E}_2(n))\}}\ \textsf{ht-rand-exp}$$

Generating and using error credits



$$\frac{\mathcal{E}_2 \colon \{0, \ldots, N\} \to \mathbb{R}^{\geq 0} \qquad \sum_{i=0}^{N} \dfrac{\mathcal{E}_2(i)}{N+1} = \varepsilon_1}{\vdash \{\lightning(\varepsilon_1)\}\ \textsf{rand}\ N\ \{n\ .\ \lightning(\mathcal{E}_2(n))\}}\ \text{ht-rand-exp}$$

Derived rules

$$\frac{X \subseteq N \qquad \varepsilon = \dfrac{|X|}{N+1}}{\vdash \{\lightning(\varepsilon)\}\ \textsf{rand}\ N\ \{n\ .\ n \notin X\}}\ \text{ht-rand-err-sub}$$

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1. Apply ht − rand − exp with $\quad \mathcal{E}_2(x) \triangleq \mathrm{if}\ (x \in X)\ \mathrm{then}\ 1\ \mathrm{else}\ 0$

Derived rules

$$\frac{X \subseteq N \qquad \varepsilon = \dfrac{|X|}{N+1}}{\vdash \{ \lightning(\varepsilon) \} \ \textsf{rand}\ N \ \{ n \,.\, n \notin X \}} \ \textsf{ht-rand-err-sub}$$

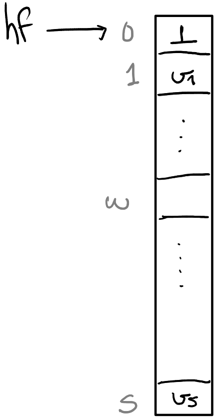
1. Apply ht – rand – exp with $\mathcal{E}_2(x) \triangleq \text{if } (x \in X) \text{ then } 1 \text{ else } 0$
2. Discard outcomes with 1 error credit

Collision-free hashing

$$
\begin{aligned}
\mathsf{query}\ hf\ w \triangleq\ & \mathsf{match}\ \mathsf{get}\ hf\ w\ \mathsf{with} \\
& \quad \mathsf{Some}(v) \Rightarrow v \\
& \mid \mathsf{None} \quad\ \ \Rightarrow \mathsf{let}\ v = \mathsf{rand}\ (S)\ \mathsf{in} \\
& \qquad\qquad\qquad\quad \mathsf{set}\ hf\ w\ v; \\
& \qquad\qquad\qquad\quad v \\
& \mathsf{end}
\end{aligned}
$$

Collision-free hashing

query $hf\ w \triangleq$ match get $hf\ w$ with
  Some$(v) \Rightarrow v$
  | None $\Rightarrow$ let $v = \text{rand}\ (S)$ in
    set $hf\ w\ v$;
    v
end



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\quad |\ \text{None} \quad \Rightarrow \text{let } v = \text{rand } (S) \text{ in} \\
\qquad\qquad\qquad \text{set } hf\,w\,v; \\
\qquad\qquad\qquad v \\
\text{end}
\end{array}
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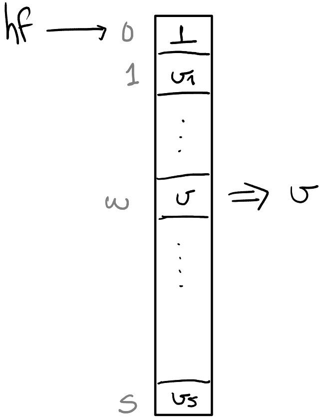

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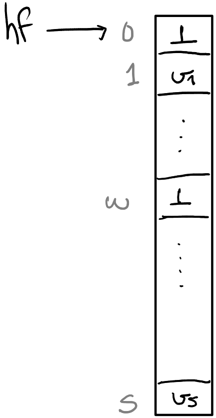

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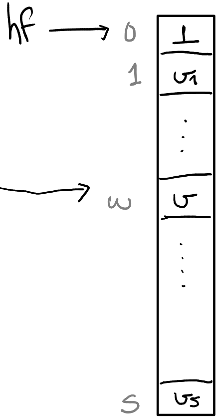

Collision-free hashing

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\text{query } hf\, w \triangleq\ & \textsf{match get } hf\, w \textsf{ with} \\
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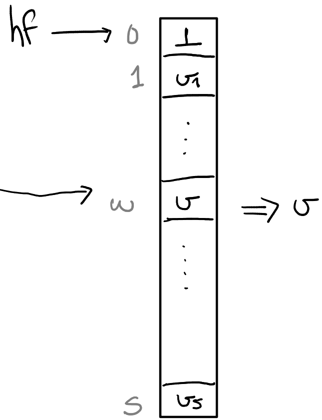

Collision-free hashing

$$\left\{n \notin \mathrm{dom}\ m * \mathsf{collFree}(hf, \mathsf{m}) * ↯\left(\frac{|\mathsf{m}|}{\mathsf{S}+1}\right)\right\}\ \mathsf{query}\ hf\ n\ \{v.\ \mathsf{collFree}(hf, \mathsf{m}[\mathsf{n} \leftarrow \mathsf{v}])\}$$

Collision-free hashing

$$\left\{ n \notin \mathsf{dom}\ m * \mathsf{collFree}(hf, \mathsf{m}) * ↯\left(\frac{|\mathsf{m}|}{\mathsf{S}+1}\right) \right\} \mathsf{query}\ hf\ n\ \{v.\ \mathsf{collFree}(hf, \mathsf{m}[\mathsf{n} \leftarrow \mathsf{v}])\}$$

$$\mathsf{collFree}(hf, \mathsf{m}) \triangleq \begin{array}{l} (\forall i < S, \\ \qquad ((hf +_{\ell} i \mapsto \mathsf{None} \wedge i \notin \mathsf{dom}(m)) \vee \\ \qquad (\exists v, hf +_{\ell} i \mapsto \mathsf{Some}\ v \wedge m[i] = v))) * \\ \mathsf{injective}\ \mathsf{m} \end{array}$$

Amortized error

Suppose we execute a sequence of M insertions. The total error will be

$$0 + \frac{1}{S+1} + \frac{2}{S+1} + \cdots + \frac{M-1}{S+1}$$

Amortized error

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We can prove a stronger specification tracking amortized error

$$\left\{ \begin{array}{c} n \notin \mathsf{dom}\ m\ * \\ |m| < M\ * \\ \mathsf{collFreeAE}(hf, \mathsf{m})\ * \\ ↯\left(\frac{M-1}{2(S+1)}\right) \end{array} \right\} \mathsf{query}\ hf\ n\ \{v.\ \mathsf{collFreeAE}(hf, \mathsf{m}[\mathsf{n} \leftarrow \mathsf{v}])\}$$

Amortized error

$$\left\{\begin{array}{c} n \notin \mathsf{dom}\ m \ * \\ |m| < M \ * \\ \mathsf{collFreeAE}(hf, \mathsf{m}) \ * \\ \lightning\left(\frac{M-1}{2(S+1)}\right) \end{array}\right\} \mathsf{query}\ hf\ \ n\ \{v.\,\mathsf{collFreeAE}(hf, \mathsf{m}[\mathsf{n} \leftarrow \mathsf{v}])\}$$

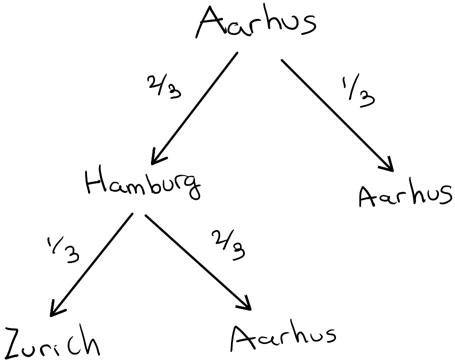
Amortized error

$$\left\{ \begin{array}{c} n \notin \mathsf{dom}\ m \ast \\ |m| < M \ast \\ \mathsf{collFreeAE}(hf, \mathsf{m}) \ast \\ ↯\left(\frac{M-1}{2(S+1)}\right) \end{array} \right\} \mathsf{query}\ hf\ n\ \{v.\, \mathsf{collFreeAE}(hf, \mathsf{m}[\mathsf{n} \leftarrow \mathsf{v}])\}$$

$$\mathsf{collFreeAE}(hf, \mathsf{m}) \quad \triangleq \quad \exists \varepsilon.\, ↯(\varepsilon) \ast \ulcorner \varepsilon \geq \sum_{\mathsf{i}=|\mathsf{m}|}^{\mathsf{M}} \frac{\mathsf{i}}{\mathsf{S}+1} \urcorner \ast \mathsf{collFree}(hf, \mathsf{m})$$

Reasoning about termination

Reasoning about termination



A Total WP

Eris also provides a total WP with the following meaning:

Assume φ pure Coq proposition, $\varepsilon \geq 0$. If

$$↯(\varepsilon) \vdash \mathsf{twp}\ e\ \{v.\varphi(v)\}$$

Then, with probability at least $1 - \varepsilon$, e will terminate and return a result v satisfying $\varphi(v)$

Refresher: Löb induction

$$\frac{\triangleright(\mathsf{wp}\ e\ \{\Phi\}) \vdash \mathsf{wp}\ e\ \{\Phi\}}{\vdash \mathsf{wp}\ e\ \{\Phi\}}\ \text{Löb}$$

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$$\frac{\triangleright(\mathsf{wp}\ e\ \{\Phi\}) \vdash \mathsf{wp}\ e\ \{\Phi\}}{\vdash \mathsf{wp}\ e\ \{\Phi\}}\ \textsf{Löb}$$

$$\frac{(\pounds(1) \mathrel{-\!\!*} \mathsf{wp}\ e\ \{\Phi\}) \vdash \mathsf{wp}\ e\ \{\Phi\}}{\vdash \mathsf{wp}\ e\ \{\Phi\}}\ \textsf{LöbLC}$$

Refresher: Löb induction

$$\frac{\triangleright(\mathsf{wp}\ e\ \{\Phi\}) \vdash \mathsf{wp}\ e\ \{\Phi\}}{\vdash \mathsf{wp}\ e\ \{\Phi\}}\ \text{Löb}$$

$$\frac{(£(1) \twoheadrightarrow \mathsf{wp}\ e\ \{\Phi\}) \vdash \mathsf{wp}\ e\ \{\Phi\}}{\vdash \mathsf{wp}\ e\ \{\Phi\}}\ \text{LöbLC}$$

- We get an induction hypothesis guarded by $£(1)$
- Later credits obtained by execution steps, but **only under partial WP**

Error induction

$$\frac{\varepsilon > 0 \qquad k > 1 \\ \forall \varepsilon', (\lightning\left(k \cdot \varepsilon'\right) \mathrel{-\!\!*} \textsf{twp}\ e\ \{\Phi\}) * \lightning\left(\varepsilon'\right) \vdash \textsf{twp}\ e\ \{\Phi\}}{\lightning(\varepsilon) \vdash \textsf{twp}\ e\ \{\Phi\}}\ \textsf{err-ind}$$

Error induction

$$\frac{\varepsilon > 0 \qquad {\color{magenta} k > 1} \qquad \forall \varepsilon', (\text{\textit{↯}}(k \cdot \varepsilon') \mathrel{-\!\!*} \mathsf{twp}\ e\ \{\Phi\}) * \text{\textit{↯}}(\varepsilon') \vdash \mathsf{twp}\ e\ \{\Phi\}}{\text{\textit{↯}}(\varepsilon) \vdash \mathsf{twp}\ e\ \{\Phi\}}\ \textsf{err-ind}$$

- We choose an amplification factor $k > 1$

Error induction

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- We choose an amplification factor $k > 1$
- We get an arbitrary amount of credits ε'

Error induction

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- We choose an amplification factor $k > 1$
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- We choose an amplification factor $k > 1$
- We get an arbitrary amount of credits ε'
- We get an induction hypothesis guarded by $\lightning(k \cdot \varepsilon')$
- Error credits can be obtained by taking probabilistic choices

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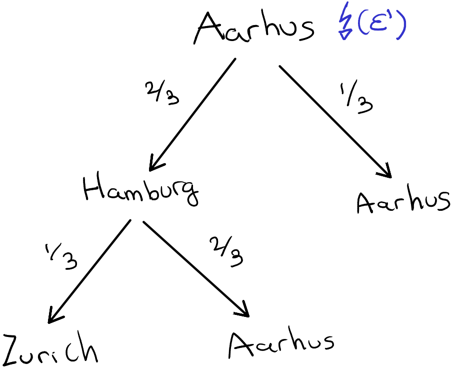
- We choose an amplification factor $k > 1$
- We get an arbitrary amount of credits ε'
- We get an induction hypothesis guarded by $\lightning(k \cdot \varepsilon')$
- Error credits can be obtained by taking probabilistic choices

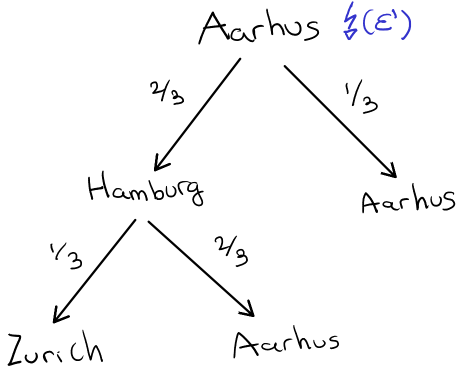
Intuitively: by amplifying ε by k enough times, we can get to $\lightning(1)$.

Proving termination



Proving termination

$K = 9/8 > 1$



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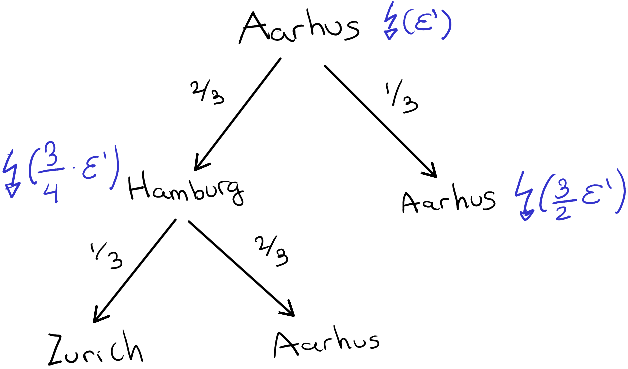

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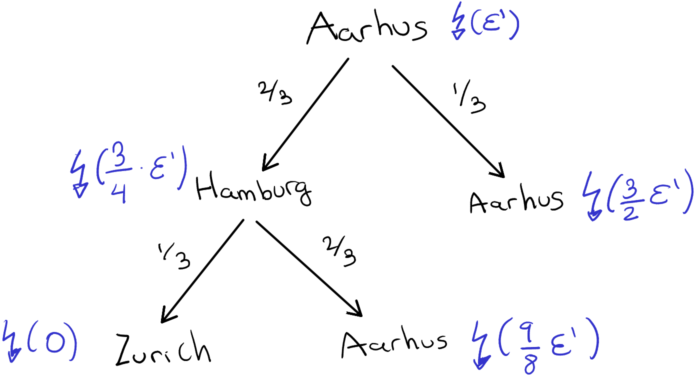

Adequacy in the limit

By the completeness of the reals, we can prove a stronger adequacy result:

Theorem (Limit adequacy for total correctness)

Let φ be a pure Coq proposition. If for every $\varepsilon > 0$ we can prove

$$↯(\varepsilon) \vdash \mathsf{twp}\; e\; \{v.\varphi(v)\}$$

then, with probability 1, e will return a result v satisfying $\varphi(v)$

Constructing the model

An error-tracking WP

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$$
\begin{aligned}
\mathsf{wp}\ e_1\ \{\Phi\} \triangleq\ & (e_1 \in \mathsf{Val} \wedge \Phi(e_1)) \\
& \vee (e_1 \notin \mathsf{Val} \wedge \forall \sigma_1, \varepsilon_1 .\, S(\sigma_1) * \lightning_\bullet(\varepsilon_1) \mathrel{-\!\!*} \\
& \quad \mathsf{GLM}(e_1, \sigma_1, \varepsilon_1,\ (\lambda\, e_2, \sigma_2, \varepsilon_2 \,.\, \triangleright(S(\sigma_2) * \lightning_\bullet(\varepsilon_2)\ * \mathsf{wp}\ e_2\ \{\Phi\}))))
\end{aligned}
$$

The Graded Lifting Modality

The GLM pairs logic and error usage with the operational semantics

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Intended meaning: $\mathsf{GLM}((e, \sigma), \varepsilon, Z)$ iff

- Starting from (e, σ) with ε error budget,
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- ensure we get (e', σ') and residual error ε' s.t. $Z((e', \sigma'), \varepsilon')$ or $\varepsilon' \geq 1$

$$\frac{\mathsf{red}(e_1, \sigma_1) \quad \varepsilon_1 + \varepsilon_2 \leq \varepsilon \quad \Pr_{\mathrm{step}(e_1,\sigma_1)}[\neg R] \leq \varepsilon_1 \\ \forall e_2, \sigma_2.\, R(e_2, \sigma_2) \twoheadrightarrow Z(e_2, \sigma_2, \varepsilon_2)}{\mathsf{GLM}((e_1, \sigma_1), \varepsilon, Z)}\ \textsf{step-simple}$$

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Using expected value composition:

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4. From R, we have to prove $1 \le \varepsilon_2$ or $Z((e_2, \sigma_2), \varepsilon_2$

Adequacy

Theorem (Adequacy for partial correctness)

Let φ be a pure Coq proposition, $\varepsilon \geq 0$. If

$$↯(\varepsilon) \vdash \mathsf{wp}\; e\; \{v.\varphi(v)\}$$

Then, with probability at least $1 - \varepsilon$, e will either diverge or return a result v satisfying $\varphi(v)$

Open questions

- Continuous WP wrt the error entirely within the logic?
- Is there a deeper induction principle behind error amplification?

Conclusions

We introduced Eris, a logic to reason about probability of errors in probabilistic programs

- Error budget represented as a separation logic resource
- New reasoning principles: amortized error, error induction, and more!
- Partial and total correctness versions of the logic

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Read our draft! `https://arxiv.org/pdf/2404.14223`
We are on Github! `https://github.com/logsem/clutch`