CS258: Information Theory

Fan Cheng
Shanghai Jiao Tong University

http://www.cs.sjtu.edu.cn/~chengfan/
chengfan@sjtu.edu.cn
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Outline

- Differential Entropy
- AEP for Continuous Random Variable
- Relative Entropy and Mutual Information
- Property of Differential Information Measures
- Information inequalities and applications

$I(X;Y)$: Correlated Gaussian

(Mutual information between correlated Gaussian random variables with correlation ρ) Let $(X,Y) \sim \mathcal{N}(0,K)$, where

$$\boldsymbol{K} = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}$$

$\boldsymbol{I(X;Y)}$?

$$h(X) = h(Y) = \frac{1}{2}\log 2\pi e\sigma^2$$

$$h(X,Y) = \frac{1}{2}\log(2\pi e)^2|K| = \frac{1}{2}\log(2\pi e)^2\sigma^4(1-\rho^2)$$

$$I(X;Y) = h(X) + h(Y) - h(X,Y) = -\frac{1}{2}\log(1-\rho^2)$$

- $\rho = 0$, X and Y are independent and I is 0
- $\rho = \pm 1$, X and Y are perfectly correlated and I is ∞

Maximum Entropy with Constraints

- Let the random variable $X \in R$ have mean μ and variance $\boldsymbol{\sigma^2}$. Then
$$\boldsymbol{h(X) \le \frac{1}{2}\log 2\pi e\sigma^2}$$
with equality iff $\boldsymbol{X \sim \mathcal{N}(\mu, \sigma^2)}$
- Let the random variable $X \in R$ satisfy $EX^2 \le \sigma^2$. Then
$$\boldsymbol{h(X) \le \frac{1}{2}\log 2\pi e\sigma^2}$$
with equality iff $X \sim \mathcal{N}(0, \sigma^2)$

1. Let $X_G \sim \mathcal{N}\left(\mu, \sigma^2\right)$. Consider
$$\boldsymbol{D(X||X_G) \ge 0}$$
Then
$$\int f \log \frac{f}{g} \ge 0$$
$$h(X) = h(f) \le -\int f \log g = -\int f \log \frac{1}{\sqrt{2\pi\sigma^2}} + f\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$
$$h(X) \le \frac{1}{2}\log 2\pi\sigma^2 + \frac{1}{2} = \frac{1}{2}\log 2\pi e\sigma^2$$
2. $\mathrm{Var}(X) = E\left(X^2\right) - E(X)^2 \le \sigma^2$. $\Rightarrow$ Case 1.

$\boldsymbol{E(X^2)}, \mathbf{Var(X)}$给定的情况下，高斯分布最大化微分熵

Maximum Entropy

Consider the following problem: Maximize the entropy $h(f)$ over all probability densities f satisfying

1. $f(x) \geq 0$, with equality outside the support
2. $\int_S f(x)dx = 1$ $\qquad (++)$
3. $\int_S f(x)r_i(x)dx = \alpha_i$ for $1 \leq i \leq m$. ($r_i(x)$ is a function of x)

Thus, f is a density on support set S meeting certain moment constraints $\alpha_1, \alpha_2, \ldots, \alpha_m$.

Theorem 12.1.1 (*Maximum entropy distribution*) *Let*

$$f^*(x) = f_\lambda(x) = e^{\lambda_0 + \sum_{i=1}^{m} \lambda_i r_i(x)}$$

$x \in S$, where $\lambda_0, \ldots, \lambda_m$ are chosen so that f^* satisfies $(++)$. Then f^* uniquely maximizes $h(f)$ over all probability densities f satisfying constraints $(++)$.

- Let $S = [a, b]$, with no other constraints. Then the maximum entropy distribution is the uniform distribution over this range.
- $S = [0, \infty)$ and $EX = \mu$. Then the entropy-maximizing distribution is

$$f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}, \qquad x \geq 0$$

- $S = (-\infty, \infty), EX = \alpha_1$, and $EX^2 = \alpha_2$. The maximum entropy distribution is $\mathcal{N}\left(\alpha_1, \alpha_2 - \alpha_1^2\right)$

Ref: Cover Ch. 12

Hadamard's Inequality

K is a nonnegative definite symmetric $n \times n$ matrix. Let $|K|$ denote the determinant of K.

Theorem (Hadamard) $|K| \le \prod K_{ii}$, with equality iff $K_{ij} = 0, \quad i \neq j$

Let $\mathrm{X} \sim \mathcal{N}(0, K)$. Then

$$\frac{1}{2}\log(2\pi e)^n |K| = \boldsymbol{h(X_1, X_2, \ldots, X_n) \le \sum h(X_i)} = \sum_{i=1}^{n} \frac{1}{2}\log 2\pi e |K_{ii}|$$

with equality iff $X_1, X_2, \ldots, X_n$ are independent (i.e., $K_{ij} = 0, i \neq j$)

- $\log|K|$ is concave
- $\log(|K_n|/|K_{n-p}|)$ is concave in K_n
- $|K_n|/|K_{n-1}|$ is concave in K_n

- A general technique to deal with nonnegative definite symmetric matrix K
- Ref. Ch. 17.9, 17.10, Cover

Balanced Information Inequality

Differences between inequalities of the discrete entropy and differential entropy

- Both $H(X,Y) \leq H(X) + H(Y)$ and $h(X,Y) \leq h(X) + h(Y)$ are valid
- $H(X,Y) \geq H(X)$ but neither $h(X,Y) \geq h(X)$ nor $h(X,Y) \leq h(X)$ is valid

Take $H(X,Y,Z) \leq \frac{1}{4}H(X) + \frac{1}{2}H(Y,Z) + \frac{3}{4}H(Z,X)$ for example.

Count the weights of random variables X, Y, Z in both sides

$$X: 1, 1;\ Y: 1, \frac{1}{2};\ Z: 1, \frac{5}{4}$$

The net weights of X, Y, Z are $0, \frac{1}{2}, -\frac{1}{4}$

Balanced: If the net weights of X, Y, Z are all zero.

$$h(X,Y) \leq h(X) + h(Y) \text{ and } h(X,Y,Z) \leq \frac{1}{2}h(X,Y) + \frac{1}{2}h(Y,Z) + \frac{1}{2}h(Z,X)$$

Let $[n] := \{1,2,\ldots,n\}$. For any $\alpha \subseteq [n]$, denote $(X_i : i \in \alpha)$ by X_α. For example, $\alpha = \{1,3,4\}$, we denote X_1, X_3, X_4 by $X_{\{1,3,4\}}$ for simplicity.

- We could write any information inequality in the form $\sum_\alpha w_\alpha H(X_\alpha) \geq 0$ or $\sum_\alpha w_\alpha h(X_\alpha) \geq 0$.
- An information inequality is called balanced if for any $i \in [n]$, the net weight of X_i is zero.
- The linear continuous inequality $\sum_\alpha w_\alpha h(X_\alpha) \geq 0$ is valid if and only if its corresponding discrete counterpart $\sum_\alpha w_\alpha H(X_\alpha) \geq 0$ is valid and balanced.

Ref: Balanced Information Inequalities, T. H. Chan, IEEE Transactions on Information Theory, Vol. 49, No. 12, December 2003

Han's Inequality

- Let $(X_1, X_2, \dots, X_n)$ have a density, and for **every $\boldsymbol{S \subseteq \{1, 2, \dots, n\}}$,** denoted by $X(S)$ the subset $\{X_i : i \in S\}$. Let

$$h_k^{(n)} = \frac{1}{\binom{n}{k}} \sum_{S:|S|=k} \frac{h\big(X(S)\big)}{k}$$

$$g_k^{(n)} = \frac{1}{\binom{n}{k}} \sum_{S:|S|=k} \frac{h(X(S)|X(S^c))}{k}$$

- When $n = 3$,

$$h_1^{(3)} = \frac{H(X_1) + H(X_2) + H(X_3)}{3} \geq h_2^{(3)} = \frac{H(X_1, X_2) + H(X_2, X_3) + H(X_3, X_1)}{3}$$

$$\geq h_3^{(3)} = H(X_1, X_2, X_3)$$

$$g_1^{(3)} = \frac{H(X_1|X_2, X_3) + H(X_2|X_1, X_3) + H(X_3|X_1, X_2)}{3}$$

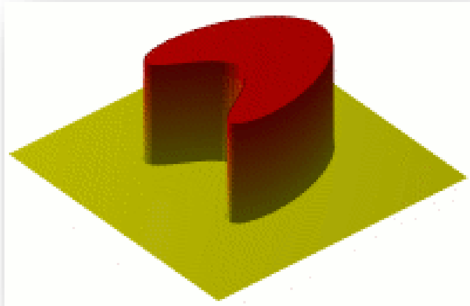
$$\leq g_2^{(3)} = \frac{H(X_1, X_2|X_3) + H(X_2, X_3|X_1) + H(X_3, X_1|X_2)}{3}$$

$$\leq g_3^{(3)} = H(X_1, X_2, X_3)$$

Han's inequality:

$$h_1^{(n)} \geq h_2^{(n)} \dots \geq h_n^{(n)} = H(X_1, X_2, \dots, X_n) = g_n^{(n)} \geq \cdots \geq g_2^{(n)} \geq g_1^{(n)}$$

Information of Heat



- Heat equation (**Fourier**): Let x be the position and t be the time,

$$\frac{\partial}{\partial t} f(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} f(x,t)$$

- Let X be any random variable with a density $f(x)$. Let Z be an independent normally distributed random variable with zero mean and unit variance, $Z \sim \mathcal{N}(0,1)$. Let

$$Y_t = X + \sqrt{t}Z$$

The **probability density function $f(y;t)$** ($f(y;t)$ is a function in y, not t) of Y_t **satisfies heat equation**

$$f(y;t) = \int f(x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dx$$

Gaussian channel ↔ Heat equation

Entropy and Fisher Information

Fisher information: Let X be any random variable with density $f(x)$. Its Fisher information is given by

$$\boldsymbol{I(X) = \int_{-\infty}^{+\infty} f(x) \left[\frac{\frac{\partial}{\partial x} f(x)}{f(x)}\right]^2 dx}$$

- Let X be any random variable with a density $f(x)$. Let Z be an independent normally distributed random variable with zero mean and unit variance. Let $Y_t = X + \sqrt{t}Z$

$$\boldsymbol{\frac{\partial}{\partial t} h(Y_t) = \frac{1}{2} I(Y_t)}$$

- Let $f(y,t)$ (or f) be the p.d.f of Y_t

$$\frac{\partial}{\partial t} h(Y_t) = \frac{1}{2} I(Y_t) = \frac{1}{2} \int \frac{f_y^2}{f} dy \geq 0$$

$$\frac{\partial^2}{\partial t^2} h(Y_t) = -\frac{1}{2} \int f \left(\frac{f_{yy}}{f} - \frac{f_y^2}{f^2}\right)^2 dy \leq 0$$

- When X is Gaussian $\mathcal{N}(0,1)$,

$$h(Y_t) = \frac{1}{2} \log 2\pi e(1+t)$$

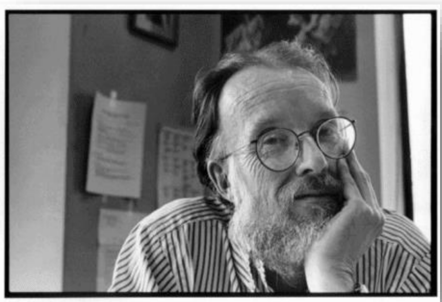
All the derivatives alternate in signs: +, -, +, -, …

Higher Order Derivatives of $h(Y_t)$

(Cheng 2015) Let X be any random variable with a density $f(x)$. Let Z be an independent normally distributed random variable with zero mean and unit variance. Let $Y_t = X + \sqrt{t}Z$ and $f(y,t)$ (or f) be the p.d.f of Y_t. Then

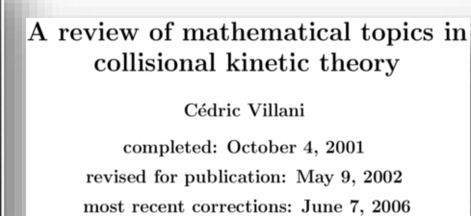
$$\frac{\partial^3}{\partial t^3} h(Y_t) \geq 0 \text{ and } \frac{\partial^4}{\partial t^4} h(Y_t) \leq 0$$

Conjecture: When n is even, $\frac{\partial^n}{\partial t^n} h(Y_t) \leq 0$, otherwise $\frac{\partial^n}{\partial t^n} h(Y_t) \geq 0$



“This suggests that……, etc., but I could not prove it” (1966)

H. P. McKean



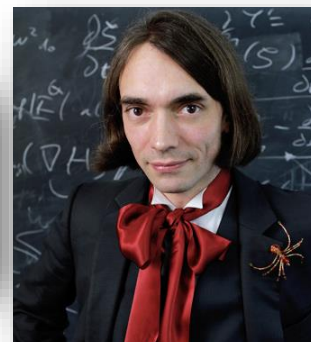
A review of mathematical topics in collisional kinetic theory

Cédric Villani

completed: October 4, 2001

revised for publication: May 9, 2002

most recent corrections: June 7, 2006



C. Villani

2010 Fields medalist

$$\frac{\partial^3}{\partial t^3} h(Y_t) = \frac{1}{2} \int f \left(\frac{f_3}{f} - \frac{f_1 f_2}{f^2} + \frac{1}{3} \frac{f_1^3}{f^3} \right)^2 + \frac{f_1^6}{45 f^5} \mathrm{d}y$$

$$\begin{aligned} &\frac{\partial^4}{\partial t^4} h(Y_t) \\ &= -\frac{1}{2} \int f \left(\frac{f_4}{f} - \frac{6}{5} \frac{f_1 f_3}{f^2} - \frac{7}{10} \frac{f_2^2}{f^2} + \frac{8}{5} \frac{f_1^2 f_2}{f^3} - \frac{1}{2} \frac{f_1^4}{f^4} \right)^2 \\ &\quad + f \left(\frac{2}{5} \frac{f_1 f_3}{f^2} - \frac{1}{3} \frac{f_1^2 f_2}{f^3} + \frac{9}{100} \frac{f_1^4}{f^4} \right)^2 \\ &\quad + f \left(-\frac{4}{100} \frac{f_1^2 f_2}{f^3} + \frac{4}{100} \frac{f_1^4}{f^4} \right)^2 \\ &\quad + \frac{1}{300} \frac{f_2^4}{f^3} + \frac{56}{90000} \frac{f_1^4 f_2^2}{f^5} + \frac{13}{70000} \frac{f_1^8}{f^7} \mathrm{d}y \end{aligned}$$

Ref: F. Cheng and Y. Geng, “Higher Order Derivatives in Costa's Entropy Power Inequality”

EPI and FII

(**Shannon 1948, Entropy power inequality (EPI)**) If X and Y are independent random n-vectors with densities, then

$$e^{\frac{2}{n}h(X+Y)} \geq e^{\frac{2}{n}h(X)} + e^{\frac{2}{n}h(Y)}$$

$$e^{2h(X+Y)} \geq e^{2h(X)} + e^{2h(Y)}$$

- Fisher information inequality (FII)

$$\frac{1}{I(X+Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}$$

- Most profound result in Shannon's 1948 paper
- EPI can imply some very fundamental results
  - Uncertainty principle
  - Young's inequality
  - Nash's inequality
  - Cramer-Rao bound

Reference

- T. Cover, "Information theoretic inequalities," 1990
- O. Rioul, "Information Theoretic Proofs of Entropy Power Inequalities," 2011

Summary

Cover: 8.9, 12.1, 17.6, 17.7, 17.9, 17.10