# CS258: Information Theory

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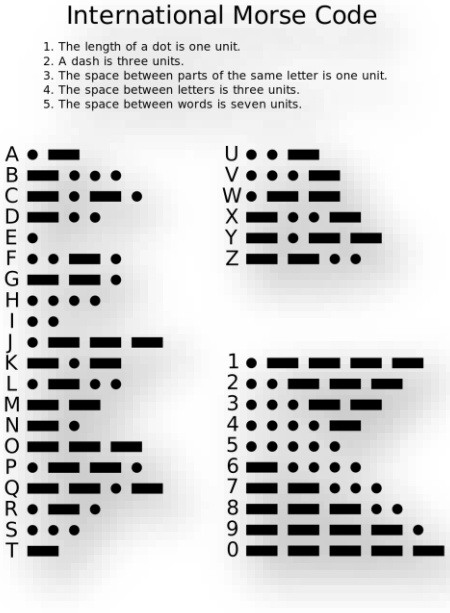
# Outline

- Kraft inequality
- Optimality of codes
- Huffman coding
- Shannon-Fano-Elias coding
- Generation of discrete distribution
- Universal source coding

# Data Compression: Existing Systems

薊镇总兵戚继光在《练兵纪实》中讲：“凡无空心台之处，即以原墩充之，有空心台所相近百步之内者，俱以空心台充墩。大约相去一、二里，梆鼓相闻为一墩.”戚继光还制定了传烽之法，编成通俗顺口的《传烽歌》让守台官兵背诵熟记。烽火台是白天点狼粪，晚上然柴草，白天烧狼粪用烟比较明显，晚上烧柴草靠火光报警。




Morse code: ‘E’, ‘I’, ‘5’, ‘4’

gzip

- “gzip” is a file format and a software application used for file compression and decompression. "g" is from "GNU“
- The gzip format is used in HTTP compression: Chrome, IE, Firefox
- Linux: gzip command
- Python: gzip module

```
import gzip
s_in = b"Lots of content here"
s_out = gzip.compress(s_in)
```

Abraham Lempel and Jacob Ziv: LZ77, LZ78

# Examples of Codes

Let $X$ be a random variable with the following distribution and codeword assignment:

$$\begin{aligned} \Pr(X = 1) = 1/2, &\quad \text{codeword } C(1) = 0 \\ \Pr(X = 2) = 1/4, &\quad \text{codeword } C(2) = 10 \\ \Pr(X = 3) = 1/8, &\quad \text{codeword } C(3) = 110 \\ \Pr(X = 4) = 1/8, &\quad \text{codeword } C(4) = 111 \end{aligned}$$

- Without of loss of generality, we can assume that the $D$-ary alphabet is $\mathcal{D} = \{0,1,\dots,D-1\}$.
- A source code (信源编码) $C$ for a random variable $X$ is a mapping from $\mathcal{X}$, the range of $X$, to $\mathrm{D}^*$, the set of finite-length strings of symbols from a **D-ary (D元组)** alphabet. Let $\boldsymbol{C(x)}$ denote the **codeword** corresponding to $x$ and let $\boldsymbol{l(x)}$ denote the length of $C(x)$.
- The **expected length** $\boldsymbol{L(C)}$ of a source code $C(x)$ for a random variable $X$ with probability mass function $p(x)$ is given by

$$L(C) = \sum_{x\in\mathcal{X}} p(x)l(x)$$

  - What is $\min L(C)$
  - How to construct such an optimal code

$$H(X) = \frac{7}{4} \text{ and } L(C) = \frac{7}{4}$$

# Nonsingular Code

- A code is said to be nonsingular if every element of the range of X maps into a different string in $D^*$; that is,
  - $x \neq x' \Rightarrow C(x) \neq C(x')$ (单射)
- The extension $C^*$ of a code $C$ is the mapping from finite length strings of $X$ to finite-length strings of $D$, defined by

$$C(x_1x_2 \ldots x_n) = C(x_1)C(x_2) \ldots C(x_n),$$

where $C(x_1)C(x_2)\ldots C(x_n)$ indicates concatenation of the corresponding codewords.

- If $C(x_1) = 00$ and $C(x_2) = 11$, then $C(x_1x_2) = 0011$.

- A code is called uniquely decodable if its extension is nonsingular.

-- In other words, any encoded string in a uniquely decodable code has only one possible source string producing it.

- A code is called a prefix code(前缀码) or an instantaneous code(即时码) if no codeword is a prefix of any other codeword.

--An instantaneous code can be decoded without reference to future codewords since the end of a codeword is immediately recognizable

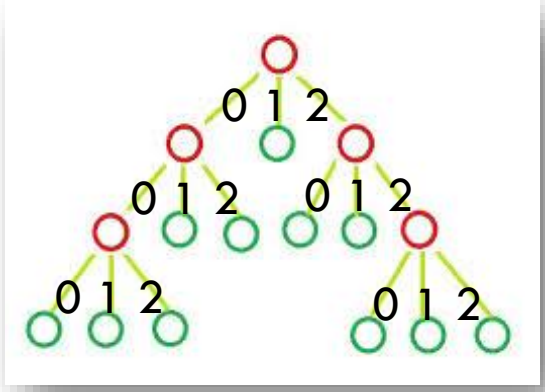
--Suffix code: no codeword is a suffix of any other codeword

**How to construct a prefix code? Enumerate?**

# Prefix Code: Representation

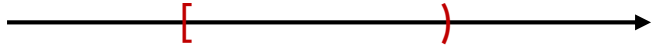
The space of $D$ −ary prefix codes admits a very special structure. To represent them is of fundamental importance

- Tree representation
- Interval representation




Start from the root:

- Each node has $D$ branches.
- Each edge corresponds one of $\{0,1,\dots,D-1\}$
- (Prefix-free) For two code words $c_1, c_2$, the corresponding paths $p_1, p_2$ will not contain each other. (A subtree will be occupied)



$$[0.d_1d_2\dots d_n,\quad 0.d_1d_2\dots d_n+\frac{1}{D^n})$$

$$0.1211 \rightarrow 0.1211xxxx \rightarrow 0.1222$$

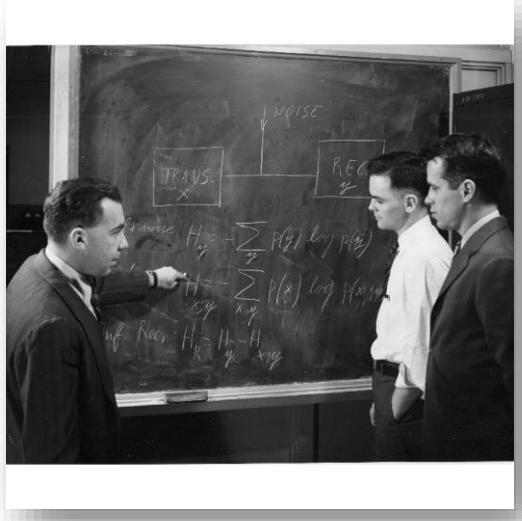
- Each codework $d_1d_2\dots d_n$ can be treated as a $D$-ary floating number $0.d_1d_2\dots d_n$
- The left closed and right open interval $[0.d_1d_2\dots d_n, 0.d_1d_2\dots d_n+\frac{1}{D^n})$
- NO other codeword is allowed in it
- NO overlap between intervals

# Kraft Inequality

**(Kraft Inequality 1949)** For any instantaneous code (prefix code) over an alphabet of size $D$, **the codeword lengths $l_1, l_2, \ldots, l_m$ must satisfy the inequality**

$$\sum_{i=1}^{m} D^{-l_i} \leq 1$$

**Conversely,** given a set of codeword lengths that satisfy this inequality, there exists an instantaneous code with these word lengths.




**Leon G. Kraft**

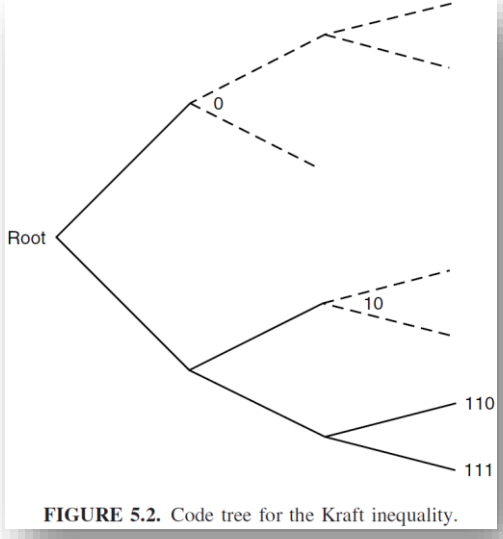



FIGURE 5.2. Code tree for the Kraft inequality.

**$D^{-l_i}$ can be treated as "percentage"**

Assume $l_1 \leq l_2 \ldots \leq l_m$ (The maximum depth is $l_m$)

- For $l_i$, it "occupied" a subtree in size $D^{l_m - l_i}$
- The aggregate size of subtrees

$$\sum_{i=1}^{m} D^{l_m - l_i}$$

- $$\sum_{i=1}^{m} D^{l_m - l_i} \leq D^{l_m} \Rightarrow "only\ if"$$
- "if": mathematical induction

# Extended Kraft Inequality

(Extended Kraft Inequality) For **any countably infinite set of codewords** that form a prefix code, the codeword lengths satisfy the extended Kraft inequality,

$$\sum_{i=1}^{\infty} D^{-l_i} \leq 1$$

Conversely, given any $l_1, l_2, \ldots$ satisfying the extended Kraft inequality, we can construct a prefix code with these codeword lengths.

- Let the D-ary alphabet be $\{0, 1, \ldots, D-1\}$. Consider the $i$th codeword $y_1 y_2 \ldots y_{l_i}$. Let $0.y_1 y_2 \ldots y_{l_i}$ be the real number given by the D-ary expansion

$$\mathbf{0.}\boldsymbol{y_1 y_2 \ldots y_{l_i}} = \sum_{j=1}^{l_i} \boldsymbol{y_{l_j} D^{-j}}$$

- **"Floating number" is more general**
- **An equation to capture "prefix"**

This codeword corresponds to the interval

$$\left[\mathbf{0.}\boldsymbol{y_1 y_2 \ldots y_{l_i}}, \mathbf{0.}\boldsymbol{y_1 y_2 \ldots y_{l_i}} + \frac{\mathbf{1}}{\boldsymbol{D^{l_i}}}\right)$$

- ✓ This is a subinterval of the unit interval [0, 1].
- ✓ By the prefix condition, these intervals are disjoint.
- Code construction: First, reorder the indexing so that $l_1 \leq l_2 \leq \ldots$ Then simply assign the intervals in order **from the low end of the unit interval**.

# Exercise

- (Slackness in the Kraft inequality.) An instantaneous code has word lengths $l_1, l_2, \ldots, l_m$, which satisfy the strict inequality

$$\sum_{i=1}^{m} D^{-l_i} < 1$$

The code alphabet is $\mathcal{D} = \{0, 1, 2, \ldots, D - 1\}$. Show that there exist arbitrarily long sequences of code symbols in $D^*$ which cannot be decoded into sequences of codewords.

- (Fix-free codes) A code is a fix-free code if it is both a prefix code and a suffix code. Let $l_1, l_2, \ldots, l_m$ be $m$ positive integers. Prove that if

$$\sum_{k=1}^{m} 2^{-l_k} \leq \frac{1}{2}$$

then there exists a binary fix-free code with codeword length $l_1, l_2, \ldots, l_m$.

- ($\frac{3}{4}$ fix-free codes) Prove that when

$$\sum_{k=1}^{m} 2^{-l_k} \leq \frac{3}{4}$$

the conclusion above holds.

# Optimal Codes: Problem Formulation

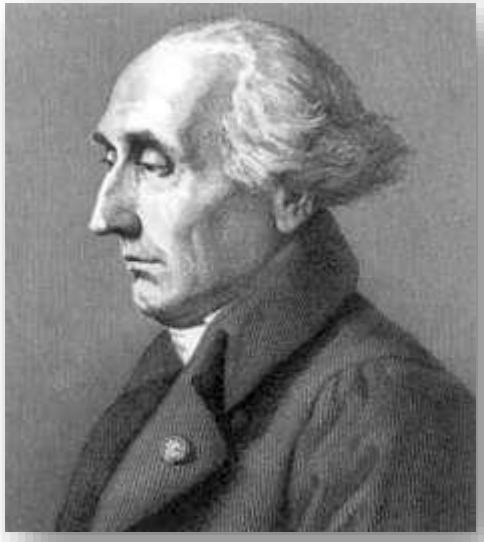
- **Kraft inequality** gives a mathematical expression on the existence of prefix code
- The problem of finding the prefix code with the minimum expected length could be formulated as a standard **optimization problem**

$$\min L = \sum p_i l_i$$

such that

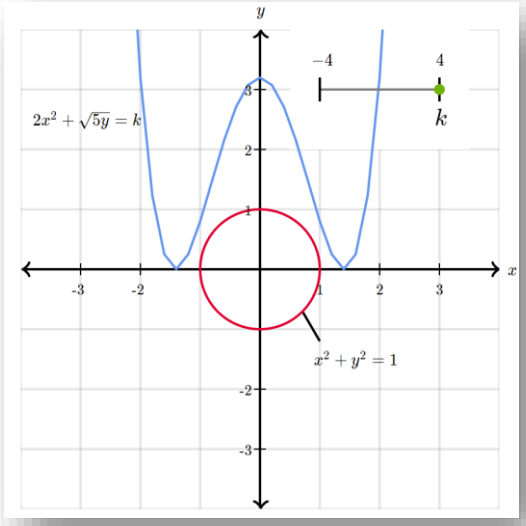
$$\sum D^{-l_i} \leq 1$$

- **How to solve it? Lagrange**



**Joseph Louis Lagrange**
**1736--1813**

$$\min \boldsymbol{f(X)}$$
$$\boldsymbol{s.t.} \quad \boldsymbol{g(X) \leq 0}$$




- Their gradient vectors are parallel

$$\nabla f(X) = \lambda \nabla g$$

- Lagrangian

$$\boldsymbol{\mathcal{L}(X, \lambda) = f(X) + \lambda g}$$

- Necessary condition

$$\boldsymbol{\nabla \mathcal{L} = 0}$$

# Optimal Codes: Solution

- The Lagrange multipliers

$$J = \sum p_i l_i + \lambda(\sum D^{-l_i} - 1)$$

- Differentiating with respect to $l_i$ , we obtain

$$\frac{\partial J}{\partial l_i} = p_i - \lambda D^{-l_i} \log_e D$$

- Setting the derivatives to 0, we obtain

$$D^{-l_i} = \frac{p_i}{\lambda \log_e D}$$

- Substituting this in the constraint to find $\lambda$, we find $\lambda = 1/\log_e D$ , and hence

$$p_i = D^{-l_i}$$

yielding optimal code lengths,

$$l_i^* = -\log_D p_i$$

- This noninteger choice of codeword lengths yields expected codeword length

$$L^* = \sum p_i l_i^* = \sum -p_i \log p_i = H_D(X)$$

- **In general, $H_D(X)$ cannot be attained**

$$L^* \geq H_D(X)$$

$$p_i = D^{-l_i}$$
$$l_i = -\log p_i$$

# Optimal Codes: Bounds

Let $l_1^*, l_2^*, \ldots, l_m^*$ be optimal codeword lengths for a source distribution $\mathbf{p}$ and $D$ −ary alphabet, and let $L^*$ be the associated expected length of an optimal code ($L^* = \sum p_i l_i^*$). Then

$$\boldsymbol{H_D(X) \leq L^* < H_D(X) + 1}$$

**+1**

- Recall that $p_i = D^{-l_i}$ and $l_i = -\log_D p_i$
- Since $\log_D \frac{1}{p_i}$ may not equal to an integer, we round it up to give integer word-length assignments,

$$l_i = \left\lceil \log_D \frac{1}{p_i} \right\rceil \Rightarrow \textbf{\textit{Shannon codes}}$$

- Check $l_i$'s satisfying Kraft inequality.

$$\log_D \frac{1}{p_i} \leq l_i < \log_D \frac{1}{p_i} + 1$$

- Take expectations

$$\boldsymbol{H_D(X) \leq L < H_D(X) + 1}$$

**Can we remove the "1" bit here?**

# Approach the Limit

- Encode $n$ symbols $X_1, X_2, \dots, X_n$ on $\mathcal{X}$ together, where $X_i's$ are i.i.d $\sim p(x)$.
- Denote the alphabet by $\mathcal{X}^n$, the expected codeword length by $L_n$, the length of codeword associated with $(x_1, x_2, \dots, x_n)$ by $l(x_1, x_2, \dots, x_n)$

$$L_n = \frac{1}{n}\sum p(x_1, x_2, \dots, x_n) l(x_1, x_2, \dots, x_n) = \frac{1}{n} El(X_1, X_2, \dots, X_n)$$

- **Treat $X_1, X_2, \dots, X_n$ as a whole and apply the lower bound aforementioned**

$$H(X_1, X_2, \dots, X_n) \le El(X_1, X_2, \dots, X_n) < H(X_1, X_2, \dots, X_n) + 1$$

- Since $X_i's$ are i.i.d, $H(X_1, X_2, \dots, X_n) = nH(X)$

$$H(X) \le L_n \le H(X) + \frac{1}{n}$$

**(Theorem.)** The minimum expected codeword length per symbol statisfies

$$\frac{H(X_1, X_2, \dots, X_n)}{n} \le L^* < \frac{H(X_1, X_2, \dots, X_n)}{n} + \frac{1}{n}$$

Moreover, if $X_1, X_2, \dots, X_n$ is a stationary stochastic process,

$$\boldsymbol{L^* \to H(\mathcal{X})}$$

Entropy rate

# Wrong code

What happens to the expected description length if the code is designed for the wrong distribution ($q(x)$). For example, the wrong distribution may be the best estimate that we can make of the unknown true distribution.

(Wrong code) The expected length under $p(x)$ of the code assignment $l(x) = \log\frac{1}{q(x)}$ satisfies

$$H(p) + D(p||q) \leq E_p l(x) < H(p) + D(p||q) + 1$$

The expected codelength is

$$\begin{aligned} El(x) &= \sum_x p(x) \left\lceil \log \frac{1}{q(x)} \right\rceil \\ &< \sum_x p(x) \left( \log \frac{1}{q(x)} + 1 \right) \\ &= \sum_x p(x) \log \frac{p(x)}{q(x)} \frac{1}{p(x)} + 1 \\ &= \sum_x p(x) \log \frac{p(x)}{q(x)} + \sum_x p(x) \log \frac{1}{p(x)} + 1 \\ &= D(p||q) + H(p) + 1 \end{aligned}$$

$\boldsymbol{D(p||q)}$

# Kraft Inequality For Uniquely Decodable Codes

**(McMillan)** The codeword lengths of **any uniquely decodable** D-ary code must satisfy the Kraft inequality

$$\sum D^{-l_i} \leq 1$$

Conversely, given a set of codeword lengths that satisfy this inequality, it is possible to construct a uniquely decodable code with these codeword lengths.

- ✓ Consider $C^k$ the $k$th extension of the code (i.e., the code formed by the concatenation of $k$ repetitions of the given uniquely decodable code $C$).
- ✓ By the definition of unique decodability, the $k$th extension of the code is nonsingular.
- ✓ Since there are only $D^n$ different $D$-ary strings of length $n$, unique decodability implies that the number of code sequences of length $n$ in the $k$th extension of the code must be no greater than $D^n$.

**The class of uniquely decodable codes does not offer any further possibilities for the set of codeword lengths than do instantaneous codes.**

# Kraft Inequality For Uniquely Decodable Codes (Cont'd)

- Let the codeword lengths of the symbols $x \in X$ be denoted by $l(x)$. For the extension code, the length of the code sequence is $l(x_1, x_2, \dots, x_k) = \sum_{i=1}^{k} l(x_i)$
- The inequality we wish to prove is $\sum_{x \in \mathcal{X}} D^{-l(x)} \leq 1$.
- Consider the $k$th power of this quantity

$$\left( \sum_{x \in \mathcal{X}} D^{-l(x)} \right)^{k} = \sum_{x_1 \in \mathcal{X}} \sum_{x_2 \in \mathcal{X}} \dots \sum_{x_k \in \mathcal{X}} D^{-l(x_1)} D^{-l(x_2)} \dots D^{-l(x_k)}$$

$$= \sum_{x_1, x_2, \dots, x_k \in \mathcal{X}^k} D^{-l(x_1)} D^{-l(x_2)} \dots D^{-l(x_k)}$$

$$= \sum_{x^k \in \mathcal{X}^k} D^{-l(x^k)} = \sum_{m=1}^{k l_{\max}} a(m) D^{-m}$$

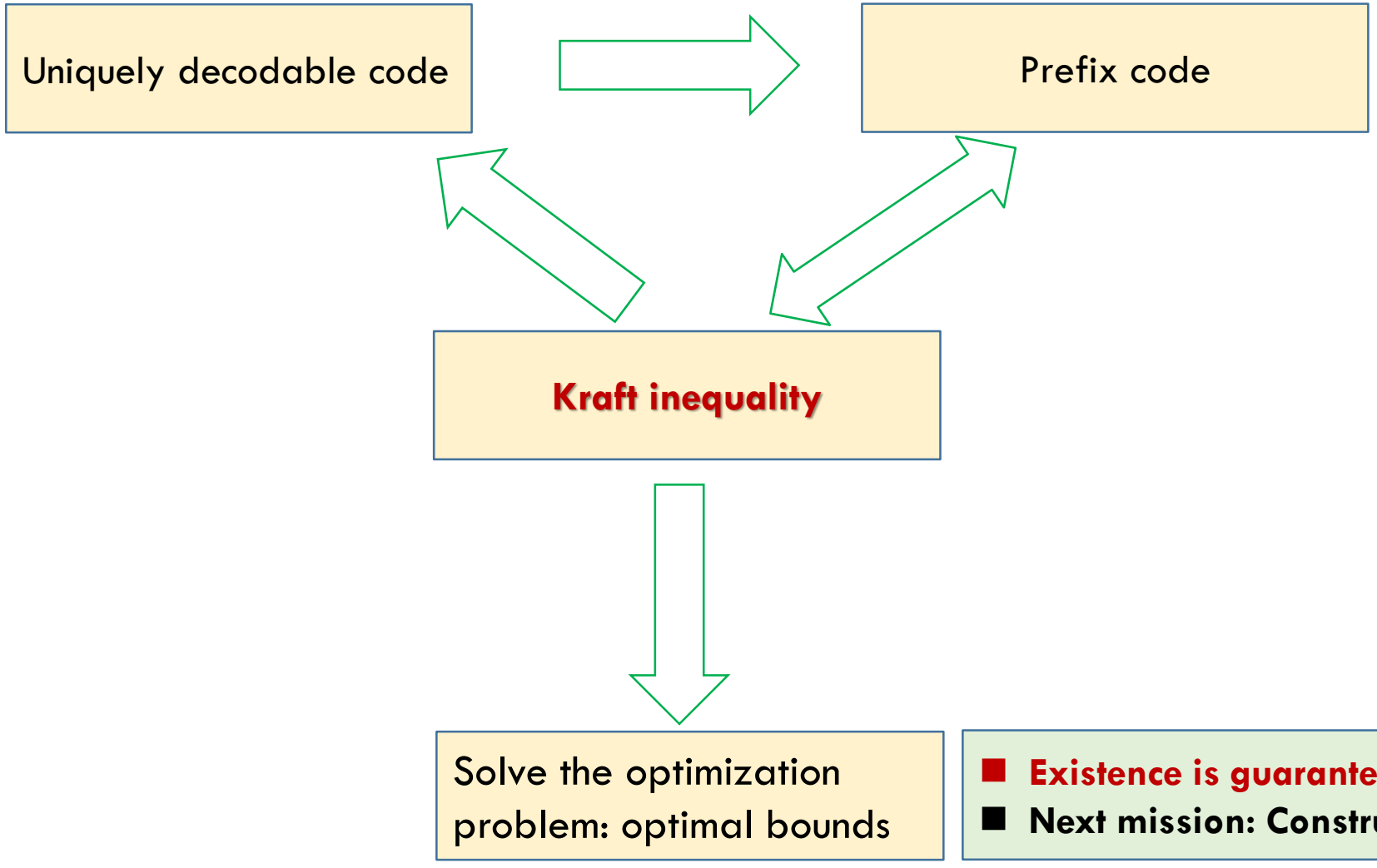
$$\leq \sum_{m=1}^{k l_{\max}} D^m D^{-m} = k l_{\max}$$

$$\sum_{x \in \mathcal{X}} D^{-l(x)} \leq (k l_{\max})^{\frac{1}{k}} \to 1, \text{as } k \to \infty$$

$l_{max}$ is the maximum codeword length and $a(m)$ is the number of source sequences $x^k$ mapping into codewords of length $m$.

# Roadmap

**Homework**
**Cover: 5.7, 5.8, 5.14, 5.16, 5.28**



# Summary

Related material: Cover, 5.1, 5.2, 5.3, 5.4, 5.5