Sum of Squares (SOS) Techniques: An Introduction

Amir Ali Ahmadi, Princeton ORFE

Sum of squares optimization is an active area of research at the interface of algorithmic algebra and convex optimization. Over the last decade, it has made significant impact on both discrete and continuous optimization, as well as several other disciplines, notably control theory. A particularly exciting aspect of this research area is that it leverages classical results from real algebraic geometry, some dating back to prominent mathematicians like Hilbert. Yet, it offers a modern, algorithmic viewpoint on these concepts, which is amenable to computation and deeply rooted in semidefinite programming. In this lecture, we give an introduction to sum of squares optimization focusing as much as possible on aspects relevant to ORF523, namely, complexity and interplay with convex optimization. A presentation of this length is naturally incomplete. The interested reader is referred to a very nice and recent edited volume by Blekherman, Parrilo, and Thomas, the PhD thesis of Parrilo or his original paper, the independent papers by Lasserre and by Nesterov, the paper by Shor (translated from Russian), and the survey papers by Laurent and by Reznick. Much of the material below can be found in these references.

Polynomial Optimization

For the purposes of this lecture, we motivate the sum of squares machinery through the polynomial optimization problem:

$$\begin{array}{ll} \text{minimize} & p(x) \\ \text{subject to} & x \in K := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, h_i(x) = 0\}, \end{array} \tag{1}$$

where p, g_i, and h_i are multivariate polynomials. A set defined by a finite number of polynomial inequalities (such as the set K above) is called *basic semialgebraic*. Of course, we can write K with polynomial inequalities only (by replacing $h_i(x) = 0$ with $h_i(x) \geq 0$ and $-h_i(x) \geq 0$), or (unlike the case of linear programming) with polynomial equalities only (by replacing $g_i(x) \geq 0$ with $g_i(x) - z_i^2 = 0$, for some new variables z_i). We prefer, however, to keep the general form above since we will later treat polynomial inequalities and equalities slightly differently.

The special case of problem (1) where the polynomials p, g_i, h_i all have degree one is of course *linear programming*, which we can solve in polynomial time. Unfortunately though, as we will review in the complexity section of these notes below, the problem quickly becomes intractable when the degrees increase from one ever so slightly. For example, unconstrained minimization of a quartic polynomial, minimization of a cubic polynomial over the sphere, or minimization of a quadratic polynomial over the simplex are all NP-hard.

The sum of squares methodology offers a hierarchy of polynomially sized semidefinite programming relaxations to cope with this computational intractability. It is quite different in philosophy from the approach taken by, say, the descent methods in nonlinear optimization. In particular, it makes absolutely *no assumptions* about convexity of the objective function p, or the constraint set K. Nevertheless, the hierarchy has a proof of asymptotic convergence to a globally optimal solution and in practice often the first few levels of the hierarchy suffice to solve the problem globally.

If we could optimize over nonnegative polynomials...

A point of departure for the sum of squares methodology is the observation that if we could optimize over the set of polynomials that take *nonnegative* values over given basic semialgebraic sets, then we could solve problem (1) globally. To see this, note that the optimal value of problem (1) is equal to the optimal value of the following problem:

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{subject to} & p(x) - \gamma \geq 0, \ \forall x \in K. \end{array} \tag{2}$$

Here, we are trying to find the largest constant γ such that $p(x) - \gamma$ is nonnegative on the set K. This formulation suggests the need to think about a few fundamental questions: given a basic semialgebraic set K as in (1), what is the structure of the set of polynomials (of, say, some fixed degree) that take only nonnegative values on K? Can we efficiently optimize a linear functional over the set of such polynomials? Can we even test membership to this set efficiently?

Observe that independent of the convexity of the set K, the set of polynomials that take nonnegative values on it form a convex set! Albeit, as we see next, this convex set is not quite tractable to work with.

Complexity considerations[1]

We first show that testing membership to the set of polynomials that take nonnegative values over a basic semialgebraic set K is NP-hard, even when $K = \mathbb{R}^n$. In order to give a very simple reduction "from scratch", we first prove this claim with the word "nonnegative" replaced by "positive".

Theorem 0.1. *Given a polynomial p of degree 4, it is strongly NP-hard to decide if it is* positive definite*, i.e., if $p(x) > 0$ for all $x \in \mathbb{R}^n$.*

Proof. We recall our reduction from ONE-IN-THREE-3SAT. (The reason why we pick this problem over the more familiar 3SAT is that an equally straightforward reduction from the latter problem would only prove hardness of positivity testing for polynomials of degree 6.) In ONE-IN-THREE 3SAT, we are given a 3SAT instance (i.e., a collection of clauses, where each clause consists of exactly three literals, and each literal is either a variable or its negation) and we are asked to decide whether there exists a $\{0, 1\}$ assignment to the variables that makes the expression true with the additional property that each clause has *exactly one* true literal.

To avoid introducing unnecessary notation, we present the reduction on a specific instance. The pattern will make it obvious that the general construction is no different. Given an instance of ONE-IN-THREE 3SAT, such as the following

$$(x_1 \vee \bar{x}_2 \vee x_4) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee x_5) \wedge (\bar{x}_1 \vee x_3 \vee \bar{x}_5) \wedge (x_1 \vee x_3 \vee x_4), \tag{3}$$

[1] You have seen some of these reductions either in previous lectures or on the homework. But I include them here for completeness/review.

we define the quartic polynomial p as follows:

$$\begin{array}{rcl} p(x) & = & \sum_{i=1}^{5} x_i^2(1-x_i)^2 \\ & & +(x_1+(1-x_2)+x_4-1)^2+((1-x_2) \\ & & +(1-x_3)+x_5-1)^2 \\ & & +((1-x_1)+x_3+(1-x_5)-1)^2 \\ & & +(x_1+x_3+x_4-1)^2. \end{array} \tag{4}$$

Having done so, our claim is that $p(x) > 0$ for all $x \in \mathbb{R}^5$ (or generally for all $x \in \mathbb{R}^n$) if and only if the ONE-IN-THREE 3SAT instance is not satisfiable. Note that p is a sum of squares and therefore nonnegative. The only possible locations for zeros of p are by construction among the points in $\{0,1\}^5$. If there is a satisfying Boolean assignment x to (3) with exactly one true literal per clause, then p will vanish at point x. Conversely, if there are no such satisfying assignments, then for any point in $\{0,1\}^5$, at least one of the terms in (4) will be positive and hence p will have no zeros. □

Deciding if a polynomial p is *nonnegative*—i.e., if $p(x) \geq 0$ for all $x \in \mathbb{R}^n$—is also NP-hard if we consider polynomials of degree 4 or higher even degree. A simple reduction is from the *matrix copositivity* problem: Given a symmetric matrix M, decide if $x^T M x \geq 0$ for all $x \geq 0$. (Note the similarity to testing matrix positive semidefiniteness, yet the drastic difference in complexity.) To see the connection to polynomial nonnegativity, observe that the quartic homogeneous polynomial

$$v(x)^T M v(x),$$

with $v(x) := (x_1^2, \ldots, x_n^2)^T$, is nonnegative if and only if M is a copositive matrix.

We already proved NP-hardness of testing matrix copositivity via a reduction from CLIQUE. If you remember, the main ingredient was the Motzkin-Straus theorem[2]: The stability number $\alpha(G)$ of a graph G with adjacency matrix A satisfies

$$\frac{1}{\alpha(G)} = \min_{x_i \geq 0, \sum x_i = 1} x^T(A+I)x.$$

A quadratic programming formulation makes sum of squares techniques directly applicable to the STABLE SET problem, and in a similar vein, applicable to *any* NP-complete problem. We end our complexity discussion with a few remarks.

- The set of nonnegative polynomials and the set of copositive matrices are both examples of *convex* sets for which optimizing a linear functional, or even testing membership, is NP-hard. In view of the common misconception about "convex problems being easy," it is important to emphasize again that the algebraic/geometric structure of the set, beyond convexity, cannot be ignored.

- Back to the polynomial optimization problem in (1), the reductions we gave above already imply that unconstrained minimization of a quartic polynomial is NP-hard. The aforementioned hardness of minimizing a quadratic form over the standard simplex follows e.g. from the Motzkin-Straus theorem above. Unlike the case of the simplex, minimizing a quadratic form over the unit sphere is easy. We have seen already that this problem (although non-convex in this formulation!) is simply an eigenvalue problem. On the other hand, minimizing forms of degree 3 over the unit sphere is NP-hard, due to a result of Nesterov.

[2]We saw this before for the clique number of a graph. This is an equivalent formulation of the theorem for the stability (aka independent set) number.

- Finally, we remark that for neither the nonnegativity problem nor the positivity problem did we claim membership in the class NP or co-NP. This is because these problems are still open! One may think at first glance that both problems should be in co-NP: If a polynomial has a zero or goes negative, simply present the vector x at which this happens as a certificate. The problem with this approach is that there are quartic polynomials, such as the following,

$$p(x) = (x_1 - 2)^2 + (x_2 - x_1^2)^2 + (x_3 - x_2^2)^2 + \cdots + (x_n - x_{n-1}^2)^2,$$

for which the only zero takes 2^n bits to write down. Membership of these two problems in the class NP is much more unlikely. Afterall, how would you give a certificate that a polynomial *is* nonnegative? Read on...

Sum of squares and semidefinite programming

If a polynomial is nonnegative, can we write it in a way that its nonnegativity becomes obvious? This is the meta-question behind Hilbert's 17th problem. As the title of this lecture suggests, one way to achieve this goal is to try to write the polynomial as a sum of squares of polynomials. We say that a polynomial p is a *sum of squares* (sos), if it can be written as $p(x) = \sum_i q_i^2(x)$ for some polynomials q_i. Existence of an sos decomposition is an *algebraic certificate* for nonnegativity. Remarkably, it can be decided by solving a single semidefinite program.

实现这个目标的一种方法是尝试把多项式写成多项式的平方和。我们说一个多项式p是一个平方和（sos），如果它可以写成p (x) = P i qi 2 (x)。sos分解的存在性是非负性的代数证明。值得注意的是，它可以通过求解一个单一的半定程序来决定。

Theorem 0.2. *A multivariate polynomial p in n variables and of degree $2d$ is a sum of squares if and only if there exists a positive semidefinite matrix Q (often called the Gram matrix) such that*

$$p(x) = z^T Q z, \tag{5}$$

where z is the vector of monomials of degree up to d

$$z = [1, x_1, x_2, \ldots, x_n, x_1 x_2, \ldots, x_n^d].$$

Proof. If (5) holds, then we can do a Cholesky factorization on the Gram matrix, $Q = V^T V$, and obtain the desired sos decomposition as

$$p(x) = z^T V^T V z = (Vz)^T (Vz) = ||Vz||^2.$$

Conversely, suppose p is sos:

$$p = \sum_i q_i^2(x),$$

then for some vectors of coefficients a_i, we must have

$$p = \sum_i (a_i^T z(x))^2 = \sum_i (z^T(x) a_i)(a_i^T z(x)) = z^T(x)(\sum_i a_i a_i^T) z(x),$$

so the positive semidefinite matrix $Q := \sum_i a_i a_i^T$ can be extracted. As a corollary of the proof, we see that the number of squares in our sos decomposition is exactly equal to the rank of the Gram matrix Q. $\square$

Note that the feasible set defined by the constraints in (5) is the intersection of an affine subspace (arising from the equality constraints matching the coefficients of p with the entries of Q) with the cone of positive semidefinite matrices. This is precisely the semidefinite programming (SDP) problem. The size of the Gram matrix Q is $\binom{n+d}{d} \times \binom{n+d}{d}$, which for fixed d is polynomial in

n. Depending on the structure of p, there are well-documented techniques for further reducing the size of the Gram matrix Q and the monomial vector z. We do not pursue this direction here but state as an example that if p is homogeneous of degree $2d$, then it suffices to place in the vector z only monomials of degree exactly d.

Example 0.1. Consider the task proving nonnegativity of the polynomial

$$\begin{aligned} p(x) \;\; = \;\; & x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_3^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 - 14x_1x_2x_3^2 + 4x_1x_3^3 \\ & +5x_3^4 - 7x_2^2x_3^2 + 16x_2^4. \end{aligned}$$

Since this is a form (i.e., a homogeneous polynomial), we take

$$z = (x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2)^T.$$

One feasible solution to the SDP in (5) is given by

$$Q = \begin{pmatrix} 1 & -3 & 0 & 1 & 0 & 2 \\ -3 & 9 & 0 & -3 & 0 & -6 \\ 0 & 0 & 16 & 0 & 0 & -4 \\ 1 & -3 & 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 2 & -6 & 4 & 2 & 0 & 5 \end{pmatrix}.$$

Upon a decomposition $Q = \sum_{i=1}^{3} a_i^T a_i$, with $a_1 = (1, -3, 0, 1, 0, 2)^T$, $a_2 = (0, 0, 0, 1, -1, 0)^T$, $a_3 = (0, 0, 4, 0, 0, -1)^T$, one obtains the sos decomposition

$$p(x) = (x_1^2 - 3x_1x_2 + x_1x_3 + 2x_3^2)^2 + (x_1x_3 - x_2x_3)^2 + (4x_2^2 - x_3^2)^2. \tag{6}$$

$\triangle$

You are probably asking yourself right now whether *every* nonnegative polynomial can be written as a sum of squares. Did we just get lucky on the above example? Well, from complexity considerations alone, we know that we should expect a gap between nonnegative and sos polynomials, at least for large n.

In a seminal 1888 paper, Hilbert was the first to show that there exist nonnegative polynomials that are not sos. In fact, for each combination of degree and dimension, he showed whether such polynomials do or do not exist. Here is his theorem.

Theorem 0.3. *All nonnegative polynomials in n variables and degree d are sums of squares if and only if*

- $n = 1$, *or*
- $d = 2$, *or*
- $n = 2, d = 4$.

The proofs of the first two cases are straightforward (we did them on the board in class). The contribution of Hilbert was to prove the last case, and to prove that these are the *only* cases where nonnegativity equals sos. These results are usually stated in the literature for forms (i.e., homogeneous polynomials). Recall that given a polynomial $p := p(x_1, \ldots, x_n)$ of degree d, we can *homogenize* it by introducing one extra variable

$$p_h(x, y) := y^d p(\frac{x}{y}),$$

and then recover p back by dehomogenizing p_h:

$$p(x) = p_h(x, 1).$$

We proved in a previous lecture the simple fact that the property of being nonnegative is preserved under both operations. It is an easy exercise to establish the same claim for the property of being sos. As a result, the result of Hilbert is equivalent to the following statement:

All nonnegative forms in n variables and degree d are sums of squares if and only if

- $n = 2$, or
- $d = 2$, or
- $n = 3, d = 4$.

Since all nonnegative ternary quartic forms are sos, we see that we did not really get lucky in the example we gave above. The same would have happened for any other nonnegative quartic form in three variables!

Hilbert's proof of existence of nonnegative polynomials that are not sos was not constructive. The first explicit example interestingly appeared nearly 80 years later and is due to Motzkin:

$$M(x_1, x_2, x_3) := x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 x_3^2 + x_3^6. \tag{7}$$

Nonnegativity of M follows from the arithmetic-geometric inequality:

$$\frac{x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6}{3} \geq x_1^2 x_2^2 x_3^2$$

Non-existence of an sos decomposition can be shown by assuming a decomposition $M = \sum q_i^2$ (with each q_i being a ternary form of degree 3), comparing coefficients, and reaching a contradiction. (We did this on the board in class). Alternatively, we could show that the Motzkin polynomial is not sos, by proving that the underlying SDP from Theorem 0.2 is infeasible.

From an application viewpoint, the good news for sum of squares optimization is that constructing polynomials of the type in (7) is not a trivial task. This is especially true if additional structure is required on the polynomial. For example, the following problem is still open.

Open problem. Construct an explicit example of a *convex*, nonnegative polynomial that is not a sum of squares.

This question is due to Parrilo. The motivation behind it is the following: we would like to understand whether sos optimization is exact for the special case of convex polynomial optimization problems. Blekherman has shown with non-constructive arguments that such "bad" convex polynomials must exist when the degree is four or larger and the number of variables goes to infinity. However, we do not know the smallest (or any reasonable) dimension for which this is possible and lack any explicit examples. For the reader interested in tackling this problem, it is known that any such convex polynomial must necessarily be not "sos-convex". Roughly speaking, sos-convex polynomials are convex polynomials whose convexity is certified by an appropriately defined sum of squares identity. Examples of convex but not sos-convex polynomials have recently appeared and a characterization of the degrees and dimensions where they exist is now available. Interestingly, this characterization coincides with that of Hilbert, for reasons that are also not fully understood; see Chapter 3 of our thesis.

Having shown that not every nonnegative polynomial is a sum of squares of polynomials, Hilbert asked in his 17th problem whether every such polynomial can be written as a sum of squares of *rational* functions. Artin answered the question in the affirmative in 1927. As we will see next, such results allow for a hierarchy of semidefinite programs that approximate the set of nonnegative polynomials better and better.

Positivstellensatz and the SOS hierarchy

Consider proving a statement that we all learned in high school:

$$\forall a, b, c, x, \ ax^2 + bx + c = 0 \Rightarrow b^2 - 4ac \geq 0.$$

Just for the sake of illustration, let us pull an algebraic identity out of our hat which *certifies* this claim:

$$b^2 - 4ac = (2ax + b)^2 - 4a(ax^2 + bx + c). \tag{8}$$

Think for a second why this constitutes a proof. The Positivstellensatz is a very powerful algebraic proof system that vastly generalizes what we just did here. It gives a systematic way of certifying infeasibility of *any* system of polynomial equalities and inequalities over the reals. Sum of squares representations play a central role in it. (They already did in our toy example above if you think about the role of the first term on the right hand side of (8)). Modern optimization theory adds a wonderfully useful aspect to this proof system: we can now use semidefinite programming to automatically find suitable algebraic certificates of the type in (8).

The Positivstellensatz is an example of a theorem of the alternative. We have already seen some results of this type, for example, the Farkas Lemma (1902) of linear programming,

"a system of linear (in)equalities $Ax + b = 0, Cx + d \geq 0$ is infeasible over the reals

$$\Updownarrow$$

there exist $\lambda \geq 0, \mu$ such that $A^T\mu + C^T\lambda = 0, b^T\mu + d^T\lambda = -1$."

or our beloved S-lemma,

"under mild regularity assumptions, a system of two quadratic inequalities

$$q_1(x) \geq 0, q_2(x) < 0$$

is infeasible over the reals

$$\Updownarrow$$

there exist a scalar $\lambda \geq 0$ and affine polynomials g_i such that $q_2 - \lambda q_1 = \sum_i g_i^2$."

Another famous theorem of this type is Hilbert's (weak) Nullstellensatz (1893),

"a system of polynomial equations $f_i(z) = 0$ is infeasible over the complex numbers

$$\Updownarrow$$

there exist polynomials $t_i(z)$ such that $\sum_i t_i(z) f_i(z) = -1$,"

All these theorems typically have an "easy" (well, trivial) direction and a "hard" direction. The same is true for the Positivstellensatz.

Theorem 0.4 (Positivstellensatz – Stengle (1974))**.** *The basic semialgebraic set*

$$K := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, k\} \quad \textit{is empty}$$

$$\Updownarrow$$

there exist polynomials $t_1, \ldots, t_k$ *and sum of squares polynomials* $s_0, s_1, \ldots, s_m$, $s_{12}, s_{13}, \ldots, s_{m-1m}$, $s_{123}, \ldots, s_{m-2m-1m}$, $\ldots, s_{12\ldots m}$ *such that*

$$\begin{array}{rcl} -1 & = & \sum_{i=1}^{k} t_i(x)h_i(x) + s_0(x) + \sum_{\{i\}} s_i(x)g_i(x) \\ & + & \sum_{\{i,j\}} s_{ij}(x)g_i(x)g_j(x) + \sum_{\{i,j,k\}} s_{ijk}(x)g_i(x)g_j(x)g_k(x) \\ & + & \cdots + s_{1\ldots m}(x)g_i(x)\ldots g_m(x). \end{array} \tag{9}$$

The number of terms in this expression is finite since we never raise any polynomial g_i to a power larger than one. The sum of squares polynomials $s_{i\ldots j}$ are of course allowed to be the zero polynomial, and in practice many of them often are. There are bounds in the literature on the degree of the polynomials t_i, $s_{i\ldots j}$, but of exponential size as one would expect for complexity reasons. There is substantial numerical evidence, however, from diverse application areas, indicating that in practice (whatever that means) the degrees of these polynomials are usually quite low. We remark that the Positivstellensatz is a very powerful result. For example, it is a good exercise to show that the solution to Hilbert's 17th problem follows as a straightforward corollary of this theorem.

Under minor additional assumptions, refined versions of the Positivstellensatz we presented are available. The two most well-known are perhaps due to Schmüdgen and Putinar. For example, Putinar's Positivstellensatz states that if the set K satisfies the so-called Archimedean property (a property slightly stronger than compactness), then emptiness of K guarantees a representation of the type (9), where the second and third line are scratched out; i.e., there is no need to take products of the constraints $g_i(x) \geq 0$. While this may look like a simplification at first, there is a tradeoff: the degree of the sos multipliers s_i may need to be higher in Putinar's representation than in Stengle's. This makes intuitive sense as the proof system needs to additionally prove statements of the type $g_i \geq 0, g_j \geq 0 \Rightarrow g_i g_j \geq 0$, while in Stengle's representation this is taken as an axiom.

SOS hierarchies. Positivstellensatz results form the basis of sos hierarchies of Parrilo and Lasserre for solving the polynomial optimization problem (1). The two approaches only differ in the version of the Positivstellensatz they use (originally, Parrilo's paper follows Stengle's version and Lasserre's follows Putinar's), and the fact that Lasserre presents the methodology from the dual (but equivalent) viewpoint of moment sequences. In either case though, the basic idea is pretty simple. We try to obtain the largest lower bound for problem (1), by finding the largest γ for which the set $\{x \in K, p(x) \leq \gamma\}$ is empty. We certify this emptiness by finding Positivstellensatz certificates. In level l of the hierarchy, the degree of the polynomials t_i and the sos polynomails s_i in (9) is bounded by l. As l increases, the quality of the lower bound monotonically increases, and for each fixed l, the search for the optimal γ, and the polynomials t_i, s_i is a semidefinite optimization problem (possibly with some bisection over γ).

Application to MAXCUT

One of the most famous applications of semidefinite programming to combinatorial optimization is the beautiful algorithm Goemans and Williamson for MAXCUT which produces an approximation ratio of 0.878. This algorithm (covered in one of our other lectures) has two steps: first we solve a semidefinite program, then we perform a rather clever randomized rounding step. In this section we focus only on the first step. We show that even low degree Positivstellensatz refutations can produce stronger bounds than the standard SDP relaxation.

Consider the 5-cycle with all edge weights equal to one. It is easy to see that the MAXCUT value of this graph is equal to 4. However, the standard SDP relaxation (i.e. the one used in the Goemans and Williamson algorithm) produces an upper bound of $\frac{5}{8}(\sqrt{5}+5) \approx 4.5225$.

The MAXCUT value of the 5-cycle is equal to minus the optimal value of the quadratic program

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_1x_5) - \frac{5}{2} \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, 5. \end{array}$$

We will find the largest constant γ such that the objective function minus γ is algebraically certified to be nonnegative on the feasible set. To do this, we solve the sos optimization problem

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{such that} & \frac{1}{2}(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_1x_5) - \frac{5}{2} - \gamma + \sum_{i=1}^{5} t_i(x)(x_i^2 - 1) \quad \text{is sos.} \end{array}$$

The decision variables of this problem are the constant γ and the coefficients of the polynomials $t_i(x)$, which in this case we parametrize to be quadratic functions. This sos program results in a polynomially sized semidefinite optimization problem via Theorem 0.2. The optimal value of the program is -4; i.e., we have solved the MAXCUT instance exactly.

You may be wondering, "can we show that a certain level of the sos hierarchy combined with an appropriately designed rounding procedure produces an approximation ratio of better than 0.878?" Let's just say that if you did this, you would probably become an overnight celebrity.

Software

There are very nice implementations of sum of squares optimization solvers that automate the process of setting up the resulting semidefinite programs. The interested reader may want to play around with SOSTOOLS, YALMIP, or GloptiPoly. We have already posted some MATLAB demo files to familiarize you with YALMIP.

Impact

While we focused in this lecture on the polynomial optimization problem, the impact of sum of squares optimization goes much beyond this area. In dynamics and control, sos optimization has enabled a paradigm shift from classical linear control to an efficient framework for design of nonlinear controllers that are provably safer, more agile, and more robust. Papers on applications of sos optimization have appeared in areas as diverse as quantum information theory, robotics, geometric theorem proving, formal verification, derivative pricing, stochastic optimization, and game theory, among others. In theoretical computer science, sos techniques are currently a subject of intense study. You will see some of these applications in future lectures, homework, or the final exam ;) If you are an ORFE student and still not convinced that this SOS business is actually useful, you may find relief in knowing that SOS is in fact the core discipline of ORFE ;)