Lecture 15: Higher-Order Schemes for Steady Convection

Last Time…

We looked more closely into the behavior of the convection operator by looking at unsteady convection

- Consider the stability of implicit and explicit time-differencing schemes for
  - » CDS
  - » UDS
  - » Lax-Wendroff scheme
- Considered the corresponding model equations and try to explain the observed behavior

This Time…

- Start looking at higher-order schemes for the steady convection operator

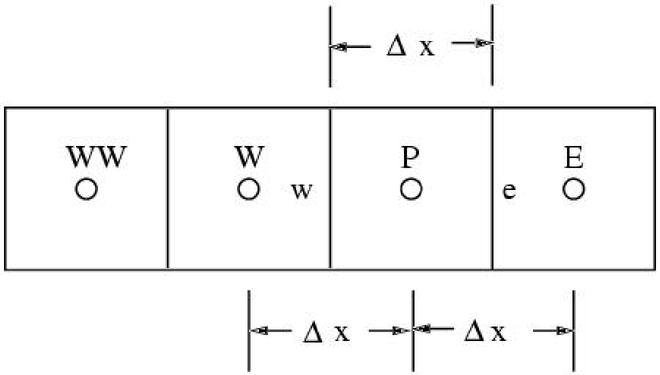
Higher-Order Schemes

- Neither UDS nor CDS are satisfactory
  - » Lot of research in devising schemes which are at least second order accurate spatially
  - » Control of spatial oscillations
- Let's first look at the basis for creating higher-order schemes using the Taylor series
- Then, over the next few lectures, we will look at approaches to controlling spatial oscillations

Taylor Series Basis

- For first-order UDS:

$$\phi_e = \phi_P$$



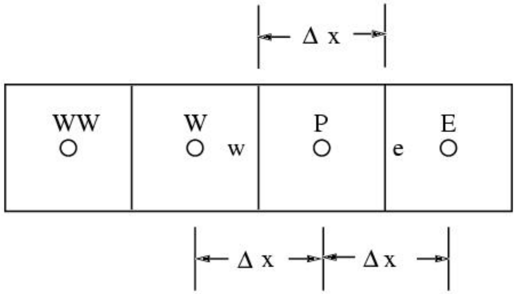
- We may consider this a truncation of a Taylor series:

$$\phi(x) = \phi_P + (x - x_P)\frac{\partial \phi}{\partial x} + \frac{(x - x_P)^2}{2!}\frac{\partial^2 \phi}{\partial x^2} + O(\Delta x)^3$$

- What if we truncated to higher order?
- Note that we are still using an upwinded expansion

Second-Order Schemes

- Taylor series about P



$$\phi(x) = \phi_P + (x - x_P)\frac{\partial \phi}{\partial x} + \frac{(x - x_P)^2}{2!}\frac{\partial^2 \phi}{\partial x^2} + O(\Delta x)^3$$

- Truncate Taylor series after second term:

$$\phi_e = \phi_P + \frac{\Delta x}{2}\frac{\partial \phi}{\partial x}$$

Truncation error : $O(\Delta x^2)$

- Many ways to write gradient
  - » Gradient must be written to at least $O(\Delta x)$
  - » Can write gradient at P using forward, backward or central difference

Basis of Fromm Scheme

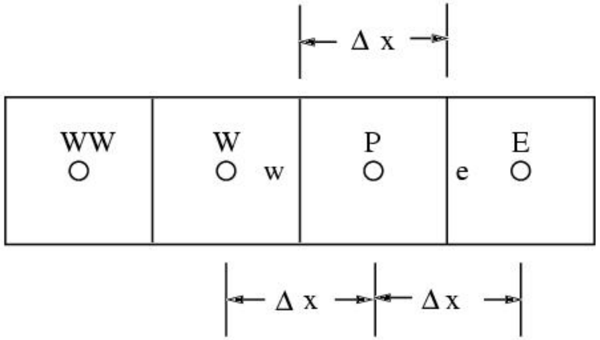
- Central difference

$$\frac{\partial \phi}{\partial x} = \frac{\phi_E - \phi_W}{2\Delta x}$$

Truncation error : $O(\Delta x^2)$

- Thus:

$$\phi_e = \phi_P + \frac{(\phi_E - \phi_W)}{4}$$



- Add and subtract $\phi_P/4$

$$\phi_e = \phi_P + \frac{(\phi_P - \phi_W)}{4} + \frac{(\phi_E - \phi_P)}{4}$$

Basis of Fromm Scheme

(Actual Fromm scheme has other terms – more on this later)

Basis of Beam-Warming Scheme

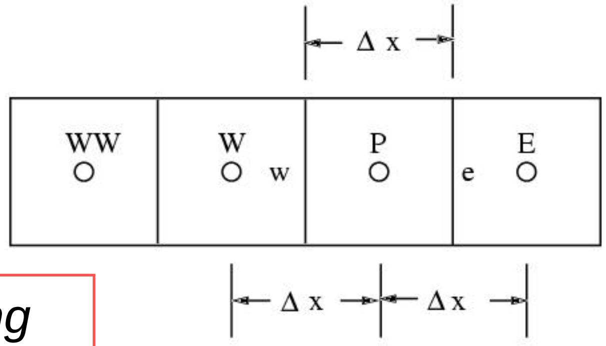
- Second-order scheme: $$\phi_e = \phi_P + \frac{\Delta x}{2}\frac{\partial \phi}{\partial x}$$
- Write gradient as:

$$\frac{\partial \phi}{\partial x} = \frac{\phi_P - \phi_W}{\Delta x}$$

Truncation error : O(Δx)

- Combining:

$$\phi_e = \phi_P + \frac{(\phi_P - \phi_W)}{2}$$



Basis of Beam-Warming scheme

Third-Order Schemes

- Truncate Taylor series to third-order:

$$\phi(x) = \phi_P + (x - x_P)\frac{\partial \phi}{\partial x} + \frac{(x - x_P)^2}{2!}\frac{\partial^2 \phi}{\partial x^2}$$

- Need to write $\left(\frac{\partial \phi}{\partial x}\right)_P$ to at least second order

- Need to write $\left(\frac{\partial^2 \phi}{\partial x^2}\right)_P$ to at least first order

Third-Order Scheme: QUICK

- Quadratic Upwind Interpolation for Convective Kinetics (QUICK):

$$\frac{\partial \phi}{\partial x} = \frac{(\phi_E - \phi_W)}{2\Delta x} + O(\Delta x^2)$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{(\phi_E + \phi_W - 2\phi_P)}{(\Delta x)^2} + O(\Delta x^2)$$

- Combining:

$$\phi_e = \phi_P + \frac{(\phi_E - \phi_W)}{4} + \frac{(\phi_E + \phi_W - 2\phi_P)}{8}$$

$O(\Delta x^3)$ accurate

QUICK (Cont'd)

- Rearranging

$$\phi_e = \frac{(\phi_E + \phi_P)}{2} - \frac{(\phi_E + \phi_W - 2\phi_P)}{8}$$

Central difference (linear)

Curvature term

$$\phi_e = \frac{1}{2}(\phi_E + \phi_P) - C(\phi_E + \phi_W - 2\phi_P)$$

Curvature factor C=1/8

Combined Representation

- The second- and third-order schemes we have seen can be combined into a single expression:

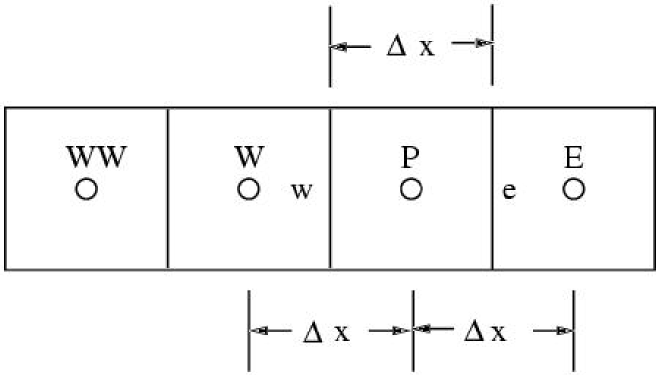
$$\phi_e = \phi_P + \frac{(1-\kappa)}{4}(\phi_P - \phi_W) + \frac{(1+\kappa)}{4}(\phi_E - \phi_P)$$

- κ = -1   Beam Warming scheme
- κ = 0   Fromm scheme
- κ = 1/2   QUICK
- κ = 1   Central difference scheme

Discussion

- If we used these schemes in steady convection problems, we would get spatial oscillations
- If used in conjunction with explicit time stepping schemes, all these schemes are *unconditionally unstable*
- Can counteract this in a variety of different ways:
  - » Use implicit schemes
  - » Multi-stage Runge-Kutta schemes
  - » Add extra terms from model equation to counteract negative diffusion coefficient

Beam-Warming Scheme



- Start with

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \frac{u^2 \Delta t}{2}\frac{\partial^2 \phi}{\partial x^2}$$

- Use face values based on:

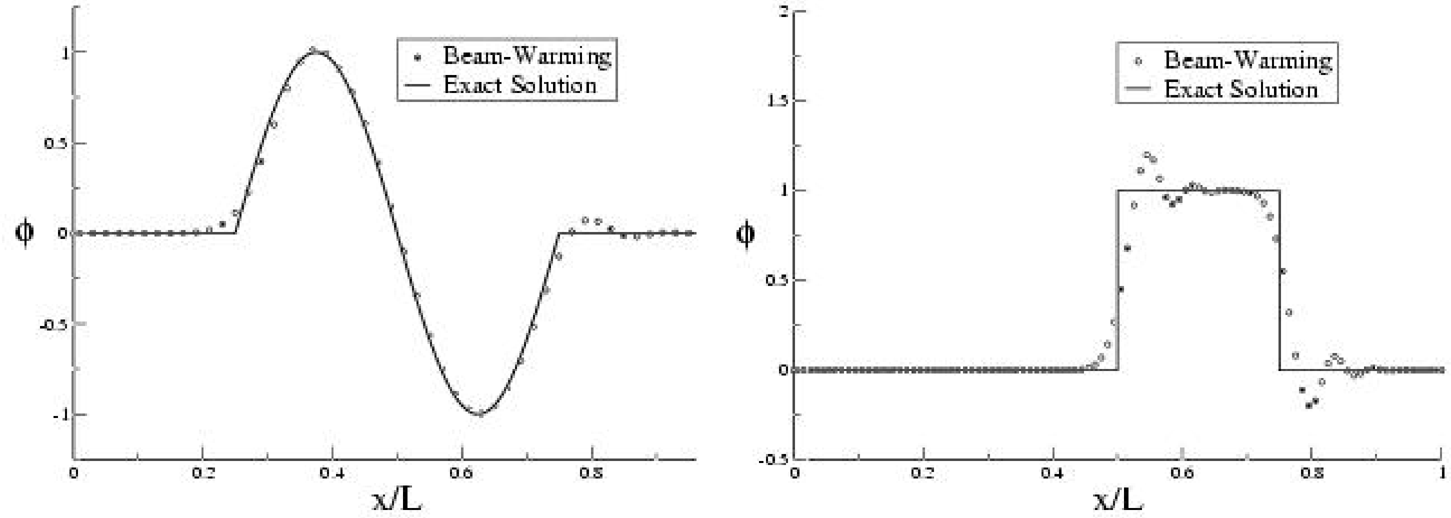
$$\phi_e = \phi_P + \frac{(\phi_P - \phi_W)}{2}$$

Second-order artificial diffusion term

- Rearrange to obtain

$$\frac{\phi_P - \phi_P^0}{\Delta t} + u\frac{(\phi_P^0 - \phi_W^0)}{\Delta x} + \frac{u(\Delta x - u\Delta t)}{2}\frac{(\phi_P^0 - 2\phi_W^0 + \phi_{WW}^0)}{(\Delta x)^2} = 0$$

Time Marching with Beam-Warming Scheme



We see the usual dispersive behavior – smooth profiles are convected relatively well, but square profile picks up wiggles

Discussion

- First-order UDS is dissipative
- CDS is dispersive
  - » unconditionally unstable in conjunction with explicit stepping
  - » Other symmetric schemes are also dispersive and unconditionally unstable when used with explicit time stepping schemes
- Higher-order upwind-weighted schemes must be stabilized if used with explicit stepping schemes
  - » Also dispersive
- Must do something to control wiggles

Added Dissipation Schemes

- We saw that the artificial dissipation in the UDS scheme stabilized it when used with explicit time stepping
- Some researchers have added an explicit diffusion term to mimic this effect.
- Want to keep truncation error O(Δx^2) if using second-order schemes.
- Add to original PDE an extra fourth-order diffusion term:

$$(\text{constant})\Delta x^3 \frac{\partial^4 \phi}{\partial x^4}$$

Note that artificial term is O(Δx^3) – preserves O(Δx^2) error

Added-Dissipation Schemes

- Corresponding face value for CDS is

$$\phi_e = \frac{\phi_P + \phi_E}{2} + \varepsilon_e^{(4)}(\phi_{EE} - 3\phi_E + 3\phi_P - \phi_W)$$

- Near shocks and discontinuities, need to add more dissipation. Usually a second-order dissipation term is necessary:

$$(\text{constant})\Delta x^2 \frac{\partial^2 \phi}{\partial x^2}$$

Destroys second-order accuracy of scheme – reduces to first-order near shocks

Added-Dissipation Schemes

- Corresponding face value for CDS is:

$$\phi_e = \frac{\phi_P + \phi_E}{2} - \varepsilon^{(2)}(\phi_E - \phi_P) + \varepsilon_e^{(4)}(\phi_{EE} - 3\phi_E + 3\phi_P - \phi_W)$$

From second-order dissipation term – $O(\Delta x)$

From fourth-order dissipation term – $O(\Delta x^2)$

- How to detect shocks to turn on second-order dissipation?
- How to choose $\varepsilon^{(2)}$ and $\varepsilon^{(4)}$?

Closure

In this lecture, we

- Started looking at higher-order spatial schemes
  - » Fromm scheme
  - » Beam-Warming
  - » QUICK
  - » Added-dissipation schemes