An Implicit Factored Scheme for the Compressible Navier-Stokes Equations

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An implicit finite-difference scheme is developed for the numerical solution of the compressible Navier-Stokes equations in conservation-law form. The algorithm is second-order-time accurate, noniterative, and spatially factored. In order to obtain an efficient factored algorithm, the spatial cross derivatives are evaluated explicitly. However, the algorithm is unconditionally stable and, although a three-time-level scheme, requires only two time levels of data storage. The algorithm is constructed in a "delta" form (i.e., increments of the conserved variables and fluxes) that provides a direct derivation of the scheme and leads to an efficient computational algorithm. In addition, the delta form has the advantageous property of a steady state (if one exists) independent of the size of the time step. Numerical results are presented for a two-dimensional shock boundary-layer interaction problem.

I. Introduction

NUMERICAL computations based on the full compressible Navier-Stokes equations first appeared slightly more than a decade ago. During the relatively brief intervening period, considerable advancement has been made in the calculation of both two- and three-dimensional flowfields. A comprehensive summary of finite-difference methods and calculations for the 1965 to 1975 period has been made by Peyret and Viviand[1] and we will not attempt to duplicate their review. Both explicit and implicit numerical methods have been successfully applied to a variety of flow calculations, and neither method has reached its full potential. Traditionally, implicit numerical methods have been praised for their improved stability and condemned for their large arithmetic operation counts. Hence, the choice of an implicit algorithm implies that the time-step limit imposed by an explicit stability bound must be significantly less than the time-step limit imposed by the accuracy bound. This situation commonly arises in the numerical solution of a time-dependent system of flow equations and results from disparate characteristic speeds and/or length scales. (Such problems are often said to be "stiff.")

Undoubtedly, the most significant efficiency achievement for multidimensional implicit methods was the introduction of the alternating-direction-implicit (ADI) algorithms by Douglas,[2] Peaceman and Rachford,[3] and Douglas and Gunn,[4] and fractional step algorithms by Yanenko.[5] Recent interest in implicit methods for systems of nonlinear partial differential equations has been spurred by the development of noniterative ADI schemes by Lindemuth and Killeen,[6] Briley and McDonald,[7] and Beam and Warming.[8] Additional impetus for the development of implicit methods is derived from the trend of current computer hardware development to be limited by data transfer speed rather than the speed of the arithmetic units.

An efficient implicit finite-difference algorithm for the Eulerian (inviscid) gasdynamic equations in conservation-law form was recently developed.[8] The purpose of this paper is to extend that algorithm to include the compressible Navier-Stokes equations (Sec. II). The extended algorithm is noniterative and retains the conservation-law form which is essential for the proper treatment of embedded shock waves ("shock capturing"). The temporal difference approximation has been generalized to retain a variety of schemes including a three-level scheme requiring only two levels of data storage. The development and final algorithm make extensive use of the "delta" form (increments of the conserved variable and flux vectors) to achieve analytical simplicity and numerical efficiency. The delta formulation also retains the advantageous property of a steady state (if one exists) independent of the time step. The method of approximate (spatial) factorization is used to implement the scheme as an ADI sequence. A three-level scheme allows the spatial cross-derivative terms to be included efficiently in a spatially factored second-order-time-accurate algorithm without upsetting the unconditional stability of the algorithm.

In Sec. III we develop an implicit time-dependent boundary-condition scheme. We consider two physical problems that provide a variety of boundary conditions. A linear stability analysis, based on model two-dimensional convective and diffusive scalar equations, is summarized in Sec. IV. The analysis indicates that the factored, second-order-accurate scheme is unconditionally stable. A method for adding numerical dissipation, when required, is presented in Sec. V.

Numerical examples in Sec. VI include the transient development of Couette flow and the oscillatory flow generated by a wall moving with sinusoidal velocity in its own plane. The purpose of these simple flow calculations was to test the algorithm and boundary conditions on unsteady problems for which the exact solutions are known. As a more severe test of the algorithm, the numerical solution of a two-dimensional shock boundary-layer interaction flow was computed. The results of the numerical examples indicate numerical stability and accuracy for Courant numbers much greater than unity.

II. Algorithm Development

The two-dimensional compressible Navier-Stokes equations can be written in the conservation-law form

$$\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}+\frac{\partial G(U)}{\partial y}=\frac{\partial V_1(U,U_x)}{\partial x}+\frac{\partial V_2(U,U_y)}{\partial x}+\frac{\partial W_1(U,U_x)}{\partial y}+\frac{\partial W_2(U,U_y)}{\partial y} \tag{1}$$

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