CANTOR SETS AND SEQUENCE SPACES

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December 13, 2022

Day 12/12/2022

0.1 Reviewing

Topics from last class: Showed that cylinders form open sets and are equivalent to open balls with the metric:

$$d_\lambda(\omega, \omega') = \sum_{n=-\infty}^{\infty} \frac{|\omega_n - \omega'_n|}{\lambda^{|n|}}$$

Found a connection between product topology and metric topology
Showed that cylinders are clopen sets
Showed that the metric above has less contributions from higher order terms.

0.2 Connectedness

On today's seminar, we are going to see a few topological concepts needed to better grasp the ideas behind the relationship between Cantor Sets and Sequence Spaces. Starting with the notion of connectedness of a space, the intuition behind is that of a "space that is a single piece." Formally, we define it as

Definition. *If X is a set such that it cannot be divided into two disjoint non-empty open sets, then X is connected.*

For our purposes, even though this definition given is the most intuitive, we are going to use an equivalent definition, namely

Definition. *A space X is connected when the only two sets that are both open and closed are X and $\emptyset$.*

The usefulness comes from the fact that since our Cylinder sets are clopen, just like it was proved on the last seminar, there is no connected subspace of X other than one-point sets, which are trivially connected. Thus the space is totally disconnected. Hence, we've cooked one ingredient in our Cantor Set homeomorphism soup.

0.3 Perfect Set

To cut it short,

Definition. *A set X is said to be perfect if it is closed and has no isolated point.*

However, we must clarify what is an isolated point. We define an element x of X to be an isolated point provided that there exists a neighborhood U of x that contains no other points besides x itself. In other words, a point is not an isolated points if every neighborhood U of x contains at least one more point p other than x.

Now, recall from calculus and analysis the definition of a limit point:

> "An element x of a set S is said to be a limit point if every neighorhood U of x contains at least one point p such that $p \neq x$."

We conclude that a set X is perfect provided that it is closed AND every point of X is a limit point!

Let's show that Ω_2^R is a perfect set, i.e., we are going to prove that given a point $\omega \in \Omega_2^R$, then $\omega \in C_{\alpha_i}^{n_i}$. However, notice that whichever other $\omega' \in \Omega$ whose i-th entry is also α_i also belongs to that neighborhood of ω. Hence, the same reasoning shows that any neighborhood of ω contains other points different from it. Thus, since our point was arbitrary, this is true for any element of Ω_2^R, showing that any point is a limit point. We conclude from this that our space is perfect.