CANTOR SETS AND SEQUENCE SPACES

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Contents

1 Preliminaries(Done during lectures)

This first part is to recall and introduce a few ideas needed to grasp the content within these notes.

Definition. *A topology τ on X is a collection of sets such that*

i) $X, \emptyset \in \tau$

ii) If $U_1, \cdots, U_\alpha$ are an arbitrary family of sets $U_\alpha \in \tau, \alpha \in J,$ then $\bigcup_{\alpha \in J} U_\alpha \in \tau$

iii) If $U_1, \cdots, U_n$ belong to $\tau,$ then $\bigcap_{i=1}^n U_i \in \tau.$

Definition. *The discrete topology on X is the one in which open sets are subsets of X.*

Definition. *A metric on a set X is a function $d : X \times X \to \mathbb{R}$ which satisfies the following*

i) $d(x, y) \geq 0, d(x, y) = 0 \iff x = y;$

ii) $d(x, y) = d(y, x);$

iii) $d(x, z) \leq d(x, y) + d(y, z).$

Definition. *A basis $\mathcal{B}$ for a topology is a collection of subsets of X such that*

i) For each x in X, there is at least one B in $\mathcal{B}$ containing x;

ii) If x belongs to the intersection of two basis elements, there is a basis element containing x and inside their intersection.

Another characterization of a basis is as follows:

Theorem. *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$. Then, C is a basis for the topology of X.*

For a product of topological spaces, we can define a topology on it based on the topologies of each individual space as follows:

Definition. *Let $X = \prod_{\alpha \in J} X_\alpha$, where each X_α is a topological space. We define the product topology on X as the one given by the basis $\mathcal{B}$ such that each basis element is defined as*

$$B = \prod_{\alpha \in J} U_\alpha, \quad U_\alpha \neq X_\alpha \textit{ for all but finitely many } \alpha$$

Each metric defines a topology themselves, known as the metric topology, in which basis elements are open balls and the topology is induced by this basis.

Definition. *Given a metric space (X, d), an open ball of radius ϵ centered at a is the set*

$$B_d(a, \epsilon) := \{x \in X : d(x, a) < \epsilon\}.$$

Definition. *Given a metric space (X, d), the metric topology τ_d induced by d on X is the one generated by the basis $\mathcal{B}$ whose basis elements are open balls.*

2 Day 24/11/2022

2.1 Introduction and Main Set

The main goal of these seminar (/notes) is to prepare whoever is using them to understand sequence spaces and their construction, with the final goal of reaching the Cantor Set construction. We start off defining our main set, known as the Sequence Space of N-1 digits, i.e.,

$$\Omega_N := \{\omega = (\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots) : \omega_i \in \{0, 1, \cdots, N-1\}\} = \{0, 1, \cdots, N-1\}^{\mathbb{Z}}$$
$$\Omega_N^R := \{\omega = (\omega_0, \cdots) : \omega_i \in \{0, 1, \cdots, N-1\}\} = \{0, 1, \cdots, N-1\}^{\mathbb{N}}$$

One way to think of this family of spaces is that it records the results from an infinite game of throwing N-sided dices. For instance, if we stay simple and assume a game of heads-and-tails, with only two possibilities, the space would be

$$\Omega_2^R = \{0, 1\}^{\mathbb{N}} = \{\omega = (\omega_0, \cdots) : \omega_i \in \{0, 1\}\}$$

This space specifically is the one we are going to work the most with, since it is going to be shown that it is equivalent to a Cantor Set, and it also makes for an easy example regarding whatever new construction we add to the space.

2.2 Structuring the Space

The next few steps are related to the structure of this space. We're going to define a topology on it, followed by a metric. Start by endowing $\{0, 1, \cdots, N-1\}$ with the discrete topology, in a way that since the space is finite, it is also compact. From here, consider the sets $X_i = \{0, 1, \cdots, N-1\}, i \in \mathbb{N}$, and define

$$B = \prod_{i=1}^{n-1} U_i \times \prod_{i=n}^{\infty} X_i,$$

where each U_i is a proper subset of X_i. Hence, the collection $\mathcal{B}$ of all the sets B form a basis for the product topology on Ω_N^R

For example, consider N = 2, i.e., the topology on $\{0, 1\}$, in which the discrete topology is $\tau = \{\emptyset, \{0, 1\}, \{0\}, \{1\}\}$. Then, the basis for the product topology is given by

$$B = \prod_{i=1}^{n-1} U_i \times \prod_{i=n}^{\infty} X_i = \prod_{i=1}^{n} \{b_i\} \times \prod_{i=n}^{\infty} U_i.$$

em que $b_i \in \{0, 1\}$ e $U_i = \{0, 1\}$. Another way of defining the topology is via what is known as cylinder sets, defined by fixing integers $n_1 < \cdots < n_k$ and $\alpha_1, \cdots, \alpha_k \in \{0, 1, \cdots, N-1\}$, and putting

$$C_{\alpha_1, \cdots, \alpha_k}^{n_1, \cdots, n_k} := \{\omega \in \Omega_N^R : \omega_{n_i} = \alpha_i\}$$

where i varies from 1 to k. Specifically, this is known as a cylinder of rank k, and the topology to be defined is such that all open sets are cylinders, hence the basis is given by them. However, to show that it is indeed a basis, take $x \in C_{\alpha_1, \cdots, \alpha_k}^{n_1, \cdots, n_k}$. Hence, for some $1 \le j \le k$,

$$x \in C_{\alpha_1}^{n_1} \cap C_{\alpha_2}^{n_2} \cap \cdots \cap C_{\alpha_k}^{n_k}$$

so that it satisfies the theorem characterizing a basis, i.e., $\mathcal{C}$ is a basis of cylinders. Moreover, notice that given a cylinder,

$$\Omega_N^R \backslash \{C_{\alpha_1, \cdots, \alpha_k}^{n_1, \cdots, n_k}\} = \{\omega \in \Omega_N^R : \omega_{n_i} \neq \alpha_i, i = 1, \cdots, k\} = C_{\sigma(\alpha_1), \cdots, \sigma(\alpha_k)}^{n_1, \cdots, n_k},$$

where σ is a permutation that excludes the j-th term $(\sigma(\alpha_i) \neq \alpha_k \forall i = 1, \cdots, k)$. As a consequence, we see that every cylinder is both open and closed in Ω_N^R.

Our next step will be defining a metric for our space and prove some of its properties.

3 Day 05/12/2022

3.1 Motivations

- Cylinders are clopen sets;
- Sequence space metric;
- Cylinders are open balls in that metric;
- Ω_N is a perfect set (First part of showing the homeomorphism with a Cantor Set).

3.2 Reviewing

Topics from last class:

- Introduced the main space we're working on;
- Defined the bases for the topology on Ω_N^R;
- We showed that each cylinder set form open sets.

3.3 Ω_N^R as a Metric Space

Our goal with this is to define a metric such that the space is perfect and can be related to the cylinder topology. For a fixed $\lambda > 1$, define $d : \Omega_N^R \times \Omega_N^R \to \mathbb{R}$ by

$$d_\lambda(\omega, \omega') = \sum_{n=-\infty}^{\infty} \frac{|\omega_n - \omega'_n|}{\lambda^{|n|}}.$$

Checking the metric axioms is not the point of this seminar, thus it's left undone on the notes. However, it hurts no one to comment that it all follows from the properties granted from the usual metric of $\mathbb{R}^n$ (i.e. the absolute value).

3.4 Open Balls in Ω_N^R

Using the definition of open balls (c.f. Section 1)k, let's analyze the case of an open ball centered at 0 with raidus r with the metric we just defined. For the sake of simplicity, take N to be 2, so that for large λ, specifically $\lambda = 10N = 20$, fix the zero sequence $\omega' = (\cdots, 0, 0, 0, \cdots)$. We want to find $r > 0$ such that $B_r(\omega') = C_0^0 = \{\nu = (\nu_n)_{n\in\mathbb{Z}} : \nu_0 = 0\}$. Equivalently, we're going to prove that

$$\{\omega \in \Omega_2 : d_\lambda(\omega, \omega') < r\} = \{\omega \in \Omega_2 : \omega_0 = 0\}.$$

Analyzing the terms of the series under these condition, one gets

$$d_\lambda(\omega, \omega') = \sum_{n=-\infty}^{\infty} \frac{|\omega_n - \omega'_n|}{\lambda^{|n|}} = \sum_{n=-\infty}^{\infty} \frac{|\omega_n|}{20^{|n|}}$$

It helps to visualize a few cases of n to understand the behavior of the series. In fact, it follows that

$$n = 0 \to \frac{|\omega_0 - \omega'_0|}{20^n} = \begin{cases} 0, & \text{if } \omega_0 = \omega'_0 = 0 \\ \frac{1}{1} = 1, & \text{if } \omega_0 \neq \omega'_0 \end{cases}$$

$$n = 1 \to \frac{|\omega_1 - \omega'_1|}{20} = \begin{cases} 0, & \text{if } \omega_1 = \omega'_1 = 0 \\ \frac{1}{20}, & \text{if } \omega_1 \neq \omega'_1 \end{cases}$$

$$\vdots$$

$$n = k \to \frac{|\omega_{\pm k} - \omega'_{\pm k}|}{20^{|k|}} = \begin{cases} 0, & \text{if } \omega_{\pm k} = \omega'_{\pm k} \\ \frac{1}{20^{|k|}}, & \text{if } \omega_{\pm k} \neq \omega_{\pm k} \end{cases}$$

In other words, we are looking for an r such that

$$d_\lambda(\omega, \omega') = \begin{cases} \sum\limits_{n=-\infty}^{\infty} \frac{1}{20^{|n|}}, & \omega_n \neq \omega_n \\ 0, & \omega_n = \omega'_n \end{cases}$$

Because $\dfrac{1}{20} < 1$, this series converges, so that it works for any $r \in \left(\sum\limits_{|n| \geq 1} 20^{-n}, 1 \right]$ works. In fact, the cases in which $\omega_n = \omega'_n$ for some $n \in \mathbb{Z}$ yield

$$d(\omega, \omega') \leq \sum_{|n| \geq 1} 20^{-n} < r$$

Notice that the case r = 1 must be included if $\omega_0 \neq \omega'_0$ but $\omega_n = \omega'_n$ for $n \in \mathbb{Z} \backslash \{0\}$. Moreover, notice that 20^{-n} is just λ^{-n}, such that we can calculate the radius of the ball for a given large λ.

One final comment should be added. On any metric space, a topology given by the metric itself can be created by setting the basis to be open balls and open sets being their union. That way, the topology induced by the metric just studied, in which open balls have been shown to be cylinders, is one generated by them (cylinders). Hence, it is equivalent to the first topology we defined on Ω_N^R!

4 Day 12/12/2022

4.1 Reviewing

Topics from last class:

- Showed that cylinders form open sets and are equivalent to open balls with the metric:

$$d_\lambda(\omega, \omega') = \sum_{n=-\infty}^{\infty} \frac{|\omega_n - \omega'_n|}{\lambda^{|n|}};$$

- Found a connection between product topology and metric topology;
- Showed that cylinders are clopen sets;
- Showed that the metric above has less contributions from higher order terms.

4.2 Connectedness

On today's seminar, we are going to see a few topological concepts needed to better grasp the ideas behind the relationship between Cantor Sets and Sequence Spaces. Starting with the notion of connectedness of a space, the intuition behind is that of a "space that is a single piece." Formally, we define it as

Definition. *If X is a set such that it cannot be divided into two disjoint non-empty open sets, then X is connected.*

For our purposes, even though this definition given is the most intuitive, we are going to use an equivalent definition, namely

Definition. *A space X is connected when the only two sets that are both open and closed are X and $\emptyset$.*

The usefulness comes from the fact that since our Cylinder sets are clopen, just like it was proved on the last seminar, there is no connected subspace of X other than one-point sets, which are trivially connected. Thus the space is totally disconnected. Hence, we've cooked one ingredient in our Cantor Set homeomorphism soup.

4.3 Perfect Set

To cut it short,

Definition. *A set X is said to be perfect if it is closed and has no isolated point.*

However, we must clarify what is an isolated point. We define an element x of X to be an isolated point provided that there exists a neighborhood U of x that contains no other points besides x itself. In other words, a point is not an isolated points if every neighborhood U of x contains at least one more point p other than x.

Now, recall from calculus and analysis the definition of a limit point:

> "An element x of a set S is said to be a limit point if every neighorhood U of x contains at least one point p such that $p \neq x$."

We conclude that a set X is perfect provided that it is closed AND every point of X is a limit point!

Let's show that Ω_2^R is a perfect set, i.e., we are going to prove that given a point $\omega \in \Omega_2^R$, then $\omega \in C_{\alpha_i}^{n_i}$. However, notice that whichever other $\omega' \in \Omega$ whose i-th entry is also α_i also belongs to that neighborhood of ω. Hence, the same reasoning shows that any neighborhood of ω contains other points different from it. Thus, since our point was arbitrary, this is true for any element of Ω_2^R, showing that any point is a limit point. We conclude from this that our space is perfect.

5 Seminar Day 30/03/2023

5.1 Motivations

- Review Topology;
- Construct the σ-algebra that will be used;
- Construct the Middle-Third Cantor set via sequences of functions randomly picked.

5.2 The Usual Construction of the Cantor Set

5.2.1 Goal of the Part

Construction of the Cantor Set via nested intervals, analytical properties of the set, its topological properties and construction via functions.

5.2.2 Main Text

Before getting on the construction forementioned, it's important to know how it's usually done, even if just a bit. For that, we consider the interval $[0,1]$. From it, we will delete the open middle third interval, i.e., $\left(\frac{1}{3},\frac{2}{3}\right)$ which will leave us with two line segments, namely $\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right]=C_1$. Now, repeat that for eacho one of these intervals and remove their open middle-third intervals, yielding

$$\begin{aligned} &\left[0,\frac{1}{9}\right], \quad \left[\frac{2}{9},\frac{1}{3}\right] \\ &\left[\frac{2}{3},\frac{7}{9}\right], \quad \left[\frac{8}{9},1\right]. \end{aligned}$$

Then, we call their union $C_2=\left[0,\frac{1}{9}\right]\cup\left[\frac{2}{9},\frac{1}{3}\right]\cup\left[\frac{2}{3},\frac{7}{9}\right]\cup\left[\frac{8}{9},1\right]$. We thus define inductively $C_n \coloneqq \frac{C_{n-1}}{3}\cup\left(\frac{2}{3}+\frac{C_{n-1}}{3}\right)$, and define the middle-third Cantor set by

$$\boxed{\mathcal{C} \coloneqq \lim_{n\to\infty} C_n = \bigcap_{n=0}^{\infty} C_n}$$

Briefly speaking about this set's properties, it has the same cardinality as the real line. To prove it, we use Bernstein-Schrœder theorem and compare the cardinality of $\mathcal{C}$ with that of $[0,1]$. Thus, $\aleph_1 \geq |\mathcal{C}|$ since $\mathcal{C} \subsetneq [0,1]$. On the other hand, to show the other side of the inequality, one constructs a surjection going from $\mathcal{C}$ to $[0,1]$. Jumping right to the definition, it is motivated by writing the points of the interval $[0,1]$ in terms of base 3, and then removing on each n step, the numbers that have 1 as their nth digit in the ternary representation.

On the first step, digits of the form $0.1aaaaaaaaaa\cdots$, where $aaaaaaaaa\cdots$ is between $00000\cdots_3$, and $22222\cdots_3$. leaving only the numbers whose first digit is either 0 or 2. The next step removes numbers with 1 on the second coordinate, leaving only those that assume the form $0.00aaaaa\cdots$, $0.02aaaaa\cdots$, or $0.22aaaaa\cdots$. We repeat it ad infinitum, and the numbers remaining on the Cantor set haver representations consisting entirely of 0's and 2's.

Hence, the function from $\mathcal{C}$ to $[0,1]$ takes the ternary numerals constructed above and replaces every 2's by 1's, in a way that we interpret the sequence obtained as a binary form of a real number in $[0,1]$, and reciprocally, any of them have a binary representation that can be translated to a ternary representation of a number in the Cantor set by replacing 1's by 2's:

$$f\left(\sum_{k\in\mathbb{N}} a_k 3^{-k}\right)=\sum_{k\in\mathbb{N}}\frac{a_k}{2}2^{-k}, \quad \forall k\in\mathbb{N}: a_k\in\{0,2\}$$

Therefore, f is surjective, so that $\aleph_1 \leq |\mathcal{C}| \leq \aleph_1$.

Some other interesting properties is that it is a set of measure 0, i.e., it is infinitesimally small, even though it has uncountably many elements. It is self-similar, a sort of primitive fractals, because it is equal to two copies of itself when translated and dilated by some factor. More specifically, dilating each copy by a factor of $\frac{1}{3}$ and translating it, and as a consequence, when you zoom in into a part of the set, it looks identical to the whole. This can be seen from the equation given before, i.e.,

$$C_n \coloneqq \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right),$$

and from here, the self-similarity follow by showing that $C\dfrac{C}{3} \cup \left(\dfrac{2}{3} + \dfrac{C}{3}\right)$. The Cantor set is also perfect, meaning that any point in the Cantor Set has an infinite sequence of points also in the set converging to that point. The proof of this statement relies on taking an element x in $\mathcal{C}$, and arbitrary $\epsilon > 0$, an $N(\epsilon) \in \mathbb{N}$ and finding a good enough M so that $x \in [M, M+1] \subseteq (x - \epsilon, x + \epsilon)$. Then, we apply the process used on the construction of $\mathcal{C}$ to the interval $[M, M+1]$, removing the interval $(\frac{3M+1}{3}, \frac{3M+2}{3})$ from it, to see that if were to belong to the removed interval or not, it'd be a limit point in both cases. From arbitrariness of the choices, it follow that $\mathcal{C}$ is indeed a perfect set.

Topologically speaking, the Cantor set has nice properties too. It's totally disconnected, compact, perfect, nonempty, and metrizable. Furthermore, any set with those properties are the Cantor Set up to homeomorphism.

5.3 The Motivation Problem

5.3.1 Goal of the Part

Review the Hutchinson Attractor, give the motivation problem behind construction the Cantor Set as a random application of functions that converge to the Attractor, set up the main problems to be studied.

5.3.2 Main Text

As Kevin talked about last week, we can construct the Cantor Middle-Third set through a function known as the Hutchinson Attractor, denoted by B, ranging from the set $\mathcal{H}(M)$ of nonempty compact subsets of a metric space M to $\mathcal{H}(M)$ itself and given by the union of the images of contractions in M. The way we construct it is, for instance, by applying B to a singleton set A = $\{x\}$ and using de previously defined metric d_H to translate the result to the Cantor Set. For each n iterations, we get an ammount of parts of the CMT set equal to 2^n. Now consider two functions $f_i : [0,1] \to [0,1]$

$$f_i = \begin{cases} \frac{x}{3}, & i = 0 \\ \frac{x}{3} + \frac{2}{3}, & i = 1 \end{cases}$$

The infinite application of these functions yield the Cantor set, which shall be denoted by $\mathcal{C}$ to make things easier. In other words, Consider now a representation of infinitely many combinations of 0's and 1's in any order, i.e., $\omega = (0, 1, 0, 0, 1, \cdots)$ and let f_ω^n be the repeated use of f_i as defined before using the indices from ω, so that

$$f_\omega^n(x) = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1} \circ f_{\omega_0}(x)$$

indicates that we start using the n-th coordinate of ω on each f. Using the example of ω from before, f_ω^n would be

$$f_\omega^n(x) = f_{\omega_{n-1}} \circ \cdots \circ f_0 \circ f_1 \circ f_0(x).$$

Using that notation, the thesis given at the start that the infinite use of f_i eventually covers all of $\mathcal{C}$ is the same as

$$\lim_{l \to \infty} \overline{\{f_\omega^n(x) : n \geq l\}} = B$$

That begs the question: Will EVERY sequence of digits $\omega \in \{0,1\}^{\mathbb{N}}$, when combined with f_i, give as its image the Cantor set? In fact, the answer is no, as the theorem is stated

Theorem. *Let ϕ be an independent and identically distributed random product of contractions $f_1, \cdots, f_k : M \to M$ and K be the corresponding Hutchinson Attractor. Given $p \in M$, we have*

$$\lim_{l\to\infty} \overline{\{f_\omega^n : n \geq l\}} = K$$

for $\mathbb{P}-$almost all ω.

On that note, to grasp what we mean by almost all sequences work, consider a sequence of a single digit, i.e., $\omega = \{0, 0, 0, \cdots\}$. Then, when we apply this to f_i, we get

$$\begin{aligned} f_\omega^n(x) &= f_0 \circ \cdots f_0 \circ f_0 \circ f_0(x) \\ &= f_0 \circ \cdots f_0(\frac{x}{3}) \\ &= f_0 \circ \cdots f_0(\frac{x}{9}) \\ &\vdots \\ &= \frac{x}{3^n}. \end{aligned}$$

Because of this, as n goes to infinity, f_ω^n tends to 0, which is not the Cantor set. In other words,

$$\lim_{l\to\infty} \overline{\{f_\omega^n(x) : n \geq l\}} \neq B.$$

a new problem arises, that is, will the number of ω's for which the initial thesis hold be way too small to be relevant? The answer is going to need quite a bit of measure theory, for the answer is, in fact, that the result holds for almost every ω.

5.4 The Meaning of "Almost Every"

5.4.1 Goals of this Part

We want to define $\sigma - algebras$, the foundation of what is a measure. Then, define measures properly speaking, what is a property that happens almost everywhere, and some probabilistc concepts, such as independence and identical distributions.

5.4.2 Main Text

Now that the goal has been set, we can start to develop the theory needed to fully grasp it. Starting with a review of $\sigma-$algebra, they are the sets in which we define measures. They are sets that have closure under countable unions and complements. Formally speaking,

Definition. *Given a set Ω, a $\sigma-$algebra $\mathcal{F}$ is a nonempty collection of subsets of Ω that satisfy*

(i) *If $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$;*

(ii) *If A_i is a countable sequence of sets, then $\bigcup_i A_i \in \mathcal{F}$;*

$(iii) \Omega \in \mathcal{F}. \square$

They are the defining ground that allow us to define measures and thus, find the meaning of "Almost Every". Without further ado, we define a measure.

Definition. *A measure is a nonnegative countably additive set function, i.e., a function $\mu : \mathcal{F} \to \mathbb{R}$ which satisfies*

i) $\mu(A) \geq \mu(\emptyset) = 0$ *for all $A \in \mathcal{F}$;*

ii) If $A_i \in \mathcal{F}$ is a countable sequence of disjoint sets, then

$$\mu(\bigcup_i A_i) = \sum_i \mu(A_i). \square$$

Now, we can formalize the meaning of almost every point satisfying something. The idea is that some property is going to hold true for every point of a set except for an ammount inside a set whose significance is nought. In other words, a set with a measure of zero, so that the set for which the property holds takes up nearly every single possibility.

Definition. *Given a measure space $(\Omega, \mathcal{F}, \mu)$, in which Ω is a set, $\mathcal{F}$ is a $\sigma-$algebra, and μ is a measure, a property P is said to hold almost everywhere if there exists a set $N \in \Sigma$ with $\mu(N) = 0$, and all $x \in \Omega/N$ satisfy that property.*

It's also important to understand what it means for a variable to be independent and identically distributed, since it allows for simplifications of results and ideas. Essentially, it means that the an element in the sequence of variables does not depend on the ones that came before it, and the value does not change with each subsequent application of it. The easiest example of such kinds of elements come up in the toss of coins, since the result of "head" or "tail" has no impact on the next tossing of coin, and neither does the probability change from 0.5 each time you flip it, meaning it is identically distributed. However, we need a mathematical definition to work better with it.

Definition. *Suppose random variables X and Y are defined to assume values in $I \subseteq \mathbb{R}$. Let $F_X(x) = P(X \leq x), F_Y(y) = P(Y \leq y)$ denote the cumulative distribution function of X and Y. Furthermore, denote their joint cumulative distributive function by*

$$F_{X,Y}(x, y) = P(X \leq x \wedge Y \leq y).$$

Definition. *Two random variables X and Y are independent if and only if*

$$F_X(x) = F_Y(x) \quad \forall x \in I \subseteq \mathbb{R}.$$

Moreover, they are said to be identically distributed if and only if

$$F_{X,Y}(x, y) = F_X(x)F_Y(y) \quad \forall x, y \in I \subseteq \mathbb{R}.$$

When both happen, we say that they are i.i.d.

The thing is that we don't work with only two variables (usually). Thankfully, we do have a definition for multiple variables

Definition. *We say that n random variables $X_1, \cdots, X_n$ are independent if and only if*

$$F_{X_1}(x) = F_{X_k}(x) \quad \forall k \in \{1, \cdots, n\} \textit{ and } \forall x \in I$$

On the other hand, they are identically distributed if and only if

$$F_{X_1,\cdots,X_n}(x) = F_{X_1}(x_1) \cdots F_{X_n}(x_n) \quad \forall x_1, \cdots, x_n \in I$$

where $F_{X_1,\cdots,X_n}(x_1, \cdots, x_n) = P(X_1 \leq x_1 \wedge \cdots \wedge X_n \leq x_n)$ denotes the joint cumulative distribution function of $X_1, \cdots, X_n$. $\square$

5.5 The Topology of Cylinder Sets and Sequence Spaces

Regarding the topology of cylinder sets,

5.5.1 Goals of the Part

Here, we shall define cylinders, working with coordinates of their elemenents, define a topology for the sequence space, and define bases for it.

5.5.2 Main Text

Definition. *Fix integers* $n_1 < \cdots < n_k$ *and* $\alpha_1, \cdots, \alpha_k \in \{0, 1, \cdots, N-1\}$. *Then, the cylinder of rank k is the set*

$$C^{n_1,\cdots,n_k}_{\alpha_1,\cdots,\alpha_k} := \{\omega \in \Omega^R_N : \omega_{n_i} = \alpha_i\}$$

where i varies from 1 to k. Another notation that may be used is

$$[\alpha_0, \cdots, \alpha_k] = C^{n_1,\cdots,n_k}_{\alpha_1,\cdots,\alpha_k}.$$

The elements of cylinders are exactly the k first coordinates of an element $\omega \in \{0, 1\}^{\mathbb{N}}$, and it should be noted that we may not know the other coordinates other than those k first ones. Cylinders are both open and closed sets simultaneously, as can be shown. The elements we are going to use as part of the definition of f will be members of the cylinders.

Furthermore, we endow the space $\{0, 1\}^{\mathbb{N}}$ with the smallest topology that makes projections continuous functions, the product topology, which has a subbasis formed exactly by the preimage of projections and a basis of cylinders.

Definition. *Let* $X = \prod_{\alpha \in J} X_\alpha$, *where each* X_α *is a topological space. We define the product topology on X as the one given by the basis* $\mathcal{B}$ *such that each basis element is defined as*

$$B = \prod_{\alpha \in J} U_\alpha, \quad U_\alpha \neq X_\alpha \textit{ for all but finitely many } \alpha$$

The product topology in question has as reference the discrete topology on $\{0, 1\}$