CANTOR SETS AND SEQUENCE SPACES

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Contents

1 Preliminaries(Done during lectures)

This first part is to recall and introduce a few ideas needed to grasp the content within these notes.

Definition. *A topology τ on X is a collection of sets such that*

i) $X, \emptyset \in \tau$

ii) If $U_1, \cdots, U_\alpha$ are an arbitrary family of sets $U_\alpha \in \tau, \alpha \in J,$ then $\bigcup_{\alpha \in J} U_\alpha \in \tau$

iii) If $U_1, \cdots, U_n$ belong to $\tau,$ then $\bigcap_{i=1}^{n} U_i \in \tau.$

Definition. *The discrete topology on X is the one in which open sets are subsets of X.*

Definition. *A metric on a set X is a function $d : X \times X \to \mathbb{R}$ which satisfies the following*

i) $d(x, y) \geq 0, d(x, y) = 0 \iff x = y;$

ii) $d(x, y) = d(y, x);$

iii) $d(x, z) \leq d(x, y) + d(y, z).$

Definition. *A basis $\mathcal{B}$ for a topology is a collection of subsets of X such that*

i) For each x in X, there is at least one B in $\mathcal{B}$ containing x;

ii) If x belongs to the intersection of two basis elements, there is a basis element containing x and inside their intersection.

Another characterization of a basis is as follows:

Theorem. *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$. Then, C is a basis for the topology of X.*

For a product of topological spaces, we can define a topology on it based on the topologies of each individual space as follows:

Definition. *Let $X = \prod\limits_{\alpha \in J} X_\alpha$, where each X_α is a topological space. We define the product topology on X as the one given by the basis $\mathcal{B}$ such that each basis element is defined as*

$$B = \prod_{\alpha \in J} U_\alpha, \quad U_\alpha \neq X_\alpha \text{ for all but finitely many } \alpha$$

Each metric defines a topology themselves, known as the metric topology, in which basis elements are open balls and the topology is induced by this basis.

Definition. *Given a metric space (X, d), an open ball of radius ϵ centered at a is the set*

$$B_d(a, \epsilon) := \{x \in X : d(x, a) < \epsilon\}.$$

Definition. *Given a metric space (X, d), the metric topology τ_d induced by d on X is the one generated by the basis $\mathcal{B}$ whose basis elements are open balls.*

2 Day 24/11/2022

2.1 Introduction and Main Set

The main goal of these seminar (/notes) is to prepare whoever is using them to understand sequence spaces and their construction, with the final goal of reaching the Cantor Set construction. We start off defining our main set, known as the Sequence Space of N-1 digits, i.e.,

$$\Omega_N := \{\omega = (\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots) : \omega_i \in \{0, 1, \cdots, N-1\}\} = \{0, 1, \cdots, N-1\}^{\mathbb{Z}}$$
$$\Omega_N^R := \{\omega = (\omega_0, \cdots) : \omega_i \in \{0, 1, \cdots, N-1\}\} = \{0, 1, \cdots, N-1\}^{\mathbb{N}}$$

One way to think of this family of spaces is that it records the results from an infinite game of throwing N-sided dices. For instance, if we stay simple and assume a game of heads-and-tails, with only two possibilities, the space would be

$$\Omega_2^R = \{0, 1\}^{\mathbb{N}} = \{\omega = (\omega_0, \cdots) : \omega_i \in \{0, 1\}\}$$

This space specifically is the one we are going to work the most with, since it is going to be shown that it is equivalent to a Cantor Set, and it also makes for an easy example regarding whatever new construction we add to the space.

2.2 Structuring the Space

The next few steps are related to the structure of this space. We're going to define a topology on it, followed by a metric. Start by endowing $\{0, 1, \cdots, N-1\}$ with the discrete topology, in a way that since the space is finite, it is also compact. From here, consider the sets $X_i = \{0, 1, \cdots, N-1\}, i \in \mathbb{N}$, and define

$$B = \prod_{i=1}^{n-1} U_i \times \prod_{i=n}^{\infty} X_i,$$

where each U_i is a proper subset of X_i. Hence, the collection $\mathcal{B}$ of all the sets B form a basis for the product topology on Ω_N^R

For example, consider N = 2, i.e., the topology on $\{0, 1\}$, in which the discrete topology is $\tau = \{\emptyset, \{0, 1\}, \{0\}, \{1\}\}$. Then, the basis for the product topology is given by

$$B = \prod_{i=1}^{n-1} U_i \times \prod_{i=n}^{\infty} X_i = \prod_{i=1}^{n} \{b_i\} \times \prod_{i=n}^{\infty} U_i.$$

em que $b_i \in \{0, 1\}$ e $U_i = \{0, 1\}$. Another way of defining the topology is via what is known as cylinder sets, defined by fixing integers $n_1 < \cdots < n_k$ and $\alpha_1, \cdots, \alpha_k \in \{0, 1, \cdots, N-1\}$, and putting

$$C_{\alpha_1, \cdots, \alpha_k}^{n_1, \cdots, n_k} := \{\omega \in \Omega_N^R : \omega_{n_i} = \alpha_i\}$$

where i varies from 1 to k. Specifically, this is known as a cylinder of rank k, and the topology to be defined is such that all open sets are cylinders, hence the basis is given by them. However, to show that it is indeed a basis, take $x \in C_{\alpha_1, \cdots, \alpha_k}^{n_1, \cdots, n_k}$. Hence, for some $1 \leq j \leq k$,

$$x \in C_{\alpha_1}^{n_1} \cap C_{\alpha_2}^{n_2} \cap \cdots \cap C_{\alpha_k}^{n_k}$$

so that it satisfies the theorem characterizing a basis, i.e., $\mathcal{C}$ is a basis of cylinders. Moreover, notice that giben a cylinder,

$$\Omega_N^R$$

3 Day 12/12/2022

3.1 Reviewing

Topics from last class: Showed that cylinders form open sets and are equivalent to open balls with the metric:

$$d_\lambda(\omega, \omega') = \sum_{n=-\infty}^{\infty} \frac{|\omega_n - \omega'_n|}{\lambda^{|n|}}$$

Found a connection between product topology and metric topology
Showed that cylinders are clopen sets
Showed that the metric above has less contributions from higher order terms.

3.2 Connectedness

On today's seminar, we are going to see a few topological concepts needed to better grasp the ideas behind the relationship between Cantor Sets and Sequence Spaces. Starting with the notion of connectedness of a space, the intuition behind is that of a "space that is a single piece." Formally, we define it as

Definition. *If X is a set such that it cannot be divided into two disjoint non-empty open sets, then X is connected.*

For our purposes, even though this definition given is the most intuitive, we are going to use an equivalent definition, namely

Definition. *A space X is connected when the only two sets that are both open and closed are X and $\emptyset$.*

The usefulness comes from the fact that since our Cylinder sets are clopen, just like it was proved on the last seminar, there is no connected subspace of X other than one-point sets, which are trivially connected. Thus the space is totally disconnected. Hence, we've cooked one ingredient in our Cantor Set homeomorphism soup.

3.3 Perfect Set

To cut it short,

Definition. *A set X is said to be perfect if it is closed and has no isolated point.*

However, we must clarify what is an isolated point. We define an element x of X to be an isolated point provided that there exists a neighborhood U of x that contains no other points besides x itself. In other words, a point is not an isolated points if every neighborhood U of x contains at least one more point p other than x.

Now, recall from calculus and analysis the definition of a limit point:

> "An element x of a set S is said to be a limit point if every neighorhood U of x contains at least one point p such that $p \neq x$."

We conclude that a set X is perfect provided that it is closed AND every point of X is a limit point!

Let's show that Ω_2^R is a perfect set, i.e., we are going to prove that given a point $\omega \in \Omega_2^R$, then $\omega \in C_{\alpha_i}^{n_i}$. However, notice that whichever other $\omega' \in \Omega$ whose i-th entry is also α_i also belongs to that neighborhood of ω. Hence, the same reasoning shows that any neighborhood of ω contains other points different from it. Thus, since our point was arbitrary, this is true for any element of Ω_2^R, showing that any point is a limit point. We conclude from this that our space is perfect.