Homework Set 1 - Networks out of Control

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Exercise 1

Claim 0.1. *Given $G(n,p)$ an Erdòs-Rény Graph. We claim that if $p \geq \frac{\log n}{\sqrt{n}}$ then, diag$(G) \leq 2$.*

Proof. The proof starts in a similar way to **theorem 3.4** from the lecture notes. Let X denote the number of vertices u, v with no common neighbor w. Observe that if $X \to 0$ a.a.s , then the claim is true (as in **theorem 3.4**). We will thus proceed by showing that $p \geq \frac{\log n}{\sqrt{n}}$ implies $X \to 0$.

First, observe that for a given pair of vertices u, v and some other vertex w, the probability that w is a common neighbor of u, v is given by p^2 (by definition of the Erdòs-Rény model). The probability that w is *not* a common neighbor is thus given by $1 - p^2$. Observe that since there are $n-2$ vertices in G (excluding u, v) the probability that u, v have no common neighbor is given by $(1-p^2)^{n-1}$.

Let $X_{u,v}$ be the indicator variable that takes value 1 if u, v, have no common neighbor and 0 else, we have that $\mathbb{E}[X_{u,v}] = \mathbb{P}[u, v \text{ have no common neighbor}] = (1-p^2)^{n-1}$. Let X be the indicator variable of the number of u, v pairs that have no common neighbors, we have $X = \sum_{u,v \in V} X_{u,v}$. By the linearity of expectation we get that $\mathbb{E}[X] = \mathbb{E}[\sum_{u,v \in V} X_{u,v}] = \sum_{u,v \in V} \mathbb{E}[X_{u,v}]$. Using the first moment method (**theorem 3.2**) we argue that showing that $\mathbb{E}[X] \to 0$ is sufficient to show that $X \to 0$ a.a.s .

$$\mathbb{E}[X] = \sum_{u,v \in V} \mathbb{E}[X_{u,v}] = \sum_{u,v \in V} (1-p^2)^{n-2} = \binom{n}{2} \cdot (1-p^2)^{n-2}$$

Which is where we start making wild upper bound guesses.

$$\begin{aligned}\binom{n}{2} \cdot (1-p^2)^{n-2} &\leq c \cdot n^2 \cdot (1-p^2)^n \\ &\leq c \cdot n^2 \cdot e^{-p^2 \cdot n}.\end{aligned}$$

We now need to show that $c \cdot n^2 \cdot e^{-p^2 \cdot n} \to 0$, to do so we take the log on both sides and argue that showing $\log\left(c \cdot n^2 \cdot e^{-p^2 \cdot n}\right) \to -\infty$ is sufficient.

$$
\begin{aligned}
\log\left(c \cdot n^2 \cdot e^{-p^2 \cdot n}\right) &\to -\infty \\
2\log(n) - n \cdot p^2 &\to -\infty
\end{aligned}
$$

Which is true if for some $c \in \mathbb{R}_+$ we have $p \geq \frac{\log n}{\sqrt{n}}$. $\square$

Exercise 2

Consider two families of random graphs $G_n = G(n,p)$ with $p = n^{\frac{-6}{5}}$ and independent $H_n = G(\log n, (\log n)^{\frac{-8}{7}})$.

Claim 0.2. *H_n does not appear in G_n for large n.*

Proof. First, we look at H_n and identify a graph that is likely to appear in it, for readability we let $m = (\log n)$ and consider:

$$H_m = G(m, m^{\frac{-8}{7}}) = H_n = G(\log n, (\log n)^{\frac{-8}{7}}).$$

Consider a tree $T = (V_T, E_T)$ with $|V_T| = 7$ and $|E_T| = 6$. Observe that by that **theorem 3.6** from the lecture notes the function $t_H(m) = m^{\frac{-7}{6}}$ is a threshold function for the apparition T in H. And that we have:

$$\frac{p_H(m)}{t(m)} = \frac{m^{\frac{-8}{7}}}{m^{\frac{-7}{6}}} \to +\infty.$$

So T is a subgraph of H a.a.s. Now using that result observe that since $G_n(n, n^{\frac{-6}{5}})$ we can use that same threshold function to show that since

$$\frac{p_G(n)}{t(n)} = \frac{n^{\frac{-6}{5}}}{m^{\frac{-7}{6}}} \to 0,$$

T is not a subgraph of G_n a.a.s., now just observe that since $T \not\subset G$ and $T \subset H$ we have $T \subset H \not\subset G$.

$\square$

Exercise 3

Fix $k \in \mathbb{Z}_+$ **theorem 3.6** implies that $t_c(n)$ provides a threshold for the appearance of some cycles (i.e. if k-cycles appear in G then G contains some cycles and $p(n)/t_c(n) \to \infty$), but we still need to show that $p(n)/t_c(n) \to 0 \; \forall \; k \in \mathbb{Z}_+ \; , k > 3$ in order to prove the claim from the datum.

Claim 0.3. *$t_c(n) = \frac{1}{n}$ is such that if $\frac{p(n)}{t_c(n)} = np(n) \to 0$ then the graph G has no cycle.*

Proof. We will prove our claim by computing the expected number of cycles (indicated by the random variable X_n) for any n. Observe that:

$$\mathbb{P}(G \text{ has a cycle}) \leq \mathbb{E}[X_n].$$

Now observe that we have:

$$\mathbb{E}[X_n] = \sum_{S \in T} \mathbb{E}[\mathbf{1}_{C_s}] = \sum_{k \geq 3} \sum_{S \in T_k} \mathbb{P}(C_s)$$

Where T denotes the set of all subsets of the vertices V of G that are distinct in the sense that a unique cycle could occur (we don't double-count cycles), T_k the set of all subsets of size k of all the vertices of V (in the same sense), C_s the event that the subset S is a cycle. Since we work with an Erdòs-Rény model we have that:

$$\mathbb{P}(C_s) = p^{|S|}.$$

Now let's look at $|T_k|$, since sets are ordered we have $|T_k| = \binom{n}{k} k! \cdot \frac{1}{2k} = \frac{(k-1)!}{2} \binom{n}{k}$. *There are $\binom{n}{k}$ sets of k different vertices we can pick out of n, we need them to be ordered so we multiply that by $k!$, but we don't care about all ordering, cycle direction and starting vertex are of no importance se we divide by $2k$.* We thus have:

$$\sum_{S \in T_k} \mathbb{P}(C_s) = \frac{(k-1)!}{2} \binom{n}{k} p^k.$$

Which we can use to get an expression for the expected number of cycles:

$$\mathbb{E}[X_n] = \sum_{k \geq 3} \frac{(k-1)!}{2} \binom{n}{k} p^k.$$

Which we now bound using that $\binom{n}{k} k! \leq n^k$:

$$\mathbb{E}[X_n] = \sum_{k \geq 3} \frac{(k-1)!}{2} \binom{n}{k} p^k \leq \sum_{k \geq 3} k! \binom{n}{k} p^k \leq \sum_{k \geq 3} n^k p^k.$$

Which allows us to bound with a geometric series:

$$\begin{aligned}\mathbb{E}[X_n] \leq \sum_{k\geq 3}(np)^k &= \sum_{k\geq 0}(np)^k - \sum_{k\geq 0}^{2}(np)^k \\ &= \frac{1}{1-np} - \frac{1-(np)^3}{1-np} = \frac{(np)^3}{1-np}.\end{aligned}$$

Now observe that $p(n)/t_c(n) \to 0$ is equivalent to $np(n) \to 0$, so $\mathbb{E}[X_n] \to 0$ which completes the proof.

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