Homework Set 2 - Networks out of Control

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Exercise 1

Claim 1. *Given that n is even, using Stirling's formula, we claim that:*

$$(n-1)!! \approx \alpha n^{n/2} e^{-n/2},$$

for some $\alpha \in \mathbb{R},\ \alpha > 0$ to be determined.

Proof. Recall that the double factorial of a number $n \in \mathbb{Z}$, denoted $n!!$, is given by the expression:

$$n!! = \prod_{k=0}^{\frac{n}{2}} (n-2k) = n \cdot (n-2) \cdot ...3 \cdot 1 \qquad \text{for an odd number,}$$
$$n!! = \prod_{k=0}^{\frac{n+1}{2}} (n-2k) = n \cdot (n-2) \cdot ...4 \cdot 2 \qquad \text{for an even number.}$$

Since n is even we have that $n-1$ is odd. Observe that the expression

$$(n-1)!! = (n-1) \cdot (n-3) \cdot ...3 \cdot 1$$

can be expressed as (let $n = 2k$, by n even $k \in \mathbb{Z}$):

$$(n-1)!! = \frac{(n-1) \cdot (n-2) \cdot (n-3) \cdot ...3 \cdot 2 \cdot 1}{(n-2) \cdot (n-4) \cdot ...4 \cdot 2}$$
$$= \frac{(2k-1) \cdot (2k-2) \cdot (2k-3) \cdot ...3 \cdot 2 \cdot 1}{(2k-2) \cdot (2k-4) \cdot ...4 \cdot 2} = \frac{(2k-1)!}{2(k-1)!}.$$

Which we reduce into:

$$(n-1)!! = \frac{(2k-1)!}{2(k-1)!} = \frac{\frac{1}{n} * (n)!}{2(n/2)! * \frac{2}{n}} = \frac{1}{4} \frac{(n)!}{(n/2)!},$$

by Stirling's formula we further get:

$$\frac{(n)!}{4(n/2)!} \approx \frac{n^n e^{-n} \sqrt{2\pi n}}{4(n^{n/2} e^{-n/2} \sqrt{\pi n})} = \frac{1}{2\sqrt{2}} n^{n/2} e^{-n/2},$$

where $1/2\sqrt{2} = \alpha$. $\square$

Exercise 2

Definition 1. *A graph $G = (V, E)$ is said to be k-connected if there are at least k vertex disjoint path between any two vertices $u, v \in V$ in G.*

Theorem 1. *(Connectivity of $G(n, r)$). For $r \geq 3$. $G(, r)$ is r-connected a.a.s. .*

Definition 2. *Partition the vertex set V of the graph the graph $G = (V, E)$ into 3 A, B, S disjoint partitions s.t. $A \cup S \cup B = V$. We say that the set $S \subset V$* ***separates*** *G if $\nexists\ (u, v) \in E$ s.t. $u \in A,\ v \in B$.*

Remark 1. *If a graph $G = (V, E)$ is k connected $\iff$ the size of the smallest set S that separates A and B is k.*

Proof. (of theorem 1) We separate the proof into two subcases, pick an arbitrary large number $a_0 \in \mathbb{Z}^+$, we distinguish the proof between two components, the **small component case** for $a = |A| < a_0$ and the **large component case** $a = |A| > a_0$.

For the **small component case** the proof is included in the lecture notes.

For the **large component case** we use a proof similar to the case $2 < a < a_0$ from the small component proof. Let $T \subseteq S$ be the subset of vertices in S adjacent to A, let $t = |T|$ and $s = |S|$. To show our result, lower-bound t and therefore s. $\square$

Exercise 3