Homework Set 3 - Networks out of Control

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Exercise 1

Question 1

Claim 1. *Consider two increasing events A_1 and A_2 having equal probabilities. Show that:*

$$\mathbb{P}_p(A_1) \geq 1 - \sqrt{1 - \mathbb{P}_p(A_1 \cup A_2)}. \tag{1}$$

Proof. Start from:

$$1 - \mathbb{P}_p(A_1 \cup A_2) = \mathbb{P}_p(\overline{A_1} \cap \overline{A_2}),$$

now observing that if A_i is increasing then $\overline{A_i}$ is decreasing, we have, by then FKG inequality (which holds for both increasing and decreasing events):

$$\begin{aligned}
\mathbb{P}_p(\overline{A_1} \cap \overline{A_2}) \geq \mathbb{P}_p(\overline{A_1})\mathbb{P}_p(\overline{A_2}) = \mathbb{P}_p(\overline{A_1})^2, \\
1 - \mathbb{P}_p(A_1 \cup A_2) = \mathbb{P}_p(\overline{A_1} \cap \overline{A_2}) \geq \mathbb{P}_p(\overline{A_1})^2, \\
\sqrt{1 - \mathbb{P}_p(A_1 \cup A_2)} \geq \mathbb{P}_p(\overline{A_1}). \\
\mathbb{P}_p(A_1) \geq 1 - \sqrt{1 - \mathbb{P}_p(A_1 \cup A_2)}.
\end{aligned}$$

□

Question 2

Suppose $\mathbb{P}_p(A_1) = \mathbb{P}_p(A_2) > 0$ but A_1 increasing while A_2 is decreasing, is (1) still true?

Pick a trivial example, consider a 2-vertices (denoted v_1 and v_2) lattice with $p = 1/2$ that an edge appears. Let A denote the event that $\exists\ v_1 \longleftrightarrow v_2$ and B $\nexists\ v_1 \longleftrightarrow v_2$. Observe that although $\mathbb{P}_p(A) = \mathbb{P}_p(B)$, A is increasing and B is decreasing. Now compute both sides of the inequality (1).

left side	$\mathbb{P}_p(A) = 1/2$
righta side	$1 - \sqrt{1 - \mathbb{P}_p(A_1 \cup A_2)} = 1 - \sqrt{1-1} = 1,$

observe that $1 \geq 1/2$. This is a counter example.

Below is the begining of a proof I had to go through to get the example, it clearly not required for the question feel free to ignore it.

One easy way to see why (1) is true is is simply to observe that the FKG inequality stops holding which breaks the bound above. Our example is the following: we will get our example starting from the proof of **Theorem 8.2 (FKG Inequality)** from the lecture notes. Consider an increasing event A and a decreasing event B. Let $X = \mathbf{1}_A$ and $X = \mathbf{1}_B$, thus X is an increasing random variable and Y a decreasing random variable.

Pick $n = 1$ so that X and Y are only function of the state $\omega(e_1)$ of the edge e_1. Pick any two states $\omega_1, \omega_2 \in \{0, 1\}$. Since X is increasing and Y is decreasing we have:

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \leq 0$$

with equality if $\omega_1 = \omega_2$. As, if $X(\omega_1) \geq X(\omega_2)$, one must have $Y(\omega_1) \leq Y(\omega_2)$ by the increasing-decreasing property (or vice-versa). Therefore:

$$\begin{aligned} 0 \geq \sum_{\omega_1=0}^{1}\sum_{\omega_2=0}^{1} \overbrace{(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2))}^{\leq 0}\overbrace{\mathbb{P}_p(\omega(e_1) = \omega_1)}^{\geq 0}\overbrace{\mathbb{P}_p(\omega(e_1) = \omega_2)}^{\geq 0} \\ = \sum_{\omega_1=0}^{1} X(\omega_1)Y(\omega_1)\mathbb{P}_p(\omega(e_1) = \omega_1) + \sum_{\omega_2=0}^{1} X(\omega_2)Y(\omega_2)\mathbb{P}_p(\omega(e_1) = \omega_2) \\ - \sum_{\omega_1=0}^{1}\sum_{\omega_2=0}^{1} (X(\omega_1) + X(\omega_2))(Y(\omega_1) - Y(\omega_2))\mathbb{P}_p(\omega(e_1) = \omega_1)\mathbb{P}_p(\omega(e_1) = \omega_2) \\ = 2(\mathbb{E}_p[XY] - \mathbb{E}_p[X]\mathbb{E}_p[Y]). \end{aligned}$$

From which we get:

$$\begin{aligned} 0 \geq \mathbb{E}_p[XY] - \mathbb{E}_p[X]\mathbb{E}_p[Y] \\ \mathbb{E}_p[X]\mathbb{E}_p[Y] \geq \mathbb{E}_p[XY], \end{aligned}$$

which gives a "reversed" FKG inequality for the $n = 1$ case.