# Homework Set 2 - Networks out of Control

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## Exercise 1

**Claim 1.** *Given that $n$ is even, using Stirling's formula, we claim that:*

$$(n-1)!! \approx \alpha n^{n/2} e^{-n/2},$$

*for some $\alpha \in \mathbb{R},\ \alpha > 0$ to be determined.*

*Proof.* Recall that the double factorial of a number $n \in \mathbb{Z}$, denoted $n!!$, is given by the expression:

$$n!! = \prod_{k=0}^{\frac{n}{2}} (n-2k) = n \cdot (n-2) \cdot ...3 \cdot 1 \qquad \text{for an odd number,}$$
$$n!! = \prod_{k=0}^{\frac{n+1}{2}} (n-2k) = n \cdot (n-2) \cdot ...4 \cdot 2 \qquad \text{for an even number.}$$

Since $n$ is even we have that $n-1$ is odd. Observe that the expression

$$(n-1)!! = (n-1) \cdot (n-3) \cdot ...3 \cdot 1$$

can be expressed as (let $n = 2k$, by $n$ even $k \in \mathbb{Z}$):

$$(n-1)!! = \frac{(n-1)\cdot(n-2)\cdot(n-3)\cdot ...3\cdot 2\cdot 1}{(n-2)\cdot(n-4)\cdot ...4\cdot 2}$$
$$= \frac{(2k-1)\cdot(2k-2)\cdot(2k-3)\cdot ...3\cdot 2\cdot 1}{(2k-2)\cdot(2k-4)\cdot ...4\cdot 2} = \frac{(2k-1)!}{2^{k-1}(k-1)!}.$$

Which we reduce into:

$$(n-1)!! = \frac{(2k-1)!}{2^{k-1}(k-1)!} = \frac{\frac{1}{n} * (n)!}{2^{k-1}(n/2)! * \frac{2}{n}} = \frac{1}{2^k}\frac{(n)!}{(n/2)!},$$

by Stirling's formula we further get:

$$\frac{(n)!}{2^{n/2}(n/2)!} \approx \frac{n^n e^{-n}\sqrt{2\pi n}}{2^{n/2}((\frac{n}{2})^{n/2} e^{-n/2}\sqrt{\pi n})} = \sqrt{2} n^{n/2} e^{-n/2},$$

where $\sqrt{2} = \alpha$. □

## Exercise 2

**Definition 1.** *A graph $G = (V, E)$ is said to be $k$-connected if there are at least $k$ vertex disjoint path between any two vertices $u, v \in V$ in $G$.*

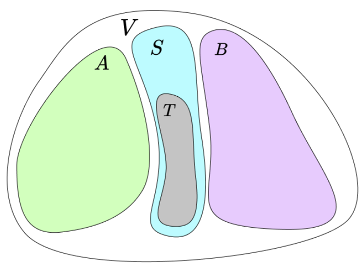
**Theorem 1.** *(Connectivity of $G(n, r)$). For $r \geq 3$. $G(, r)$ is $r$-connected a.a.s. .*

**Definition 2.** *Partition the vertex set $V$ of the graph the graph $G = (V, E)$ into 3 $A, B, S$ disjoint partitions s.t. $A \cup S \cup B = V$. We say that the set $S \subset V$* ***separates*** *$G$ if $\nexists\ (u, v) \in E$ s.t. $u \in A,\ v \in B$.*

**Remark 1.** *If a graph $G = (V, E)$ is $k$ connected $\iff$ the size of the smallest set $S$ that separates $A$ and $B$ is $k$.*

*Proof.* (of theorem 1) We separate the proof into two subcases, pick an arbitrary large number $a_0 \in \mathbb{Z}^+$, we distinguish the proof between two components, the **small component case** for $a = |A| < a_0$ and the **large component case** $a = |A| > a_0$.

For the **small component case** the proof is included in the lecture notes. For the **large component case** we use a proof similar to the case $2 < a < a_0$ from the small component proof.




Partition $V$ into three disjoint subsets $A,\ B,\ S$ s.t. $A \cup B \cup S = V$ and $S$ separates $A$ and $B$. Let $T \subseteq S$ be the subset of vertices in $S$ adjacent to $A$, let $t = |T|$ and $s = |S|$ and $a = |A|$. To show our result, we lower-bound $t$ and therefore $s$, by *remark 1*, this is equivalent to showing that $G$ is $r$-connected. Let $H$ be the subgraph of $G$ that spans $T \cup A$. As in the small component proof, we have that $v(H) = a + t$ and $e(H) \geq \frac{ra+t}{2}$. To show that $G$ is $r$-connected we proceed by contradiction. We use the result from *theorem 4.2 from the lecture notes* that states that a subgraph $H$ has a number of copies $\theta(n^{v(H)-e(H)})$. We will show that if $r > t$, then we can choose $a_0$ s.t. the number of copies of $H$ is $O(n^{-2})$, which will allow us to prove the $r$-connected claim. Suppose that $r > t$, then we have that:

$$e(H) - v(H) \leq (a + t) - \frac{ra + t}{2}$$

We want to show that $e(H) - v(H) \leq -2$, so with a bit of algebra we have:

$$\begin{aligned}
(a+t) - \frac{ra+t}{2} &\leq -2 \\
2a + 2t - ra + t &\leq -4 \\
a(2-r) + t &\leq -4 \\
a &\geq \frac{\overbrace{-4-t}^{<0}}{\underbrace{2-r}_{<0}}.
\end{aligned}$$

So letting $a_0 > \frac{-4-t}{2-r}$ gives us that # of occurrences of $H$ is $O(n^{-2})$. From that we get that:

$$\begin{aligned}
&\mathbb{P}\left[\exists \text{ the smallest } S \text{ has } |S| < r \text{ for any } n\right] = \\
&\sum_{a_0}^{n} \mathbb{P}\left[\exists \text{ the smallest } S \text{ has } |S| < r \text{ for some } n\right] \\
&\leq n \cdot O(n^{-2}) = O(n^{-1}) \to 0,
\end{aligned}$$

which completes the proof. $\square$

## Exercise 3

### Question 1.

Let $X$ be generated from the $G(n,p)$ model and $Y$ be generated from the $G(n,r)$ model, i.e. sampled uniformly from the set pf $r$-regular graphs $\mathcal{G}(n,r)$ graphs. If $G$ is a $r$-regular is $\mathbb{P}\left[X = G | X \text{ is } r\text{-regular}\right]$ the same as $\mathbb{P}\left[Y = G\right]$ ?

Let $\boldsymbol{G_{r,n}}$ be the set of all possible $r$-regular graphs on $n$ vertices, observe that sampling from $G(n,p)$ and restricting to $r$-regular graphs or sampling $G(n,r)$ has in both case the effect of uniformly sampling through $\boldsymbol{G_{r,n}}$. Observe that:

$$\mathbb{P}\left[X = G | X \text{ is } r\text{-regular}\right] = \mathbb{P}\left[Y = G\right] = \frac{1}{|\boldsymbol{G_{r,n}}|}.$$

The probability doesn't depend on the sampling method.

### Question 2.

We use the $G^*(n,r)$ matching construction to prove r-regular graph properties instead of the Erdòs-Rény model since properties that hold from $G^*(n,r)$ hold a.a.s for $G(n,r)$, which is not true of the $G(n,p)$ model.