4. MARKOV NETWORKS

RENATO LUI GEH
NUSP: 8536030

ABSTRACT. This document shows the solutions to the proposed exercises.

1. EXERCISES

Exercise 1. *Prove that the following statements are equivalent for two nodes X and Y in an acyclic directed graph G:*

   i X and Y are adjacent.
  ii There is no set $\mathbf{Z}$ that d-separated X and Y.
 iii X and Y are not d-separated by $An(X) \cup An(Y)$.
 iv X and Y are not d-separated by $Pa(X) \cup Pa(Y)$.

Proof. Since X and Y are adjacent, than there exists an edge $X \to Y$ or $Y \to X$. From the definition of d-separation, we can conclude that X and Y are d-connected (or unblocked). This equates to saying states (i) and (ii). For statements (iii) and (iv), consider the following graphs:
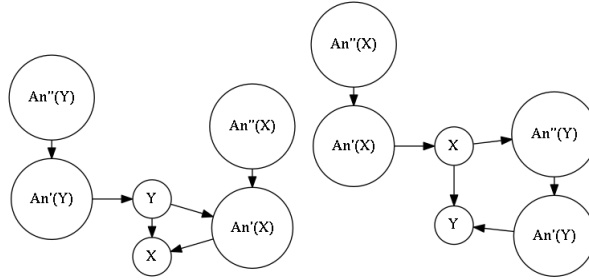

FIGURE 1. Graph (a) on the left and graph (b) on the right.

    These two graphs summarize all possible situations in which the ancestrals of X and Y could be connected to X and Y respectively. Consider graph (a). Note that we cannot form any connection $(X, Y, An(X), An(Y))$ such that X and Y are d-separated. For instance, consider the path $Y \to An'(X) \to X$. In this case $An'(X)$ indeed blocks X and Y; however there exists a path $Y \to X$ that is unblocked. From the definition of d-separation, if there exists an unblocked trail, then variables X and Y are not d-separated. This explains the equivalence of statement (iv). Now consider graph (b). Again, the path $X \to An'(Y) \to An''(y) \to Y$ is a serial connection where $\mathcal{B} = \{An'(Y), An''(Y)\}$ blocks X and Y. Similar to our previous

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conjecture, for X and Y to be d-separated all trails must be blocked. Yet, $X \to Y$ is obviously unblocked, proving that X and Y must be d-connected.

The resulting decisions are presented below:

: $B \perp C|A$?
      true: $\Rightarrow$ Graph (1)
      false: $\Rightarrow$ Graph (2) or (3)
      : $B \perp C|\emptyset$?
            true: Graph (3)
            false: Graph (2)

$\square$

Exercise 2. *Suppose you have an independency oracle I which answers independency queries "is $X \perp Y|Z$?". Discuss how you can use the result in the previous exercise and the oracle to decide which of the graphs below represent the independency relation induced by the oracle. Make an effort to find the most computationally efficient way, that is, one which calls the oracle the least number of times.*
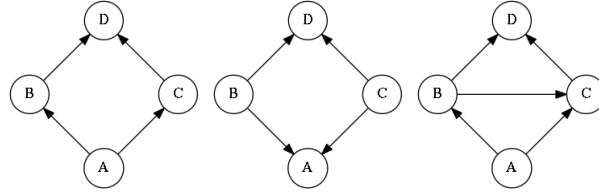

FIGURE 2. Possible graphs as solutions. Numbered 1 to 3 from left to right.

Solution. First we ask the query $I(B, A, C)$ (i.e. $B \perp C|A$) to the oracle. If the result is true, then we have that the corresponding graph is (1). Otherwise it may either be (2) or (3). This condition is true since, if the statement $I(B, A, C)$ is true, then B and C must be d-separated. For graph (1) we have two connections: convergent and divergent. Since we were given A, then both connections have A block B and C. Thus we conclude B and C are d-separated in graph (1). Graph (2) has two convergent connections: $B \to A \leftarrow C$ and $B \to D \leftarrow C$. We only have A as evidence, so the first path is indeed unblocked; therefore, even though $B \to D \leftarrow C$ is blocked, there still exists a trail that unblocks B and C, making them d-separated. For graph (3) we have the trivial case where B and C are adjacent. From the previous exercise — item (iv) — we know that $B \not\perp C|A$.

Now consider the query $I(B, \emptyset, C)$. Graph (2) has two convergent connections: $B \to A \leftarrow C$ and $B \to D \leftarrow C$. We were given no evidence, which means B and C are blocked on both trails, which means $I(B, \emptyset, C) = true$ in graph (2). B and C are adjacent in graph (3). From item (ii) of our previous exercise we know that there is no set $\mathbf{Z}$ in which B and C are d-separated. Therefore $B \not\perp C|(\mathbf{Z} = \emptyset) \equiv I(B, \mathbf{Z} = \emptyset, C) = false$. $\square$