# Two Perspectives to Learning with Circuits

# Motivation

Given a selection of sushi...

...and people's preferences...

**Alice:** 

**Bob:** 

**Carol:** 

...how can we model this as a probability distribution...

$p(1^{\text{st}} =$  $, 3^{\text{rd}} =$  $)$

$p(2^{\text{nd}} =$  $| 1^{\text{st}} =$  $)$

$\arg\max p(1^{\text{st}} = ?, 2^{\text{nd}} = ?, 3^{\text{rd}} = ?, 4^{\text{th}} =$  $, 5^{\text{th}} =$  $)$

$p((3^{\text{rd}} =$  $\rightarrow 1^{\text{st}} =$  $) \vee 2^{\text{nd}} =$  $)$

...and extract meaningful queries from it?

Kamishima [2003]

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$p(1^{\text{st}} =$  $, 3^{\text{rd}} =$  $)$ **Marginals**

$p(2^{\text{nd}} =$  $| 1^{\text{st}} =$  $)$ **Conditionals**

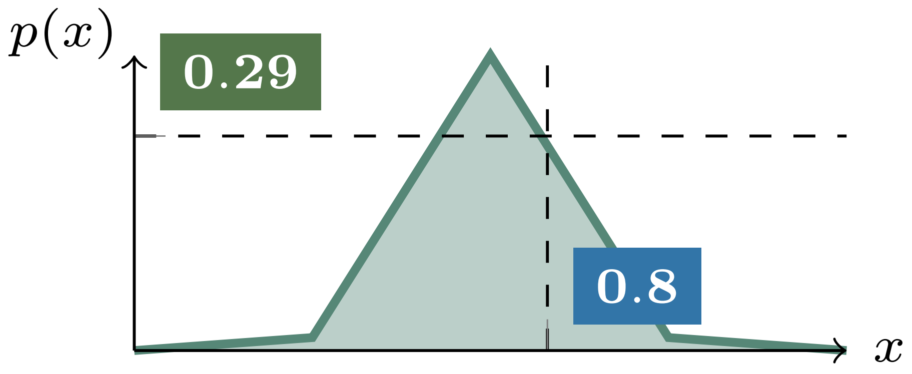
$\arg\max p(1^{\text{st}} = ?, 2^{\text{nd}} = ?, 3^{\text{rd}} = ?, 4^{\text{th}} =$  $, 5^{\text{th}} =$  $)$ **MPE**

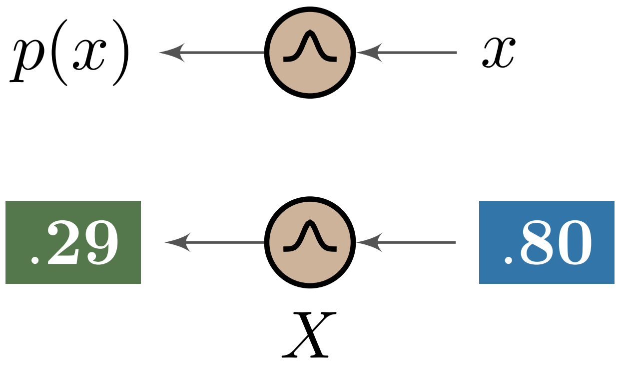
$p((3^{\text{rd}} =$  $\rightarrow 1^{\text{st}} =$  $) \vee 2^{\text{nd}} =$  $)$ **Logical events**

...and extract meaningful queries from it?

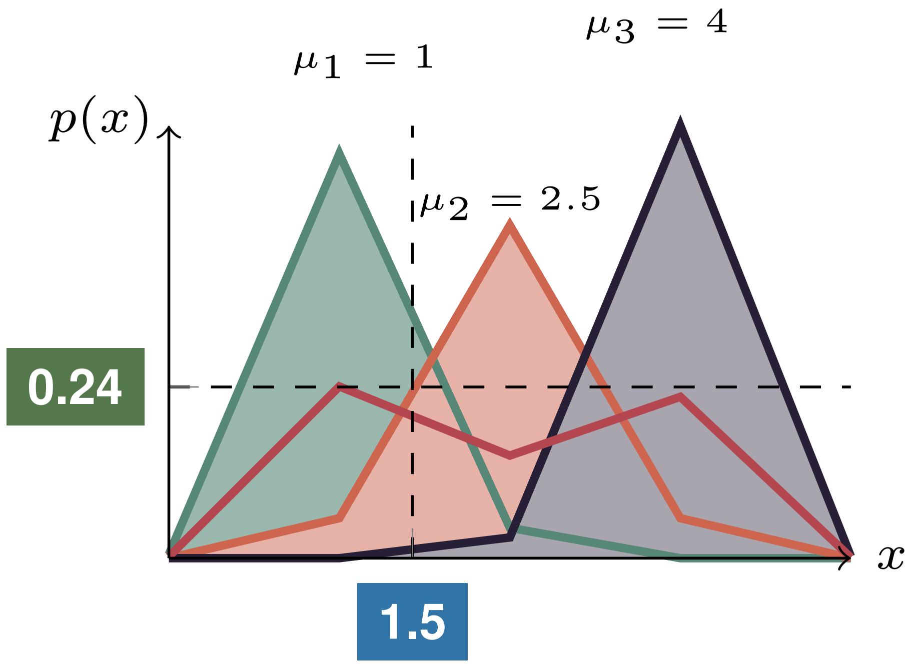
Kamishima [2003]

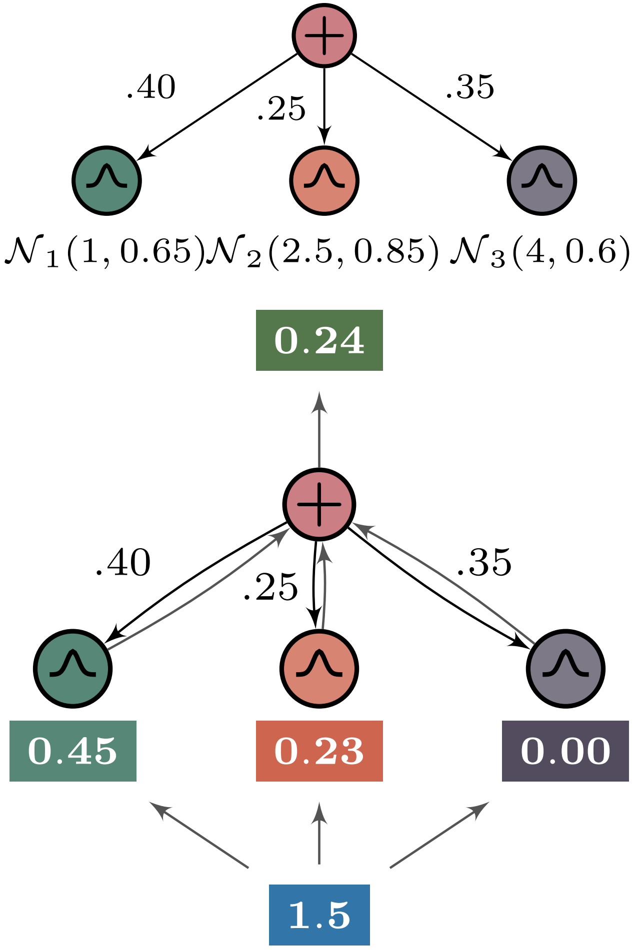
# Probabilistic Circuits – Inputs



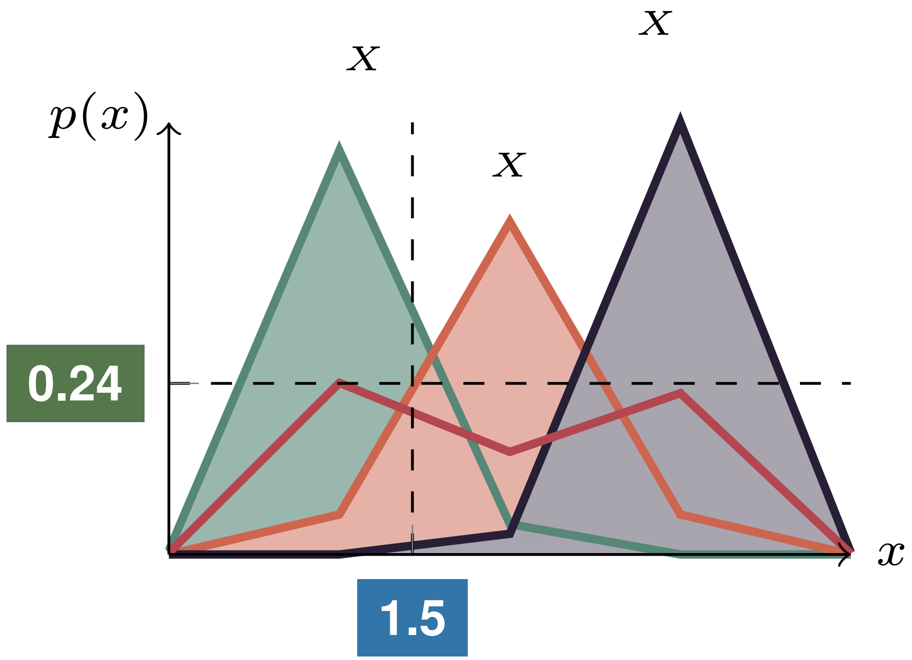



# Probabilistic Circuits – Sums

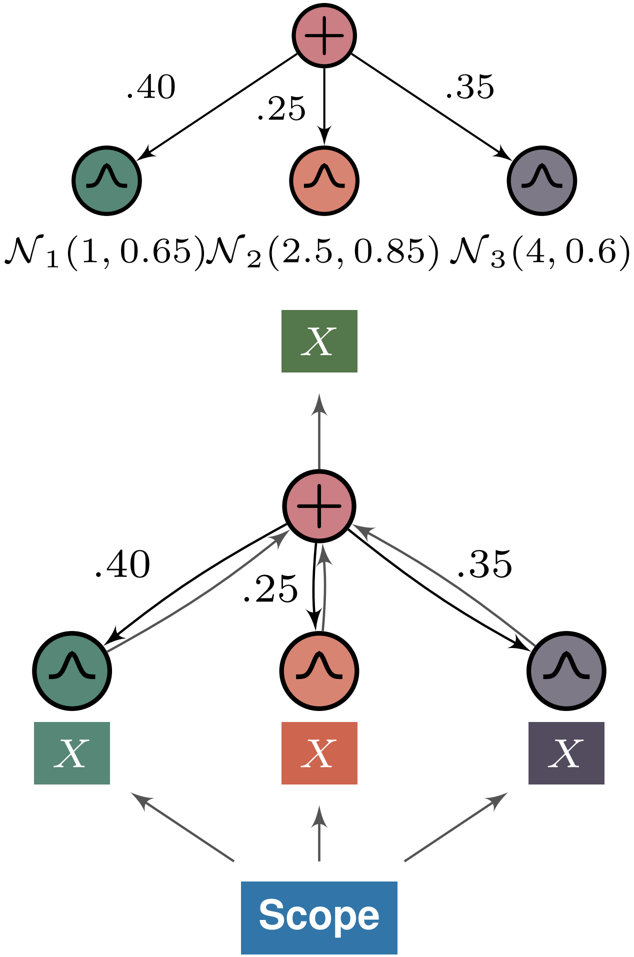


# Probabilistic Circuits – Smoothness

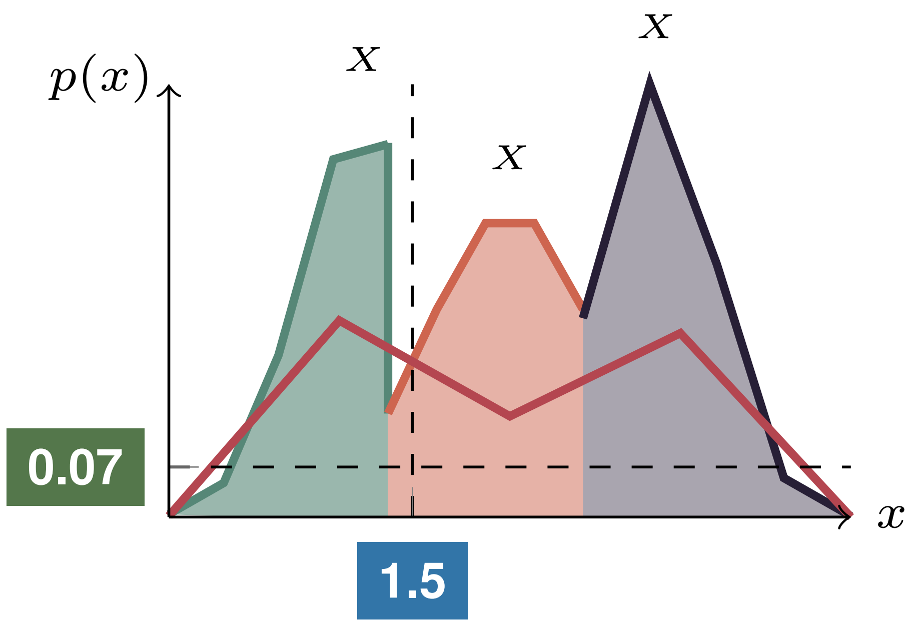


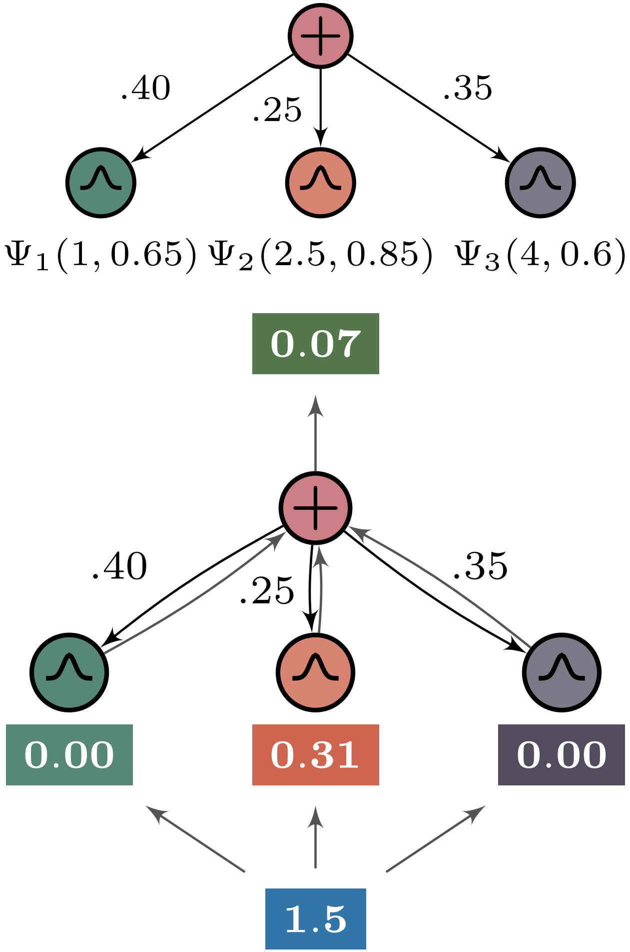

**Definition 1** (Smoothness)**.**
*Every sum node child mentions the <u>same</u> variables.*




Darwiche [2001a]

# Probabilistic Circuits – Determinism

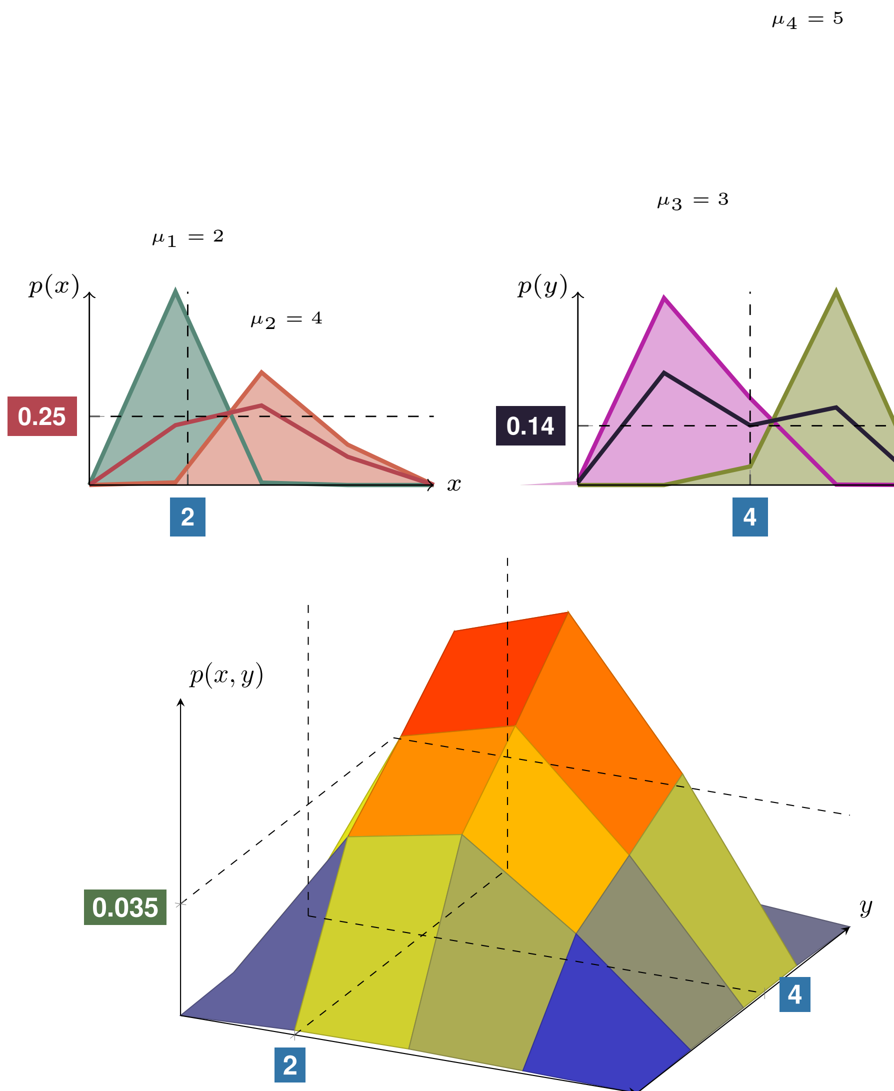


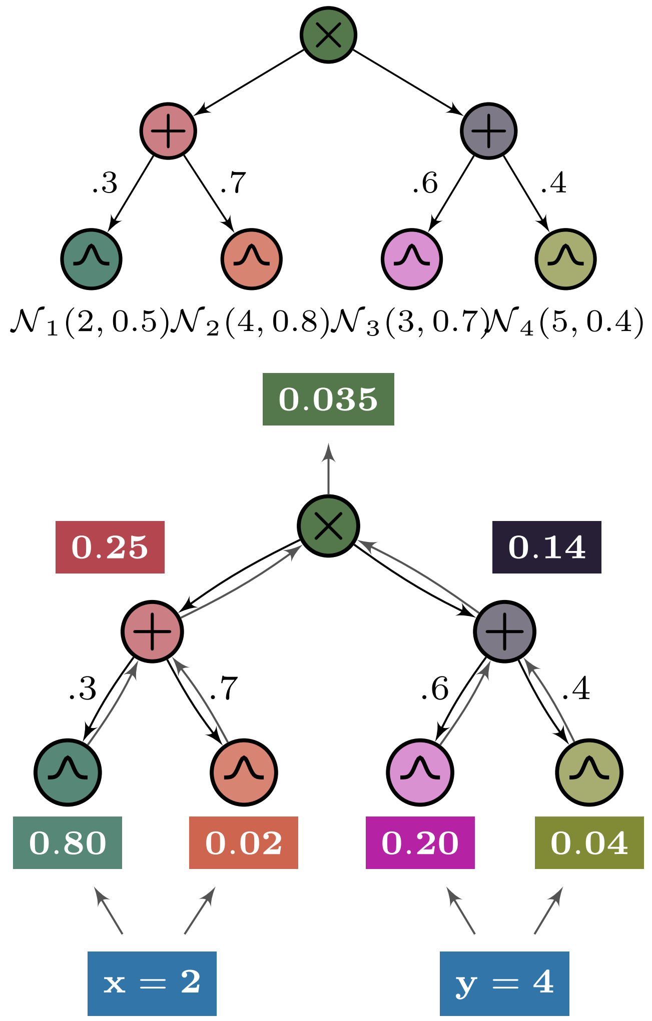




**Definition 2** (Determinism)**.**
*<u>At most one</u> sum node child has a positive value.*

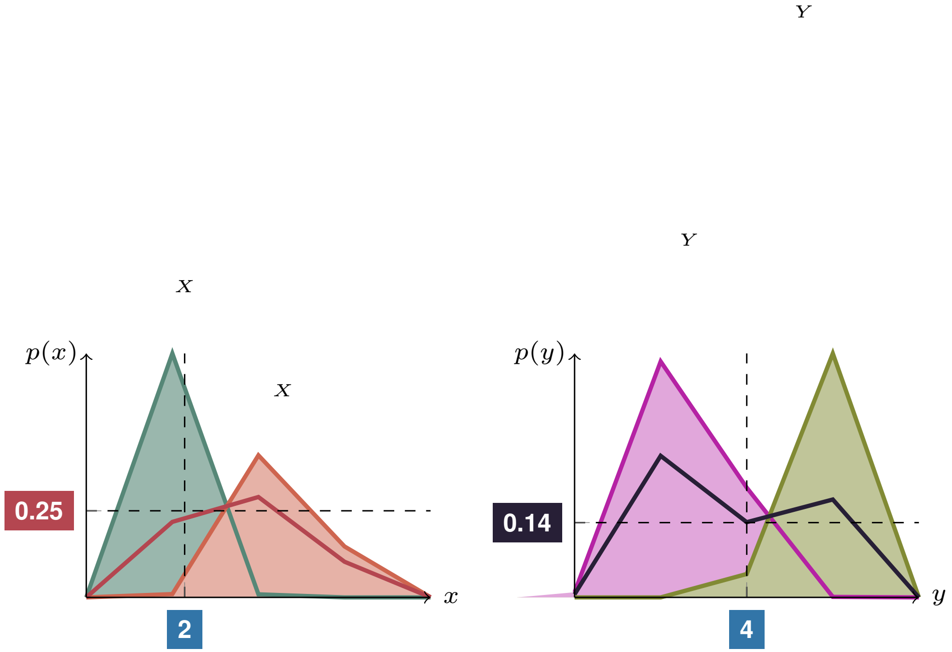
Darwiche [2001a]

# Probabilistic Circuits – Products

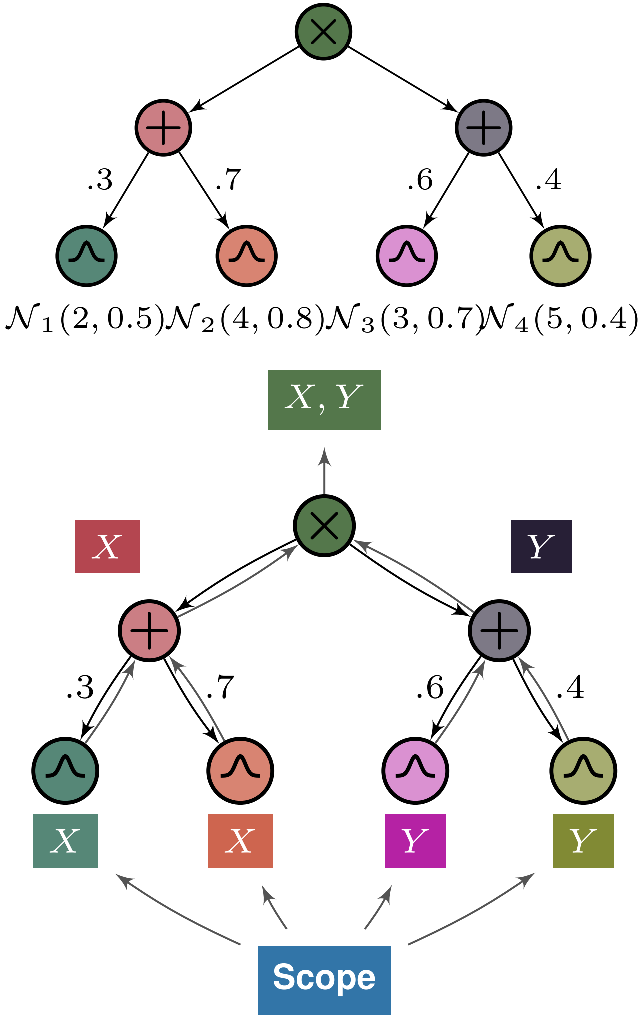


# Probabilistic Circuits – Decomposability

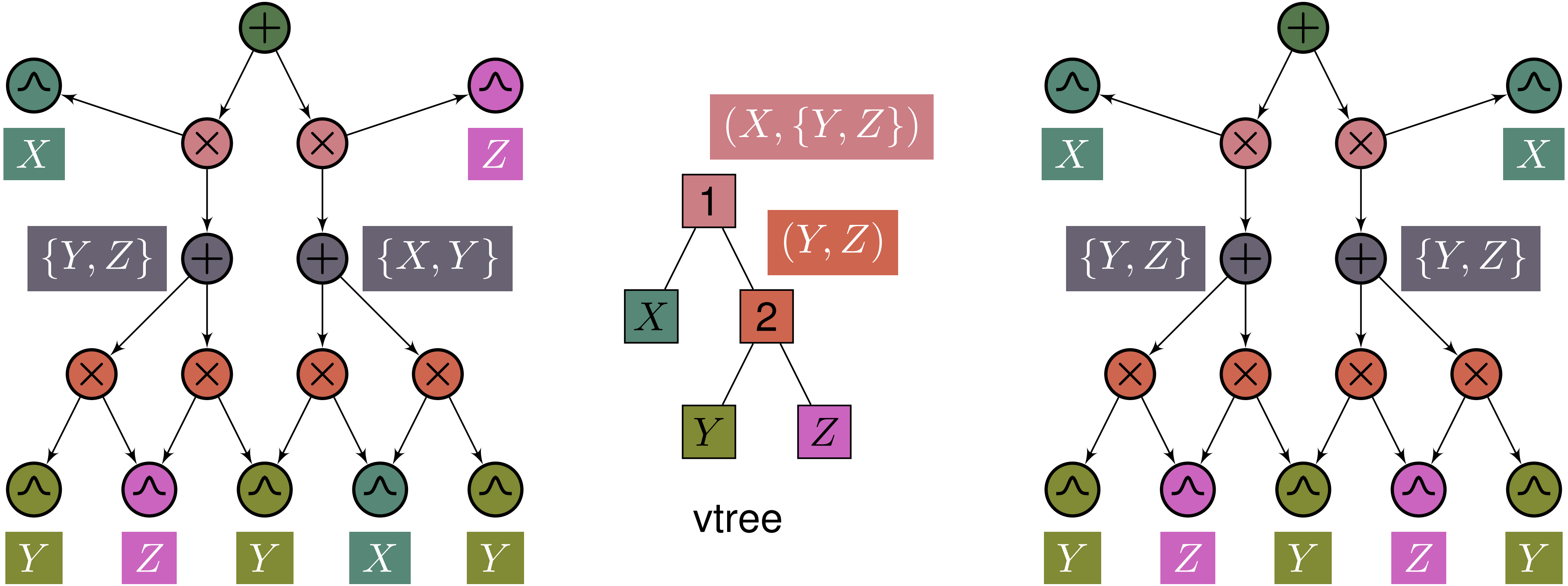



**Definition 3** (Decomposability)**.**
*Every product node child mentions <u>different</u> variables.*




Darwiche [1999, 2001b]

# Probabilistic Circuits – Structured Decomposability




**Definition 4** (Structured decomposability)**.** *Every product node follows a vtree decomposition.*

Pipatsrisawat and Darwiche [2008]

# Probabilistic Circuits – Tractability

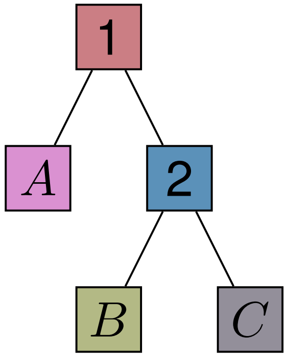
| Query | +Sm? | +Dec? | +Det? | +Str Dec? |
|---|---|---|---|---|
| Evidence | ✓ | ✓ | ✓ | ✓ |
| Marginals | ✗ | ✓ | ✓ | ✓ |
| Conditionals | ✗ | ✓ | ✓ | ✓ |
| MPE | ✗ | ✗ | ✓ | ✓ |
| Shannon Entropy | ✗ | ✗ | ✓ | ✓ |
| Rényi Entropy | ✗ | ✗ | ✓ | ✓ |
| Cross Entropy | ✗ | ✗ | ✗ | ✓ |
| Kullback-Leibler Div | ✗ | ✗ | ✗ | ✓ |
| Rényi's Alpha Div | ✗ | ✗ | ✗ | ✓ |
| Cauchy-Schwarz Div | ✗ | ✗ | ✗ | ✓ |
| Logical Events | ✗ | ✗ | ✗ | ✓ |
| Mutual Information | ✗ | ✗ | ✗ | ✓ |

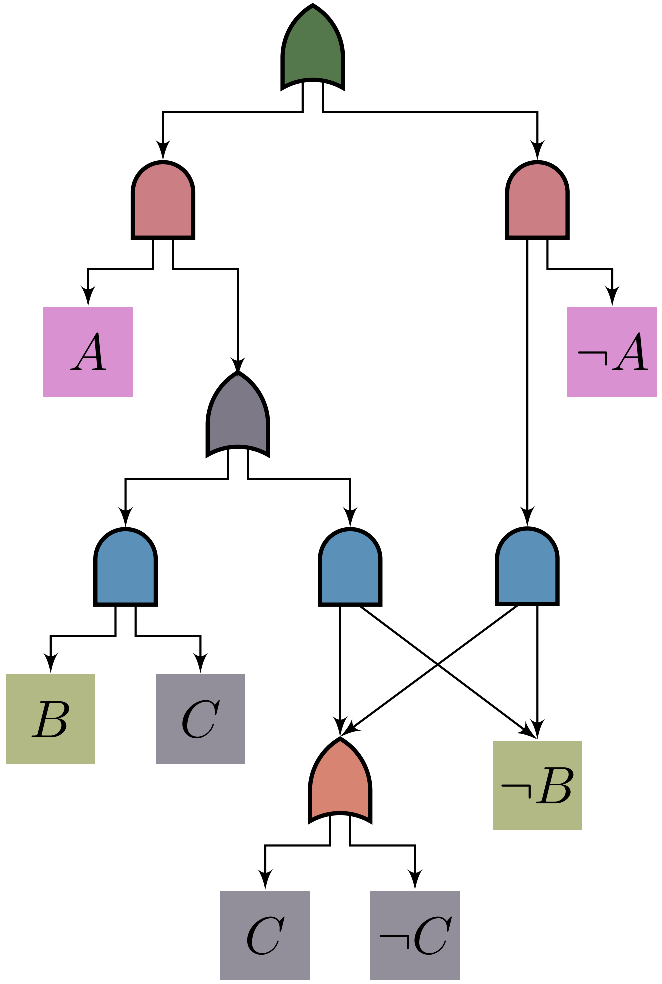
Vergari et al. [2021], Poon and Domingos [2011], Peharz et al. [2016]

# Probabilistic Circuits – Logic Circuits

| $A$ | $B$ | $C$ | $\phi(\mathbf{x})$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 1 |

$$\phi(A, B, C) = (A \vee B) \wedge (\neg B \vee C)$$




Darwiche [2011]

# Probabilistic Circuits – Support

| $A$ | $B$ | $C$ | $\phi(\mathbf{x})$ | $p(\mathbf{x})$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0.140 |
| 1 | 0 | 0 | 1 | 0.024 |
| 0 | 1 | 0 | 0 | 0.000 |
| 1 | 1 | 0 | 0 | 0.000 |
| 0 | 0 | 1 | 1 | 0.560 |
| 1 | 0 | 1 | 1 | 0.096 |
| 0 | 1 | 1 | 0 | 0.000 |
| 1 | 1 | 1 | 1 | 0.180 |

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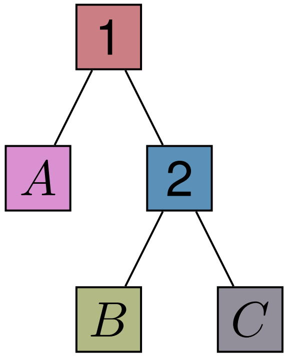
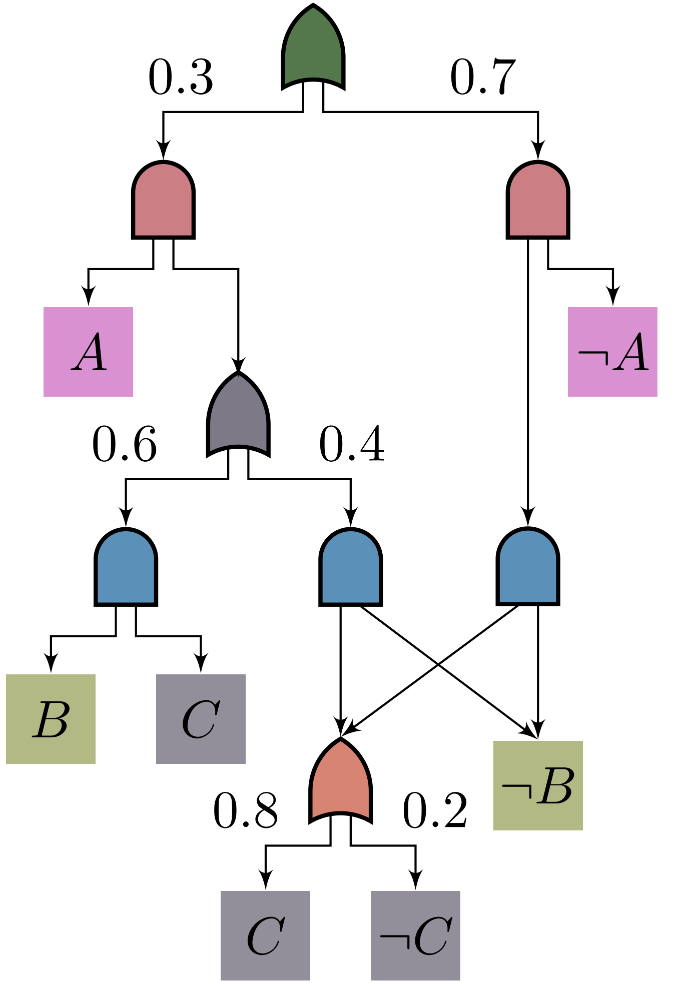


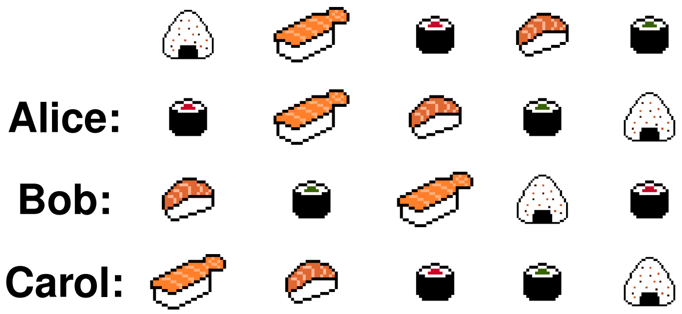

Kisa et al. [2014]

# Learning Probabilistic Circuits – Where are we right now?

| Name | Class | Time Complexity | # hyperparams | Accepts logic? | Sm? | Dec? | Det? | Str Dec? | $\{0,1\}$? | $\mathbb{N}$? | $\mathbb{R}$? | Reference |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| LearnSPN | DIV | $\begin{cases}\mathcal{O}(nkmc) & \text{, if sum}\\ \mathcal{O}(nm^3) & \text{, if product}\end{cases}$ | $\geq 2$ | ✗ | ✓ | ✓ | ✗ | ✗ | ✓ | ✓ | ✓ | Gens and Domingos [2013] |
| ID-SPN | DIV | $\begin{cases}\mathcal{O}(nkmc) & \text{, if sum}\\ \mathcal{O}(nm^3) & \text{, if product}\\ \mathcal{O}(ic(rn+m)) & \text{, if input}\end{cases}$ | $\geq 2+3$ | ✗ | ✓ | ✓ | ✗ | ✗ | ✓ | ✓ | ✗ | Rooshenas and Lowd [2014] |
| Prometheus | DIV | $\begin{cases}\mathcal{O}(nkmc) & \text{, if sum}\\ \mathcal{O}(m(\log m)^2) & \text{, if product}\end{cases}$ | $\geq 1$ | ✗ | ✓ | ✓ | ✗ | ✗ | ✓ | ✓ | ✓ | Jaini et al. [2018] |
| LearnPSDD | INCR | $\begin{cases}\mathcal{O}(m^2) & \text{, top-down vtree}\\ \mathcal{O}(m^4) & \text{, bottom-up vtree}\\ \mathcal{O}(i\lvert\mathcal{C}\rvert^2) & \text{, circuit structure}\end{cases}$ | $1$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✗ | ✗ | Liang et al. [2017] |
| Strudel | INCR | $\begin{cases}\mathcal{O}(m^2 n) & \text{, CLT + vtree}\\ \mathcal{O}(i(\lvert\mathcal{C}\rvert n+m^2)) & \text{, circuit structure}\end{cases}$ | $1$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✗ | ✗ | Dang et al. [2020] |
| RAT-SPN | RAND | $\mathcal{O}(rd(s+l))$ | $4$ | ✗ | ✓ | ✓ | ✗ | ✗ | ✓ | ✓ | ✓ | Peharz et al. [2020] |
| XPC | RAND | $\mathcal{O}(i(t+kn)+ikm^2n)$ | $3$ | ✗ | ✓ | ✓ | ✓ | ✓ | ✓ | ✗ | ✗ | Mauro et al. [2021] |
| SamplePSDD | RAND | $\begin{cases}\mathcal{O}(m) & \text{, random vtree}\\ \mathcal{O}(kc\log c+\log_2^2 k) & \text{, per call}\end{cases}$ | $1$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✗ | ✗ | Geh and Mauá [2021] |
| LearnRP | RAND | $\begin{cases}\mathcal{O}(m^2) & \text{, top-down vtree}\\ \mathcal{O}(m^4) & \text{, bottom-up vtree}\\ \mathcal{O}(knm) & \text{, per call}\end{cases}$ | $0$ | ✗ | ✓ | ✓ | ✗ | ✓ | ✓ | ✓ | ✓ | To appear |

# A Logical Perspective

# Motivation




If we assume

- $n$ sushi types,
- $k$ sized rankings with $k \leq n$,
- $X_{ij}$ binary variables; $i$ is sushi type, $j$ is position in ranking;

then the total number of possible assignments of the $n \cdot k$ variables is $2^{nk}$...

...but many of these are zero probability assignments!

If we can embed total ranking constraints...

...we go down to k! total assignments!

**Takeaway:** models which exploit domain knowledge are much more efficient!

**Example:**

$n = 3,\ k = 3$

| $X_{11}$ | $X_{12}$ | $X_{13}$ | $X_{21}$ | $\cdots$ | $X_{33}$ | $p(\mathbf{x}) > 0$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 0 | 1 | 1 | 1 | 1 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 |

Assignments: $2^{3 \cdot 3} = 512$

Positive assignments: $3! = 6$

Choi et al. [2015]

# Motivation

**Existing approaches:**

LearnPSDD (Liang et al. [2017]):

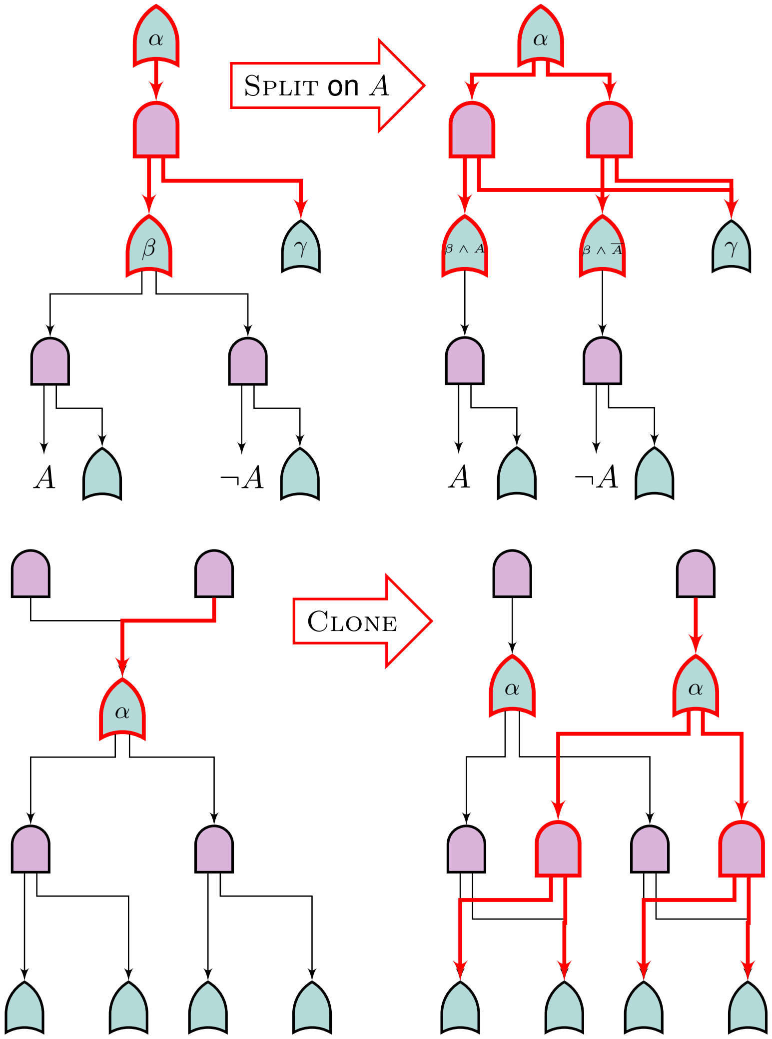
- ☒ Requires initial logic circuit encoding the support...
- ☒ Scales poorly to complex formulae and/or high dimension...
- ☒ Costly whole circuit evaluation at every iteration...
- ☑ Very good performance!

Strudel (Dang et al. [2020]):

- ☑ Constructs an initial structure (from a CLT)!
- ☒ But does not encode constraints...
- ☑ Scales to high dimension!
- ☒ As long as the circuit doesn't get too big...

SamplePSDD (Geh and Mauá [2021]):

- ☑ Scales to high dimension and complex formulae!
- ☑ Constructs a structure consistent with constraints!
- ☒ But does so by relaxing the formula...
- ☒ Performance varies on set bounds and vtree structure...

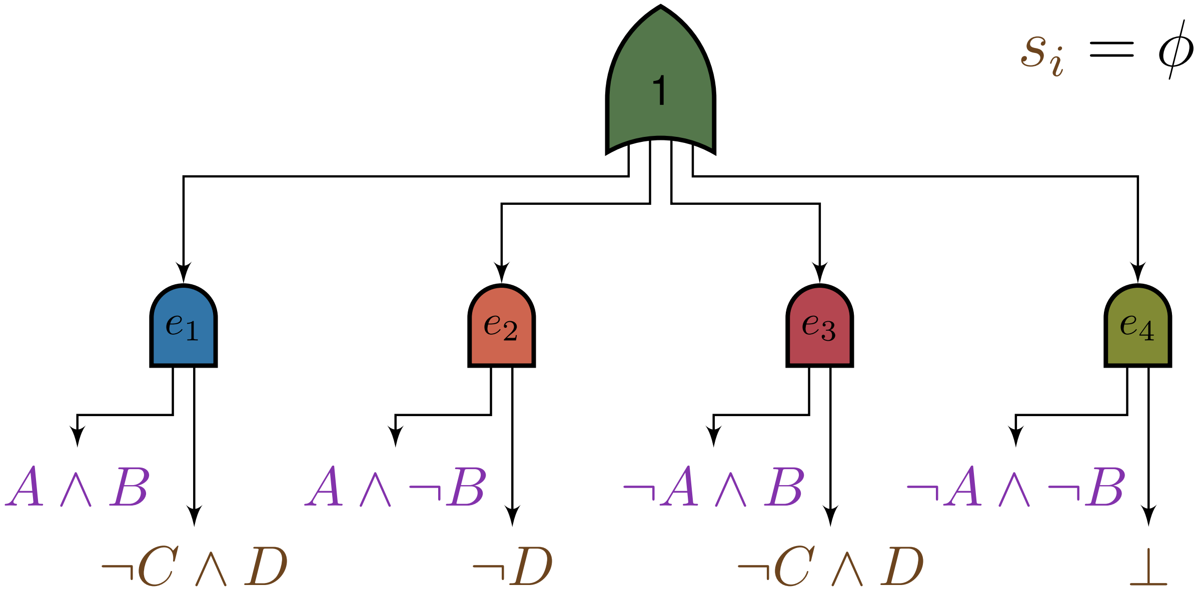


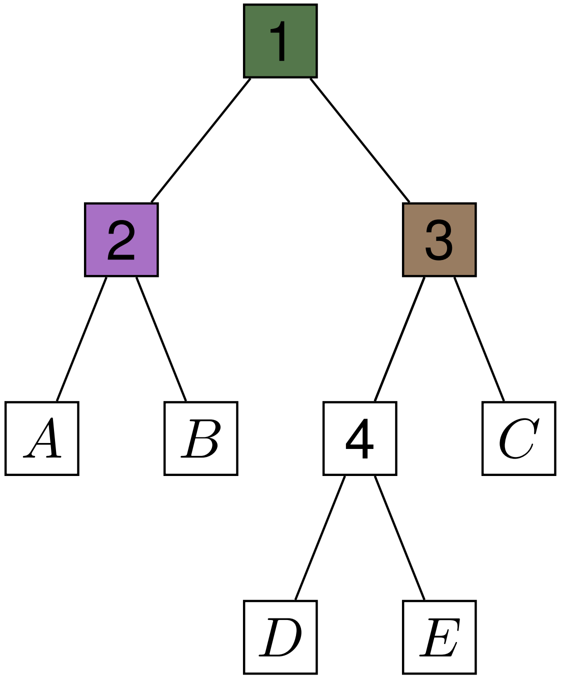
# SamplePSDD

Common assumption: $p_i$ are conjunctions of literals.

$$\phi(A, B, C, D) = (A \land \neg B \land \neg D) \lor (B \land \neg C \land D)$$

$$s_i = \phi|_{p_i}$$



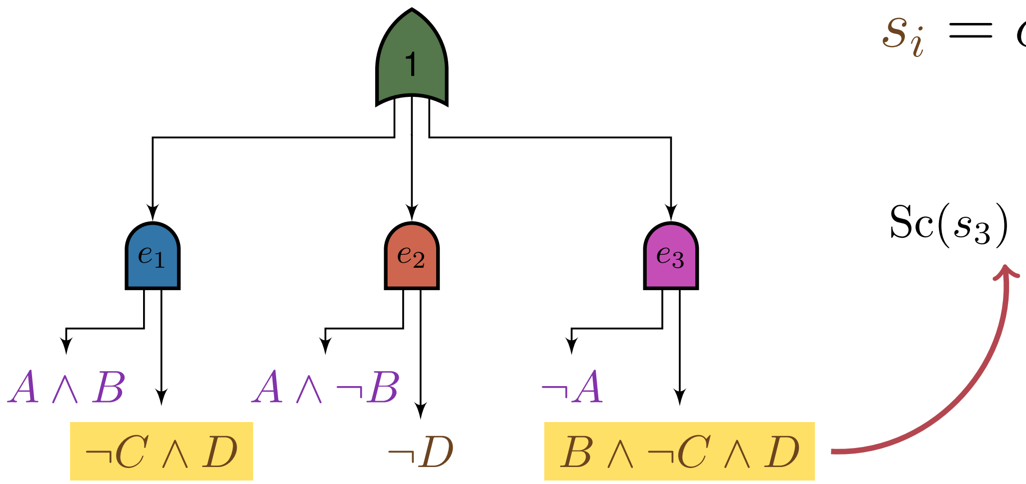




**Problem:** size of circuit is exponential in the size of $p_i$'s scope.

# SamplePSDD

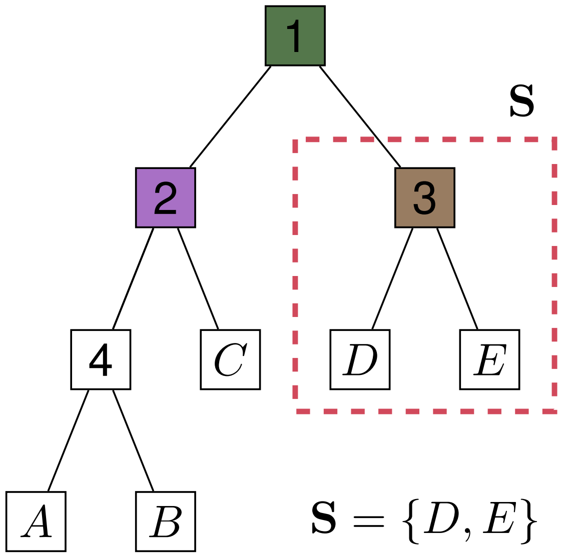
**Solution:** randomly sample a bounded number ($k$) of $p_i$

$$\phi(A, B, C, D) = (A \land \neg B \land \neg D) \lor (B \land \neg C \land D)$$




$$s_i = \phi|_{p_i}$$

$\mathrm{Sc}(s_3) \not\subseteq \mathbf{S}$




$\mathbf{S} = \{D, E\}$

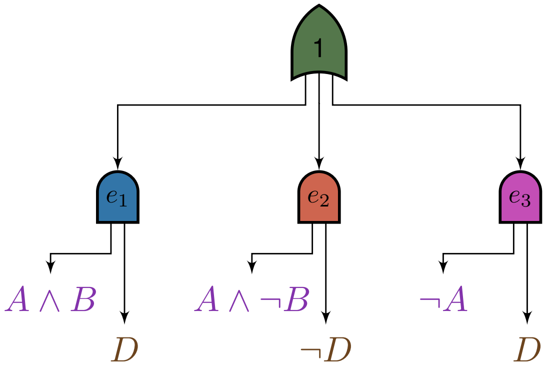
**But:** this violates structured decomposability:

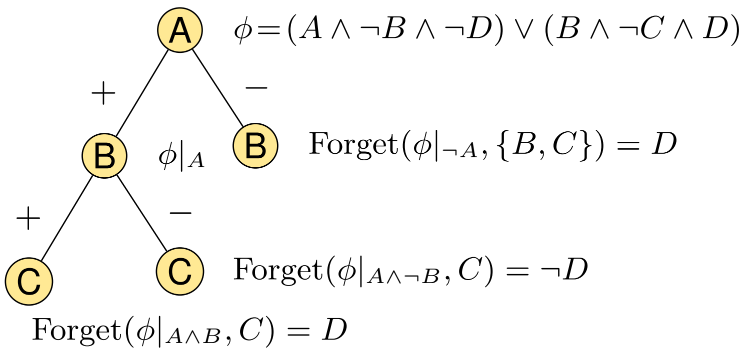
$\neg C \land D$ contains $C$, and $C \notin \mathbf{S}$

$\neg B \land \neg C \land D$ contains $B$ and $C$, and $B, C \notin \mathbf{S}$

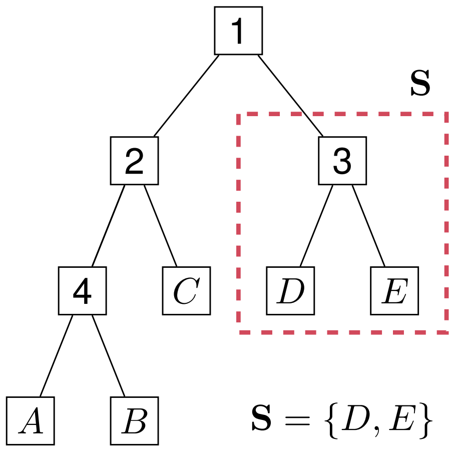
# SamplePSDD

**New solution:** relax logical constraints $\phi$



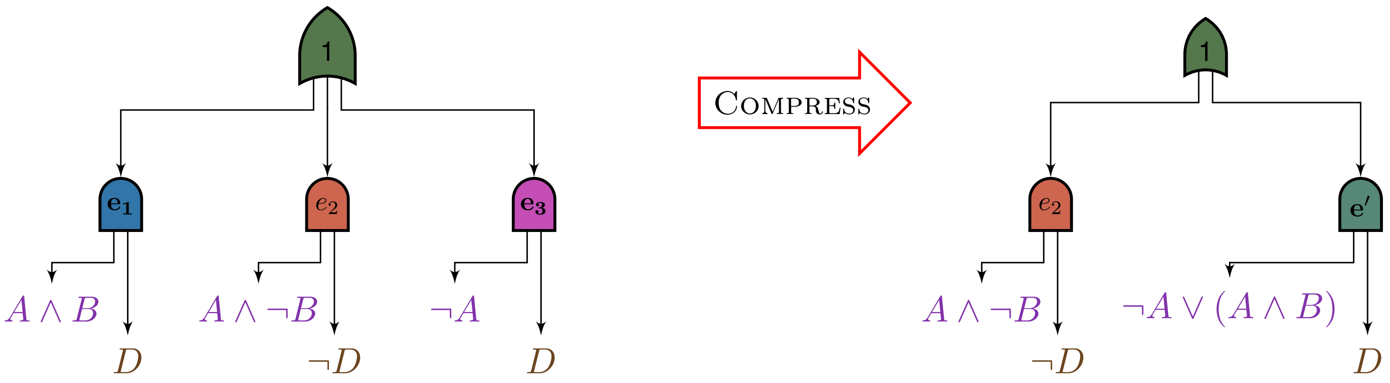



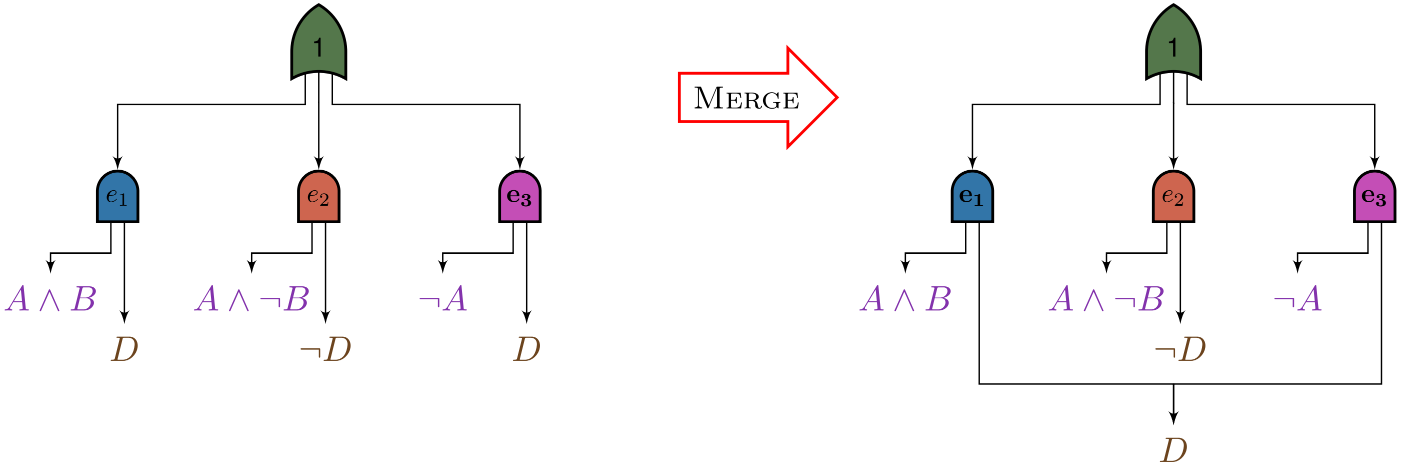

Now all $s_i$ respect S



# SamplePSDD

Apply **local transformations** for variety and size reduction




Darwiche [2011]

# Experiments

**Evaluation:** we sample 30 PSDDs and use 5 ensemble strategies:

- ● **Likelihood weighting (LLW)**,
- ■ **Uniform weights**,
- ◆ **Expectation Maximization (EM)**,
- ▲ **Stacking**,
- ▼ **Bayesian Model Combination (BMC)**;

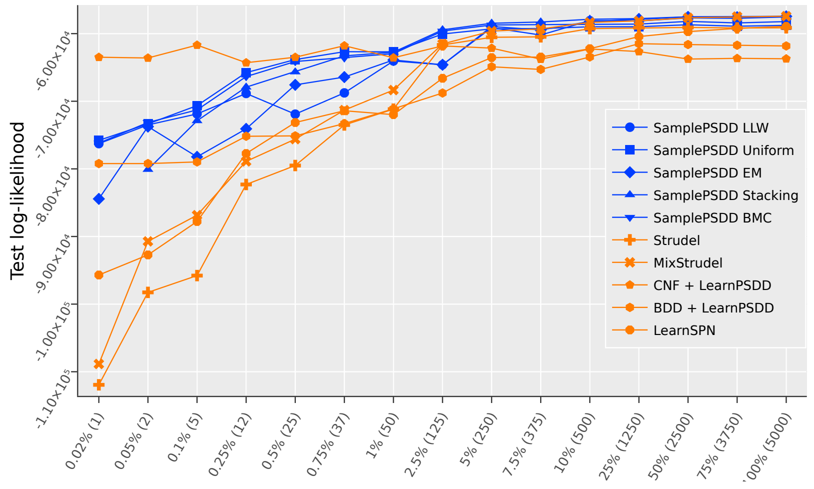
comparing against Strudel, LearnPSDD and LearnSPN.

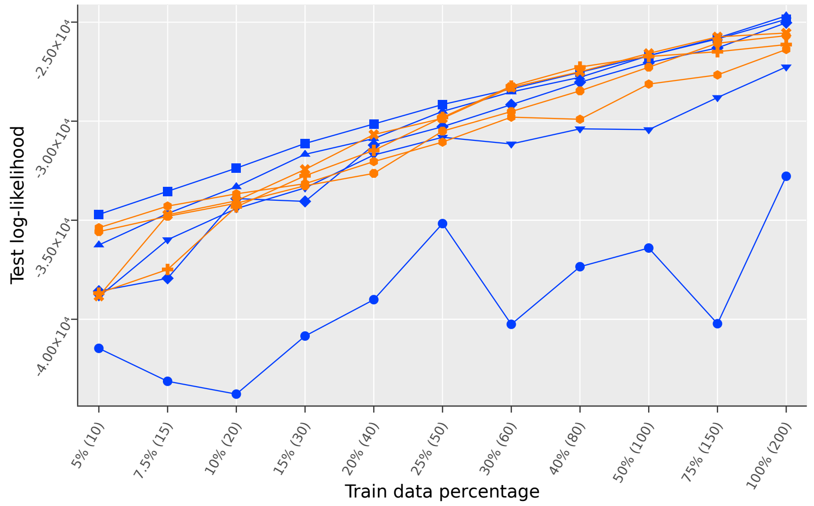
**Datasets:** we evaluate with 5 data + knowledge as logic constraints:

| | Dataset | #vars | #train | $\phi$'s size |
|---|---|---|---|---|
| $\Rightarrow$ | LED | 14 | 5000 | 23 |
| $\Rightarrow$ | LED + Images | 157 | 700 | 39899 |
| | Sushi Ranking | 100 | 3500 | 17413 |
| | Sushi Top 5 | 10 | 3500 | 37 |
| | Dota 2 Games | 227 | 92650 | 1308 |

Our approach fares **better** with **fewer** data, yet remains **competitive** under **lots of data**.

Mattei et al. [2020], Kamishima [2003], Shen et al. [2017], Choi et al. [2015], Gens and Domingos [2013], Dang et al. [2020]



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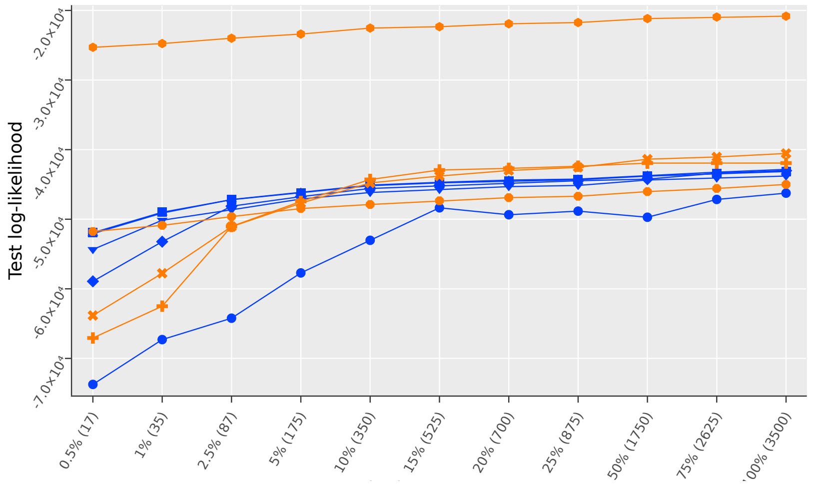


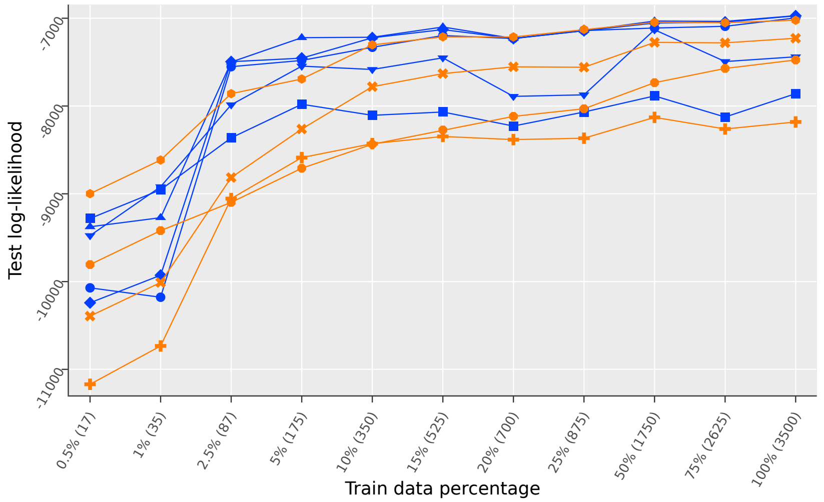

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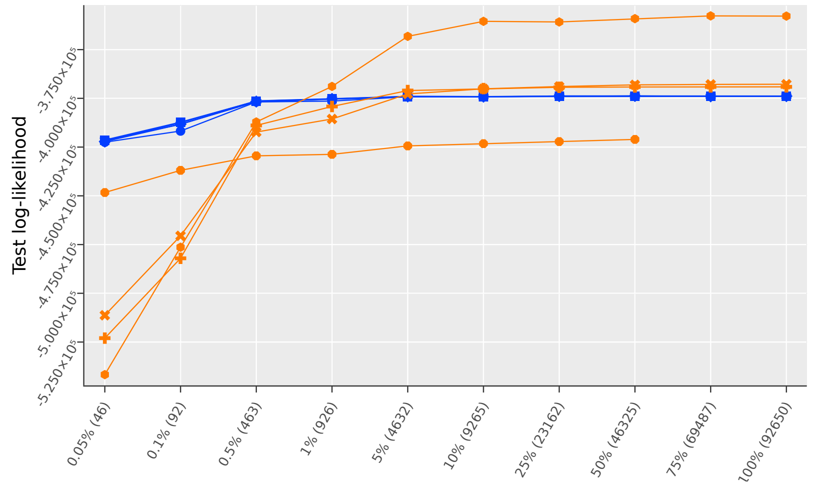
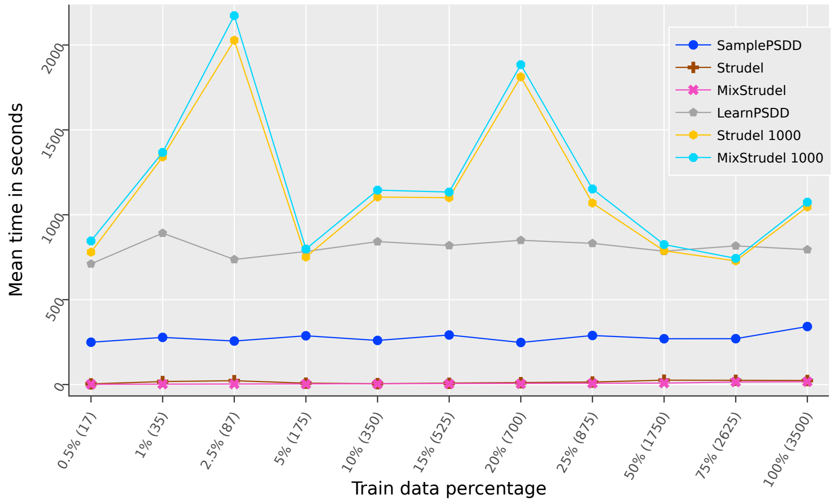


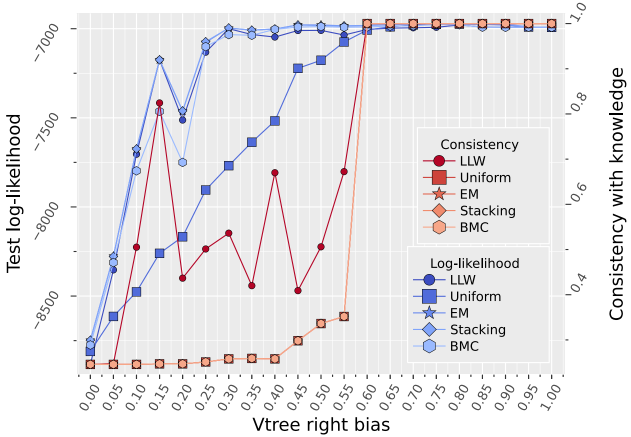



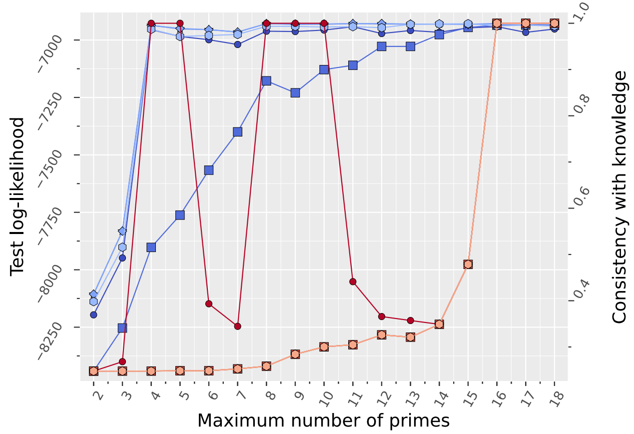

Mattei et al. [2020], Kamishima [2003], Shen et al. [2017], Choi et al. [2015], Gens and Domingos [2013], Dang et al. [2020]

# Experiments

What is the impact of higher $k$'s and right-leaning vtrees

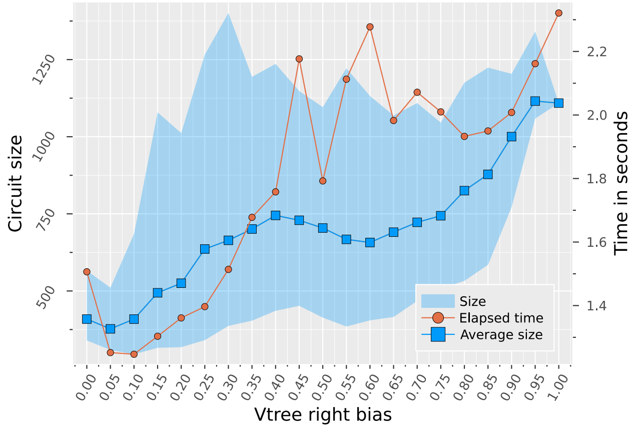
in log-likelihood and consistency?

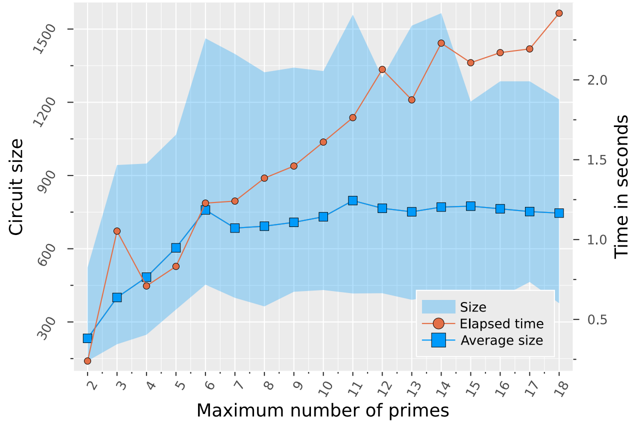



Samples perform **better** with higher $k$'s and right-leaning vtrees...

...but at a **cost** to complexity.



# What do we gain from this?

**Available queries:**

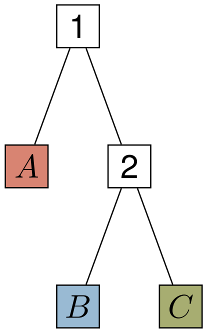
- ☑ Probability of Evidence;
- ☑ Marginal Probability;
- ☑ Conditional Probability;
- ☑ Most Probable Explanation;
- ☑ Shannon Entropy;
- ☑ Cross Entropy;
- ☑ Kullback-Leibler Divergence;
- ☑ Rényi's Alpha Divergence;
- ☑ Cauchy-Schwarz Divergence;
- ☑ Probability of Logical Events;
- ☑ Mutual Information.

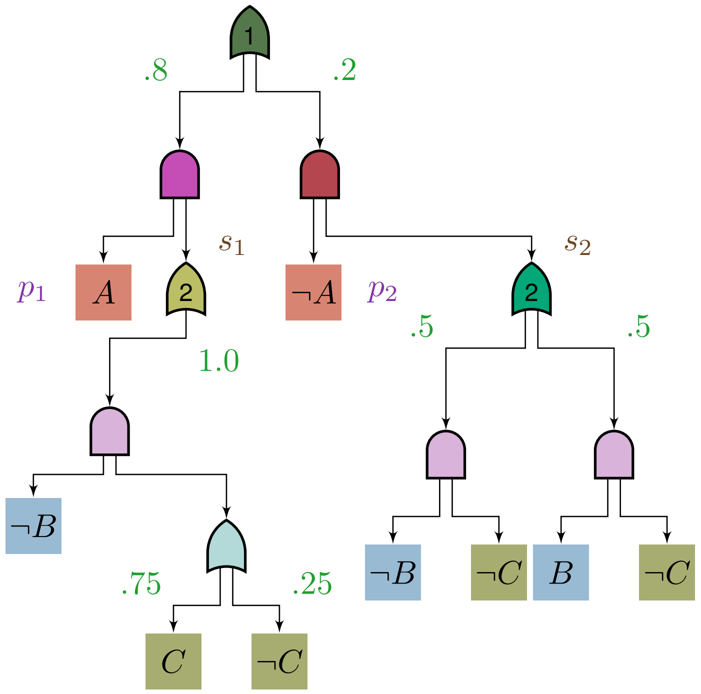
**Support:**

- ☑ Defineable as a logic formula;
- ☒ Consistent with a relaxation;
- ☑ Ensembles mitigate relaxation.

| $A$ | $B$ | $C$ | $p(\mathbf{x})$ |
|---|---|---|---|
| 0 | 0 | 0 | 0.1 |
| 0 | 1 | 0 | 0.1 |
| 1 | 0 | 0 | 0.2 |
| 1 | 0 | 1 | 0.6 |

$$\phi(A,B,C) = (A \to \neg B) \land (C \to A)$$



# A Data Perspective

# Motivation

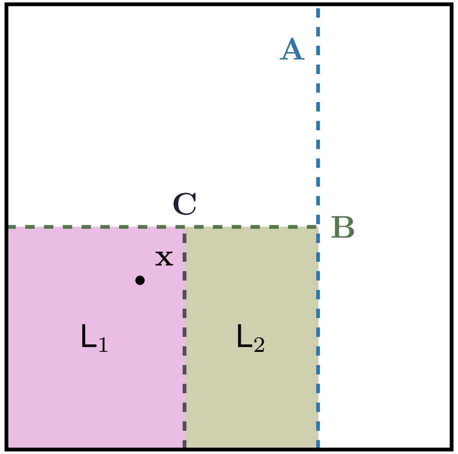
**Density Estimation Trees...**

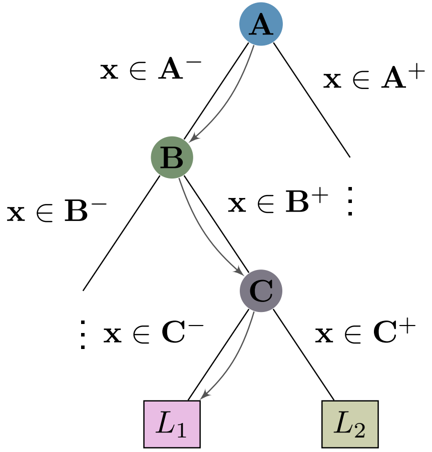
- ☑ ...are fast;
- ☑ ...are interpretable;
- ☑ ...are (somewhat) explainable;
- ☑ ...have extensive literature coverage;
- ☒ ...are not so expressive;
- ☒ ...only accept marginalization queries;
- ☒ ...are not so accurate;

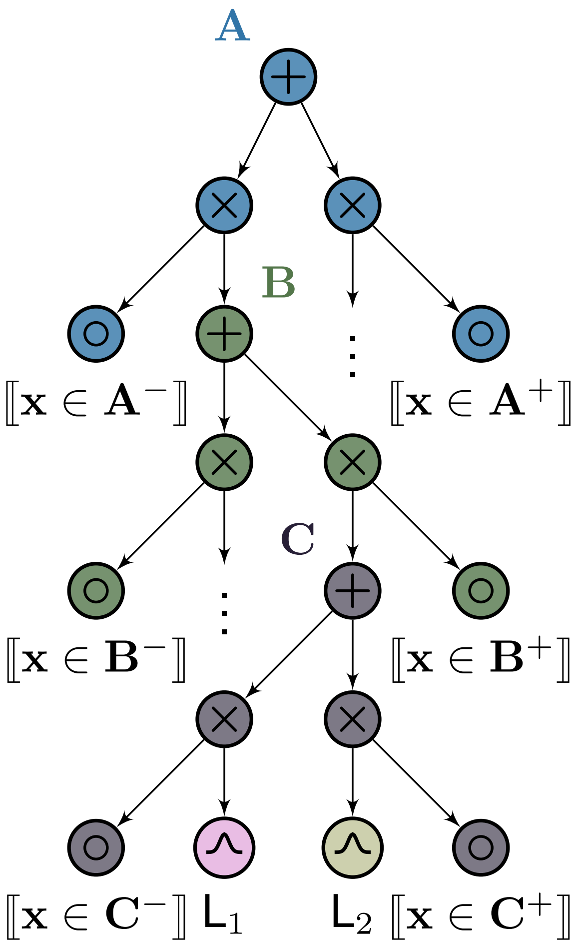
**...but are subsumed by circuits!**

Learn DETs $\subseteq$ Learn PCs?

**Can we take advantage of known learning procedures in DETs and transplant them to more general circuits?**

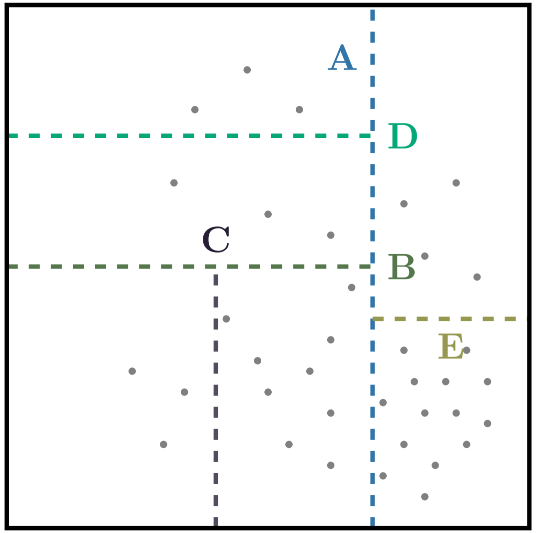


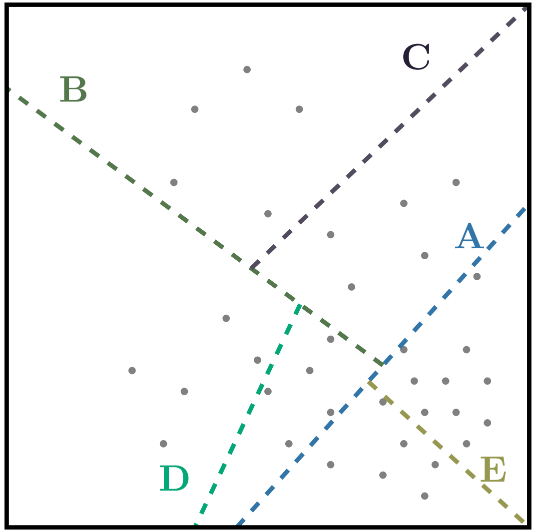




Correia et al. [2020]

# Random Projections




*Axis-aligned projections*

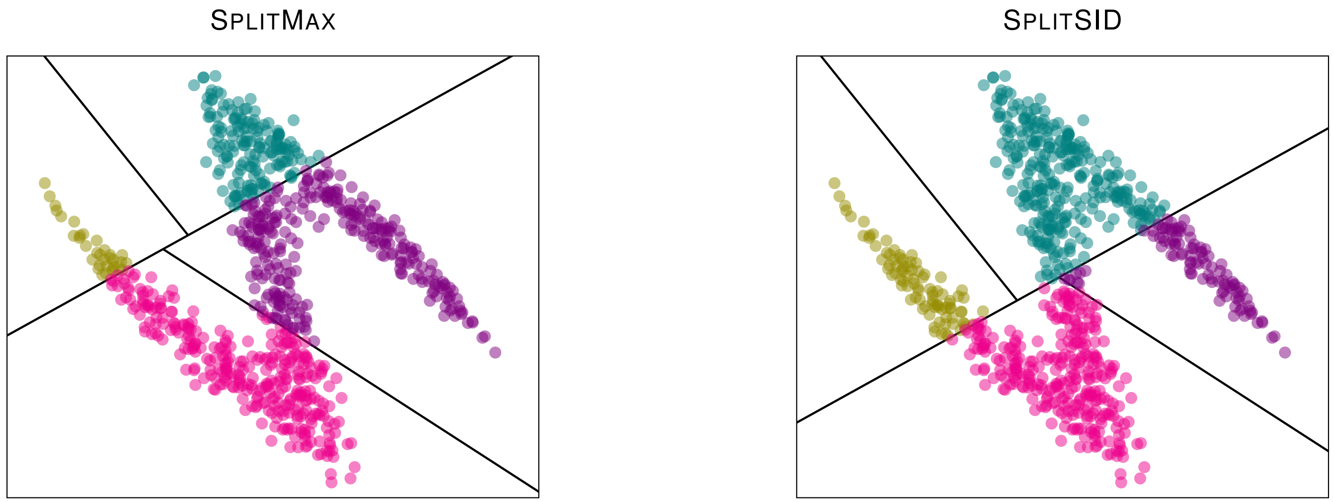



*Random projections*

If the data has *intrinsic dimension* $d$, then with constant probability the part of the data at level $d$ or higher of the tree has average diameter less than half of the data.

Freund et al. [2008], Dasgupta and Freund [2008]

# Random Projections

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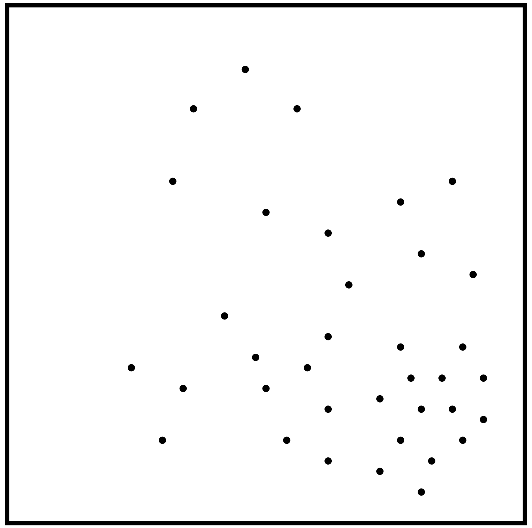
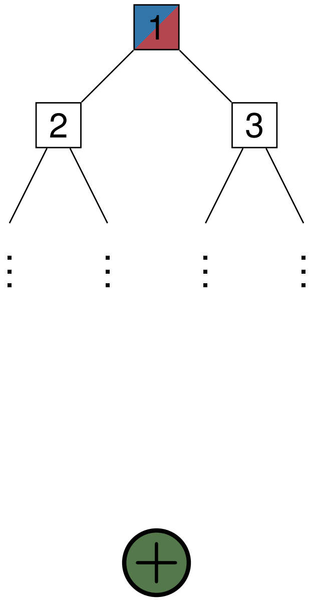
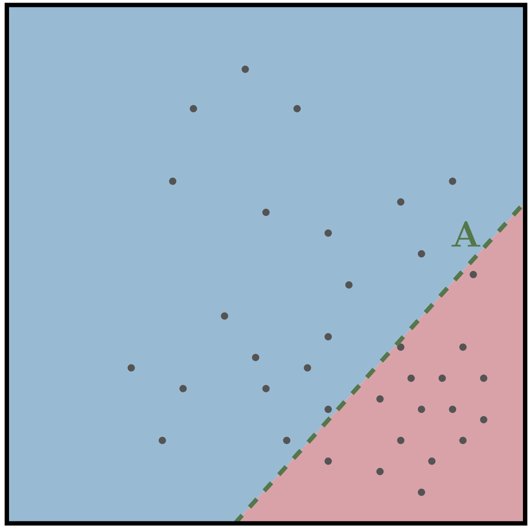
Freund et al. [2008], Dasgupta and Freund [2008]

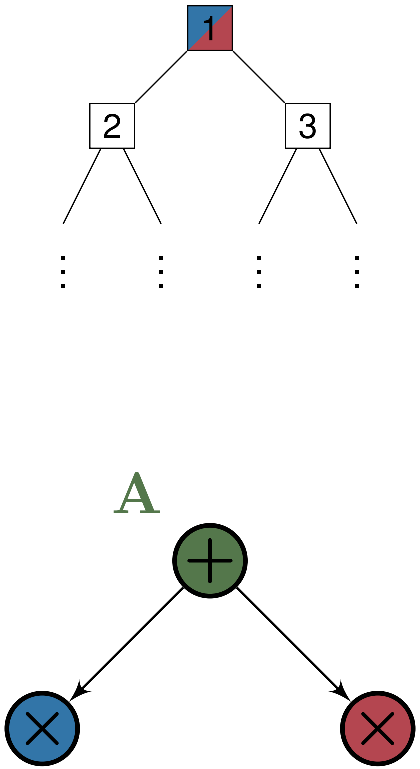
# LearnRP





# LearnRP



# LearnRP