

Two Perspectives to Learning with Circuits



Motivation

Given a selection of sushi...



...and people's preferences...

Alice:     

Bob:     

Carol:     

...how can we model this as a probability distribution...

$$p(1^{\text{st}} = \text{salmon nigiri}, 3^{\text{rd}} = \text{salmon nigiri})$$

$$p(2^{\text{nd}} = \text{salmon nigiri} \mid 1^{\text{st}} = \text{maki roll})$$

$$\arg \max p(1^{\text{st}} = ?, 2^{\text{nd}} = ?, 3^{\text{rd}} = ?, 4^{\text{th}} = \text{maki roll}, 5^{\text{th}} = \text{maki roll with red sauce})$$

$$p((3^{\text{rd}} = \text{salmon nigiri} \rightarrow 1^{\text{st}} = \text{maki roll}) \vee 2^{\text{nd}} = \text{salmon nigiri})$$

...and extract meaningful queries from it?

Motivation

Given a selection of sushi...



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Alice:     

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...how can we model this as a probability distribution...

$$p(1^{\text{st}} = \text{salmon nigiri}, 3^{\text{rd}} = \text{tuna nigiri})$$

$$p(2^{\text{nd}} = \text{tuna nigiri} \mid 1^{\text{st}} = \text{white rice ball})$$

$$\arg \max p(1^{\text{st}} = ?, 2^{\text{nd}} = ?, 3^{\text{rd}} = ?, 4^{\text{th}} = \text{white rice ball}, 5^{\text{th}} = \text{maki roll})$$

$$p((3^{\text{rd}} = \text{salmon nigiri} \rightarrow 1^{\text{st}} = \text{white rice ball}) \vee 2^{\text{nd}} = \text{tuna nigiri})$$

Marginals

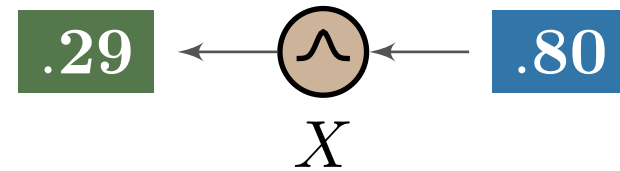
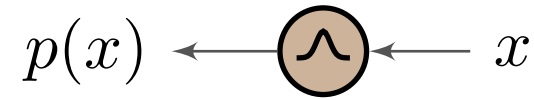
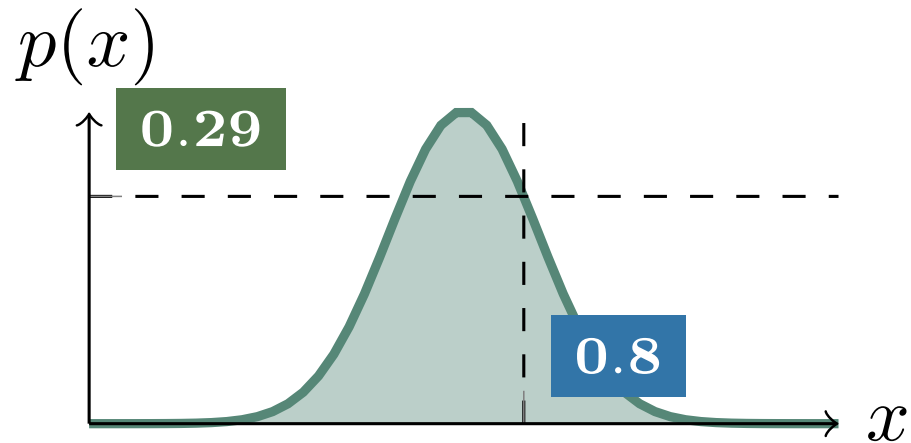
Conditionals

MPE

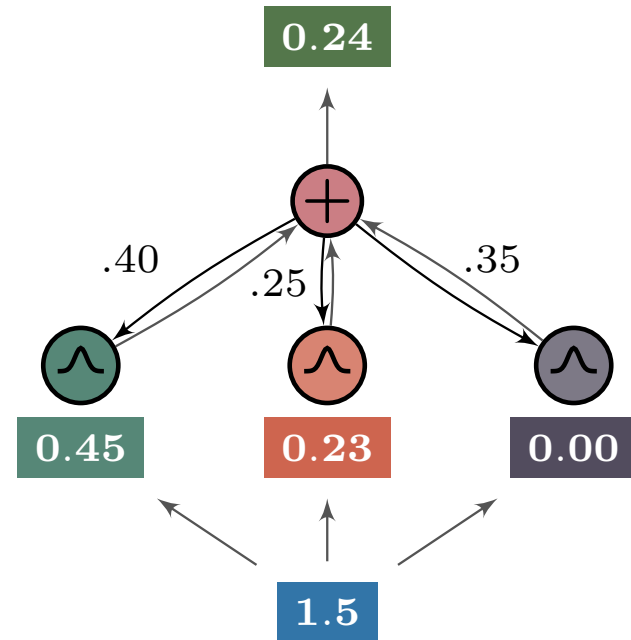
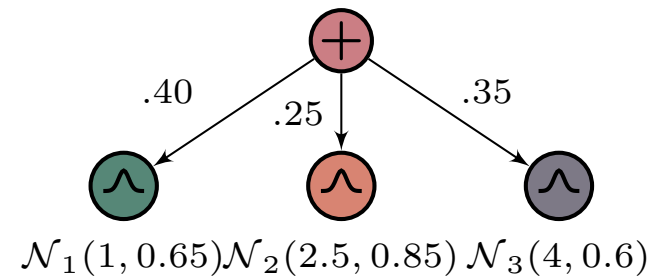
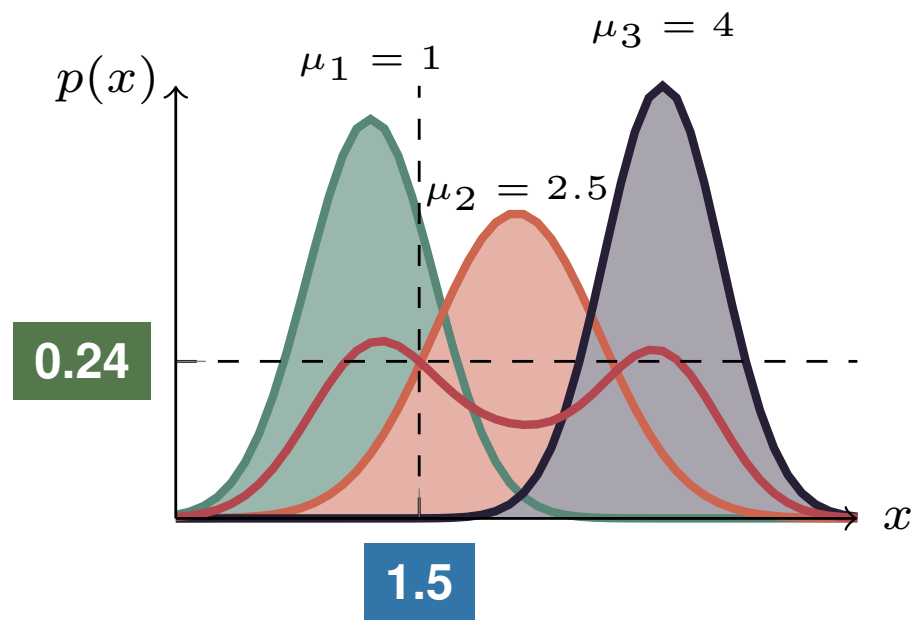
Logical events

...and extract meaningful queries from it?

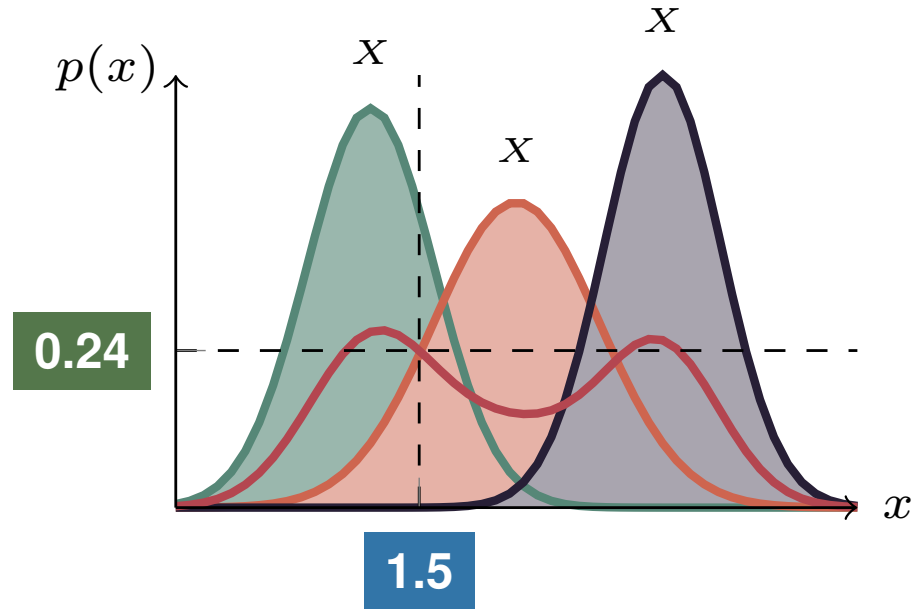
Probabilistic Circuits – Inputs



Probabilistic Circuits – Sums

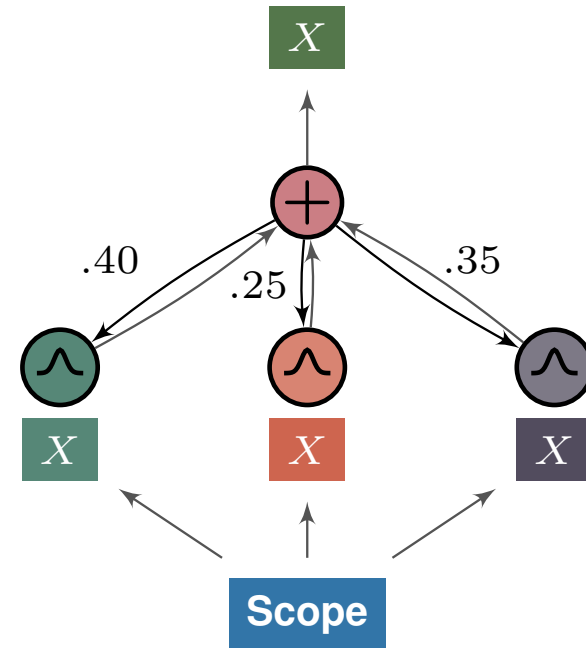
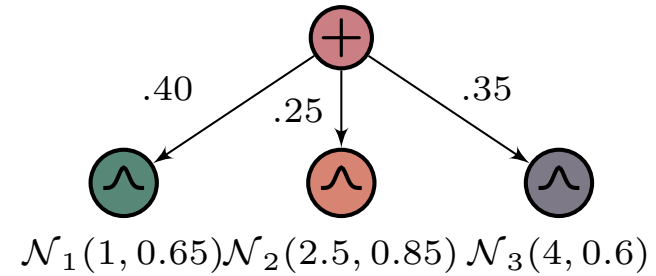


Probabilistic Circuits – Smoothness

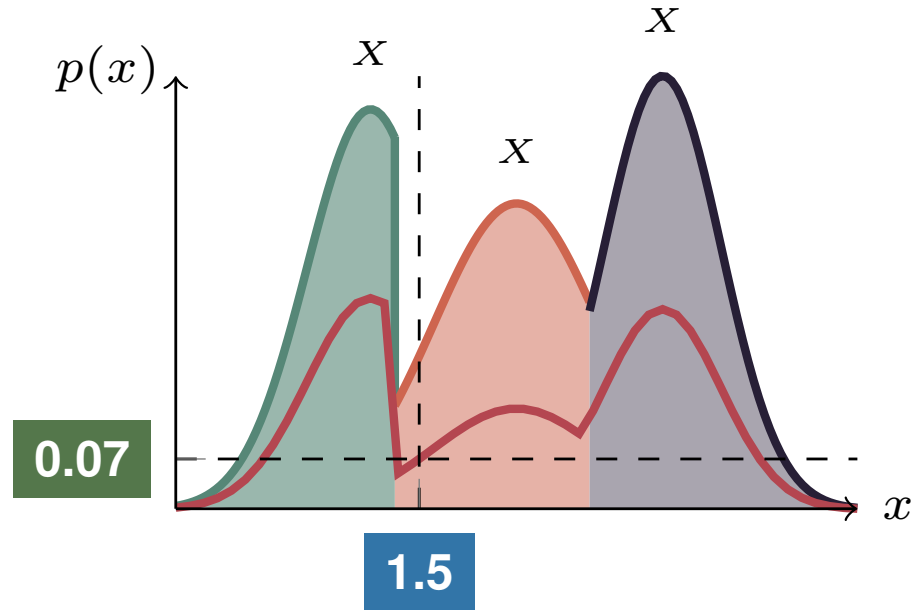


Definition 1 (Smoothness).

Every sum node child mentions the same variables.

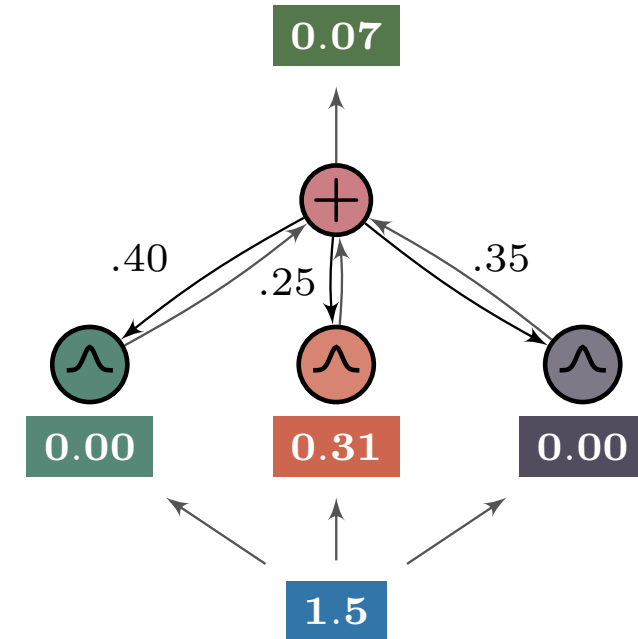
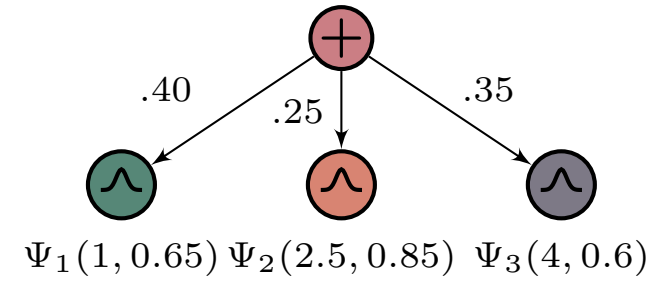


Probabilistic Circuits – Determinism

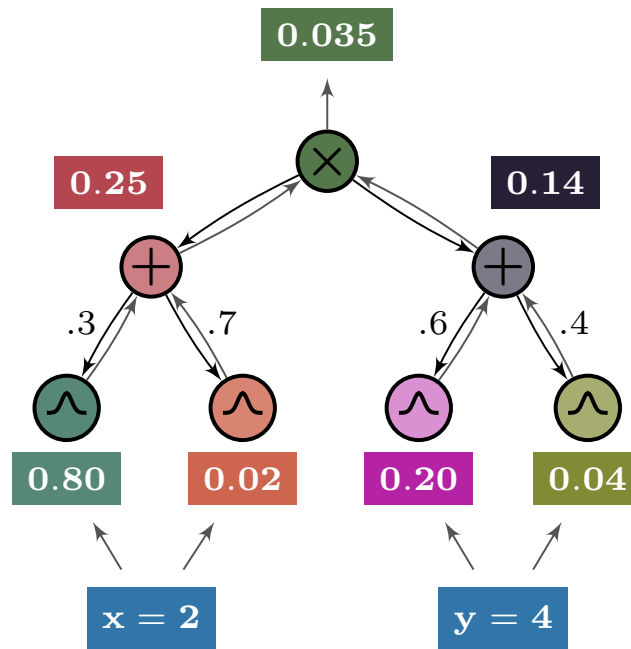
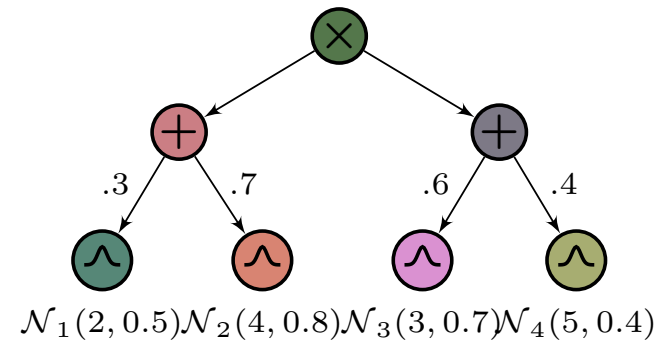
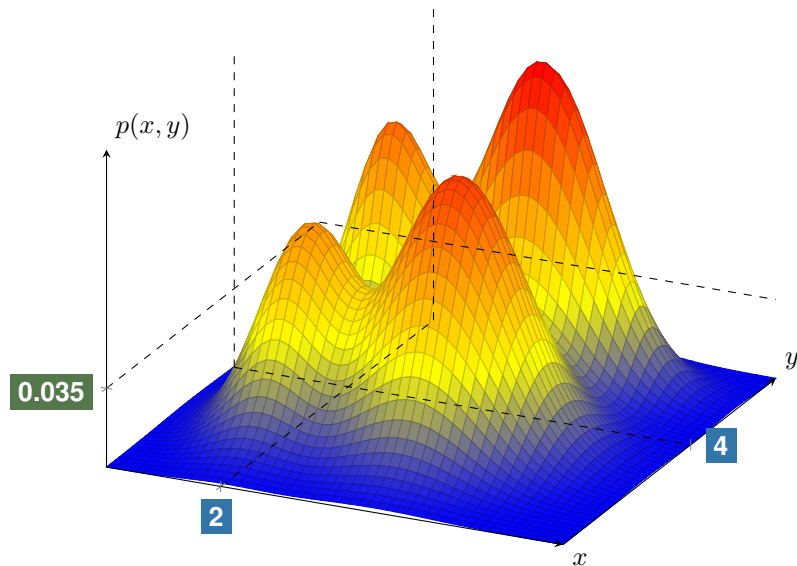
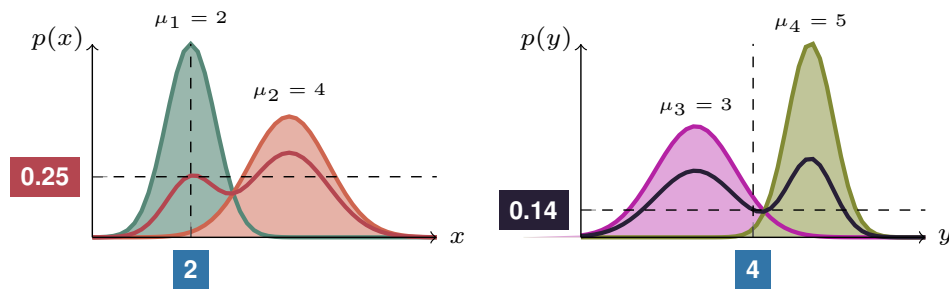


Definition 2 (Determinism).

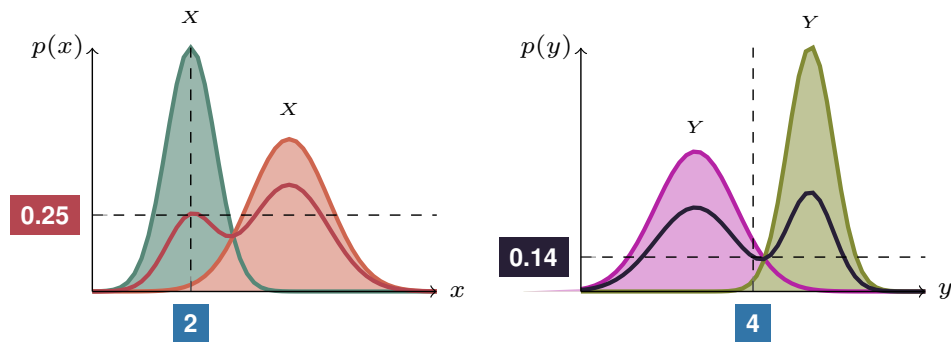
At most one sum node child has a positive value.



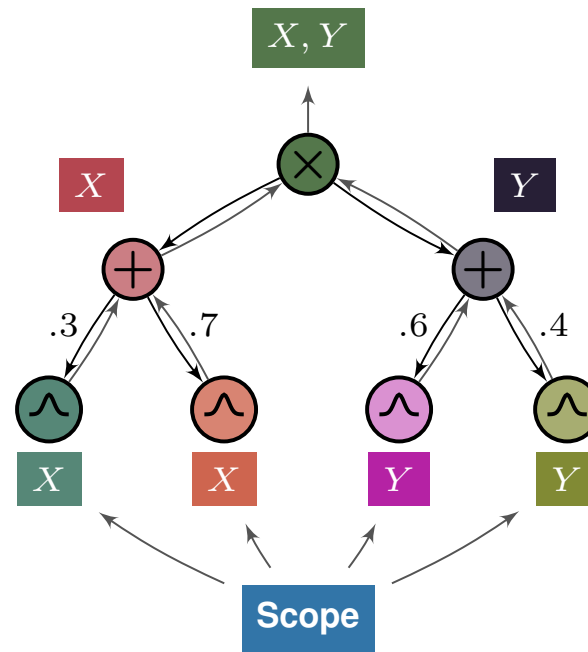
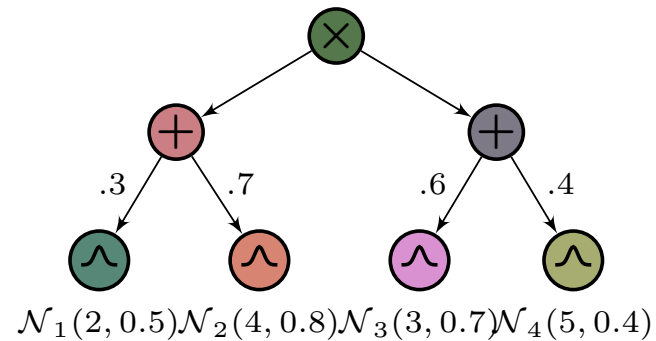
Probabilistic Circuits – Products



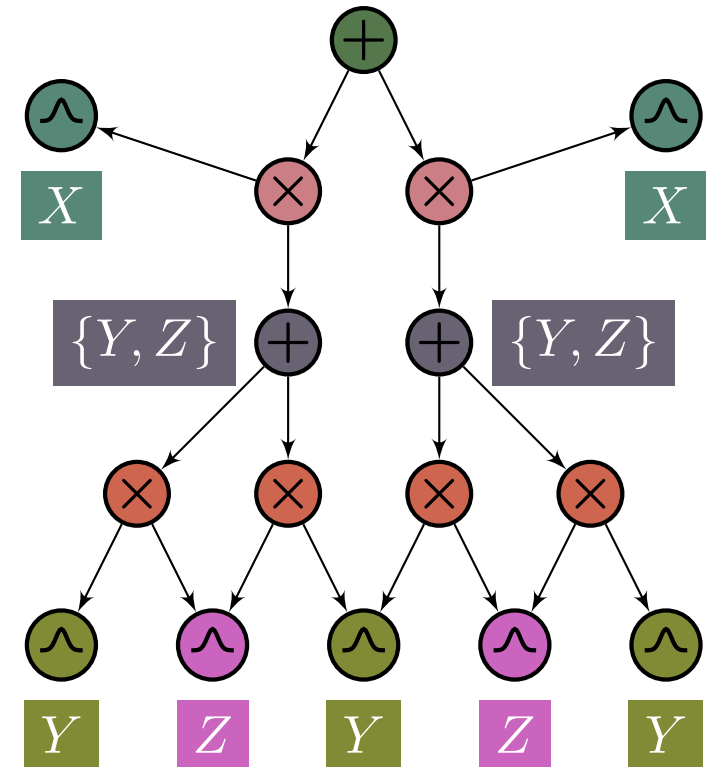
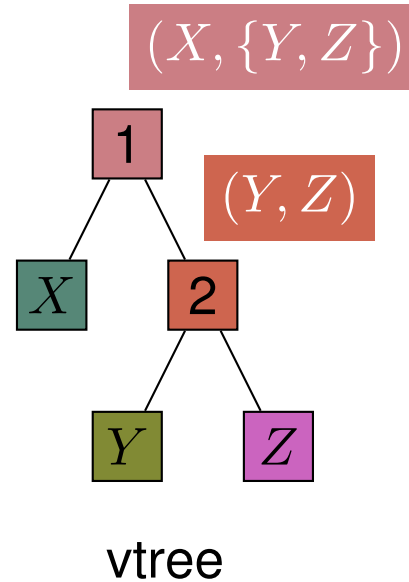
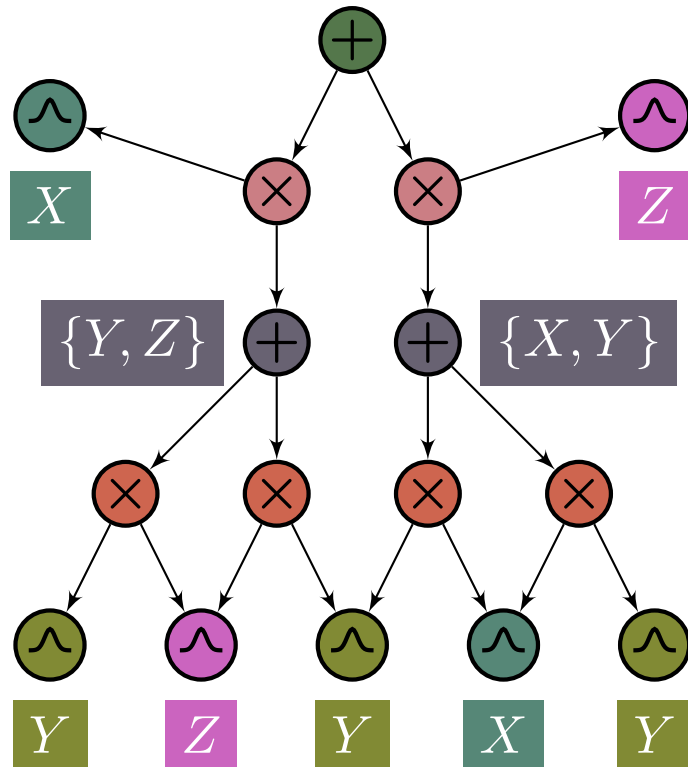
Probabilistic Circuits – Decomposability



Definition 3 (Decomposability).
Every product node child mentions different variables.



Probabilistic Circuits – Structured Decomposability



Definition 4 (Structured decomposability). *Every product node follows a vtree decomposition.*

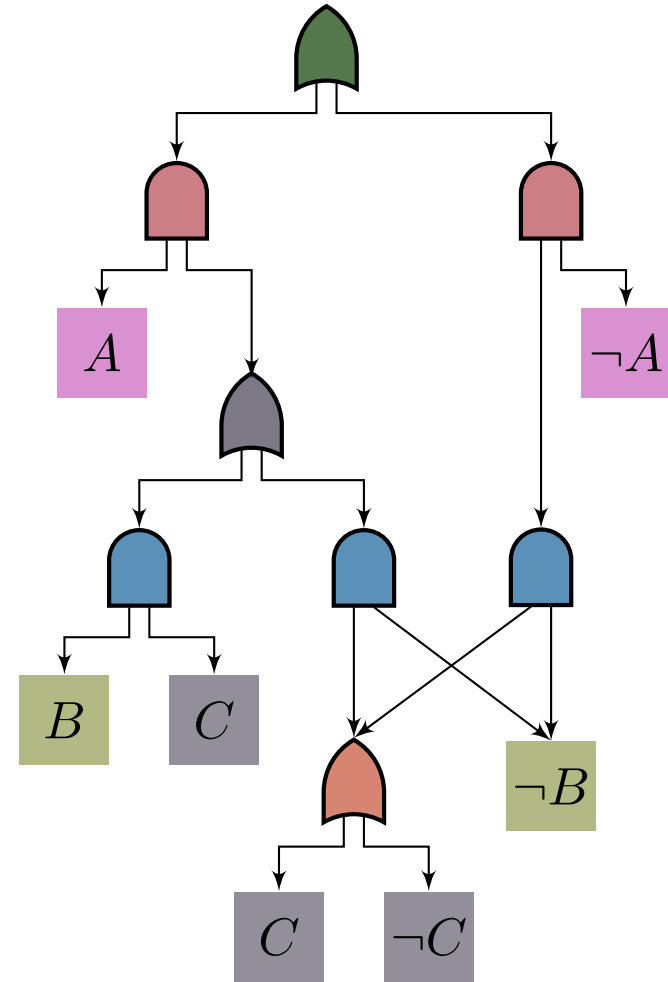
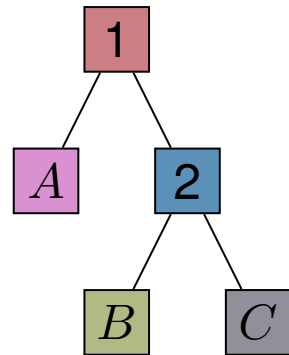
Probabilistic Circuits – Tractability

| Query | +Sm? | +Dec? | +Det? | +Str Dec? |
|----------------------|------|-------|-------|-----------|
| Evidence | ✓ | ✓ | ✓ | ✓ |
| Marginals | ✗ | ✓ | ✓ | ✓ |
| Conditionals | ✗ | ✓ | ✓ | ✓ |
| MPE | ✗ | ✗ | ✓ | ✓ |
| Shannon Entropy | ✗ | ✗ | ✓ | ✓ |
| Rényi Entropy | ✗ | ✗ | ✓ | ✓ |
| Cross Entropy | ✗ | ✗ | ✗ | ✓ |
| Kullback-Leibler Div | ✗ | ✗ | ✗ | ✓ |
| Rényi's Alpha Div | ✗ | ✗ | ✗ | ✓ |
| Cauchy-Schwarz Div | ✗ | ✗ | ✗ | ✓ |
| Logical Events | ✗ | ✗ | ✗ | ✓ |
| Mutual Information | ✗ | ✗ | ✗ | ✓ |

Probabilistic Circuits – Logic Circuits

| A | B | C | $\phi(\mathbf{x})$ |
|-----|-----|-----|--------------------|
| 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 1 |

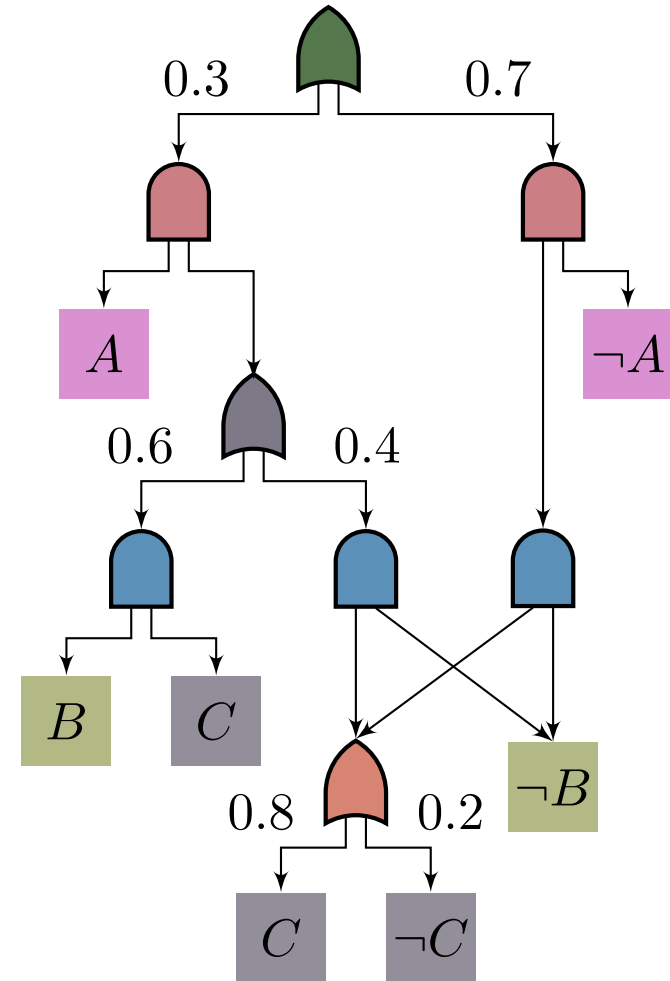
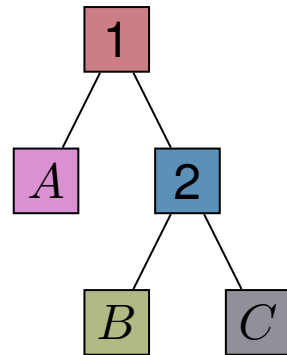
$$\phi(A, B, C) = (A \vee B) \wedge (\neg B \vee C)$$



Probabilistic Circuits – Support

| A | B | C | $\phi(\mathbf{x})$ | $p(\mathbf{x})$ |
|-----|-----|-----|--------------------|-----------------|
| 0 | 0 | 0 | 1 | 0.140 |
| 1 | 0 | 0 | 1 | 0.024 |
| 0 | 1 | 0 | 0 | 0.000 |
| 1 | 1 | 0 | 0 | 0.000 |
| 0 | 0 | 1 | 1 | 0.560 |
| 1 | 0 | 1 | 1 | 0.096 |
| 0 | 1 | 1 | 0 | 0.000 |
| 1 | 1 | 1 | 1 | 0.180 |

$$\phi(A, B, C) = (A \vee B) \wedge (\neg B \vee C)$$



Learning Probabilistic Circuits

Divide-and-Conquer Approaches (DIV)

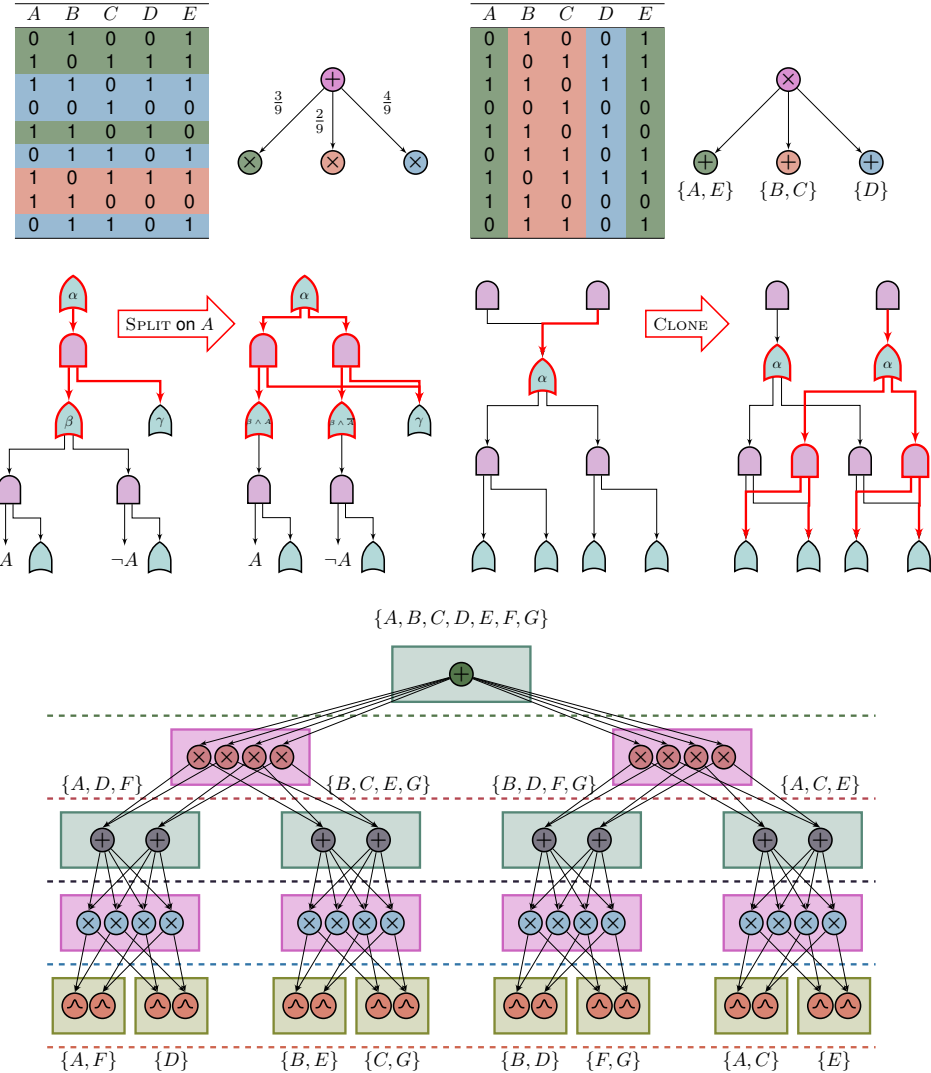
- Usually recursive;
- Splits data by similarity and stat dep;
- Stat dep usually costly;
- Usually tree-shaped.

Incremental Approaches (INCR)

- Requires an initial circuit;
- Grows from local transformations;
- Local transformations preserve properties;
- Searching for candidates to transform is costly.

Random Approaches (RAND)

- Fast;
- Randomly generates circuits;
- Data blind and data guided approaches exist;
- Usually relies on many hyperparams;
- Worse performance.



Learning Probabilistic Circuits – Where are we right now?

| Name | Class | Time Complexity | # hyperparams | Accepts logic? | Sm? | Dec? | Det? | Str Dec? | $\{0, 1\}$? | \mathbb{N} ? | \mathbb{R} ? | Reference |
|------------|-------|--|---------------|----------------|-----|------|------|----------|--------------|----------------|----------------|---------------------------|
| LEARNSPN | DIV | $\begin{cases} \mathcal{O}(nkmc) & , \text{ if sum} \\ \mathcal{O}(nm^3) & , \text{ if product} \end{cases}$ | ≥ 2 | ✗ | ✓ | ✓ | ✗ | ✗ | ✓ | ✓ | ✓ | Gens and Domingos [2013] |
| ID-SPN | DIV | $\begin{cases} \mathcal{O}(nkmc) & , \text{ if sum} \\ \mathcal{O}(nm^3) & , \text{ if product} \\ \mathcal{O}(ic(rn + m)) & , \text{ if input} \end{cases}$ | $\geq 2 + 3$ | ✗ | ✓ | ✓ | ✗ | ✗ | ✓ | ✓ | ✗ | Rooshenas and Lowd [2014] |
| PROMETHEUS | DIV | $\begin{cases} \mathcal{O}(nkmc) & , \text{ if sum} \\ \mathcal{O}(m(\log m)^2) & , \text{ if product} \end{cases}$ | ≥ 1 | ✗ | ✓ | ✓ | ✗ | ✗ | ✓ | ✓ | ✓ | Jaini et al. [2018a] |
| LEARNPSSD | INCR | $\begin{cases} \mathcal{O}(m^2) & , \text{ top-down vtree} \\ \mathcal{O}(m^4) & , \text{ bottom-up vtree} \\ \mathcal{O}(i \mathcal{C} ^2) & , \text{ circuit structure} \end{cases}$ | 1 | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✗ | ✗ | Liang et al. [2017] |
| STRUDEL | INCR | $\begin{cases} \mathcal{O}(m^2n) & , \text{ CLT + vtree} \\ \mathcal{O}(i(\mathcal{C} n + m^2)) & , \text{ circuit structure} \end{cases}$ | 1 | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✗ | ✗ | Dang et al. [2020] |
| RAT-SPN | RAND | $\mathcal{O}(rd(s + l))$ | 4 | ✗ | ✓ | ✓ | ✗ | ✗ | ✓ | ✓ | ✓ | Peharz et al. [2020] |
| XPC | RAND | $\mathcal{O}(i(t + kn) + ikm^2n)$ | 3 | ✗ | ✓ | ✓ | ✓ | ✓ | ✓ | ✗ | ✗ | Mauro et al. [2021] |
| SAMPLEPSDD | RAND | $\begin{cases} \mathcal{O}(m) & , \text{ random vtree} \\ \mathcal{O}(kc \log c + \log_2^2 k) & , \text{ per call} \end{cases}$ | 1 | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✗ | ✗ | Geh and Mauá [2021] |
| LEARNRP | RAND | $\begin{cases} \mathcal{O}(m^2) & , \text{ top-down vtree} \\ \mathcal{O}(m^4) & , \text{ bottom-up vtree} \\ \mathcal{O}(knm) & , \text{ per call} \end{cases}$ | 0 | ✗ | ✓ | ✓ | ✗ | ✓ | ✓ | ✓ | ✓ | To appear |

A Logical Perspective

Motivation



Example:

$$n = 3, k = 3$$

| X_{11} | X_{12} | X_{13} | X_{21} | \dots | X_{33} | $p(\mathbf{x}) > 0$ |
|----------|----------|----------|----------|----------|----------|---------------------|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |
| 0 | 1 | 1 | 1 | 1 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 |

Assignments: $2^{3 \cdot 3} = 512$

Positive assignments: $3! = 6$

If we assume

n sushi types,

k sized rankings with $k \leq n$,

X_{ij} binary variables; i is sushi type, j is position in ranking;

then the total number of possible assignments of the $n \cdot k$ variables is $2^{nk} \dots$

...but many of these are zero probability assignments!

If we can embed total ranking constraints...

...we go down to $k!$ total assignments!

Takeaway: models which exploit domain knowledge are much more efficient!

Motivation

Existing approaches:

LEARNPSDD (Liang et al. [2017]):

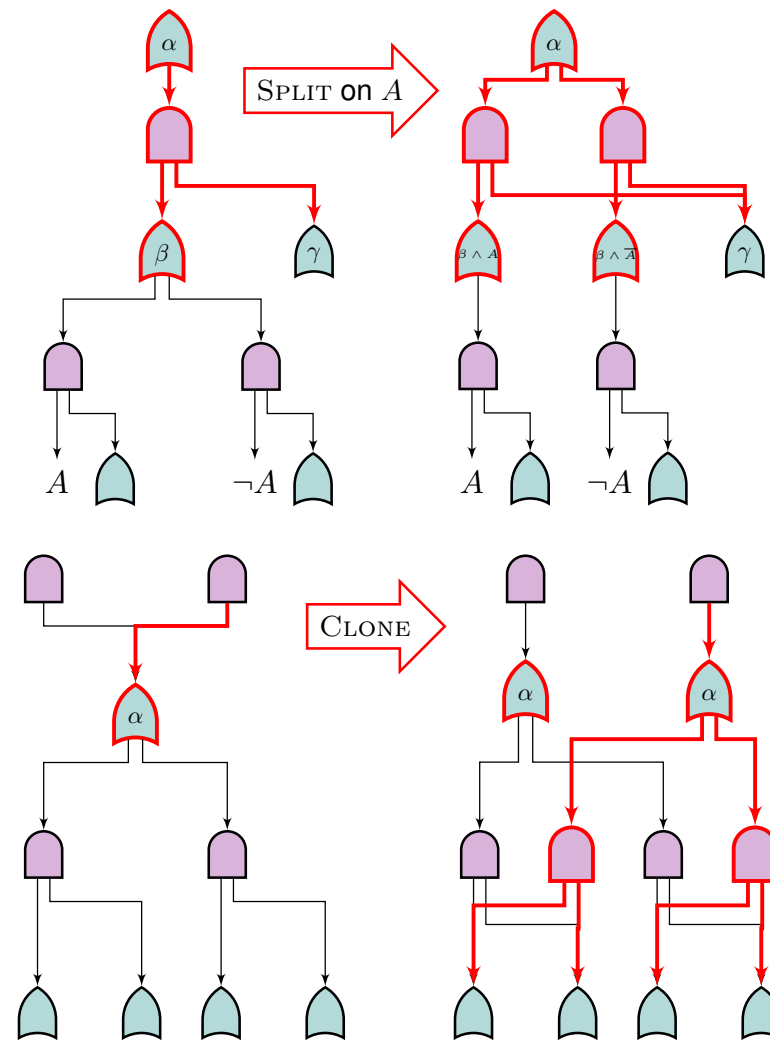
- ✗ Requires initial logic circuit encoding the support...
- ✗ Scales poorly to complex formulae and/or high dimension...
- ✗ Costly whole circuit evaluation at every iteration...
- ✓ Very good performance!

STRUDEL (Dang et al. [2020]):

- ✓ Constructs an initial structure (from a CLT)!
- ✗ But does not encode constraints...
- ✓ Scales to high dimension!
- ✗ As long as the circuit doesn't get too big...

SAMPLEPSDD (Geh and Mauá [2021]):

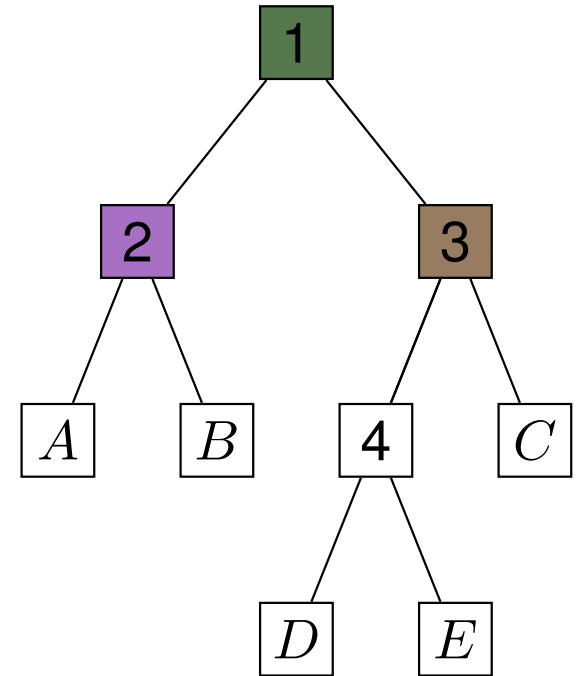
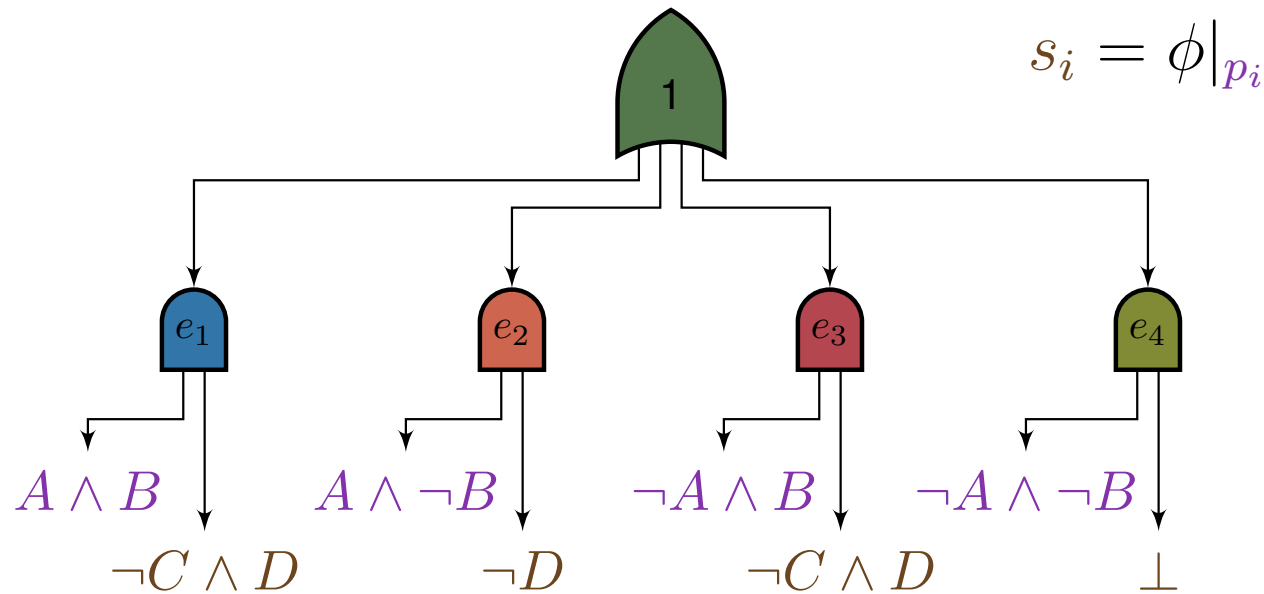
- ✓ Scales to high dimension and complex formulae!
- ✓ Constructs a structure consistent with constraints!
- ✗ But does so by relaxing the formula...
- ✗ Performance varies on set bounds and vtree structure...



SAMPLEPSDD

Common assumption: p_i are conjunctions of literals.

$$\phi(A, B, C, D) = (A \wedge \neg B \wedge \neg D) \vee (B \wedge \neg C \wedge D)$$

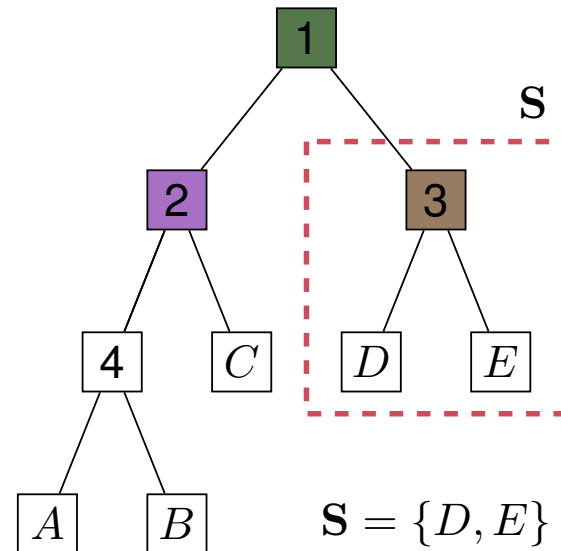
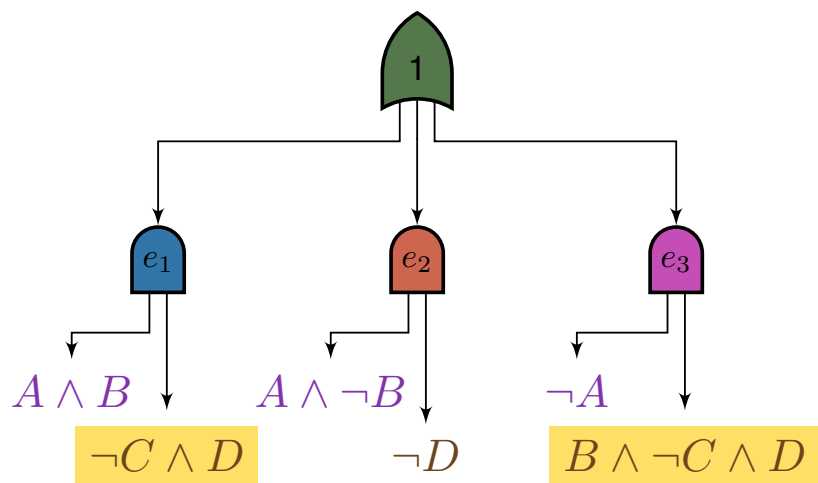


Problem: size of circuit is **exponential** in the size of p_i 's scope.

SAMPLEPSDD

Solution: randomly sample a bounded number (k) of p_i

$$\phi(A, B, C, D) = (A \wedge \neg B \wedge \neg D) \vee (B \wedge \neg C \wedge D)$$



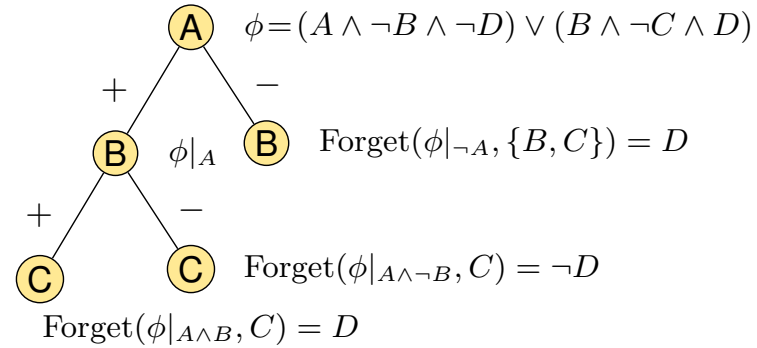
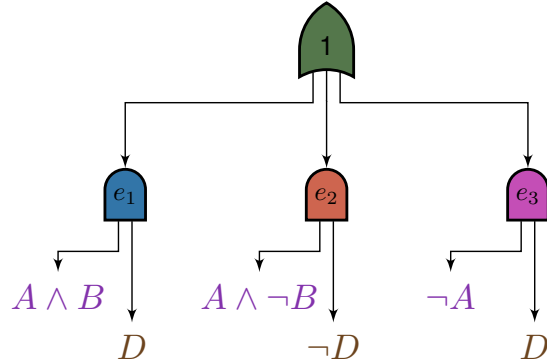
But: this violates structured decomposability:

$\neg C \wedge D$ contains C , and $C \notin S$

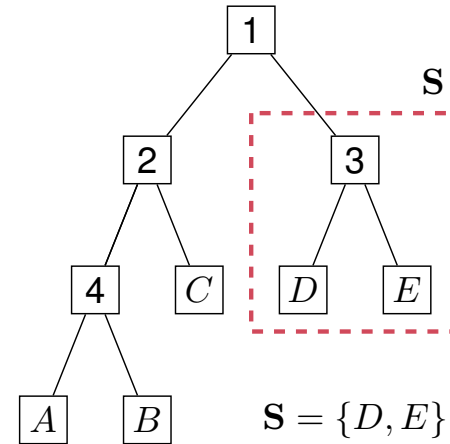
$\neg B \wedge \neg C \wedge D$ contains B and C , and $B, C \notin S$

SAMPLEPSDD

New solution: relax logical constraints ϕ

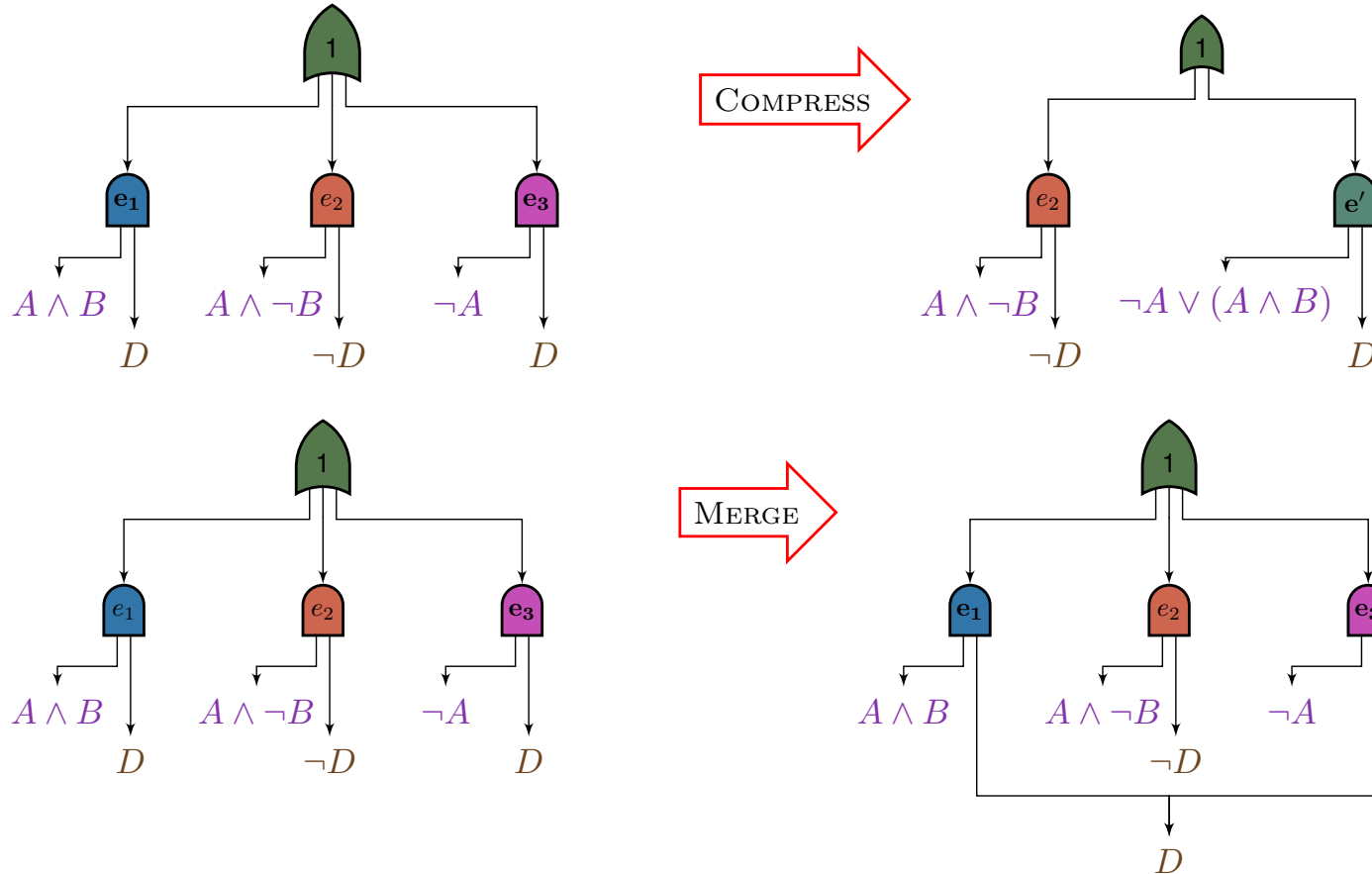


Now all s_i respect S



SAMPLEPSDD

Apply **local transformations** for variety and size reduction



Experiments

Evaluation: we sample 30 PSDDs and use 5 ensemble strategies:

- Likelihood weighting (LLW),
- Uniform weights,
- ◆ Expectation-Maximization (EM),
- ▲ Stacking,
- ▼ Bayesian Model Combination (BMC);

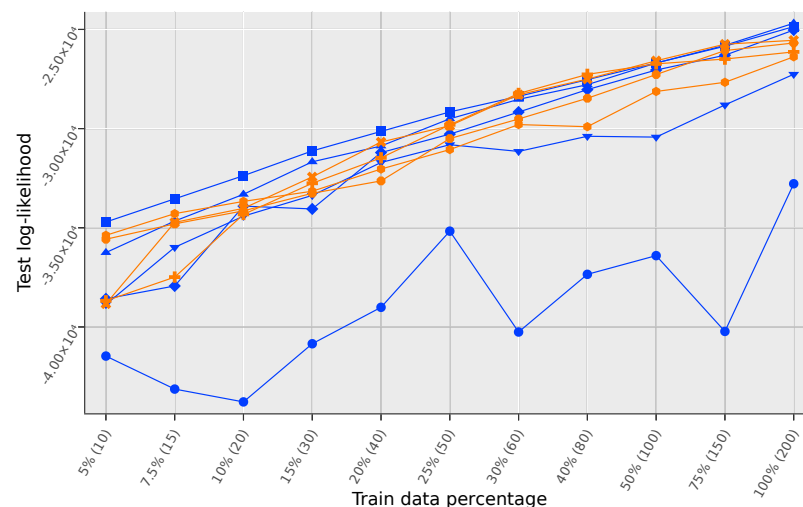
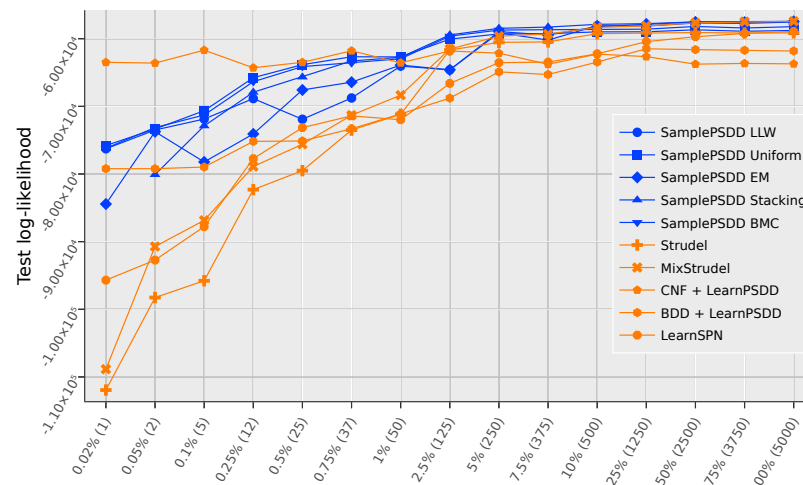
comparing against **STRUDEL**, **LEARNPSDD** and **LEARNSPN**.

Datasets: we evaluate with 5 data + knowledge as logic constraints:

| | Dataset | #vars | #train | ϕ 's size |
|---|---------------|-------|--------|----------------|
| ⇒ | LED | 14 | 5000 | 23 |
| ⇒ | LED + IMAGES | 157 | 700 | 39899 |
| | SUSHI RANKING | 100 | 3500 | 17413 |
| | SUSHI TOP 5 | 10 | 3500 | 37 |
| | DOTA 2 GAMES | 227 | 92650 | 1308 |

Our approach fares **better** with **fewer** data, yet
remains **competitive** under **lots of data**.

Mattei et al. [2020], Kamishima [2003], Shen et al. [2017],
Choi et al. [2015], Gens and Domingos [2013], Dang et al. [2020]



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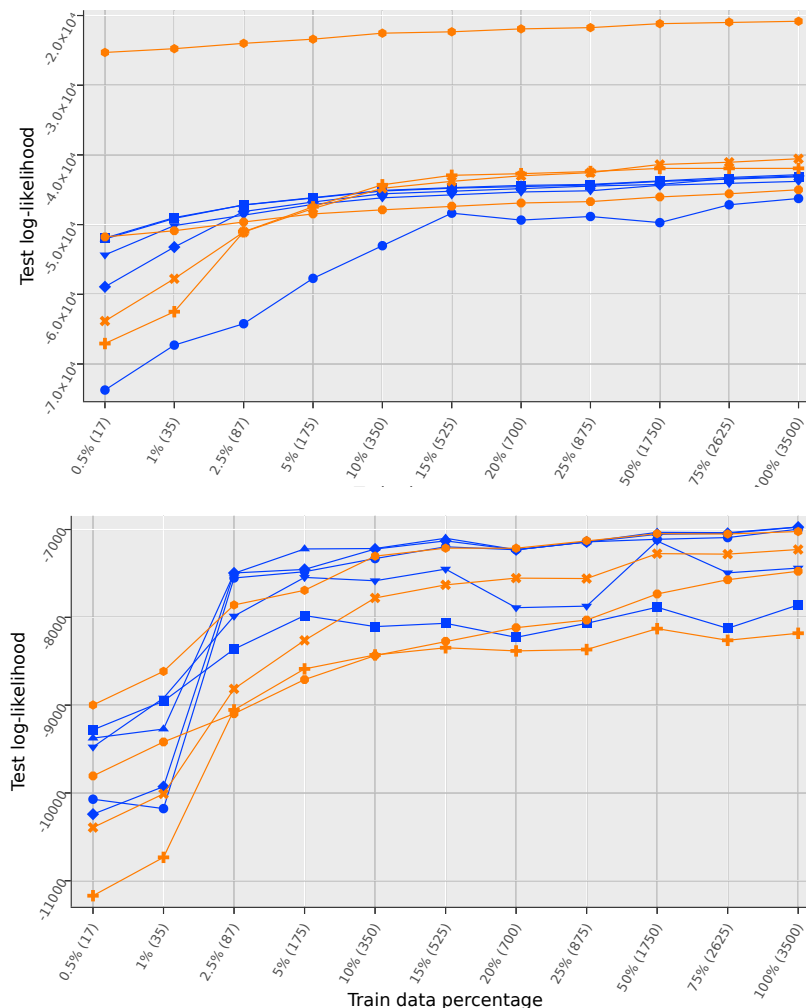
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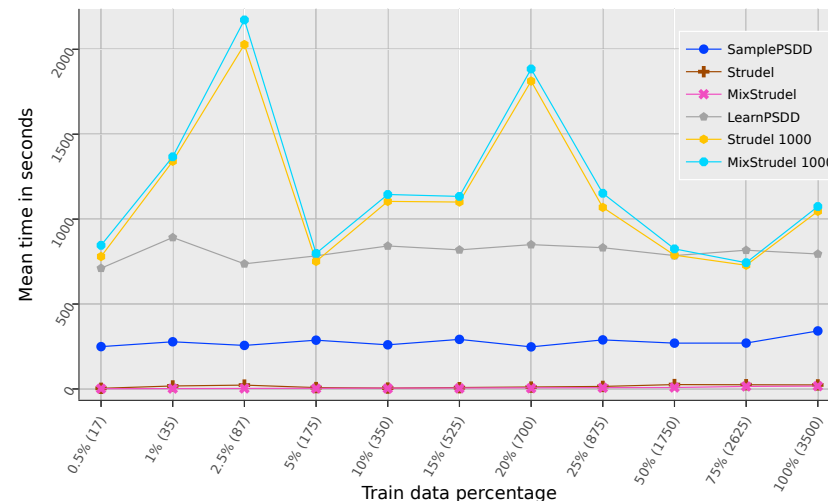
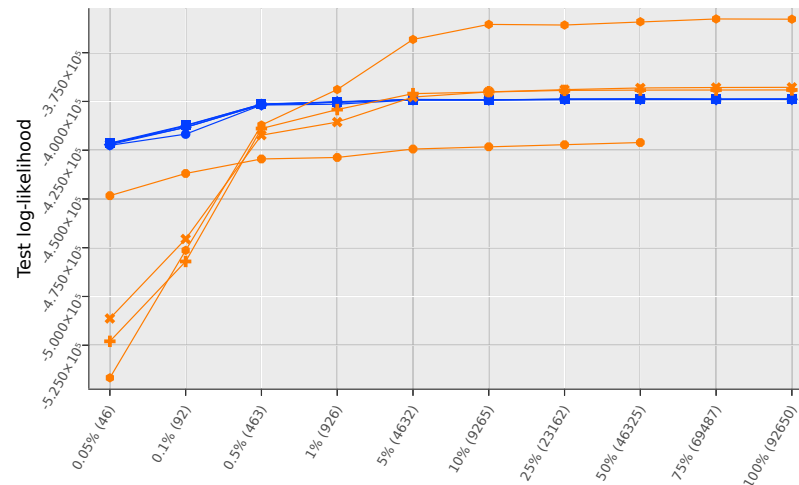
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|----------------|-------|--------|----------------|
| LED | 14 | 5000 | 23 |
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A Data Perspective

Motivation

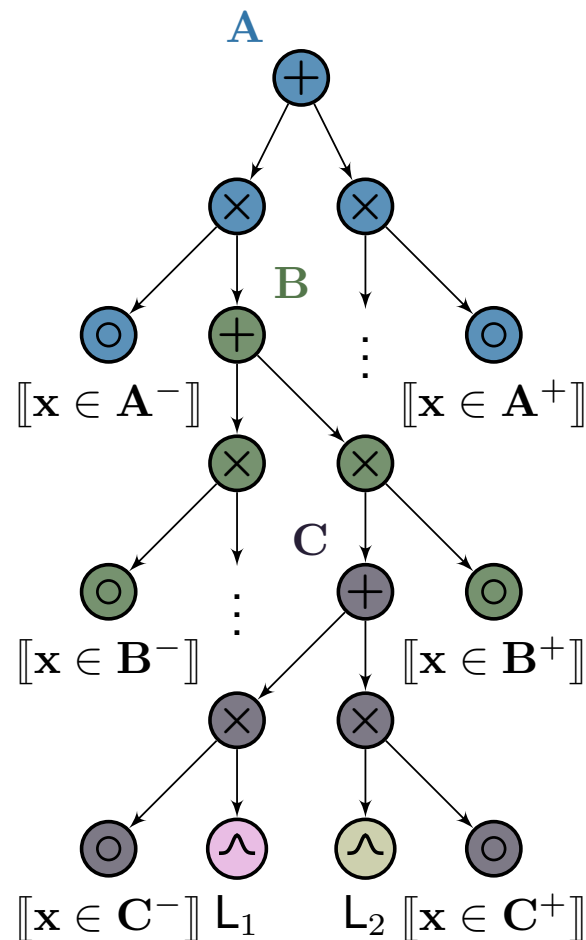
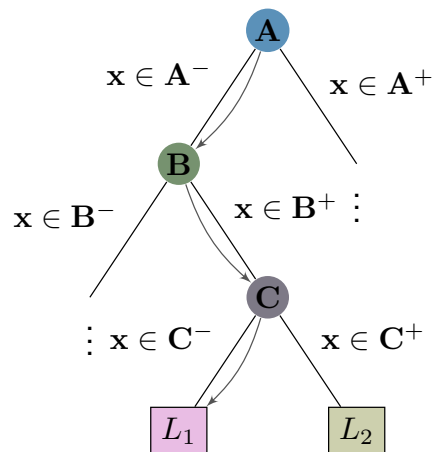
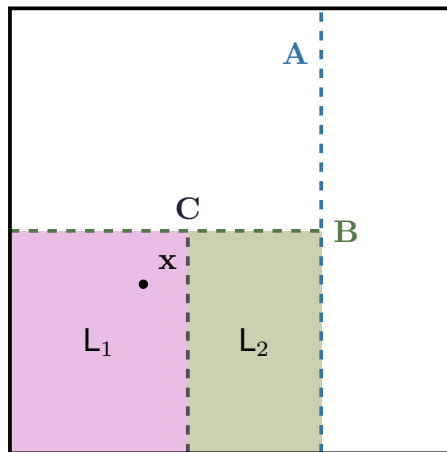
Density Estimation Trees...

- ✓ ...are fast;
- ✓ ...are interpretable;
- ✓ ...are (somewhat) explainable;
- ✓ ...have extensive literature coverage;
- ✗ ...are not so expressive;
- ✗ ...only accept marginalization queries;
- ✗ ...are not so accurate;

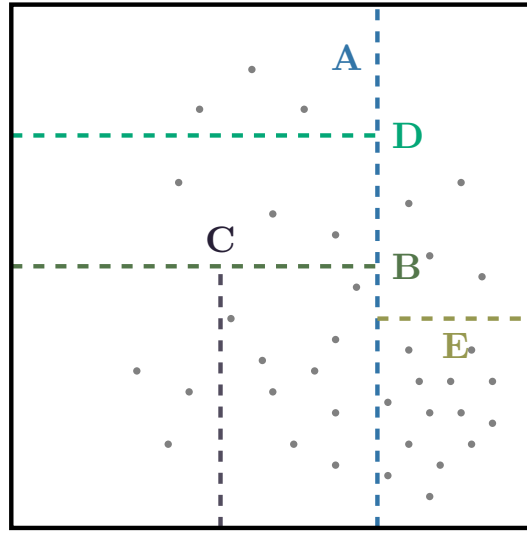
...but are subsumed by circuits!

Learn DETs \subseteq Learn PCs?

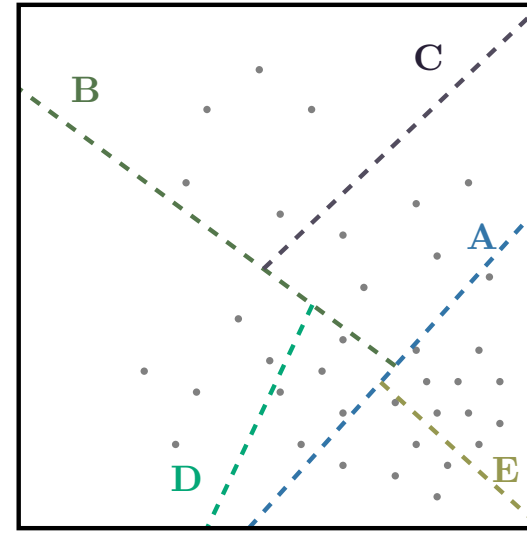
Can we take advantage of known learning procedures in DETs and transplant them to more general circuits?



Random Projections



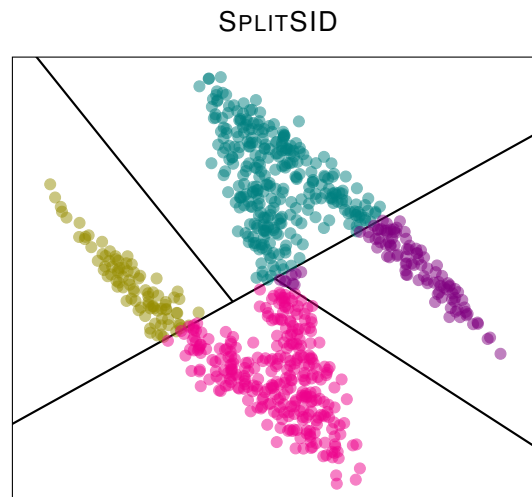
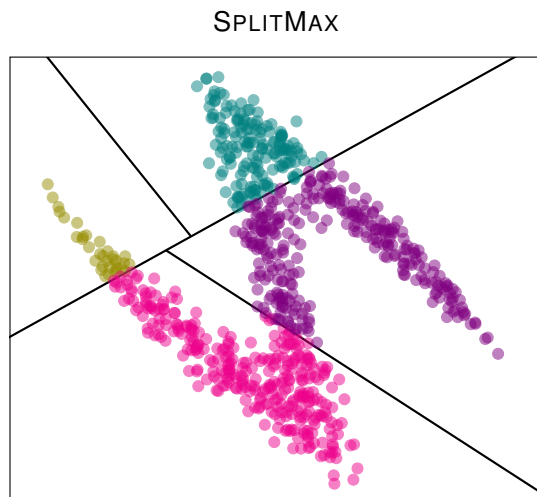
Axis-aligned projections



Random projections

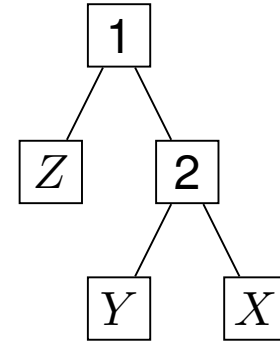
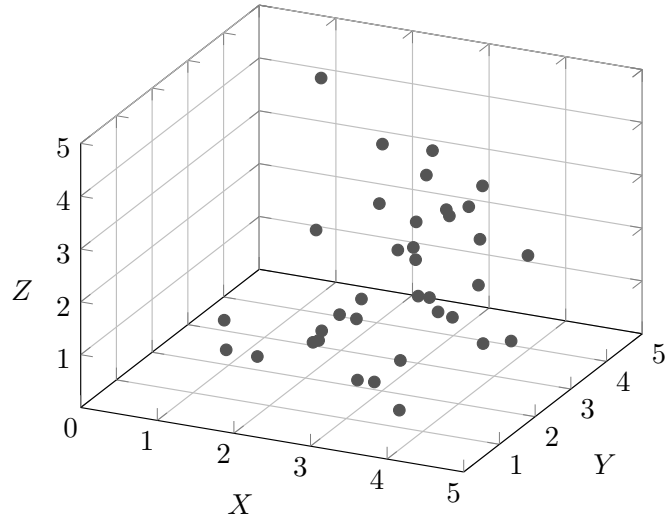
If the data has *intrinsic dimension* d , then with constant probability the part of the data at level d or higher of the tree has average diameter less than half of the data.

Random Projections

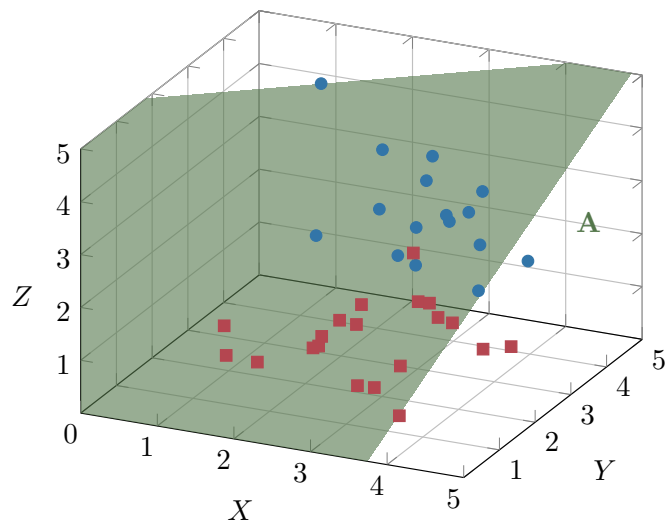


If the data has *intrinsic dimension* d , then with constant probability the part of the data at level d or higher of the tree has average diameter less than half of the data.

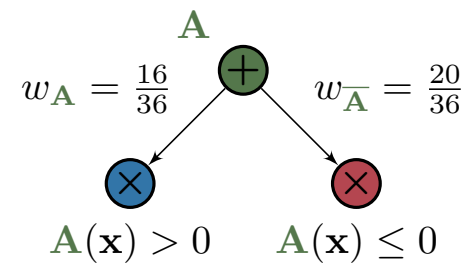
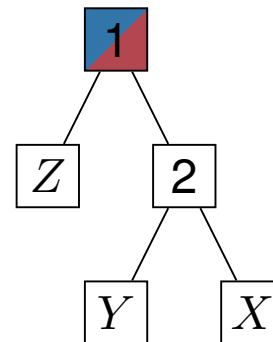
LearnRP



LearnRP

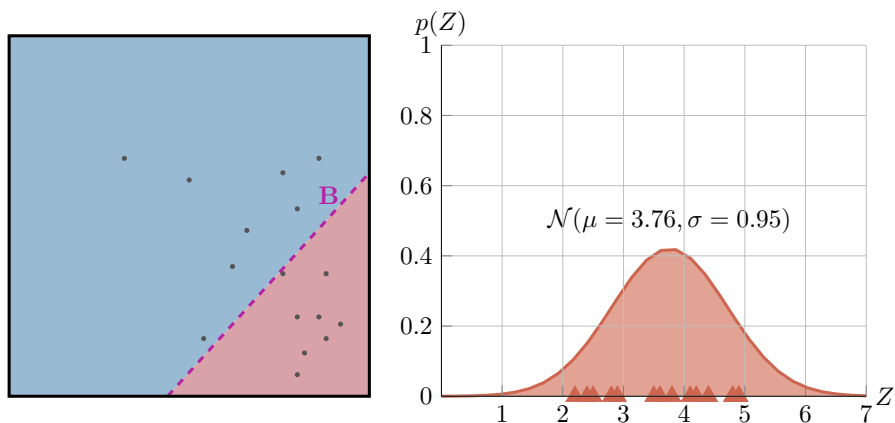
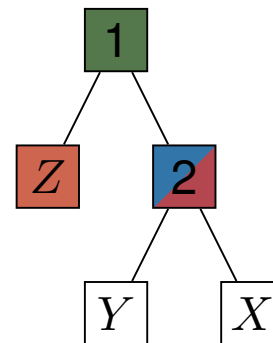
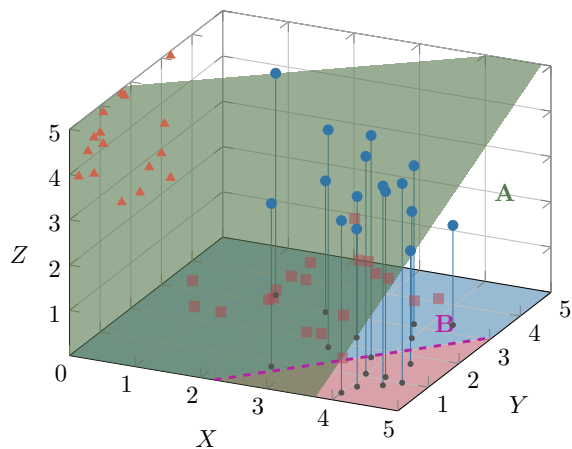


$$\mathbf{A}(x, y, z) = \begin{bmatrix} x & y & z \end{bmatrix} \cdot \begin{bmatrix} -0.31 \\ -0.40 \\ 0.85 \end{bmatrix} + 1$$

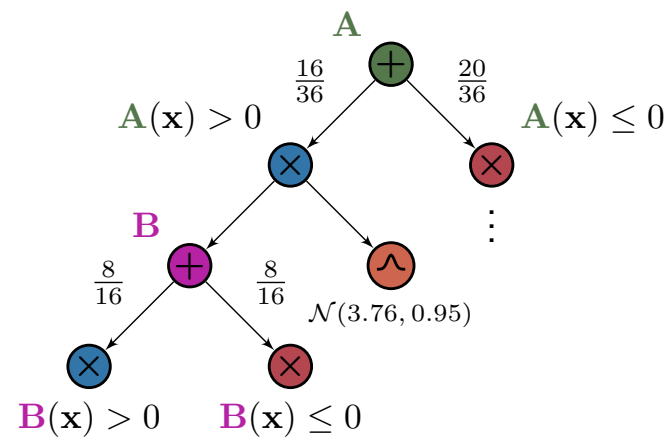


$w_{\mathbf{A}}$: probability of $\mathbf{A}(\mathbf{x}) > 0$

LearnRP



$$\mathbf{B}(x, y) = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 1.10 \\ -1.00 \end{bmatrix} - 2.43$$



Parameter Optimization

Expectation-Maximization (EM)

- Minibatch EM

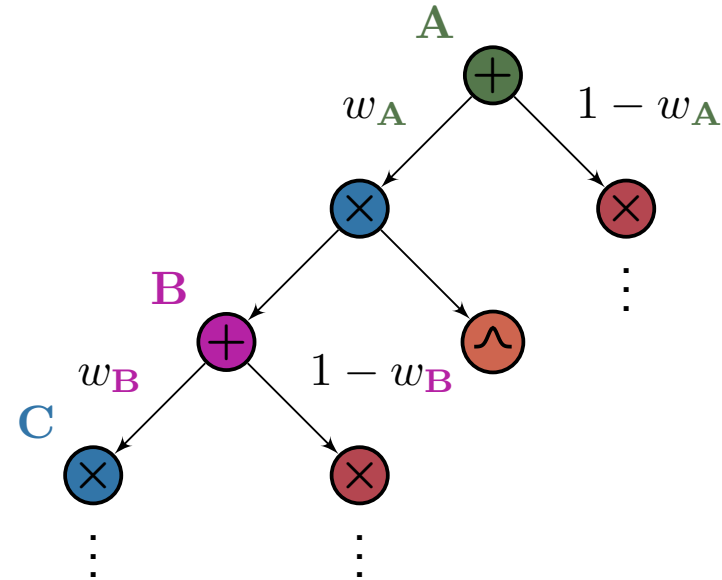
$$w_{\mathbf{B}} \propto \sum_{\mathbf{x} \in \mathbf{M}} \frac{1}{p_{\mathbf{A}}(\mathbf{x})} \cdot \frac{\partial p_{\mathbf{A}}(\mathbf{x})}{\partial p_{\mathbf{B}}(\mathbf{x})} \cdot p_{\mathbf{C}} \cdot w_{\mathbf{B}}$$

- Full EM

$$w_{\mathbf{B}} \propto \sum_{\mathbf{x} \in \mathbf{D}} \frac{1}{p_{\mathbf{A}}(\mathbf{x})} \cdot \frac{\partial p_{\mathbf{A}}(\mathbf{x})}{\partial p_{\mathbf{B}}(\mathbf{x})} \cdot p_{\mathbf{C}} \cdot w_{\mathbf{B}}$$

LEARNRP-100: LEARNRP + 100 itrs of minibatch

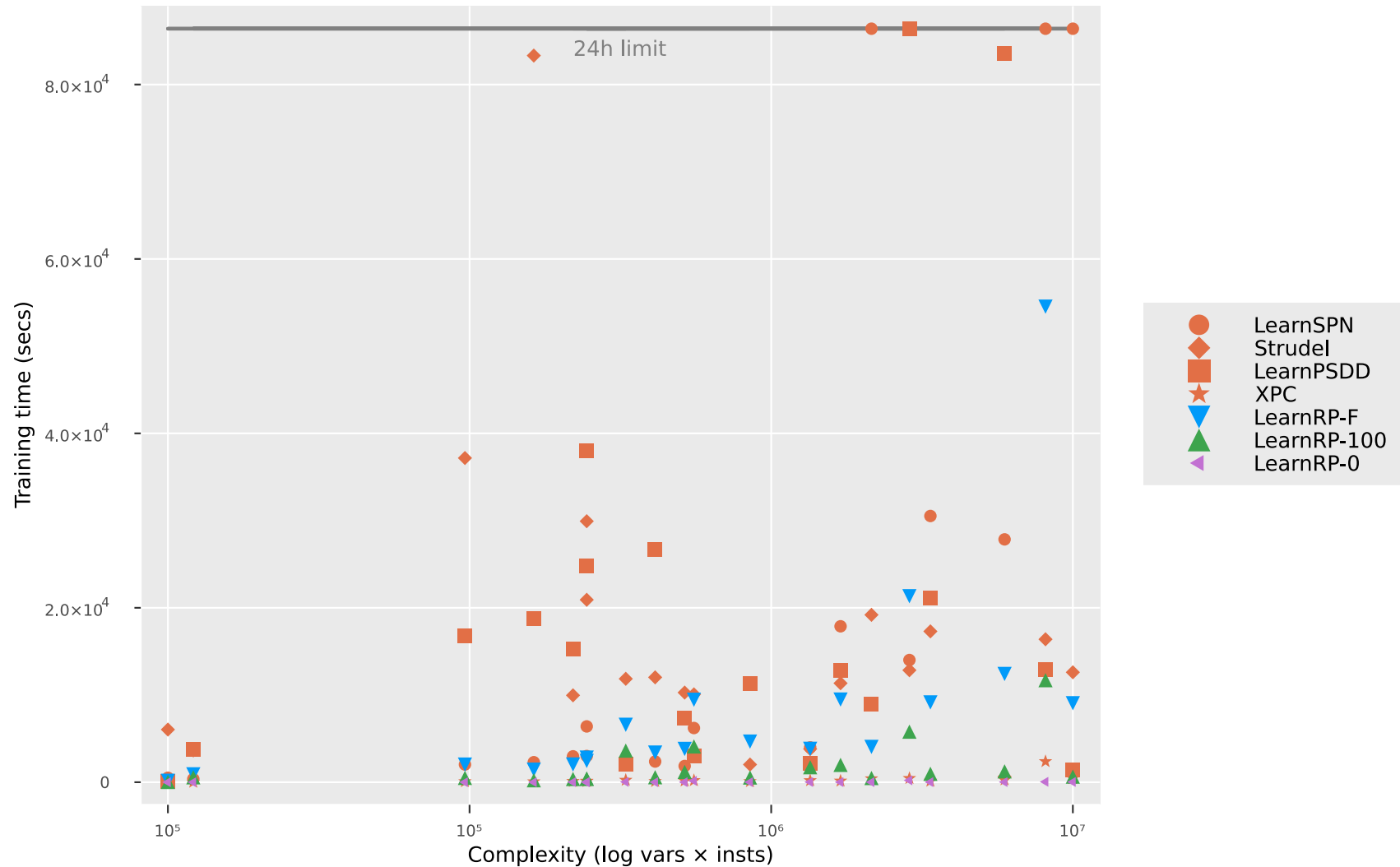
LEARNRP-F: LEARNRP-100 + 30 itrs of full



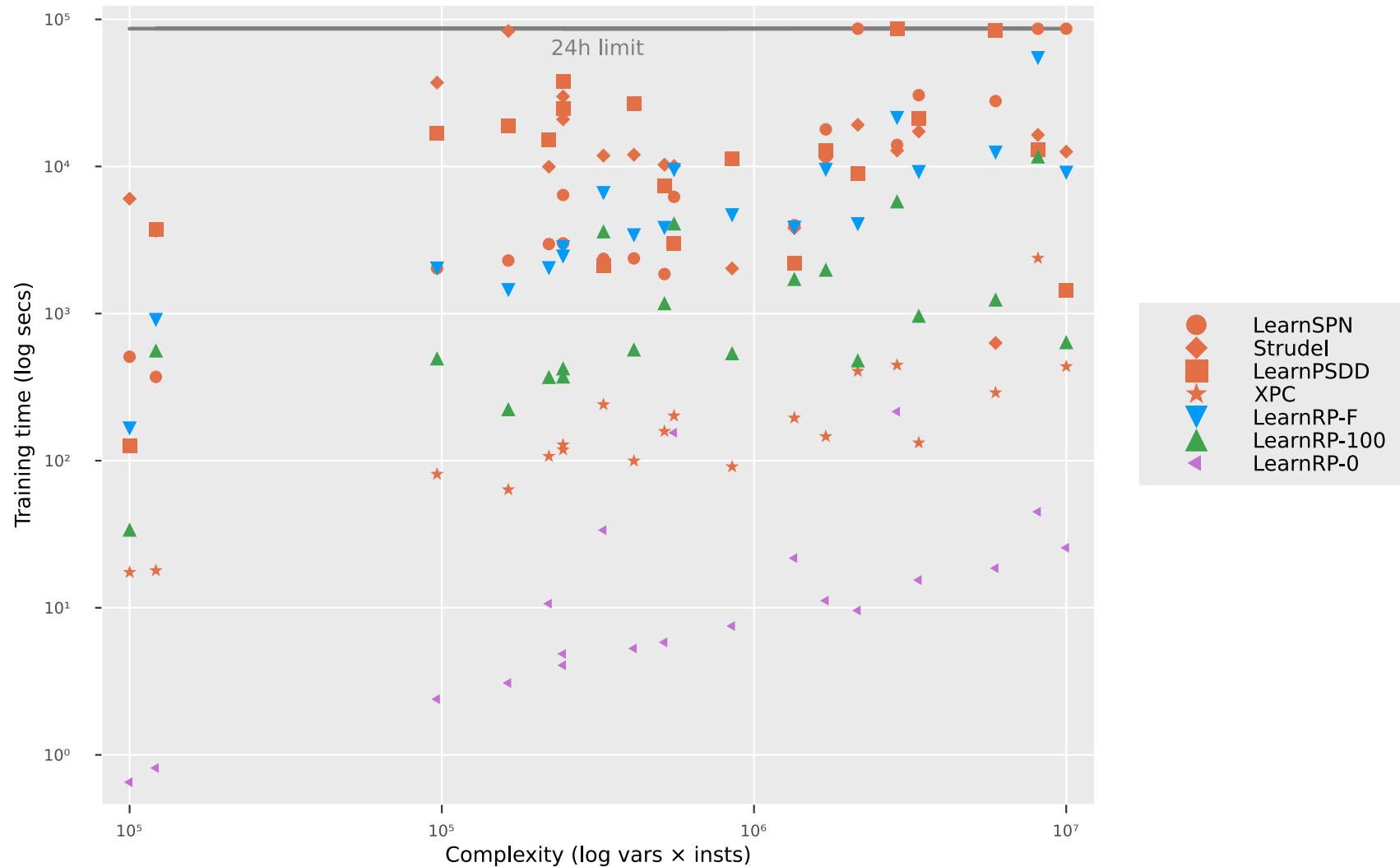
Experiments

| Dataset | LEARNSPN | STRUDEL | LEARNPSTD | XPC | PROMETHEUS | LEARNRP-F | LEARNRP-100 |
|--------------------|----------------|---------------|---------------|----------------|--------------------|--------------------|----------------|
| ACCIDENTS | -30.03 | <u>-28.73</u> | -30.16 | -31.02 | -27.91 | <u>-28.65</u> | -28.87 |
| AD | -19.73 | <u>-16.38</u> | -31.78 | -15.50 | -23.96 | <u>-19.20</u> | -20.32 |
| AUDIO | -40.50 | -41.50 | <u>-39.94</u> | -40.91 | -39.80 | <u>-40.18</u> | -40.23 |
| BBC | <u>-250.68</u> | -254.41 | -253.19 | -248.34 | <u>-248.50</u> | -254.97 | -255.55 |
| NETFLIX | <u>-57.02</u> | -58.69 | -55.71 | -57.58 | <u>-56.47</u> | -57.07 | -57.05 |
| BOOK | -35.88 | -34.99 | -34.97 | -34.75 | <u>-34.40</u> | <u>-33.57</u> | -33.52 |
| 20-NEWSGRP | -155.92 | -154.47 | -155.97 | <u>-153.75</u> | -154.17 | <u>-152.78</u> | -152.76 |
| REUTERS-52 | <u>-85.06</u> | -86.22 | -89.61 | <u>-84.70</u> | -84.59 | -85.73 | -85.47 |
| WEBKB | -158.20 | -155.33 | -161.09 | <u>-153.67</u> | -155.21 | <u>-154.43</u> | -152.60 |
| DNA | -82.52 | -86.22 | -88.01 | -86.61 | -84.45 | <u>-83.03</u> | <u>-83.85</u> |
| JESTER | -75.98 | -55.03 | -51.29 | -53.43 | <u>-52.80</u> | -52.92 | <u>-52.89</u> |
| KDD | -2.18 | <u>-2.13</u> | -2.11 | -2.15 | <u>-2.12</u> | <u>-2.13</u> | -2.14 |
| KOSAREK | -10.98 | -10.68 | -10.52 | -10.77 | <u>-10.59</u> | <u>-10.65</u> | -10.67 |
| MSNBC | <u>-6.11</u> | -6.04 | -6.04 | <u>-6.18</u> | -6.04 | -6.31 | -6.36 |
| MSWEB | -10.25 | -9.71 | -9.89 | -9.93 | <u>-9.86</u> | <u>-9.85</u> | -9.97 |
| NLTCS | -6.11 | -6.06 | -5.99 | <u>-6.05</u> | <u>-6.01</u> | -6.35 | -6.23 |
| PLANTS | <u>-12.97</u> | <u>-12.98</u> | -13.02 | -14.19 | -12.81 | -13.68 | -14.00 |
| PUMSB-STAR | <u>-24.78</u> | <u>-24.12</u> | -26.12 | -26.06 | -22.75 | -25.88 | -26.19 |
| EACHMOVIE | -52.48 | -53.67 | -58.01 | -54.82 | <u>-51.49</u> | <u>-51.37</u> | -51.06 |
| RETAIL | -11.04 | <u>-10.81</u> | -10.72 | -10.94 | -10.87 | <u>-10.85</u> | -10.86 |
| Avg. Rank | 4.80 ± 1.91 | 4.22 ± 1.81 | 4.05 ± 2.56 | 4.60 ± 1.93 | 2.55 ± 1.43 | <u>3.62 ± 1.56</u> | 4.15 ± 2.03 |
| Pos. (mean) | 7th | 5th | <u>3rd</u> | 6th | 1st | <u>2nd</u> | 4th |

Experiments



Experiments



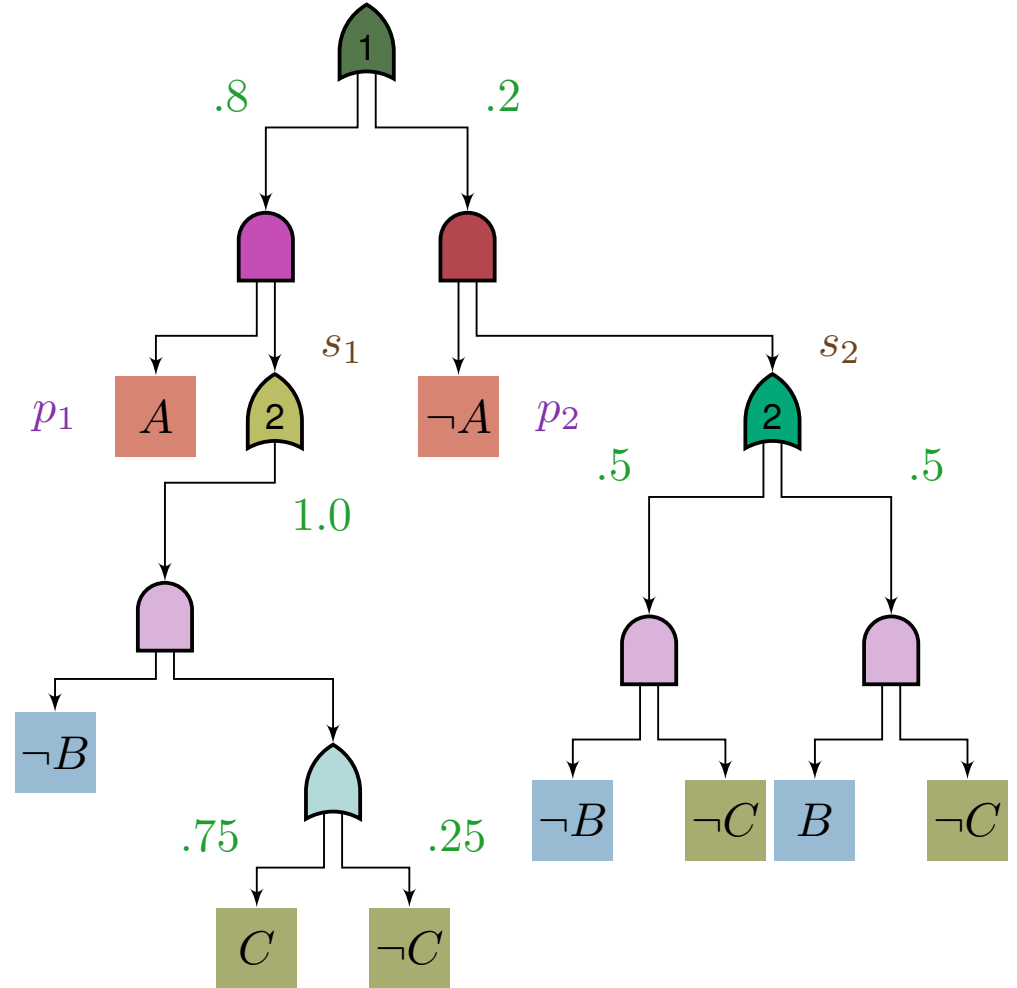
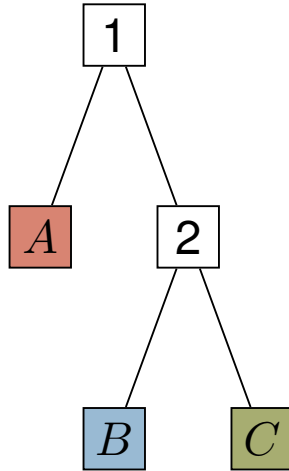
Experiments

| Dataset | Vars | SRBMs | oSLRAU | GBMMs | iGMMs | GMMs | PROMETHEUS | iSPTs | LEARNRP | Size |
|------------|------|--------------|--------------|--------|-------|--------------|--------------|--------------|---------|-------|
| ABALONE | 8 | -2.28 | -0.94 | -1.17 | — | -0.59 | <u>-0.85</u> | — | -6.13 | 317 |
| CA | 22 | -4.95 | <u>21.19</u> | 3.42 | — | -1.08 | 27.82 | — | -5.84 | 2765 |
| QUAKE | 4 | -2.38 | <u>-1.21</u> | -3.76 | — | -0.58 | -1.50 | — | -3.76 | 79 |
| SENSORLESS | 48 | -26.91 | <u>60.72</u> | 8.56 | — | -1.39 | 62.03 | — | -38.46 | 12589 |
| BANKNOTE | 4 | -2.76 | <u>-1.39</u> | -4.64 | — | -1.05 | -1.96 | — | -6.06 | 79 |
| FLOWSIZE | 3 | -0.79 | <u>15.32</u> | 5.72 | — | -36.50 | 18.03 | — | 2.20 | 49 |
| KINEMATICS | 8 | -5.55 | -11.13 | -11.20 | — | <u>-6.11</u> | -11.12 | — | -11.02 | 319 |
| IRIS | 4 | — | — | — | -3.94 | 0.20 | <u>-1.06</u> | -3.74 | -3.47 | 79 |
| OLDFAITH | 2 | — | — | — | -1.73 | -2.09 | -1.48 | <u>-1.70</u> | -4.33 | 19 |
| CHEMDIABET | 3 | — | — | — | -3.02 | -0.58 | <u>-2.59</u> | -2.88 | -18.68 | 48 |

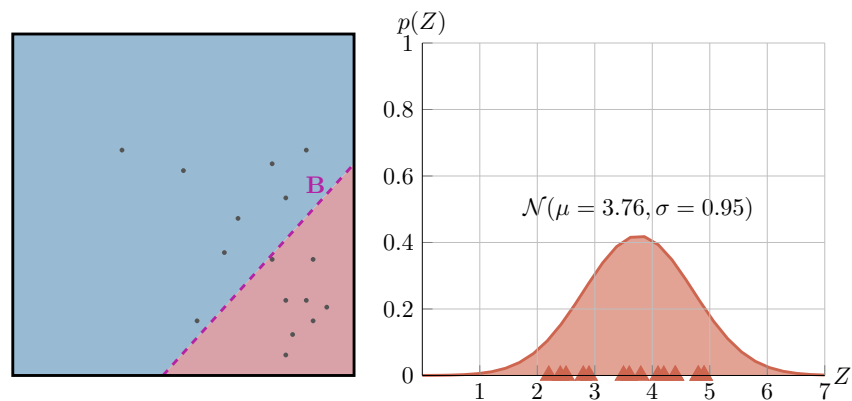
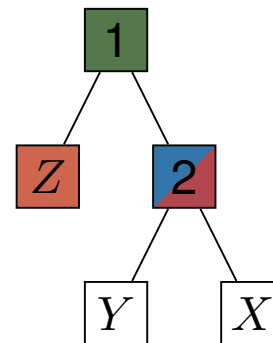
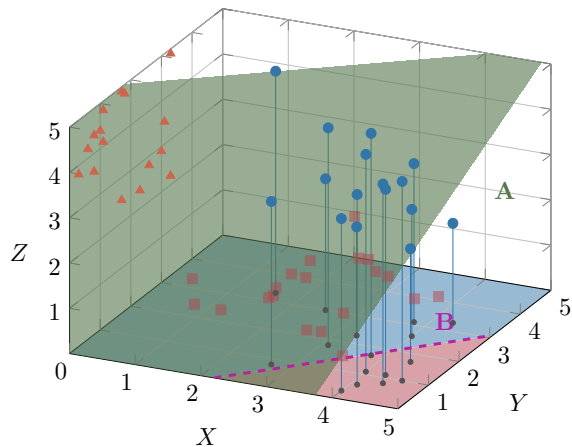
In summary

| A | B | C | $p(\mathbf{x})$ |
|-----|-----|-----|-----------------|
| 0 | 0 | 0 | 0.1 |
| 0 | 1 | 0 | 0.1 |
| 1 | 0 | 0 | 0.2 |
| 1 | 0 | 1 | 0.6 |

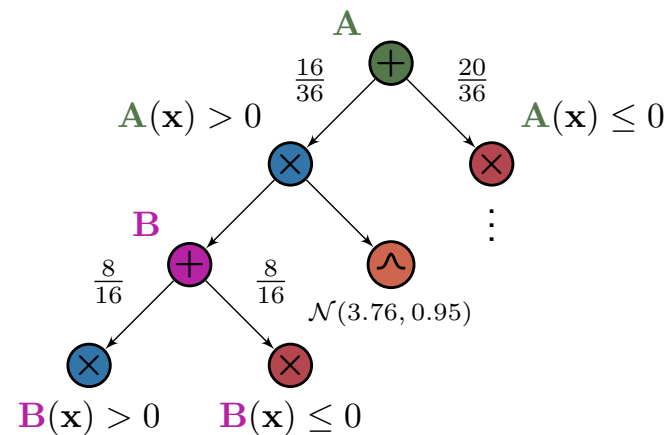
$$\phi(A, B, C) = (A \rightarrow \neg B) \wedge (C \rightarrow A)$$



In summary



$$\mathbf{B}(x, y) = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 1.10 \\ -1.00 \end{bmatrix} - 2.43$$

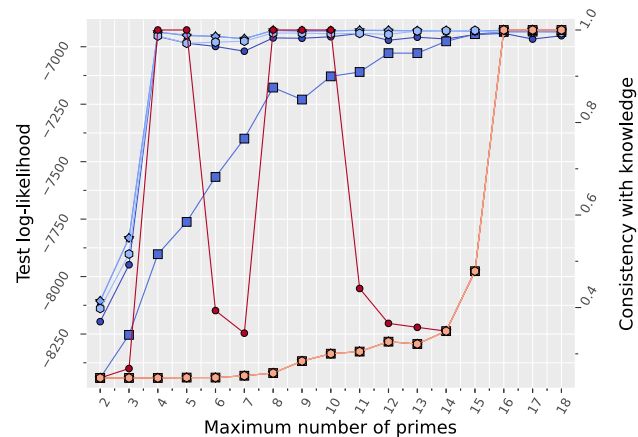
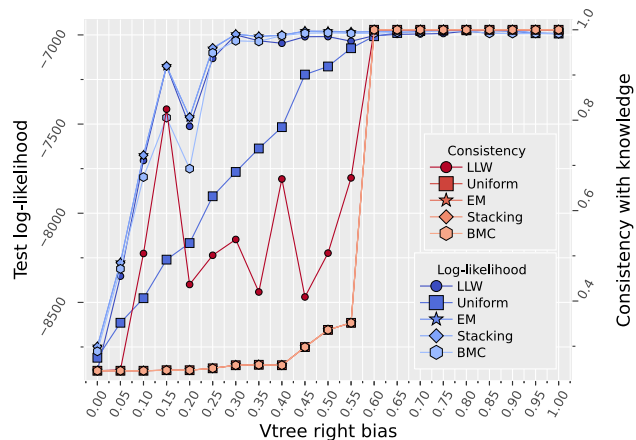


Supplemental Material

SAMPLEPSDD – Experiments

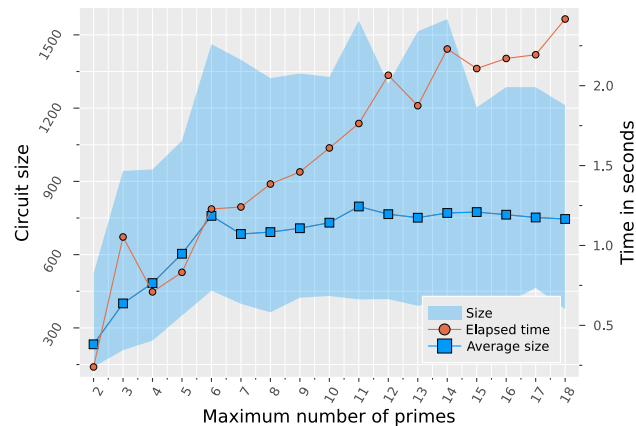
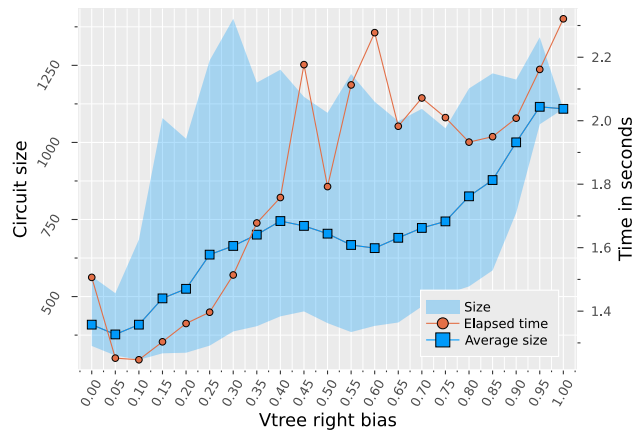
What is the impact of higher k 's and right-leaning vtrees

in log-likelihood and consistency?



Samples perform better with higher k 's and right-leaning vtrees ...

...but at a cost to complexity.



SAMPLEPSDD – What do we gain from this?

Available queries:

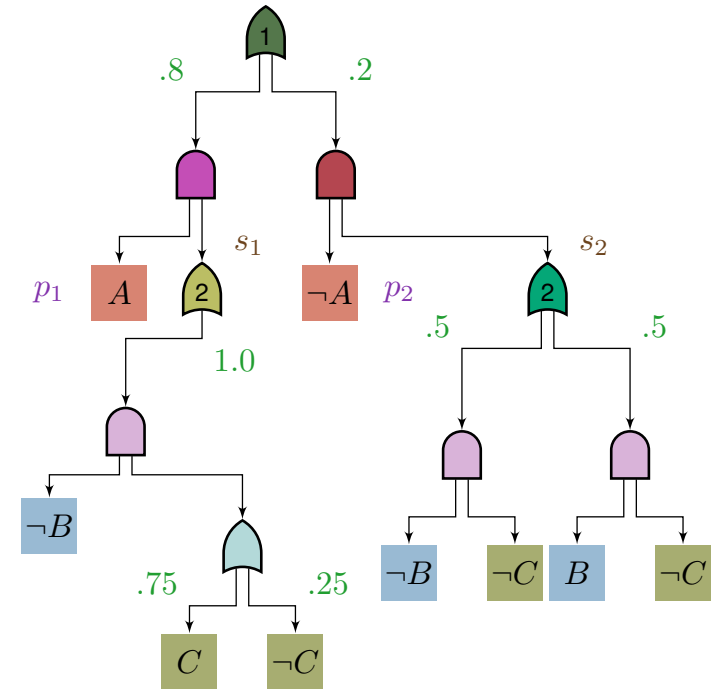
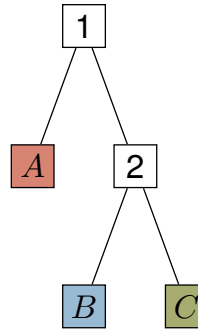
- ✓ Probability of Evidence;
- ✓ Marginal Probability;
- ✓ Conditional Probability;
- ✓ Most Probable Explanation;
- ✓ Shannon Entropy;
- ✓ Cross Entropy;
- ✓ Kullback-Leibler Divergence;
- ✓ Rényi's Alpha Divergence;
- ✓ Cauchy-Schwarz Divergence;
- ✓ Probability of Logical Events;
- ✓ Mutual Information.

Support:

- ✓ Defineable as a logic formula;
- ✗ Consistent with a relaxation;
- ✓ Ensembles mitigate relaxation.

| A | B | C | $p(\mathbf{x})$ |
|-----|-----|-----|-----------------|
| 0 | 0 | 0 | 0.1 |
| 0 | 1 | 0 | 0.1 |
| 1 | 0 | 0 | 0.2 |
| 1 | 0 | 1 | 0.6 |

$$\phi(A, B, C) = (A \rightarrow \neg B) \wedge (C \rightarrow A)$$



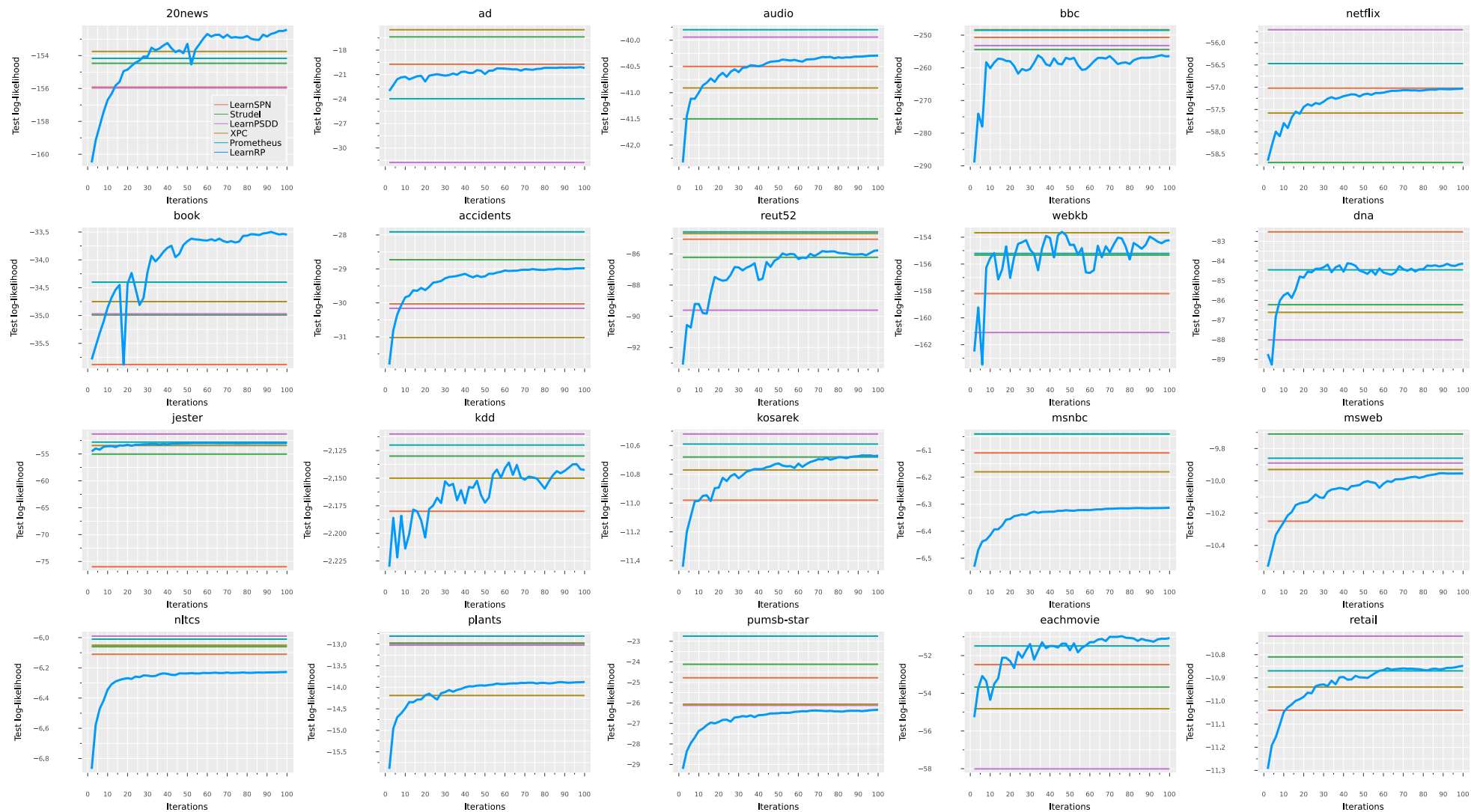
LEARNRP – Datasets

| Dataset | Vars | Train | Test | Domain | Dataset | Vars | Train | Test | Domain |
|------------|------|--------|-------|------------|------------|------|---------|--------|--------------|
| ACCIDENTS | 111 | 12758 | 2551 | $\{0, 1\}$ | NLTCS | 16 | 16181 | 3236 | $\{0, 1\}$ |
| AD | 1556 | 2461 | 491 | $\{0, 1\}$ | PLANTS | 69 | 17412 | 3482 | $\{0, 1\}$ |
| AUDIO | 100 | 15000 | 3000 | $\{0, 1\}$ | PUMSB-STAR | 163 | 12262 | 2452 | $\{0, 1\}$ |
| BBC | 1058 | 1670 | 330 | $\{0, 1\}$ | EACHMOVIE | 500 | 4524 | 591 | $\{0, 1\}$ |
| NETFLIX | 100 | 15000 | 3000 | $\{0, 1\}$ | RETAIL | 135 | 22041 | 4408 | $\{0, 1\}$ |
| BOOK | 500 | 8700 | 1739 | $\{0, 1\}$ | ABALONE | 8 | 3760 | 417 | \mathbb{R} |
| 20-NEWSGRP | 910 | 11293 | 3764 | $\{0, 1\}$ | CA | 22 | 7373 | 819 | \mathbb{R} |
| REUTERS-52 | 889 | 6532 | 1540 | $\{0, 1\}$ | QUAKE | 4 | 1961 | 217 | \mathbb{R} |
| WEBKB | 839 | 2803 | 838 | $\{0, 1\}$ | SENSORLESS | 48 | 52659 | 5850 | \mathbb{R} |
| DNA | 180 | 1600 | 1186 | $\{0, 1\}$ | BANKNOTE | 4 | 1235 | 137 | \mathbb{R} |
| JESTER | 100 | 9000 | 4116 | $\{0, 1\}$ | FLOWSIZE | 3 | 1358674 | 150963 | \mathbb{R} |
| KDD | 65 | 180092 | 34955 | $\{0, 1\}$ | KINEMATICS | 8 | 7373 | 819 | \mathbb{R} |
| KOSAREK | 190 | 33375 | 6675 | $\{0, 1\}$ | IRIS | 4 | 90 | 10 | \mathbb{R} |
| MSNBC | 17 | 291326 | 58265 | $\{0, 1\}$ | OLDFAITH | 2 | 245 | 27 | \mathbb{R} |
| MSWEB | 294 | 29441 | 5000 | $\{0, 1\}$ | CHEMDIABET | 3 | 131 | 14 | \mathbb{R} |

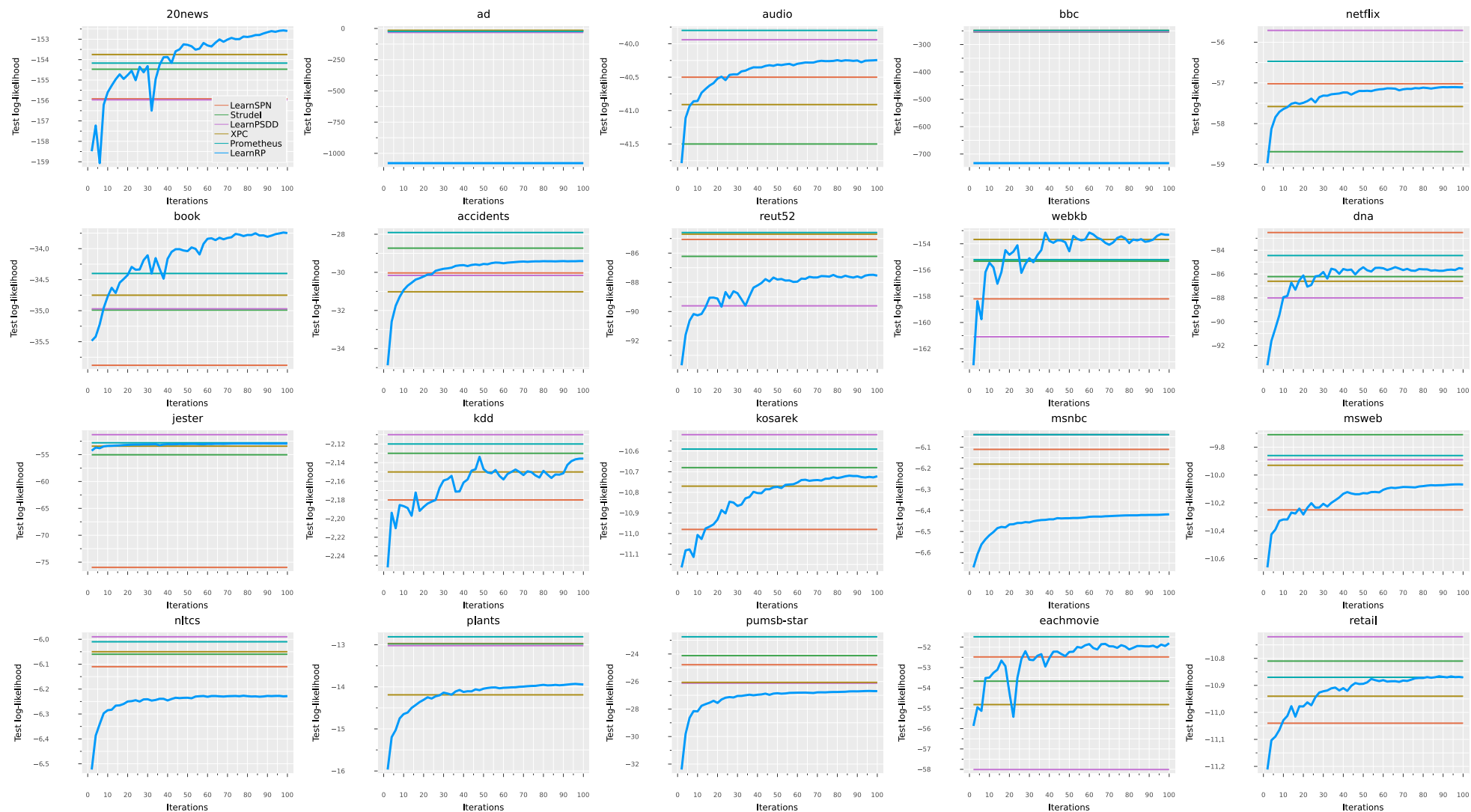
LEARNRP – Binary Benchmark

| Dataset | LEARNSPN | STRUDEL | LEARNSDD | XPC | PROMETHEUS | LEARNRP-F | LEARNRP-100 | LEARNRP-30 | LEARNRP-20 | LEARNRP-10 |
|-------------|------------------|-----------------|--------------------|------------------|--------------------|--------------------|--------------------|-------------|------------------|-------------|
| ACCIDENTS | -30.03 | <u> -28.73 </u> | -30.16 | -31.02 | -27.91 | <u> -28.65 </u> | -28.87 | -29.38 | -29.58 | -29.99 |
| AD | -19.73 | <u> -16.38 </u> | -31.78 | -15.50 | -23.96 | <u> -19.20 </u> | -20.32 | -21.42 | -21.44 | -21.94 |
| AUDIO | -40.50 | -41.50 | <u> -39.94 </u> | -40.91 | -39.80 | <u> -40.18 </u> | -40.23 | -40.46 | -40.63 | -40.94 |
| BBC | <u> -250.68 </u> | -254.41 | -253.19 | -248.34 | <u> -248.50 </u> | -254.97 | -255.55 | -262.35 | -257.67 | -262.39 |
| NETFLIX | <u> -57.02 </u> | -58.69 | -55.71 | -57.58 | <u> -56.47 </u> | -57.07 | -57.05 | -57.29 | -57.48 | -57.66 |
| BOOK | -35.88 | -34.99 | -34.97 | -34.75 | -34.40 | <u> -33.57 </u> | -33.52 | -34.34 | <u> -34.24 </u> | -34.73 |
| 20-NEWSGRP | -155.92 | -154.47 | -155.97 | <u> -153.75 </u> | -154.17 | <u> -152.78 </u> | -152.76 | -154.32 | -155.03 | -156.26 |
| REUTERS-52 | <u> -85.06 </u> | -86.22 | -89.61 | <u> -84.70 </u> | -84.59 | -85.73 | -85.47 | -87.41 | -87.05 | -89.26 |
| WEBKB | -158.20 | -155.33 | -161.09 | <u> -153.67 </u> | -155.21 | -154.43 | -152.60 | -154.83 | <u> -154.33 </u> | -158.01 |
| DNA | -82.52 | -86.22 | -88.01 | -86.61 | -84.45 | <u> -83.03 </u> | <u> -83.85 </u> | -84.77 | -84.98 | -85.40 |
| JESTER | -75.98 | -55.03 | -51.29 | -53.43 | <u> -52.80 </u> | -52.92 | <u> -52.89 </u> | -53.23 | -53.22 | -53.54 |
| KDD | -2.18 | <u> -2.13 </u> | -2.11 | -2.15 | <u> -2.12 </u> | <u> -2.13 </u> | -2.14 | -2.17 | -2.16 | -2.20 |
| KOSAREK | -10.98 | -10.68 | -10.52 | -10.77 | <u> -10.59 </u> | <u> -10.65 </u> | -10.67 | -10.79 | -10.86 | -11.00 |
| MSNBC | <u> -6.11 </u> | -6.04 | -6.04 | <u> -6.18 </u> | -6.04 | -6.31 | -6.36 | -6.40 | -6.41 | -6.44 |
| MSWEB | -10.25 | -9.71 | -9.89 | -9.93 | <u> -9.86 </u> | <u> -9.85 </u> | -9.97 | -10.06 | -10.21 | -10.27 |
| NLTCS | -6.11 | -6.06 | -5.99 | <u> -6.05 </u> | -6.01 | -6.35 | -6.23 | -6.25 | -6.27 | -6.32 |
| PLANTS | <u> -12.97 </u> | <u> -12.98 </u> | -13.02 | -14.19 | -12.81 | -13.68 | -14.00 | -14.26 | -14.40 | -14.70 |
| PUMSB-STAR | <u> -24.78 </u> | <u> -24.12 </u> | -26.12 | -26.06 | -22.75 | -25.88 | -26.19 | -26.36 | -26.54 | -27.17 |
| EACHMOVIE | -52.48 | -53.67 | -58.01 | -54.82 | <u> -51.49 </u> | <u> -51.37 </u> | -51.06 | -51.55 | -52.86 | -52.21 |
| RETAIL | -11.04 | <u> -10.81 </u> | -10.72 | -10.94 | -10.87 | <u> -10.85 </u> | -10.86 | -10.93 | -10.97 | -11.04 |
| Avg. Rank | 6.08 ± 3.03 | 5.28 ± 2.97 | 5.20 ± 3.86 | 5.55 ± 2.76 | 2.90 ± 2.07 | <u>3.83 ± 1.98</u> | <u>4.15 ± 2.03</u> | 6.35 ± 1.50 | 6.95 ± 1.70 | 8.72 ± 1.50 |
| | 4.80 ± 1.91 | 4.22 ± 1.81 | <u>4.05 ± 2.56</u> | 4.60 ± 1.93 | 2.55 ± 1.43 | <u>3.62 ± 1.56</u> | 4.15 ± 2.03 | | | |
| Pos. (mean) | 7th | 5th | 4th | 6th | 1st | <u>2nd</u> | <u>3rd</u> | 8th | 9th | 10th |
| | 7th | 5th | <u>3rd</u> | 6th | 1st | <u>2nd</u> | 4th | | | |

LEARNRP – Learning Curves



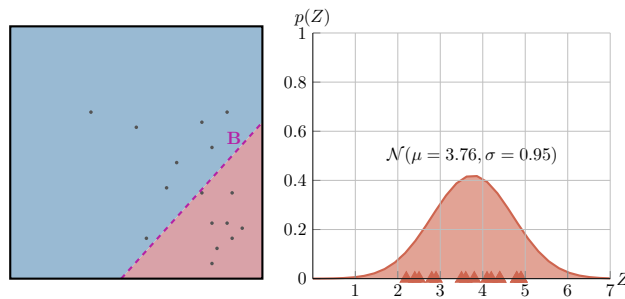
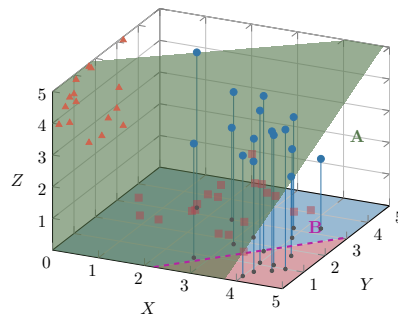
LEARNRP – Random Initializations



LEARNRP – What do we gain from this?

Available queries:

- ✓ Probability of Evidence;
- ✓ Marginal Probability;
- ✓ Conditional Probability;
- ✗ Most Probable Explanation;
- ✗ Shannon Entropy;
- ✗ Cross Entropy;
- ✗ Kullback-Leibler Divergence;
- ✗ Rényi's Alpha Divergence;
- ✓ Cauchy-Schwarz Divergence;
- ✓ Probability of Logical Events;
- ✗ Mutual Information.



$$\mathbf{B}(x, y) = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 1.10 \\ -1.00 \end{bmatrix} - 2.43$$

