LIÈGE université
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MATH0461-2 Introduction to numerical optimization

Project report - Investment Portfolio Management

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1 Linear Model

1. In order to build the optimal portfolio based on the given data, we can build the following linear problem :
   — **Decision variable :** $parts_i$, which is the fraction of the total capital invested in stock i.
   — The **objective function** that needs to be maximized is therefore :
   $$max \quad \frac{1}{567}\sum_{i=1}^{462}\sum_{t=2}^{568} parts_i\, r_{i,t}$$
   where $r_{i,t}$ is the weekly return of stock i at week t, not taken as a percentage.
   — Since we have only one variable, we can only express the **constraints** on $parts_i$ to ensure that we respect the demands of the statement :
   - We can invest up to 100% of the entire capital :
   $$\sum_{i=1}^{462} parts_i \leq 1$$
   - We can invest up to 20% of our capital in a single sector :
   $$\sum_{i \in sector_j} parts_i \leq 0.2$$
   where $sector_j \in Sectors$ contains the stocks that belongs in the j^{th} of the 10 sectors given in the data and $Sectors$ contains the 10 sectors.
   - We can not invest negative parts :
   $$parts_i \in \mathbb{R}^+, \quad \forall i$$

   To summarize, here is the problem that we need to optimize in order to find the best portfolio based on the given dataset :
   $$\begin{aligned} max \quad & \frac{1}{567}\sum_{i=1}^{462}\sum_{t=2}^{568} parts_i\, r_{i,t} \\ s.t. \quad & \sum_{i=1}^{462} parts_i \leq 1 \\ & \sum_{i \in sector_j} parts_i \leq 0.2 \quad \forall sector_j \in Sectors\ (j \in \{0, ..., 9\}) \\ & parts_i \in \mathbb{R}^+, \quad \forall i \end{aligned}$$

2. The optimal portfolio is composed of stocks from Tesla, Advanced Micro Devices (more commonly known as AMD), Netflix, Enphase Energy,and Nvidia.

   Here is a summary table of the portfolio :

Name of the stock	Number of shares	Part of the capital invested	Average weekly return	Sector
TSLA (Tesla)	43680.84302707339	0.2	1.1106%	7
AMD (Advanced Micro Devices)	40160.64241646517	0.2	0.9176%	2
NFLX (Netflix)	7607.868454433332	0.2	0.8191%	8
ENPH (Enphase Energy)	26809.651337438736	0.2	1.1354%	6
NVDA (Nvidia)	34057.20074485186	0.2	1.0549%	0

These results shows that it is optimal to invest in a single stock per sector to maximize the objective function, which is to maximize the historical average weekly return while limiting sector exposure. The selected stocks are likely those with the highest average weekly returns within their respective sectors, as well as the best-performing top stocks across all sectors.The number of shares purchased varies according to the price of each individual stock.

3. Let p and $q_j \in \mathbb{R}^+$ be dual variables associated with the first, and second to eleventh constraints of the primal problem, respectively,

$$
\begin{aligned}
max \quad & \frac{1}{567}\sum_{i=1}^{462}\sum_{t=2}^{568} parts_i,\, r_{i,t} \\
s.t. \quad & \sum_{i=1}^{462} parts_i \leq 1 : p \\
& \sum_{i \in sector_j} parts_i \leq 0.2 : q_j \quad \forall sector_j \in Sectors\ (j \in \{0, ..., 9\}) \\
& parts_i \in \mathbb{R}^+ \forall i
\end{aligned}
$$

Dualising these constraints yields the following problem

$$
\begin{aligned}
d(p, q_j) = max \quad & \frac{1}{567}\sum_{i=1}^{462}\sum_{t=2}^{568} parts_i\, r_i + p(1 - \sum_{i=1}^{462} parts_i) + \sum_{j=0}^{9} q_j(0.2 - \sum_{i \in sector_j} parts_i) \\
s.t.\ & parts_i \in \mathbb{R}^+ \forall i
\end{aligned}
$$

Then, after rearranging terms in the objective function, the problem becomes

$$
\begin{aligned}
d(p, q_j) = & max \quad \sum_{i=1}^{462}(\frac{1}{567}\sum_{t=2}^{568} r_{i,t} - p - \sum_{j : i \in sector_j} q_j) parts_i + p + 0.2\sum_{j=0}^{9} q_j \\
& s.t. \quad parts_i \in \mathbb{R}^+ \forall i
\end{aligned}
$$

Inspecting the first term reveals that the following conditions must be satisfied to guarantee that the relaxed problem above is bounded,

$$
\sum_{t=2}^{568} r_{i,t} - p - q_j \leq 0 \ \forall i \text{ where j is the sector of the stock i}
$$

Therefore leads to the problem

$$
\begin{aligned}
min \quad & p + 0.2\sum_{j=0}^{9} q_j \\
s.t. \quad & p + q_j \geq \frac{1}{567}\sum_{t=2}^{568} r_{i,t} \ \ \forall i \text{ where j is the sector of the stock i} \\
& p \in \mathbb{R}^+, q_j \in \mathbb{R}^+ \ \ \forall j
\end{aligned}
$$

Theoretically, the dual variables are shadow prices. They quantify how much the value of the optimal solution would change if the RHS (right-hand side) of the related constraint were adjusted. In this project, the primal objective has a clear economic interpretation, as it represents the expected historical average weekly return from investing in different stocks. Thus, the dual variable of a given constraint can be interpreted as the marginal gain or loss as the RHS coefficient is updated. Specifically, for the first constraint, it represents the impact of having more or less capital available. For the other constraints, it represents the impact of increasing or decreasing the sector exposure limit.

4. Let the following primal problem

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

And the corresponding dual problem

$$\begin{aligned} \max \quad & b^T p \\ s.t. \quad & A^T p = c \\ & p \geq 0 \end{aligned}$$

If the primal has an optimal solution x^*, then the dual has an optimal solution p^* such that

$$c^T x^* = b^T p^*$$

The table shows the values of the dual variables at optimality

p	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9
0.73730	0.3195	0	0.1819	0	0	0	0.4001	0.3753	0.0833	0

The higher the average weekly return, the greater the dual variables associated with that sector. This result was expected, as at optimality, relaxing a constraint involving a stock or sector with a higher average weekly return is better. This allows additional investment in sectors with the greatest return potential, thereby maximizing overall portfolio performance.

The dual variables for sectors 1, 3, 4, 5, and 9 are equal to zero because these sectors are not used in the optimal portfolio. So, relaxing these constraints would not have any impact on the optimal solution.

5. Both methods required only one iteration to find the optimal solution. The dual simplex is slightly faster but need the Markowitz tolerance tighten to 0.5. When a constraint is change but still in the same optimal basis (see next question), the dual simplex still need one only iteration whereas the primal requires 2 iterations.

6. The interval for l_6, within which the optimal basis remains unchanged, is $[0, 0.2]$. When l_6 is decreased within this interval, the optimal basis does not change, but a variable from another sector, which was previously set to zero, become non-zero while the variable associated to the sector 6 also decreased. This indicates a degenerate solution, where more than n (number of variables)-m (number of constraints) variables are set to 0.

   Since l_6 is defined as non negative, it obviously cannot take negative values ; therefore, setting it to a negative value is not feasible. If we increase the value of l_6 its original value, the optimal basis will change, as the constraint on sector 6 will be relaxed. This relaxation allows for more investment in sector 6, which may cause other constraints to become non-binding, leading to a different optimal allocation and thus a change in the basis.

2 Non-Linear Model : Risk Management

1. In order to take into account the risk in our maximization problem, we will use the same same constraints as in the linear part, but the objective function becomes :

$$max \sum_{i=1}^{462} \sum_{t=2}^{568} parts_i \, r_{i,t} - \gamma \sum_{i=1}^{462} \sum_{j=1}^{462} x_i \Sigma_{ij} x_j$$

   where $\gamma \sum_{i=1}^{462} \sum_{j=1}^{462} x_i \Sigma_{ij} x_j = \gamma x^T \Sigma x$ represents the weighted risk of the portfolio. The double sum computes de covariance between every pair of stocks which tells us how each pair contributes to the entire portfolio.

2. After solving this new problem, we obtain the following figure :

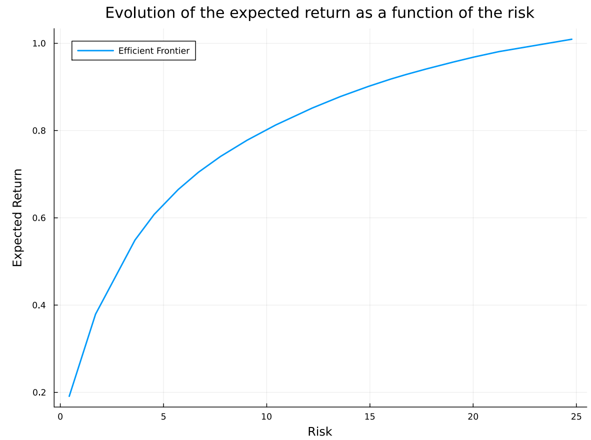

FIGURE 1 – Efficient Frontier

The efficient frontier plot reveals the relationship between risk and expected return for the portfolio. The most obvious observation is that the expected return increases with risk. This is intuitive because investing in riskier, uncertain assets typically offers greater rewards if this is a successful. The positive correlation shown in the graph supports the idea that, to achieve higher returns, investors must be willing to accept higher risks.

The relationship between risk and expected return is not linear ; instead, it looks like a logarithmic curve. In the initial stages, from Risk ≈ 0 to Risk ≈ 10, the expected return increases rapidly with relatively small increases in risk. This suggests that the initial risk is the most impactful, providing the greatest gain in return relative to the risk taken. After Risk ≈ 10, the curve begins to flatten, indicating diminishing returns while increasing risk. At this point, although the expected return continues to grow, it does so at a slower rate compared to the initial increase. This means that the investor must take bigger risk for only a modest increase in expected return.

The diminishing increase in expected return for additional units of risk reflects the classic principle of diminishing marginal returns in portfolio management. Initially, diversifying into riskier assets contributes significantly to increasing returns, but over time, each additional unit of risk yields less benefit. This reinforces the importance of carefully evaluating how much risk to take beyond a certain point.

The efficient frontier essentially outlines the best possible portfolios for a given level of risk. Portfolios below the efficient frontier are inefficient, as they offer a lower return for the same level of risk. The frontier helps investors visualize how they can optimize their investments based on the risk that they are willing to take—either minimizing risk for a given return or maximizing return for a given risk level. This graph could be used as a tool for investors to determine their optimal point on the frontier, aligning their risk tolerance with their return goals. The efficient use of risk in the initial segment is crucial, while the later parts of the curve remind us of the potential inefficiencies in pursuing higher returns, especially as risks grow disproportionately compared to the incremental gains in return.