Easy Problems

Problem 1: Given 100 fair coin flips, what is the probability that the number of heads is even? What about 99 coin clips? What about 101 coin flips?

The answer to all three questions is 1/2. The parity of the number of heads is ultimately decided by the final coin flip which has 1/2 probability of being heads. This is the warmup.

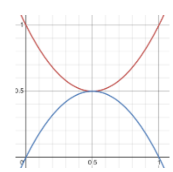
Problem 2: Two decks of cards. One deck has 52 cards, the other has 104. You pick two cards separately from a same pack. If both of two cards are red, you win. Choose a pack.

Intuitively we want to take from the 104 pack since taking a red diminishes the chance that we will get another red less. Formally, we represent this idea with the following inequality

$$\left(\frac{52}{104}\right)\left(\frac{51}{103}\right) > \left(\frac{26}{52}\right)\left(\frac{25}{51}\right) \Leftrightarrow \frac{51}{103} > \frac{25}{51} \Leftrightarrow 2601 > 2575$$

This question shows that for small number of trials from a large population without replacement it is simple justified to use independence as an approximation.

Problem 3: For best 2 out of 3 games, would you bet on them finishing in 2 rounds or 3 rounds? Does it depend on the strength of the teams or no?



Let team 1 have probability p of winning and team 2 have probability $1-p$. Then the probability that we finish in 2 rounds is $p^2+(1-p)^2$ and the probability that we finish in 3 rounds is $2p(1-p)$. We can show that $p^2 + (1-p)^2 \geq 2p(1-p)$. Observe that:

$$\begin{aligned} p^2+(1-p)^2 \geq 2p(1-p) &\Leftrightarrow p^2+1-2p+p^2 \geq 2p-2p^2 \\ &\Leftrightarrow 4p^2-4p+1=(2p-1)^2 \geq 0 \end{aligned}$$

There is a certain elegance to the structure of the problem. The symmetry of the probability of the game finishing early and late is revealed in the graph.

Problem 4: You have a 3 gallon jug and 5 gallon jug, and a bucket. How do you put exactly 4 gallons in the bucket? Is this possible? What about $\boldsymbol{n}$ gallons?

Fill up the 3 gallon jug. Then, pour the liquid into the 5 gallon jug. Fill the 3 gallon jug again, and then fill the 5 gallon jug until it is full. We now have 1 gallon remaining in the 3 gallon jug. We can put this into the bucket and indeed we can put n gallons in the bucket.

The generalization of this problem to having a jugs of x gallons, b jugs of y gallons, and filling buckets with multiples of d gallons where $d = \gcd(a,b)$, is Bézout's identity, which is one of the most fundamental equations in number theory.

Intermediate Problems

Problem 5: You have a coin which comes heads with probability p, and you toss it n times. If there are multiple coins in a row (could be just 1), we call them/it a streak. For example, if we get HTHTT, we have 4 streaks: first H, second T, third H, and last two T. What is the expected number of streaks in n tosses for a coin with bias p?

Let $\mathbb{I}_i$ for $i \in [2, n]$ be the indicator variable on if the ith coin is different form the last (i.e. 1 if the are different and 0 if they are the same). These indicators are identically distributed (note that independence is irrelevant here). Note that $\mathbb{E}(\mathbb{I}_i) = \mathbb{P}(\mathbb{I}_i = 1) = 2p(1-p)$ so the total number of expected switches is

$$\mathbb{E}\left(\sum_{i=2}^{n} \mathbb{I}_i\right) = \sum_{i=2}^{n} \mathbb{E}(\mathbb{I}_i) = (n-1)2p(1-p)$$

This is a powerful application of the linearity of expectation and indicator variables. The two are often combined to solve interesting problems.

Problem 6: A line of 100 passengers is waiting to board a plane. They each hold a ticket to one of the 100 seats on that flight (for convenience, let's say that the nth passenger in line has a ticket for the seat number n). Unfortunately, the first person in line is crazy, and will ignore the seat number on their ticket, picking a random seat to occupy. All of the other passengers are quite normal and will go to their proper seat unless it is already occupied. If it is occupied, they will then find a free seat to sit in, at random. What is the probability that the last (100th) person to board the plane will sit in their proper seat (#100)?

The fate of the last passenger is determined when either the first or last seat on the plane is taken. This statement is true because the last person will either get the first seat or the last seat. All other seats will necessarily be taken by the time the last passenger gets to pick their seat. Since at each choice step, the first or last seat has an equal probability of being taken, the last person will get either the first or last with equal probability: 0.5.

This is another application of symmetry similar to what we did in problem 1. Note that symmetric structure in problems can often lead to a solution involving 0.5. However, a rigorous and convincing argument is still needed to reveal the structure of the problem.

Problem 7: I have a deck of n cards which I shuffle many times. What is the expected number of cards that still occupy their original place in the deck after my shuffles?

Let $\mathbb{I}_i$ for $i \in [1, n]$ be the indicator variable on if the ith card is in its former place. These indicators are identically distributed (note that independence is irrelevant here). Note that $\mathbb{E}(\mathbb{I}_i) = \mathbb{P}(\mathbb{I}_i = 1) = 1/n$ so the total number of expected fixed-points is

$$\mathbb{E}\left(\sum_{i=1}^{n} \mathbb{I}_i\right) = \sum_{i=1}^{n} \mathbb{E}(\mathbb{I}_i) = n \times \frac{1}{n} = 1$$

Harder Problems

Problem 8: Find all sets of positive integers $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ such that

$$\frac{\mathbf{1}}{\boldsymbol{a}}+\frac{\mathbf{1}}{\boldsymbol{b}}+\frac{\mathbf{1}}{\boldsymbol{c}}=\mathbf{1}$$

Let us take, without any loss of generality, $a \leq b \leq c$. How small can a be? First set a = 1. This gives no solutions, because it leaves nothing for 1/b + 1/c. Next set a = 2 and try values of b (with $b \geq 2$, by assumption) in order:

if b = 2, then 1/c = 0, which is no good;
if b = 3, then c = 6, which works;
if b = 4, then c = 4, which works;
if b ≥ 5, then b ≥ c, so we need not consider this.

Then set a = 3, and try values of b (with $b \geq 3$) in order:

if b = 3, then c = 3, which works;
if b ≥ 4, then b ≥ c, so we need not consider this.

Finally, if a a4c 4, then at least one of b and c must be ≤ a, so we need look no further.
The only possibilities are therefore (2, 3, 6), (2, 4, 4) and (3, 3, 3) and permutations.

Problem 9: My two friends, who shall remain nameless, but whom I shall refer to as P and Q, both told me this afternoon that there is a body in my fridge. I'm not sure what to make of this, because P tells the truth with a probability of p, while Q (independently) tells the truth with a probability of only q. I haven't looked in the fridge for some time, so if you had asked me this morning, I would have said that there was just as likely to be a body in the fridge as not. In view of what my friends have told me, I must revise my estimate. Explain what my new estimate of the probability of there being a body in the fridge should be.

Here, we take the events A and B to be

A = P and Q both say that there is a body in the fridge
B = there is a body in the fridge

From the information given in the question, P(B) = 1/2, so

$$P(B|A)=\frac{P(B)P(A|B)}{P(A)}=\frac{0.5\times pq}{P(A)}$$

Now, we deduce what $P(A)$ is by considering

$$\begin{aligned} P(A) &= P(\text{body}) \times \mathrm{P}(\text{they say there is a body}) + \mathrm{P}(\text{no body}) \times P(\text{They say there is a body}) \\ &= \frac{1}{2}pq+\frac{1}{2}(1-p)(1-q) \end{aligned}$$

So the final expression that we desire is

$$\frac{\frac{1}{2}\times pq}{\frac{1}{2}pq+\frac{1}{2}(1-p)(1-q)}=\frac{pq}{1-p-q+2pq}$$

Challenge Problems

Problem 10: Can the polynomial

$$\boldsymbol{1 + xy + x^2y^2}$$

be factored into the form

$$\boldsymbol{a(x)c(y) + b(x)d(y)}$$

where $\boldsymbol{a, b, c, d}$ have real coefficients?

Hint: Set $y = -1,0,1$ so that $c(y), d(y) \in \mathbb{R}$ and consider the resulting polynomials a vectors. Consider also $a(x)$ and $b(x)$ as vectors. Can they span the polynomials generated?

Using the hint we see that $a(x)$ and $b(x)$ must span a space containing the points $\{1 - x + x^2, 1, 1 + x + x^2\}$. However, these vectors are linearly independent by inspecting the determinant

$$\begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -2 \neq 0$$

So it is not possible for (x) and $b(x)$ to span the desired space. The factoring cannot exist.

Problem 11: Consider the system of equations:

$$\boldsymbol{2yz + zx - 5xy = 2}$$
$$\boldsymbol{yz - zx + 2xy = 1}$$
$$\boldsymbol{yz - 2zx + 6xy = 3}$$

By substitution or otherwise, find the possible values for the product of solutions. Hence find all integer solutions for $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$.

We try substitution with the following assignments: $u = yx, v = zx, w = xy$. Our non-linear system hence reduces to the following linear system:

$$2u + v - 5w = 2$$
$$u - v + 2w = 1$$
$$u - 2v + 6w = 3$$

Using your favorite method of solving linear systems, we get the following: $u = 2, v = 6,$ $w = 3$. The product of uvw actually gives us the square of the product of xyz. Observe that $uvw = (yx)(zx)(xy) = (xyz)^2 = 36$. The possible values for the product of solutions is hence $xyz = \pm 6$. Consider the case of $xyz = 6$ and $xyz = -6$ indepdendently. If $xyz = 6$: $yz = 3$ so $x = 2$; $yx = 2$ so $y = 1$; $zx = 6$ so $z = 3$. If $xyz = -6$: then by symmetry deduce $x = -2, y = -1, z = -3$. The two solutions are hence $(2,1,3)$ and $(-2, -1, -3)$.

The three original equations actually describe hyperbolas in 3D space instead of planes which would be the case for linear systems. Unlike linear systems of equation where we can only have 0,1, or ∞ solutions, in this specific case we have exactly 2 solutions.