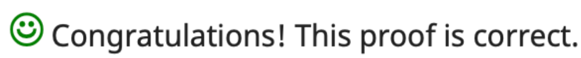
Homework 2: Propositional Logic, Continued

CS 81: Computability and Logic

Due: 11:59pm, Tuesday, September 12, 2023

Instructions (read carefully)

1. Although we encourage you to work out your natural deduction proofs on paper first, in your submission please typeset all natural deduction proofs using the online natural deduction proof checker at `proofs.openlogicproject.org` . Take a screenshot of the whole proof *including the smiley face*

   and embed this picture in your PDF submission.

   On a Mac, Shift-Command-4 will let you select a portion of the screen and create a picture on the Desktop; or you can use the Screenshot app in `/Applications/Utilities`. Other operating systems will provide different methods.

2. For this assignment, your proofs may use the following rules:
   - "Basic Rules" R through **IP** (new!)
   - "Derived Rules" R, **DNE**, and **LEM** (new!)
   - "Rules for Cambridge" $\bot I$ and $\bot E$

   These include both the constructive rules and the classical rules discussed in Lecture 3 (though beware of the proof checker's nonstandard LEM rule).

3. The easier-to-type ASCII equivalents for logical operators are shown on the left side of the proof checker page under "Instructions."

4. On the last page of this assignment, there are some general hints (more than last week!) for finding Natural Deduction proofs.

Have fun!

1 Proof or No Proof? [40 points]

For each of the following, either show there is no proof (by finding a model that makes the assumptions true and the conclusion false), or provide a natural deduction proof (via screenshot). If you provide a counterexample, briefly explain why the model makes each assumption true and the premise false, or identify *one* model that is a counterexample.=

1. $F \lor G,\;\; G \to \neg F \;\;\vdash\;\; F \to G$
2. $F \to G,\; \neg F \to G \;\vdash\; G$
3. $F \to (G \to H) \;\vdash\; G \to (F \to H)$
4. $(F \to G) \to H \;\vdash\; F \to (G \to H)$.
5. $F \to G,\; C \to D \;\vdash\; (F \lor C) \to (G \land D)$

2 Proof! [24 points]

Give (screenshots of) natural deduction proofs for each of the following. Prove the first two constructively (i.e., without LEM, IP, or DNE).

1. $\neg(F \to G) \;\vdash\; G \to F$ (This is deduction, not logical equivalence!)
2. $A \lor B,\; \neg B \lor C \;\vdash\; A \lor C$
3. $\neg(F \land G) \;\vdash\; \neg F \lor \neg G$

   [Hint: one strategy is to use LEM and prove $\neg F$ and hence $\neg F \lor \neg G$) in one case and prove $\neg G$ (and hence $\neg F \lor \neg G$) in the other.]

3 A Crisis in Classical Logic? [14 points]

1. Give a (screenshotted) natural deduction proof of $E \to D, \neg(K \to D) \vdash E \to K$
2. Dr. Read claims Classical Logic is broken. Identify the two errors in his argument—at least from the standpoint of Classical Logic. [Hint: This is tricky, but think truth tables!]

   "Suppose Prof. Dodds claims the number of students currently enrolled at HMC is even, and Prof. Kuenning claims this number is odd.

   Consider two statements about this situation (which we formalize using E, D, and K):

   - If the number of HMC students is even, then Prof. Dodds is right $(E \to D)$
   - If Prof. Kuenning is right, then Prof. Dodds is right $(K \to D)$

   The first statement $(E \to D)$ is obviously true.
   The second statement is obviously false, and so its negation $\neg(K \to D)$ is true.

   But from $E \to D$ and $\neg(K \to D)$, Classical Logic lets us prove:

   - If the number of HMC students is even, then Prof. Kuenning is right $(E \to K)$

   which is clearly false. Thus, Classical Logic lets us derive false conclusions from true assumptions, and it's not a trustworthy model of reasoning."

4 OS Wars [20 points]

Consider the following argument:

- **Premise 1**. If my computer runs Windows, then Microsoft got my money.
- **Premise 2**. If my computer runs macOS, then Apple got my money.
- **Conclusion**. At least one of the following statements is true:
  - If my computer runs macOS, then Microsoft got my money
  - If my computer runs Windows, then Apple got my money

1. Clearly define four variables to stand for relevant propositions in this proof, and show how to represent the premises and conclusion as WFFs.
2. Surprisingly, this is a valid argument! Demonstrate this by providing either a complete truth table or by providing a screenshot of a checked Natural Deduction proof.

   [Hint: if you decide to try Natural Deduction, be advised that there is no proof using just the constructive rules; you'll need to use a rule like LEM to prove the conclusion.]

5 Survey [2 points]

Please wait until you're done with the rest of the assignment to answer this quick survey:

1. How long (in hours) did you spend working on this assignment?
2. What was the most interesting thing you learned while doing this assignment? (We're sure there was *something* you learned.)

Hints for Building Natural Deduction Proofs

1. Here's what I'd try first (unless there's an obviously simpler or better way):
   - If you have a conjunction $\mathcal{A} \wedge \mathcal{B}$, immediately use $\wedge E$ to get $\mathcal{A}$ and $\mathcal{B}$ separately.
   - If you have an implication $\mathcal{A} \to \mathcal{B}$, see if you have (or can prove) $\mathcal{A}$ so you can use $\to E$.
   - If you have a disjunction $\mathcal{A} \vee \mathcal{B}$ and are trying to prove $\mathcal{C}$, apply proof by cases ($\vee E$) to two subproofs: one assuming $\mathcal{A}$ and proving $\mathcal{C}$, and one assuming $\mathcal{B}$ and proving $\mathcal{C}$.
   - If you have a negation $\neg A$, see if you have (or can prove) $\mathcal{A}$, because a contradiction $\perp$ would let us conclude anything we want.

   - To prove a conjunction $\mathcal{A} \wedge \mathcal{B}$, try to prove $\mathcal{A}$ and $\mathcal{B}$ separately and use $\wedge I$.
   - To prove an implication $\mathcal{A} \to \mathcal{B}$, apply $\to I$ to a subproof assuming $\mathcal{A}$ and concluding $\mathcal{B}$.
   - To prove a disjunction $\mathcal{A} \vee \mathcal{B}$, you can only use $\vee I$ if you know which of $\mathcal{A}$ or $\mathcal{B}$ is true. Otherwise, you have to work harder. For example, you might have a proof-by-cases ($\vee E$) where $\mathcal{A}$ (and hence $\mathcal{A} \vee \mathcal{B}$) is true in one case and $\mathcal{B}$ (and hence $\mathcal{A} \vee \mathcal{B}$) is true in the other.
   - To prove a negation $\neg\mathcal{A}$, apply $\neg I$ to a subproof assuming $\mathcal{A}$ and concluding $\perp$.

   - **If all else fails (and you're not asked for a constructive proof), pick a propositional variable A and use LEM, i.e., show that your desired conclusion is true when we make the extra assumption A, and show that your desired conclusion is true when we make the extra assumption $\neg A$.**

2. Don't make new assumptions (add sub-proofs) inside your proof unless you know why you're doing so!

   This will *only* be the case when working backwards; you have a WFF you want to prove, and you realize a sub-proof with a specific assumption and specific conclusion would let you use $\to I$ or $\neg I$ or $\vee E$ to immediately justify that WFF.

3. **Natural Deduction is sound and complete, so truth-table reasoning can be helpful for intuition. You can prove everything entailed by your assumptions and nothing else.**

   For example, if your current assumptions are $\neg P$ and Q, then there's no point trying to prove $(P \vee R)$ or $(Q \to P)$, because these are not entailed by your assumptions.

   More interestingly, if you have already proved or assumed Q to be true, then you know you can prove $(R \to Q)$ for any R because the implication must be true. In fact, you can prove the implication in just three more lines using $\to I$ and the R rule.

   Similarly, if you already know $\neg P$, then $(P \to R)$ is easily provable because it's logically true (whether or not R is).