Natural Deduction, Continued

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CS 81: Computability and Logic

> While mathematical logic and set theory indeed make up the language spanning all fields of mathematics, mathematicians rarely speak it. To borrow notions from computers, basic mathematical logic can be regarded as the "machine language" for mathematicians who usually use much higher-level languages
>
> —Gil Kalai

Warmup: Show that $(P \to Q) \to P \models P$

P	Q	$P \to Q$	$(P \to Q) \to P$	P	
t	t	t	t	t	✓
t	f	f	t	t	✓
f	t	t	f	f	✓
f	f	t	f	f	✓

Show that $\models A \vee \neg A$

A	$\neg A$	$A \vee \neg A$
t	f	t
f	t	t

Show that $\neg A, A \vDash \bot$

$\neg A$	A	$\bot$
t	f	f
f	t	f

assumptions never true!

Show that $\perp \models A$

$\perp$	A	
f	t	✓
f	f	✓

Truth vs. Proof

We say that $\mathcal{A} \models \mathcal{B}$ if $\mathcal{B}$ is true whenever $\mathcal{A}$ is.

We say that $\mathcal{A} \vdash \mathcal{B}$ if $\mathcal{B}$ is provable (step by step) from $\mathcal{A}$.

Ideally, we will have

- If $\mathcal{A} \vdash \mathcal{B}$, then $\mathcal{A} \models \mathcal{B}$ (soundness).
- If $\mathcal{A} \models \mathcal{B}$, then $\mathcal{A} \vdash \mathcal{B}$ (completeness).

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Being able to check soundness and completeness requires us to precisely define

- What it means for $\mathcal{B}$ and $\mathcal{A}$ to be *formulas*. ✓
- What it means for $\mathcal{B}$ and $\mathcal{A}$ to be *true*. ✓
- What counts as a *proof* of $\mathcal{B}$ from $\mathcal{A}$.

Proof Systems

A *proof system* is a small toolkit of rules (proof steps) that allowed in a formal proof.

Logicians have studied many *different* proof systems.

- Natural Deduction
- Resolution
- Hilbert Systems, Sequent Calculus, Tableaux, ...

Technically, the notation $\mathcal{A} \vdash \mathcal{B}$ is ambiguous, and we should say

- $\mathcal{A} \vdash_{\mathsf{ND}} \mathcal{B}$ if $\mathcal{B}$ is provable from $\mathcal{A}$ using Natural Deduction steps
- $\mathcal{A} \vdash_{\mathsf{R}} \mathcal{B}$ if $\mathcal{B}$ is provable from $\mathcal{A}$ using Resolution steps
- *etc.*

but context usually tells us what proof system we mean.

Why Are There Multiple Proof Systems?

In terms of what you can prove, it doesn't matter.

- All the systems mentioned are sound and complete so, e.g.,

$$\mathcal{A} \vdash_{\mathsf{ND}} \mathcal{B} \quad \text{iff} \quad \mathcal{A} \models \mathcal{B} \quad \text{iff} \quad \mathcal{A} \vdash_{\mathsf{R}} \mathrm{B}$$

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But different systems organize proofs differently, e.g.,

- The 12 rules of Natural Deduction mimic how humans write proofs.
- Resolution uses 1 rule over and over, and is easier for computers.

Recall: The "Constructive" Natural Deduction Rules

$$\frac{\mathcal{A} \qquad \mathcal{B}}{\mathcal{A} \wedge \mathcal{B}}\ \wedge\mathrm{I} \qquad \frac{\mathcal{A} \wedge \mathcal{B}}{\mathcal{A}}\ \wedge\mathrm{E} \qquad \frac{\mathcal{A} \wedge \mathcal{B}}{\mathcal{B}}\ \wedge\mathrm{E} \qquad \frac{\mathcal{A} \to \mathcal{B} \qquad \mathcal{A}}{\mathcal{B}}\ {\to}\mathrm{E}$$

$$\frac{\mathcal{A}}{\mathcal{A} \vee \mathcal{B}}\ \vee\mathrm{I} \qquad \frac{\mathcal{B}}{\mathcal{A} \vee \mathcal{B}}\ \vee\mathrm{I} \qquad \frac{\neg\mathcal{A} \qquad \mathcal{A}}{\bot}\ \neg\mathrm{E} \qquad \frac{\bot}{\mathcal{A}}\ \bot\mathrm{E}$$

$$\frac{\begin{array}{|l} \mathcal{A} \\ \hline \vdots \\ \mathcal{B} \end{array}}{\mathcal{A} \to \mathcal{B}}\ {\to}\mathrm{I} \qquad \frac{\mathcal{A} \vee \mathcal{B} \qquad \begin{array}{|l} \mathcal{A} \\ \hline \vdots \\ \mathcal{C} \end{array} \qquad \begin{array}{|l} \mathcal{B} \\ \hline \vdots \\ \mathcal{C} \end{array}}{\mathcal{C}}\ \vee\mathrm{E} \qquad \frac{\begin{array}{|l} \mathcal{A} \\ \hline \vdots \\ \bot \end{array}}{\neg\mathcal{A}}\ \neg\mathrm{I}$$

Justifying $\neg$E and $\bot$E

We showed that according to the truth-table definition of entailment,

- $\neg A,\ A \models \bot$
- $\bot \models A$.

Because Natural Deduction should be complete (prove all correct conclusions) we demand

- $\neg A,\ A \vdash_{\mathsf{ND}} \bot$
- $\bot \vdash_{\mathsf{ND}} A$.

To make these provable, we need the rules $\neg$E and $\bot$E.

$$\frac{\neg \mathcal{A} \qquad \mathcal{A}}{\bot}\ \neg\mathrm{E} \qquad\qquad \frac{\bot}{\mathcal{A}}\ \bot\mathrm{E}$$

Prove $\vdash \neg(P \land \neg P)$ using Natural Deduction

1	$(P \land \neg P)$	
2	P	$\land$E on 1
3	$\neg P$	$\land$E on 1
4	$\bot$	$\neg$E on 2 and 3
5	$\neg(P \land \neg P)$	$\neg$I on 1–4

Terminology

A *formal proof* involves WFFs and a steps in a proof system (e.g., Natural Deduction)

- Purely a matter of rule-following and symbol manipulation
- *"Mathematics is the science in which we do not know what we are talking about, and do not care whether what we say about it is true." —Bertrand Russell*

So, a proof written by humans for humans is technically an *informal proof.*

- Informal *doesn't* mean sloppy or careless!
- Combination of English (or another natural language) and symbolic notation.
- Assumes the reader can fill in any missing steps or justifications.

Burgess: A *rigorous* proof is something that convinces us a formal proof exists.

Can you see $\rightarrow$I, $\wedge$E, and $\rightarrow$E moves?

Theorem: If $x \in S \cap T$ and $S \subseteq U$ and $T \subseteq V$, then $x \in U \cap V$.

Proof. Suppose $x \in S \cap T$ and $S \subseteq U$ and $T \subseteq V$

Then $x \in S$, and so $x \in U$.

Similarly, $x \in T$, and so $x \in V$.

Therefore, $x \in U \cap V$.

$x \in (S \cap T)$ means $(x \in S) \wedge (x \in T)$
$S \subseteq U$ means $(x \in S) \rightarrow (x \in U)$
$T \subseteq V$ means $(x \in T) \rightarrow (x \in V)$

Can you see $\to$I, $\vee$E, $\to$E, and $\vee$I moves?

Theorem: If $x \in S \cup T$, $S \subseteq U$, and $T \subseteq V$ then $x \in U \cup V$.

Proof. Suppose $x \in S \cup T$, $S \subseteq U$, and $T \subseteq V$.

Since $x \in S \cup T$, we consider two cases:

- If $x \in S$, then $x \in U$ and so $x \in U \cup V$.
- If $x \in T$, then $x \in V$ and so $x \in U \cup V$.

Since one of these two cases must hold, $x \in U \cup V$.

$x \in (S \cup T)$ means $(x \in S) \vee (x \in T)$
$S \subseteq U$ means $(x \in S) \to (x \in U)$
$T \subseteq V$ means $(x \in T) \to (x \in V)$
$x \in (U \cup V)$ means $(x \in U) \vee (x \in V)$

Can you see $\neg$I and $\neg$E moves?

Theorem: $\sqrt{2}$ is irrational (*i.e.*, not *equal* to any fraction).

Proof. Suppose $\sqrt{2}$ were rational. Then $\sqrt{2} = \frac{a}{b}$ for a simplified fraction with $b \neq 0$.

Then $2b^2 = a^2$, so a^2 is *even*, so a is *even*.

But if a is *even* then a^2 is a multiple of 4,

so $2b^2$ is a multiple of 4, which means b is a multiple of 2.

Then a and b are both *even*, but $\frac{a}{b}$ is simplified so a and b aren't both *even*, a contradiction.

Therefore, $\sqrt{2}$ is not rational. QED.

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Natural Deduction So Far

The introduction and elimination rules give us *Constructive Propositional Logic*

There are valid entailments we still can't prove, including

$$\neg\neg P \;\vdash\; P$$

$$P \to Q \;\vdash\; \neg P \vee Q$$

$$(P \to Q) \to P \;\vdash\; P$$

That is, the rules so far are *sound*, but not *complete*.

To get full *Classical Propositional Logic*, we need at least one more rule.

A Non-Constructive Argument

What's non-constructive about it?
Why do we need a new rule?

Theorem. There exist irrational a and b such that a^b is rational.

Proof. We saw that $\sqrt{2}$ is irrational.

Either $\sqrt{2}^{\sqrt{2}}$ is rational or it is not.

- If it is rational, we can take $a = \sqrt{2}$ and $b = \sqrt{2}$.
- If it is irrational, then we can take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$, because

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^{2} = 2.$$

QED.

Three *Classical* Rules

Add *any one* of these to ND, and the other two become theorems:

$$\dfrac{}{\mathcal{A} \vee \neg\mathcal{A}}\ \text{LEM} \qquad \dfrac{\neg\neg\mathcal{A}}{\mathcal{A}}\ \text{DNE} \qquad \dfrac{\begin{array}{|l} \neg\mathcal{A} \\ \hline \vdots \\ \bot \end{array}}{\mathcal{A}}\ \text{IP}$$

Warning

`proofs.openlogicproject.org` uses a non-standard LEM rule:

$$
\dfrac{\begin{array}{|l} \mathcal{A} \\ \hline \vdots \\ \mathcal{B} \end{array} \qquad \begin{array}{|l} \neg\mathcal{A} \\ \hline \vdots \\ \mathcal{B} \end{array}}{\mathcal{B}}\ \text{LEM}
$$

Combines our LEM with an immediate $\vee$E proof-by-cases.

Prove: $P \to Q \vdash \neg P \vee Q$

Hint: this is not *constructively* provable.

1	$P \to Q$	
2	$\quad$ P	
3	$\quad$ Q	$\to$E on 1, 2
4	$\quad$ $\neg P \vee Q$	$\vee$I on 3
5	$\quad$ $\neg P$	
6	$\quad$ $\neg P \vee Q$	$\vee$I on 5
7	$\neg P \vee Q$	LEM 2-4, 5-6

PROVE: $\vdash \neg\neg P \rightarrow P$

Hint: this is not constructively provable

1	$\quad \neg\neg P$	
2	$\quad P$	DNE on 1
3	$\neg\neg P \rightarrow P$	$\rightarrow$I on 1-2