Predicate Logic

September 11, 2023

CS 81: Computability and Logic

> "Todd, trust math. As in Matics, Math E. First-order predicate logic. Never fail you."
>
> –David Foster Wallace

Formalizing Arguments

We argued that natural language could be misleading.

We wanted to think about valid arguments abstractly.

Water is wet or grass is orange	$P \vee Q$
Grass is not orange	$\neg Q$
Therefore, water is wet.	Therefore, P.

But:

All men are mortal	P
Socrates is a man	Q
Therefore, Socrates is mortal	Therefore, R ?

We need more than just propositional logic!

The Solution: Predicate Logic

Also known as *Predicate Calculus* or *First-Order Logic*.

Extends Propositional Logic with

- *Terms* that describe specific things (*individuals*)
- *Relations* on Terms
- Quantifiers that range over individuals

$$\forall i \, (\, i \in \mathbb{N} \rightarrow \exists j \, (\, j \leqslant 0 \wedge i + j = 1 \,) \,)$$

Re-Formalizing Arguments

All men are mortal

Socrates is a man

Therefore, Socrates is mortal

$\forall x \; (\; \mathbf{man}(x) \rightarrow \mathbf{mortal}(x) \;)$

$\mathbf{man}(s)$

Therefore, $\mathbf{mortal}(s)$

What We Need To Know

1. What are the well-formed formulas of Predicate Logic?
2. What are the models? When is a formula true in a model?
3. How can we reason correctly?

Terms

A *term* is an expression whose value is an *individual*

Terms can involve constants, variables, and/or function symbols.

canada   max(x, sqrt(7))   fatherOf(mary)

Terms, Formally

The terms of Predicate Logic are defined inductively as follows:

- Any constant symbol $\mathbf{c}$, $\mathbf{d}$, … is a term.
- Any variable x, y, … is a term.
- If $\mathbf{f}$ is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $\mathbf{f}(t_1, \ldots, t_n)$ is a term.

Well-Formed Formulas

Formulas extend propositional logic with predicates/relations on terms, and universal/existential quantifiers

- $\neg \mathrm{inhabited}(\mathrm{canada})$
- $7 = \mathrm{max}(x, \mathrm{sqrt}(7))$
- $\exists x\ (P(x) \wedge Q(x^2 + 1))$
- $\forall y\ (\exists z\ (z > y))$
- $\forall x\ (\mathrm{knows}(x, \mathrm{bob}) \ \rightarrow\ \mathrm{age}(x) < 99)$

Formulas, Formally

The wffs of Predicate Logic are defined inductively as follows:

- If P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are **terms**, then $P(t_1, \ldots, t_n)$ is a wff.
- $\top$ and $\bot$ are wffs.
- If A is some wff, then so is $\neg A$
- If A and B are wffs, so is $(A \wedge B)$
- If A and B are wffs, so is $(A \vee B)$
- If A and B are wffs, so is $(A \rightarrow B)$
- If A is a wff, so are $\forall x\, A$ and $\exists x\, A$. for any variable x.

Interpret as English

$\forall x \; A(x,y)$

$G(x)$	x is a grutor
$S(x)$	x is a student
$L(x)$	x is a lecture
$A(x,y)$	x attended y
m	Mary

1. $S(m) \land \neg G(m)$

   Mary is a student but not a grutor
   (and)

2. $\exists x\,(L(x) \land A(m,x))$

   Mary attended some lecture.
   (a)

   Mary attended at least one lecture.

3. $\forall x (\exists y \; A(x,y))$

   Everything attended something.

   For each thing, there's something it attended.

4. $\exists y \; \forall x \; A(x,y)$

   There's something that everything attended.

Represent as Formulas

$K(x, y)$	x knows y
$A(x, y)$	x attended y
$G(x)$	x is a grutor
$S(x)$	x is a student
$L(x)$	x is a lecture
m	Mary

1. Mary knows herself.

   $K(m, m)$

2. Every grutor knows Mary.

   $\forall x\ (G(x) \rightarrow K(x, m))$

3. Some grutor knows Mary.

   $\exists x\ (G(x) \wedge K(x, m))$

4. Mary knows every grutor.

   $\forall x\ (G(x) \rightarrow K(m, x))$

$K(\forall x G(x), m)$ X

$\underbrace{(G(x))}_{\text{"x is a grutor"}} \rightarrow \underbrace{(\forall x\ K(x, m))}_{\text{everything knows Mary.}}$ X

$\exists x\ (G(x) \rightarrow K(x, m))$

Represent as Formulas

$\neg(x \vee y) \equiv \neg x \wedge \neg y$

$K(x, y)$	x knows y
$A(x, y)$	x attended y
$G(x)$	x is a grutor
$S(x)$	x is a student
$L(x)$	x is a lecture
m	Mary

5. Mary attended every lecture.

$$\forall x\,(L(x) \to A(m,x))$$

$$\forall \ell\,(L(\ell) \to A(m,\ell))$$

6. No student attended every lecture.

$$\neg\left(\exists x\left[S(x) \wedge \underbrace{\forall y\,(L(y) \to A(x,y))}_{\text{"x attended every lecture."}}\right]\right)$$

7. No lecture was attended by every student.

$$\neg\left(\exists x\,(L(x) \wedge \underbrace{\forall s\,(S(s) \to A(s,x)}_{\text{"x was attended by every student}}\right)$$

Free and Bound Variables

A use of a variable in a term is said to be *bound* if it inside a quantifier for that variable, and *free* otherwise.

free! free!

$$P(\underline{x}) \land \forall x\,(R(\underline{x}) \to T(\underline{x}, \underline{y}))$$

bound

bound

Why We Care (1)

Formulas with no free variables are true or false, depending on the model.

$$\forall x\, \exists s\, (x \in s)$$

(a proposition)

Formulas with free variables are true or false, depending on the model *and the values of the free variables*

free

$$\exists x\, (x \in s)$$

(a proposition "about s")

Why We Care (2)

The names of *bound* variables (mostly) don't matter

$$\exists x\,(x > y) \equiv \exists z\,(z > y)$$

$$\exists x\,(x > y) \equiv \exists k\,(k > y)$$

Why We Care (2)

The names of *bound* variables (mostly) don't matter

$$\boxed{\exists x\,(x > y)} \equiv \exists z\,(z > y)$$

$$\exists x\,(x > y) \equiv \exists k\,(k > y)$$

$$\exists x\,(x > y) \not\equiv \exists x\,(x > z)$$

$$\exists x\,(x > y) \not\equiv \exists x\,(x > k)$$

Why We Care (2)

The names of *bound* variables (mostly) don't matter

$$\exists x\,(x > y) \;\equiv\; \exists z\,(z > y)$$

$$\exists x\,(x > y) \;\equiv\; \exists k\,(k > y)$$

$$\exists x\,(x > y) \;\not\equiv\; \exists x\,(x > z)$$

$$\exists x\,(x > y) \;\not\equiv\; \exists x\,(x > k)$$

$$\exists x\,(x > y) \;\not\equiv\; \exists y\,(y > y)$$

Digression: Why is this *First-Order* Logic?

In first-order logic, we can quantify over individuals.

$$\forall x\,(P(x) \vee \neg P(x))$$

But When is a Formula *True*?

The truth of $\forall x\, P(c, x)$ depends on our model:

$\{0, 1, 2, 3, \ldots\}$

- Model 1: the domain is natural numbers, c is the integer 0, and P is the "less-than-or-equal" relation. t

- Model 2: the domain is people, c is Prof. Stone's cousin Ken, and P is the "lives-next-door-to" relation? f

Ingredients of a Model, Formally

1. The *domain* (**nonempty** set of individuals being considered);
2. For each function/constant symbol, a corresponding mathematical function/constant;
3. For each relation symbol, a corresponding mathematical relation on individuals.

(We require $=$ be interpreted as equality.)

Truth in a Model

$P(t)$ is true iff the individual t has property P (according to the model)

$R(t, u)$ is true iff the individuals t and u are in relation R (according ...)

$\forall x \, P(x)$ is true iff *every* individual in the domain has property P

$\exists x \, P(x)$ is true iff *at least one* individual in the domain has property P

The propositional operators $(\vee, \wedge, \rightarrow, \neg)$ follow the same truth tables.

END