

Predicate Logic

September 11, 2023
CS 81: Computability and Logic

“Todd, trust math. As in Matics, Math E.
First-order predicate logic. Never fail you.”
–David Foster Wallace

FORMALIZING ARGUMENTS

We argued that natural language could be misleading.

We wanted to think about valid arguments abstractly.

Water is wet or grass is orange

Grass is not orange

Therefore, water is wet.

$P \vee Q$

$\neg Q$

Therefore, P .

But:

All men are mortal

Socrates is a man

Therefore, Socrates is mortal

P

Q

Therefore, R ?

We need more than just propositional logic!

THE SOLUTION: PREDICATE LOGIC

Also known as *Predicate Calculus* or *First-Order Logic*.

Extends Propositional Logic with

- *Terms* that describe specific things (*individuals*)
- *Relations* on Terms
- Quantifiers that range over individuals

$$\forall i (i \in \mathbb{N} \rightarrow \exists j (j \leq 0 \wedge i + j = 1))$$

RE-FORMALIZING ARGUMENTS

All men are mortal

Socrates is a man

Therefore, Socrates is mortal

$\forall x (\text{man}(x) \rightarrow \text{mortal}(x))$

$\text{man}(s)$

Therefore, $\text{mortal}(s)$

WHAT WE NEED TO KNOW

1. What are the well-formed formulas of Predicate Logic?
2. What are the models? When is a formula true in a model?
3. How can we reason correctly?

TERMS

A term is an expression whose value is an individual

Terms can involve constants, variables, and/or function symbols.

canada max(x, sqrt(7)) fatherOf(mary)

TERMS, FORMALLY

The terms of Predicate Logic are defined inductively as follows:

- Any constant symbol c, d, \dots is a term.
- Any variable x, y, \dots is a term.
- If f is an n -ary function symbol and t_1, \dots, t_n are terms, then $f(t_1, \dots, t_n)$ is a term.


WELL-FORMED FORMULAS

Formulas extend propositional logic with predicates/relations on terms, and universal/existential quantifiers

- $\neg \text{inhabited}(\text{canada})$
- $7 = \max(x, \text{sqrt}(7))$
- $\exists x (P(x) \wedge Q(x^2 + 1))$
- $\forall y (\exists z (z > y))$
- $\forall x (\text{knows}(x, \text{bob}) \rightarrow \text{age}(x) < 99)$

FORMULAS, FORMALLY

The wffs of Predicate Logic are defined inductively as follows:

- If P is an n -ary predicate symbol and t_1, \dots, t_n are **terms**, then $P(t_1, \dots, t_n)$ is a wff.
- \top and \perp are wffs.
- If A is some wff, then so is $\neg A$
- If A and B are wffs, so is $(A \wedge B)$
- If A and B are wffs, so is $(A \vee B)$
- If A and B are wffs, so is $(A \rightarrow B)$
- If A is a wff, so are $\forall x A$ and $\exists x A$.  *for any variable x .*

INTERPRET AS ENGLISH

$$\forall x A(x, y)$$

$G(x)$	x is a grutor
$S(x)$	x is a student
$L(x)$	x is a lecture
$A(x, y)$	x attended y
m	Mary

$$1. S(m) \wedge \neg G(m)$$

Mary is a student but not a grutor
(and)

$$2. \exists x (L(x) \wedge A(m, x))$$

Mary attended some lecture.
(a)

Mary attended at least one lecture.

$$3. \forall x (\exists y A(x, y))$$

Everything attended something.

For each thing, there's something it attended.

$$4. \exists y \forall x A(x, y)$$

There's something that everything attended.

REPRESENT AS FORMULAS

1. Mary knows herself.

$$K(m, m)$$

2. Every grutor knows Mary.

$$\forall x (G(x) \rightarrow K(x, m))$$

3. Some grutor knows Mary.

$$\exists x (G(x) \wedge K(x, m))$$

4. Mary knows every grutor.

$$\forall x (G(x) \rightarrow K(m, x))$$

$K(x, y)$	x knows y
$A(x, y)$	x attended y
$G(x)$	x is a grutor
$S(x)$	x is a student
$L(x)$	x is a lecture
m	Mary

$K(\forall x G(x), m)$ X

$(G(x)) \rightarrow (\forall x K(x, m))$ X

" x is a grutor"

everything knows Mary.

$$\exists x (G(x) \rightarrow K(x, m))$$

REPRESENT AS FORMULAS

$$\neg(x \vee y) \equiv \neg x \wedge \neg y$$

$K(x, y)$	x knows y
$A(x, y)$	x attended y
$G(x)$	x is a grutor
$S(x)$	x is a student
$L(x)$	x is a lecture
m	Mary

5. Mary attended every lecture.

$$\forall x (L(x) \rightarrow A(m, x))$$

$$\forall l (L(l) \rightarrow A(m, l))$$

6. No student attended every lecture.

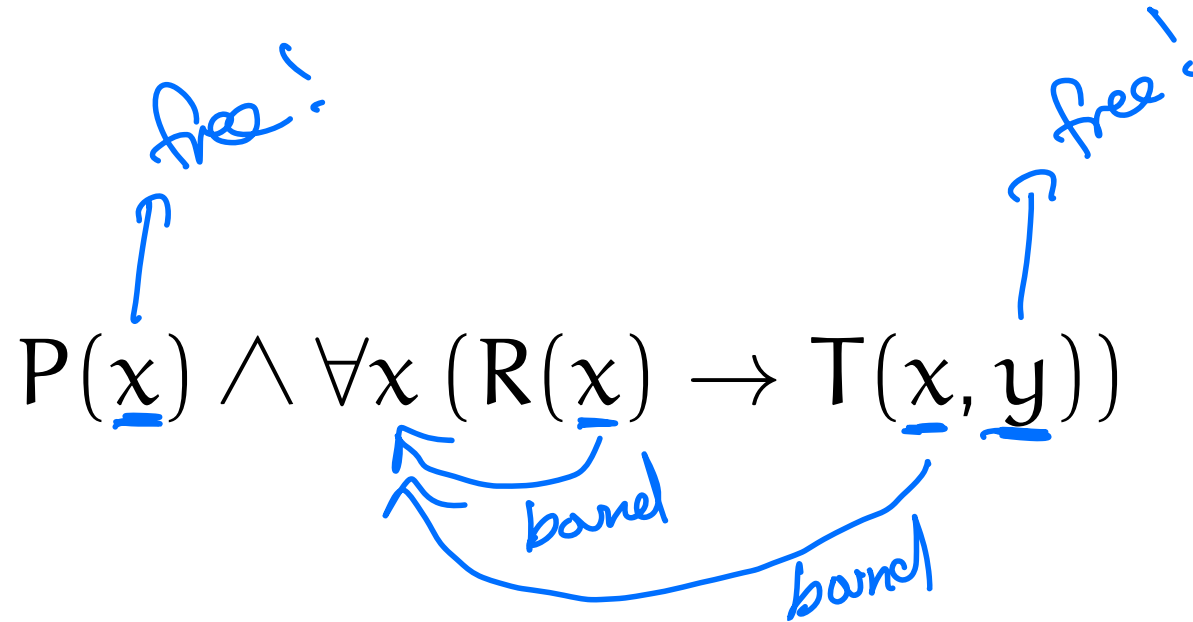
$$\neg \left(\exists x \left[S(x) \wedge \underbrace{\forall y (L(y) \rightarrow A(x, y))}_{\text{"x attended every lecture."}} \right] \right)$$

7. No lecture was attended by every student.

$$\neg \left(\exists x \left(L(x) \wedge \underbrace{\forall s (S(s) \rightarrow A(s, x))}_{\text{"x was attended by every student"}} \right) \right)$$


FREE AND BOUND VARIABLES

A use of a variable in a term is said to be *bound* if it is inside a quantifier for that variable, and *free* otherwise.




WHY WE CARE (1)

Formulas with no free variables are true or false, depending on the model.

$$\forall x \exists s (x \in s)$$


(a proposition)

Formulas with free variables are true or false, depending on the model *and the values of the free variables*

$$\exists x (x \in s)$$


(a proposition “about s”)

WHY WE CARE (2)

The names of *bound* variables (mostly) don't matter

$$\begin{aligned}\exists x (x > y) &\equiv \exists z (z > y) \\ \exists x (x > y) &\equiv \exists k (k > y)\end{aligned}$$

WHY WE CARE (2)

The names of *bound* variables (mostly) don't matter

$$\begin{aligned} \boxed{\exists x (x > y)} &\equiv \exists z (z > y) \\ \exists x (x > y) &\equiv \exists k (k > y) \\ \exists x (x > y) &\not\equiv \exists x (x > z) \\ \exists x (x > y) &\not\equiv \exists x (x > k) \end{aligned}$$

WHY WE CARE (2)

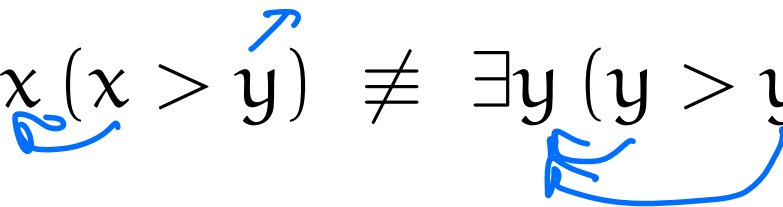
The names of *bound* variables (mostly) don't matter

$$\exists x (x > y) \equiv \exists z (z > y)$$

$$\exists x (x > y) \equiv \exists k (k > y)$$

$$\exists x (x > y) \not\equiv \exists x (x > z)$$

$$\exists x (x > y) \not\equiv \exists x (x > k)$$

$$\exists x (x > y) \not\equiv \exists y (y > y)$$


DIGRESSION: WHY IS THIS *FIRST-ORDER* LOGIC?

In first-order logic, we can quantify over individuals.

$$\forall x (P(x) \vee \neg P(x))$$

BUT WHEN IS A FORMULA TRUE?

The truth of $\forall x P(c, x)$ depends on our model:

- Model 1: the domain is natural numbers, c is the integer 0, and P is the “less-than-or-equal” relation. $\{0, 1, 2, 3, \dots\}$ t
- Model 2: the domain is people, c is Prof. Stone’s cousin Ken, and P is the “lives-next-door-to” relation? f

INGREDIENTS OF A MODEL, FORMALLY

1. The *domain* (**nonempty** set of individuals being considered);
2. For each function/constant symbol, a corresponding mathematical function/constant;
3. For each relation symbol, a corresponding mathematical relation on individuals.

(We require $=$ be interpreted as equality.)

TRUTH IN A MODEL

$P(t)$ is true iff the individual t has property P (according to the model)

$R(t, u)$ is true iff the individuals t and u are in relation R (according ...)

$\forall x P(x)$ is true iff *every* individual in the domain has property P

$\exists x P(x)$ is true iff *at least one* individual in the domain has property P

The propositional operators ($\vee, \wedge, \rightarrow, \neg$) follow the same truth tables.

END