Only one handout today!
(But first: finish previous handout)

# Classical Propositional Logic: Natural Deduction

August 30, 2023

CS 81: Computability and Logic

> Euclid taught me that without assumptions there is no proof. Therefore, in any argument, examine the assumptions.
>
> —E. t. Bell

# The Story So Far...

We were investigating *Classical Propositional Logic*.

- Language of Propositional Logic
  - We defined a special set of strings (containing $\mathbf{P}$'s, $\mathbf{Q}$'s, $\wedge$'s, $\perp$'s, *etc.*).
  - We called the strings in this set *Well-Formed Formulas* (WFFs).
- Classical Semantics (meaning/truth)
  - We defined a *model* to be a value of t or f for each propositional variable.
  - The truth of a WFF *relative to a model* is calculated using truth-table lookups.

| $\mathcal{A}$ | $\mathcal{B}$ | $(\mathcal{A} \wedge \mathcal{B})$ | $(\mathcal{A} \vee \mathcal{B})$ | $(\mathcal{A} \rightarrow \mathcal{B})$ |
|---|---|---|---|---|
| t | t | t | t | t |
| t | f | f | t | f |
| f | t | f | t | t |
| f | f | f | f | t |

| $\mathcal{A}$ | $\neg\mathcal{A}$ |
|---|---|
| t | f |
| f | t |

# Translating to Classical Propositional Logic

# Encode Symbolically

1. **I have vanilla, and I have chocolate.**
   $(P \wedge Q)$
   P is "I have vanilla"
   Q is "I have chocolate."
2. **The weather today is hot and humid.**
   $(P \wedge Q)$
   P is "The weather today is hot"
   Q is "The weather today is humid" (not: Q is "humid")
3. **Prof Dodds and Prof O'Neill like coffee and tea (respectively).**
   $(P \wedge Q)$
   P is "Prof Dodds likes coffee"
   Q is "Prof O'Neill likes tea"

# Encode Symbolically

4. I don't like chocolate.

$\neg P$ P= "I like chocolate"

5. The book is on the first or second shelf.

Q

P

$P \vee Q$

6. Neutral molecules have no positive or negative charge.

$\neg P \wedge \neg Q$ $\neg(P \vee Q)$

P is "Neutral molecules have a positive charge"
Q is "Neutral molecules have a negative charge"

# Encode Symbolically

7. If $\underbrace{\text{the weather is warm}}_{P}$, then $\underbrace{\text{I drink soda}}_{Q}$. $P \to Q$

8. $\underbrace{\text{I drink soda}}_{P}$ $\overset{\leftarrow}{\text{if}}$ $\underbrace{\text{the weather is warm}}_{Q}$. $Q \to P$

9. $\underbrace{\text{I drink soda}}_{P}$ $\overset{\longrightarrow}{\text{only if}}$ $\underbrace{\text{the weather is warm}}_{Q}$. $P \to Q$

10. $\underbrace{\text{I drink soda}}_{P}$ if and only if $\underbrace{\text{the weather is warm}}_{Q}$. $P \leftrightarrow Q$ $(P \to Q) \wedge (Q \to P)$

11. $\underbrace{\text{I drink soda}}_{P}$ iff $\underbrace{\text{the weather is warm}}_{Q}$. $P \leftrightarrow Q$ $(P \to Q) \wedge (Q \to P)$

# Encode Symbolically

12. (Definition) $n$ is *even* if $n$ is divisible by 2 with no remainder.

$\underbrace{\text{n is even}}_{P}$ iff $\underbrace{\text{n is divisible by 2 with no remainder}}_{Q}$

$P \leftrightarrow Q$

13. If $\underbrace{\text{you clean your room}}_{P}$, $\underbrace{\text{we will have ice cream}}_{Q}$.

$P \to Q$

(or $P \leftrightarrow Q$)

14. $\underbrace{\text{The spirit is willing}}_{P}$, but $\underbrace{\text{the flesh is weak}}_{Q}$.

$P \wedge Q$

# Recall: The *Entailment* Relation

Assumptions $\mathcal{A}_1, \ldots, \mathcal{A}_n$ *entail* a conclusion $\mathcal{B}$

when

in *every* situation where all the assumptions are true,
the conclusion is true.

**Equivalently:**

there's *no* counterexample situation where
the assumptions are true, but the conclusion is false.

In this case we write

$$\mathcal{A}_1, \ldots, \mathcal{A}_n \vDash \mathcal{B}$$

and say the inference from $\mathcal{A}_1, \ldots, \mathcal{A}_n$ to $\mathcal{B}$ is *valid*.

# Important Equivalences

Definition: $\mathcal{A}$ and $\mathcal{B}$ are logically *equivalent* if $\mathcal{A} \vDash \mathcal{B}$ and $\mathcal{B} \vDash \mathcal{A}$.

That is, $\mathcal{A} \equiv \mathcal{B}$ if they have the same truth value in *every* model.

Memorize the following *equivalences*:

$$
\begin{array}{lcll}
\neg\neg\mathcal{A} & \equiv & \mathcal{A} & \\
\mathcal{A} \to \mathcal{B} & \equiv & \neg\mathcal{A} \lor \mathcal{B} & \\
\mathcal{A} \to \mathcal{B} & \equiv & \neg\mathcal{B} \to \neg\mathcal{A} & \text{(Contrapositive)} \\
\neg(\mathcal{A} \land \mathcal{B}) & \equiv & (\neg\mathcal{A} \lor \neg\mathcal{B}) & \text{(DeMorgan's Law)} \\
\neg(\mathcal{A} \lor \mathcal{B}) & \equiv & (\neg\mathcal{A} \land \neg\mathcal{B}) & \text{(DeMorgan's Law)} \\
\neg(\mathcal{A} \to \mathcal{B}) & \equiv & \mathcal{A} \land \neg\mathcal{B} & \\
\neg\mathcal{A} & \equiv & \mathcal{A} \to \bot &
\end{array}
$$

# Checking for Entailment

Suppose I want to know whether

$$(P \to Q),\ \neg Q \models \neg P$$

We can consider all (four) relevant models.

Check: is the conclusion true in *every* model where no assumption is false?

| | P | Q | $(P \to Q)$ | $\neg Q$ | $\neg P$ | |
|---|---|---|---|---|---|---|
| (Model 1) | t | t | t | f | f | ✓ |
| (Model 2) | t | f | f | t | f | ✓ |
| (Model 3) | f | t | t | f | t | ✓ |
| (Model 4) | f | f | t | t | t | ✓ |

# The Problem with Entailment

If *we want to show*

$$(P \to Q), (Q \to R), (R \to S), (S \to T), (T \to U) \models (P \to U)$$

it is annoying *to enumerate* all $2^5 = 32$ truth assignments.

[When *we* later *add* quantifiers ($\forall$ *and* $\exists$) there will *be* ***infinitely*** many *models to check*!]

# THE *PROVABILITY* RELATION

Assumptions $\mathcal{A}_1, \ldots, \mathcal{A}_n$ *prove* a conclusion $\mathcal{B}$

if

*we can validate the conclusion using*

*the assumptions and a fixed set of rules.*

In this case we write

$$\mathcal{A}_1, \ldots, \mathcal{A}_n \vdash \mathcal{B}$$

Formal proof is pure symbol manipulation. No need for truth tables!

# Example

Suppose we want to verify that $P,\ P \rightarrow Q,\ Q \rightarrow R \vDash R$.

MP: if $A \rightarrow B$ and $A$ then $B$

- **Option 1.** Check all models.

| P | Q | R | P | P → Q | Q → R | R |
|---|---|---|---|---|---|---|
| t | t | t | t | t | t | t |
| t | t | f | t | t | f | f |
| t | f | t | t | f | t | t |
| t | f | f | t | f | t | f |
| f | t | t | f | t | t | t |
| f | t | f | f | t | f | f |
| f | f | t | f | t | t | t |
| f | f | f | f | t | t | f |

- **Option 2.** Show $P,\ P \rightarrow Q,\ Q \rightarrow R \vdash R$ using *sound* (truth-preserving) rules.

| | | |
|---|---|---|
| 1. | $P$ | assumption |
| 2. | $P \rightarrow Q$ | assumption |
| 3. | $Q \rightarrow R$ | assumption |
| 4. | $Q$ | Modus Ponens (lines 1 and 2) |

# Example

Suppose we want to verify that $P,\ P \to Q,\ Q \to R \vDash R$.

- **Option 1.** Check all models.

| $P$ | $Q$ | $R$ | $P$ | $P \to Q$ | $Q \to R$ | $R$ |
|---|---|---|---|---|---|---|
| t | t | t | t | t | t | t |
| t | t | f | t | t | f | f |
| t | f | t | t | f | t | t |
| t | f | f | t | f | t | f |
| f | t | t | f | t | t | t |
| f | t | f | f | t | f | f |
| f | f | t | f | t | t | t |
| f | f | f | f | t | t | f |

- **Option 2.** Show $P,\ P \to Q,\ Q \to R \vdash R$ using *sound* (truth-preserving) rules.

| | | |
|---|---|---|
| 1. | $P$ | assumption |
| 2. | $P \to Q$ | assumption |
| 3. | $Q \to R$ | assumption |
| 4. | $Q$ | Modus Ponens (lines 1 and 2) |
| 5. | $R$ | Modus Ponens (lines 3 and 4) |

# Which Rules Should be "Built In" as a single step?

- Modus Ponens: "if $\mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{A}$, then $\mathcal{B}$". Seems useful.
- What about "if $\mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{B} \rightarrow \mathcal{C}$, then $\mathcal{A} \rightarrow \mathcal{C}$" ?
- What about "if $\mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{B} \rightarrow \mathcal{C}$ and $\mathcal{C} \rightarrow \mathcal{D}$, then $\mathcal{A} \rightarrow \mathcal{D}$" ?
- What about "if $(\mathcal{A} \vee \mathcal{D}) \rightarrow (\mathcal{B} \vee \mathcal{C})$ and $\neg\mathcal{B}$ and $\neg\mathcal{C}$, then $(\mathcal{A} \rightarrow \neg\mathcal{D})$" ?

Logicians study various *proof systems*.

- Small sets of built-in rules that never steer us wrong.
- Lots of room for choice.
  E.g., is $\vdash \mathcal{P} \rightarrow (\mathcal{Q} \rightarrow \mathcal{P})$ built-in, or something to be proved?

# Relating Proof and Entailment

A proof system is said to be *sound* if

$$(\mathcal{A}_1, \ldots, \mathcal{A}_n \vdash \mathcal{B}) \quad \text{implies} \quad (\mathcal{A}_1, \ldots, \mathcal{A}_n \vDash \mathcal{B})$$

(i.e., if all provable conclusions are correct conclusions).

A proof system is said to be *complete* if

$$(\mathcal{A}_1, \ldots, \mathcal{A}_n \vDash \mathcal{B}) \quad \text{implies} \quad (\mathcal{A}_1, \ldots, \mathcal{A}_n \vdash \mathcal{B})$$

(i.e., if all correct conclusions things are provable conclusions).

# Natural Deduction

Small toolkit of proof rules that mimic human reasoning!

**Today**: the *constructive rules* of Natural Deduction

- Organized into *introduction* and *elimination* rules:

  | | |
  |---|---|
  | How do we prove a conjunction? | $\wedge$I |
  | What can we conclude from a conjunction? | $\wedge$E |
  | How do we prove a disjunction? | $\vee$I |
  | What can we conclude from a disjunction? | $\vee$E |
  | $\vdots$ | $\vdots$ |

- Rules are (mostly) independent: the $\wedge$ rules only mention $\wedge$, the $\to$ rules only mention $\to$, etc.

**Next Week**: the *non-constructive* (or *classical*) rules of Natural Deduction.

# Inference Rule Notation

It is traditional to describe "single step" inferences as follows:

$$\frac{\text{Premise}_1 \qquad \cdots \qquad \text{Premise}_n}{\text{Conclusion}}$$

meaning, "if the specified premises are true, we can immediately claim the specified conclusion is true."

$$\frac{A \to B \qquad A}{B}\ \text{MP}$$

# Conjunction: $\wedge$E and $\wedge$I

$$\frac{\mathcal{A} \quad \mathcal{B}}{\mathcal{A} \wedge \mathcal{B}} \wedge\mathrm{I} \qquad \frac{\mathcal{A} \wedge \mathcal{B}}{\mathcal{A}} \wedge\mathrm{E} \qquad \frac{\mathcal{A} \wedge \mathcal{B}}{\mathcal{B}} \wedge\mathrm{E}$$

PROVE: $P \wedge Q \vdash Q \wedge P$

| | | |
|---|---|---|
| 1. | $P \wedge Q$ | Assumption |
| 2. | $P$ | $\wedge E$ 1 |
| 3. | $Q$ | $\wedge E$ 1 |
| 4 ~~3~~. | $Q \wedge P$ | $\wedge I$ 2,3 |

PROVE: $P \wedge (Q \wedge R) \vdash (P \wedge Q) \wedge R$

| | | |
|---|---|---|
| 1. | $P \wedge (Q \wedge R)$ | |
| 2. | $P$ | $\wedge E$ 1 |
| 3. | $Q \wedge R$ | $\wedge E$ 1 |
| 4. | $Q$ | $\wedge E$ 3 |
| 5. | $R$ | $\wedge E$ 3 |
| 6. | $P \wedge Q$ | $\wedge I$ 2, 4 |
| 7. | $(P \wedge Q) \wedge R$ | $\wedge I$ 6, 5 |

# Implication: $\to$E and $\to$I

$$\dfrac{\begin{array}{|l} \mathcal{A} \\ \hline \vdots \\ \mathcal{B} \end{array}}{\mathcal{A} \to \mathcal{B}}\ {\to}\mathrm{I}$$

$$\dfrac{\mathcal{A} \to \mathcal{B} \qquad \mathcal{A}}{\mathcal{B}}\ {\to}\mathrm{E}$$

PROVE: $P \to Q,\ Q \to R \vdash P \to R$

| | | |
|---|---|---|
| 1. | $P \to Q$ | |
| 2. | $Q \to R$ | |
| 3. | $P$ | |
| 4 | $Q$ | $\to E$ 1,3 |
| ~~4~~ 5 | $R$ | $\to E$ 2,4 |
| 6 ~~5~~ | $P \to R$ | $\to I$ 3-5 |

PROVE: $\vdash ((P \wedge Q) \to R) \to (P \to (Q \to R))$

| | | |
|---|---|---|
| 1. | $(P \wedge Q) \to R$ | |
| 2. | $P$ | |
| 3. | $Q$ | |
| 4. | $P \wedge Q$ | $\wedge I$ 2,3 |
| 5. | $R$ | $\to E$ 1,4 |
| 6. | $Q \to R$ | $\to I$ 3-5 |
| 7. | $P \to (Q \to R)$ | $\to I$ 2-6 |
| 8. | $((P \wedge Q) \to R) \to (P \to (Q \to R))$ | $\to I$ 1-7 |

# Disjunction: $\lor$I and $\lor$E

$$\frac{A}{A \lor B}\ \lor\text{I} \qquad \frac{B}{A \lor B}\ \lor\text{I} \qquad \frac{A \lor B \qquad \begin{array}{|l} A \\ \hline \vdots \\ C \end{array} \qquad \begin{array}{|l} B \\ \hline \vdots \\ C \end{array}}{C}\ \lor\text{E}$$

PROVE: $P \vee Q \vdash Q \vee P$

| | | | |
|---|---|---|---|
| 1. | $P \vee Q$ | | |
| 2. | $P$ | | |
| | ⋮ | | |
| 3 | $Q \vee P$ | $\vee I$ | 2 |
| 4. | $Q$ | | |
| | ⋮ | | |
| 5. | $Q \vee P$ | $\vee I$ | 4 |
| n. | $Q \vee P$ | $\vee E$ | 1, 2-3, 4-5 |

# Negation and Contradiction: ¬I, ¬E, ⊥E

$$\dfrac{\begin{array}{|l} A \\ \hline \vdots \\ \bot \end{array}}{\neg A}\ \neg\mathrm{I} \qquad \dfrac{\neg A \qquad A}{\bot}\ \neg\mathrm{E}\ \&\ \bot\mathrm{I} \qquad \dfrac{\bot}{A}\ \bot\mathrm{E}$$

$\neg A \equiv A \to \bot$

PROVE: $P \vee Q, \neg P \vdash Q$

| | | |
|---|---|---|
| 1. | $P \vee Q$ | |
| 2. | $\neg P$ | |
| 3. | $\quad \vert\ \underline{P}$ | |
| 4. | $\quad \vert\ \bot$ | $\neg E$ 2,3 |
| 5 | $\quad \vert\ Q$ | $\bot E$ 4 |
| 6. | $\quad \vert\ \underline{Q}$ | |
| 7 | $\quad \vert\ Q$ | R 6 |
| 8 | $Q$ | |

PROVE: $\vdash P \rightarrow \neg\neg P$

PROVE: $\neg(P \lor Q) \vdash \neg P \land \neg Q$