Natural Deduction, Continued

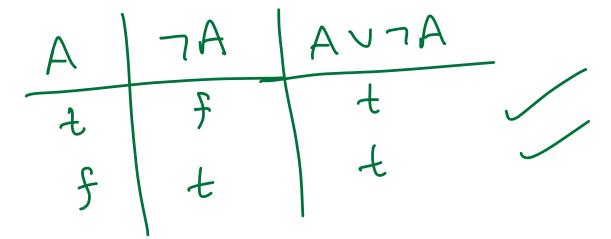
September 7, 2023 CS 81: Computability and Logic

While mathematical logic and set theory indeed make up the language spanning all fields of mathematics, mathematicians rarely speak it. To borrow notions from computers, basic mathematical logic can be regarded as the "machine language" for mathematicians who usually use much higher-level languages

—Gil Kalai

Warmup: Sh	IOW THA	AT $(P \rightarrow$		
P	Q	P->Q	(P-1Q)->P	P
+	1	7	+ +	\ t
t	f	f	£	\ f /
5	 	t	F 1	\ f \
f	}	\ t	\	

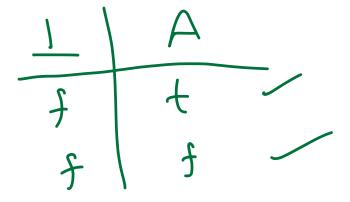
Show that $\models A \lor \neg A$



Show that $\neg A, A \models \bot$

assumptions never true!

Show that $\bot \models A$



TRUTH VS. PROOF

We say that $A \models B$ if B is true whenever A is.

We say that $A \vdash B$ if B is provable (step by step) from A.

Ideally, we will have

- If $A \vdash B$, then $A \models B$ (soundness).
- If $A \models B$, then $A \vdash B$ (completeness).

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Being able to check soundness and completeness requires us to precisely define

- What it means for ${\mathbb B}$ and ${\mathcal A}$ to be formulas. \checkmark
- What it means for $\mathcal B$ and $\mathcal A$ to be true. \checkmark
- What counts as a proof of ${\mathfrak B}$ from ${\mathcal A}$.

PROOF SYSTEMS

A *proof system* is a small toolkit of rules (proof steps) that allowed in a formal proof. Logicians have studied many *different* proof systems.

- Natural Deduction
- Resolution
- Hilbert Systems, Sequent Calculus, Tableaux, ...

Technically, the notation $\mathcal{A} dash \mathfrak{B}$ is ambiguous, and we should say

- $\mathcal{A} \vdash_{\mathsf{ND}} \mathcal{B}$ if \mathcal{B} is provable from \mathcal{A} using Natural Deduction steps
- $A \vdash_R B$ if B is provable from A using Resolution steps
- · etc.

but context usually tells us what proof system we mean.

WHY ARE THERE MULTIPLE PROOF SYSTEMS?

In terms of what you can prove, it doesn't matter.

• All the systems mentioned are sound and complete so, e.g.,

$$A \vdash_{ND} B$$
 iff $A \vDash B$ iff $A \vdash_{R} B$

WHY ARE THERE MULTIPLE PROOF SYSTEMS?

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All the systems mentioned are sound and complete so, e.g.,

$$\mathcal{A} \vdash_{\mathsf{ND}} \mathcal{B} \quad \mathsf{iff} \quad \mathcal{A} \models \mathcal{B} \quad \mathsf{iff} \quad \mathcal{A} \vdash_{\mathsf{R}} \mathsf{B}$$

But different systems organize proofs differently, e.g.,

- The 12 rules of Natural Deduction mimic how humans write proofs.
- Resolution uses 1 rule over and over, and is easier for computers.

RECALL: THE "CONSTRUCTIVE" NATURAL DEDUCTION RULES

$$\frac{\mathcal{A}}{\mathcal{A} \wedge \mathcal{B}} \wedge \mathbf{I}$$

$$\frac{A \wedge B}{A} \wedge E$$

$$\frac{A \wedge B}{B} \wedge E$$

$$\frac{\mathcal{A} \quad \mathcal{B}}{\mathcal{A} \wedge \mathcal{B}} \wedge I \qquad \frac{\mathcal{A} \wedge \mathcal{B}}{\mathcal{A}} \wedge E \qquad \frac{\mathcal{A} \wedge \mathcal{B}}{\mathcal{B}} \wedge E \qquad \frac{\mathcal{A} \rightarrow \mathcal{B}}{\mathcal{B}} \rightarrow E$$

$$\frac{\mathcal{A}}{\mathcal{A}\vee\mathcal{B}}\vee I$$

$$\frac{\mathcal{B}}{\mathcal{A}\vee\mathcal{B}}\vee I$$

$$\frac{\mathcal{A}}{\mathcal{A} \vee \mathcal{B}} \vee I$$
 $\frac{\mathcal{B}}{\mathcal{A} \vee \mathcal{B}} \vee I$ $\frac{\neg \mathcal{A}}{\mathcal{A}} \rightarrow E$ $\frac{\bot}{\mathcal{A}} \perp E$

$$\frac{\perp}{A}$$
 \perp E

$$\frac{\begin{vmatrix} \mathcal{A} \\ \vdots \\ \mathcal{B} \\ \mathcal{A} \to \mathcal{B} \end{vmatrix} \to I$$

$$\frac{\begin{vmatrix} \mathcal{A} \\ \vdots \\ \bot \\ \neg \mathcal{A} \end{vmatrix}}{\neg \mathcal{A}} \neg \mathbf{I}$$

JUSTIFYING TE AND LE

We showed that according to the truth-table definition of entailment,

- ¬A, A ⊨ ⊥
- ⊥ ⊨ A.

Because Natural Deduction should be complete (prove all correct conclusions) we demand

- $\neg A$, $A \vdash_{ND} \bot$
- \perp \vdash_{ND} A.

To make these provable, we need the rules $\neg E$ and $\bot E$.

$$\frac{\neg A}{\bot} \frac{A}{\bot} \neg E \qquad \frac{\bot}{A} \bot$$

Prove $\vdash \neg (P \land \neg P)$ using Natural Deduction

$$\begin{array}{c|cccc}
1 & P & NE & on & 1 \\
\hline
2 & P & NE & on & 1 \\
\hline
3 & P & NE & on & 1 \\
\hline
4 & I & NE & on & 2 & and & 3 \\
\hline
5 & 7(P \land P) & 7I & on & 1-4
\end{array}$$

TERMINOLOGY

A formal proof involves WFFs and a steps in a proof system (e.g., Natural Deduction)

- Purely a matter of rule-following and symbol manipulation
- "Mathematics is the science in which we do not know what we are talking about, and do not care whether what we say about it is true." —Bertrand Russell

So, a proof written by humans for humans is technically an *informal proof*.

- Informal doesn't mean sloppy or careless!
- Combination of English (or another natural language) and symbolic notation.
- Assumes the reader can fill in any missing steps or justifications.

Burgess: A rigorous proof is something that convinces us a formal proof exists.

Can you see $\rightarrow I$, $\land E$, and $\rightarrow E$ moves?

Theorem: If $x \in S \cap T$ and $S \subseteq U$ and $T \subseteq V$, then $x \in U \cap V$.

Proof. Suppose $x \in S \cap T$ and $S \subseteq U$ and $T \subseteq V$

Then $x \in S$, and so $x \in U$.

Similarly, $x \in T$, and so $x \in V$.

Therefore, $x \in U \cap V$.

$$x \in (S \cap T)$$
 means $(x \in S) \land (x \in T)$
 $S \subseteq U$ means $(x \in S) \rightarrow (x \in U)$
 $T \subseteq V$ means $(x \in T) \rightarrow (x \in V)$

Can you see \rightarrow I, \bigvee E, \rightarrow E, and \bigvee I moves?

Theorem: If $x \in S \cup T$, $S \subseteq U$, and $T \subseteq V$ then $x \in U \cup V$.

Proof. Suppose $x \in S \cup T$, $S \subseteq U$, and $T \subseteq V$.

Since $x \in S \cup T$, we consider two cases:

- If $x \in S$, then $x \in U$ and so $x \in U \cup V$.
- If $x \in T$, then $x \in V$ and so $x \in U \cup V$.

Since one of these two cases must hold, $x \in U \cup V$.

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x \in (S \cup T) means (x \in S) \lor (x \in T)

S \subseteq U means (x \in S) \to (x \in U)

T \subseteq V means (x \in T) \to (x \in V)

x \in (U \cup V) means (x \in U) \lor (x \in V)
```

CAN YOU SEE I AND E MOVES?

Theorem: $\sqrt{2}$ is irrational (i.e., not equal to any fraction).

Proof. Suppose $\sqrt{2}$ were rational. Then $\sqrt{2}=\frac{a}{b}$ for a simplified fraction with $b\neq 0$.

Then $2b^2=\alpha^2$, so α^2 is even, so α is even.

But if α is even then α^2 is a multiple of 4,

so $2b^2$ is a multiple of 4, which means b is a multiple of 2.

Then α and b are both even, but $\frac{\alpha}{b}$ is simplified so α and b aren't both even, a contradiction.

Therefore, $\sqrt{2}$ is not rational. QED.

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NATURAL DEDUCTION SO FAR

The introduction and elimination rules give us Constructive Propositional Logic

There are valid entailments we still can't prove, including

$$\neg\neg P \vdash P$$

$$P \to Q \vdash \neg P \lor Q$$

$$(P \to Q) \to P \vdash P$$

That is, the rules so far are sound, but not complete.

To get full Classical Propositional Logic, we need at least one more rule.

A Non-Constructive Argument

Theorem. There exist irrational α and b such that α^b is rational.

Proof. We saw that $\sqrt{2}$ is irrational.

Either $\sqrt{2}^{\sqrt{2}}$ is rational or it is not.

- If it is rational, we can take $\alpha = \sqrt{2}$ and $b = \sqrt{2}$.
- If it is irrational, then we can take $\alpha=\sqrt{2}^{\sqrt{2}}$ and $b=\sqrt{2}$, because

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2.$$

QED.

THREE CLASSICAL RULES

Add any one of these to ND, and the other two become theorems:

$$\frac{1}{A \vee \neg A} \text{ LEM} \qquad \frac{\neg \neg A}{A} \text{ DNE} \qquad \frac{\bot}{A} \text{ IP}$$

WARNING

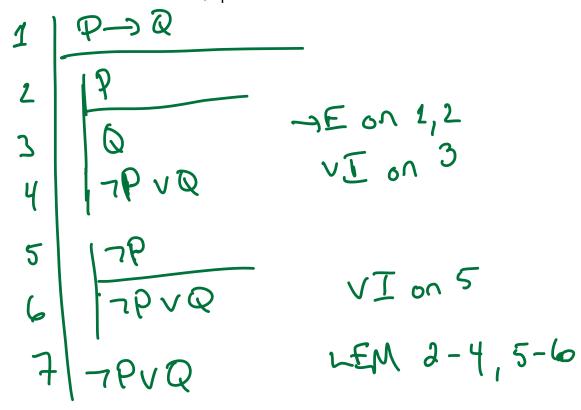
proofs.openlogicproject.org uses a non-standard LEM rule:

$$\begin{array}{c|c}
 & \neg A \\
 \vdots & \vdots \\
 & \mathcal{B} \\
\hline
 & \mathcal{B}
\end{array}$$
LEM

Combines our LEM with an immediate $\bigvee E$ proof-by-cases.

Prove: $P \rightarrow Q \vdash \neg P \lor Q$

Hint: this is not constructively provable.



PROVE: $\vdash \neg \neg P \rightarrow P$

Hint: this is not constructively provable

