Homework 6

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Problem 1.

Solution: Compute the log likelihood of the data:

$$\ell\left(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k\right) = \sum_k \sum_i r_{ik} \log \mathbb{P}\left(\mathbf{x}_i \mid \boldsymbol{\theta}_k\right) = -\frac{1}{2} \sum_i r_{ik} \left(\log \left|\boldsymbol{\Sigma}_k\right| + \left(\mathbf{x}_i - \boldsymbol{\mu}_k\right)^\top \boldsymbol{\Sigma}_k^{-1} \left(\mathbf{x}_i - \boldsymbol{\mu}_k\right)\right)$$

Taking the gradient w.r.t μ_k:

$$\frac{\partial \ell}{\partial \mu_k} = \sum_i r_{ik} \boldsymbol{\Sigma}_k^{-1} \left(\mathbf{x}_i - \boldsymbol{\mu}_k\right)$$

Since $\boldsymbol{\Sigma}_k^{-1}$ is linear:

$$\frac{\partial \ell}{\partial \mu_k} = \boldsymbol{\Sigma}_k^{-1} \sum_i r_{ik} \left(\mathbf{x}_i - \boldsymbol{\mu}_k\right) = 0 \leftrightarrow \sum_i r_{ik} \mathbf{x}_i = \boldsymbol{\mu}_k \sum_i r_{ik}$$

as expected.

Now, taking the gradient w.r.t Σ_k:

$$\frac{\partial \ell}{\partial \Sigma_k} = -\frac{1}{2} \sum_i r_{ik} \left(\boldsymbol{\Sigma}_k^{-1} - \Sigma_k^{-1} \left(\mathbf{x}_i - \boldsymbol{\mu}_k\right) \left(\mathbf{x}_i - \boldsymbol{\mu}_k\right)^\top \boldsymbol{\Sigma}_k^{-1}\right) = 0$$

$$\leftrightarrow \sum_i r_{ik} I = \left(\sum_i r_{ik} \left(\mathbf{x}_i - \mu_k\right) \left(\mathbf{x}_i - \mu_k\right)^\top\right) \Sigma_k^{-1}$$

Multiplying by Σ_k on the right and dividing by $r_k = \sum_i r_{ik}$, we have:

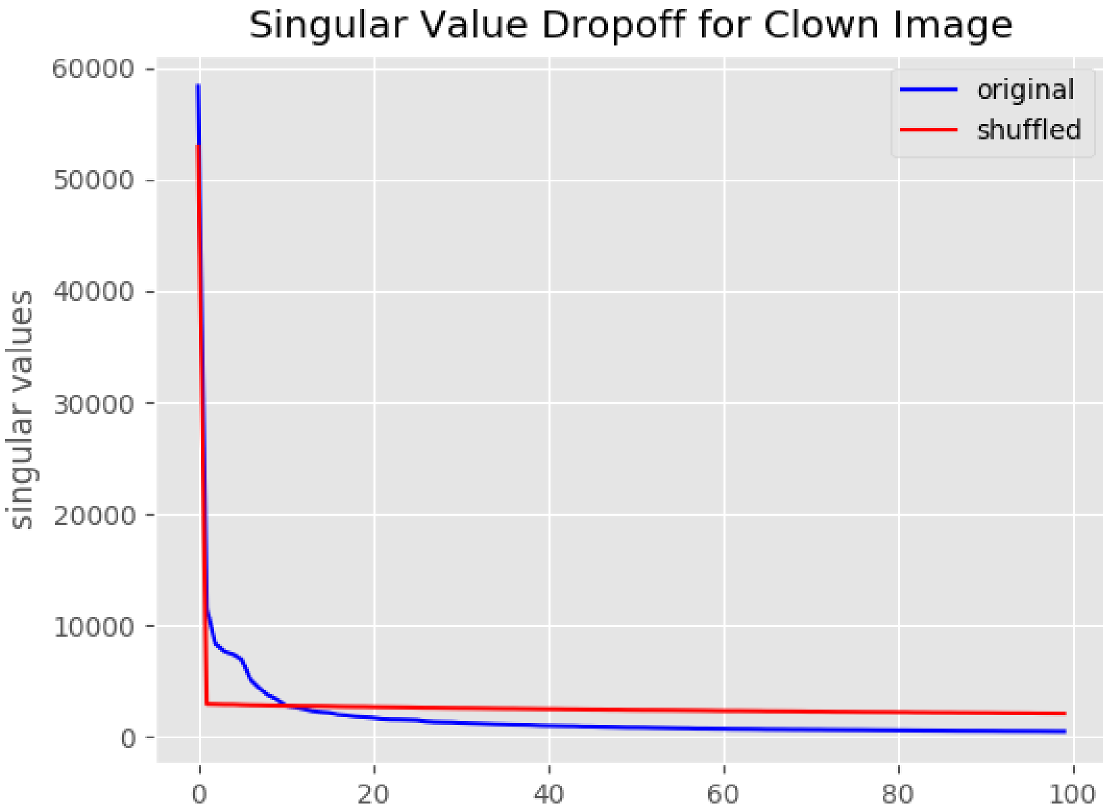
$$\boldsymbol{\Sigma}_k = \frac{1}{r_k} \sum_i r_{ik} \mathbf{x}_i \mathbf{x}_i^\top - r_k \boldsymbol{\mu}_k \boldsymbol{\mu}_k^\top$$

as expected. □

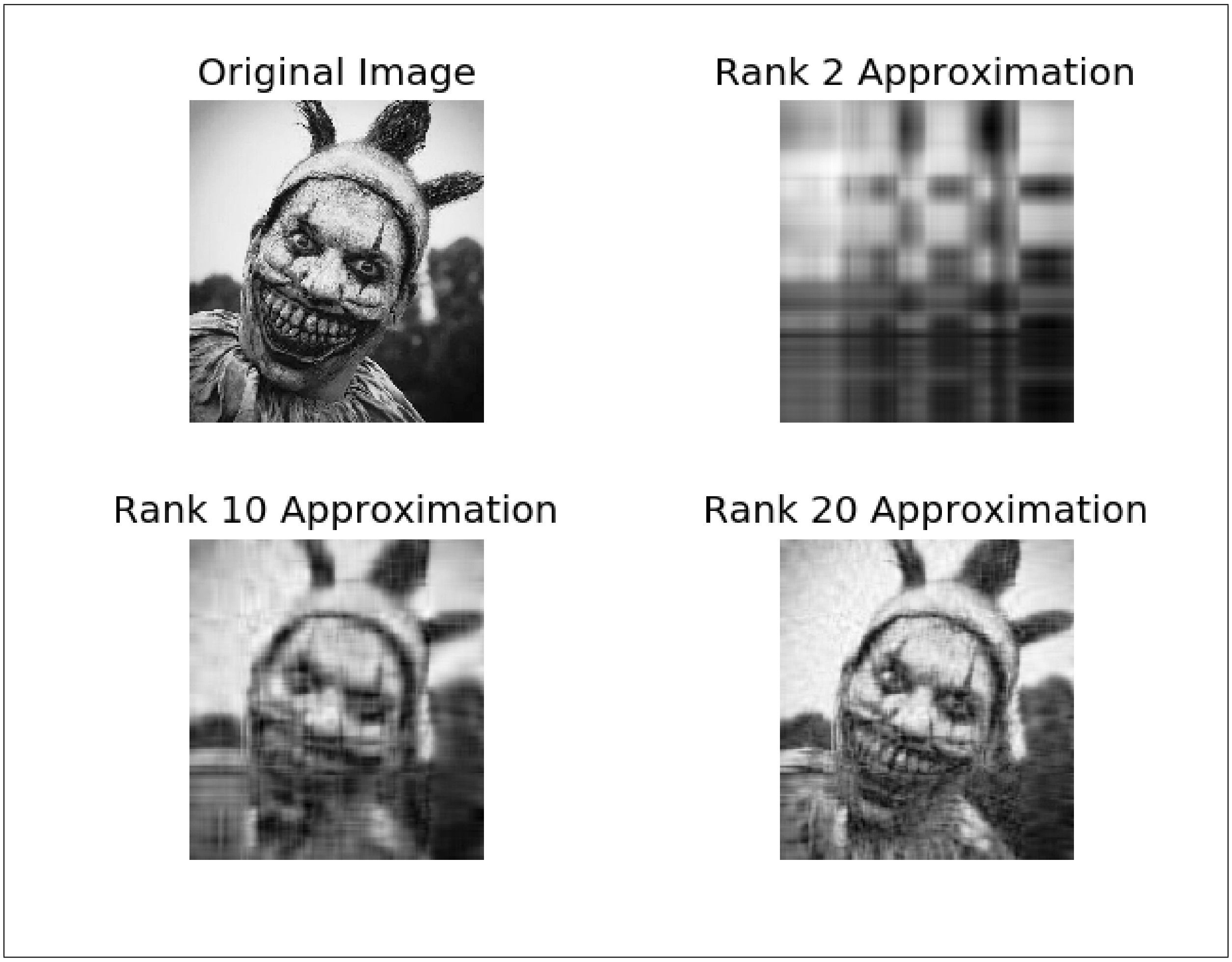
Problem 2.

Solution: See Github repo for code.

Plot the progression of the 100 largest singular values for the original image and a randomly shuffled version of the same image (all on the same plot).



In a single figure plot a grid of four images: the original image, and a rank k truncated SVD approximation of the original image for $k \in \{2, 10, 20\}$.


Original Image
Rank 2 Approximation
Rank 10 Approximation
Rank 20 Approximation