Strategic Exercise in Dynamics: Overbuilding, Government Intervention and Hesitating Follower in Real Estate Market

In the past decades, game-theoretic approach to option exercise has been widely developed to explain the strategic interactions between players under dynamic uncertainty. Most decision making could be generalized with a real-option, decision-maker has to consider the uncertainty not in a static view, but to consider it as an uncertain path on the way, and decide whether to exercise an opportunity in her hand with an irreversible cost immediately. These kinds of decisions often build upon an underlying random variable whose dynamics over time could be modeled with a stochastic process incorporating Brwonian motion or Jump. As a result, there is often a trigger value of that underlying variable upon which it is reasonable for the decision maker to exercise her option. With more than one decision maker, it is often found that their decision individually will affect the behavior of the other. These games generally will produce equilibrium exercise of options, because they face identical industry-wide uncertainty, and their asymmetric characteristics will pick up the leader and follower.

In this article, I demonstrate that a dynamic-system approach could be used to describe how the equilibrium strategies drive players to evolve with time. Instead of emphasizing the impact of uncertainty on players' behaviors, I try to build my model upon equilibrium strategies developed by others. I focus on the dynamics of the equilibrium strategies. Different from the option-exercise approach where people concerned with usually one option, my approach assumes identical options over time, an identical option is often acquired by the players time and time again, and each time players got the same game to play, the equilibrium strategy will thus produce a continuous path of the players' behavior. In order to provide more applicability and intuition, I will use the real estate market as my real-world example. The approach I used will make the implication of the equilibrium strategies easier to spot, and facilitate the discussion of market disturbance and long-term equilibrium. For example, real estate market often witness several years of idle development but could suddenly burst into development cascades, during such periods, government intervention often has little power. Likewise, such development cascades could last for years and market gradually restore its peace, before the final settlement, fluctuating market feature often rises, real estate developers hesitate to build more or less, and such hesitation shrinks stage by stage. At last, winner and loser are picked up, a so-called "reshuffling" phenomenon. The model I uses could also incorporate the case of downward jumping of the market, and it is seen that a probability of downward jumping will not change the market feature but just slow down the development speed, which could possibly explain why real estate competitors still build more in the face of weak economy.

These market features, as I will show, are mainly driven by the fear of preemption by competitors. Real-option and game theory researches on this aspect have provided vital background for my analysis. Among those researches, Grenadier(1996) is a significant cornerstone. Grenadier analyzes the case of two symmetric real estate developers who lease their existing properties and each holds the option to develop a new, superior building with a construction cost and a period to build, during which no rental income comes in. With new building in the market, the leader will have a monopolistic power and the follower's old building will face a decreased rental income. Grenadier has shown that under an industry-wide uncertain demand, both sequential and simultaneous exercise is possible. There are two trigger value of underlying demand: the first triggers the development of new building of the leader. The second, which is higher than the first, triggers the follower's development of building. Between the two trigger values, sequential exercise is realized. Upon the second trigger value, both developers are indifferent from building new and sit tight, because any exercise will cause the other to exercise immediately. Such equilibriums are infinite on a continuous region and are featured by an idle development of the real estate market. On the other hand, these equilibriums are often unstable, under market disturbance, one could starts to build and the other will follow immediately. With a decreased market demand, where a leader will not be followed immediately, it is reasonable for both to be the leader, and the leader will exercise her option while the follower has to wait the market demand climbs up to the second trigger value again.

Pawlina and Kort(2002) has developed the model to allow sunk cost asymmetry in the game. They proved that the first-mover advantage grows with cost advantage. They have also shown that a cost disadvantage decrease follower's intention to compete but could possibly increase her relative value to the leader and her decision could have bigger effect on the leader's decision. Kong and Kwok(2006) further the research to incorporate revenue flow asymmetry. They have shown that preemptive and simultaneous strategies could be easily presented using the asymmetry as a parameter.

Following other approach, DeCoster and Strange(2010) have two possible reasons for overbuilding —— statistical and reputation-based. The first arises because competitors tend to learn from other's decisions and this kind of emulating and herding is often uninformed and irrational, followers choose to ignore their signals and rather to follow immediately in a fear of preemption. Reputation-based, which in the contrast is an informed choice, arises because developers want to make it harder for banks to discern their real quality or they are afraid that banks could draw wrong conclusions of their well-beings. Both reasons reflect the feature of preemption and immediate following of developers. Wang and Zhou (2000) followed a purely game-theoretic approach. They argue that the vacant land does not produce income and it's natural for developers to build with development opportunity. Under competition, developers fight for such opportunities and their fight lead to oversupply. Under oversupply, developers wait for the market demand to grow and absorb the remaining units, and then they start to overbuild again. Such phenomenon will appears periodically.

In this article, I try to use a dynamic system to describe how these strategies evolving with time. In particular, the feature that fears of preemption leads to immediate exercise of competitors is shared by all these articles. Using a dynamic system, it is possible to consider all these strategies more generally with the help of phase space. Every state could be easily visualized in the phase space, a state vector is determined by the system, and such vector could denote both the direction and the magnitude of the change at that state. The past and future of an evolutionary process are uniquely determined by its present state. Instead of focusing on a single solution, we can easily extend our discussion to more general case and visualize the long-term behavior of a system. We will have a symmetric system to describe the competition, and we still have chance to discuss the asymmetry.

More specially, I model the evolutionary processes with a first order 2 by 2 differential system[1]. For each developer, the derivative of her development, namely her tendency to change is affected by both his current state and her competitor's current state. Her tendency to change with respect to her competitors state reflects the feature that she is afraid of being preempted; her tendency to change with respect to the state of herself reflects her idiosyncrasy, which could be affected by the government intervention and her cost or revenue disadvantage. Besides these, both idiosyncratic and industry-wide uncertainty could be included to form a 2 by 2 inhomogeneous differential system.

The article is organized as follows. In section I we will give a summary of the model we use and gives the background of the game, including the competition the two developers have and how initial market equilibrium is reached. In section II we analyze the basic model about overbuilding where developers will fight for preemption. In section III we extend our model to include the industry-wide feedback for developers, featured with the case of government intervention to the market. In section IV we discuss how comparable advantage would help one developer to be winner in the long-run and discuss how the loser could use mixed strategy to maximize her value. Section V will make comments on the results, give some possible extensions and draw a conclusion for the paper.

I. Summary of The Model

The differential system is used to illustrate the case of real estate market. At the beginning, there are two distinct real estate developers in local real estate market. They have options to build new units of buildings or redevelop existing buildings into new and superior ones. And there are more options to come in the future. Due to their own characteristic, they would have a normal level of developing, under such level they have a stable income and have no incentive to build more or less.

If one choose to build first and become the leader. Normally, the other, for the fear of preemption would follow her move instantaneously thereafter and become the follower. Such equilibrium is assumed to be simultaneous exercise. But in general, the follower has a chance to maximize her value when leader starts to build. And under disturbance, it might be optimal for the follower to wait. In the case that both wants to be leader, it is assumed that the leader will be picked up randomly with coin tossing. Such random toss could be backed up by the situation that a real estate development needs the approval of local government and such approval usually comes in a line.

New buildings face a downward-sloping demand. New buildings will increase expected cash flow of the developer if her competitor does not follow. But as long as two developers exercise their options simultaneously, the value of their exercised options will decrease due to increased supply. And it's reasonable to assume that during the idle period where no developers choose to exercise her option, the value of the exercised options is equal to the cost of exercising. (Building cost, advertisement cost etc.)

[1] With matrix language, it's easy to extend a 2 by 2 system to n by n. But a duopoly case is good enough to capture the market feature. More essentially, a 2 by 2 system could produce a 2 dimensional phase portrait that is easier to relate.

Under positive demand shock, it could be possible that simultaneous exercise still produce positive cash flow for both developers. And they will exercise their options simultaneously as long as such opportunity is recognized. On the other hand, a negative demand shock could make the follower to have a negative cash flow if she follows and she would choose to wait until the demand climbs up to normal level. (Of course if she exercise, she and the leader will both face negative cash flows, but she is maximizing her own value so she chose to wait.) Both competitors then fight to be the leader since they know the follower would wait, and she would have a period to enjoy her monopoly power. Such new development under a decreased demand is featured by so-called "Recession-induced Construction Boom". Grenadier(1996) provides a rational underpinning for such phenomenon.

We choose x and y to denote respectively the departure from normal developing level of the first developer and the second. Therefore, a positive x means that the first developer builds more than she normally does, and negative x means she builds less than normal.

Thus the dynamics of their development could be described by the following system:

$$\begin{cases} \dot{x} = \mathrm{f}(x, y) \\ \dot{y} = \mathrm{g}(x, y) \end{cases} \tag{1.1}$$

where $\dot{x}$ is the derivative of x with respect to time. $\mathrm{f}(x, y)$ and $\mathrm{g}(x, y)$ could be nonlinear functions. And if they are linear, which would be our main discussion, the system could be written with matrix language:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{r}(t) \tag{1.2}$$

where $\mathbf{x} = [x, y]^{tr}$. $\mathbf{A}$ is a 2 by 2 matrix, it describes how $\dot{\mathbf{x}}$ responds to current level of $\mathbf{x}$. $\mathbf{r}(t)$ is a 2 by 1 vector input to the system. Furthermore, we could write the matrix $\mathbf{A}$ as follows:

$$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \tag{1.3}$$

this is a symmetric matrix where the elements on the back-diagonal are the same. Specially, the value on the up-right corner corresponds to the impact of y on $\dot{x}$, and the value of the down-left corner corresponds to the impact of x on $\dot{y}$. These two values are same to capture developers' behavior of simultaneous exercise: both of them would build as much as the other has built. The value a and b, on the other hand, are the feedback to developers, they are supposed to be negative. For example, if x is positive, which means the first developer build more than she normally does, she could face the punishment from government, and that force her to return to her normal level of development. In such case, where feedback are introduced by government intervention, a and b should be same since government intervention is industry-wide. But each developers could have different cost and expected revenue as we mentioned. Then a and b are different to reflect the asymmetry.

Our discussion in following sections are mainly based on the analysis to matrix $\mathbf{A}$. Specially, I use phase portrait to show how different disturbance lead to different outcome, and since our system is deterministic, any current state not only describes its future but also uniquely determine its past. And we are then able to read the story form the graph. By analyzing stability of the system, we are able to know how different current state lead to different long-term behavior. Especially, I decoupled the fundamental case to introduce two new variables, different from x and y on the surface, those two variables are more to the essence of the system, and their behaviors contain more information. We will spend much on graphics, and we will extend the basic model step by step.

II. Emulating Competitors and Overbuilding

This section will discuss the basic case. At the outset, both developers stay calm and stick to their normal level of development. Such state are often called an idle period. Their normal level of development could be 0. In such case, both of them have options to develop new buildings but they both have no incentive to do that.

With disturbance, either positive or negative demand shock, developers will start to build. In this section we discuss how different disturbance lead to different story.

A. Solutions to the System

In basic situation, we neglect government intervention and developers' idiosyncrasies, as well as possible input to the system, but rather we focus on their characteristic to emulate each other. Such emulation, as we discussed, is mainly driven by the fear of preemption.

The system could be written like the following:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = c \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{2.1}$$

where c is a scalar. The 1s on the off-diagonal simply implies that each would build as much as the other has done. Since we define x and y as the departure from normal level, this system says that each developer is ready to deviate as much as the other does. Since c is a constant, it is just a measure of how quickly each developer will respond, and thus is not essence to the state of this system[2]. We will focus on the 2 by 2 matrix.

Solving this system gives:

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \tag{2.2}$$

where c_1 and c_2 are constants to be determined by initial condition or boundary condition. But we don't consider any particular solution in this article. Now a simple look-at would good enough

[2] When we draw the pahse portraits, we will regard c as 1.

to illustrate that: as time approaches infinity, the vector $\begin{bmatrix}1 & 1\end{bmatrix}^{tr}$ will dominate; on the contrary, when time approaches to negative infinity, the vector $\begin{bmatrix}1 & -1\end{bmatrix}^{tr}$ dominates.

Figure 1 has shown the phase portrait of this system. The arrows are phase vectors, their direction denote the tendency of change at that point, and their length denote the magnitude of change. There is an equilibrium point in this system at $(0,0)$, at which both x and y have derivatives 0 and no tendency to move. This point could be the beginning of our story. At first, both developers has no incentive to build more, they are indifferent from exercising their options to sitting tight. And the value of x and y imply they remain their normal level of building. There are two asymptotic lines: $y = x$ and $y = -x$, they are determined by the two eigenvectors for matrix $\mathbf{A}$, namely $\begin{bmatrix}1 & 1\end{bmatrix}^{tr}$ and $\begin{bmatrix}1 & -1\end{bmatrix}^{tr}$. Notice they are the two independent vectors that span the solutions' space. Plus, the direction of the vectors alone $\begin{bmatrix}1 & 1\end{bmatrix}^{tr}$ approaches infinity, while the direction of vectors alone $\begin{bmatrix}1 & -1\end{bmatrix}^{tr}$ approaches original point. Therefore, the system is unstable alone the direction of vector $\begin{bmatrix}1 & 1\end{bmatrix}^{tr}$, while it gets to equilibrium point at original point alone the direction of $\begin{bmatrix}1 & -1\end{bmatrix}^{tr}$. Therefore original point is also a saddle point since it is the maximum value in one direction but the minimum at the other direction. Several integral curves are drew up to illustrate the dynamics of this system. As time approaches negative infinity, the vector $\begin{bmatrix}1 & -1\end{bmatrix}^{tr}$ dominates; as time approaches positive infinity, the vector $\begin{bmatrix}1 & 1\end{bmatrix}^{tr}$ dominates. In particular, this system has two groups of solutions, one of them approaches to $(+\infty, +\infty)$, and the other approaches to $(-\infty, -\infty)$. We will focus on the former one since they tell the story. The other one, although is also the dynamics of this system, is not possible from the intuition: this group of solutions means that both developers build less and less as time pass by, and this is because both of them try to build less as the other has done. This situation will not happen since if one of the developer decides to build less, the other would have chance to exercise her options without having a competitor. This would boost her value and she would not follow the action of the other.

Only temporary disadvantage is possible and the system implies that most of these disturbances will result in overbuilding in the long-term.

First, think about the case where one chooses to build less, and the other then has the incentive to exercise exactly as many more options as the other has chosen not to follow. This is the case where an initial point is at the line of $y = -x$. Based on our discussion above, we know that as time goes by, the equilibrium will restore and each developer will goes back to their initial normal level of development. In this case there is no overbuilding in the long-term, the disturbance only gives a temporary advantage for the developer who chooses to exercise, as time goes by, the other developer recovers from the disturbance and choose to follow, the leader then become indifferent from exercising to sitting tight again, and the initial equilibrium is restored.

Then, let's consider the case where the developer who has the opportunity to exercise without competition somehow choose to exercise more than the other has now incapable to follow. This case could be depicted by the curves that start from region above the curve $y = -x$ in quadrants 2 and 4. The one who has the temporary advantage, or the leader, chose to exercise more since she knows her competitor has no capability to follow in the short future and she could exploit the market as a monopolist. The follower, though has no choice but waiting in the short future, will finally regain her capability. During this period, the leader somehow chooses to exercise a little less then she initially does. First, this is because her competitor has gradually got more power to follow, this hurts her market power. Second, she knows that the other developer will not follow, and without the fear of preemption, the cost of exercising force her to exercise less. This could be explained by the fact that any development calls for months even years to complete, and during this time there's no income. The leader, though do has the wiliness to build more, has to consider her own ability to bear both risk and costs. In the graph, let's look at the two curves that goes through point and $(1,0)$ and $(3,0)$. For the curve that goes through $(1,0)$, it started from a place a little above the line $y = -x$, while the other curve started from a place well above the line $y = -x$. The leader who has wiliness to build more build more as she could, and thus the more capable she is, the more she chooses to build. These two curves has shown that the leader who has more ability to build will retreat less until her competitor recovered to initial state(namely the level 0). As soon as the follower has recovered, she would chase the leader since she now faces a market where the demand for old building has shrank. The leader notice that her competitor now has the ability to follow will try to build more again: she knows that all the developments need time to complete, and while the follower tries to make up for the buildings that the leader has built, the leader will try to take her last advantage to exercise the options first, because in foreseeable future, the follower will still follow the leader sequentially, instead of simultaneously. As time goes by, in the long-term, they again exercise options simultaneous, and this is captured by the characteristic that the integral curve approaches the line $y = x$ as t approaches infinity. The more powerful the leader, the longer the time of sequential exercise would be, as easily spot in the phase portrait. In the long-term, both developers has built more than they initially dose, and since each of them has the incentive to preempt, they just build more and more. Ironically, in the long-term, though both of them has gone much further, they difference between their numbers of exercising is growing little. And that means none of them has got the advantage, they share the market equally, and as we discussed before, none of them gets better.

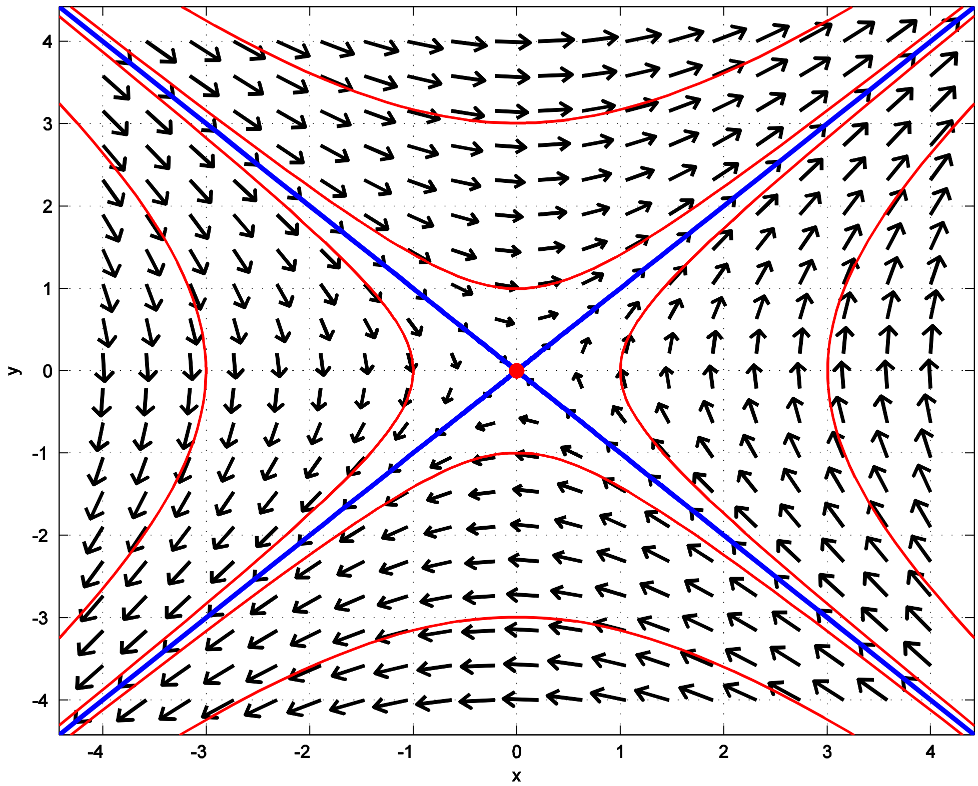

Figure 1

B. Decouple the System

We try to give a whole story in the last part. But actually the initial disturbance could lead to a state in the first quadrant and the second half of our story could then apply. The point is that, as time pass by, the fear of preemption and the advantage of preemption could lead developers to overbuild. Of course, the cost and risk could force one of them to stop for a while, but in the long-term as they gradually gain ability to lead or follow, they would as the system has shown.

In this part, we try to analyze the essence of this system. More specially, we try to find two new variable that are more to the essence of the story. We try to decouple the system. In other words, we try to find two new variables u and v:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

they change the original system into two independent first-order differential equations that has no interaction:

$$\begin{cases} \dot{u} = \theta_1 u \\ \dot{v} = \theta_2 v \end{cases}$$

Before formal discussion, it's helpful for us to consider this question intuitively. We need to find a linear transformation from variables x and y to two new variables u and v. New variables change will time exponentially and has no interaction. In other words, new variables govern the dynamics of the system independently.

What are the two variables that govern the story? We should guess them to be:

$$\begin{cases} u = x + y \\ v = x - y \end{cases} \tag{3.1}$$

the first is the total departure from the initial equilibrium, and the second evaluate the comparable advantage between two developers. Plug equation (3.1) into (2.1) gives:

$$\begin{cases} \dot{u} = cu \\ \dot{v} = -cv \end{cases} \tag{3.2}$$

the solution to this system is obvious to be:

$$u = u_0 e^{ct} \quad v = v_0 e^{-ct}$$

In other words, the total departure of the two developers u will goes to infinity as time pass by, and the comparable advantage between two developers will decrease exponentially. And notice the power of the exponential are just eigenvalues we found in last part. Think about the stories we discussed in the last part. Under disturbance, if the developer with advantage choose to build exactly more as her competitor has choose not to follow, then initial total departure is 0 and the system goes back to normal equilibrium with time; if the leader has the decide to build more, no matter how little more than the former case, the initial total departure is positive, and this will goes to infinity as time goes by. The other variable v, the comparable advantage, will decrease to 0 as time approaches infinity. This is what we have learned in last part, the advantage gradually disappeared, and no matter how much the two developers have departed from the original equilibrium; none of them has got better.

Formally, the two new variables are found by using eigenvectors[3]. Here we give a further note: the two new variables we have found gives a new orthogonal basis for the system (that why they have no interaction), and the geometrically the two eigenvectors have become the new axis. Using equation (3.1) to transform the two eigenvectors gives:

$$\begin{cases} \begin{bmatrix} 1 & 1 \end{bmatrix}_{xy} \to \begin{bmatrix} 1 & 0 \end{bmatrix}_{uv} \\ \begin{bmatrix} 1 & -1 \end{bmatrix}_{xy} \to \begin{bmatrix} 0 & 1 \end{bmatrix}_{uv} \end{cases} \tag{3.3}$$

This geometric feature is true for general first order linear system. And it will help us to consider the system more essentially. We will focus on eigenvalues and eigenvectors.

[3] http://book.douban.com/subject/3216955/

III. Government Intervention: Industry-wide Negative Feedback

In this section we will extend the basic model to incorporate industry-wide negative feedback for developers. First, we will add an industry-wide negative feedback to the developers, such case could be justified by government intervention or negative economic expectation.

A. Government Intervention

If we introduce the industry-wide negative feedback for the developers, the system is like following:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = c \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{3.4}$$

where s captures the magnitude of feedback. And again the constant k is not of importance. Now let's consider how s would affect the dynamics of this system and the relevant implications. As is well known, the dynamics of a first-order differential system is governed by the eigenvalues of the matrix and we have a characteristic quadratic:

$$\lambda^2 - 2s\lambda + s^2 - 1 = 0$$

and the solutions for it is:

$$\lambda_1 = s + 1 \quad \lambda_2 = s - 1 \tag{3.5}$$

Before Analyzing the dynamics, we give an proposition.

Proposition 1 : Industry-wide feedback does not change the eigenvectors of the system. And equation (3.1) will still decouple the system.

Proof. Multiplying the matrix on the right-side with the two vectors $\begin{bmatrix} 1 & 1 \end{bmatrix}^{tr}$ and $\begin{bmatrix} 1 & -1 \end{bmatrix}^{tr}$ gives:

$$\begin{gathered} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} s+1 \\ s+1 \end{bmatrix} = (s+1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} s-1 \\ 1-s \end{bmatrix} = (s-1) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{gathered} \tag{3.6}$$

So the eigenvectors are kept same under the effect of industry-wide feedback. Plug equation (3.1) into equation (3.4) to find:

$$\begin{cases} \dot{u} = k(s+1)u \\ \dot{v} = k(s-1)v \end{cases} \tag{3.7}$$

Four cases for the dynamics of the system are possible(figure 2 shows the four possibilities):

i. $\lambda_1 > \lambda_2 \geq 0$

If both eigenvalues are bigger than 0, the original point will be an unstable nodal source. And any disturbance will drive the system to goes to infinity. And based on equation (3.5) we know this implies $s \geq 1$. This is not possible in our context since s is used to measure the negative feedback from government intervention. It is depicted by the graph in the first quadrant of figure 2.

ii. $\lambda_1 > 0 > \lambda_2$

In this case, our system is almost the same as we have discussed in section II. Based on proposition 1and equation (3.6), the solutions are given by:

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{(s+1)t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{(s-1)t}$$

where the only difference between this system and that in the section II is the eigenvalues. They appear on the powers of the exponentials and could be considered as the speed of adjustment. The phase portrait of this system is similar to that of in section II, which is the graph in the third quadrant to figure 2. Since in this case s is negative. The decoupled system in equation (3.7) simply says that: under government light intervention (in this case $-1 < s < 0$), the dynamics of the system in the long-term will not change, but the total departure will increase in a lower rate, and the comparable advantage will decrease in a higher rate. Compare this graph with figure 1, you could see that the integral curve in this graph approaches to line $y = x$ relatively faster. Thus the leader in this case gets worse off because she has shorter time of preemption with government intervention. The story has not changed under soft government intervention.

iii. $0 > \lambda_1 > \lambda_2$

This is the case where $s < -1$, and its phase portrait is that in the second quadrant of figure 2. In this kind of system, the original point is a stable nodal sink, all integral curves approach to it in the long-term. In other words, any disturbance of the market will be temporary, as time goes by the market will restores its peace and two developers will be forced by to where they initially are. Nobody gains and there's no overbuilding. The intuition is that the government intervention has been increased to a level at which both developers would find the punishment from government brings more negative effect than the positive effect of following competitor's move. This is hardly realized in the reality because it also means that the government has to be more powerful than the market. The fear of preemption is actually a fear of losing market power, and it's a utility-driving natural move of developers to compete for more market share. A severe government intervention is both costly and against the market discipline, this might be the reason why government intervention often only does not change the market feature. Most government intervention are like that in case ii, government intervention makes the leader worse off, but the competition of developers still leads to overbuilding.

iv. $0 = \lambda_1 > \lambda_2$

This is a special case mathematically, but it has no big difference to that of former stories. Mathematically speaking, when one of the eigenvalues of this system is 0, it means that the matrix itself is singular, and the two rows of the matrix is dependent. Plus the 1s on the off-diagonal, and our assumption that government intervention brings negative feedback to developers. The only matrix that fits all these criticisms is:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

this is the case where the power of government intervention is just as that of the desire to preempt and follow. In this special case the solutions of the system is given by:

$$x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} \tag{3.8}$$

where one of the independent solution does not change with time and is a constant vector. The phase portrait for this system is in the fourth quadrant of figure 2. Instead of having a single isolated equilibrium point at the origin, the system now has a line of critical points on line of $y = x$, a disturbance will force the state to approach line $y = x$ with a trajectory of line in the direction of $y = -x$. At long run, the system will settled on a point of line $y = x$. This point is not necessarily the origin. Think about the decoupled system we discussed before, there are two points to be explained: first, a final settlement away from the origin means that the two developers have deviate to a new equilibrium point. In this case both positive and negative deviation is reasonable; second, the final settlement on the line $y = x$ means that still no one gets any better, they share the market equally.

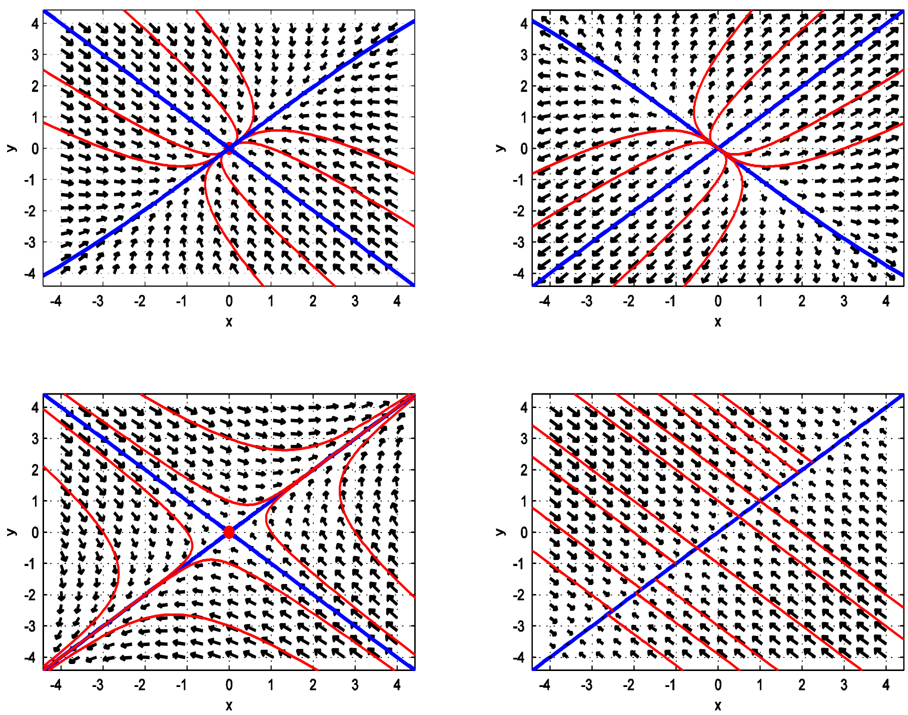

Figure 2

IV. Asymmetry and Hesitating Follower: Reshuffling

In section III we analyzed the situation where the developers are under the influence of industry-wide negative feedback, and we choose government intervention as the realistic context for such system. Now we try to consider the case where developers have different feedback, one of them would have comparable advantage, and we will see in long-term this will help her to gain more market share. Then, we consider a special case where the follower seems to be hesitating about whether to follow the leader or not, such hesitation could be modeled with a periodic function: when the departure of the leader is low, she follows; when the leader's departure still increase, the follower's own incapability would force her to wait, but if the leader choose to exercise even more, the follower would have to follow again since the demand for old building shrank a lot to a dilapidated level and exercising options is the best she could do, even such exercise may incur negative cash flow.

A. Cost and Revenue Asymmetry

As we discussed before, developers could have different cost and revenue from same option exercise. This asymmetry could be captured by giving different feedback for two developers. Since cost decrease developers' desire to developer new buildings, cost is a negative feedback. On the other hand, revenue is positive feedback, and should give developers positive feedback. The system for asymmetry modeling is:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = c \begin{bmatrix} \theta_1 & 1 \\ 1 & \theta_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{4.1}$$

this is a pretty general system, where the matrix is the same as that in equation (1.3). An assumption is important:

$$|a| < c, |b| < c \tag{4.2}$$

this assumption means that the effect of asymmetry feedback is weaker than the desire to preempt or follow, the asymmetry we are modeling are the individual consideration. Think about what if the cost feedback is bigger than developers' desire to preempt, this means that cost is larger than the expected income from exercising the options and in such case we will no competition because any exercise of option is against developers' utility. Similarly, if revenue feedback is bigger than the desire to preempt or follow, then revenue is the dominant consideration of exercising option. Based on our discussion in section I, the initial equilibrium exists because simultaneous exercise makes developers from exercising to sitting tight. If revenue feedback is the dominant force, developers would not wait or consider any strategic exercise because any exercise produces a positive income less cost, and it's always reasonable for developers to move regardless what the other is doing. These two cases will have phase portrait respectively like that in second and first quadrant of figure 2: high cost leads to a stable nodal sink where any disturbance will vanish over time; high revenue leads to an unstable nodal source where the system always goes to infinity.

Of course the phase portraits are still a little bit different. More importantly, it is reasonable for us to consider the case where developers are under the feedback from both government intervention and their own cost considerations, this may take them to a place where negative feedback is bigger than 1.

We will analyze the system analytically now, again we may ignore the constant c. Equation (4.1) is governed by the characteristic equation:

$$\lambda^2 - (\theta_1 + \theta_2)\lambda + \theta_1\theta_2 - 1 = 0$$

and the solutions of the system are given by:

$$\mathbf{x} = c_1\mathbf{x_1}e^{\lambda_1 t} + c_2\mathbf{x_2}e^{\lambda_2 t} \tag{4.3}$$

where:

$$\lambda_1 = \frac{(\theta_1 + \theta_2) + \sqrt{(\theta_1 - \theta_2)^2 + 4}}{2}$$

$$\lambda_2 = \frac{(\theta_1 + \theta_2) - \sqrt{(\theta_1 - \theta_2)^2 + 4}}{2}$$

$$\mathbf{x_1} = \begin{bmatrix} \dfrac{(\theta_1 - \theta_2) + \sqrt{(\theta_1 - \theta_2)^2 + 4}}{2} \\ 1 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} \dfrac{(\theta_1 - \theta_2) - \sqrt{(\theta_1 - \theta_2)^2 + 4}}{2} \\ 1 \end{bmatrix}$$

with assumption (4.2) we will also find that:

$$\lambda_1 > 0, \lambda_2 < 0$$

recall that in this case the origin will be an saddle point. When time approaches negative infinity the line represented by vector $\mathbf{x_2}$ is dominant, while vector $\mathbf{x_1}$ is dominant when time approaches infinity. And though we won't give a proof but a system that will treat the two eigenvectors as the orthogonal basis will decouple the system (4.1).

Look at equation (4.3), we have found that the feature of the system has not change. $\theta_1 + \theta_2$ and $\theta_1 - \theta_2$ are the two variables that govern the system. The former is the total feedback of the two developers; the latter one is more essential, it perfectly reflects the comparable advantage between the two developers.

We now focus on the long-term behaviors of this asymmetry system, and the vector $\mathbf{x_1}$ is the focus because as time goes by the system approaches this vector, and of course we narrowed down our discussion to the positive direction of vector $\mathbf{x_1}$. We consider the case where $\theta_1 - \theta_2 > 0$, this might be situation that developers has a lower cost or she has bigger revenue from developing, in either case she has comparable advantage. It's easily seen that in this case:

$$\begin{cases} W = \dfrac{(\theta_1 - \theta_2) + \sqrt{(\theta_1 - \theta_2)^2 + 4}}{2} > 1 \\ \dfrac{\delta W}{\delta g} > 0 \\ g = \theta_1 - \theta_2 > 0 \end{cases}$$

and vice versa if $\theta_1 - \theta_2 < 0$. Recall that we expect the state of the system approaches to the line though vector $\mathbf{x_1}$ in long-run. The second entry of vector $\mathbf{x_1}$ is 1, and thus if $\theta_1 - \theta_2 > 0$ the first entry will be bigger. Different from the system we have discussed before where the feedbacks for developers are symmetry, this asymmetry system expects the developer who has comparable advantage to gain more market power in long-run. As time evolves, vector $\mathbf{x_1}$ says that $x - y > 0$ in the case where the first developer has comparable advantage. And notice that the cost advantage where developer has lower cost and the revenue advantage where developer has higher revenue could both be generalized with this system, because the absolute value of the feedback does not concern, it's their difference that govern the dynamics of this system.

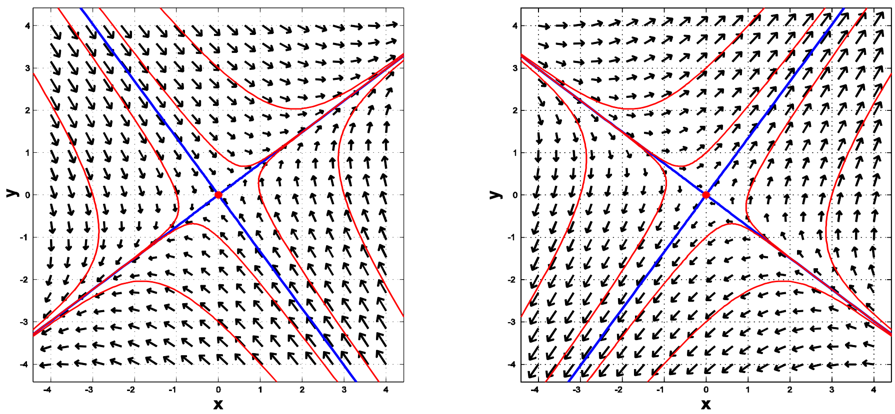

Figure 3

Figure 3 are produced by choosing $c = 1$ and taking respectively $\theta_1 - \theta_2 = 0.6$ as cost asymmetry and $\theta_1 - \theta_2 = -0.6$ as revenue asymmetry. First, as we expected, the phases in the first graph approach to a line that has slope less than 1 and the limit state has $x > y$, while the phases is exactly at the contrary; Second, notice that the integral curves in the first graph have approached the asymptotic line faster than those in the second graph, this is due to the different asymmetry we address: in the first graph, we have cost asymmetry and cost is a negative feedback it decreases the leader's income from preempt and this force the system to approach to its limit state faster; on the other hand, the revenue asymmetry in the second graph is a positive feedback for both developers and it specially increases leader's income from preemption, which has made the system approach to limit state at a lower rate. Both discussions above are compatible with our discussions in section II and section III.

Figure 4 is an exploration of how the magnitude of asymmetry affects the dynamics. In both graph, we calculate the solution for the initial condition $\mathbf{x}_0 = (0,1)$ both forward and backward. In particular, these two solutions are from system of cost asymmetry and the cost asymmetry are respectively 0.6 and 0.3. The initial condition is a possible disturbance where the developers who has comparable disadvantage happen to lead. We can see from the graphs that the value of x gradually exceed that of y. Take a look at time $t = 5$, the difference in the first graph is bigger than that in the second, this is due to a higher cost advantage endows more power to the first developer. But the values in first graph are lower than that in the second, a lower cost advantage means her competitor is more capable to follow her and this together has make the system grows at a higher rate.

The long-term state will make give the developer with advantage more power and this completion brings the system away from the initial equilibrium. This is a case where over-building still features the market and the new market shares that developers have reflect a so-called reshuffling phenomenon.

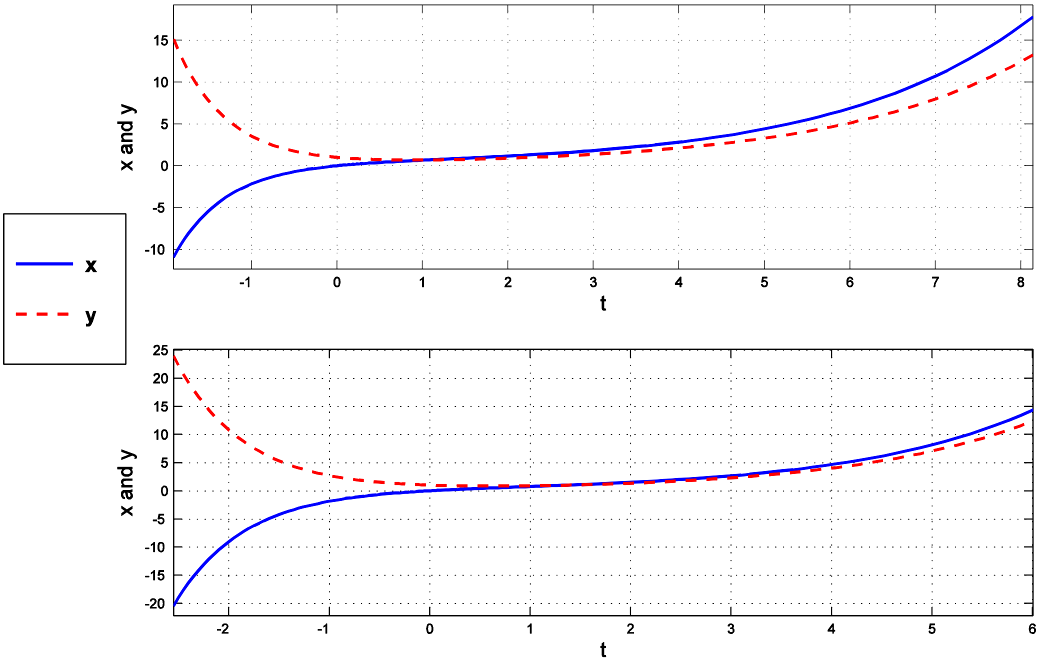

Figure 4

B. Hesitating Follower : a Mixed Strategy

Sometimes the asymmetry isn't consistent and the follower hesitates whether she should follow or not. Assume one of the developers has cost disadvantage, let's call her the loser, this puts her into an inferior condition. As we analyzed in the last part, regardless the initial condition, the state of this system will approaches to a limit state where the developer who has comparable advantage will gain more market share, no matter how big the temporary market power the initial market disturbance has grant to the follower, she will finally lose it. Plus, the endless will continually put the follower in an inferior situation, no matter how much she tries, the best she could do is to keep her share in the market, unfortunately this share is lower. Suppose at the start of the story, the two developers share the market, a continual competition means both continual cost and a loss of market share. So, why should she follow if she has comparable disadvantage? If she never follows, her competitor will choose to exercise as much as possible and enjoy the monopoly power, so she should follow her competitor to gain some market share in the market of new buildings and this could cover her loss of income from the market for old building due to shrinking market share. Somehow she should not always follow, this does her no good. She should follow her competitor when her competitor's departure is low, this will help her to cover her loss and gain some share in the new market, but she should chooses to build less if her competitor has deviated a lot from the initial equilibrium. When that happens, more competition will make her even worse, this force her to constrict her desire to follow. Noting this, her competitor would also decrease the number of option exercises because her competitor is no longer under the threat of a follower and she could find her own maximizing point. The difference is: the winner of long-run will not fear about a follower or be happy to lose market share, she has the desire and capability to follow and lead. The loser on the contrary could not bear a continual competition, her comparable disadvantage force her move more wisely, she follows when her competitor deviates a little but retreats when that departure grows to certain level. If the winner does not decrease the number of option exercises, this means the winner still has capability to preempt more when the follower have to retreat, this will dramatically dilapidate the market for the old buildings. As a result, instead of following wisely, she loser is forced to follow, otherwise suffer from the low market share in new market and big loos even negative income from old market. As a result, the loser's response to the winner could be periodic: positive when the winner deviates a little, negative because she have to retreat from a continual completion when winner deviates a lot, positive again as she is forced to compete and so on. To model this hesitation from cost disadvantage, we use the following system:

$$\begin{cases} x = y \\ y = \sin x - \omega y \end{cases} \tag{4.4}$$

where ω is negative and captures the cost disadvantage for the second developer. We don't give a cost feedback for the first developer since it's the comparable disadvantage that matters.

This is a non-linear first-order differential system. Matrix could not be used directly and we calculate its solutions numerically. A differential system like this models a damping-pendulum. We will focus on its phase portrait instead of analytical solutions. The way to find the phase portrait is to find the critical points of the system (where the value of variables stay constant) and take Jacobian matrix as the linearization for the system.

Solve the system:

$$\begin{cases} y = 0 \\ \sin x - \omega y = 0 \end{cases}$$

gives the critical points for the system as:

$$\begin{cases} x = 0, \pm\pi, \pm 2\pi, \cdots \\ y = 0 \end{cases} \tag{4.5}$$

notice the critical points are independent from ω, no matter how big the comparable disadvantage is the system will have critical points given by equation (4.5). Since the function $\sin x$ is periodic, we expect the linearization of the system to be periodic, too. Calculating the linearization near the critical points with Jacobian matrix gives the following two system:

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\omega \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & at \ (2k\pi, 0) \\ \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\omega \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & at \ ((2k+1)\pi, 0) \end{cases} \quad k = 0, \pm 1, \pm 2, \cdots$$

where the first system is just a special case we discussed before and it has an linearization like that in figure 3. The characteristic quadratic of the second system is:

$$\lambda^2 + \omega\lambda + 1 = 0$$

this will produce complex eigenvalues. Also the real part of its eigenvalues is negative because $\omega < 0$. Then the linearization for the second system is featured with an asymptotically stable spiral sink. Figure 5 has shown the big picture for system (4.4) taking $\omega = 0.5$.

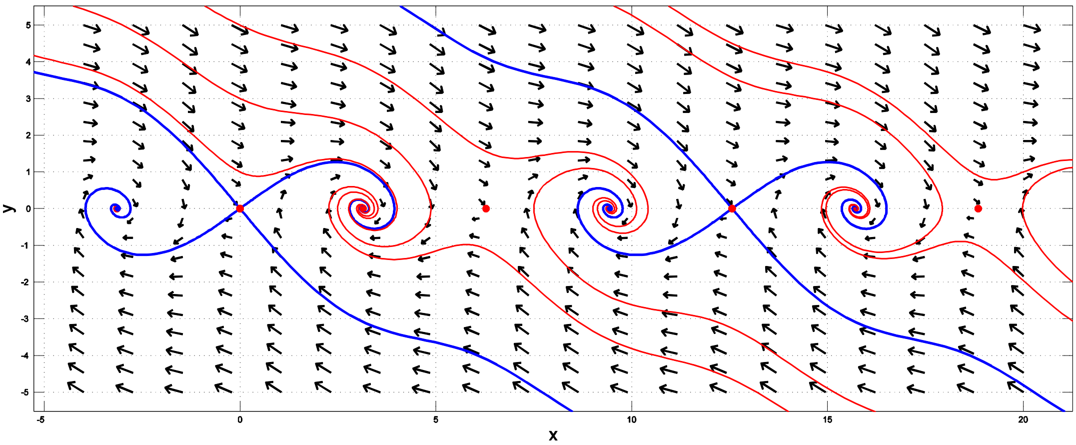

Figure 5

Notice that we have limit curves instead of limit lines in the case of linear system. But the regions around the critical points behave like we expected. Especially, look at the critical points at origin and $(\pi, 0)$, they are respectively saddle point and spiral sink, and such feature appears periodically alone the x axis.

We shall consider the market disturbance now. The easy case is that the winner, denoted with x starts to exercise more options than she normally does, this will be a point alone the positive x axis, and since all the critical points are on x axis, this move will soon come to an end. New equilibrium entitles the winner more market share since she now has more options to exercise, the loser on the other hand remains her initial number of option exercises. Every stable point is reached with a spiral. During such period, fluctuating market is seen but as the phase portrait has shown, the magnitude of fluctuations should decrease over time. These fluctuations could last for quite a long time. During this period, the winner and loser has been picker out, but both of them have incentive to maximize their utility. The winner need to consider the point that entitles her greatest market power without competition of a follower, she tries and see that the other developer will do, and as time goes by she has more knowledge about the other's move and then the fluctuations is smaller and smaller and finally come to an end. At the same time, the loser chooses to follow or not, as a follower she has no knowledge about what the winner would do next. The loser's problem is easier: she takes the actions of winner as a signal, and chooses the strategy to maximize her own value. Since the follower's strategy is not consistent, the leader cannon expect her move as first, instead she winner choose to figure out the loser's mixed strategy by making different signal and as she gradually realizes loser's move she could have make her own best choice. These periods with fluctuating market feature is often seen in reality where the supply and price of real estate fluctuate. Similar case is a disturbance that gives the winner temporary advantage, the loser gradually recovers from failure like in previous sections, and the system comes to an spiral sink again and same story happens again.

An interesting story is about how the loser uses her mixed strategy to maximize her value if the market disturbance gives her temporary market power. This could be a case that has initial state alone the positive y axis. The loser happens to be the temporary leader, and she knows that her competitor has the capability and desire to follow and win the market power back. Her strategy would be to lead as long as she can and when her temporary strategy has gradually diminished, she retreats. The winner notice that the loser has retreated would become leader again and the same fluctuations appear again. The loser who has already earned temporary extra income now take the winner's move as signal to maximize her own value, the leader takes time to figure out the loser's strategy. The interesting part is that, the temporary market power has benefit the loser for a while, but at last she still remains her initial level of option exercise, the winner on the other hand got more market power in long-term. Nobody seems to lose in this kind of disturbance, the loser takes as much as she can from the temporary advantage, the winner though has to follow at first gradually win more market power because she has the cost comparable advantage. Even more interesting, the bigger the initial temporary market power the loser has, she bigger the final market power the winner has. This could be seen by looking at different integral curves that pass through the positive y axis: the higher the initial y is, the higher the final x is. This might be a game that developers are equally happy. But it's not, the initial state still concerns. There are only finite critical points in the system, and there are infinite integral curves that end up with the same critical point. Different integral curves passes through difference value of y at the positive y axis, they entitles different temporary market power to the loser, but they could lead to same final settlement. The loser does not really come to draw with the winner, she just takes the advantage of her luck and tries to extract as much as possible from it, at last the cost comparable disadvantage would still force her to go back to her initial level of development. Nevertheless, the loser is smarter by taking the mixed strategy, she knows that in the long-run the winner will get more market power, so she is happy to be lucky but she never fights with the winner.

As a result, there could be temporary more developments in the market, but there will be no lasting over-building in the market since in the long-run the market will come to a new equilibrium. At the new equilibrium, the winner got more market power while the loser stays market share. This is also a kind of reshuffling in the real-estate market. As time goes by, when the follower become more capable, the new equilibrium could become an origin as the loser now has the power to follow the winner at a higher level. This is a game stage by stage. Any disturbance of the market gives the winner more market power, and as time goes by, the loser could get stronger and be able to take advantage of her luck, though she still has no power to get more market share at last. The mixed strategy makes the loser always stay alive in the market, while if she chooses to compete to infinity as that in part A, she could be swept out by the winner because she has comparable disadvantage.

A final point is that the phase portrait will now change by different value of ω as we have shown in equation (4.5), the market feature is independent from ω. In others words, no matter how little the comparable disadvantage is, the loser will always lose. But fortunately, the smaller the comparable disadvantage is, the longer it takes the market to reach a new equilibrium, that means the loser could extract more from her temporary market power.

V. Conclusion