

Integer Programming

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Abstract

We shall conquer at all costs!

What is it?

Def LP problems with at least one variable that has to be an integer.

Types

- Pure - All decision vars have to be integers. $X \in \text{int}$
- Mixed - Some ints and some not (Even int + binary).
- Binary - Decision vars can only take on the values 0 or 1.
 $X \in \{0; 1\}$

Characteristics

- The change from a normal LP to an integer one changes the problem drastically; Making it more complex (Might not even have a solution).

Example 1

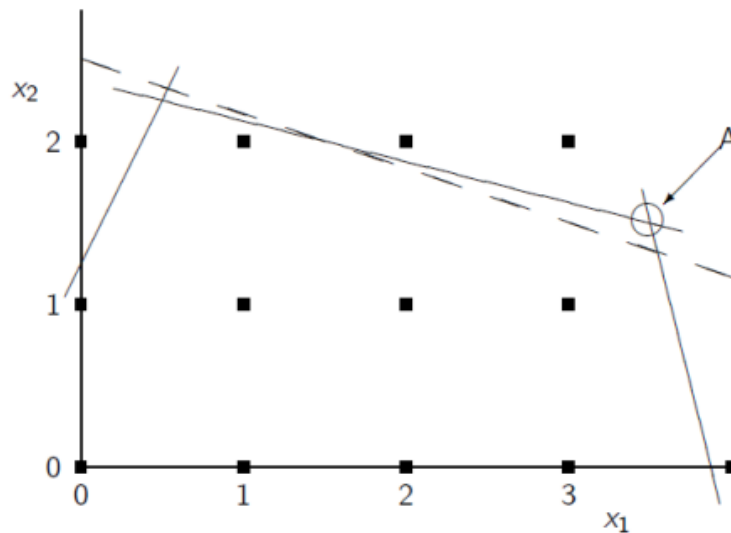


Figure 1: Example 1

$\text{Max } X_1 + 3X_2$
 $S.t :$
 $2X_1 + 8X_2 \leq 9$
 $4X_1 + X_2 \leq 15.5$
 $8X_1 - 4X_2 \geq -5$
 $X_1, X_2 \geq 0$ and Integer

- Point A would be the optimal solution point if this was a normal LP problem.
- Trying to use the same point rounded up would result in one using a point that's actually not in the feasible region.
- If you round down, the solution would be feasible but not optimal.
- Adding an int constraint can only restrict the solution.
- Therefore A is the best solution we could ever hope to achieve having restricted things.
- So shifting the isocost line down from A (Restricting solution), the first int point it touches will be the optimal point.
- There's an algorithm we'll use to find these solutions.

Example 2 - Project selection problem

Suppose we have identified four investment opportunities and we have R14000 to invest. Project 1 requires an investment of R5,000 and has a present value (a time-discounted value) of R8,000; Project 2 requires R7,000 and has a value of R11,000; Project 3 requires R4,000 and has a value of R6,000; and Project 4 requires R3,000 and has a value of R4,000. The question is: into which projects should we place our money in order to maximize our total present value?

Figure 2: Project selection problem

$$Let X_j = \begin{cases} 1 & \text{if project } j \text{ is selected; } j = 1, 2, 3, 4 \\ 0 & \text{Otherwise} \end{cases} \quad (1)$$

Max $8000X_1 + 11000X_2 + 6000X_3 + 4000X_4$
 St:
 $5000X_1 + 7000X_2 + 4000X_3 + 3000X_4 \leq 14000$
 $X_1, X_2, X_3, X_4 \in \{0, 1\}$

- Using simplex, $Z = \text{R}22000$ ($X_1 = X_2 = 1, X_3 = 0.5, X_4 = 0$)
- This obviously doesn't satisfy the last constraint.
- If you round down, $Z = \text{R}19000$
- Better solution - $X_1 = 0, X_2 = X_3 = X_4 = 1$
- This is found using the algorithm we'll learn later.
- We could have added constraints:
 - Only 2 projects can be selected ($X_1 + X_2 + X_3 + X_4 = 2$).
 - If project 1 is selected, then project 3 must also be selected ($X_1 \leq X_3$)
 - If project 2 is selected, then project 4 can't be selected ($X_2 + X_4 \leq 1$) - He acknowledged the fact that this condition goes the other way around as well.

Example 3 - Set covering problem

Decide the number of hospitals to build in order to meet the requirement of the city

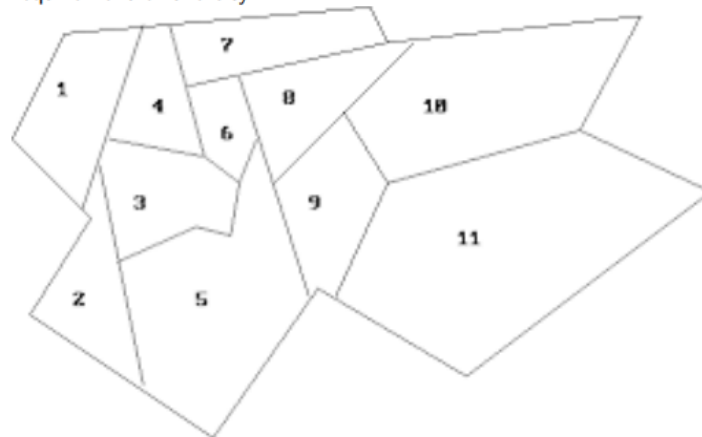


Figure 3: Set covering problem

Find min number of hospitals that can be built. Trying to give each city access to a hospital(They have to be adjacent to simulate access).

$$Let X_j = \begin{cases} 1 & \text{if hospital built in } j; j = 1, 2, 3, \dots, 11 \\ 0 & \text{Otherwise} \end{cases} \quad (2)$$

$$\text{Min } X_1 + X_2 + \dots + X_{11}$$

St:

$$X_1 + X_2 + X_3 + X_4 \geq 1 \dots (X1)$$

$$X_1 + X_2 + X_3 + X_5 \geq 1 \dots (X2)$$

$$X_1 + X_2 + X_3 + X_5 + X_6 \geq 1 \dots (X3)$$

$$X_1 + X_3 + X_6 + X_7 \geq 1 \dots (X4)$$

$$X_2 + X_3 + X_6 + X_8 + X_9 \geq 1 \dots (X5)$$

$$X_3 + X_4 + X_5 + X_7 + X_8 + X_9 \geq 1 \dots (X6)$$

$$X_4 + X_6 + X_8 \geq 1 \dots (X7)$$

$$X_5 + X_6 + X_7 + X_9 + X_{10} \geq 1 \dots (X8)$$

$$X_5 + X_6 + X_8 + X_{10} + X_{11} \geq 1 \dots (X9)$$

$$X_8 + X_9 + X_{11} \geq 1 \dots (X10)$$

$$X_9 + X_{10} \geq 1 \dots (X11)$$

$$X_1, X_2, X_3, \dots, X_{11} \in \{0, 1\}$$

- To get the constraints, we go from X_1 to X_{11} - For each X , the sum of it and all adjacent (touching) X 's must at least 1.
- This means we build at least one hospital to accomodate that group of cities. This will eventually cover all combinations of X 's thus we'll be sure we ran through every possible hospital location (aka covered all cities).
- Optimal solution - $X_3 = X_8 = X_9 = 1$

Example 4 - Fixed charge problem

Operation	Prod 1	Prod 2	Prod 3	Hours available
Machining	2	3	6	600
Grinding	6	3	4	300
Assembly	5	6	2	400
Unit profit	R48	R55	R50	
Setup cost	1000	800	900	

Figure 4: Fixed charge problem

Let X_j be the number of product j produced.

$$Let Y_j = \begin{cases} 1 & \text{if product is produced; } j = 1, 2, 3 \\ 0 & \text{Otherwise} \end{cases} \quad (3)$$

$$\text{Max } 48X_1 + 55X_2 + 50X_3 - 1000Y_1 - 800Y_2 - 900Y_3$$

St:

$$2X_1 + 3X_2 + 6X_3 \leq 600$$

$$6X_1 + 3X_2 + 4X_3 \leq 300$$

$$5X_1 + 6X_2 + 2X_3 \leq 400$$

$$X_1 \leq M_1Y_1$$

$$X_2 \leq M_2Y_2$$

$$X_3 \leq M_3Y_3$$

$$Y_1, Y_2, Y_3 \in \{0, 1\}$$

$$X_1, X_2, X_3 \geq 0 \text{ and Integer}$$

- Manufacturing where you're making paint
- **Setup cost (fixed charge)** - Cost of cleaning machine after making red paint to prepare to make yellow for example.
- **Variable cost** - e.g Hiring trucks will cost a certain, fixed, amount (fixed cost) and the cost of using them will vary (hence variable cost).
- The 4th to 6th constraints are added to make sure that each cost Y_j is only applied if X_j is actually produced.

- M is a big number greater than 0.
- if X is 0, Y can be 0 or 1.
- BUT if X is 1, Y **HAS** to be 1 as well (Since MY has to be at least equal to X).
- This is to say, surely if we produce at least one (X), there has to be some cost involved (Y).
- In fact, the moment X is slightly above 0, Y is 1.
- M is acting as an upper bound for X .
- In this case, We can find a reasonable value for M 1:
 - * Method - For each X , in the first 3 constraints, if the other X values are 0, M will be the max value X can take:
 - For X_1 ,
 - For constraint 1: $X_1 = 300$

- For constraint 2: $X_1 = 50$
- For constraint 3: $X_1 = 80$
- Therefore, the max it can be will be the min of the 3 values
(**We do this so that all constraints are satisfied;**
Therefore, $M_1=50$).

Relaxation of ILP

► Original ILP

$$\begin{aligned}
 & \max 2x_1 + 3x_2 \\
 & \text{subject to :} \\
 & \quad 4x_1 + 12x_2 \leq 33 \\
 & \quad 10x_1 + 4x_2 \leq 35 \\
 & \quad x_1, x_2 \geq 0 \\
 & \quad x_1, x_2 \text{ must be integers}
 \end{aligned}$$

► LP Relaxation, drop integer constraints:

$$\begin{aligned}
 & \max 2x_1 + 3x_2 \\
 & \text{subject to :} \\
 & \quad 4x_1 + 12x_2 \leq 33 \\
 & \quad 10x_1 + 4x_2 \leq 35 \\
 & \quad x_1, x_2 \geq 0
 \end{aligned}$$

Figure 5: Relaxation of ILP

- When you drop the integer constraints, you relax the integer constraint, hence LP relaxation.
- If the solution we find just happens to be an integer solution, we're done, else, we have to do something else.
- The feasible region of the relaxed problem is a **superset** (contains) of the associated integer linear programming solution feasible region. This makes sense as restricting things can only make the feasible region smaller.
- The objective func value for the integer solution can't be greater than that of the relaxed (normal simplex) version in a maximization.

- The optimal solution of the relaxed solution is the upper bound of the ILP solution for a maximization.
- The optimal solution of the relaxed solution is the lower bound of the ILP solution for a minimization.
- At each iteration in our search for a solution, the objective function value for any feasible solution ILP is a lower bound for an maximization (We can't accept anything lower since we already have at least that value as a potential solution).
- At each iteration in our search for a solution, the objective function value for any feasible solution ILP is an upper bound for an minimization (We can't accept anything above since we already have at most that value as a potential solution).
- This is the **Branch and Bound** algorithm.
- It involves solving a series of LP problems.
- It can solve any ILP; In fact, any optimization (It's just that some take forever).
- In practice, the branch and bound can take a lot of computational effort and time.

Example using the Branch and Bound algorithm

► Original ILP

$$\begin{array}{ll}\max & 10x_1 + x_2 \\ \text{subject to :} & \\ & 2x_1 + 5x_2 \leq 11 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \text{ must be integers}\end{array}$$

► LP Relaxation, drop integer constraints:

$$\begin{array}{ll}\max & 10x_1 + x_2 \\ \text{subject to :} & \\ & 2x_1 + 5x_2 \leq 11 \\ & x_1, x_2 \geq 0\end{array}$$

Figure 6: Branch and Bound example 1

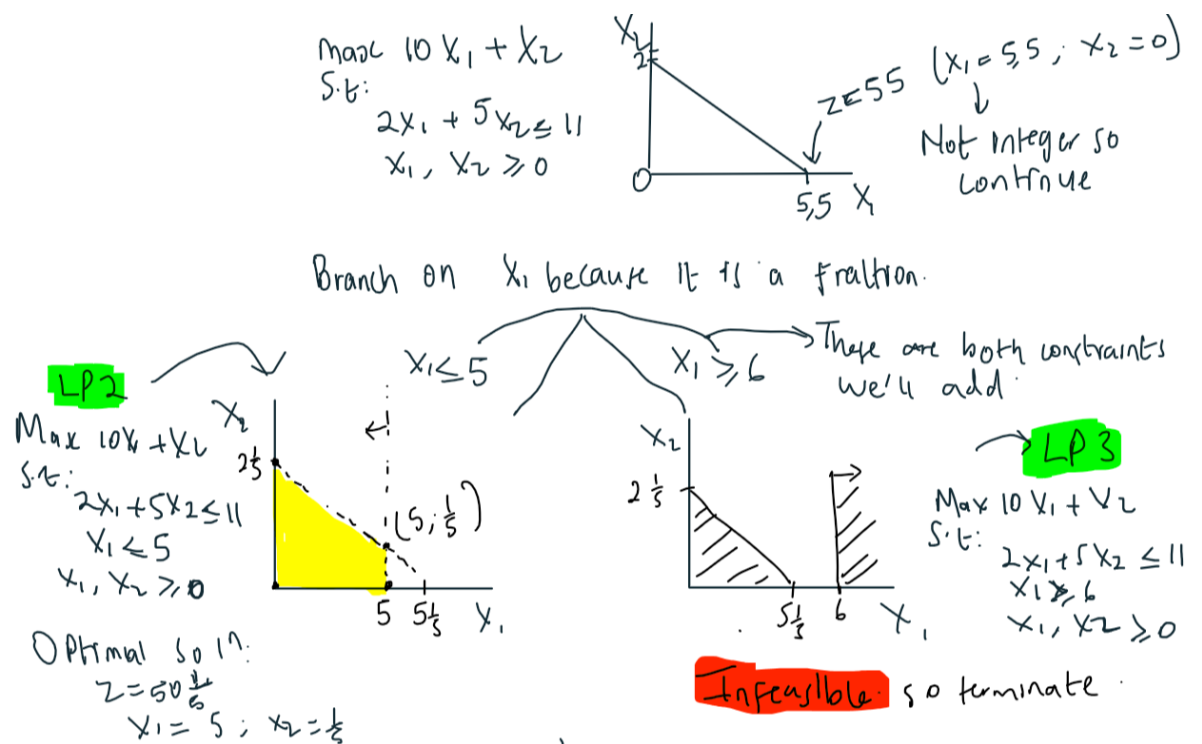


Figure 7: Branch and Bound worked example part 1

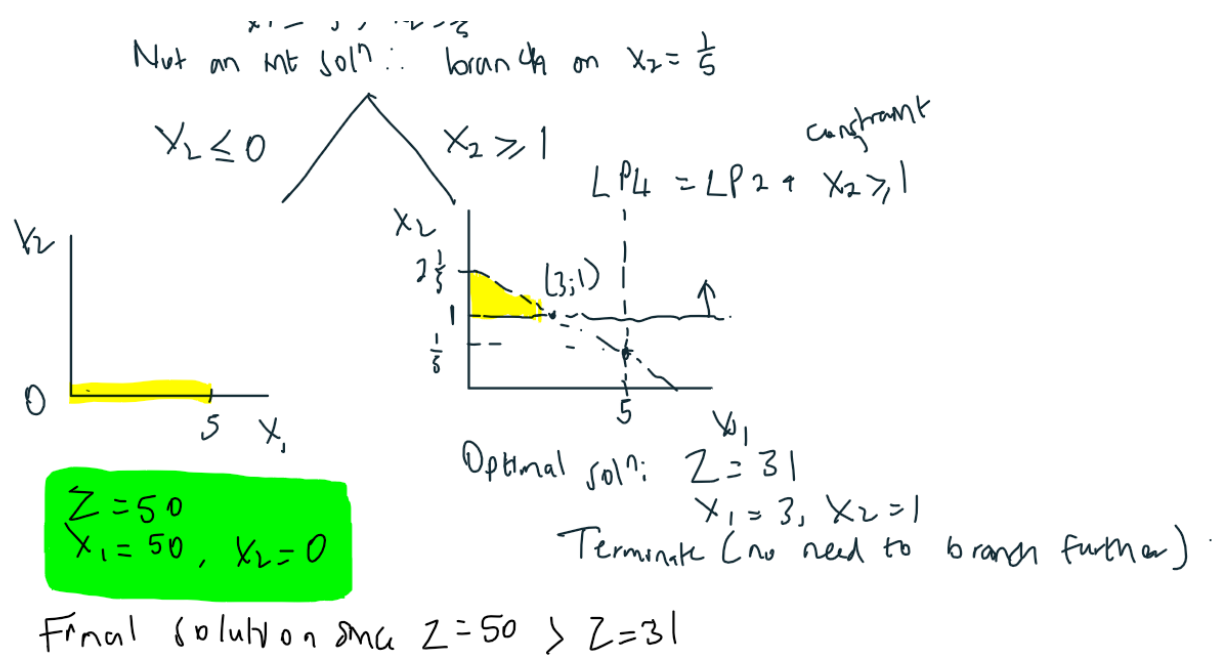


Figure 8: Branch and Bound worked example part 2

- If both decision variables are not integers, branch arbitrarily on any one of them. You will get to the same solution eventually.

