AM216 Stochastic Differential Equations

Lecture 12
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List of topics in this lecture

- Differential operator, Inner product, adjoint operator
- An alternative derivation of the forward equation, average amount of reward for an ensemble, time invariability of the reward for an intact ensemble
- Absorbing boundary and reflecting boundary for the forward equation
- Exit time, escape problem

Review

Conservation form:

In general, the evolution of the quantity governed by the backward equation is non-conservative; the evolution of the forward equation is conservative.

Meaning of the backward equation

$$\begin{cases} q_t = b(z)q_z + \frac{1}{2}a(z)q_{zz} \\ q(z,t)\big|_{t=0} = u(z) \end{cases}$$

$q(z, t)$ = The conditional average of position-dependent reward determined at a fixed future time T given starting at position z at time $(T-t)$.

$u(z)$: amount of reward if $X(T) = z$.

Variable t in the backward equation = *real time* $(T-t)$.

Meaning of the forward equation

$$\begin{cases} p_t = -\left(b(x)p\right)_x + \frac{1}{2}\left(a(x)p\right)_{xx} \\ p(x,t)\big|_{t=0} = v(x) \end{cases}$$

Consider an ensemble of X with density $v(x)$ at time 0.

$p(x, t)$ = ensemble density of X at time t.

Variable t in the forward equation = *real time t*.

End of review

Forward equation and backward equation in terms of differential operators

We consider the autonomous case:

$a(z, s) = a(z)$, $\quad b(z, s) = b(z)$

We introduce linear differential operator L_z.

$$L_z = b(z)\frac{\partial \bullet}{\partial z} + \frac{1}{2}a(z)\frac{\partial^2 \bullet}{\partial z^2}$$

which means $L_z[u] = b(z)\frac{\partial u}{\partial z} + \frac{1}{2}a(z)\frac{\partial^2 u}{\partial z^2}$

Short story:

- Backward equation in terms of L_z:

  $$q_t = L_z\left[q\right]$$

- Forward equation in terms of L_z:

  $$p_t = L_z^*\left[p\right]$$

  where L_z^* is the adjoint operator of L_z.

- An alternative derivation of forward equation that is more intuitive and much simpler than the mathematically elaborated derivation we did previously.

Long story:

Backward equation can be written as

$$q_t = L_z\left[q\right].$$

This follows directly from the definition of L_z.

To write out the operator for the forward equation, we need to define a few things.

Definition (inner product)

The inner product of two functions is defined as

$$\langle u_1, u_2 \rangle = \int u_1(z) u_2(z) dz$$

Definition (adjoint operator)

The adjoint operator of L is denoted by L^* and is defined by the condition

$$\langle u, L^*[v] \rangle = \langle L[u], v \rangle$$

for all functions $u(z)$ and $v(z)$ that decay to zero rapidly as $|z| \to \infty$.

Example:

Let A be a square matrix. We view matrix A as a linear operator

$$u \longrightarrow Au$$

For vectors, the inner product is defined as

$$\langle u, v \rangle = u^T v = \sum u_i v_i$$

We find A^*, the adjoint operator of A. We use the definition of A^*.

$$\langle u, A^* v \rangle = \langle Au, v \rangle = (Au)^T v = u^T A^T v = \langle u, A^T v \rangle$$

==> $A^* = A^{\mathrm{T}}$

(That is, the adjoint operator of A is A^{T})

Example:

Consider differential operator

$$D_x = a(x) \frac{\partial^2 \bullet}{\partial x^2}$$

We find D_x^*, the adjoint operator of D_x. We use the definition of D_x^*.

$$\langle u, D_x^*[v] \rangle = \langle D_x[u], v \rangle = \int a(x) \frac{\partial^2 u}{\partial x^2} v(x) dx$$

Integration by parts twice

$$= -\int \frac{\partial u}{\partial x} \cdot \frac{\partial}{\partial x}\big(a(x) v(x)\big) dx = \int u(x) \cdot \frac{\partial^2}{\partial x^2}\big(a(x) v(x)\big) dx = \left\langle u, \frac{\partial^2}{\partial x^2}\big(a(x)\, v\,\big) \right\rangle$$

==> $D_x^* = \dfrac{\partial^2}{\partial x^2}\big(a(x) \bullet \big)$

Example:

Consider differential operator

$$L_z = b(z)\frac{\partial \bullet}{\partial z} + \frac{1}{2}a(z)\frac{\partial^2 \bullet}{\partial z^2}$$

Show that the adjoint operator is

$$L_z^* = -\frac{\partial}{\partial z}\big(b(z)\bullet\big) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\big(a(z)\bullet\big)$$

(The derivation is in your homework)

A more intuitive derivation of the forward equation

We discard the previous derivation of the forward equation. So logically,

- we don't know the mathematical expression of the forward equation;
- we know that the ensemble density is governed by the forward equation.

First, we look at the IVP of the backward equation

$$\begin{cases} q_t = A[q] \\ q(x,0) = u(x) \end{cases} \qquad \text{where } A = L_x.$$

Function $q(x, t)$ has the meaning:

- Variable t in the backward equation = *Real time* $(T-t)$.
- The amount of reward is determined at *real time* T as $u(X(T))$.
- $q(x, t)$ = conditional average of position-dependent reward given $X(T-t) = x$.

$$q(x,t) = E\Big(u(X(T))\,\Big|\,X(T-t) = x\Big)$$

Next, we look at the IVP of the forward equation

$$\begin{cases} p_t = B[p] \\ p(x, T-t_0) = v(x) \end{cases}$$

where operator B is to be determined. We are going to show $B = A^*$.

Function $p(x, t)$ has the meaning:

- $p(x, T-t_0) = v(x)$ is the starting density of ensemble X at *real time* $(T-t_0)$.
- $p(x, t)$ = density of ensemble X at *real time* t.

Key observation:

Consider ensemble X starting with density $p(x, T-t_0)$ at *real time* $(T-t_0)$.

The reward occurs at real time T. Given the starting time $(T-t_0)$ and the starting density $p(x, T-t_0)$, the average amount of reward that the ensemble gets at real time T is completely determined!

Example:

Consider an ensemble of (independent) random walks governed by

$$dX = dW$$

Suppose the reward is determined at *real time* T as

$$\text{Reward} = \begin{cases} 1, & X(T) > c_0 \\ 0, & \text{otherwise} \end{cases}$$

Start the ensemble at real time $(T-t_0)$ with density $p(x, T-t_0) \propto \delta(x-x_0)$.

The average amount of reward for the ensemble is completely determined by the starting time $(T-t_0)$ and the starting density $p(x, T-t_0)$.

$$\text{Average reward} = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{x_0 - c_0}{\sqrt{2t_0}}\right)$$

(See Appendix A for the derivation)

Example:

The same ensemble and the same time evolution as above.

Now suppose the reward is determined at *real time* T as

$$\text{Reward} = \begin{cases} X(T) - c_0, & X(T) > c_0 \\ 0, & \text{otherwise} \end{cases}$$

The average amount of reward for the ensemble is

$$\text{Average reward} = \sqrt{\frac{t_0}{2\pi}}\exp\left(\frac{-(x_0 - c_0)^2}{2t_0}\right) + \frac{x_0 - c_0}{2} + \frac{x_0 - c_0}{2}\text{erf}\left(\frac{x_0 - c_0}{\sqrt{2t_0}}\right)$$

(See Appendix A for the derivation)

Mathematical formulation:

Let $X(T-\tau) \sim p(x, T-\tau)$ denote the situation where ensemble X has density $p(x, T-\tau)$ at real time $(T-\tau)$.

Using the definition of $q(x, t)$, the average amount of reward for the ensemble based on the density at real time $(T-t_0)$ is

$$E\left(u(X(T))\middle| X(T-t_0)\sim p(x,T-t_0)\right)=\int \underbrace{q(x,t_0)}_{\substack{\text{expected reward}\\ \text{start at position } x\\ \text{at real time } (T-t_0)}} \cdot \underbrace{p(x,T-t_0)}_{\substack{\text{ensemble density}\\ \text{of position } x\\ \text{at real time } (T-t_0)}} dx$$

The average reward for the ensemble based on the density at real time $(T-t_2)$ is

$$E\left(u(X(T))\middle| X(T-t_2)\sim p(x,T-t_2)\right)=\int \underbrace{q(x,t_2)}_{\substack{\text{expected reward}\\ \text{start at position } x\\ \text{at real time } (T-t_2)}} \cdot \underbrace{p(x,T-t_2)}_{\substack{\text{ensemble density}\\ \text{of position } x\\ \text{at real time } (T-t_2)}} dx$$

If $p(x, T-t_0)$ and $p(x, T-t_2)$ are the densities of the same ensemble at real times $(T-t_0)$ and $(T-t_2)$, then these two amounts of reward should be the same.

$$\int q(x,t_0)p(x,T-t_0)dx=\int q(x,t_2)p(x,T-t_2)dx \quad \text{for all } t_0\ge 0, t_2\ge 0$$

In summary, if $p(x, \tau)$ describes the density vs τ of ensemble X, then

$$\int q(x,t)p(x,T-t)dx \text{ is independent of } t.$$

In particular, we have

$$\int q(x,0)p(x,T)dx=\int q(x,\Delta t)p(x,T-\Delta t)dx \qquad \text{(EQ01)}$$

(EQ01) provides a way of deriving the forward equation (operator B).

Writing out operator B

We set $q(x,0)=u(x)$ and $p(x,T-\Delta t)=v(x)$.

That is, we set the reward function to $u(x)$ and start the ensemble at real time $(T-\Delta t)$ with density $v(x)$.

We expand $q(x, \Delta t)$ around $(x, 0)$.

$$\begin{aligned} q(x,\Delta t)&=q(x,0)+\Delta t\cdot q_t\big|_{t=0}+o(\Delta t)=u(x)+\Delta t\cdot A\left[q\big|_{t=0}\right]+o(\Delta t)\\ &=u(x)+\Delta t\cdot A[u]+o(\Delta t)\end{aligned}$$

We expand $p(x, T)$ around $(x, T-\Delta t)$.

$$\begin{aligned} p(x,T)&=p(x,T-\Delta t)+\Delta t\cdot p_t\big|_{t=(T-\Delta t)}+o(\Delta t)=v(x)+\Delta t\cdot B\left[p\big|_{t=(T-\Delta t)}\right]+o(\Delta t)\\ &=v(x)+\Delta t\cdot B[v]+o(\Delta t)\end{aligned}$$

Substituting the expansions into (EQ01), we obtain

$$\int u(x)\left(v(x)+\Delta t\cdot B[v]\right)dx=\int\left(u(x)+\Delta t\cdot A[u]\right)v(x)dx+o(\Delta t)$$

==> $$\Delta t\int u(x)B[v]dx=\Delta t\int A[u]v(x)dx+o(\Delta t)$$

Dividing by Δt and taking the limit as $\Delta t\to 0$, we arrive at

$$\int u(x)B[v]dx=\int A[u]v(x)dx$$

==> $$\left\langle u,B[v]\right\rangle=\left\langle A[u],v\right\rangle$$

Since this is valid for all $u(x)$ and $v(x)$, we conclude

$$B = A^*$$

Therefore, the forward equation is

$$p_t=L_x^*\left[p\right]$$

Caution:

In the above, the time invariability refers to the average amount of reward for the entire ensemble. During the time evolution, while the distribution density of the ensemble changes, the membership remains the same: it does not gain any new member nor does it lose any member.

For a random sample (member) in the ensemble, the expected amount of reward for the sample is the same as the average amount of reward for the ensemble.

When the realized position X of the sample becomes known, this additional information changes the situation. It is no longer a random sample of the original ensemble. Effectively, the additional condition narrows the ensemble in consideration. Consequently, the conditional expected amount of reward for the sample given the additional information would be different.

Example:

Suppose I am viewed as a random sample of a sub-population (an ensemble).

In my doctor's eye, Prob(developing cancer by age A) is well defined.

Without examining me or doing any test on me, that probability is unchanged over the time because in my doctor's eye, the representative sub-population remains the same.

Any new test result has the potential of narrowing the representative sub-population and changing the probability of cancer.

Example:

Consider a call option of a stock.

A call option is the right (not obligation) to buy a certain number of shares of the stock at a specified price at the expiration date.

The amount of reward for the call option holder at the expiration date

= (market price of the stock at the expiration date) – (specified price).

The expected amount of reward is affected by

- current stock price
- time to the expiration date
- volatility of the stock

The price of the call option reflects the expected amount of reward.

On each day, when the realized stock price becomes known (big jump upward or downward), the expected amount of reward is influenced and the price of the call option changes accordingly.

We will discuss the stock option pricing later.

The end of intuitive derivation of the forward equation

Boundary conditions for the forward equation

Absorbing boundary at $x = L$

- When a particle gets to $x = L$, it is removed from the ensemble; it no longer participate in the evolution of ensemble beyond that time.
- When a game reaches $x = L$, it is stopped and removed from the ensemble; it no longer participate in the evolution of ensemble beyond that time.

Mathematically, the absorbing boundary is described by

$$p(x,t)\Big|_{x=L} = 0$$

Reflecting boundary at $x = L$

- When a particle gets to $x = L$, it is not allowed to pass through; it is not removed from the ensemble; it is "turned back".
- When a game reaches $x = L$, it is not allowed to go over that; it is not removed from the ensemble; it is "turned back".

Mathematically, the reflecting boundary is described by

$$J(x,t)\Big|_{x=L} = 0$$

where $J(x,t) \equiv b(x)p - \frac{1}{2}\big(a(x)p\big)_x$ is the flux at position x at time t.

==> $$\left.\left(b(x)p-\frac{1}{2}\left(a(x)p\right)_x\right)\right|_{x=L}=0$$

Next we look at applications of the forward and the backward equations.

Exit time (escape problem):

Suppose $X(t)$ is governed by the SDE

$$dX=b(X)dt+\sqrt{a(X)}\,dW$$

Consider the problem of exiting (i.e., escaping from) a prescribed region.

We study the time until escape, also called exit time or escape time.

Probability of exit by time t

Let Y = the exit time (a random variable).

Let $u(x, t)$ = probability of exiting the region by time t given starting at x at time 0.

$$u(x,t)\equiv \Pr\left(Y\le t\,\middle|\,X(0)=x\right)$$

Governing equation for $u(x, t)$

Note that when deriving the governing differential equation for u, we don't need to know the region. The specifications of the region will be reflected in the boundary conditions for the differential equation.

For x inside the region, when dt is small enough, we have

$$u(x,t)=E\left(u(x+dX,t-dt)\right)+o(dt)$$

Use Taylor expansion to show that u satisfies the backward equation

$$u_t=\frac{1}{2}a(x)u_{xx}+b(x)u_x$$

(Derivation is in your homework).

Average exit time

Let $T(x)$ be the average exit time given that $X(0) = x$.

$$T(x)=E\left(Y\,\middle|\,T(0)=x\right)$$

For x inside the region, when dt is small enough, we have

$$T(x) = E\big(T(x+dX)\big) + dt + o(dt)$$

Using Taylor expansion, we can derive the ODE for $T(x)$.

$$-1 = \frac{1}{2}a(x)T_{xx} + b(x)T_x$$

We look at a few examples before discussing "escape of a Brownian particle".

Example:

$a(x) = 1, \quad b(x) = 0$

The region is $[0, L]$. Exit can occur at both $x = 0$ and $x = L$.

We have seen this problem before. The exit time $T(x)$ here has the meaning of the time until breaking the bank or bankrupt in gambler's ruin problem where

x: your initial cash; L: total cash of casino + you

The governing equation for $T(x)$ is

$$-1 = \frac{1}{2}T_{xx}$$

The boundary value problem (BVP) for $T(x)$ is

$$\begin{cases} T_{xx} = -2 \\ T(0) = 0, \quad T(L) = 0 \end{cases}$$

The solution is

$T(x) = x\,(L-x)$

In particular, we have

$T(L/2) = L^2/4$

Example:

$a(x) = 1, \quad b(x) = b$

The region is $[L_1, L_2]$. Exit can occur at both L_1 and L_2.

The boundary value problem (BVP) for $T(x)$ is

$$\begin{cases} T_{xx} + 2bT_x = -2 \\ T(L_1) = 0, \quad T(L_2) = 0 \end{cases}$$

We solved a similar problem before, in the gambler's ruin problem.

The solution is

$$T(x)=\frac{1}{b}(L_2-x)-\frac{1}{b}(L_2-L_1)\cdot\frac{\exp(2b(L_2-x))-1}{\exp(2b(L_2-L_1))-1}$$

(Derivation is in your homework).

For $b > 0$, as $L_1 \to -\infty$, we have

$$T(x)\to\frac{1}{b}(L_2-x) \quad \text{as } L_1\to-\infty\,.$$

This is consistent with the picture of a deterministic escape.

In a deterministic escape, we always have

$T_{\text{deterministic}}(x) = (L_2 - x)/b$ independent of L_1.

<u>A model for casino:</u>

As a player, casino has $b > 0$. All gamblers in casino are collectively viewed as the other player with new arrivals replacing the departing ones.

Suppose casino has no cash ($x = 0$) but has a line of credit. It is solvent as long as the balance is above L_1 which is negative ($L_1 < 0$).

The end of game is defined as either casino's balance dropping below L_1 (which is very unlikely) or casino winning L_2 amount from the other player, the collection of gamblers. The average time until the end of game is

$$T(0)\to\frac{1}{b}L_2 \quad \text{as } L_1\to-\infty$$

Basically, for the casino "the end of each game" is an earning of L_2 cash.

Appendix

Start X at real time $T-t_0$ with ensemble density $p(x, T-t_0) \propto \delta(x-x_0)$.

$X(t)$ evolves according to $dX = dW$.

==> $X(T-t_0+\Delta t) \sim N(x_0, \Delta t)$

==> $X(T) \sim N(x_0, t_0)$

Case 1: Suppose the reward is determined at *real time T* as

$$\text{Reward} = \begin{cases} 1, & X(T) > c_0 \\ 0, & \text{otherwise} \end{cases}$$

The average amount of reward for the ensemble is

$$\text{Average reward} = \int_{c_0}^{+\infty} \rho_{N(x_0,t_0)}(x)dx = \int_{c_0-x_0}^{+\infty} \rho_{N(0,t_0)}(\xi)d\xi$$

$$= \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{x_0-c_0}{\sqrt{2t_0}}\right)$$

Case 2: Suppose the reward is determined at *real time T* as

$$\text{Reward} = \begin{cases} X(T)-c_0, & X(T) > c_0 \\ 0, & \text{otherwise} \end{cases}$$

The average amount of reward for the ensemble is

$$\text{Average reward} = \int_{c_0}^{+\infty} (x-c_0)\rho_{N(x_0,t_0)}(x)dx = \int_{c_0-x_0}^{+\infty} \left(\xi-(c_0-x_0)\right)\rho_{N(0,t_0)}(\xi)d\xi$$

$$= \sqrt{\frac{t_0}{2\pi}}\exp\left(\frac{-(x_0-c_0)^2}{2t_0}\right) + \frac{x_0-c_0}{2} + \frac{x_0-c_0}{2}\text{erf}\left(\frac{x_0-c_0}{\sqrt{2t_0}}\right)$$