AM216 Stochastic Differential Equations

Lecture 09

List of topics in this lecture

- Convergence in probability, sufficient condition for convergence in probability:

  $$\lim_{N\to\infty} E\big(Q_N(\omega)\big)=0 \;\text{ and }\; \lim_{N\to\infty} \text{var}\big(Q_N(\omega)\big)=0$$

- Ito's interpretation, Stratonovich's interpretation of stochastic integrals, the relation between the two, proof of Ito's lemma
- Stochastic integrals based on axioms, the λ-chain rule

Recap

Ornstein-Uhlenbeck process:

$Y(t)$: particle velocity; $X(t)$: particle position

Diffusion coefficient of particle position:

$$D \equiv \lim_{t\to\infty} \frac{1}{2t}\text{var}\big(X(t)-X(0)\big)$$

Einstein-Smoluchowski relation: $D=\dfrac{k_B T}{b}$

It connects the diffusion coefficient and the drag coefficient.

Going backward in time in equilibrium

Key point: Scaling $t_{\text{new}} = -t_{\text{old}}$ does not work in stochastic differential equation!

Tool: Bayes theorem

$$\big(Y(-t)\big|Y(0)=y_2\big) \sim N\left(e^{-\beta t}y_2,\, \frac{\gamma^2}{2\beta}(1-e^{-2\beta t})\right) \quad \text{for } t>0$$

Comparing with going forward in time

$$\big(Y(t)\big|Y(0)=y_0\big) \sim N\left(e^{-\beta t}y_0,\, \frac{\gamma^2}{2\beta}(1-e^{-2\beta t})\right) \quad \text{for } t>0$$

Time reversibility of equilibrium

Samples of $Y(t)$ are indistinguishable from samples of $\tilde{Y}(t)=Y(-t)$. Given a set of samples, we cannot tell if they are from $Y(t)$ or $\tilde{Y}(t)=Y(-t)$.

Different interpretations of stochastic integrals

$$\int_0^t f(s,W(s))dW(s)=\lim_{N\to\infty}\sum_{j=0}^{N-1} f(\tilde{s}_j,W(\tilde{s}_j))\Delta W_j$$

where

$$\Delta s=\frac{t}{N}, \qquad s_j=j\Delta s, \qquad \tilde{s}_j\in[s_j\ s_{j+1}]$$
$$W_j=W(s_j), \qquad \Delta W_j=W_{j+1}-W_j$$

Note that

- $f(s, W(s))$ is not a deterministic function of s.
- $f(\tilde{s}_j, W(\tilde{s}_j))$ is a random variable, correlated with ΔW_j.
- As a result, different choices of $\tilde{s}_j \in [s_j, s_{j+1}]$ lead to different results.
- Thus, integral $\int_0^t f(s,W(s))dW(s)$ is subject to different interpretations.

Ito's interpretation (Kiyosi Ito):

$\tilde{s}_j = s_j$

$$\int_0^t f(s,W(s))dW(s)=\lim_{N\to\infty}\sum_{j=0}^{N-1} f(s_j,W(s_j))\Delta W_j$$

Stratonovich's interpretation (Ruslan Stratonovich):

(based on the trapezoidal rule)

$$\tilde{s}_j=\frac{1}{2}(s_j+s_{j+1})$$

$$\int_0^t f(s,W(s))dW(s)=\lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}\Big(f(s_j,W(s_j))+f(s_{j+1},W(s_{j+1}))\Big)\Delta W_j$$

Note: Stratonovich's interpretation is defined based on the trapezoidal rule; it is not exactly the Riemann sum with $\tilde{s}_j=(s_j+s_{j+1})/2$.

Road map of the discussion:

1. We show that the trapezoidal rule expression of Stratonovich's interpretation is equivalent to the Riemann sum with $\tilde{s}_j = (s_j + s_{j+1})/2$.

$$\lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}\Big(f(s_j,W(s_j))+f(s_{j+1},W(s_{j+1}))\Big)\Delta W_j$$
$$=\lim_{N\to\infty}\sum_{j=0}^{N-1} f\Big(s_{j+1/2},W(s_{j+1/2})\Big)\Delta W_j\,,\qquad s_{j+1/2}=\frac{s_j+s_{j+1}}{2} \tag{E01}$$

   In the process, we introduce convergence in probability and prove a theorem regarding the general case of (E01).
2. We demonstrate the relation between the Ito interpretation and the Stratonovich interpretation in a simple example.
3. To connect the Ito interpretation and the Stratonovich interpretation in the general situation, we prove the integral form of Ito's lemma (a theorem).
4. We write out the relation between Ito and Stratonovich interpretations.

Item 1 of the road map: (average of Riemann sums)

Convergence in probability

Definition

Let $\{Q_N(\omega)\}$ be a sequence of random variables. We say $\{Q_N(\omega)\}$ converges to 0 in probability if for any $\varepsilon > 0$,

$$\lim_{N\to\infty}\Pr\Big(|Q_N(\omega)|>\varepsilon\Big)=0$$

Note: The background and motivation of this definition and the connection to the convergence of a deterministic sequence is discussed in your homework.

Next, we state a sufficient condition for convergence in probability

Theorem:

Let $\{Q_N(\omega)\}$ be a sequence of random variables. Suppose

$$\lim_{N\to\infty}E\Big(Q_N(\omega)\Big)=0 \ \text{ and } \ \lim_{N\to\infty}\mathrm{var}\Big(Q_N(\omega)\Big)=0\,.$$

Then, $\{Q_N(\omega)\}$ converges to 0 in probability.

Proof: in your homework.

We are ready to state the theorem regarding the average of Riemann sums.

Theorem:

For any $0 \le \alpha \le 1$, we have (in probability) that

$$\lim_{N\to\infty}\sum_{j=0}^{N-1}\Big(\alpha f(s_j,W(s_j))+(1-\alpha)f(s_{j+1},W(s_{j+1}))\Big)\Delta W_j$$

$$=\lim_{N\to\infty}\sum_{j=0}^{N-1} f(\tilde{s}_j,W(\tilde{s}_j))\Delta W_j \quad \text{where } \tilde{s}_j=\alpha s_j+(1-\alpha)s_{j+1}$$

Note: Both sides are still random variables.

Proof: Let

$$Q_N \equiv \sum_{j=0}^{N-1}\Big[\alpha f(s_j,W(s_j))+(1-\alpha)f(s_{j+1},W(s_{j+1}))-f(\tilde{s}_j,W(\tilde{s}_j))\Big]\Delta W_j$$

$\{ Q_N \}$ is a sequence of random variables. We only need to show

$$\lim_{N\to\infty} E\big(Q_N(\omega)\big)=0 \quad\text{and}\quad \lim_{N\to\infty}\text{var}\big(Q_N(\omega)\big)=0$$

The rest of proof (a reduced version) is in your homework.

Item 2 of the road map: (an example)

Key point:

Ito interpretation and Stratonovich interpretation yield different values!

Example:

$$I=\int_0^t W(s)dW(s)$$

Discretization and notations:

$$\Delta s=\frac{t}{N}, \qquad s_j=j\Delta s, \qquad W_j=W(s_j), \qquad \Delta W_j=W_{j+1}-W_j$$

Stratonovich interpretation:

$$I_{\text{Stratonovich}}=\lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}(W_j+W_{j+1})(\Delta W_j)=\lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}(W_j+W_{j+1})(W_{j+1}-W_j)$$

$$=\lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}\Big((W_{j+1})^2-(W_j)^2\Big)=\frac{1}{2}\Big((W_N)^2-(W_0)^2\Big)=\frac{1}{2}W(t)^2$$

Ito interpretation:

$$I_{\text{Ito}}=\lim_{N\to\infty}\sum_{j=0}^{N-1}W_j(\Delta W_j)$$

$$= \lim_{N\to\infty} \sum_{j=0}^{N-1} \left(\frac{1}{2}(W_j + W_{j+1}) - \frac{1}{2}(W_{j+1} - W_j) \right) \Delta W_j$$

$$= \underbrace{\lim_{N\to\infty} \sum_{j=0}^{N-1} \frac{1}{2}(W_j + W_{j+1}) \Delta W_j}_{\text{Stratonovich}} - \lim_{N\to\infty} \underbrace{\sum_{j=0}^{N-1} \frac{1}{2}(\Delta W_j)^2}_{\text{Converging to } \frac{1}{2}\int_0^t (dW)^2}$$

$$= I_{\text{Stratonovich}} - \frac{1}{2}\int_0^t (dW)^2$$

In homework, you showed $\int_0^t (dW)^2 = t$. Thus, we obtain.

$$I_{\text{Ito}} = I_{\text{Stratonovich}} - \frac{1}{2} t$$

<u>Caution:</u> This is only for the example $\int_0^t W(s) dW(s)$.

<u>Item 3 of the road map:</u>

<u>Theorem</u> (integral form of Ito's lemma)

Suppose $g(s, w)$ is a smooth function of (s, w). We have

$$\lim_{N\to\infty} \sum_{j=0}^{N-1} g(s_j, W_j)(\Delta W_j)^2 = \lim_{N\to\infty} \sum_{j=0}^{N-1} g(s_j, W_j)\Delta s$$

<u>Remarks:</u>

- The Riemann sums converge to the integrals
$$\int_0^t g(s, W(s))(dW(s))^2 = \int_0^t g(s, W(s)) ds$$
- Both the Riemann sums and the integrals are still random variables.
- This is equivalent to Ito's lemma in that $(dW(s))^2$ can be replaced with ds.

<u>Proof:</u> Let

$$Q_N \equiv \sum_{j=0}^{N-1} g(s_j, W_j)(\Delta W_j)^2 - \sum_{j=0}^{N-1} g(s_j, W_j)\Delta s = \sum_{j=0}^{N-1} g(s_j, W_j)\left((\Delta W_j)^2 - \Delta s\right)$$

$\{ Q_N \}$ is a sequence of random variables. We only need to show

$$\lim_{N\to\infty} E\left(Q_N(\omega)\right) = 0 \quad \text{and} \quad \lim_{N\to\infty} \text{var}\left(Q_N(\omega)\right) = 0$$

<u>Step 1:</u>

Verifying $E(Q_N) = 0$

Notice the fact that ΔW_j is independent of W_j. The independence gives us

$$E(Q_N) = \sum_{j=0}^{N-1} E\Big(g(s_j, W_j)\big((\Delta W_j)^2 - \Delta s\big)\Big)$$

$$= \sum_{j=0}^{N-1} E\big(g(s_j, W_j)\big) E\big((\Delta W_j)^2 - \Delta s\big) = 0$$

Step 2:

Verifying $\text{var}(Q_N) \to 0$ as $N \to +\infty$

Since $E(Q_N) = 0$, we have

$$\text{var}(Q_N) = E(Q_N{}^2) = E\left(\left[\sum_{j=0}^{N-1} g(s_j, W_j)\big((\Delta W_j)^2 - \Delta s\big)\right]^2\right)$$

$$= E\left(2\sum_{k=0}^{N-1}\sum_{j>k} g(s_k, W_k) g(s_j, W_j)\big((\Delta W_k)^2 - \Delta s\big)\big((\Delta W_j)^2 - \Delta s\big)\right)$$
$$+ E\left(\sum_{j=0}^{N-1} g(s_j, W_j)^2 \big((\Delta W_j)^2 - \Delta s\big)^2\right)$$

For $j > k$, ΔW_j is independent of ΔW_k, W_k, and W_j.

Using this fact, we write the variance as

$$\text{var}(Q_N) = 2\sum_{k=0}^{N-1}\sum_{j>k} E\Big(g(s_k, W_k) g(s_j, W_j)\big((\Delta W_k)^2 - \Delta s\big)\Big)\cdot \underbrace{E\big((\Delta W_j)^2 - \Delta s\big)}_{=0}$$
$$+ \left(\sum_{j=0}^{N-1} E\big(g(s_j, W_j)^2\big) E\big(((\Delta W_j)^2 - \Delta s)^2\big)\right)$$

Using $(\Delta W_j)^2 = O(\Delta s)$, we have $((\Delta W_j)^2 - \Delta s)^2 = O((\Delta s)^2)$

$$= \sum_{j=0}^{N-1} E\big(g(s_j, W_j)^2\big) O\big((\Delta s)^2\big) = O(\Delta s) \to 0 \quad \text{as } N \to \infty$$

End of proof of the theorem

Item 4 of the road map: (Relation between Ito and Stratonovich interpretations)

Now we look at the general case: $\int_0^t f(s, W(s)) dW(s)$.

We start with the Stratonovich interpretation:

$$I_{\text{Stratonovich}} = \lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}\Big(f(s_j,W_j)+f(s_{j+1},W_{j+1})\Big)\Delta W_j$$

$$= \lim_{N\to\infty}\sum_{j=0}^{N-1}\left[f(s_j,W_j)\Delta W_j + \frac{1}{2}\Big(f(s_{j+1},W_{j+1})-f(s_j,W_j)\Big)\Delta W_j\right]$$

We do Taylor expansion of the second term around s_j.

We neglect all o(Δs) terms in the summand. Recall that $\Delta W \sim O(\sqrt{\Delta s})$. We have

$$\Big(f(s_{j+1},W_{j+1})-f(s_j,W_j)\Big)\Delta W_j$$

$$= \left((f_t)_j\Delta s + (f_w)_j\Delta W_j + \frac{1}{2}(f_{ww})_j(\Delta W_j)^2 + o(\Delta s)\right)\Delta W_j$$

$$= (f_w)_j(\Delta W_j)^2 + o\big(\Delta s\big), \qquad (f_w)_j \equiv f_w(s_j,W_j)$$

Substituting into the summation of Stratonovich interpretation, we get

$$I_{\text{Stratonovich}} = \underbrace{\lim_{N\to\infty}\sum_{j=0}^{N-1}f(s_j,W_j)\Delta W_j}_{\text{Ito}} - \lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}(f_w)_j(\Delta W_j)^2 + \underbrace{\lim_{N\to\infty}\sum_{j=0}^{N-1}o(\Delta s)}_{\substack{\text{Converging}\\ \text{to zero}}}$$

The second term has a simple expression:

$$\lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}(f_w)_j(\Delta W_j)^2 = \lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}(f_w)_j\Delta s = \frac{1}{2}\int_0^t f_w(s,W(s))ds$$

which follows directly from the theorem above (integral form of Ito's lemma).

Combining these results, we obtain the main theorem connecting the Stratonovich interpretation and the Ito interpretation.

Theorem:

For integral $\int_0^t f(s,W(s))dW(s)$, we have

$$I_{\text{Stratonovich}} = I_{\text{Ito}} + \frac{1}{2}\int_0^t f_w(s,W(s))ds$$

Stochastic integrals based on axioms

In the above, we interpreted stochastic integrals as limits of Riemann sums. Different choices of Riemann sums lead to different interpretations. Alternatively, we can calculate stochastic integrals based on a set of axioms.

Two axioms:

1) Fundamental theorem of calculus (FTC)

$$\int_a^b dH(t,W(t)) = H(t,W(t))\Big|_a^b$$

2) λ-chain rule

$$dH(t,W(t)) = H_t dt + H_w dW(t) + \left(\frac{1}{2} - \lambda\right) H_{ww} dt$$

Ito's interpretation: $\lambda = 0$

Stratonovich's interpretation: $\lambda = 0.5$

Meaning of the λ-chain rule

Case 1: deterministic function $u(t)$

Consider the increment

$$\Delta H(t,u(t)) \equiv H(t+\Delta t, u(t+\Delta t)) - H(t,u(t))$$

Taylor expansion around t

$$\Delta H(t,u(t)) = H_t(t,u(t))\Delta t + H_u(t,u(t)) \underbrace{\Delta u}_{O(\Delta t)} + o(dt) \qquad \text{(E02A)}$$

where $\Delta u = u(t+\Delta t) - u(t) = O(\Delta t)$

Taylor expansion around $(t + \Delta t)$

$$\Delta H(t,u(t)) = H_t(t+\Delta t, u(t+\Delta t))\Delta t + H_u(t+\Delta t, u(t+\Delta t)) \underbrace{\Delta u}_{O(\Delta t)} + o(dt) \qquad \text{(E02B)}$$

Notice that (E02A) and (E02B) have the same form.

Therefore, we can write it as a differential with no ambiguity.

$$dH(t,u(t)) = H_t dt + H_u du$$

- We don't need to treat dt as finite Δt,
- We don't need to distinguish $H_u(t,u(t))\Delta u$ and $H_u(t+\Delta t, u(t+\Delta t))\Delta u$

That is the beauty of deterministic calculus.

Case 2: stochastic process $W(t)$

Consider the increment

$$\Delta H(t,W(t)) \equiv H(t+\Delta t, W(t+\Delta t)) - H(t,W(t))$$

Ito's interpretation:

Taylor expansion around t

$$\Delta H(t,W(t)) = H_t(t,W(t))\Delta t + H_w(t,W(t))\Delta W + \frac{1}{2}H_{ww}(t,W(t))(\Delta W)^2 + o\left(\Delta t\right)$$

Replacing $(\Delta W)^2$ by Δt, We write it symbolically as a differential:

$$dH(t,W(t)) = H_t(t,W(t))dt + H_W(t,W(t))dW + \frac{1}{2}H_{WW}(t,W(t))dt$$

with the understanding $H_w = H_w(t,W(t))$.

This is the λ-chain rule with $\lambda = 0$.

Stratonovich's interpretation:

Taylor expansion around $(t + \Delta t/2)$

$$\Delta H(t,W(t)) = \underbrace{H\big|_{t+\Delta t} - H\big|_{t}}_{\text{Short notation}} = \left(H\big|_{t+\Delta t} - H\big|_{t+\Delta t/2}\right) - \left(H\big|_{t} - H\big|_{t+\Delta t/2}\right)$$

$$= \left(H_t\big|_{t+\Delta t/2}\frac{\Delta t}{2} + H_w\big|_{t+\Delta t/2}\Delta W^+ + \frac{1}{2}H_{ww}\big|_{t+\Delta t/2}(\Delta W^+)^2\right)$$
$$- \left(H_t\big|_{t+\Delta t/2}\left(\frac{-\Delta t}{2}\right) + H_w\big|_{t+\Delta t/2}(-\Delta W^-) + \frac{1}{2}H_{ww}\big|_{t+\Delta t/2}(-\Delta W^-)^2\right)$$

where

$$H_w\big|_{t+\Delta t/2} \equiv H_w\left(t + \frac{\Delta t}{2}, W\left(t + \frac{\Delta t}{2}\right)\right)$$

$$\Delta W^- \equiv W\left(t + \frac{\Delta t}{2}\right) - W(t), \qquad \Delta W^+ \equiv W(t+\Delta t) - W\left(t + \frac{\Delta t}{2}\right)$$

Combining similar terms, we write ΔH as

$$\Delta H = H_t\big|_{t+\Delta t/2}\Delta t + H_w\big|_{t+\Delta t/2}\left(\Delta W^+ + \Delta W^-\right) + \frac{1}{2}H_{ww}\big|_{t+\Delta t/2}\left((\Delta W^+)^2 - (\Delta W^-)^2\right)$$

Replacing each of $(\Delta W^+)^2$ and $(\Delta W^-)^2$ by $\Delta t/2$, and using $\Delta W^+ + \Delta W^- = \Delta W$, we obtain

$$\Delta H = H_t\big|_{t+\Delta t/2}\Delta t + H_w\big|_{t+\Delta t/2}\Delta W$$

We write it symbolically as a differential:

$$dH(t,W(t)) = H_t dt + H_w dW(t)$$

with the understanding $H_w = H_w\left(t+\frac{\Delta t}{2}, W\left(t+\frac{\Delta t}{2}\right)\right)$.

This is the λ-chain rule with λ = 0.5.

Remark:

The purpose of the λ-chain rule is to implicitly distinguish

$H_w(t, W(t))\Delta W$ and $H_w\left(t+\frac{\Delta t}{2}, W\left(t+\frac{\Delta t}{2}\right)\right)\Delta W$

Use the axioms to calculate $\int_a^b f(t, W(t))dW(t)$

Strategy:

Use the two axioms to capture the tricky part of integral $\int_a^b f(t, W(t))dW(t)$.

Specifically, we use the λ-chain rule to write the integrand as

$$f(t, W(t))dW(t) = dH - g(t, W(t))dt$$

$$\Longrightarrow \quad \int_a^b f(t, W(t))dW(t) = \underbrace{\int_a^b dH}_{\substack{\text{Fundamental} \\ \text{theorem of calculus}}} - \underbrace{\int_a^b g(t, W(t))dt}_{\substack{\text{This part is} \\ \text{not tricky}}}$$

Remarks:

- Integral $\int_a^b g(t, W(t))dt$ is not subject to different interpretations.
- Different interpretations are reflected in the integrand $g(t, W(t))$, calculated using the λ-chain rule.

Procedure:

Step 1:

From the integrand, identify $f(t, w)$, a regular function of 2 variables.

Set $H(t, 0) = 0$ and $H_w(t, w) = f(t, w)$. Integrate with respect to w to obtain

$H(t, w) = \int_0^w f(t, u)du$ which is just a regular integral.

Step 2:

Differentiate to calculate $H_t(t, w)$ and $H_{ww}(t, w)$.

Again, both are regular derivatives and regular functions.

Step 3:

Use the definition of $H(t, w)$ to write the integrand as

$$f(t,W(t))dW(t)=H_w(t,W(t))dW(t)$$

Step 4:

Use the λ-chain rule $dH=H_t dt+H_w dW+\left(\tfrac{1}{2}-\lambda\right)H_{ww}dt$, we write

$$H_w dW=dH-\left(H_t+\left(\frac{1}{2}-\lambda\right)H_{ww}\right)dt$$

$$f(t,W(t))dW(t)=dH-\left(H_t+\left(\frac{1}{2}-\lambda\right)H_{ww}\right)dt$$

This step is subject to different interpretations (different values of λ).

Step 5:

Use the fundamental theorem of calculus to calculate the integral

$$\int_a^b f(t,W(t))dW(t)=H(t,W(t))\Big|_a^b-\int_a^b\left(H_t+\left(\frac{1}{2}-\lambda\right)H_{ww}\right)dt$$

Example:

$$\int_a^b tW(t)^2\,dW(t)$$

1. Identify $f(t, w) = t\,w^2$

   Set $H(t,0)=0$ and $H_w(t,w)=t\,w^2$

$$H(t,w)=\int_0^w t\,u^2\,du=t\frac{w^3}{3}$$

2. Differentiate to calculate $H_t(t,w)$ and $H_{ww}(t,w)$.

$$H_t(t,w)=\frac{w^3}{3},\qquad H_{ww}(t,w)=2t\,w$$

3. Write the integrand as

$$f(t,W(t))dW(t)=H_w(t,W(t))dW(t)$$

4. Use the λ-chain rule to write

$$f(t,W(t))dW(t) = dH - \left(H_t + \left(\frac{1}{2} - \lambda\right)H_{ww}\right)dt$$

$$\Longrightarrow \quad tW(t)^2 dW(t) = \underbrace{d\left(t\frac{W(t)^3}{3}\right)}_{dH} - \left(\frac{W(t)^3}{3} + \left(\frac{1}{2} - \lambda\right)2tW(t)\right)dt$$

<u>5.</u> Use the fundamental theorem of calculus to calculate the integral

$$\int_a^b tW(t)^2 dW(t) = t\frac{W(t)^3}{3}\bigg|_a^b - \int_a^b \left(\frac{W(t)^3}{3} + \left(\frac{1}{2} - \lambda\right)2tW(t)\right)dW(t)$$

$$= b\frac{W(b)^3}{3} - a\frac{W(a)^3}{3} - \int_a^b \frac{W(t)^3}{3}dt - 2\left(\frac{1}{2} - \lambda\right)\int_a^b tW(t)dt$$

Ito: $\lambda = 0$; Stratonovich: $\lambda = 0.5$

All terms in the result are random variables.