AM216 Stochastic Differential Equations

Lecture 04
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List of topics in this lecture

- Properties of Wiener process, $dW = O(\sqrt{dt})$
- Discrete version of $W(t)$; arc length of $W(t)$ over finite time is infinity!
- Ito's lemma; $(dW)^2$ can be replaced by dt.
- The problem of gambler's ruin, survival probability vs time

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Recap

Stochastic differential equation

$$dX = b(X,t)dt + \sqrt{a(X,t)}\,dW$$

The Wiener process, denoted by $W(t)$ satisfies

1) $W(0) = 0$
2) For $t_2 \geq t_1 \geq 0$, $\quad W(t_2)-W(t_1) \sim N(0, t_2-t_1)$
3) For $t_4 \geq t_3 \geq t_2 \geq t_1 \geq 0$,

   increments $W(t_2)-W(t_1)$ and $W(t_4)-W(t_3)$ are independent.

Note:

$W(t)$ is a random process (a function with randomness).

The full notation is $W(t, \omega)$ where ω is the (random) outcome of an experiment.

Notation:

$$dW = dW(t) \equiv W(t+dt) - W(t)$$

$$dX = dX(t) \equiv X(t+dt) - X(t)$$

Shifting and scaling of a normal RV

- If $X \sim N(\mu, \sigma^2)$, then $(X-\mu) \sim N(0, \sigma^2)$.
- If $X \sim N(0, \sigma^2)$, then $X/\sigma \sim N(0, 1)$.
- If $X \sim N(0, 1)$, then X is called a standard normal RV.

General approach:

In discussion of stochastic differential equations, we work with finite value of time increment dt. Then at the end, we take the limit as $dt \to 0$.

The correct notation for finite time increment should be Δt. But most of the time, for convenience, we just use notation dt both before taking the limit and after.

Properties of Wiener process:

1) $dW \sim N(0, dt)$ ==> $dW = \sqrt{dt}\, X$ where $X \sim N(0,1)$
2) $E(dW) = 0$
3) $E((dW)^2) = dt$
4) $dW(t_1)$ and $dW(t_2)$ are independent if the time increments are disjoint.
5) $dW = O(\sqrt{dt})$

Reasoning:

Recall the 95%, 99% and 99.9% probability intervals of a normal RV

$$\begin{cases} \Pr(|X-\mu| \le 1.96\sigma) = 0.95, \\ \Pr(|X-\mu| \le 2.58\sigma) = 0.99 \\ \Pr(|X-\mu| \le 3.29\sigma) = 0.999 \end{cases}$$

$$\Longrightarrow \quad \begin{cases} |dW| \le 1.96\sqrt{dt} \text{ with } \Pr = 0.95 \\ |dW| \le 2.58\sqrt{dt} \text{ with } \Pr = 0.99 \\ |dW| \le 3.29\sqrt{dt} \text{ with } \Pr = 0.999 \end{cases}$$

==> Approximately we have $dW = O(\sqrt{dt})$

A discrete version of $W(t)$

The discrete version is easy to understand conceptually, and is practical to work with computationally in simulations.

We consider $W(t)$ at a set of discrete times.

$$\left\{ W_j = W(j\Delta t), \quad j = 0, 1, \ldots, n \right\}, \qquad \Delta t = \frac{t_f}{n}$$

Question: How to generate a sample path of $\{W_j, j = 0, 1, 2, \ldots\}$?

By definition, we have

$$(W_j - W_{j-1}) \sim N(0, \Delta t)$$

We express it in terms of a standard normal RV

$$(W_j - W_{j-1}) = \sqrt{\Delta t}\, X_j\,, \quad X_j \sim N\left(0,1\right)$$

In Matlab, "randn" generates independent samples of $N(0, 1)$.

With $W_0 = 0$, we write W_{j} as

$$W_j = \sqrt{\Delta t}\sum_{k=1}^{j} X_k\,, \quad j = 1, 2, \ldots, n$$

Method: Generate n independent samples of $X \sim N(0, 1)$

$$\left\{X_j,\ j = 1, 2, \ldots, n\right\} \sim \text{ i.i.d. } N(0,1)$$

$$W_0 = 0\,, \qquad W_j = \sqrt{\Delta t}\sum_{k=1}^{j} X_k\,, \quad j = 1, 2, \ldots, n$$

Remarks:

This method completely specifies the random experiment for generating a sample path of $\{W_{\mathrm{j}}, j = 0, 1, 2, \ldots\}$, the discrete version of $W(t)$. The experiment can be repeated a large number of times. All probabilities associated with $\{W_{\mathrm{j}}, j = 0, 1, 2, \ldots\}$ can be interpreted in the framework of repeated experiments.

It is difficult to write out a concrete experiment for generating a continuous sample path of $W(t)$. Any algorithm that can be implemented in a computer must be discrete, and as a result, cannot generate a full path of $W(t)$.

Mathematically, we can view $W(t)$ as the limit of $\{W_{\mathrm{j}}, j = 0, 1, 2, \ldots\}$ as $\Delta t \to 0$.

If you think $W(t)$ is somewhat unusual and different from the functions we are familiar with, it indeed is. Below we illustrate one aspect of $W(t)$.

A peculiar feature of $W(t)$

The arc length of $W(t)$ over the time interval $t \in [0, t_f]$ is infinity.

Derivation:

We start with the arc length of $\{W_{\mathrm{j}}, j = 0, 1, 2, \ldots\}$.

$$\text{Discrete arc length} = \sum_{j=1}^{n}\left|(t_j, W_j) - (t_{j-1}, W_{j-1})\right|$$

$$= \sum_{j=1}^{n} \left| (\Delta t,\ \sqrt{\Delta t}\, X_j) \right| = \sum_{j=1}^{n} \sqrt{(\Delta t)^2 + (\sqrt{\Delta t}\, X_j)^2}$$

$$> \sum_{j=1}^{n} \sqrt{\Delta t}\, |X_j| = n\sqrt{\Delta t} \left(\frac{1}{n} \sum_{j=1}^{n} |X_j| \right)$$

Use $\Delta t = \frac{t_f}{n}$, $\frac{1}{n}\sum_{j=1}^{n}\left|X_j\right| \approx E(|X|)$

$$\approx n\sqrt{\frac{t_f}{n}}\, E(|X|), \qquad X \sim N(0,1)$$

$$= \sqrt{n t_f}\sqrt{\frac{2}{\pi}} \longrightarrow +\infty \qquad \text{as } n \to +\infty$$

Here we used the result:

If $X \sim N(0, 1)$, then $E(|X|) = \sqrt{\frac{2}{\pi}}$

(Homework problem)

Therefore, we conclude that the arc length of the continuous $W(t)$ over a finite time interval $t \in [0, t_f]$ is infinity!

<u>Ito's lemma</u>:

Suppose $f(t, w)$ is a smooth function of variables t and w.

Replacing w by $W(t)$ gives us $f(t, W(t))$, a non-smooth random function of t. The randomness comes from the Wiener process $W(t, \omega)$.

We examine the increment of $f(t, W(t))$ using the expansion of $f(t, w)$.

$$\begin{aligned} f(t+dt, w+dw) &= f(t, w) + f_t dt + f_w dw \\ &+ \frac{1}{2}\left[f_{tt}(dt)^2 + 2 f_{tw}(dt)(dw) + f_{ww}(dw)^2 \right] \\ &+ O\left((dt)^3 + (dt)^2(dw) + (dt)(dw)^2 + (dw)^3 \right) \end{aligned}$$

Here dt and dw are small but finite. Now we apply the expansion to $f(t, W(t))$.

$$\begin{aligned} f(t+dt, W+dW) &= f(t, W) + f_t dt + f_w dW \\ &+ \frac{1}{2}\left[f_{tt}(dt)^2 + 2 f_{tw}(dt)(dW) + f_{ww}(dW)^2 \right] \qquad \text{(E01)} \\ &+ O\left((dt)^3 + (dt)^2(dW) + (dt)(dW)^2 + (dW)^3 \right) \end{aligned}$$

Using $dW = O(\sqrt{dt})$, we write $df(t,W(t)) \equiv f(t+dt, W+dW) - f(t,W)$ as

$$df(t,W(t)) = f_t dt + f_w dW + \frac{1}{2} f_{ww} (dW)^2 + o(dt)$$

Claim: we can approximately replace $(dW)^2$ by dt and write df as

$$df(t,W(t)) = f_t dt + f_w dW + \frac{1}{2} f_{ww} dt + o(dt)$$
$$= \left(f_t + \frac{1}{2} f_{ww} \right) dt + f_w dW + o(dt)$$

For any finite t_f, it takes t_f/dt steps to reach t_f. Over time interval $[0, t_f]$, the cumulative error converges to zero as $dt \to 0$.

This is called Ito's lemma.

We will look at the justification in subsequent lectures/assignments.

The averaged version:

The average of $df(t, W(t))$ can be calculated rigorously without any complication.

Averaging (E01) with respect to dW, and using $E(dW)=0$ and $E((dW)^2)=dt$, we have

$$E_{dW}\Big(f(t+dt, W+dW)\Big) = f(t,W) + f_t dt + \frac{1}{2} f_{tt} (dt)^2 + \frac{1}{2} f_{ww} dt + o(dt)$$

Below we will use this averaged version of Ito's lemma to study the problem of Gambler's ruin.

Gambler's ruin (applications of Ito's lemma)

Notation and modeling approach:

C: total cash = the sum of your cash and casino's cash
(assuming you are the only one playing with the casino).

$X(t)$: your cash at time t.

In practice, $C \gg X(0)$.

"Breaking the bank" means "$X(t)$ hits C before hitting 0".

Case 1: we first consider a fair game

$$dX = dW$$

which means $X(t+dt) = X(t) + dW$

It is a fair game because

$$E_{dW}(dX) = E(dW) = 0$$

We study the two questions below.

Question #1: How long can you play?

Question #2: What is the chance that you break the bank?

Note that for stochastic differential equations, it is not enough just to calculate $X(t)$ at a given time t_j. We also want to know quantities that are not readily determined from $\{X(t_j), j = 0, 1, 2, \ldots\}$, such as answers to questions above.

Answer to Question #2 (we address Question #1 after this)

Let

$$u(x) = \Pr\Big(X(t) \text{ hits } C \text{ before } 0 \Big| X(0) = x \Big)$$

Strategy:

Find a boundary value problem (BVP) governing $u(x)$.

Boundary condition:

$u(C) = 1$ and $u(0) = 0$.

Differential equation:

Start with $X(0) = x$. After a short time dt, we have

$X(dt) = x + dW$

Recall that $dW = O(\sqrt{dt})$. For a fixed x in (0, C), when dt is small enough, the probability of $X(t)$ hitting 0 or C in time interval $[0, dt]$ is exponentially small. Here the magnitude of dt depends on x. If x is very close to one of the boundaries, dt has to be extremely small to make the probability of hitting a boundary in time interval $[0, dt]$ negligible.

For a fixed x in (0, C), when dt is small enough, we have

$$u(x) = E_{dW}\big(u(x + dW)\big) + o(dt)$$

$$= E_{dW}\left(u(x) + u_x dW + \frac{1}{2} u_{xx} (dW)^2 \right) + o(dt)$$

$$= u(x) + \frac{1}{2} u_{xx} dt + o(dt)$$

Divide by dt and then take the limit as $dt \to 0$, we obtain

$$u_{xx} = 0$$

This is the differential equation governing $u(x)$. Thus, function $u(x)$ satisfies the boundary value problem (BVP)

$$\begin{cases} u_{xx}(x)=0 & \text{differential equation} \\ u(0)=0, \quad u(C)=1 & \text{boundary conditions} \end{cases}$$

Solving the BVP, we obtain

$$u(x)=\frac{x}{C}$$

The probability of breaking the bank is proportional to your initial cash and inversely proportional to the total cash.

Answer to Question #1

Let

$$T(x)=E\Big(\text{time until } X(t)=C \text{ or } X(t)=0 \Big| X(0)=x\Big)$$

Strategy:

Find a boundary value problem (BVP) governing $T(x)$.

Boundary condition:

$T(0) = 0$ and $T(C) = 0$.

Differential equation:

Start with $X(0) = x$. After a short time dt, we have

$X(dt) = x + dW$

For a fixed x in (0, C), when dt is small enough (depending on x), we have

$$T(x)=dt+E_{dW}\Big(T(x+dW)\Big)+o\Big(dt\Big)$$

$$=dt+E_{dW}\left(T(x)+T_x dW+\frac{1}{2}T_{xx}(dW)^2\right)+o(dt)$$

$$=dt+T(x)+\frac{1}{2}T_{xx}dt+o(dt)$$

Divide by dt and then take the limit as $dt \to 0$, we obtain

$$T_{xx}=-2$$

This is the differential equation governing $T(x)$. Thus, function $T(x)$ satisfies the boundary value problem (BVP)

$$\begin{cases} T_{xx}(x) = -2 & \text{differential equation} \\ T(0)=0, \quad T(C)=0 & \text{boundary conditions} \end{cases}$$

Solving the BVP, we obtain

$$T(x) = x(C-x)$$

A more detailed answer to Question #1:

The average time does not give us the full picture!

$T(x)$ is the average time until going bankrupt or breaking the bank. However, this average does not give us the full picture of how long we can play with initial cash x.

In particular, when $C = \infty$ (when the casino has infinite amount of cash), we have

$T(x) = x(C-x) = \infty$.

This certainly does not mean we can play forever with initial cash x.

To have a more detailed picture of how long we can play when $C = \infty$, we look at the probability that we can play beyond a certain time.

Assume $C = \infty$. Consider

$$P(x,t) = \Pr\Big(X(\tau) > 0 \text{ for } \tau \in [0,t] \,\Big|\, X(0) = x \Big)$$

$P(x, t)$ is the conditional probability of surviving (at least) to time t given $X(0) = x$.

Strategy:

Find an initial boundary value problem (IBVP) governing $P(x, t)$.

Initial and boundary conditions:

$P(x, 0) = 1$ and $P(0, t) = 0$

Differential equation:

Start with $X(0) = x$. After a short time dt, we have

$X(dt) = x + dW$

For a fixed x in (0, C), when dt is small enough (depending on x), we have

$$P(x,t) = E_{dW}\Big(P(x+dW, t-dt)\Big) + o(dt)$$

$$= E_{dW}\left(P(x,t) + P_t(-dt) + P_x dW + \frac{1}{2} P_{xx} (dW)^2 \right) + o(dt)$$

$$= P(x,t) + P_t(-dt) + \frac{1}{2} P_{xx} dt + o(dt)$$

Divide by dt and then take the limit as $dt \to 0$, we obtain

$$P_t = \frac{1}{2} P_{xx}$$

This is the PDE governing $P(x, t)$. Thus, function $P(x, t)$ satisfies the initial boundary value problem (IBVP)

$$\begin{cases} P_t = \frac{1}{2} P_{xx} & \text{partial differential equation} \\ P(0,t) = 0 & \text{boundary condition} \\ P(x,0) = 1 & \text{initial condition} \end{cases}$$

To solve this IBVP, we use an odd extension to convert it to an IVP.

<u>Odd extension</u>:

$$P(-x, t) = -P(x, t)$$

The extended function $P(x, t)$ satisfies the IVP

$$\begin{cases} P_t = \frac{1}{2} P_{xx} \\ P(x,0) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \end{cases}$$

<u>Solution of a general IVP of the heat equation:</u>

$$\begin{cases} u_t = a u_{xx} \\ u(x,0) = f(x) \end{cases} \qquad \text{(E02)}$$

The solution of (E02) has the expression:

$$u(x,t) = \frac{1}{\sqrt{4\pi a t}} \int_{-\infty}^{+\infty} \exp\left(\frac{-\xi^2}{4at}\right) f(x-\xi)\, d\xi$$

Using this general formula to solve for $P(x, t)$, we obtain

$$f(x-\xi) = \begin{cases} 1, & \xi < x \\ -1, & \xi > x \end{cases}$$

$$P(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left(\frac{-\xi^2}{2t}\right) f(x-\xi)\, d\xi$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x} \exp\left(\frac{-\xi^2}{2t}\right) d\xi - \frac{1}{\sqrt{2\pi t}} \int_{x}^{\infty} \exp\left(\frac{-\xi^2}{2t}\right) d\xi$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x} \exp\left(\frac{-\xi^2}{2t}\right) d\xi - \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{-x} \exp\left(\frac{-\xi^2}{2t}\right) d\xi$$

$$= \frac{2}{\sqrt{2\pi t}} \int_{0}^{x} \exp\left(\frac{-\xi^2}{2t}\right) d\xi$$

Change of variables: $d\xi = \sqrt{2t}\, ds$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{x}{\sqrt{2t}}} \exp\left(-s^2\right) ds = \operatorname{erf}\left(\frac{x}{\sqrt{2t}}\right)$$

Thus, probability $P(x, t)$ has the expression

$$P(x,t) = \operatorname{erf}\left(\frac{x}{\sqrt{2t}}\right)$$

<u>Scaling property</u> of $P(x, t)$:

Start with initial cash x. The survival probability p and the time t are related by

$$p = \operatorname{erf}\left(\frac{x}{\sqrt{2t}}\right)$$

==> $\qquad \dfrac{x}{\sqrt{2t}} = \operatorname{erfinv}(p)$

==> $\qquad t = \dfrac{x^2}{2\operatorname{erfinv}(p)^2}$

For a prescribed probability p, the time t is proportional to x^2.

<u>A few example values:</u>

$p = 0.1$	<==>	$t = 63.33\, x^2$
$p = 0.3$	<==>	$t = 6.735\, x^2$
$p = 0.5$	<==>	$t = 2.198\, x^2$
$p = 0.7$	<==>	$t = 0.931\, x^2$
$p = 0.9$	<==>	$t = 0.370\, x^2$