AM216 Homework #5

Problem 1:

i) Show that $\int_0^T W(t)dt \sim N\left(0, \frac{T^3}{3}\right)$

ii) Use the same method to calculate $\int_0^T t\,W(t)dt$

Hint:

Using $W(t)=\int_0^t dW(s)$, we write

$$\int_0^T W(t)dt = \int_0^T \int_0^t dW(s)dt = \int_0^T \int_s^T dt\, dW(s) = \int_0^T (T-s)dW(s)$$

This is a sum of <u>independent Gaussians</u> …

Problem 2:

Let $X(s)$ be the stochastic process governed by the SDE (Ito interpretation)

$$dX(s)=b(X,s)ds+\sqrt{a(X,s)}\,dW(s)$$

Consider the probability $u(z,s)\equiv \Pr\left(X(T)\geq x_C \middle| X(s)=z\right), \qquad T>s.$

Use $u(z,s)=E\left(u(z+dX,s+ds)\right)$ and the moments of dX to derive the governing partial differential equation for $u(z, s)$.

Problem 3:

Consider a fair game between players A and B. (you are not one of them)

$dX = dW$,

$X(t)$ = player A's cash at time t.

Suppose player B has infinite amount of cash (so won't run out of cash!).

You bet that player A's cash will be more than x_C at the specified time $T > s$.

Consider the probability

$$v(z,s)\equiv \Pr\left(X(T)\geq x_C \text{ and } \boxed{X(t)>0 \text{ for all } t\in[s,T]} \middle| X(s)=z\right), \qquad T>s$$

Part 1:

Derive the governing partial differential equation for $v(z, s)$.

Part 2:

Write out the final condition for $v(z, s)$ at $s = T$.

Write out the boundary condition for $v(z, s)$ at $z = 0$.

Part 3:

Apply the change of variables:

$$s = T - \tau$$

$$\psi(z,\tau) = v(z, T-\tau)$$

Write out the initial boundary value problem (IBVP) for $\psi(z, \tau)$.

Compare with problem 5 of homework 2.

Problem 4:

Continue with the game-bet problem in Problem 3.

Now consider the probability

$$w(z,s) \equiv \Pr\left(\begin{array}{c} \boxed{X(t_1) \geq x_c \text{ at some } t_1 \in [s,T]} \\ \text{and } \boxed{X(t) > 0 \text{ for all } t \in [s,t_1]} \end{array} \middle| X(s) = z \right), \qquad T > s$$

Part 1:

Write out the governing partial differential equation for $w(z, s)$.

Part 2:

Write out the boundary conditions for $w(z, s)$, at $z = 0$ and at $z = x_C$.

Part 3:

Apply the change of variables:

$$s = T - \tau$$

$$\psi(z,\tau) = w(z, T-\tau)$$

Write out the initial boundary value problem (IBVP) for $\psi(z, \tau)$.

Problem 5:

Part 1:

Solve the IVP

$$\begin{cases} \phi_\tau = \dfrac{1}{2}\phi_{zz} \\ \phi(z,0) = \begin{cases} 1, & z \ge x_C \\ 0, & z < x_C \end{cases} \end{cases}$$

Write the solution in terms of erf( ).

Part 2:

In lecture, $\phi(z, \tau)$ is related to probability $u(z, s)$ via the change of variables:

$$s = T - \tau, \qquad \phi(z,\tau) = u(z, T-\tau)$$

Reverse the change of variables to write out $u(z, s)$.

Remark:

The solution for $v(z,s) \equiv \Pr\left(X(T) \ge x_C \text{ and } \boxed{X(t) > 0 \text{ for all } t \in [s,T]} \,\middle|\, X(s) = z \right)$ is

$$v(z,s) = \frac{1}{2}\mathrm{erf}\left(\frac{z + x_c}{\sqrt{2(T-s)}}\right) + \frac{1}{2}\mathrm{erf}\left(\frac{z - x_c}{\sqrt{2(T-s)}}\right) \qquad \text{(derivation not included)}$$

Comparing the analytical solutions of $u(z, s)$ and $v(z, s)$, we can see the effect of imposing the condition $X(t)>0$ for $t \in [s, T]$.