AM216 Homework #3

Problem 1:

Solve the BVP

$$\begin{cases} T_{xx} - 2mT_x = -2 & \text{differential equation} \\ T(0) = 0, \quad T(C) = 0 & \text{boundary conditions} \end{cases}$$

to derive the solution

$$T(x) = \frac{x}{m} - \frac{C}{m}\left(\frac{e^{2mx} - 1}{e^{2mC} - 1}\right)$$

Hint:

- First find a particular solution of the non-homogeneous ODE, by trying $T(x) = a\,x$.
- Next, write out the general solution of the homogeneous ODE, using the two roots of the characteristic equation.
- Then combine the two together and enforce the two boundary conditions.

Problem 2:

Let $X \sim N(0, \sigma^2)$. Show that

$E(X^2) = \sigma^2$ and

$\text{var}(X^2) = 2\sigma^4$

Hint:

Write X as $X = \sigma Y$ where $Y \sim N(0, 1)$.

Use the results obtained in homework #2 to show $E(Y^2) = 1$ and $\text{var}(Y^2) = 2$.

Problem 3:

Consider the Wiener process $W(t)$. Define quantity Q_N as follows

$$\Delta t = \frac{T}{N}, \qquad t_k = k\Delta t, \qquad W_k = W(t_k), \qquad \Delta W_k = W_{k+1} - W_k$$

$$Q_N = \sum_{k=0}^{N-1} (\Delta W_k)^2$$

Since $W(t)$ is random, Q_N is a random variable. Show that

- $E(Q_N) = T$ and

- $\text{var}(Q_N) = 2N(\Delta t)^2 = \dfrac{2T^2}{N}$

Hint:

Use the result obtained in problem 2 above.

Remark:

It follows that $\int_0^T (dW)^2 \equiv \lim_{N\to\infty} \sum_{k=0}^{N-1} (\Delta W_k)^2 = T$ is deterministic.

This is consistent with Ito's lemma, which gives $\int_0^T (dW)^2 = \int_0^T dt = T$.

Thus, the results obtained here "support" Ito's lemma.

Problem 4:

Prevalence of leukemia in US is about 76 per 100,000 population = 0.76×10^{-3}.

p_{leuk} = Pr(leukemia) = 0.76×10^{-3}

Consider a test for detecting leukemia. Suppose the test has the false positive rate and the false negative rate given below

r_{FP} = Pr(tested positive | no leukemia) = 0.005 (0.5%)

r_{FN} = Pr(tested negative | leukemia) = 0.05 (5%)

Part1: Use Bayes' Theorem to calculate

Pr(leukemia | tested positive)

Pr(no leukemia | tested negative)

Hint:

To facilitate the calculation, introduce

L = leukemia, L^C = no leukemia

P = tested positive, P^C = tested negative

Use the law of total probability to write out Pr(P) and Pr(P^C).

Part 2: Suppose we test n = 100,000 random persons. Let

N_{FP} = # of false positive incidents

N_{FN} = # of false negative incidents

Calculate

$E(N_{\text{FP}})$ and

$E(N_{\text{FN}})$

Hint:

$N_{\text{FP}} \sim \text{bino}(n, p_{\text{FP incident}})$ where

$p_{\text{FP incident}}$ = Pr(false positive incident in a random person)

= Pr(a random person having no leukemia and testing positive)

= Pr(L^C and P) ...

$N_{\text{FN}} \sim \text{bino}(n, p_{\text{FN incident}})$ where

$p_{\text{FN incident}}$ = Pr(false negative incident in a random person)

= Pr(a random person having leukemia and testing negative)

= Pr(L^C and P) ...

Problem 5:

Suppose t_1, t_2 and t_3 are all positive. Use Bayes' Theorem to show that

$$\rho\left(W(t_1)=x \middle| W(t_1+t_2)=y_2 \text{ and } W(t_1+t_2+t_3)=y_3 \right) = \rho\left(W(t_1)=x \middle| W(t_1+t_2)=y_2 \right)$$

Hint:

Notice that in the target PDF, x is the independent variable. Any factor that does not contain x will be part of the normalizing factor. As we discussed in lecture, we do not need to keep track of the normalizing factor. In particular

$$\rho\left(W(t_1+t_2+t_3)=y_3 \middle| W(t_1)=x \text{ and } W(t_1+t_2)=y_2 \right) \sim N(y_2, t_3)$$

is independent of x.

Remark:

The result tells us that any additional constraint beyond the time of an already prescribed constraint will not change the conditional distribution.