AM216 Homework #2

Problem 1:

Consider the scaling of a RV.

Let $Y = \alpha X$ where X is a RV and $\underline{\alpha > 0}$ is a constant.

a) Express $F_Y(y)$ (the CDF of Y) in terms of $F_X(x)$ (the CDF of X).

b) Express $\rho_Y(y)$ (the PDF of Y) in terms of $\rho_X(x)$ (the PDF of X).

Hint:

- Write $\Pr(Y \le y)$ in terms of a probability on X …
- Differentiate the CDF to obtain the PDF …

Problem 2:

Suppose X and Y are independent, and $X \sim N(\mu_1, \sigma_1^2)$ and $(X+Y) \sim N(\mu_2, \sigma_2^2)$.

Show that

$$Y \sim N(\mu_2 - \mu_1, \sigma_2^2 - \sigma_1^2)$$

Hint:

Calculate the characteristic function of Y.

Problem 3:

Let $X \sim N(0, 1)$. Show that

a) $E(|X|) = \sqrt{\dfrac{2}{\pi}}$

Hint: Use symmetry and then calculate $\int_0^{+\infty} x \exp\left(\dfrac{-x^2}{2}\right) dx$.

b) $E(|X|^3) = 2\sqrt{\dfrac{2}{\pi}}$

Hint: Use symmetry and then integrate by parts to connect to part a).

c) $E(X^4) = 3$

Hint: Use integration by parts to connect to $E(X^2)$, which is 1.

Problem 4:

Consider another version of Monty Hall's game.

Suppose the rules of the game are set as follows.

i) The host puts a card of \$200 in one of the 3 boxes without you looking.

ii) After your initial selection, the host must ask you to randomly open one of the two boxes you did not pick.

iii) If the box you open contains the card, then the game is invalid. You and the host will just start all over again.

If the box you open is empty, the host must offer you the option of paying \$5 to switch to the box you did not open.

Question: When the box you open is empty and you are offered the option of paying \$5 to switch, should you switch?

Hint:

Let

C = "your initial selection is correct"

E = "the box you open is empty" <==> "the game is valid"

Calculate

Pr(C and E), Pr(C^C and E), Pr(C and E^C), Pr(C^C and E^C)

Pr(E), and Pr(C | E)

Pr(C | E) is the conditional probability of correct initial selection given that the subsequent box opening results in a valid game.

Then make your decision based on the value of Pr(C | E).

Problem 5:

Use the odd extension to solve the initial boundary value problem (IBVP)

$$\begin{cases} u_t = \frac{1}{2} u_{xx} & \text{partial differential equation} \\ u(0,t) = 0 & \text{boundary condition} \\ u(x,0) = \begin{cases} 1, & x > c_0 \\ 0, & 0 < x < c_0 \end{cases} & \text{initial condition, } c_0 > 0 \end{cases}$$

Hint:

- In the IBVP, the initial condition is specified only for $x > 0$.
- After the odd extension, the initial condition becomes

$$u(x,0)=\begin{cases} 1\,, & x>c_0 \\ 0\,, & -c_0<x<c_0 \\ -1 & x<-c_0 \end{cases}$$

- Using the solution formula of a general IVP to write out the solution.
  Express the solution in terms of the error function erf(·).