AM216 Stochastic Differential Equations

Lecture 11
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List of topics in this lecture

- In general, transition probability density $q(x, t \mid z, s)$ is NOT a density of z.
- Derivation of Kolmogorov forward equation, method of test function
- Meaning of backward equation, the average of position-dependent reward at time T given starting at position z at time $(T-t)$
- Meaning of forward equation, ensemble density at time t given the starting ensemble density at time 0

Recap

Different interpretations of stochastic differential equation

The Stratonovich interpretation of $dX = b(X,t)dt + \sqrt{a(X,t)}\,dW$

is equivalent to the Ito interpretation of a modified equation

$$dX = \left(b(X,t) + \frac{1}{4} a_x(X,t) \right) dt + \sqrt{a(X,t)}\,dW$$

Transition probability density:

$$q(\underbrace{x,t}_{\substack{\uparrow \\ \text{end time}}} \mid \underbrace{z,s}_{\substack{\uparrow \\ \text{starting time}}}) \equiv \frac{1}{dx} \Pr\left(x \le X(t) < x + dx \,\middle|\, X(s) = z \right), \qquad t > s$$

Backward view

$$\underbrace{q(x,t \mid z,s)}_{q(\bullet,s)} = \int \underbrace{q(x,t \mid z+y, s+ds)}_{q(\bullet,s+ds)} \, \underbrace{q(z+y, s+ds \mid z,s)}_{\text{density of } dX} \, dy$$

We move the starting time backward from $(s+ds)$ to s

$$\xrightarrow{\quad s \qquad (s+ds) \qquad\qquad\qquad t \qquad\qquad \text{Time} \quad}$$

Backward equation and the final value problem (FVP)

$$\begin{cases} q_s = -b(z,s)q_z - \dfrac{1}{2}a(z,s)q_{zz} \\ q(x,T|z,s)\Big|_{s=T} = \delta(z-x) \end{cases}$$

We solve it backward from $s = T > 0$ to $s = 0$.

Mathematically, we can convert it to an IVP using change of variables

$\tau = T - s$

An application:

$$u(z,s) \equiv \Pr\Big(X(T) \ge x_C \Big| X(s) = z\Big)$$

= probability of winning the bet $X(T) \ge x_C$ given that $X(s) = z$.

End of recap

We look at the two interpretations in a modeling setting.

Example 1:

Consider a fair game between you and a casino.

Let $X(t)$ = your cash at time t.

Each round, you bet a small fixed percentage of your current cash (assuming the casino has no lower limit on the amount of bet).

The governing SDE is

$dX = \alpha X dW$

with the Ito interpretation by the design of game

$dX = \alpha X(t) dW$

Example 2:

We study the Stratonovich interpretation of $dX = \alpha X dW$.

$$dX = \alpha \frac{X(t) + (X(t) + dX)}{2} dW$$

which is equivalent to the Ito interpretation of

$$dX = \frac{1}{2}\alpha X dt + \alpha X dW$$

Let $Y = \log(X)$. We have

$X(t) = \exp(Y(t))$, $\quad X(t) + dX = \exp(Y(t) + dY)$

In terms of Y, the Stratonovich interpretation is

$$\exp(Y(t)+dY)-\exp(Y(t))=\alpha\frac{\exp(Y(t))+\exp(Y(t)+dY)}{2}dW$$

Dividing by $\exp(Y(t)+dY/2)$, expanding in dY, and neglecting $o(dt)$ terms, we get

$$\exp(dY/2)-\exp(-dY/2)=\alpha\frac{\exp(-dY/2)+\exp(-dY/2)}{2}dW$$

==> $\quad dY+O\Big((dY)^3\Big)=\alpha\Big(1+O\Big((dY)^2\Big)\Big)dW$

==> $\quad dY=\alpha dW$

==> $\quad d\big(\log(X)\big)=\alpha dW$

which is suitable for modeling the growth of a bacteria population when the surrounding environment temperature is stochastic.

In summary, the interpretation is determined in the modeling stage.

Now back to the transition probability density.

Caution:

In general, $q(x, t \mid z, s)$ is NOT a density of z.

Example:

Ornstein-Uhlenbeck process

$$dY=-\beta Y dt+\sqrt{\gamma^2}\,dW$$

Recall that previously we derived

$$\Big(Y(t)\Big|Y(0)=y_0\Big)\sim N\left(e^{-\beta t}y_0,\ \frac{\gamma^2}{2\beta}(1-e^{-2\beta t})\right)$$

We apply the result to time interval $[s, t]$ with $t > s$.

$$\Big(Y(t)\Big|Y(s)=z\Big)\sim N\left(e^{-\beta(t-s)}z,\ \frac{\gamma^2}{2\beta}(1-e^{-2\beta(t-s)})\right)$$

The transition probability density of Y is

$$q(x,t\,\big|\,z,s)\equiv\frac{1}{dx}\Pr\Big(x\le Y(t)<x+dx\Big|Y(s)=z\Big)$$

$$=\frac{1}{\sqrt{\pi(1-e^{-2\beta(t-s)})\gamma^2/\beta}}\exp\left(\frac{-(x-e^{-\beta(t-s)}z)^2}{(1-e^{-2\beta(t-s)})\gamma^2/\beta}\right)$$

As a function of x, it is a probability density.

As a function of z, at $x = 0$, we have

$$q(0,t\,|\,z,s) = \frac{1}{\sqrt{\pi(1-e^{-2\beta(t-s)})\gamma^2/\beta}}\exp\left(\frac{-e^{-2\beta(t-s)}z^2}{(1-e^{-2\beta(t-s)})\gamma^2/\beta}\right)$$

$$= e^{\beta(t-s)}\cdot\frac{1}{\sqrt{\pi(e^{2\beta(t-s)}-1)\gamma^2/\beta}}\exp\left(\frac{-z^2}{(e^{2\beta(t-s)}-1)\gamma^2/\beta}\right)$$

$$= e^{\beta(t-s)}\cdot\rho_{N\left(0,\,(e^{2\beta(t-s)}-1)\gamma^2/(2\beta)\right)}(z)$$

Key observation:

- We should not expect $\int q(x, t\,|\,z, s)dz = 1$
- We should not expect $\int q(x, t\,|\,z, s)dz$ to be conserved with respect to s.

Derivation of the forward equation

We fix (z, s) and view q as a function of (x, t):

$q(x, t) \equiv q(x, t\,|\,z, s)$

Forward view:

$$\underbrace{q(x,t+dt\,|\,z,s)}_{q(\bullet,t+dt)} = \int \underbrace{q(x,t+dt\,|\,y,t)}_{\substack{\text{density of}\\ X(t+dt)|X(t)=y}}\ \underbrace{q(y,t\,|\,z,s)}_{q(\bullet,t)}\,dy$$

We move the end time forward from t to $(t+dt)$.

$$s \qquad\qquad t \qquad (t+dt) \qquad \text{Time} \longrightarrow$$

Key for the derivation:

As $dt \to 0$, the integral is dominated by contribution from small $(y-x)$.

For $(y-x)$ not small, $q(x, t+dt\,|\,y, t)$ is exponentially small as $dt \to 0$.

The old approach does not work

Expanding $q(y, t) = q(x+(y-x), t)$ around x and substituting it directly into the master equation won't work!

$$q(y,t) = q(x+(y-x),t) = \cdots + q_x(x,t\,|\,z,s)(y-x) + \cdots$$

$$\int q(x,t+dt\,|\,y,t)q(y,t)dy = \cdots + \int \underbrace{q\left(x,t+dt\,|\,y,t\right)}_{\substack{\text{This is NOT}\\ \text{a density of } y!}}\ \underbrace{q_x(x,t)}_{\substack{\text{This is fine. It is}\\ \text{independent of } y}}\ (y-x)dy + \cdots$$

Integrating a density leads to moments.

$$\int (x-y)\underbrace{q(x,t+dt\,|\,y,t)}_{\text{This is a density of } x.}dx = E(dX)$$

Integrating a non-density leads to nowhere.

$$\int (y-x)\underbrace{q(x,t+dt\,|\,y,t)}_{\substack{\text{This is NOT}\\ \text{a density of } y!}}dy = \text{unknown}$$

Strategy:

- We need an integral with respect to *x*.
- We can have it by integrating the master equation with respect to *x*.

$$\int q(x,t+dt)dx = \iint q(y,t)q(x,t+dt\,|\,y,t)dy\,dx$$

  But we will lose information in the integration.
- Remedy: we multiply it by a test function $h(x)$ and then integrate

Implementation

Let $h(x)$ be a smooth function with compact support.

Definition:

Function $h(x)$ has compact support if there exists M such that

$$h(x)=0 \quad \text{for} \quad |x|>M$$

We multiply both sides of the master equation by $h(x)$ and integrate

$$\int q(x,t+dt)h(x)dx = \int\Big[\int q(y,t)q(x,t+dt\,|\,y,t)dy\Big]h(x)dx$$

Changing the order of integration leads to

$$\int q(x,t+dt)h(x)dx = \int q(y,t)\left[\int \underbrace{q(x,t+dt\,|\,y,t)}_{\text{This is a density of } x.}h(x)dx\right]dy \qquad \text{(E01)}$$

The inner integral is dominated by contribution from small $(x-y)$.

We expand $h(x) = h(y+(x-y))$ around y.

$$h(x)=h(y+(x-y))=h(y)+h_y(y)(x-y)+\frac{h_{yy}(y)}{2}(x-y)^2+O\big((x-y)^3\big)$$

We substitute the expansion into the inner integral and use the moments of dX,

$$\int q(x,t+dt\,|\,y,t)dx = 1$$

$$\int q(x,t+dt\,|\,y,t)(x-y)dx = b(y,t)dt+o(dt)$$

$$\int q(x,t+dt\,|\,y,t)(x-y)^2\,dx = a(y,t)dt + o(dt)$$

$$\int q(x,t+dt\,|\,y,t)(x-y)^3\,dx = o(dt)$$

The inner integral becomes

$$\int q(x,t+dt\,|\,y,t)h(x)dx = h(y)+h_y(y)E(dX)+\frac{h_{yy}(y)}{2}E\left((dX)^2\right)+o(dt)$$

$$= h(y)+h_y(y)b(y,t)dt+\frac{h_{yy}(y)}{2}a(y,t)dt+o\left(dt\right)$$

- Substituting this result into (E01),
- carrying out integration by parts, and
- using the compactness of function $h(y)$, we obtain

$$\text{RHS} = \int q(y,t)\left[h(y)+h_y(y)b(y,t)dt+\frac{h_{yy}(y)}{2}a(y,t)dt\right]dy+o(dt)$$

$$= \int\left[q-\left(b(y,t)q\right)_y dt+\frac{1}{2}\left(a(y,t)q\right)_{yy}dt\right]h(y)dy+o(dt), \qquad q\equiv q(y,t\,|\,z,s)$$

$$\text{LHS} = \int q(x,t+dt)h(x)dx = \int\left[q(x,t)+q_t dt+o(dt)\right]h(x)dx$$

- on RHS, renaming the integration variable from y to x,
- subtracting $\int q(x,t)h(x)dx$ from both sides,
- dividing by dt, and taking the limit as $dt \to 0$, we arrive at

$$\text{LHS} = \int q_t h(x)dx$$

$$\text{RSH} = \int\left[-\left(b(x,t)q\right)_x+\frac{1}{2}\left(a(x,t)q\right)_{xx}\right]h(x)dx, \qquad q\equiv q(x,t\,|\,z,s)$$

Since LHS = RHS for all test function $h(x)$, we conclude

$$q_t = -\left(b(x,t)q\right)_x+\frac{1}{2}\left(a(x,t)q\right)_{xx}$$

This is called the Fokker-Planck equation or the Kolmogorov forward equation.

Conservation form:

The forward equation has the conservation form

$$q_t = -\frac{\partial}{\partial x}J(x,t)$$

where $J(x, t)$ is the probability flux given by

$$J(x,t)=b(x,t)q-\frac{1}{2}\big(a(x,t)q\big)_x$$

Note:

flux ≡ flow per unit time

Remarks:

- Solution of $q_t=-\frac{\partial}{\partial x}J(x,t)$ is conserved:

  change in $\int_a^b q(x,t)dx$ is attributed to in-flow at $x = a$ and out-flow at $x = b$.

$$\int_a^b q(x,t_2)dx-\int_a^b q(x,t_1)dx=\underbrace{\int_{t_1}^{t_2}J(a,t)dt}_{\text{In-flow}}-\underbrace{\int_{t_1}^{t_2}J(b,t)dt}_{\text{Out-flow}}$$

- In contrast, the backward equation is not in the conservation form.

$$q_s=-b(z,s)q_z-\frac{1}{2}a(z,s)q_{zz}$$

- In general, solution of the backward equation is not conserved.

The initial value problem (IVP) for $q(x, t) \equiv q(x, t \mid z, 0)$

$$\begin{cases} q_t=-\big(b(x,t)q\big)_x+\frac{1}{2}\big(a(x,t)q\big)_{xx} \\ q(x,t|z,0)\big|_{t=0}=\delta(x-z) \end{cases}$$

We solve it forward from $t = 0$ forward to $t = T$.

Remarks:

- The initial condition specifies the system state at time 0.
- The forward equation describes the forward time evolution of system state.

More on the backward equation:

The autonomous case:

$$a(z, s) = a(z)\ , \qquad b(z, s) = b(z)$$

- There is no explicit dependence on time.
- If we shift everything together in time, the problem remains the same.

Backward equation in the autonomous case:

When we shift everything in time by $(T-t)$, we have

$q(x, T \mid z, T-t) = q(x, t \mid z, 0)$

The IVP for $q(x, t \mid z, 0)$ as a function of (z, t) is

$$\begin{cases} q_t = b(z) q_z + \dfrac{1}{2} a(z) q_{zz} \\ q(x,t \mid z,0)\Big|_{t=0} = \delta(z-x) \end{cases}$$

Important:

In applications, end time T is fixed and t in $q(x, t)$ refers to real time $(T-t)$.

Meaning of the backward equation with a *general initial condition*

$$\begin{cases} q_t = b(z) q_z + \dfrac{1}{2} a(z) q_{zz} \\ q(z,t)\Big|_{t=0} = u(z) \end{cases} \qquad \text{(BE\_IVP1)}$$

It is straightforward to verify that the solution of (BE_IVP1) is

$$q(z,t) = \int q(x,t \mid z,0) u(x) dx = \underbrace{\int q(x,T \mid z,T-t) u(x) dx}_{\substack{\text{end time is fixed} \\ \text{at real time } T}} \qquad \text{(B01)}$$

Observations:

- $q(x, T \mid z, T-t)$ is the transition probability density.
- Real time T is a future time, for example, the expiration date of an option.
- Variable t in the backward equation is the time until the end time. It corresponds to *real time* $(T-t)$.

Meaning of $q(z, t)$

Consider the stochastic process X governed by

$$dX = b(X) dt + \sqrt{a(X)}\, dW$$

For example, $X(t)$ = the price of a stock at time t.

Suppose the amount of reward is determined by position $X(T)$ at *real time T*.

Let $u(x)$ denote the reward function, which calculates the reward at time T.

The amount of reward = $u(X(T))$

Suppose X starts at position z at *real time* $(T-t)$. The conditional distribution of X at *real time* T given $X(T-t) = z$ is described by the transition PD

$q(x, T \mid z, T-t)$

The conditional expected amount of reward given $X(T-t) = z$ is

$$E\Big(u(X(T))\Big|X(T-t)=z\Big)=\int \underbrace{q(x,T\,|\,z,T-t)}_{\substack{\text{transition density of}\\ X(T)=x|X(T-t)=z}}\ \underbrace{u(x)}_{\substack{\text{reward at}\\ \text{position } x}}\ dx \qquad \text{(B02)}$$

This is exactly the same as the solution $q(z, t)$ given in (B01).

Summary:

Suppose the reward is determined at *real time T* based on $X(T)$ as $u(X(T))$.

The expected amount of position-dependent reward starting at $X(T-t) = z$ is

$$q(z,t)=E\Big(u(X(T))\Big|X(T-t)=z\Big)$$
$$=\text{solution of the backward equation}$$

The backward equation describes the backward time evolution of the expected amount of reward. The end time is fixed at T. The backward time evolution refers to the start time. It means that we move the start time gradually backward from T to $(T-t)$.

In general, the expected amount of reward $q(z, t)$ is not conserved.

$$\int q(z,t_1)dz \neq \int q(z,t_2)dz$$

This is related to that the backward equation is not in the conservation form.

More on the forward equation

Forward equation in the autonomous case:

The IVP for $q(x, t \mid z, 0)$ as a function of (x, t) is

$$\begin{cases} q_t = \big(b(x)q\big)_x + \dfrac{1}{2}\big(a(x)q\big)_{xx} \\ q(x,t\,|\,z,0)\Big|_{t=0} = \delta(x-z) \end{cases}$$

Meaning of the forward equation with a *general initial condition*

$$\begin{cases} p_t = \big(b(x)p\big)_x + \dfrac{1}{2}\big(a(x)p\big)_{xx} \\ p(x,t)\Big|_{t=0} = v(x) \end{cases} \qquad \text{(FE\_IVP1)}$$

It is straightforward to verify that the solution of (FE_IVP1) is

$$p(x,t) = \underbrace{\int q(x,t\,|\,z,0)v(z)dz}_{\substack{\text{start time is fixed}\\ \text{at real time 0}}} \tag{F01}$$

<u>Observations</u>:

- We use $p(\ )$ to denote the solution of forward equation, to distinguish it from the solution of backward equation.
- Here $q(x, t \mid z, 0)$ is the transition probability density.
- Variable t in the forward equation is the time elapsed since the start time.

<u>Meaning of solution $p(x, t)$</u>

Let $X(t)$ be the stochastic process governed by

$$dX = b(X)dt + \sqrt{a(X)}\,dW$$

For example, X = position of a small particle in water.

Consider <u>an ensemble</u> of X.

Let $v(x)$ be the <u>ensemble</u> density of X at position x at time 0.

In general, ensemble density $v(x)$ is an <u>unnormalized</u> density.

The <u>ensemble</u> density of X at position x at time t is

$$\int \underbrace{q(x,t\,|\,z,0)}_{\substack{\text{transition density}\\ \text{of } X(t)=x|X(0)=z}} \cdot \underbrace{v(z)}_{\substack{\text{ensemble}\\ \text{density at } z}} dz \tag{F02}$$

This is exactly the same as the solution $p(x, t)$ given in (F01).

<u>Summary:</u>

Consider <u>an ensemble</u> of X with ensemble density $v(x)$ at time 0.

$p(x, t)$ = ensemble density of X at time t.

The forward equation describes the forward time evolution of ensemble density.

The ensemble density $p(x, t)$ is conserved.

$$\int_a^b p(x,t_2)dx - \int_a^b p(x,t_1)dx = \left(\text{In-flow}\right) - \left(\text{Out-flow}\right)$$

This is related to that the forward equation is in the conservation form.