AM216 Homework #9

Problem 1:

Consider the hypothetical situation where the stock price is a linear Brownian motion

$$dS = \mu dt + \sigma dW$$

Assume that the interest rate is zero: $r = 0$.

Let $C(s, t)$ denote the price of option with strike price K and expiry T.

Consider the same delta hedge portfolio

1 unit of delta hedge of time t

= owning (−1) unit of call option and $C_S(S(t), t)$ shares of stock.

We carry out the same scheme of maintaining a portfolio of $F(S(t), t)$ units delta hedge of time t, over time period $[0, T]$. Specifically, over each $[t, t+dt]$, i), we do

- start with no position at time t,
- at time t, buy $F(S(t), t)$ units of delta hedge of time t;
- hold it during $[t, t+dt]$;
- at time $(t+dt)$, sell $F(S(t), t)$ units of delta hedge of time t;
- end with no position at time $(t+dt)$.

Follow the same procedure in Lectures to calculate the net gain over $[0, T]$.

Problem 2:

Continue with the set up of Problem 1.

Derive the governing equation for option price function $C(s, t)$.

Write out the final condition at time T and write out the FVP.

Convert the FVP to an initial value problem.

You should obtain the IVP

$$\begin{cases} u_\tau(s,\tau) = \frac{1}{2}\sigma^2 u_{ss}(s,\tau) \\ u(s,\tau)\Big|_{\tau=0} = \max(s-K, 0) \end{cases}$$

Problem 3:

Continue with Problem 2.

Solve the initial value problem obtained in Problem 2.

Express the solution using elementary functions and the error function.

Problem 4:

Continue with the set up of Problem 1.

Suppose an investor purchases the option at time t when the stock price is $S(t) = s$.

Calculate the expected reward at time T (of owning the option)

$$w(s,\tau) \equiv E\left(\max(S(T)-K,0)\middle| S(t)=s\right) \qquad \tau = T-t, \text{ time until expiry}$$

Problem 5:

For $\mu > 0$, show that $w(s, \tau) > u(s, \tau)$.