AM216 Stochastic Differential Equations

Lecture 05

List of topics in this lecture

- Non-dimensionalization, advantage of working with $\frac{dW(t)}{\sqrt{dt}}$
- Case 2 of gambler's ruin: a biased game
- White noise dW/dt, not a regular function in the conventional sense
- The delta function, how to make sense of an invalid limit or operation?
- Fourier transform (FT), properties of FT

Recap

Gambler's ruin

Case 1: fair game

$dX = dW$ which means $X(t+dt) = X(t) + dW$

When the game is fair, on average the casino cannot make any money.

Case 2: a biased game

$$dX = -m dt + dW$$

It is a biased game because

$$E_{dW}(dX) = -m dt < 0$$

Before we solve case 2, let us study the process of non-dimensionalization.

Non-dimensionalization is a key step in modeling, analysis and simulations.

Non-dimensionalization

Back to the stochastic differential equation (SDE) of case 1.

$$dX = dW$$

In this simple equation, we treat all quantities as non-dimensional. Otherwise the physical units of LHS and RHS will not match.

$[X] = \$$, X has the dimension of money

$E(W^2) = t$ ==> $[W^2] = [t] = \text{time} \equiv T$

W has the dimension of $\sqrt{T}$

Case 1: the original dimensional equation (before non-dimensionalization) looks like

$$dX = \sqrt{\sigma^2}\, dW$$

where σ^2 has the dimensional of $\$^2/T$.

$$[\sigma^2] = \frac{\$^2}{T}, \quad [\sigma] = \frac{\$}{\sqrt{T}}, \quad [dW] = \sqrt{T}$$

Case 2: the original dimensional equation of case 2 looks like

$$dX = -m dt + \sqrt{\sigma^2}\, dW$$

where

$$[X] = \$, \quad [m] = \frac{\$}{T}, \quad [dt] = T, \quad [\sigma] = \frac{\$}{\sqrt{T}}, \quad [dW] = \sqrt{T}$$

We write the dimensional equation as

$$dX = -m dt + \sqrt{\sigma^2 dt}\, \frac{dW(t)}{\sqrt{dt}}$$

Advantage of working with $\dfrac{dW(t)}{\sqrt{dt}}$

$$dW(t) \sim N(0, dt) = \sqrt{dt}\, N(0,1)$$

==> $\dfrac{dW(t)}{\sqrt{dt}} \sim N(0,1)$ is dimensionless and independent of dt and t.

This property is especially useful in non-dimensionalization!

Caution:

$$\frac{dW(t)}{\sqrt{dt}} \text{ is not } \frac{dW(t)}{dt}$$

Short notation for being dimensionless

$$\left[\frac{X}{\$}\right] = \text{one}, \quad \left[\frac{dW}{\sqrt{dt}}\right] = \text{one}$$

Caution: "one" means "dimensionless". It does not mean numerical value 1.

$$\frac{\$8}{\$}=8\,,\qquad \left[\frac{\$8}{\$}\right]=\text{one}$$

We non-dimensionalize the equation in case 2.

Objectives of non-dimensionalization

- A dimensionless equation
- In the dimensionless equation, getting rid of σ.

Scales for non-dimensionalization

Time scale: $[t_0] = T$

In this problem, we pick the time scale, for example, 20 seconds or 1 minute.

Money scale: $[\sigma^2 t_0]=\$^2$ ==> $[\sqrt{\sigma^2 t_0}]=\$$

The money scale is derived from the given σ and the selected time scale t_0.

Non-dimensional quantities

Non-dimensional time: $t_{\text{ND}}=\dfrac{t}{t_0}$

Non-dimensional money: $X_{\text{ND}}=\dfrac{X}{\sqrt{\sigma^2 t_0}}$

Non-dimensional equation

Start with the dimensional equation

$$dX=-m\,dt+\sqrt{\sigma^2 dt}\,\frac{dW(t)}{\sqrt{dt}}$$

where $\boxed{\dfrac{dW(t)}{\sqrt{dt}} \sim N(0,1) \text{ is dimensionless and independent of } dt \text{ and } t.}$

We write all dimensional quantities in terms of non-dimensional quantities and then substitute them in the dimensional equation

$$t=t_0\,t_{\text{ND}}\,,\qquad X=\sqrt{\sigma^2 t_0}\,X_{\text{ND}}$$

$$dt=t_0\,dt_{\text{ND}}\,,\qquad dX=\sqrt{\sigma^2 t_0}\,dX_{\text{ND}}\,,\qquad \frac{dW(t)}{\sqrt{dt}}=\frac{dW(t_{\text{ND}})}{\sqrt{dt_{\text{ND}}}}$$

==> $\sqrt{\sigma^2 t_0}\, dX_{\text{ND}} = -m t_0 dt_{\text{ND}} + \sqrt{\sigma^2 t_0 dt_{\text{ND}}}\,\dfrac{dW(t_{\text{ND}})}{\sqrt{dt_{\text{ND}}}}$

Divide the equation by $\sqrt{\sigma^2 t_0}$, we get

$$dX_{\text{ND}} = -m\frac{t_0}{\sqrt{\sigma^2 t_0}}dt_{\text{ND}} + \sqrt{dt_{\text{ND}}}\,\frac{dW(t_{\text{ND}})}{\sqrt{dt_{\text{ND}}}}$$

We obtain the dimensionless equation

$$dX_{\text{ND}} = -m_{\text{ND}} dt_{\text{ND}} + dW(t_{\text{ND}}), \qquad m_{\text{ND}} = m\sqrt{\frac{t_0}{\sigma^2}}$$

Once we have the dimensionless equation, we can drop the subscript "ND" and revert back to the simple notation $(X,\ t,\ C,\ m)$.

Summary

When we work with the dimensionless equation

$$dX = -m dt + dW$$

we need to keep in mind

$$X = \frac{X_{\text{phy}}}{\sqrt{\sigma^2 t_0}}, \qquad C = \frac{C_{\text{phy}}}{\sqrt{\sigma^2 t_0}}, \qquad t = \frac{t_{\text{phy}}}{t_0}, \qquad m = m_{\text{phy}}\sqrt{\frac{t_0}{\sigma^2}}$$

where $(*)_{\text{phy}}$ denotes the physical quantity before non-dimensionalization.

Solutions of case 2

For the biased game, we again study the two questions.

Question #1: How long can you play?

Question #2: What is the chance that you break the bank?

Answer to Question #2

The strategy we use is the same as that in case 1.

Let

$$u(x) = \Pr\Big(X(t) \text{ hits C before } 0 \Big| \text{X}(0) = x\Big)$$

Strategy:

Find a boundary value problem (BVP) governing $u(x)$.

Boundary condition:

$u(C) = 1$ and $u(0) = 0$.

Differential equation:

Start with $X(0) = x$. After a short time dt, we have

$X(dt) = x + dX, \quad dX = -m\,dt + dW$

dX satisfies

$$E_{dW}(dX) = E_{dW}(-mdt + dW) = -mdt$$

$$E_{dW}\left((dX)^2\right) = E_{dW}\left((-mdt)^2 + 2(-mdt)dW + (dW)^2\right) = dt + o(dt)$$

$$dX = O(\sqrt{dt})$$

For a fixed x in (0, C), when dt is small enough (depending on x), we have

$$u(x) = E_{dW}\left(u(x+dX)\right) + o\left(dt\right)$$

$$= E_{dW}\left(u(x) + u_x dX + \frac{1}{2}u_{xx}(dX)^2\right) + o\left(dt\right)$$

$$= u(x) - u_x mdt + \frac{1}{2}u_{xx}dt + o\left(dt\right)$$

Divide by dt and then take the limit as $dt \to 0$, we obtain a 2nd order ODE for $u(x)$

$$u_{xx} - 2mu_x = 0$$

Function $u(x)$ satisfies the boundary value problem (BVP)

$$\begin{cases} u_{xx} - 2mu_x = 0 & \text{differential equation} \\ u(0) = 0, \quad u(C) = 1 & \text{boundary conditions} \end{cases}$$

Solution of the BVP

We look at its characteristic equation

$$\lambda^2 - 2m\lambda = 0$$

It has two roots

$$\lambda_1 = 2m, \qquad \lambda_2 = 0$$

A general solution of the ODE has the form

$$u(x) = c_1 e^{2mx} + c_2$$

Enforcing the two boundary conditions, we get

$$u(x) = \frac{e^{2mx} - 1}{e^{2mC} - 1} = \frac{e^{-2mC}(e^{2mx} - 1)}{1 - e^{-2mC}}$$

When mC is moderately large (for example, $mC \geq 5$), we have

$$u(x) = \frac{e^{-2mC}(e^{2mx}-1)}{1-e^{-2mC}} \approx e^{-2mC}(e^{2mx}-1)$$

which decays exponentially with $2mC$.

Comparison of fair game vs biased game

We look at u(C/2). The game we consider here is a two-player game between you and the casino. u(C/2) is the probability of winning all the cash when you and the casino start with the equal amount of cash (C/2).

Fair game:

$$u(x) = x/C \qquad \Longrightarrow \qquad u\left(\frac{C}{2}\right) = \frac{1}{2}$$

Biased game:

$$u\left(\frac{C}{2}\right) \approx e^{-2mC}(e^{2mC/2}-1) \approx e^{-mC}$$

which is exponentially small when mC is moderately large.

For example, $mC = 5$ ==> $e^{-mC} = e^{-5} = 0.0067$.

Answer to Question #1

The strategy we use is the same as that in case 1.

Let

$$T(x) = E\Big(\text{time until } X(t) = C \text{ or } X(t) = 0 \,\Big|\, X(0) = x\Big)$$

Strategy:

Find a boundary value problem (BVP) governing $T(x)$.

Boundary condition:

$T(C) = 0$ and $T(0) = 0$.

Differential equation:

Start with $X(0) = x$. After a short time dt, we have

$X(dt) = x + dX$, $\quad dX = -m\,dt + dW$

dX satisfies

$$E_{dW}(dX) - mdt\,, \qquad E_{dW}\left((dX)^2\right) = dt + o(dt)\,, \qquad dX = O(\sqrt{dt})$$

For a fixed x in (0, C), when dt is small enough (depending on x), we have

$$T(x) = dt + E_{dW}\Big(T(x+dX)\Big) + o(dt)$$

$$= dt + E_{dW}\left(T(x) + T_x dX + \frac{1}{2}T_{xx}(dX)^2 \right) + o(dt)$$

$$= dt + T(x) - T_x m dt + \frac{1}{2}T_{xx}dt + o(dt)$$

Divide by dt and then take the limit as $dt \to 0$, we obtain an ODE for $T(x)$

$$T_{xx} - 2mT_x = -2$$

Function $T(x)$ satisfies the boundary value problem (BVP)

$$\begin{cases} T_{xx} - 2mT_x = -2 & \text{differential equation} \\ T(0) = 0, \quad T(C) = 0 & \text{boundary conditions} \end{cases}$$

<u>The solution of the BVP</u> is

$$T(x) = \frac{x}{m} - \frac{C}{m}\left(\frac{e^{2mx} - 1}{e^{2mC} - 1} \right)$$

(Derivation is in your homework).

Suppose mC is moderately large (for example, $mC \geq 5$) and $x \leq C/2$. We have

$$T(x) = \frac{x}{m} \cdot \left(1 - \frac{C}{x}\left(\frac{e^{2mx} - 1}{e^{2mC} - 1} \right) \right) \approx \frac{x}{m} \qquad \text{for } x \leq \frac{C}{2}$$

Here we have used $\frac{C}{x}\left(\frac{e^{2mx} - 1}{e^{2mC} - 1} \right) << 1$, which is derived in Appendix A.

The result of $T(x) \approx x/m$ is consistent with the intuitive picture that if your cash decreases with speed m, then your initial cash x will last a time period of (x/m).

<u>Meaning of</u> $mC \geq 5$

We examine this condition in terms of physical quantities.

$$m = m_{\text{phy}}\sqrt{\frac{t_0}{\sigma^2}}\,, \qquad C = \frac{C_{\text{phy}}}{\sqrt{\sigma^2 t_0}}\,, \qquad x = \frac{x_{\text{phy}}}{\sqrt{\sigma^2 t_0}}$$

==> $$mC = \frac{m_{\text{phy}}C_{\text{phy}}}{\sigma^2}$$

$mC \geq 5$ corresponds to $\quad \dfrac{m_{\text{phy}}C_{\text{phy}}}{\sigma^2} \geq 5$.

An example (with physical quantities)

Consider a biased game with physical parameters below.

$$t_0 = 1 \text{ min}, \quad \sigma = 5\frac{\$}{\sqrt{\text{min}}}, \quad m_{\text{phy}} = 0.25\frac{\$}{\text{min}}$$

$$C_{\text{phy}} = 1000\$, \quad x_{\text{phy}} = 500\$$$

The scales and the dimensionless quantities are

$$\sqrt{\sigma^2 t_0} = 5\$, \quad m = m_{\text{phy}}\sqrt{\frac{t_0}{\sigma^2}} = 0.05$$

$$C = \frac{C_{\text{phy}}}{\sqrt{\sigma^2 t_0}} = 200, \quad mC = 10, \quad x = \frac{x_{\text{phy}}}{\sqrt{\sigma^2 t_0}} = 100, \quad mx = 5$$

Since mC = 10, the approximate expressions for $u(x)$ and $T(x)$ are valid.

Probability of breaking the bank:

$$u(x) \approx e^{-2mC}(e^{2mx} - 1) = e^{-20}(e^{10} - 1) \approx e^{-10} = 4.54 \times 10^{-5}$$

The chance of breaking the bank is virtually zero even though you and the casino start with the same amount \$500.

Average time until the end of game:

$$T(x) \approx \frac{x}{m} = \frac{100}{0.05} = 2000$$

The physical time is

$T_{\text{phy}} = T\, t_0 = 2000$ minutes.

White noise

While $\dfrac{dW}{\sqrt{dt}}$ is nice to work with in non-dimensionalization, we do need to study $\dfrac{dW}{dt}$.

Consider the stochastic differential equation (SDE)

$$dX = -m\,dt + dW$$

We write the "formal" derivative of X as

$$\frac{dX}{dt} = -m + \frac{dW}{dt}$$

Recall that in SDE, dt is finite until we take the limit as $dt \to 0$.

Here the limit $\lim\limits_{dt\to 0}\dfrac{dW}{dt}$ does not exist in the conventional sense if we take the limit directly. The key strategy in working around this obstacle is that we take the limit only after manipulations and interactions with other functions.

The short story

1) Let $Z(t)\equiv\dfrac{dW}{dt}$

$$Z(t)\equiv\frac{dW}{dt}=\frac{1}{\sqrt{dt}}\cdot\frac{dW}{\sqrt{dt}}, \qquad \frac{dW}{\sqrt{dt}}\sim N\left(0,1\right) \text{ independent of } dt$$

$Z(t)$ diverges to $\pm\infty$ as $dt\to 0$. $Z(t)$ is not a regular function.

2) $E\left(Z(t)Z(s)\right)=\delta(t-s)$

3) $\int e^{-i2\pi\xi t}E\left(Z(t)Z(0)\right)dt=1$

4) $Z(t)$ is a white noise (we will clarify the meaning of white noise).

Before we discuss the details in the long story of white noise, we review some of the mathematical tools/methods we will use.

Mathematical preparations

We first review an unconventional limit that we may be already familiar with.

Delta function (Dirac's delta function):

Definition 1:

Consider the limit of a rectangular function:

$$\lim_{d\to 0}\Pi_d(x)$$

where $\Pi_d(x)=\begin{cases}\dfrac{1}{d}, & \text{for } x\in\left(\dfrac{-d}{2},\dfrac{d}{2}\right)\\ 0, & \text{otherwise}\end{cases}$

This limit does not exist in the conventional sense.

However, for any smooth function $g(x)$, we have

$$\lim_{d\to 0}\int\Pi_d(x)g(x)dx=g(0)$$

We define the delta function $\delta(x)$ as an entity satisfying

$$\int \delta(x) g(x) dx = g(0) \qquad \text{for all smooth functions } g(x).$$

Then "formally" we write

$$\lim_{d\to 0} \Pi_d(x) = \delta(x)$$

Its true meaning is $\displaystyle \lim_{d\to 0} \int \Pi_d(x) g(x) dx = \int \delta(x) g(x) dx$

Here the <u>key strategy</u> in making sense of $\lim_{d\to 0} \Pi_d(x)$ is that we do the integration with $g(x)$ before we take the limit as $d \to 0$, not the other way around.

<u>Definition 2:</u>

In a similar way, we can define $\delta(x)$ as the limit of a normal distribution

$$\delta(x) = \lim_{\sigma\to 0} \rho_{N(0,\sigma^2)}(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

These two definitions are equivalent to each other. Each definition is mathematically more convenient in some situations.

Both definitions are based on the probability density of a scaled random variable with the multiplier converging to zero.

<u>Definition 1:</u> $\displaystyle \delta(x) = \lim_{\alpha\to 0} \rho_{\alpha X}(x), \qquad X \sim \text{uniform}\left(\frac{-1}{2}, \frac{1}{2}\right)$

<u>Definition 2:</u> $\displaystyle \delta(x) = \lim_{\alpha\to 0} \rho_{\alpha X}(x), \qquad X \sim N(0,1)$

<u>Fourier transform:</u>

Forward transform:

$$\underbrace{\hat{y}(\xi)}_{\text{Notation}} \equiv \underbrace{F[y(t)]}_{\text{New notation}} \equiv \int_{-\infty}^{+\infty} \exp(-i2\pi\xi t) y(t) dt$$

inverse transform:

$$y(t) = F^{-1}[\hat{y}(\xi)] \equiv \int_{-\infty}^{+\infty} \exp(i2\pi\xi t) \hat{y}(\xi) d\xi$$

<u>Remarks</u>:

- Here we introduce a new operator notation for the Fourier transform. This new notation will be convenient in the discussion of dW/dt.

- Here we use t as the independent variable instead of x. This choice of notation is for a better connection to the discussion of dW/dt.

Properties of Fourier transform:

1) Fourier transform of a normal PDF

$$F\left[\rho_{N(0,\sigma^2)}(t)\right]=F\left[\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-t^2}{2\sigma^2}\right)\right]=\exp\left(-2\pi^2\sigma^2\xi^2\right)$$

Proof:

Use the characteristic function (CF) of a normal RV we derived in Lecture 3. Then use the connection between the CF and the Fourier transform of PDF.

2) Fourier transform of the delta function

$F[\delta(t)] = 1$

Proof:

We view the delta function as the limit of normal distribution

$$\delta(t)=\lim_{\sigma\to 0}\rho_{N(0,\sigma^2)}(t)$$

We apply the Fourier transform and then take the limit as $\sigma \to 0$.

$$F\left[\delta(t)\right]=\lim_{\sigma\to 0}F\left[\rho_{N(0,\sigma^2)}(t)\right]=\lim_{\sigma\to 0}\exp\left(-2\pi^2\sigma^2\xi^2\right)=1$$

3) Fourier transform of $y(t) \equiv 1$

$F[1] = \delta(\xi)$

Proof:

Fourier transform of $y(t) \equiv 1$ does not converge in the conventional sense.

$$F[1]=\int_{-\infty}^{+\infty}\exp(-i2\pi\xi t)dt \quad \text{does not converge!}$$

We view 1 as the limit of

$$1=\lim_{\sigma\to\infty}\exp\left(\frac{-t^2}{2\sigma^2}\right)$$

$$=\lim_{\sigma\to\infty}\sqrt{2\pi\sigma^2}\rho_{N(0,\sigma^2)}(t) \quad \text{where} \quad \rho_{N(0,\sigma^2)}(t)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-t^2}{2\sigma^2}\right)$$

We apply the Fourier transform and before taking the limit as $\sigma \to \infty$.

$$F[1]=\lim_{\sigma\to\infty}\sqrt{2\pi\sigma^2}F\left[\rho_{N(0,\sigma^2)}(t)\right]=\lim_{\sigma\to\infty}\sqrt{2\pi\sigma^2}\exp\left(-2\pi^2\sigma^2\xi^2\right)$$

We write the RHS as the PDF of a normal RV

$$\sqrt{2\pi\sigma^2}\exp\left(-2\pi^2\sigma^2\xi^2\right)=\frac{1}{\sqrt{2\pi\left(\frac{1}{4\pi^2\sigma^2}\right)}}\exp\left(\frac{-\xi^2}{2\left(\frac{1}{4\pi^2\sigma^2}\right)}\right)=\rho_{N\left(0,\frac{1}{4\pi^2\sigma^2}\right)}(\xi)$$

In the limit of $\sigma\to\infty$, we have $1/(4\pi^2\sigma^2)\to 0$ and

$$F[1]=\lim_{\sigma\to\infty}\rho_{N\left(0,\frac{1}{4\pi^2\sigma^2}\right)}(\xi)=\delta(\xi)$$

Key observation:

If an operator acting on the limit of a function is invalid in the conventional sense, we can try to make sense of it by delaying taking the limit. That is, we first apply the operator and then we take the limit afterwards. That is why in the discussion of stochastic differential equations, *dt* is finite until we take the limit.

Example:

"Formally" we can conveniently write

$$\int\lim_{\sigma\to 0}\rho_{N(0,\sigma^2)}(t)g(t)dt=\int\delta(t)g(t)dt=g(0)$$

The true mathematical meaning is

$$\lim_{\sigma\to 0}\int\rho_{N(0,\sigma^2)}(t)g(t)dt=g(0)$$

With this order of operations, everything makes sense and is mathematically rigorous.

Appendix A

We show that $\frac{C}{x}\left(\frac{e^{2mx}-1}{e^{2mC}-1}\right) << 1$ when mC is moderately large and $x \le C/2$.

Lemma:

Function $f(s) \equiv \frac{e^s - 1}{s}$ increases monotonically for $s > 0$.

Proof:

$$\frac{e^s - 1}{s} = \frac{1}{s}\left(\sum_{n=0} \frac{1}{n!} s^n - 1\right) = \sum_{n=1} \frac{1}{n!} s^{n-1}$$

Each term in the summation is positive, and increases monotonically for $s > 0$.

End of proof

Apply the lemma to $\frac{e^{2mx}-1}{x}$ for $x \le C/2$,we get

$$\frac{e^{2mx}-1}{x} = 2m \cdot \frac{(e^{2mx}-1)}{2mx} \le 2m \cdot \frac{(e^{2m(C/2)}-1)}{2m(C/2)} = \frac{2(e^{mC}-1)}{C}$$

Using this inequality, we write $\frac{C}{x}\left(\frac{e^{2mx}-1}{e^{2mC}-1}\right)$ as

$$\frac{C}{x}\left(\frac{e^{2mx}-1}{e^{2mC}-1}\right) = \frac{C}{(e^{2mC}-1)}\left(\frac{e^{2mx}-1}{x}\right) \le \frac{C}{(e^{2mC}-1)} \frac{2(e^{mC}-1)}{C}$$

$$= \frac{2(e^{mC}-1)}{(e^{2mC}-1)} = \frac{2}{e^{mC}+1} \approx 2e^{-mC} << 1 \qquad \text{for moderately large } mC$$

Therefore, we conclude that $\frac{C}{x}\left(\frac{e^{2mx}-1}{e^{2mC}-1}\right) << 1$ when mC is moderately large and $x \le C/2$.