AM216 Stochastic Differential Equations

Lecture 17

List of topics in this lecture

- Analytical expression of option price $C(s, t)$ at time t when $S(t) = s$
- Expected reward at time T for paying $C(s, t)$ for the option
- Nominal value at time T of amount $C(s, t)$ at time t
- Effect of interest rate: option price increase with interest rate
- Effect of volatility: option price increase with volatility

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Review

Black-Scholes option pricing model

Evolution of the stock price

$dS = \mu S\,dt + \sigma S\,dW$ with Ito interpretation

Options associated with a stock

1 unit of call option = the right to buy 1 share of ABC at price K at time T.

1 unit of put option = the right to sell 1 share of ABC at price K at time T.

Assumption on the price of an option

The option price at time t is a deterministic function of the current stock price $S(t)$ and the current time t.

Option price function: $C(s, t)$,

the deterministic function connecting the stock price and the option price

The key question:

Suppose I am a market maker and I am required to set and publish $C(s, t)$.

How should I set function $C(s, t)$ to avoid a guaranteed loss?

Delta hedging portfolio

1 unit of delta hedging of time t

= owning (-1) unit of call option and $C_S(S(t), t)$ shares of stock.

Caution: the composition of portfolio varies with t.

Net gain/loss in time period [0, T]

Suppose that over time period [0, T], we maintain a portfolio of $F(S(t), t)$ units delta hedging of time t, at time t by carrying out transactions needed to adjust the portfolio.

We set $F(s,t) \equiv \Big(C(s,t) - C_s(s,t)s\Big)r - C_t(s,t) - \frac{1}{2}C_{ss}(s,t)\sigma^2 s^2$

…

$$G_{\text{Total}} = \int \left(\Big(C(s,t) - C_s(s,t)s\Big)r - C_t(s,t) - \frac{1}{2}C_{ss}(s,t)\sigma^2 s^2 \right)^2 \Bigg|_{s=S(t)} dt$$

G_{Total} would be a risk-free gain unless $(\,\cdot\,) \equiv 0$.

Governing equation of $C(s, t)$

$$\begin{cases} C_t(s,t) + \frac{1}{2}\sigma^2 s^2 C_{ss}(s,t) = r\Big(C(s,t) - sC_s(s,t)\Big) \\ C(s,t)\Big|_{t=T} = \max(s-K, 0) \end{cases}$$

End of review

Analytical expression of *C*(*s*, *t*)

We solve for $C(s, t)$ from the PDE and the final condition.

We use a change of variables to write it as a simple initial value problem.

Change of variables

New time variable

$\tau = T - t$ time to expiration

==> $t = T - \tau$

New spatial (price) variable

$$x = \log\frac{s}{K} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)$$

$$\Longrightarrow \quad s = K\exp\left(x - \left(r - \frac{1}{2}\sigma^2\right)\tau\right)$$

New price function

$$u(x,\tau) = e^{r(T-t)}C(s,t)$$

$$\Longrightarrow \quad C(s,t) = e^{-r\tau}u(x,\tau)$$

Derivatives of $C(s, t)$

We start with the derivatives of (τ, x) with respect to (t, s).

$$\frac{\partial\tau}{\partial t}=-1, \qquad \frac{\partial\tau}{\partial s}=0$$

$$\frac{\partial x}{\partial t}=\frac{\partial}{\partial t}\left(\log\frac{s}{K}+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)=-\left(r-\frac{1}{2}\sigma^2\right)$$

$$\frac{\partial x}{\partial s}=\frac{\partial}{\partial s}\left(\log\frac{s}{K}+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)=\frac{1}{s}$$

We express derivatives of $C(s, t)$ in terms of those of $u(x, \tau)$ using the chain rule.

$$\frac{\partial}{\partial t}C(s,t)=\frac{\partial}{\partial\tau}\left[e^{-r\tau}u(x,\tau)\right]\cdot\frac{\partial\tau}{\partial t}+\frac{\partial}{\partial x}\left[e^{-r\tau}u(x,\tau)\right]\cdot\frac{\partial x}{\partial t}$$

$$=re^{-r\tau}u(x,\tau)-e^{-r\tau}\frac{\partial}{\partial\tau}u(x,\tau)-\left(r-\frac{1}{2}\sigma^2\right)e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)$$

$$\frac{\partial}{\partial s}C(s,t)=\frac{\partial}{\partial x}\left[e^{-r\tau}u(x,\tau)\right]\cdot\frac{\partial x}{\partial s}=e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)\cdot\frac{1}{s}$$

$$\frac{\partial^2}{\partial s^2}C(s,t)=\frac{\partial}{\partial s}\left[e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)\cdot\frac{1}{s}\right]$$

$$=\frac{\partial}{\partial s}\left[e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)\right]\cdot\frac{1}{s}-e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)\cdot\frac{1}{s^2}$$

$$=e^{-r\tau}\frac{\partial^2}{\partial x^2}u(x,\tau)\cdot\frac{1}{s^2}-e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)\cdot\frac{1}{s^2}$$

Equation for $u(x, \tau)$

Substituting these derivatives into the PDE for $C(s, t)$, we obtain the PDE for $u(x, \tau)$.

$$\underbrace{re^{-r\tau}u(x,\tau)-e^{-r\tau}\frac{\partial}{\partial\tau}u(x,\tau)-\left(r-\frac{1}{2}\sigma^2\right)e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)}_{C_t(s,t)\equiv T_1+T_2+T_3}$$

$$\underbrace{+\frac{1}{2}\sigma^2\left[e^{-r\tau}\frac{\partial^2}{\partial x^2}u(x,\tau)-e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)\right]}_{\frac{1}{2}\sigma^2s^2C_{ss}(s,t)\equiv T_4+T_5}=\underbrace{r\left(e^{-r\tau}u(x,\tau)-e^{-r\tau}\frac{\partial}{\partial x}u(x,\tau)\right)}_{r(C(s,t)-sC_s(s,t))\equiv T_6+T_7}$$

Combining T_1 with T_6, first part of T_3 with T_7, second pat of T_3 with T_5, we obtain

$$-e^{-r\tau}\frac{\partial}{\partial \tau}u(x,\tau)+\frac{1}{2}\sigma^2 e^{-r\tau}\frac{\partial^2}{\partial x^2}u(x,\tau)=0$$

==> $$u_\tau(x,\tau)=\frac{1}{2}\sigma^2 u_{xx}(x,\tau)$$

where $u(x, \tau)$ is related to $C(s, t)$ by

$$u(x,\tau)=e^{r\tau}C(s,t)\,,\quad x=\log\frac{s}{K}+\left(r-\frac{1}{2}\sigma^2\right)\tau\,,\quad \tau=T-t\,.$$

Initial condition for $u(x, \tau)$

We use $s=K\exp\left(x-\left(r-\frac{1}{2}\sigma^2\right)\tau\right)$ to write out $u(x, \tau)|_{\tau=0}$.

$$u(x,\tau)\Big|_{\tau=0}=C(s,t)\Big|_{t=T}=\max\big(s-K,0\big)=K\max\big((e^x-1),0\big)$$

The initial value problem (IVP) for $u(x, \tau)$

$$\begin{cases} u_\tau(x,\tau)=\frac{1}{2}\sigma^2 u_{xx}(x,\tau) \\ u(x,\tau)\Big|_{\tau=0}=K\begin{cases}(e^x-1)\,, & x>0\\ 0\,, & x<0\end{cases}\end{cases}$$

Solution of $u(x, \tau)$

$$u(x,\tau)=\int_{-\infty}^{\infty}u(y,0)\frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left(\frac{-(y-x)^2}{2\sigma^2\tau}\right)dy$$

$$=\frac{K}{\sqrt{2\pi\sigma^2\tau}}\int_0^{\infty}(e^y-1)\exp\left(\frac{-(y-x)^2}{2\sigma^2\tau}\right)dy\equiv K(I_2-I_1)$$

We express I_1 and I_2 in terms of the error function.

Recall that the normal CDF has the expression

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{x}\exp\left(\frac{-\xi^2}{2\sigma^2\tau}\right)d\xi=\frac{1}{2}\left(1+\operatorname{erf}\left(\frac{x}{\sqrt{2\sigma^2\tau}}\right)\right)$$

We write out integral I_1 in terms of the error function.

$$I_1 = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_0^{\infty} \exp\left(\frac{-(y-x)^2}{2\sigma^2\tau} \right) dy$$

change of variables $\xi = x - y$

$$= \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{x} \exp\left(\frac{-\xi^2}{2\sigma^2\tau} \right) d\xi = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x}{\sqrt{2\sigma^2\tau}} \right) \right)$$

For integral I_2, we first complete the square in the exponent.

$$I_2 = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_0^{\infty} \exp\left(\frac{-(y-x)^2}{2\sigma^2\tau} + y \right) dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_0^{\infty} \exp\left(\frac{-\left[y^2 - 2(x+\sigma^2\tau)y + (x+\sigma^2\tau)^2 \right]}{2\sigma^2\tau} + x + \frac{\sigma^2\tau}{2} \right) dy$$

$$= \exp\left(x + \frac{\sigma^2\tau}{2} \right) \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_0^{\infty} \exp\left(\frac{-(y-x-\sigma^2\tau)^2}{2\sigma^2\tau} \right) dy$$

We then use change of variables $\xi = x + \sigma^2\tau - y$ to write I_2 as

$$I_2 = \exp\left(x + \frac{\sigma^2\tau}{2} \right) \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{(x+\sigma^2\tau)} \exp\left(\frac{-\xi^2}{2\sigma^2\tau} \right) dy$$

$$= \exp\left(x + \frac{\sigma^2\tau}{2} \right) \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x+\sigma^2\tau}{\sqrt{2\sigma^2\tau}} \right) \right)$$

Combining I_1 and I_2, we obtain an analytical expression for $u(x, \tau)$

$$\boxed{u(x,\tau) = \frac{K}{2}\exp\left(x + \frac{\sigma^2\tau}{2} \right)\left(1 + \mathrm{erf}\left(\frac{x+\sigma^2\tau}{\sqrt{2\sigma^2\tau}} \right) \right) - \frac{K}{2}\left(1 + \mathrm{erf}\left(\frac{x}{\sqrt{2\sigma^2\tau}} \right) \right)}$$

<u>Solution of</u> $C(s, t)$

$$C(s,t) = e^{-r\tau} u(x,\tau), \qquad x = \log\frac{s}{K} + \left(r - \frac{1}{2}\sigma^2 \right)\tau, \quad \tau = T - t$$

From the expression of x, we have

$$x + \frac{\sigma^2\tau}{2} = \log\frac{s}{K} + r\tau, \qquad \exp\left(x + \frac{\sigma^2\tau}{2} \right) = \frac{s}{K} e^{r\tau}$$

Using these results, we write $C(s, t) = e^{-r\tau} u(x, \tau)$ as

$$\boxed{C(s,t)=\frac{s}{2}\left[1+\mathrm{erf}\left(\frac{\log\frac{s}{K}+\left(r+\frac{\sigma^2}{2}\right)\tau}{\sqrt{2\sigma^2\tau}}\right)\right]-\frac{e^{-r\tau}K}{2}\left[1+\mathrm{erf}\left(\frac{\log\frac{s}{K}+\left(r-\frac{\sigma^2}{2}\right)\tau}{\sqrt{2\sigma^2\tau}}\right)\right]}$$

where $\tau = T - t$

Function $\phi(\eta)$ and its derivative

We re-write $C(s, t)$ as

$$C(s,t)=\frac{e^{-r\tau}K}{2}\left(\exp(\underbrace{\log\frac{s}{K}+r\tau}_{\eta})\left[1+\mathrm{erf}\left(\frac{\log\frac{s}{K}+r\tau+\frac{\sigma^2}{2}\tau}{\sqrt{2\sigma^2\tau}}\right)\right]-\left[1+\mathrm{erf}\left(\frac{\log\frac{s}{K}+r\tau-\frac{\sigma^2}{2}\tau}{\sqrt{2\sigma^2\tau}}\right)\right]\right)$$

$$\Longrightarrow \quad \boxed{C(s,t)=\frac{e^{-r\tau}K}{2}\phi(\eta,\omega)\,,\qquad \eta=\log\frac{s}{K}+r\tau\,,\quad \omega=\frac{1}{2}\sigma^2\tau} \qquad \text{(C-1)}$$

where function $\phi(\eta, \omega)$ is defined as

$$\phi(\eta,\omega)=e^{\eta}\left[1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right]-\left[1+\mathrm{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right] \qquad \text{(F-1)}$$

We calculate the derivative of $\phi(\eta, \omega)$.

$$\frac{d}{dz}\mathrm{erf}(z)=\frac{2}{\sqrt{\pi}}\exp(-z^2)$$

$$\frac{\partial}{\partial\eta}\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)=\frac{2}{\sqrt{\pi}}\exp\left(-\frac{(\eta^2+2\eta\omega+\omega^2)}{4\omega}\right)\frac{1}{\sqrt{4\omega}}$$

$$\frac{\partial}{\partial\eta}\mathrm{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)=\frac{2}{\sqrt{\pi}}\exp\left(-\frac{(\eta^2-2\eta\omega+\omega^2)}{4\omega}\right)\frac{1}{\sqrt{4\omega}}$$

$$\Longrightarrow \quad e^{\eta}\frac{\partial}{\partial\eta}\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)=\frac{\partial}{\partial\eta}\mathrm{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)$$

$$\Longrightarrow \quad \boxed{\frac{\partial}{\partial\eta}\phi(\eta,\omega)=e^{\eta}\left(1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right)>0} \qquad \text{(DF-1)}$$

Function $C_s(s, t)$

We use (DF-1) to calculate $C_s(s, t)$, which is needed in the delta hedging.

$$C(s,t)=\frac{e^{-r\tau}K}{2}\phi(\eta,\omega)\,,\qquad \eta=\log\frac{s}{K}+r\tau\,,\quad \omega=\frac{1}{2}\sigma^2\tau$$

$$\Longrightarrow\qquad \frac{\partial}{\partial s}C(s,t)=\frac{e^{-r\tau}K}{2}\frac{\partial}{\partial\eta}\phi(\eta,\omega)\frac{d\eta}{ds}=\frac{e^{-r\tau}K}{2}\mathrm{e}^{\eta}\left(1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right)\frac{1}{s}$$

$$=\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right),\qquad \mathrm{e}^{\eta}=\frac{s}{K}e^{r\tau}$$

We arrive at

$$C_s(s,t)=\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right),\qquad \eta=\log\frac{s}{K}+r\tau\,,\quad \omega=\frac{1}{2}\sigma^2\tau$$

Expected reward for paying $C(s, t)$ for the option

We compare the rewards of buying the option vs not buying.

Nominal value at time T of amount $C(s, t)$ at time t

$\mathrm{e}^{r(T-t)}C(s, t)$

= the nominal value at time T of amount $C(s, t)$ at time t

where $C(s, t)$ is the amount needed to buy the option at time t when the stock price is s.

We write out $\mathrm{e}^{r(T-t)}C(s, t)$ using equation (C-1)

$$\mathrm{e}^{r\tau}C(s,t)=\frac{K}{2}\phi(\eta,\omega)\,,\qquad \eta=\log\frac{s}{K}+r\tau\,,\quad \omega=\frac{1}{2}\sigma^2\tau$$

Next we calculate the expected reward at time T for owning the option.

Evolution of $Y = \log(S)$

$$dS=\mu S\,dt+\sigma S\,dW\,,\qquad \text{starting at } S(t)=s$$

The Ito interpretation of this SDE corresponds to

$$dY=\left(\mu-\frac{\sigma^2}{2}\right)dt+\sigma\,dW\,,\qquad Y=\log(S)\quad \text{starting at } Y(t)=\log(s)$$

$$\Longrightarrow\qquad Y(T)=Y(t)+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)+\sigma\big(W(T)-W(t)\big)$$

$$\Longrightarrow\qquad Y(T)=\log(s)+\left(\mu-\frac{\sigma^2}{2}\right)\tau+N\big(0,\sigma^2\tau\big),\qquad \tau=T-t$$

The probability density of $Y(T)$ is

$$\Longrightarrow \quad \rho_Y(y)=\frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left(\frac{-\left(y-\log(s)-\left(\mu-\frac{\sigma^2}{2}\right)\tau\right)^2}{2\sigma^2\tau}\right)$$

<u>Expected reward</u> at time T

$$E\Big(\max(S(T)-K,0)\Big)=E\Big(\max(\exp(Y(T))-K,0)\Big)$$

$$=\int_{-\infty}^{\infty}\max\left(e^y-K,0\right)\rho_Y(y)dy=\int_{\log(K)}^{\infty}\left(e^y-K\right)\rho_Y(y)dy$$

$$=\frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{\log(K)}^{\infty}\left(e^y-K\right)\exp\left(\frac{-\left(y-\log(s)-\mu\tau+\frac{\sigma^2}{2}\tau\right)^2}{2\sigma^2\tau}\right)dy\equiv J_2-J_1$$

We calculate J_1 and J_2 similar to what did previously for I_1 and I_2.

In J_1, we use change of variables: $y=-\xi$

$$J_1=\frac{K}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{-\log(K)}\exp\left(\frac{-\left(\xi+\log(s)+\mu\tau-\frac{\sigma^2}{2}\tau\right)^2}{2\sigma^2\tau}\right)d\xi$$

$$=\frac{K}{2}\left(1+\operatorname{erf}\left(\frac{\eta_\mu-\omega}{\sqrt{4\omega}}\right)\right),\qquad \eta_\mu\equiv\log\frac{s}{K}+\mu\tau,\quad \omega=\frac{1}{2}\sigma^2\tau$$

In J_2, we complete square and use change of variables: $y=-\xi$

$$J_2=\frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{\log(K)}^{\infty}\exp\left(\frac{y2\sigma^2\tau-\left(y-\log(s)-\mu\tau+\frac{\sigma^2}{2}\tau\right)^2}{2\sigma^2\tau}\right)dy$$

$$=\frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left(\log(s)-\mu\tau\right)\int_{\log(K)}^{\infty}\exp\left(\frac{-\left(y-\log(s)-\mu\tau-\frac{\sigma^2}{2}\tau\right)^2}{2\sigma^2\tau}\right)dy$$

$$=\frac{K}{\sqrt{2\pi\sigma^2\tau}}\exp\left(\log\frac{s}{K}+\mu\tau\right)\int_{-\infty}^{-\log(K)}\exp\left(\frac{-\left(\xi+\log(s)+\mu\tau+\frac{\sigma^2}{2}\tau\right)^2}{2\sigma^2\tau}\right)d\xi$$

$$= \frac{K}{2}\exp(\eta_\mu)\left(1+\text{erf}\left(\frac{\eta_\mu+\omega}{\sqrt{4\omega}}\right)\right), \qquad \eta_\mu \equiv \log\frac{s}{K}+\mu\tau\,, \quad \omega=\frac{1}{2}\sigma^2\tau$$

The expected reward at time T is

$$E\big(\max(S(T)-K,0)\big)$$
$$= \frac{K}{2}\exp(\eta_\mu)\left(1+\text{erf}\left(\frac{\eta_\mu+\omega}{\sqrt{4\omega}}\right)\right)-\frac{K}{2}\left(1+\text{erf}\left(\frac{\eta_\mu-\omega}{\sqrt{4\omega}}\right)\right)$$

Recall the definition of $\phi(\eta, \omega)$ in (F-1).

We write the expected reward at time T as

$$E\big(\max(S(T)-K,0)\big)=\frac{K}{2}\phi(\eta_\mu,\omega)\,, \qquad \eta_\mu \equiv \log\frac{s}{K}+\mu\tau\,, \quad \omega=\frac{1}{2}\sigma^2\tau$$

We compare it with the <u>nominal value</u> at time T of amount $C(s, t)$ at time t

$$e^{r\tau}C(s,t)=\frac{K}{2}\phi(\eta_r,\omega)\,, \qquad \eta_r \equiv \log\frac{s}{K}+r\tau\,, \quad \omega=\frac{1}{2}\sigma^2\tau$$

<u>The two values</u> have the same form with r corresponding to μ.

<u>Expected reward of owning the option vs nominal value at time T</u> of amount $C(s, t)$

The reward at time T of owning the option is not risk-free (in fact, it has very high risk!). It is a random variable with the average

$$E\big(\max(S(T)-K,0)\big)=\frac{K}{2}\phi(\eta_\mu,\omega)$$

The nominal value at time T of amount $C(s, t)$ at time t is risk-free.

$$e^{r\tau}C(s,t)=\frac{K}{2}\phi(\eta_r,\omega)$$

The basic principle of risk-reward tells us that we should have

$$\frac{K}{2}\phi(\eta_\mu,\omega)>\frac{K}{2}\phi(\eta_r,\omega)$$

(DF-1) implies that $\phi(\eta, \omega)$ is an increasing function of η.

==> $\eta_\mu > \eta_r$ ==> $\mu > r$

<u>Remarks:</u>

- For the underlying stock, an investment is not risk-free.

$$dS=\mu S\,dt+\sigma S\,dW$$

==> $dE(S) = \mu E(S) dt$

==> $\frac{dE(S)}{dt} = \mu E(S)$ which is an exponential growth with rate = μ

$\mu > r$ corresponds to the principle that the expected reward of a risky investment should be higher than the risk-free reward (based on interest rate).

- The risk-reward principle is true in the broader sense when we include rewards of all forms received from all sources.

  Example:

  Buying a lottery ticket.

  I may assign a significant monetary value to the excitement of possibly winning

  or I may believe my number selection scheme will increase my chance so that my perceived average reward is significantly larger than the lottery ticket price.

The effect of interest rate *r* on $C(s, t)$

We write $C(s, t)$ as

$$C(s,t) = \frac{K}{2} e^{-r\tau} \phi(\eta,\omega) = \frac{s}{2} e^{-\eta} \phi(\eta,\omega), \qquad \eta \equiv \log\frac{s}{K} + r\tau, \quad \omega = \frac{1}{2}\sigma^2 \tau$$

The effect of interest rate r is contained in variable η.

Differentiating $e^{-\eta}\phi(\eta, \omega)$ and using (F-1) and (DF-1), we have

$$\frac{\partial}{\partial \eta}\left(e^{-\eta}\phi(\eta,\omega)\right) = -e^{-\eta}\phi + e^{-\eta}\frac{\partial}{\partial \eta}\phi = e^{-\eta}\left(1 + \text{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right) > 0$$

==> $$\frac{\partial}{\partial r} C(s,t) = \frac{s}{2}\frac{\partial}{\partial \eta}\left(e^{-\eta}\phi\right)\frac{d\eta}{dr} = \frac{s\tau}{2} e^{-\eta}\left(1 + \text{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right) > 0$$

Conclusion:

Option price $C(s, t)$ increases with interest rate r.

Interpretation:

When interest rate r is higher, the perceived future drift μ for the stock price must be higher in one of the two ways below:

- The hike of interest rate r is in response to the increase in the perceived future drift μ. That is, the increase in the perceived future drift precedes the hike of interest rate.
- When the interest rate r is raised, it makes the stock less attractive as an investment. In response, the stock drops to a lower price to increase the future

percentage-wise gain so as to attract investors. When the perceived future drift is large enough, the stock price drop stops.

Thus, a higher interest rate r must correspond to a higher perceived future drift μ, in one way or the other. A higher perceived future drift μ increases the average reward at time T of owning the option and makes the option price higher.

The effect of volatility σ

The effect of volatility σ is contained in variable ω.

We differentiate $\phi(\eta, \omega)$ with respect to σ.

$$\phi(\eta,\omega)=e^{\eta}\left[1+\operatorname{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right]-\left[1+\operatorname{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right], \qquad \eta\equiv\log\frac{s}{K}+r\tau\,, \quad \omega=\frac{1}{2}\sigma^2\tau$$

$$\begin{aligned}\frac{\partial}{\partial\omega}\phi(\eta,\omega) &= e^{\eta}\exp\left(\frac{-(\eta^2+2\eta\omega+\omega^2)}{4\omega}\right)\left(-\frac{\eta}{4\omega^{3/2}}+\frac{1}{4\omega^{1/2}}\right)\\ &\quad -\exp\left(\frac{-(\eta^2-2\eta\omega+\omega^2)}{4\omega}\right)\left(-\frac{\eta}{4\omega^{3/2}}-\frac{1}{4\omega^{1/2}}\right)\\ &= \exp\left(\frac{-(\eta^2-2\eta\omega+\omega^2)}{4\omega}\right)\frac{1}{2\omega^{1/2}}>0\end{aligned}$$

$$\frac{\partial}{\partial\sigma}\phi(\eta,\omega)=\frac{\partial}{\partial\omega}\phi(\eta,\omega)\cdot\frac{d\omega}{d\sigma}=\exp\left(\frac{-(\eta^2-2\eta\omega+\omega^2)}{4\omega}\right)\frac{1}{2\omega^{1/2}}\cdot\sigma\tau>0$$

$$\frac{\partial}{\partial\sigma}C(s,t)=\frac{K}{2}e^{-r\tau}\frac{\partial}{\partial\sigma}\phi(\eta,\omega)\ >\ 0$$

Conclusion:

Option price $C(s, t)$ increases with volatility σ.

Interpretation:

A higher volatility increases the average reward at time T of owning the option and makes the option price higher.

The case of unknown σ

We can estimate σ from the past history of stock price and then use the estimated σ to predict the option price $C(s, t)$.

Conversely, we can use the current market price $C(s, t)$ of the option to estimate investors' perceived future volatility of the underlying stock.

- $C(s, t)$ increases with σ monotonically.

- For each realized sample of market price $C(s, t)$, there is a corresponding estimated value of future volatility σ.