AM216 Stochastic Differential Equations

Lecture 16
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List of topics in this lecture

- Black-Scholes option pricing model: stochastic evolution of the stock price, call option, put option, strike price, expiration date
- Delta hedging portfolio: 1 unit of delta hedging of time t, continuously revised delta hedging, mathematical view of delta hedging, net gain of delta hedging in a small time interval, the net gain of delta hedging until the expiry
- Governing equation of the option price function

Review

Feynman-Kac formula for the forward equation

Stochastic differential equation (SDE)

$$dX = b(X,t)dt + \sqrt{a(X,t)}\,dW$$

$u(x, t)$ is defined using path integral

$$u(x,t) = E\left(\delta\Big(X(t)-x\Big)\exp\left(-\int_0^t \psi(X(s),s)ds\right)\right)$$

Meaning of $u(x, t)$

= ensemble density at time t of the surviving ensemble.

$u(x, t)$ satisfies the forward equation with a fatality/growth term

$$u_t = -\Big(b(x,t)u\Big)_x + \frac{1}{2}\Big(a(x,t)u\Big)_{xx} - \psi(x,t)u$$

An application: estimating potential from non-equilibrium measurements

A particle diffusing in potential $V(x)$, under applied force $F(t)$.

What we can measure: a set of sample paths $\{X^{(j)}(t)$, j = 1, 2, …, $N\}$.

Goal: To determine potential $V(x)$.

We consider a "hypothetical" density defined as

$$\rho^{(F)}(x,t) \equiv \frac{1}{Z}\exp\left(-V(x)+F(t)x\right), \qquad Z=\int \exp\left(-V(x)\right)dx \qquad \text{(E01)}$$

$\rho^{(F)}(x, t)$ satisfies the forward equation with a fatality term

$$\rho_t^{(F)} = L_{\{H\}}\left[\rho^{(F)}\right] - \psi(x,t)\cdot\rho^{(F)}, \qquad \psi(x,t) = -F'(t)x$$

Feynman-Kac formula

$$\rho^{(F)}(x,t) = E\left(\delta(X(t)-x)\exp\left(\int_0^t F'(s)X(s)ds\right)\right) \qquad \text{(E02)}$$

(E02) allows us to calculate $\rho^{(F)}(x, t)$ from measured sample paths.

(E01) allows us to construct potential $V(x)$ from the calculated $\rho^{(F)}(x, t)$.

End of review

Black-Scholes option pricing model

The underlying stock:

Consider the stock of ABC company:

One share of stock = a fraction of ownership of ABC company.

Example:

There are 1,805 million shares of Walt Disney Company stock.

Each share = $1/(1.8\times10^9)$ fraction of Walt Disney Company.

Evolution of the stock price

Let $S(t)$ = stock price of ABC company at time t

(price per share of ABC company at time t).

Assumption:

$S(t)$ is a geometric Brownian motion with a drift.

Specifically, stock price $S(t)$ is governed by the SDE

$$\frac{1}{S}dS = \mu\, dt + \sigma\, dW$$

$$\Longrightarrow \qquad dS = \mu S\, dt + \sigma S\, dW$$

with the Ito interpretation.

Notation:

Volatility: σ

Drift: μ

Interest rate: r

r is important although it does not appear in the SDE of $S(t)$.

Remark:

It is debatable whether the stock price should be modeled as

$$dS = \mu S\,dt + \sigma S\,dW \qquad \text{(S01A)}$$

or $$d(\log S) = \mu\,dt + \sigma\,dW \qquad \text{(S01B)}$$

SDEs (S01A) and (S01B) are different.

While (S01A) is subject to different interpretations, (S01B) is not.

The Stratonovich interpretation of (S01A) is equivalent to the Ito interpretation of

$$dS = \left(\mu + \frac{1}{2}\sigma^2\right) S\,dt + \sigma S\,dW$$

which is equivalent to (S01B).

Even in the absence of an apparent drift term ($\mu = 0$), (S01B) produces a drift.

$$\log(S + dS) - \log(S) = \sigma\,dW$$

==> $S + dS = S\exp(\sigma\,dW)$

==> $dS = S\big(\exp(\sigma\,dW) - 1\big) = S\left(\sigma dW + \frac{1}{2}\sigma^2 (dW)^2 + o(dt)\right)$

==> $dS = \frac{1}{2}\sigma^2 S\,dt + \sigma S\,dW$ with the Ito interpretation.

We will use (S01A) with the Ito interpretation.

Options associated with a stock

An option is a contract which

- gives the buyer (the owner or holder of the option) the right, but not the obligation,
- to buy or sell an underlying stock
- at a specified strike price
- prior to or on a specified date

Call options

1 contract of call option

= the right to buy 100 shares of ABC at price K at time T.

We introduce 1 "unit" of call option.

1 unit of call option = 1/100 contract of call option

= the right to buy 1 share of ABC at price K at time T.

In the analysis below, 1 unit of option is mathematically more convenience to work with since it corresponds to 1 share of the stock.

Put options

1 contract of put option

= the right to sell 100 shares of ABC at price K at time T.

1 unit of put option = 1/100 contract of put option

= the right to sell 1 share of ABC at price K at time T.

Terminology:

Strike price: K, the specified purchase price (call option) or specified sell price (put option).

Exercising an option: exercising the right to buy or sell.

Expiry (expiration date): Time T at which the option can be exercised.

Important points:

- Call option = the right to buy …
- Put option = the right to sell …
- The right ≠ the obligation

  (The option holder has no obligation to buy or sell the stock).
- This is the European style option.

  American style option = the right … at any time before or at T.

  We study the European style option because it is mathematically simpler.
- Options have no assets of their own. Their values are solely derived from the underlying stock. For this reason, options are called financial derivatives.
- Options are not issued by the company of the underlying stock. Options are underwritten by other investors (financial gamblers). Option buyers pay money to own the right. Option writers (financial gamblers who sell options) receive money from buyers in exchange for giving option holders the right to buy or sell.

- Options are much more volatile than the underlying stock because the risk of options is much higher than that of the stock. The option buyer may win big if the underlying stock moves significantly in one direction, and may lose all money spent on buying the options if the stock moves the other direction. For the option underwriter, the risk is even worse; the loss may be unlimited.
- Options are not limited to stocks. Options can be written on anything that has a market (and thus, a market price), for example, oil, natural gas, wheat, corn, soy bean, gold, silver, copper, lumber, …
- In particular, options can be written on other options. They are the derivatives of derivatives. They are even more volatile (more risky!).

Price of an option

Consider an option of a specified type (call or put), a specified strike price and a specified expiration date. For example, a call option to buy Disney stock at $120/per share on August 20th.

Assumption:

The option price at time t is a deterministic function of the current stock price $S(t)$ and the current time t.

Let $C(s, t)$ denote the price per unit of the call option at time t when $S(t) = s$.

$C(s, t)$ is a deterministic function of two variables (s, t).

At time t, both the stock price and the option price are stochastic. But the two are related by a deterministic function: $C(S(t), t)$.

Note:

- Options are traded by number of contracts.
- The option price is quoted as price per unit (per 1/100th contract).
- The randomness in the option price is solely caused by the stock price $S(t)$.

List of variables and parameters:

$S(t)$: stock price at time t

$C(S(t), t)$: option price at time t

σ: volatility

μ: drift in the SDE of stock price

r: interest rate

K: the strike price

T: the expiry (expiration date)

List of assumptions

- The stock does not pay a dividend.

  This is true for some stocks.

- The stock price $S(t)$ is a geometric Brownian motion with volatility (σ) and drift (μ).

  $$\frac{1}{S}dS = \mu\,dt + \sigma\,dW$$

- Volatility (σ) and interest rate (r) are known.

  As we will see, drift (μ) does not appear in the PDE for the option price.

- The option price at time t is $C(S(t), t)$ where $C(s, t)$ is a deterministic function of (s, t).
- Function $C(s, t)$ is set by a market maker (a dealer).

  We will visit this issue shortly.

- We can buy/sell any amount of option/stock, including short selling.

  Short selling = selling something we don't own.

- There is no bid-ask spread. At any time, the price we can buy (dealer's ask price) is the same as the price we can sell (dealer's bid price).

  In reality, there is always a bid-ask spread.

  This unrealistic assumption is just to make the model simple.

- There is a single interest rate. We can borrow/lend any amount at this known rate.

  In reality, the interest rate of borrowing (car loan, mortgage) is significantly higher than the interest rate of bank saving accounts.

  This unrealistic assumption is just to make the model simple.

- There is no transaction fee associated with buying/selling stock/option, no transaction fee associated with borrowing/lending.

  For large financial firms, this is not completely unrealistic.

Key question:

Suppose I am a market maker and I am required to set and publish $C(s, t)$, the deterministic function connecting the stock price and the option price.

How should I set function $C(s, t)$ to avoid guaranteed loss?

Could someone design a scheme of trading stocks/options based on the published function $C(s, t)$ to make a guaranteed gain?

Answer:

We study the delta hedging portfolio.

Delta hedging portfolio

We consider the delta hedging involving a call option.

The delta hedging involving a put option can be discussed in a similar way.

Composition of the delta hedging

Consider the call option with strike price K and expiry T.

1 unit of delta hedging of time t

= being short one unit of call option and long $C_S(S(t), t)$ shares of stock

= owning (−1) unit of call option and $C_S(S(t), t)$ shares of stock.

Note:

$C_S(s, t)$ is the derivative of $C(s, t)$.

$$C_s\left(S(t),t\right)=\left.\frac{\partial C\left(s,t\right)}{\partial s}\right|_{s=S(t)}$$

Since we can buy/sell any amount (positive or negative, small or large), we can have (−1) unit of delta hedging, or any positive or negative fraction of 1 unit.

(−1) unit of delta hedging of time t

= owning (+1) unit of call option and $-C_S(S(t), t)$ shares of stock.

Important points:

- The composition of 1 unit delta hedging of time t is determined by the stock price at time t and the published function $C(s, t)$.
- The composition of 1 unit delta hedging of time t is not static; it varies with t. To maintain 1 unit delta hedging over time, buying/selling stocks is needed.
- Specifically, to change 1 unit delta hedging of time t to 1 unit delta hedging of time $(t+dt)$, we need to

  buy $[C_S(S(t+dt), t+dt) - C_S(S(t), t)]$ shares of stock at time $(t+dt)$.

  This is called continuously revised delta hedging.
- To change 1 unit delta hedging of time t to 1 unit delta hedging of time $(t+dt)$, mathematically, it is more convenient to do it by

  selling 1 unit delta hedging of time t at time $(t+dt)$, and

  buying 1 unit delta hedging of time $(t+dt)$ at time $(t+dt)$.

  "of time t" refers to the composition of 1 unit delta hedging.

"at time (t+dt)" refers to the time of buying/selling.

- In a general setting, suppose we like to have a portfolio of $F(S(t), t)$ units delta hedging of time t, at time t, over a time period. We need to update the portfolio by

    selling $F(S(t), t)$ units delta hedging of time t at time (t+dt), and

    buying $F(S(t+dt), t+dt)$ units delta hedging of time (t+dt) at time (t+dt).

    These are the two actions at each time (t+dt) over a time period.

- We view the selling action as associated with time interval [t, t+dt] and view the buying action as associated with time interval [t+dt, t+2dt].

    Mathematically, it is more convenient to start a time interval with no position, end it with no position, and calculate the net gain/loss in that time interval.

    In this mathematical view, the actions over time interval [t, t+dt] are

    start with no position at time t,

    at time t, buy $F(S(t), t)$ units of delta hedging of time t;

    hold it during [t, t+dt];

    at time (t+dt), sell $F(S(t), t)$ units of delta hedging of time t;

    end with no position at time (t+dt).

- This mathematical view involves a lot of "hypothetical" trading transactions. They are for the purpose of facilitating the calculation of net gain/loss in [t, t+dt]. Many of these "hypothetical" trading transactions cancel each other. The actual trading transactions needed in real operation are a lot less.

Net gain/loss in time period [0, T]

Suppose that over time period [0, T], we maintain a portfolio of $F(S(t), t)$ units delta hedging of time t, at time t by carrying out the actions described above in each time interval. Here $F(s, t)$ is a function to be specified.

We first calculate the net gain/loss in each time interval [t, t+dt]. Since we are concerned with only the net gain/loss, we start each time interval with 0 cash.

Cash balance at time t after buying $F(S(t), t)$ units of delta hedging of time t,

$$B(t)=\underbrace{F\big(S(t),t\big)}_{\substack{\text{\# of units of}\\ \text{delta hedging}}}\Bigg[\underbrace{1}_{\substack{\text{\# of units}\\ \text{of option}}}\times\underbrace{C\big(S(t),t\big)}_{\substack{\text{option}\\ \text{price}}}-\underbrace{C_s\big(S(t),t\big)}_{\substack{\text{\# of shares}\\ \text{of stock}}}\times\underbrace{S(t)}_{\substack{\text{stock}\\ \text{price}}}\Bigg]$$

Note: The cash balance may be positive or negative.

The interest earned in [t, t+dt] from the cash balance

$I = B(t) \times r\, dt$

Note:

The interest earned may be positive or negative, depending on the cash balance.
Negative interest = interest cost for borrowing money.

Change in cash balance at time (t+dt) after selling $F(S(t), t)$ unit of delta hedging of time t

$$\Delta B = \underbrace{F\big(S(t),t\big)}_{\substack{\text{\# of units of}\\ \text{delta hedging}}}\left[\underbrace{-1}_{\substack{\text{\# of units}\\ \text{of option}}} \times \underbrace{C\big(S(t+dt),t+dt\big)}_{\substack{\text{option price}\\ \text{at time } t+\Delta t}} + \underbrace{C_s\big(S(t),t\big)}_{\substack{\text{\# of shares}\\ \text{of stock}}} \times \underbrace{S(t+dt)}_{\substack{\text{stock price}\\ \text{at time } t+\Delta t}}\right]$$

Note:

The composition of delta hedging of time t is unchanged in [t, t+dt].

The prices of stock and option do vary in [t, t+dt].

To facilitate the analysis, we introduce short notation:

$S \equiv S(t), \qquad dS \equiv S(t+\Delta t)-S(t)$

==> $\quad S(t+\Delta t) = S+dS$

We write the change in cash balance as

$$\Delta B = F(S,t)\big[-C(S+dS,t+dt)+C_s(S,t)(S+dS)\big]$$

Taylor expansion in terms of dS and dt

$$= F(S,t)\begin{bmatrix} -C(S,t)-\underbrace{C_s(S,t)dS}_{dS\text{ term}}-C_t(S,t)dt-\dfrac{1}{2}C_{ss}(S,t)(dS)^2 \\ +C_s(S,t)S+\underbrace{C_s(S,t)dS}_{dS\text{ term}}+o(dt) \end{bmatrix}$$

By the special design of delta hedging, two dS terms cancel each other.

$$= F(S,t)\left[-C(S,t)+C_s(S,t)S-C_t(S,t)dt-\frac{1}{2}C_{ss}(S,t)(dS)^2+o(dt)\right]$$

The net gain/loss in [t, t+dt]

Let $G(t)$ denote the rate of net gain in [t, t+dt] (the net gain per time).

The net gain in [t, t+dt] is

$$Gdt = B(t)+I+\Delta B = B(t)(1+rt)+\Delta B$$

$$= \underbrace{F(S,t)\big[C(S,t)-C_s(S,t)S\big]}_{\text{cash balance after buying at time } t}(1+r\,dt)$$

$$+\underbrace{F(S,t)\left[-C(S,t)+C_s(S,t)S-C_t(S,t)dt-\frac{1}{2}C_{ss}(S,t)(dS)^2\right]}_{\text{change in cash balance when selling at time } t+dt}$$

$$=F(S,t)\left[\Big(C(S,t)-C_s(S,t)S\Big)r\,dt-C_t(S,t)dt-\frac{1}{2}C_{ss}(S,t)(dS)^2\right]$$

where $\quad S\equiv S(t), \quad dS = S(t+\Delta t)-S(t)$

We use the SDE to express $(dS)^2$ in terms of dW.

$$dS=\mu S\,dt+\sigma S\,dW$$

==> $\quad (dS)^2=\sigma^2S^2(dW)^2+o(dt)$

We write the net gain in $[t, t+dt]$ as

$$Gdt=F(s,t)\left[\Big(C(s,t)-C_s(s,t)s\Big)r\,dt-C_t(s,t)dt-\frac{1}{2}C_{ss}(s,t)\sigma^2s^2(dW)^2\right]\Bigg|_{s=S(t)}$$

<u>The total gain</u> in $[0, T]$ is

$$G_{\text{Total}}=\int G\,dt=\int F(s,t)\left[\Big(C(s,t)-C_s(s,t)s\Big)r\,dt-C_t(s,t)dt-\frac{1}{2}C_{ss}(s,t)\sigma^2s^2(dW)^2\right]\Bigg|_{s=S(t)}$$

Recall that in an integral we can replace $(dW)^2$ by dt. We have

$$G_{\text{Total}}=\int F(s,t)\left[\Big(C(s,t)-C_s(s,t)s\Big)r-C_t(s,t)-\frac{1}{2}C_{ss}(s,t)\sigma^2s^2\right]\Bigg|_{s=S(t)}dt$$

<u>Specifying function</u> $F(s, t)$ based on $C(s, t)$

Based on the published function $C(s, t)$, we select $F(s, t)$ as

$$F(s,t)\equiv\Big(C(s,t)-C_s(s,t)s\Big)r-C_t(s,t)-\frac{1}{2}C_{ss}(s,t)\sigma^2s^2$$

<u>The total gain</u> in $[0, T]$ with the specified $F(s, t)$

$$G_{\text{Total}}=\int\left(\Big(C(s,t)-C_s(s,t)s\Big)r-C_t(s,t)-\frac{1}{2}C_{ss}(s,t)\sigma^2s^2\right)^2\Bigg|_{s=S(t)}dt \qquad \text{(G01)}$$

Governing equation of $C(s, t)$

Observations:

- The integrand in the total gain (G01) is always non-negative. That means no matter where $S(t)$ goes, we will never lose money. The trading scheme is risk-free!
- If the integrand is non-zero in a region of (s, t), then we have a gain whenever $S(t)$ goes through the region. Since $S(t)$ is stochastic, there is always a small probability of $S(t)$ going through such a region.
- If the integrand is non-zero in a region of (s, t), we have certain probability of making a gain with no risk.

Therefore, the integrand in the total gain (G01)) must be identically zero.

$$\big(C(s,t)-C_s(s,t)s\big)r-C_t(s,t)-\frac{1}{2}C_{ss}(s,t)\sigma^2 s^2=0$$

It follows that $C(s, t)$ satisfies the PDE

$$C_t(s,t)+\frac{1}{2}\sigma^2 s^2 C_{ss}(s,t)=r\big(C(s,t)-sC_s(s,t)\big)$$

At end time T (expiration date), the price of the call option is simply the amount of money the call option holder can save when buying the stock by exercising the option instead of buying in the open market. The amount of saving is

$$=\begin{cases}(\text{stock price})-(\text{strike price}), & \text{if stock price} > \text{strike price}\\ 0, & \text{otherwise}\end{cases}$$

The end/final condition for $C(s, t)$ is

$$C(s,t)\big|_{t=T}=\max(s-K,0)$$

The FVP for $C(s, t)$ is

$$\begin{cases}C_t(s,t)+\frac{1}{2}\sigma^2 s^2 C_{ss}(s,t)=r\big(C(s,t)-sC_s(s,t)\big)\\ C(s,t)\big|_{t=T}=\max(s-K,0)\end{cases}$$

Notice that the drift term μ is not in the FVP.