AM216 Stochastic Differential Equations

Lecture 04 Supplemental
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Computational demonstration of Ito's lemma

Let $f(t, w)$ be a smooth function of (t, w). Replacing w with $W(t)$ yields $f(t, W(t))$, a non-smooth random function of t. We select a numerical grid on t.

$$\Delta t = \frac{t_f}{N}, \qquad t_j = j\Delta t, \qquad W_j = W(t_j), \qquad dW_j = W_{j+1} - W_j$$

Let Δf_{j} denote the increment of f. We consider 3 different versions of Δf_{j}.

- <u>The exact increment</u>

$$\left(\Delta f_{\text{exact}}\right)_j = f(t_{j+1}, W(t_{j+1})) - f(t_j, W(t_j))$$

- <u>Approximate increment based on Taylor expansion</u>

$$\left(\Delta f_{\text{Taylor}}\right)_j = (f_t)_j \Delta t + (f_w)_j \Delta W_j + \frac{1}{2}(f_{ww})_j (\Delta W_j)^2$$

- <u>Approximate increment based on Ito's lemma</u>

$$\left(\Delta f_{\text{Ito}}\right)_j = (f_t)_j \Delta t + (f_w)_j \Delta W_j + \frac{1}{2}(f_{ww})_j \Delta t$$

With the 3 versions of increment Δf_{j} and the initial value $f(0, 0)$, we construct 3 versions of approximation for random function $f(t_f, W(t_f))$.

- <u>The exact</u>

$$\left(f_{\text{exact}}\right)\Big|_{t_f} = f(0,0) + \sum_{j=0}^{N-1} \left(\Delta f_{\text{exact}}\right)_j = f(t_f, W(t_f))$$

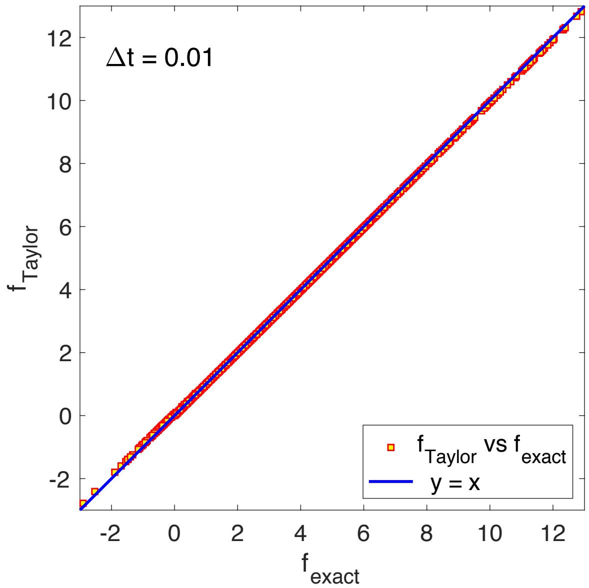
- <u>Approximation based on Taylor expansion</u>

$$\left(f_{\text{Taylor}}\right)\Big|_{t_f} = f(0,0) + \sum_{j=0}^{N-1} \left(\Delta f_{\text{Taylor}}\right)_j$$

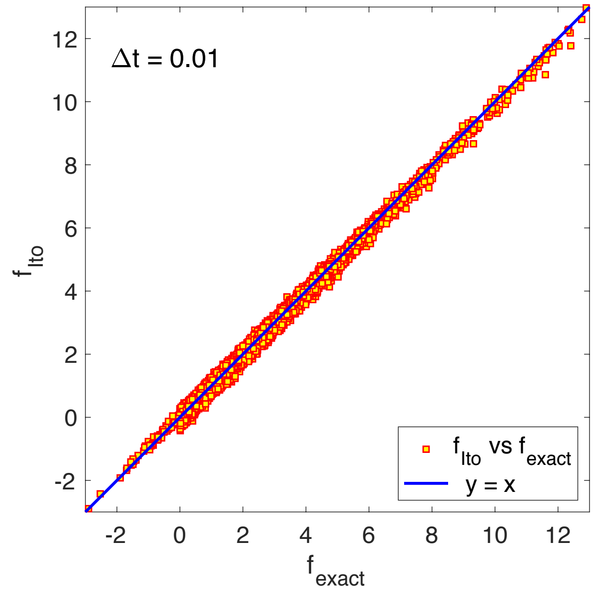
- <u>Approximation based on Ito's lemma</u>

$$\left(f_{\text{Ito}}\right)\Big|_{t_f} = f(0,0) + \sum_{j=0}^{N-1} \left(\Delta f_{\text{Ito}}\right)_j$$

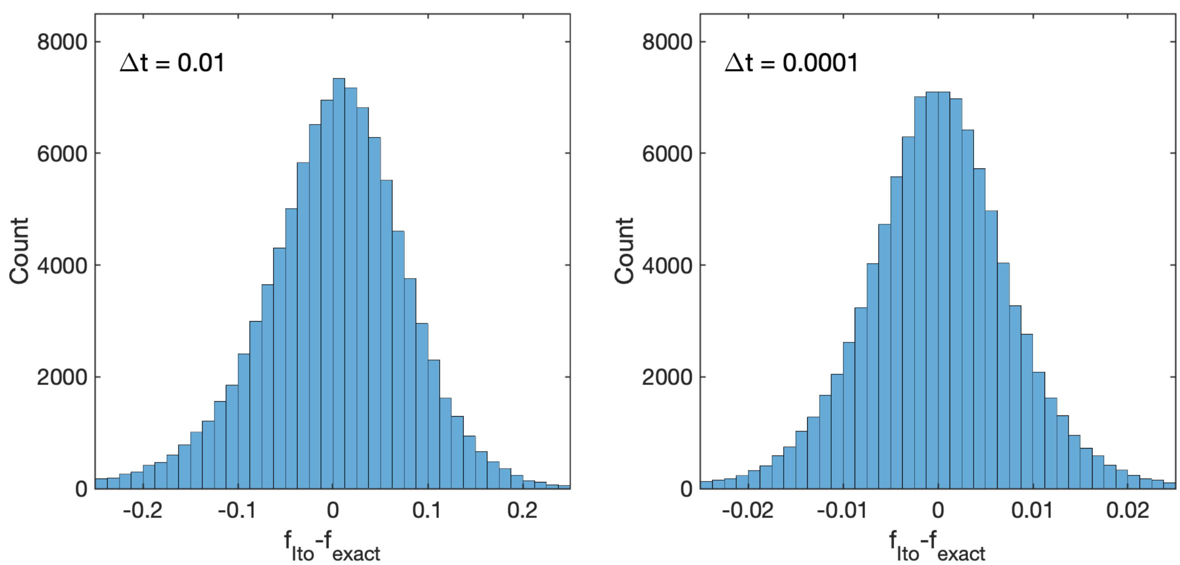
For $f(t,w) = \frac{1}{2}w^2 + \frac{t}{6}w^3$, we compute the 3 versions of approximations. We use $t_f = 1$ and $\Delta t = 0.01$, and we generate 100,000 samples of $(f_{\text{exact}}, f_{\text{Taylor}}, f_{\text{Ito}})$.



f_{Taylor} vs f_{exact}



f_{Ito} vs f_{exact}



Histogram of $(f_{\text{Ito}} - f_{\text{exact}})$