AM216 Stochastic Differential Equations

Lecture 18
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List of topics in this lecture

- The case of time-dependent drift
- Estimating volatility from market prices of options
- Scaling laws of option price: effect of volatility, effect of interest rate
- In-the-money vs out-of-the-money options
- Properties of option price

Review

Black-Scholes option pricing model

Option price function: $C(s, t)$, option price at time t when stock price $S(t) = s$.

Analytical expression of $C(s, t)$

$$C(s,t) = \frac{e^{-r\tau}K}{2}\phi(\eta,\omega), \qquad \eta = \log\frac{s}{K} + r\tau, \quad \omega = \frac{1}{2}\sigma^2\tau$$
$$\phi(\eta,\omega) = e^{\eta}\left[1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right] - \left[1+\mathrm{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right]$$

The basic principle of risk-reward

Expected reward of buying option at price $C(s, t)$

> (Risk-free) nominal value at time T of amount $C(s, t)$ at time t

Effect of interest rate r

$C(s, t)$ increases with r

Effect of volatility σ

$C(s, t)$ increases with σ

End of review

The case of time-dependent drift $\mu(t)$

We study the general case of time-dependent drift $\mu(t)$.

$$dS = \mu(t)Sdt + \sigma SdW$$

Key observation:

We re-visit the process of deriving the PDE for $C(s, t)$.

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At time t, buy $F(S(t), t)$ unit of delta hedging of time t.

At time $(t+dt)$, sell $F(S(t), t)$ unit of delta hedging of time t purchased at time t.

Change in cash balance at time $(t+dt)$ after selling

$$\Delta B = \underbrace{F\big(S(t),t\big)}_{\substack{\text{\# of units of}\\ \text{delta hedge}}}\left[\underbrace{-1}_{\substack{\text{\# of units}\\ \text{of option}}} \times \underbrace{C\big(S(t+dt),t+dt\big)}_{\substack{\text{option price}\\ \text{at time } t+\Delta t}} + \underbrace{C_s\big(S(t),t\big)}_{\substack{\text{\# of shares}\\ \text{of stock}}} \times \underbrace{S(t+dt)}_{\substack{\text{stock price}\\ \text{at time } t+\Delta t}}\right]$$

written in short notation

$$= F(S,t)\big[-C(S+dS,t+dt)+C_s(S,t)(S+dS)\big]$$

Taylor expansion in terms of dS and dt

$$= F(S,t)\left[\begin{array}{l} -C(S,t)-\underbrace{C_s(S,t)dS}_{dS\text{ term}}-C_t(S,t)dt-\dfrac{1}{2}C_{ss}(S,t)(dS)^2 \\ +C_s(S,t)S+\underbrace{C_s(S,t)dS}_{dS\text{ term}}+o(dt) \end{array}\right]$$

By the special design of delta hedging portfolio, the two dS terms cancel each other.

$$dS = \mu(t)Sdt + \sigma SdW$$

==> $\quad (dS)^2 = \big(\mu(t)Sdt + \sigma SdW\big)^2 = \sigma^2 S^2 (dW)^2 + o(dt)$

The effect of $\mu(t)$ is buried in the $o(dt)$ term, which disappears as $dt \to 0$.

...

Conclusion:

$\mu(t)$ does not appear in the expression of $C(s, t)$.

In reality, $\mu(t)$ is unknown and unpredictable.

Clarification:

The conclusion above says that the expression of price function $C(s, t)$ is not affected by the drift term. The stochastic option price $C(S(t), t)$ is highly influenced by the drift term via the underlying stock price $S(t)$.

$C(S(t), t)$: stochastic option price at time t, corresponding to stock price $S(t)$

$C(s, t)$: a function of s mapping stock price to option price at time t

Estimating volatility from market prices of options

Recall that $C(s, t)$ is an increasing function of σ. Based on the observed market stock price $S(t)$ and market option price $C(S(t), t)$, we can estimate the perceived volatility.

Example:

Current time: t = the closing of May 26th, 2020.

Current stock price of Walt Disney Co: $s \equiv S(t) = \$120.95$.

Interest rate: $r = 2\%/\text{year}$; μ is not needed.

Table 1: Market prices of call options expiring June 5th, and the predicted volatility.

Expiry: T = June 5th 2020, τ = 8 trading days.

K, strike price	$C(s, t)$, market price of the option,	Predicted volatility σ^2 [1/year]
\$121	\$2.91	0.1140
\$121	\$2.47	0.1158
\$123	\$1.98	0.1092
\$124	\$1.63	0.1096
\$125	\$1.35	0.1118
\$126	\$1.05	0.1081

$E(\sigma^2) = 0.1114/\text{year}$ ==> $\sqrt{E(\sigma^2)} = 0.334/\sqrt{\text{year}}$

==> volatility = 33.4% fluctuation in a year

Table 2: Market prices of call options expiring June 19th, and the predicted volatility.

Expiry: T = June 19th 2020, τ = 18 trading days.

K, strike price	$C(s, t)$, market price of the option,	Predicted volatility σ^2 [1/year]
\$121	\$4.40	0.1138
\$121	\$3.90	0.1121
\$123	\$3.45	0.1112
\$124	\$3.05	0.1109
\$125	\$2.61	0.1067
\$126	\$2.31	0.1081

$E(\sigma^2) = 0.1105/\text{year}$ ==> $\sqrt{E(\sigma^2)} = 0.332/\sqrt{\text{year}}$

==> volatility = 33.2% fluctuation in a year

Scaling laws

Interest-rate-adjusted strike price at time t

To exercise the option at time T (buying the stock at strike price K), we need to allocate cash $e^{-r\tau}K$ at time t, which will grow to K at time T. Here $\tau = T - t$.

(Price K at time T) <====> (price $e^{-r\tau}K$ at time t)

We define the interest-rate-adjusted-strike price as $K_c \equiv e^{-r\tau}K$ and use K_c to represent the strike price in the mathematical formulation.

We write $C(s, t)$ in terms of K_c.

$$C(s,t) = \frac{K_c}{2}\phi(\eta,\omega), \quad \eta = \log\frac{s}{K_c}, \quad \omega = \frac{1}{2}\sigma^2\tau$$

Remarks:

In this formulation, for an option with a fixed nominal strike price K, its rate-adjusted strike price K_c varies with time.

In this formulation, when K_c and s are fixed, the option price $C(s, t)$ depends only on the combination $\sigma^2\tau = \sigma^2(T-t)$.

Non-dimensional ratios

We normalize the option price and the rate-adjusted strike price by the stock price.

$$q \equiv \frac{K_c}{s} = \frac{e^{-r\tau}K}{s}, \qquad Q \equiv \frac{C(s,t)}{s}$$

We write Q as a function of q, and $\omega = \sigma^2\tau/2$.

$$Q(q,\omega) = \frac{e^{-\eta}}{2}\phi(\eta,\omega), \quad \eta = \log\frac{1}{q}, \quad \omega = \frac{1}{2}\sigma^2\tau$$

Scaling property 1: behavior of $C(S(t), t)/S(t)$

At given t, given $K/S(t)$ and given σ, the option price is proportional to the stock price.

$$q = \frac{e^{-r(T-t)}K}{S(t)} = \text{fixed}, \quad \omega = \frac{1}{2}\sigma^2(T-t) = \text{fixed}$$

$$C(S(t),t) = S(t)\cdot Q(q,\omega) \propto S(t)$$

Example

Stock 1 is at $S_1(t) = 45$ and the call option on stock 1 with $K = 50$ is at $C_1 = 2$.

Suppose stock 2 has the same volatility and is at $S_2(t) = 90$.

The call option on stock 2 with $K = 100$ and the same T should be at $C_2 = 4$.

Scaling property 2: effect of volatility σ

At given $K_c = e^{-r\tau}K$ and given $S(t)$, the option price depends only on the combination $\omega = \sigma^2\tau/2$. Here $\tau = (T-t)$ is the time to expiry.

$$q = \frac{K_c}{S(t)} = \text{fixed}$$

$$C(S(t),t) = S(t)\cdot Q\left(q, \frac{\sigma^2\tau}{2}\right)$$

Suppose stock 1 has volatility σ_1 and stock 2 has volatility $\sigma_2 = 2\sigma_1$. Suppose both stocks are currently at the same price $S_1(t) = S_2(t) = S(t)$.

We consider two options with times to expiry (τ_1, τ_2) and strike prices (K_1, K_2).

The call option on stock 1 with τ_1 and $K_1 = \exp(r\tau_1)K_c$.

The call option on stock 2 with $\tau_2 = \tau_1/4$ and $K_2 = \exp(r\tau_2)K_c = \exp(-3r\tau_2)K_1$.

These two options should have the same price.

Observations:

- When the volatility is doubled, the option will have the same price if
    i) the time to expiry is shorten by a factor of 4 and
    ii) the strike price is slightly decreased.
- When the volatility is halved, the option will have the same price if
    i) the time to expiry is stretched by a factor of 4 and
    ii) the strike price is slightly increased.

Scaling property 3: effect of interest rate r

The effect of interest rate r is completely contained in the rate-adjusted strike price $K_c \equiv e^{-r\tau}K$, and thus, in the ratio $q = e^{-r\tau}K/S(t)$.

$$C(S(t),t) = S(t)\cdot Q(q,\omega)$$

Mathematically, the option price stays the same when both the interest rate r and the nominal strike price K are increased to maintain the same rate-adjusted strike price.

If $K_1 \exp(-r_1\tau) = K_c = K_2 \exp(-r_2\tau)$, then $C(s,t)\Big|_{(r_1,K_1)} = C(s,t)\Big|_{(r_2,K_2)}$.

In-the-money vs out-of-the-money options, short-term vs long-term

Recall that we are discussing call options.

Definitions of in-the-money and out-of-the-money

In the terminology of option trading market, in-the-money and out-of-the-money call options are defined as follows.

In-the-money option:

$K < S(t)$ (strike price < current stock price)

If the option is exercised immediately, there is a gain (not counting the premium already paid for the option)

Out-of-the-money option:

$K > S(t)$ (strike price > current stock price)

If the option is exercised immediately, there is no gain.

Mathematically, we define in-the-money and out-of-the-money slightly differently

In-the-money option (mathematical definition)

$K_c \equiv e^{-r\tau}K < S(t)$ (rate-adjusted strike price < current stock price)

There is a gain if the option holder does the followings

i) sells the stock at time t,

ii) set aside the proceeds (which will grow to $S(t)e^{r\tau}$ at time T), and

iii) at time T, use cash K to exercises the option to buy back the stock,

which yields a cash balance of $S(t)e^{r\tau} - K > 0$.

Out-of-the-money option (mathematical definition)

$K_c \equiv e^{-r\tau}K > s$ (rate-adjusted strike price > current stock price)

There is no gain if the option holder does the steps described above.

Remarks:

- The first set of definitions is motivated in the context of American style options where options can be exercised any time before or at the expiry.
- The second set of definitions is mathematically more relevant.
- The two sets of definitions are essentially the same for short-term.

Short-term in-the-money option:

$$C(s,t)=\frac{K_c}{2}\phi(\eta,\omega)\,,\qquad \eta=\log\frac{s}{K_c}\,,\quad \omega=\frac{1}{2}\sigma^2\tau$$

$$\phi(\eta,\omega)=e^{\eta}\left[1+\text{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right]-\left[1+\text{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right]$$

Mathematically, <u>short-term</u> means ω is small and <u>in-the-money</u> means η > 0.

We use the expansion of the complementary error function erfc(·)

$$\text{erfc}(z)=\frac{\exp(-z^2)}{z\sqrt{\pi}}\left(1-\frac{1}{2z^2}+\cdots\right)\approx 0\qquad \text{for } z>0 \text{ and large}$$

$$\text{erf}(z)=1-\text{erfc}(z)\approx 1-\frac{\exp(-z^2)}{z\sqrt{\pi}}\qquad \text{for } z>0 \text{ and large}$$

$$\text{erf}(-z)=-\text{erf}(z)=\text{erfc}(z)-1\approx\frac{\exp(-z^2)}{z\sqrt{\pi}}-1\qquad \text{for } z>0 \text{ and large}$$

We expand ϕ(η, ω) for small ω and η > 0.

$$\frac{\eta+\omega}{\sqrt{4\omega}}>0 \text{ and large}\,,\qquad \frac{\eta-\omega}{\sqrt{4\omega}}>0 \text{ and large}$$

$$\text{erfc}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\approx\frac{\sqrt{4\omega}}{(\eta+\omega)\sqrt{\pi}}\exp\left(\frac{-(\eta+\omega)^2}{4\omega}\right)$$

$$\text{erfc}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\approx\frac{\sqrt{4\omega}}{(\eta-\omega)\sqrt{\pi}}\exp\left(\frac{-(\eta-\omega)^2}{4\omega}\right)$$

$$=\frac{\sqrt{4\omega}}{(\eta-\omega)\sqrt{\pi}}e^{\eta}\cdot\exp\left(\frac{-(\eta+\omega)^2}{4\omega}\right)$$

$$\phi(\eta,\omega)=e^{\eta}\left[1+\text{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right]-\left[1+\text{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right]$$

$$=e^{\eta}\left[2-\text{erfc}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right]-\left[2-\text{erfc}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right]$$

$$=2e^{\eta}-2+e^{\eta}\left(\frac{1}{(\eta-\omega)}-\frac{1}{(\eta+\omega)}\right)\frac{\sqrt{4\omega}}{\sqrt{\pi}}\exp\left(\frac{-(\eta+\omega)^2}{4\omega}\right)$$

$$= 2e^{\eta} - 2 + e^{\eta} \frac{1}{(\eta^2 - \omega^2)} \frac{4\omega^{3/2}}{\sqrt{\pi}} \exp\left(\frac{-(\eta+\omega)^2}{4\omega} \right)$$

$$\approx 2e^{\eta} - 2 + e^{\eta} \frac{1}{\eta^2} \frac{4\omega^{3/2}}{\sqrt{\pi}} \exp\left(\frac{-\eta^2}{4\omega} \right)$$

We use the expansion of $\phi(\eta, \omega)$ to write out the option price $C(s, t)$

$$C(s,t) = \frac{K_c}{2} \phi(\eta,\omega), \qquad \eta = \log \frac{s}{K_c}, \quad \omega = \frac{1}{2}\sigma^2 \tau$$

$$\approx \frac{K_c}{2} \left[2e^{\eta} - 2 + e^{\eta} \frac{1}{\eta^2} \frac{4\omega^{3/2}}{\sqrt{\pi}} \exp\left(\frac{-\eta^2}{4\omega} \right) \right]$$

$$\approx s - K_c + s \frac{\sigma^3 \tau^{3/2}}{\eta^2 \sqrt{2\pi}} \exp\left(\frac{-\eta^2}{2\sigma^2 \tau} \right) \approx s - K_c$$

Interpretation:

When the current stock price s is higher than the rate-adjusted strike price K_c and quantity $\left| \eta / \sqrt{\omega} \right|$ is moderately large, the option price is essentially $(s - K_c)$. The small amount above $(s - K_c)$ represents the small chance of big reward for the option holder if the stock price goes much higher.

Short-term out-of--the-money option:

Mathematically, short-term means ω is small and out-the-money means $\eta < 0$.

We expand $\phi(\eta, \omega)$ for small ω and $\eta < 0$.

$$\frac{-(\eta+\omega)}{\sqrt{4\omega}} > 0 \text{ and large}, \qquad \frac{-(\eta-\omega)}{\sqrt{4\omega}} > 0 \text{ and large}$$

$$\operatorname{erfc}\left(\frac{-(\eta+\omega)}{\sqrt{4\omega}} \right) \approx \frac{\sqrt{4\omega}}{(-\eta-\omega)\sqrt{\pi}} \exp\left(\frac{-(\eta+\omega)^2}{4\omega} \right)$$

$$\operatorname{erfc}\left(\frac{-(\eta-\omega)}{\sqrt{4\omega}} \right) \approx \frac{\sqrt{4\omega}}{(-\eta+\omega)\sqrt{\pi}} \exp\left(\frac{-(\eta-\omega)^2}{4\omega} \right)$$

$$= \frac{\sqrt{4\omega}}{(-\eta+\omega)\sqrt{\pi}} e^{\eta} \cdot \exp\left(\frac{-(\eta+\omega)^2}{4\omega} \right)$$

$$\phi(\eta,\omega) = e^{\eta}\left[1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right] - \left[1+\mathrm{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right]$$

$$= e^{\eta}\,\mathrm{erfc}\left(\frac{-(\eta+\omega)}{\sqrt{4\omega}}\right) - \mathrm{erfc}\left(\frac{-(\eta-\omega)}{\sqrt{4\omega}}\right)$$

$$= e^{\eta}\left(\frac{1}{(-\eta-\omega)} - \frac{1}{(-\eta+\omega)}\right)\frac{\sqrt{4\omega}}{\sqrt{\pi}}\exp\left(\frac{-(\eta+\omega)^2}{4\omega}\right)$$

$$= e^{\eta}\frac{1}{(\eta^2-\omega^2)}\frac{4\omega^{3/2}}{\sqrt{\pi}}\exp\left(\frac{-(\eta+\omega)^2}{4\omega}\right)$$

$$\approx e^{\eta}\frac{1}{\eta^2}\frac{4\omega^{3/2}}{\sqrt{\pi}}\exp\left(\frac{-\eta^2}{4\omega}\right)$$

We use the expansion of $\phi(\eta, \omega)$ to write out the option price $C(s, t)$

$$C(s,t) = \frac{K_c}{2}\phi(\eta,\omega), \qquad \eta = \log\frac{s}{K_c}, \quad \omega = \frac{1}{2}\sigma^2\tau$$

$$\approx \frac{K_c}{2}\left[e^{\eta}\frac{1}{\eta^2}\frac{4\omega^{3/2}}{\sqrt{\pi}}\exp\left(\frac{-\eta^2}{4\omega}\right)\right]$$

$$\approx s\frac{\sigma^3\tau^{3/2}}{\eta^2\sqrt{2\pi}}\exp\left(\frac{-\eta^2}{2\sigma^2\tau}\right) \approx 0$$

Interpretation:

When the current stock price s is lower than the rate-adjusted strike price K_c and quantity $\left|\eta/\sqrt{\omega}\right|$ is moderately large, the option price is essentially 0. The small amount above 0 represents the small chance of reward for the option holder if the stock price goes above the strike price.

Long-term options:

Mathematically, long-term means $\sigma^2\tau \gg 1$, which almost does not exist in the real world of option trading market because $\sigma^2\tau \gg 1$ corresponds to a really long time.

Example:

$\sigma^2 = 0.25$/year (50% fluctuation in a year, a significant fluctuation)

$\sigma^2\tau \geq 20$ requires $\tau \geq 80$ years

Even at $\sigma^2\tau = 20$, we have $\dfrac{\sigma^2\tau/2}{\sqrt{2\sigma^2\tau}} = \dfrac{10}{\sqrt{40}} \approx 1.58$, not a large number.

In the real world of option trading market, "long-term" refers to the intermediate-term $\sigma^2\tau \sim O(1)$ in the mathematical sense, which does not have a simple asymptotic expression.

Properties of $C(s, t)$

Price of option shall never be negative

$C(S(t), t) < 0$ is absolutely impossible.

Otherwise, we can make a risk-free gain by "buying" the option and doing nothing.

It makes no sense to exercise before expiry T

Exercising at time t requires cash K at time t.

Exercising at time T requires cash $e^{-r\tau}K < K$ at time t.

American style options and European style options are essentially the same (for stocks that do not pay a dividend).

Price of option shall never be below $S(t)-e^{-r\tau}K$

$C(S(t), t) < S(t)-e^{-r\tau}K$ is absolutely impossible.

If $C(S(t), t) < S(t)-e^{-r\tau}K$, we can make a risk-free gain by doing the steps below.

- Sell the stock at $S(t)$ at time t.
- Buy the option at $C(S(t), t)$ at time t.
- The two actions above yield a cash balance of $S(t)- C(S(t), t)$, which will grow to $e^{r\tau}(S(t)-C(S(t), t)) > K$ at time T.
- At time T, use cash K to exercise the option to close the position.
- We end with a positive cash balance: $e^{r\tau}(S(t)-C(S(t), t)) - K > 0$.

(The analytical proof of $C(s, t) > s - e^{-r\tau}K$ is not straightforward.)

The option price shall never exceed the stock price

Recall that $C(s, t)$ has the expression

$$C(s,t) = s\frac{e^{-\eta}}{2}\phi(\eta,\omega), \qquad \eta = \log\frac{s}{K_c}, \quad \omega = \frac{1}{2}\sigma^2\tau$$

The coefficient satisfies

$$\frac{e^{-\eta}}{2}\phi(\eta,\omega)=\frac{1}{2}\left[1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right]-\frac{e^{-\eta}}{2}\left[1+\mathrm{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right]$$

$$<\frac{1}{2}\left[1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right]<1$$

which implies $C(s, t) < s$.

Percentage-wise increase in the option price

We revisit the derivative $C(s, t)$ with respect to s.

$$\frac{\partial}{\partial s}C(s,t)=\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right)>\frac{e^{-\eta}}{2}\phi(\eta,\omega)=\frac{C(s,t)}{s}$$

$$\Longrightarrow \quad \frac{C(s+\Delta s,t)-C(s,t)}{\Delta s}>\frac{C(s,t)}{s}$$

$$\Longrightarrow \quad \left|\frac{C(s+\Delta s,t)-C(s,t)}{C(s,t)}\right| > \left|\frac{\Delta s}{s}\right|$$

Percentage-wise, the option price is more volatile than the stock price.

Change in the option price vs change in the strike price

We calculate $\partial C/\partial K$.

$$C(s,t)=s\frac{e^{-\eta}}{2}\phi(\eta,\omega), \qquad \eta=\log\frac{s}{K}+r\tau, \quad \omega=\frac{1}{2}\sigma^2\tau$$

$$\Longrightarrow \quad \frac{\partial}{\partial K}C(s,t)=\frac{s}{2}\frac{\partial}{\partial\eta}\left(e^{-\eta}\phi(\eta,\omega)\right)\frac{\partial\eta}{\partial K}=\frac{-s}{2K}\frac{\partial}{\partial\eta}\left(e^{-\eta}\phi(\eta,\omega)\right)$$

Recall that previously we derived

$$\frac{\partial}{\partial\eta}\left(e^{-\eta}\phi(\eta,\omega)\right)=e^{-\eta}\left(1+\mathrm{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right) < 2e^{-\eta}$$

$$\Longrightarrow \quad 0 < -\frac{\partial C}{\partial K} < \frac{s}{K}e^{-\eta}=e^{-r\tau}$$

$$\Longrightarrow \quad -\frac{\left(C\big|_{K+\Delta K}-C\big|_{K}\right)}{(\Delta K)}<e^{-r\tau}, \qquad \Delta K>0$$

==> $\underbrace{C\Big|_K - C\Big|_{K+\Delta K}}_{\text{drop in option price}} < e^{-r\tau} \underbrace{\Delta K}_{\text{increase in strike price}}$

The drop in the option price is less than the increase in the strike price.

The effect of strike price decreases as the time to expiry increases.

Price of the at-the-money option

$$C(s,t) = K_c \frac{1}{2}\phi(\eta,\omega), \qquad \eta = \log\frac{s}{K_c}, \quad \omega = \frac{1}{2}\sigma^2\tau$$

At-the-money means $S(t) = K_c$ and thus $\eta = 0$.

$$\phi(\eta,\omega)\Big|_{\eta=0} = \left[1+\mathrm{erf}\left(\frac{\sqrt{\omega}}{2}\right)\right] - \left[1-\mathrm{erf}\left(\frac{\sqrt{\omega}}{2}\right)\right] = 2\mathrm{erf}\left(\frac{\sqrt{\omega}}{2}\right)$$

When the stock price is $S(t) = K_c \equiv e^{-r\tau}K$, the option price is

$$C(s,t) = K_c \cdot \frac{1}{2}\phi(\eta,\omega)\Big|_{\eta=0} = e^{-r\tau}K\cdot\mathrm{erf}\left(\frac{\sqrt{\sigma^2\tau}}{2\sqrt{2}}\right), \qquad \text{when } s = e^{-r\tau}K$$

Price of a near-the-money option

$$C(s,t) = K_c \frac{1}{2}\phi(\eta,\omega), \qquad \eta = \log\frac{s}{K_c}, \quad \omega = \frac{1}{2}\sigma^2\tau$$

Near-the-money means $S(t) \approx K_c$ and thus $\eta \approx 0$.

$$\eta \approx (e^{\eta} - 1) = \frac{s}{K_c} - 1$$

We expand $C(s, t)$ around $\eta = 0$.

$$\phi(\eta,\omega)\Big|_{\eta=0} = \left[e^{\eta}\left(1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right) - \left(1+\mathrm{erf}\left(\frac{\eta-\omega}{\sqrt{4\omega}}\right)\right)\right]\Bigg|_{\eta=0} = 2\mathrm{erf}\left(\frac{\sqrt{\omega}}{2}\right)$$

$$\frac{\partial}{\partial\eta}\phi(\eta,\omega)\Big|_{\eta=0} = e^{\eta}\left(1+\mathrm{erf}\left(\frac{\eta+\omega}{\sqrt{4\omega}}\right)\right)\Bigg|_{\eta=0} = 1+\mathrm{erf}\left(\frac{\sqrt{\omega}}{2}\right)$$

$$\phi(\eta,\omega) \approx \phi(\eta,\omega)\Big|_{\eta=0} + \eta\frac{\partial}{\partial\eta}\phi(\eta,\omega)\Big|_{\eta=0}$$

We use the expansion of $\phi(\eta, \omega)$ to approximate $C(s, t)$.

$$
\begin{aligned}
C(s,t) &\approx \frac{K_c}{2}\left[\phi(\eta,\omega)\Big|_{\eta=0} + \eta \frac{\partial}{\partial \eta}\phi(\eta,\omega)\Big|_{\eta=0} \right] \\
&\approx \frac{K_c}{2}\left[2\mathrm{erf}\left(\frac{\sqrt{\omega}}{2} \right) + \left(\frac{s}{K_c} - 1 \right)\left(1 + \mathrm{erf}\left(\frac{\sqrt{\omega}}{2} \right) \right) \right] \\
&\approx K_c \mathrm{erf}\left(\frac{\sqrt{\omega}}{2} \right) + (s - K_c)\frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{\sqrt{\omega}}{2} \right) \right) \\
&\approx \mathrm{e}^{-r\tau} K \cdot \mathrm{erf}\left(\frac{\sqrt{\sigma^2 \tau}}{2\sqrt{2}} \right) + (s - \mathrm{e}^{-r\tau} K)\frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{\sqrt{\sigma^2 \tau}}{2\sqrt{2}} \right) \right), \qquad \text{for } s \approx \mathrm{e}^{-r\tau} K
\end{aligned}
$$