AM216 Homework #4

Problem 1:

Show that

$$F\left[\exp(-\beta|t|)\right]=\frac{2\beta}{\beta^2+(2\pi\xi)^2}$$

<u>Hint:</u>

Break the integral into $\int_{-\infty}^{\infty}(\,)dt=\int_{0}^{\infty}(\,)dt+\int_{-\infty}^{0}(\,)dt$. Then calculate each piece directly

$$\int_0^{+\infty}\exp\left((-\beta+i\lambda)t\right)dt=\left.\frac{\exp\left((-\beta+i\lambda)t\right)}{(-\beta+i\lambda)}\right|_0^{\infty}=\frac{-1}{(-\beta+i\lambda)}$$

Problem 2:

Show that

$$E\left(\left[W(t+h)-W(t)\right]\left[W(s+h)-W(s)\right]\right)=\begin{cases}0, & |t-s|>h\\ h-|t-s|, & |t-s|\le h\end{cases}$$

<u>Hint:</u>

Discuss separately the case of $(t-s) > h$ and the case of $0 \le (t-s) \le h$.

Use the symmetry of t and s to write out the results for $(t-s) < -h$ and $-h \le (t-s) \le 0$.

Problem 3:

<u>Background:</u>

Recall the convergence of a regular sequence (a sequence of deterministic numbers).

Sequence $\{q_n\}$ converges to 0 if for any $\varepsilon > 0$,

$\left|q_n\right|>\varepsilon$ is impossible when n is large enough.

For a sequence of random variables, we have the <u>convergence in probability</u>.

<u>Definition</u>

We say sequence $\{Q_N(\omega)\}$ converges to 0 in probability if for any $\varepsilon > 0$,

$$\lim_{N\to\infty}\Pr\left(\left|Q_N(\omega)\right|>\varepsilon\right)=0$$

<u>**Your task:**</u>

Suppose $\lim_{N\to\infty} E\left(Q_N(\omega)\right)=0$ and $\lim_{N\to\infty} \mathrm{var}\left(Q_N(\omega)\right)=0$

Use Chebyshev-Markov inequality to show that $\{Q_N(\omega)\}$ converges to 0 in probability.

Hint:

$\lim_{N\to\infty} E\left(Q_N(\omega)\right)=0$ implies that for any $\varepsilon > 0$, when N is large enough, we have

$$| E(Q_N(\omega)) | < \varepsilon/2$$

$$\Longrightarrow \quad \left|Q_N(\omega)\right| < \left|Q_N(\omega)-E\left(Q_N(\omega)\right)\right| + \frac{\varepsilon}{2}$$

$$\Longrightarrow \quad \Pr\left(\left|Q_N(\omega)\right|>\varepsilon\right) \le \Pr\left(\left|Q_N(\omega)-E\left(Q_N(\omega)\right)\right|>\frac{\varepsilon}{2}\right)$$

Then apply Chebyshev-Markov inequality as we did in lecture.

Problem 4:

Let $f(w)$ be a smooth (infinitely differentiable) function of w.

Consider random variable Q_N defined as

$$Q_N \equiv \sum_{j=0}^{N-1}\left(\frac{1}{2}f(W(s_j))+\frac{1}{2}f(W(s_{j+1}))-f(W(s_{j+1/2}))\right)\Delta W_j$$

where

$$\Delta s=\frac{t}{N}, \quad s_j = j\Delta s, \quad s_{j+1/2}=\frac{s_j+s_{j+1}}{2}, \quad \Delta W_j = W(s_{j+1})-W(s_j)$$

Expand the summand around $W(s_j)$ and neglect $o(\Delta s)$ terms. Treat random variable ΔW as $(\Delta W)^2 = O(\Delta s)$ when deciding if a term is $o(\Delta s)$.

Use the expansion results to show that

$$\lim_{N\to\infty} E\left(Q_N\right)=0$$

Hint:

We introduce short notations:

$$W_j = W(s_j), \qquad W_{j+1} = W(s_{j+1}), \qquad W_{j+1/2} = W(s_{j+1/2})$$
$$\Delta W_j = W_{j+1}-W_j, \qquad \Delta W_j^+ = W_{j+1}-W_{j+1/2}, \qquad \Delta W_j^- = W_{j+1/2}-W_j$$

$$f_j = f(W_j), \qquad f'_j = f'(W_j)$$

Carrying out expansions around $W(s_j)$, we have

$$f(W(s_{j+1}))\Delta W_j = f(W_j+\Delta W_j)\Delta W_j = \left(f_j(W_j)+f_j'\Delta W_j + O(\Delta s)\right)\Delta W_j$$

$$f(W(s_{j+1/2}))\Delta W_j = f(W_j+\Delta W_j^-)\Delta W_j = \left(f_j+f_j'\Delta W_j^- + O(\Delta s)\right)\Delta W_j$$

==> $$\left(f(W(s_{j+1}))+f(W(s_j))-2f(W(s_{j+1/2}))\right)\Delta W_j$$

$$=\left(f_j'\Delta W_j - 2f_j'\Delta W_j^- + O(\Delta s)\right)\Delta W_j = f_j'\left((\Delta W_j^+)^2-(\Delta W_j^-)^2\right)+o(\Delta s)$$

==> $$E\left(Q_N\right)=\frac{1}{2}\sum_{j=0}^{N-1}E\left(f_j'\left((\Delta W_j^+)^2-(\Delta W_j^-)^2\right)\right)+o(1)$$

Use the properties of $W(t)$

- ΔW_j^- and ΔW_j^+ are independent of W_j.
- $E\left((\Delta W_j^-)^2\right)=\Delta s$ and $E\left((\Delta W_j^+)^2\right)=\Delta s$ …

Problem 5:

Continue with the random variable Q_N defined in Problem 4. Show that

$$\lim_{N\to\infty}\mathrm{var}\left(Q_N\right)=0$$

Hint:

Based on results from Problem 4, we have $\mathrm{var}\left(Q_N\right)=E\left({Q_N}^2\right)$. We write

$$E\left({Q_N}^2\right)=\frac{1}{4}\sum_{j=0,\,k=0}^{N-1}E\left(f_j'\left((\Delta W_j^+)^2-(\Delta W_j^-)^2\right)f_k'\left((\Delta W_k^+)^2-(\Delta W_k^-)^2\right)\right)+o(1)$$

Use the properties of $W(t)$

- For $j>k$, ΔW_j^- and ΔW_j^+ are independent of $W_j, W_k, \Delta W_k^-$ and ΔW_k^+.

==> $$E\left(f_j'\left((\Delta W_j^+)^2-(\Delta W_j^-)^2\right)f_k'\left((\Delta W_k^+)^2-(\Delta W_k^-)^2\right)\right)$$

$$=E\left((\Delta W_j^+)^2-(\Delta W_j^-)^2\right)E\left(f_j'f_k'\left((\Delta W_k^+)^2-(\Delta W_k^-)^2\right)\right)\qquad \text{for } j>k$$

…

==> $$E\left({Q_N}^2\right)=\frac{1}{4}\sum_{k=0}^{N-1}E\left((f_k')^2\right)E\left(\left((\Delta W_k^+)^2-(\Delta W_k^-)^2\right)^2\right)+o(1)$$

Use $(\Delta W_k^+)^2=O(\Delta s)$, $(\Delta W_k^-)^2=O(\Delta s)$ …