AM216 Stochastic Differential Equations

Lecture 07

List of topics in this lecture

- Wiener process is continuous in probability
- Ornstein-Uhlenbeck Process (OU), Stokes' law, thermal excitations
- Solution of particle velocity, colored noise, convergence to a white noise
- Fluctuation-dissipation theorem, Maxwell-Boltzmann distribution
- Solution of particle position

Recap

dW/dt is a white noise (stationary stochastic process, $R(t)$, PSD)

Constrained Wiener process (Bayes theorem)

$$\rho\Big(W(a+h)=x \,\Big|\, W(a)=y_a \text{ and } W(a+2h)=y_b\Big) \sim N\left(\frac{y_a+y_b}{2},\ \frac{h}{2}\right)$$

Wiener process is continuous in probability

Recall the continuity of a regular function $f(t)$.

Function $f(t)$ is continuous at t if for any $\varepsilon > 0$,

$\big|f(t+h)-f(t)\big| \geq \varepsilon$ is impossible when h is small enough.

Theorem: (Continuity of $W(t)$ in probability)

Intuitively, for any $\varepsilon > 0$,

$\big|W(t+h)-W(t)\big| \geq \varepsilon$ is almost impossible when h is small enough.

More precisely, for any $\varepsilon>0$, we have

$$\lim_{h\to 0} \Pr\Big(\big|W(t+h)-W(t)\big| \geq \varepsilon\Big) = 0$$

Note: This property/condition is the definition of continuity in probability.

Proof of the theorem:

To prove the theorem, we need

Chebyshev-Markov inequality:

For a random variable X, we write $E(|X|^\alpha)$ as

$$E\left(|X|^\alpha\right) = \int |x|^\alpha \rho(x)dx \ge \int_{|x|\ge\varepsilon} |x|^\alpha \rho(x)dx \ge$$

$$\ge \varepsilon^\alpha \int_{|x|\ge\varepsilon} \rho(x)dx = \varepsilon^\alpha \Pr\left(|X| \ge \varepsilon\right)$$

==> $$\boxed{\Pr\left(|X| \ge \varepsilon\right) \le \frac{1}{\varepsilon^\alpha} E\left(|X|^\alpha\right)}$$

This is called the Chebyshev-Markov inequality.

Applying the Chebyshev-Markov inequality to $X = W(t+h)-W(t)$ with $\alpha = 2$, we have

$$\Pr\left(|W(t+h)-W(t)| \ge \varepsilon\right) \le \frac{E\left(|W(t+h)-W(t)|^2\right)}{\varepsilon^2}$$

$$= \frac{h}{\varepsilon^2} \to 0 \quad \text{as } h \to 0$$

Thus, the Wiener process $W(t)$ is continuous in probability.

Ornstein-Uhlenbeck Process

Consider the stochastic motion of a small particle in water (as Robert Brown observed the motion of pollen particles in water under a microscope)

For simplicity, we discuss the one-dimensional motion.

In the 3-dimensional motion, each dimension is described by this model.

Let

X = position of the particle

Y = velocity of the particle

m = mass of the particle

Newton's second law (governing the motion)

$$m\frac{dY}{dt} = \text{viscous drag} + \text{Brownian force}$$

Stokes law (for the viscous drag)

viscous drag = $-b\,Y$

where b is the drag coefficient. For a spherical particle, the drag coefficient is

$b = 6\pi\,\eta\, a$

a = radius of the particle

η = viscosity of the fluid media

A short digression: Pollution particles suspended in air

When a solid ball is dropped in mid-air, it first accelerates, driven by the gravity. Then it reaches a constant velocity (called the terminal velocity or settling velocity) when the drag force balances the gravitational force.

The settling velocity satisfies

$$\underbrace{(6\pi\eta a)V_{\text{settling}}}_{\text{Drag force}} = \underbrace{\left(\frac{4}{3}\pi a^3 \rho_{\text{mass}}\right) g}_{\text{Gravity}}$$

$$\Longrightarrow \quad V_{\text{settling}} = \left(\frac{2\rho_{\text{mass}}\, g}{9\eta}\right) a^2 \propto a^2$$

where the air viscosity is $\eta = 1.8\times10^{-4}\ \text{g(cm)}^{-1}\text{s}^{-1}$.

Consider budesonide, a drug used in treating asthma. It has $\rho_{\text{mass}} = 1.26\ \text{g/cm}^3$.

For a ball of 0.1 mm in diameter

$a = 50\ \mu\text{m} \quad \Longrightarrow \quad V_{\text{settling}} = 38\ \text{cm/s}$

For a ball of 10 μm in diameter (PM_{10} particles)

$a = 5\ \mu\text{m} \quad \Longrightarrow \quad V_{\text{settling}} = 0.38\ \text{cm/s}$

For a ball of 2.5 μm in diameter ($PM_{2.5}$ particles)

$a = 1.25\ \mu\text{m} \quad \Longrightarrow \quad V_{\text{settling}} = 0.024\ \text{cm/s}$

With this tiny settling velocity, it takes more than 1 hour for a 2.5 μm particle to descend 1 meter with respect to the surrounding air.

Remark:

Small pollution particles are more dangerous for two reasons:

- They stay in air much longer (virtually forever)
- They can pass the filtration system of human body to enter the lung and the circulatory system.

End of digression

Thermal excitations (Brownian force)

We model the Brownian force as a white noise.

$$\text{Brownian force} = q\frac{dW}{dt}$$

where the coefficient q is to be determined later.

The governing equation

$$mdY = \underbrace{-bY\,dt}_{\text{dissipation}} + \underbrace{q\,dW}_{\text{fluctuation}}$$

$$dX = Y\,dt$$

Remark:

Both the viscous drag and the Brownian force on the particle are results from the particle colliding with surrounding fluid molecules: the viscous drag is the mean and the Brownian force is the fluctuation. As a result, the fluctuation coefficient (q) and the dissipation coefficient (b) are related by the fluctuation-dissipation theorem.

Four goals of the discussion

1) Solve for $Y(t)$
2) Show that

   A) $Y(t)$ is a colored noise and

   B) $Y(t)$ converges to a white noise as m converges to zero (when the effect of inertia is negligible)
3) Relate q to b (fluctuation-dissipation theorem)
4) Study the behavior of $X(t)$

Goal #1: We solve for $Y(t)$.

For mathematical convenience, we divide the equation by m

$$mdY = -bY\,dt + q\,dW$$

==> $$dY = -\beta Y\,dt + \gamma\,dW\,, \qquad \beta = \frac{b}{m}, \quad \gamma = \frac{q}{m}$$

We use the integrating factor method. Multiply by $e^{\beta t}$

$$e^{\beta t}dY + \beta e^{\beta t}Y\,dt = \gamma e^{\beta t}dW$$

==> $$d\left(e^{\beta t}Y(t)\right) = \gamma e^{\beta t}dW$$

==> $$e^{\beta t}Y(t)-Y(0)=\int_0^t \gamma e^{\beta s}\,dW(s)\equiv G(t)$$

where the integral $G(t)$ is defined as the limit of Riemann sum.

$$\Delta s=\frac{t}{N},\qquad s_j=j\Delta s,\qquad dW_j=W(s_{j+1})-W(s_j)$$

$$G(t)=\lim_{N\to\infty}\sum_{j=0}^{N-1}\gamma e^{\beta s_j}dW_j$$

Recall that the sum of independent Gaussians (normal RVs) is a Gaussian.

$\{dW_j,\ \ j = 0, 1, \ldots, N-1\}$ are independent Gaussians.

==> $G(t)$ is a Gaussian.

The mean and variance of $G(t)$ are

$$E\Big(G(t)\Big)=\lim_{N\to\infty}\sum_j \gamma e^{\beta s_j}E\Big(dW_j\Big)=0$$

$$\mathrm{var}\Big(G(t)\Big)=\lim_{N\to\infty}\sum_j(\gamma e^{\beta s_j})^2\,\mathrm{var}(dW_j)=\lim_{N\to\infty}\sum_j(\gamma e^{\beta s_j})^2\Delta s$$

$$=\int_0^t(\gamma e^{\beta s})^2\,ds=\gamma^2\int_0^t e^{2\beta s}\,ds=\frac{\gamma^2}{2\beta}(e^{2\beta t}-1)$$

<u>Caution:</u>

ds in integral comes from Δs in Riemann sum, which comes from var(dW).

<u>This works well for $t > 0$</u> in $\int_0^t \gamma e^{\beta s}\,dW(s)$.

For $t < 0$, increments $\{dW_j,\ \ j = 0, 1, \ldots, N-1\}$ are backwards in time and are no longer independent. The situation with dW is different from that of a regular ODE.

We will discuss the case of $t < 0$ later. For the time being, we work with $t > 0$.

<u>Summary:</u>

$$G(t)\equiv\int_0^t\gamma e^{\beta s}\,dW(s)\ \sim\ N\left(0,\ \frac{\gamma^2}{2\beta}(e^{2\beta t}-1)\right)\qquad \text{for } t>0$$

In the above, we just derived a theorem.

<u>Theorem:</u>

$$\int_0^L f(t)dW(t)\ \sim\ N\left(0,\ \int_0^L f(t)^2\,dt\right)$$

Now we write out $Y(t)$

$$e^{\beta t}Y(t)-Y(0)=G(t)$$

$$\Longrightarrow \quad Y(t)=e^{-\beta t}Y(0)+e^{-\beta t}G(t) \qquad \text{for } t>0$$

When $Y(0)$ is fixed, $Y(t)$ is a Gaussian with mean and variance given by

$$E\big(Y(t)\big|Y(0)\big)=e^{-\beta t}Y(0) \qquad \text{for } t>0$$

$$\text{var}\big(Y(t)\big|Y(0)\big)=e^{-2\beta t}\,\text{var}\big(G(t)\big)=\frac{\gamma^2}{2\beta}(1-e^{-2\beta t}) \qquad \text{for } t>0$$

<u>Summary:</u>

$$\big(Y(t)\big|Y(0)=y_0\big) \sim N\left(e^{-\beta t}y_0,\ \frac{\gamma^2}{2\beta}(1-e^{-2\beta t})\right) \qquad \text{for } t>0$$

As t increases, $Y(t)$ relaxes to the equilibrium.

$$E\big(Y(t)\big)=e^{-\beta t}Y(0)\to 0 \qquad \text{as } t\to+\infty$$

$$\text{var}\big(Y(t)\big)=\frac{\gamma^2}{2\beta}(1-e^{-2\beta t})\to\frac{\gamma^2}{2\beta} \qquad \text{as } t\to+\infty$$

For large t, $Y(t)$ reaches an equilibrium Gaussian distribution.

$$Y(\text{large } t) \sim N\left(0,\ \frac{\gamma^2}{2\beta}\right)$$

<u>Goal #2</u>: we show that

A) $Y(t)$ is a colored noise and

B) $Y(t)$ converges to a white noise as m converges to zero.

We assume that the equilibrium has been reached long time ago and $Y(t)$ is already a stationary process. Under this assumption, $Y(t)$ has the equilibrium distribution.

$$Y(t) \sim N\left(0,\ \frac{\gamma^2}{2\beta}\right) \qquad \text{for all } t$$

<u>Goal #2A:</u> We show that $Y(t)$ is a colored noise.

We calculate the autocorrelation function.

$$R(t)\equiv E\big(Y(t)Y(0)\big)$$

We use <u>the law of total expectation</u>.

$$E(Z_1)=E\Big(E(Z_1\big|Z_2)\Big)$$

We select $Z_1 = Y(t)\,Y(0)$ and $Z_2 = Y(0)$. We consider the case of $t > 0$.

$$E\Big(Y(t)Y(0)\Big)=E\Big(E\Big(Y(t)Y(0)\Big|Y(0)\Big)\Big)=E\Big(Y(0)\cdot E\Big(Y(t)\Big|Y(0)\Big)\Big)$$

using $E\Big(Y(t)\Big|Y(0)\Big)=e^{-\beta t}Y(0)$ for $t > 0$ and $Y(t)\sim N\left(0,\ \dfrac{\gamma^2}{2\beta}\right)$ for all t

$$=E\Big(Y(0)\cdot e^{-\beta t}Y(0)\Big)=e^{-\beta t}E\Big(Y(0)^2\Big)=\frac{\gamma^2}{2\beta}e^{-\beta t}\qquad \text{for } t>0$$

==> $\quad R(t)=\dfrac{\gamma^2}{2\beta}e^{-\beta t}\qquad$ for $t>0$

From the definition of auto-correlation function, $R(t)$ is an even function of t:

$$R(-t)\equiv E\Big(Y(s-t)Y(s)\Big)\qquad \text{for all } s$$
$$=E\Big(Y(0)Y(t)\Big)=R\big(t\big)$$

Therefore, we obtain

$$R(t)=\frac{\gamma^2}{2\beta}\exp(-\beta|t|)\qquad \text{for } t\in(-\infty,\ +\infty)$$

The corresponding power spectrum density is

$$s(\xi)=\frac{\gamma^2}{2\beta}F\Big[\exp(-\beta|t|)\Big]=\frac{\gamma^2}{2\beta}\cdot\frac{2\beta}{\beta^2+4\pi^2\xi^2}=\frac{\gamma^2}{\beta^2+4\pi^2\xi^2}$$

(Homework problem)

<u>In conclusion</u>, $Y(t)$ is a colored noise.

<u>Goal #2B:</u> We show that $Y(t)$ converges to a white noise as $m \to 0$ (when the effect of inertia is negligible).

<u>A simplified story:</u> $m \to 0$ (while b and q stay unchanged)

Recall that $\beta=\dfrac{b}{m}$, $\gamma=\dfrac{q}{m}$.

$$R(t)=\frac{\gamma^2}{2\beta}\exp(-\beta|t|)=\frac{q^2}{m^2}\cdot\frac{m}{2b}\exp\left(-\frac{b}{m}|t|\right)=\frac{q^2}{b^2}\cdot\underbrace{\frac{b}{m}}_{1/h}\cdot\underbrace{\frac{1}{2}\exp\left(-\frac{b}{m}|t|\right)}_{f()}$$

$$=\frac{q^2}{b^2}\cdot\frac{1}{h}f\left(\frac{t}{h}\right),\qquad h\equiv\frac{m}{b},\qquad f(u)\equiv\frac{1}{2}\exp\left(-|u|\right)$$

$f(u)$ given above is a density function. For a density function, we have

$$\lim_{h\to 0}\frac{1}{h}f\left(\frac{t}{h}\right)=\delta(t)$$

As $m\to 0$, we have $h\equiv m/b\to 0$ and

$$R(t)=\frac{q^2}{b^2}\cdot\frac{1}{h}f\left(\frac{t}{h}\right)\ \to\ \frac{q^2}{b^2}\cdot\delta\left(t\right)$$

Therefore, $\lim\limits_{m\to 0}Y(t)$ is a white noise.

<u>The real story:</u>

Mathematically, the limit above is rigorous.

In physics, the situation is a bit complicated. Coefficients m, b and q are all related. We cannot make $m\to 0$ without changing b and q.

Consider a spherical particle. The mass of the particle is

$$m=\frac{4\pi}{3}\rho_{\text{mass}}a^3$$

where ρ_{mass} is the mass density and a the radius of particle.

In physics, $m\to 0$ is achieved by $a\to 0$.

We need to consider the effect of radius a on coefficients b and q.

$$m=\frac{4\pi}{3}\rho_{\text{mass}}a^3=O(a^3)\to 0$$

$$b=6\pi\eta a=O(a)\to 0$$

$$h\equiv\frac{m}{b}=O(a^2)\to 0$$

$$q=\sqrt{2k_BTb}=O(\sqrt{a})\to 0$$ (we will derive this shortly)

$$\frac{q^2}{b^2}=O(a^{-1})\to\infty$$

$$a\frac{q^2}{b^2}=O(1)$$ independent of a.

Consider $\sqrt{a}\,Y(t)$. We have

$$R_{\sqrt{a}Y}(t) = aR_Y(t) = \left(a\frac{q^2}{b^2}\right)\cdot\frac{1}{h}f\left(\frac{t}{h}\right) \to \left(a\frac{q^2}{b^2}\right)\cdot\delta(t) \qquad \text{as } a \to 0$$

==> $\sqrt{a}\,Y(t)$ is a white noise.

In physics, as radius $a \to 0$, $Y(t)$ converges to a white noise of magnitude $\dfrac{1}{\sqrt{a}}$.

Goal #3: We relate fluctuation coefficient q to drag coefficient b

To connect b and q, we need the Maxwell-Boltzmann distribution

Maxwell-Boltzmann distribution

$$\rho(Y = y) \propto \exp\left(\frac{-\text{Energy}(Y = y)}{k_B T}\right)$$

where

$\rho(y)$ is the equilibrium distribution of velocity Y,

k_B is the Boltzmann constant and

T is the absolute temperature.

Maxwell-Boltzmann distribution is a universal law applicable to all thermodynamic systems. In our system, y = velocity and

$$\text{Energy}(Y = y) = \frac{1}{2}m\,y^2$$

The Maxwell-Boltzmann distribution gives us

$$\rho(Y = y) \propto \exp\left(\frac{-\text{Energy}}{k_B T}\right) = \exp\left(\frac{-\frac{1}{2}my^2}{k_B T}\right)$$

writing it into the form of a Gaussian $\exp\left(\dfrac{-y^2}{2\sigma^2}\right)$

$$= \exp\left(\frac{-y^2}{2\left(\frac{k_B T}{m}\right)}\right) \sim N\left(0,\ \frac{k_B T}{m}\right)$$

We have two equilibrium distributions:

- The equilibrium dictated by the Maxwell-Boltzmann distribution is

$$Y(t) \sim N\left(0, \frac{k_B T}{m}\right)$$

- The equilibrium derived from the Ornstein-Uhlenbeck Process is

$$Y(t) \sim N\left(0, \frac{\gamma^2}{2\beta}\right) \qquad \text{for all } t$$

Comparing these two equilibrium distributions, we obtain

$$\frac{\gamma^2}{2\beta} = \frac{k_B T}{m}$$

Recall that $\beta = \frac{b}{m}$, $\gamma = \frac{q}{m}$.

==> $\frac{q^2}{m^2} \cdot \frac{m}{2b} = \frac{k_B T}{m}$

==> $q^2 = 2 k_B T b$

Therefore, we conclude

$$\boxed{q = \sqrt{2 k_B T b}}$$

This is called the fluctuation dissipation relation (theorem).

With the fluctuation dissipation relation, the governing equation becomes.

$$m dY = \underbrace{-bY dt}_{\text{dissipation}} + \underbrace{\sqrt{2 k_B T b}\, dW}_{\text{fluctuation}}$$

Remark:

Now all coefficients in the governing equation are determined.

Goal #4: we study the behavior of $X(t)$.

First, we solve for $X(t)$.

$$Y(t) = e^{-\beta t} Y(0) + e^{-\beta t} G(t) \qquad \text{for } t > 0, \qquad G(t) \equiv \int_0^t \gamma e^{\beta s}\, dW(s)$$

$$X(t) - X(0) = \int_0^t Y(\tau) d\tau = \int_0^t \left(e^{-\beta\tau} Y(0) + e^{-\beta\tau} G(\tau)\right) d\tau$$

$$= \int_0^t \left(e^{-\beta\tau} Y(0) + e^{-\beta\tau} \int_0^\tau \gamma e^{\beta s}\, dW(s) \right) d\tau$$

$$= \frac{1}{\beta}(1 - e^{-\beta t}) Y(0) + \gamma \int_0^t \int_0^\tau e^{-\beta\tau} e^{\beta s}\, dW(s) d\tau$$

Change the order of integration

$$= \frac{1}{\beta}(1 - e^{-\beta t}) Y(0) + \gamma \int_0^t \left(\int_s^t e^{-\beta\tau}\, d\tau \right) e^{\beta s}\, dW(s)$$

$$= \frac{1}{\beta}(1 - e^{-\beta t}) Y(0) + \frac{\gamma}{\beta} \cdot \underbrace{\int_0^t (1 - e^{-\beta(t-s)}) dW(s)}_{G_2(t)}$$

$G_2(t) \equiv \int_0^t (1 - e^{-\beta(t-s)}) dW(s)$ is a sum of independent Gaussians.

==> $G_2(t)$ is a Gaussian.

Therefore, $(X(t) - X(0))$ is a Gaussian. We will look into it in more detail.