AM216 Homework #1

Problem 1:

i) Derive $\mathrm{var}(\alpha X) = \alpha^2 \mathrm{var}(X)$

ii) Prove that if X and Y are independent, then we have

$\mathrm{var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y)$

Problem 2:

Let $X \sim \text{binomial}(n, p)$. We know

$E(X) = np$ and $\mathrm{var}(X) = np(1-p)$.

Use these results to calculate $E(X^2)$. Write the result as a polynomial of n.

Problem 3:

Let $X \sim N(\mu, \sigma^2)$. Fill in the details in the derivation

$$\Pr\left(\mu - \eta\sigma \le X \le \mu + \eta\sigma\right) = F_X(\mu + \eta\sigma) - F_X(\mu - \eta\sigma) = \cdots = \mathrm{erf}\left(\frac{\eta}{\sqrt{2}}\right)$$

Problem 4:

Consider a coin with prob(“head”) = p.

In the first part of an experiment, flip the coin n times. Let

Y = # of heads in the sequence of flips in the first part

In the second part of the experiment, flips the coin Y times (depending on the outcome of the first part). Let

X = # of heads in the sequence of flips in the second part.

Use the law of total expectation to calculate

i) $E(X)$ and

ii) $E(XY)$

Hint:

- $Y \sim \text{Bino}(n, p)$
- $(X \mid Y = m) \sim \text{Bino}(m, p)$
- Calculate $E(X \mid Y = m)$. Then use the law of total expectation …

Problem 5:

Draw a data set of $\underline{n=10}$ independent samples of $X \sim N(\mu, \sigma^2)$ with

$\mu = 0.6, \quad \sigma = 1.3$

Repeat this M = 500,000 times.

i) For each data set, calculate the exact 95% confidence interval using $\sigma = 1.3$.

Out of 500,000 exact confidence intervals, calculate the fraction of confidence intervals that contain the true value $\mu^{(\text{True})}$. Report the fraction.

ii) For each data set, calculate the approximate 95% confidence interval using the sample standard deviation

$$\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{j=1}^{n}(X_j - \hat{\mu})^2}\,, \qquad \hat{\mu} = \frac{1}{n}\sum_{j=1}^{n} X_j$$

Out of 500,000 approximate confidence intervals, calculate the fraction of confidence intervals that contain the true value $\mu^{(\text{True})}$. Report the fraction.