AM216 Homework #6

Problem 1:

Consider differential operator

$$L_z = b(z)\frac{\partial \bullet}{\partial z} + \frac{1}{2}a(z)\frac{\partial^2 \bullet}{\partial z^2}$$

Show that the adjoint operator is

$$L_z^* = -\frac{\partial}{\partial z}\left(b(z)\bullet\right) + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(a(z)\bullet\right)$$

Problem 2

Let $X(t)$ be the stochastic process governed by the Ito interpretation of

$$dX = b(X)dt + \sqrt{a(X)}\,dW$$

Consider the probability of exiting a region by time t.

$$u(x,t) \equiv \Pr\left(\text{exit by time } t \middle| X(0) = x\right)$$

Use $u(x,t) = E\left(u(x+dX, t-dt)\right)$, carry out Taylor expansion, and use moments of X, to show that u satisfies the backward equation

$$u_t = b(x)u_x + \frac{1}{2}a(x)u_{xx}$$

Problem 3:

For $b \neq 0$, solve the boundary value problem

$$\begin{cases} T_{xx} + 2bT_x = -2 \\ T'(L_1) = 0, \quad T(L_2) = 0 \end{cases}$$

to derive the solution

$$T(x) = \frac{1}{b}(L_2 - x) - \frac{1}{2b^2}\cdot\frac{\exp(2b(L_2 - x)) - 1}{\exp\left(2b(L_2 - L_1)\right)}$$

<u>Note:</u>

This is an exit problem. $T(x)$ is the average exit time given $X(0) = x$. The left end ($x = L_1$) is a reflecting boundary while the right end ($x = L_2$) is an absorbing boundary.

Problem 4:

Let $X(t)$ be the stochastic process governed by

$$dX = -Xdt + dW$$

Suppose the reward is determined at time T as

$$u(X(T)) \equiv \begin{cases} 1, & X(T) > c_0 \\ 0, & \text{otherwise} \end{cases}$$

Let $q(x, t)$ be the average amount of reward given $X(T-t) = x$.

$$q(x,t) = E\left(u(X(T)) \middle| X(T-t) = x\right)$$

<u>Find an analytical expression of</u> $q(x, t)$.

<u>Hint:</u>

$X(t)$ is an Ornstein-Uhlenbeck process.

Use the analytical solution of the Ornstein-Uhlenbeck process.

Express the CDF of normal distribution in terms of erf( ), as we did in Lecture 2.

Problem 5:

This problem is a derivation of the Maxwell-Boltzmann distribution.

Consider the sphere of radius $\sqrt{N}$, centered at the origin, in $\mathbb{R}^N$.

$$S \equiv \left\{ (y_1, y_2, \cdots, y_N) \middle| \sum_{j=1}^{N} y_j^{\,2} = N \right\}$$

Suppose $\vec{Y} = (Y_1, Y_2, \cdots, Y_N)$ is uniformly distributed over sphere S.

<u>Background:</u>

This mathematical formulation models the ensemble of N particles in a <u>closed system</u> where Y_j is the velocity of particle j. The system is closed; there is no energy exchange with “outside”; and as a result, the total energy of N particles is conserved. The total energy $\sum_{j=1}^{N} Y_j^{\,2}$ of the system is proportional to the system size, N.

a) Show that the marginal probability density of Y_1 satisfies

$$\rho_{Y_1}(y_1) \propto \left(1-\frac{y_1^{\,2}}{N}\right)^{\frac{N-3}{2}}$$

b) Use $\lim\limits_{N\to\infty}\left(1+\frac{\alpha}{N}\right)^N = \exp(\alpha)$ to show that

$$\lim_{N\to\infty}\rho_{Y_1}(y_1) \propto \exp\left(\frac{-y_1^{\,2}}{2}\right)$$

which is the Boltzmann distribution of particle 1, in thermal equilibrium with the rest of ensemble in a closed system.

Hint:

Consider the sphere of radius R in the n-dimensional space:

$$S(R;n) \equiv \left\{ (x_1, x_2, \cdots, x_n) \,\middle|\, \sum_{j=1}^{n} x_j^{\,2} = R^2 \right\}$$

Let $A(R;\, n)$ denote the area of $S(R;\, n)$. We have.

$A(R;\, n) = C_n\, R^{(n-1)}$

Here, we don't need to know the value of constant C_n.

The cross-section of $S(R;\, n)$ at $x_1 = y_1$ is a lower dimensional sphere:

$$S(r(y_1);\ (n-1)), \qquad r(y) = \sqrt{R^2 - y^2}$$

The section of $S(R;\, n)$ in the range of $y_1 \le x_1 < y_1 + dy$ is a strip along the lower dimensional sphere $S(r(y_1);\ (n-1))$ with width given by

$$\text{width} = \sqrt{\left(r'(y_1)dy\right)^2 + (dy)^2} = \frac{R\,dy}{\sqrt{R^2 - y_1^{\,2}}}$$

The area of the section is

$$\begin{aligned} \text{Area} &= A(\sqrt{R^2 - y_1^{\,2}};\ (n-1)) \times \text{width} \\ &= C_{(n-1)}(\sqrt{R^2 - y_1^{\,2}})^{(n-2)} \frac{R\,dy}{\sqrt{R^2 - y_1^{\,2}}} \\ &\propto (R^2 - y_1^{\,2})^{\frac{n-3}{2}}\, dy \propto \left(1 - \frac{y_1^{\,2}}{R^2}\right)^{\frac{n-3}{2}} dy \end{aligned}$$

Applying this result to sphere $S \equiv \left\{ (y_1, y_2, \cdots, y_N) \middle| \sum_{j=1}^{N} y_j^{\,2} = N \right\}$, we obtain the marginal probability density of Y_1.

$$\rho_{Y_1}(y_1) = \frac{\Pr(y_1 \le Y_1 < y_1 + dy)}{dy} \propto \left(1 - \frac{y_1^{\,2}}{N}\right)^{\frac{N-3}{2}}$$