AM216 Homework #7

Problem 1:

Consider $X(t) = (W(t))^3$ where $W(t)$ is the Wiener process.

Part 1:

Expand $dX = (W(t)+dW)^3 - W(t)^3$ to derive the SDE for $X(t)$ (Ito interpretation)

Part 2:

An alternative way of deriving the equation for $X(t)$ is to study the backward equation.

Let $Y(t) = (X(t))^{1/3} = W(t)$. The backward equation for Y is

$$q_t = \frac{1}{2} q_{yy}$$

$q(y, t)$ = average reward at time T given starting at $Y(T-t) = y$.

For $X(t)$, the corresponding quantity is

$Q(x, t)$ = average reward at time T given starting at $X(T-t) = x$

$q(y, t)$ and $Q(x, t)$ are related by a change of variables

$q(y, t)$ = average reward at time T given starting at $X(T-t) = y^3$.

$$\implies \quad q(y,t) = Q(x,t)\Big|_{x=y^3}$$

Starting from the backward for $q(y, t)$, use the chain rule to derive the backward equation for $Q(x, t)$.

Problem 2:

Let $X(t) = W(t)$. Consider the escape from region [0, 1].

$x = 0$: reflecting boundary

$x = 1$: absorbing boundary

Let $u(x, t) = \Pr(\text{exiting by time } t \mid X(0) = x)$.

The initial boundary value problem for $u(x, t)$ is

$$\begin{cases} u_t = \frac{1}{2} u_{xx} \\ u_x(0,t) = 0, \quad u(1,t) = 1 \\ u(x,0) = 0 \end{cases} \qquad \text{(IBVP\_1)}$$

Let

$$f(x,t) = \operatorname{erf}\left(\frac{x}{\sqrt{2t}}\right)$$

$$v(x,t;n) = f(x-(-2n+1),t) + f(x-(2n+1),t)$$

$$w(x,t) = 1 + f(x-1,t) + \sum_{n=1}^{\infty} (-1)^n v(x,t;n)$$

a) Verify that $f(x, t)$ satisfies the PDE $u_t = u_{xx}/2$.

Note: a) implies that $v(x, t; n)$ satisfies $u_t = u_{xx}/2$ for all n.

b) Verify that $f(x-1, t)$ and $v(x, t; n)$ satisfy the BC $u(1, t) = 0$.

Note: This is not the same condition as the one in the IBVP.

c) Verify that $1+ f(x-1, t)$ and $v(x, t; n)$ satisfy the IC

$$\lim_{t\to 0+} u(x,t) = 0 \qquad \text{for } x \in (0,1)$$

d) Verify that $w(x, t)$ satisfies the BC $u_x(0, t) = 0$.

Hint:

In the expression of $w(x, t)$, each $v(x, t; n)$ is the sum of a pair. Break the current pairing and form new pairing so that each new pair satisfies the BC $u_x(0, t) = 0$.

Remark:

a)-d) above demonstrate that $w(x, t)$ is the solution of (IBVP_1).

This way of solving (IBVP_1) is called the method of mirror images.

Problem 3:

In Lecture, we derived the exact solution for the average escape time

$$T(x) = \int_x^1 dy \exp(V(y)) \int_0^y ds \exp(-V(s))$$

This is valid for any potential. Here we study $V(x) = \alpha x$.

Part 1:

Carry out the integrations to write out $T(x)$.

Part 2:

Consider the case of large α (deep potential well). Use the method of capturing the dominant contributions in inner integral and outer integral, as we discussed in lecture.

For example, $\int_0^y ds \exp(-\alpha s) \approx \int_0^\infty ds \exp(-\alpha s)$ for $y > 0$.

For $x < 1$, find an approximation of $T(x)$ that is independent of x.

Problem 4:

Continue with Problem 3.

Suppose the starting position $X(0)$ has the Boltzmann distribution

$$\rho_{X(0)}(x) = \frac{e^{-V(x)}}{\int_0^1 dx\, e^{-V(x)}} = \frac{\alpha e^{-\alpha x}}{1 - e^{-\alpha}}$$

Integrate the exact $T(x)$ from Problem 3 with the Boltzmann distribution to calculate the overall average exit time

$$T^{(\text{Boltzmann})} = \int_0^1 dx\, T^{(\text{exact})}(x) \rho_{X(0)}(x)$$

In the calculation, you may neglect terms that are transcendentally small <u>relative to the leading term</u>. Is the <u>relative difference</u> between $T^{(\text{Boltzmann})}$ and the approximation from Problem 3 transcendentally small as $\alpha \to \infty$?

Problem 5:

Let $Y \geq 0$ be the random escape time of an unspecified stochastic process.

Let $\rho(t)$ be the probability density of random variable Y.

Suppose Y satisfies the specific memoryless property described below:

$$\frac{1}{\int_{t_0}^{\infty} \rho(t) dt} \int_{t_0}^{\infty} (t - t_0)^2 \rho(t) dt = E(Y^2) \qquad \text{independent of } t_0$$

Show that Y has an exponential distribution.

<u>Hint:</u> Consider $G(t) \equiv \int_t^{\infty} \int_u^{\infty} \rho(s) ds\, du$. Show that

$$G''(t) = \alpha^2 G''(t)$$

Use $\lim_{t \to +\infty} G(t) = 0$ to solve this second order linear ODE …