AM216 Stochastic Differential Equations

Lecture 10
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List of topics in this lecture

- Ito's interpretation, Stratonovich's interpretation of stochastic differential equations, the relation between the two
- Transition probability density, moments of dX, backward view, forward view
- Derivation of Kolmogorov backward equation, final value problem (FVP), meaning of the FVP, applications of the backward equation

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Recap

Convergence in probability:

Sufficient condition: $\lim_{N\to\infty} E\left(Q_N(\omega)\right)=0$ and $\lim_{N\to\infty} \mathrm{var}\left(Q_N(\omega)\right)=0$

Different interpretations of stochastic integral $\int_a^b f(s,W(s))dW(s)$

Ito interpretation:

$$I_{\text{Ito}}=\lim_{N\to\infty}\sum_{j=0}^{N-1} f(s_j,W(s_j))\Delta W_j$$

Stratonovich interpretation:

$$I_{\text{Stratonovich}}=\lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{1}{2}\Big(f(s_j,W(s_j))+f(s_{j+1},W(s_{j+1}))\Big)\Delta W_j$$

The two are related by

$$I_{\text{Stratonovich}}=I_{\text{Ito}}+\frac{1}{2}\int_0^t f_w(s,W(s))ds$$

Stochastic integrals based on axioms:

1) Fundamental theorem of calculus
2) λ–chain rule

Different interpretations of stochastic differential equations

Recall the stochastic differential equation (SDE) for a biased game

$$dX = -m\,dt + \sqrt{\sigma^2}\,dW$$

Now we consider a more general situation

$$dX = b(X,t)dt + \sqrt{a(X,t)}\,dW$$

This SDE is not meaningful until properly interpreted.

We treat Δt as finite and write out increment ΔX explicitly.

<u>Ito interpretation:</u>

$$\Delta X = b(X(t),t)\Delta t + \sqrt{a(X(t),t)}\,\Delta W + o(\Delta t)$$

<u>Note:</u> Ito interpretation as an algorithm is <u>explicit</u>.

<u>Stratonovich interpretation:</u>

$$\Delta X = \frac{1}{2}\Big(b(X(t),t) + b(X(t+\Delta t),t+\Delta t)\Big)\Delta t + \frac{1}{2}\Big(\sqrt{a(X(t),t)} + \sqrt{a(X(t+\Delta t),t+\Delta t)}\Big)\Delta W + o(\Delta t)$$

<u>Note:</u> Stratonovich interpretation as an algorithm is <u>implicit</u>.

On the RHS, $X(t+\Delta t) = X(t) + \Delta X$ contains ΔX.

<u>Theorem:</u>

The Stratonovich interpretation of equation

$$dX = b(X,t)dt + \sqrt{a(X,t)}\,dW$$

<u>is the same as</u> the Ito interpretation of the <u>modified equation</u>

$$dX = \left(b(X,t) + \frac{1}{4}a_x(X,t)\right)dt + \sqrt{a(X,t)}\,dW$$

<u>Proof:</u>

We start with the Stratonovich interpretation and try to express it in the form of Ito interpretation of a modified equation.

For conciseness, we write $X(t+\Delta t) = X(t) + \Delta X$ as $X + \Delta X$.

<u>Stratonovich interpretation:</u>

$$\begin{aligned}\Delta X &= \frac{1}{2}\Big(b(X,t)+b(X+\Delta X,t+\Delta t)\Big)\Delta t \\ &\quad +\frac{1}{2}\Big(\sqrt{a(X,t)}+\sqrt{a(X+\Delta X,t+\Delta t)}\Big)\Delta W+o(\Delta t)\end{aligned} \tag{E01}$$

Recall that ΔW and ΔX have the magnitude: $\Delta W \sim O(\sqrt{\Delta t})$ and $\Delta X \sim O(\sqrt{\Delta t})$.

We expand $a(\ )$ and $b(\ )$ around t. In (E01), we neglect o(Δt) terms since over a finite time interval, the sum of o(Δt) terms will disappear in the limit of $\Delta t \to 0$.

$$\sum_{j=0}^{N-1} o(\Delta t)=o(1)\to 0 \qquad \text{as } \Delta t\to 0$$

Expansion of $b(\ )\Delta t$:

$$b(X+\Delta X,t+\Delta t)\Delta t=b(X,t)\Delta t+o(\Delta t)$$

Expansion of $\sqrt{a(\ )}\,\Delta W$:

$$\sqrt{a(X+\Delta X,t+\Delta t)}\,\Delta W=\left(\sqrt{a(X,t)}+\frac{a_x(X,t)}{2\sqrt{a(X,t)}}\Delta X\right)\Delta W+o(\Delta t)$$

In the above, we have used the regular chain rule: $\dfrac{d}{du}\sqrt{a(u)}=\dfrac{a_u(u)}{2\sqrt{a(u)}}$.

Substitute these two expansions into (E01), we get

$$\Delta X=b(X,t)\Delta t+\sqrt{a(X,t)}\,\Delta W+\frac{a_x(X,t)}{4\sqrt{a(X,t)}}\Delta X\cdot\Delta W+o(\Delta t)$$

Moving the ΔX term to the LHS, we get

==> $$\left(1-\frac{a_x(X,t)}{4\sqrt{a(X,t)}}\Delta W\right)\Delta X=b(X,t)\Delta t+\sqrt{a(X,t)}\,\Delta W+o(\Delta t)$$

Using the expansion $(1-\varepsilon)^{-1}=1+\varepsilon+o(\varepsilon)$, we obtain

$$\begin{aligned}\Delta X &=\left(1-\frac{a_x(X,t)}{4\sqrt{a(X,t)}}\Delta W\right)^{-1}\Big(b(X,t)\Delta t+\sqrt{a(X,t)}\,\Delta W\Big)+o(\Delta t) \\ &=\left(1+\frac{a_x(X,t)}{4\sqrt{a(X,t)}}\Delta W+O(\Delta t)\right)\Big(b(X,t)\Delta t+\sqrt{a(X,t)}\,\Delta W\Big)+o(\Delta t)\end{aligned}$$

$$= b(X,t)\Delta t + \sqrt{a(X,t)}\,\Delta W + \frac{a_x(X,t)}{4}(\Delta W)^2 + o(\Delta t)$$

Recall Ito's lemma that (ΔW) can be replaced by Δt. We arrive at

$$\Delta X = \left(b(X(t),t) + \frac{1}{4} a_x(X(t),t) \right)\Delta t + \sqrt{a(X(t),t)}\,\Delta W + o(\Delta t)$$

This is the Ito interpretation of the modified equation

$$dX = \left(b(X,t) + \frac{1}{4} a_x(X,t) \right) dt + \sqrt{a(X,t)}\,dW$$

End of proof

Remarks:

1) A stochastic different equation is not meaningful when the interpretation is not specified.
2) Ito interpretation and Stratonovich interpretation are different models.
3) It is not possible to determine a "proper" interpretation, based on the "formal" stochastic differential equation. This has to be decided in the modeling stage. A model consists of the SDE and the specified interpretation.
4) Since Stratonovich interpretation can be viewed as Ito interpretation of a modified equation, from now on, we focus on Ito interpretation.

Backward equation and forward equation

Let $X(t)$ be the stochastic process governed by the SDE

$$dX = b(X,t)dt + \sqrt{a(X,t)}\,dW$$

Transition probability density:

$$q(x,\underbrace{t}_{\substack{\uparrow \\ \text{end time}}} \,|\, z,\underbrace{s}_{\substack{\uparrow \\ \text{starting time}}}) = \frac{1}{dx}\Pr\left(x \le X(t) < x + dx \,\middle|\, X(s) = z \right), \qquad t > s$$

The special case of $X(t) = W(t)$.

For $t > s$, we have

$$W(t) = W(s) + (W(t) - W(s))$$

==> $(W(t) \mid W(s)=z) \sim N(z, t-s)$

==> $$q(x,t \,|\, z,s) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(\frac{-(x-z)^2}{2(t-s)} \right)$$

Governing equations of $q(x, t|z, s)$ in the special case of $W(t)$

- We view $q(x, t|z, s)$ as a function of (x, t). It is straightforward to verify

$$q_t = \frac{1}{2} q_{xx}$$

  This is called the forward equation or Kolmogorov forward equation or Fokker-Planck equation.

- We view $q(x, t|z, s)$ as a function of (z, s). It is straightforward to verify

$$q_s = -\frac{1}{2} q_{zz}$$

  This is called the backward equation or Kolmogorov backward equation.

We consider the general case and our goal is to derive the backward equation and the forward equation of transition PD $q(x, t|z, s)$.

Mathematical preparations for the general case

We focus on Ito interpretation.

$$X(s+ds) = X(s) + dX$$

$$dX = b(X(s), s)ds + \sqrt{a(X(s), s)}\, dW(s) + o(ds)$$

where ds = finite $\to 0$

The moments of dX:

We calculate the moments of dX based on Ito interpretation.

$$(dX|\, X(s) = z) \sim N\big(b(z,s)ds,\ a(z,s)ds\big) + o(ds)$$

$$E\big(dX|\, X(s) = z\big) = b(z,s)ds + o(ds)$$

$$E\big((dX)^2|\, X(s) = z\big) = E\big((b(z,s)ds + a(z,s)dW)^2\big)$$

$$= E\Big(b(z,s)^2(ds)^2 + 2b(z,s)\sqrt{a(z,s)}\, dW\, ds + a(z,s)(dW)^2\Big)$$

$$= a(z,s)ds + o(ds)$$

$$E\big((dX)^n|\, X(s) = z\big) = E\big((b(z,s)ds + a(z,s)dW)^n\big) = o(ds), \qquad \text{for } n \geq 3$$

Definition of Ito interpretation based on moments

Alternatively, Ito interpretation can be defined by specifying the moments of dX.

$$X(s+ds) = X(s) + dX$$

$$E\left(dX \middle| X(s)=z\right) = b(z,s)ds + o(ds)$$

$$E\left((dX)^2 \middle| X(s)=z\right) = a(z,s)ds + o(ds)$$

$$E\left((dX)^n \middle| X(s)=z\right) = o(ds), \qquad \text{for } n \geq 3$$

<u>Moments of transition PD</u> $q(z+y, s+ds|z, s)$

$q(z+y, s+ds|z, s)$, as a function of y, is the probability density of increment dX.

We write out the moments of $q(z+y, s+ds|z, s)$ in integral forms.

0) $\int q(z+y, s+ds | z,s)dy = 1$

Equivalently $\int q(x, s+ds | z,s)dx = 1$

1) $\int q(z+y, s+ds | z,s)\, y\, dy = E\left(dX \middle| X(s)=z\right) = b(z,s)ds + o(ds)$

Equivalently $\int q(x, s+ds | z,s)(x-z)dx = b(z,s)ds + o(ds)$

2) $\int q(z+y, s+ds | z,s)\, y^2\, dy = E\left((dX)^2 \middle| X(s)=z\right) = a(z,s)ds + o(ds)$

Equivalently $\int q(x, s+ds | z,s)(x-z)^2 dx = a(z,s)ds + o(ds)$

3) $\int q(z+y, s+ds | z,s)\, y^n\, dy = E\left((dX)^n \middle| X(s)=z\right) = o(ds), \qquad \text{for } n \geq 3$

Equivalently $\int q(x, s+ds | z,s)(x-z)^n dx = o(ds), \qquad \text{for } n \geq 3$

These results will be very useful in the derivations of the forward equation and the backward equation.

We explore two views on the transition PD $q(z+y, s+ds|z, s)$: backward view and forward view. They will be used, respectively, in the derivation of backward equation and the derivation of forward equation.

<u>"Backward view"</u>

We fix (x, t) and view q as a function of (z, s):

$q(z, s) \equiv q(x, t \mid z, s)$

Recall that t is the end time and s is the starting time.

$[s \to t]$ is divided into $[s \to s+ds]$ and $[s+ds \to t]$.

$q(\cdot, s)$ and $q(\cdot, s+ds)$ are related by the law of total probability:

$$\underbrace{q(x,t\,|\,z,s)}_{\substack{q(\bullet,s)\\ [s\to t]}} = \int \underbrace{q(x,t\,|\,z+y,s+ds)}_{\substack{q(\bullet,s+ds)\\ [s+ds\to t]}}\ \underbrace{q(z+y,s+ds\,|\,z,s)}_{\substack{\text{density of } dX\\ [s\to s+ds]}}\, dy$$

In physics, this is called the master equation.

In mathematics, this is called Chapman-Kolmogorov equation.

We treat $q(z+y, s+ds|z, s)$ as known since we have all of its moments.

Using the master equation, (in principle,) we can update q as follows

$$q(\,\cdot\,, s+ds) \longrightarrow q(\,\cdot\,, s)$$

That is, we move the starting time backward from $(s+ds)$ to s.

(Draw a geometric illustration of backward view.)

s $(s+ds)$ t Time

That is why this is called the backward view.

"Forward view"

We fix (z, s) and view q as a function of (x, t):

$$q(x, t) \equiv q(x, t \mid z, s)$$

$[s \to t+dt]$ is divided into $[s \to t]$ and $[t \to t+dt]$.

$q(\cdot, t)$ and $q(\cdot, t+dt)$ are related by the law of total probability:

$$\underbrace{q(x,t+dt\,|\,z,s)}_{\substack{q(\bullet,t+dt)\\ [s\to t+dt]}} = \int \underbrace{q(x,t+dt\,|\,y,t)}_{\substack{\text{density of } X(t+dt)|X(t)=y\\ [t\to t+dt]}}\ \underbrace{q(y,t\,|\,z,s)}_{\substack{q(\bullet,t)\\ [s\to t]}}\, dy$$

This is also called the master equation.

We treat $q(x, t+dt|y, t)$ as known since we have all of its moments.

Using the master equation, (in principle,) we can update q as follows

$$q(\,\cdot\,, t) \longrightarrow q(\,\cdot\,, t+dt)$$

That is, we move the end time forward from t to $(t+dt)$.

(Draw a geometric illustration of forward view)

s t $(t+dt)$ Time

That is why this is called the forward view.

Derivation of the backward equation

We fix (x, t) and view q as a function of (z, s):

$q(z, s) \equiv q(x, t \mid z, s)$

Backward view:

$$\underbrace{q(x,t|z,s)}_{q(z,s)} = \int \underbrace{q(x,t|z+y,s+ds)}_{q(z+y,s+ds)} \underbrace{q(z+y,s+ds|z,s)}_{\text{density of } dX} dy$$

Key for the derivation:

As $ds \to 0$, $dX = O(\sqrt{ds})$ the integral is dominated by contribution from small y.

For y not small, $q(z+y, s+ds \mid z, s)$ is exponentially small as $ds \to 0$.

Strategy:

Since we only need $q(z+y, s+ds)$ for small y, we expand it around (z, s).

$$q(z+y,s+ds) = q(z,s) + q_s(z,s)ds + q_z(z,s)y + \frac{1}{2}q_{zz}(z,s)y^2 + O(y^3) + o(ds)$$

Substituting into the master equation, we have

$$q(z,s) = \int \left[q(z,s) + q_s(z,s)ds + q_z(z,s)y + \frac{1}{2}q_{zz}(z,s)y^2 + O(y^3) + o(ds) \right] \times q(z+y,s+ds|z,s)dy$$

Using moments of $q(z+y, s+ds|z, s)$ listed in 0), 1), 2), and 3)

$$= q(z,s) + q_s(z,s)ds + q_z(z,s)b(z,s)ds + \frac{1}{2}q_{zz}(z,s)a(z,s)ds + o(ds)$$

Dividing by ds and taking the limit as $ds \to 0$, we obtain

$$0 = q_s + b(z,s)q_z + \frac{1}{2}a(z,s)q_{zz}$$

This is called the Kolmogorov backward equation.

It is a diffusion type equation with negative diffusion coefficient.

It is ill-posed if we solve it forward from $s = 0$ to $s = T > 0$.

In applications, we solve it backward from $s = T > 0$ to $s = 0$.

We impose an end/final condition on $q(z, s)$

The final value problem (FVP) for $q(z, s) \equiv q(x, T|z, s)$

$$\begin{cases} q_s = -b(z,s)q_z - \dfrac{1}{2}a(z,s)q_{zz} \\ q(x,T|z,s)\Big|_{s=T} = \delta(z-x) \end{cases}$$

Remarks:

- The end condition specifies the system state at the end time if the system starts at the end time with the given state.
- The backward equation describes the change of system state at the end time when we move the starting time gradually backward.

An application of backward equation

Besides the transition PD, other quantities are also governed by the backward equation. Now we study one of these quantities. We will see more examples later.

Let $X(t)$ be the stochastic process governed by the SDE

$$dX = b(X,t)dt + \sqrt{a(X,t)}\,dW$$

We consider the probability

$$u(z,s) \equiv \Pr\left(X(T) \ge x_C \middle| X(s) = z\right)$$

Meaning of $u(z, s)$:

Suppose a stock price is described by $dX = b(X,t)dt + \sqrt{a(X,t)}\,dW$.

You make a bet that $X(T) \ge x_C$.

$u(z, s)$ = probability of your winning the bet given that $X(s) = z$.

Governing equation for $u(z, s)$

We write $u(z, s)$ as

$$u(z,s) = \int_{x_C}^{\infty} q(x,T|z,s)dx$$

Taking derivatives of u, we have

$$u_s = \int_{x_C}^{\infty} q_s(x,T|z,s)dx\,, \quad u_z = \int_{x_C}^{\infty} q_z(x,T|z,s)dx\,, \quad u_{zz} = \int_{x_C}^{\infty} q_{zz}(x,T|z,s)dx$$

Since $q(z, s)$ satisfies the backward equation, $u(z, s)$ satisfies the same equation.

$$u_s + b(z,s)u_z + \frac{1}{2}a(z,s)u_{zz} = 0$$

Alternatively, we can derive the equation for u directly from moments of dX.

$$u(z,s) = E\Big(u(z+dX, s+ds)\Big)$$

$$= E\left(u(z,s) + u_s(z,s)ds + u_z(z,s)dX + \frac{1}{2}u_{zz}(z,s)(dX)^2 + o(ds) \right)$$

...

(The derivation is in your homework.)

<u>The final value problem</u> (FVP) for $u(z, s)$:

$$\begin{cases} u_s = -b(z,s)u_z - \frac{1}{2}a(z,s)u_{zz} \\ u(z,s)\big|_{s=T} = \begin{cases} 1, & z \ge x_C \\ 0, & z < x_C \end{cases} \end{cases}$$

We solve it backward from $s = T$ to $s = 0$.

<u>Convert it to an IVP</u>

Mathematically, numerically, going backward in starting time s may be a bit awkward. We are more comfortable working with a forward time evolution.

We can change it to an IVP. Let

$$s = T - \tau$$

$$\phi(z,\tau) = u(z, T-\tau), \quad \alpha(z,\tau) = a(z, T-\tau), \quad \beta(z,\tau) = b(z, T-\tau)$$

τ is the time until a specified end time, for example, the time until the expiration of a call option (a financial derivative).

<u>The initial value problem</u> (IVP) for $\phi(z, \tau)$

$$\begin{cases} \phi_\tau = \beta(z,\tau)\phi_z + \frac{1}{2}\alpha(z,\tau)\phi_{zz} \\ \phi(z,0) = \begin{cases} 1, & z \ge x_C \\ 0, & z < x_C \end{cases} \end{cases}$$

We solve it forward from $\tau = 0$ to $\tau = T$.

Example 1:

Consider a fair game between players A and B (you are not one of them).

$dX = dW$

$X(t)$ = player A's cash at time t

You bet that player A's cash will be more than x_c at a specified time T.

Let $u(z,s) \equiv \Pr\left(X(T) \ge x_C \middle| X(s) = z\right)$

= probability of your winning the bet given that $X(s) = z$.

We apply change of variables:

$s = T - \tau$

$\phi(z,\tau) = u(z, T-\tau)$

$\phi(z, \tau)$ satisfies the initial value problem.

$$\begin{cases} \phi_\tau = \dfrac{1}{2}\phi_{zz} \\ \phi(z,0) = \begin{cases} 1, & z \ge x_C \\ 0, & z < x_C \end{cases} \end{cases}$$

Solution:

$$\phi(z,\tau) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{z - x_c}{\sqrt{2\tau}}\right)$$

$$\Longrightarrow \quad u(z,s) = \phi(z, T-s) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{z - x_c}{\sqrt{2(T-s)}}\right)$$

(Derivation is in your homework.)

Caution:

This is for the simplified situation where the game keeps going even when one of the players runs out of cash, for example, when $X(t) < 0$!

Example 2 (a more realistic version of Example 1)

Suppose player B has infinite amount of cash (so won't run out of cash!).

Consider the probability

$$v(z,s) \equiv \Pr\left(X(T) \ge x_C \text{ and } \boxed{X(t) > 0 \text{ for all } t \in [s,T]} \middle| X(s) = z\right)$$

This is the probability of player A surviving to time T and $X(T) > x_C$.

Governing equation:

Start at time s with $X(s) = z$. For $z > 0$, when ds is small enough (depending on z), the probability of $X(s+ds)$ hitting 0 is exponentially small and is negligible.

$$v(z,s) = E\big(v(z+dX, s+ds)\big)$$
$$= E\left(v(z,s) + v_z(z,s)dX + v_s(z,s)ds + \frac{1}{2}v_{zz}(z,s)(dX)^2 + o(ds) \right)$$

…

Remarks:

- The governing equation of $v(z, s)$ is not affected by the condition $X > 0$.
- Probability $v(z, s)$ is affected by the condition $X(t) > 0$ for all $t \in [s, T]$.
- The condition $X > 0$ affects $v(z, s)$ via a boundary condition.

…

(Your homework problem)

Example 3 (another realistic version of Example 1)

Consider the probability

$$w(z,s) \equiv \Pr\left(\begin{array}{c} \boxed{X(t_1) \geq x_c \text{ at some } t_1 \in [s,T]} \\ \text{and } \boxed{X(t) > 0 \text{ for all } t \in [s, t_1]} \end{array} \middle| X(s) = z \right)$$

This is the probability of player A surviving to time t_1 and having cash above threshold x_C at time t_1 for some t_1 before the end time.

In other words, $w(z, s)$ is the probability of player A reaching cash position x_C before hitting 0 and within the specified time limit $[s, T]$.

…

(Your homework problem)