AM216 Stochastic Differential Equations

Lecture 20
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List of topics in this lecture

- A mathematical derivation of Smoluchowski-Kramers approximation
- Scaling of X, Y and $F(X, t)$
- Solvability condition

Review

Smoluchowski-Kramers approximation

Consider a small particle in water governed by the Langevin equation

$$dX = Y\,dt$$

$$m dY = -bY\,dt + F(X,t)dt + b\sqrt{2D}\,dW$$

where

X : position; Y : velocity; a : radius of the particle

$m = 4\pi/3\ a^3 = O(a^3)$: mass of the particle

$b = 6\pi\ \eta\ a = O(a)$: drag coefficient of the particle

$D = \dfrac{k_B T}{b} = O(a^{-1})$: diffusion coefficient

Time scale of inertia and thermal excitation: $t_0 = \dfrac{m}{b} = O(a^2)$

Equipartition of energy: $\dfrac{1}{2} m E\left(Y^2\right) = \dfrac{1}{2} k_B T$

Root-mean-square velocity of a particle:

$$\sqrt{E(Y^2)} = \sqrt{\frac{k_B T}{m}} = \sqrt{\frac{3 k_B T}{4\pi\rho a^3}} \propto a^{-3/2}$$

The case of $F(X, t) = F_0$

For $dt/t_0 \gg 1$ (i.e., on a "coarse" grid), $X(t)$ satisfies

$$dX = \frac{F(X,t)}{b}dt + \sqrt{2D}\,dW$$

End of review

A mathematical derivation of Smoluchowski-Kramers approximation

Let $\varepsilon = \sqrt{\frac{m}{b}} = \sqrt{O(a^2)} = O(a)$ be the small parameter.

Scaling for *X*

The over-damped Langevin equation for *X* is

$$dX = \frac{F(X,t)}{b}dt + \sqrt{2D}\,dW\,, \qquad D = O(a^{-1}) = O(\varepsilon^{-1})$$

This equation is not free of ε. When $F = 0$ and $t = O(1)$, we have

$$X(1) - X(0) \sim \sqrt{2D} = O(\varepsilon^{-1/2})$$

We select the scaling for *X* to make it O(1)

$$\hat{X} = \sqrt{\varepsilon}\,X\,, \qquad \Rightarrow \qquad \hat{X}(1) - \hat{X}(0) = O(1)$$

We consider the case where for $t = O(1)$, the effect of external driving force is comparable to that of the diffusion

$$\frac{F(X,t)}{b} = O(\sqrt{D}) = O(\varepsilon^{-1/2})\,, \qquad b = O(a) = O(\varepsilon)$$

==> $\quad F(X,t) = O(\sqrt{\varepsilon})$

We define

$$b_0 \equiv \frac{b}{\varepsilon} = O(1)\,, \qquad D_0 \equiv \varepsilon D = O(1)$$

$$f(\hat{X},t) \equiv \frac{F(X,t)}{\sqrt{\varepsilon}\,b_0} = O(1)$$

Scaling for *Y*

When $F = 0$, the equipartition of energy gives us

$$\sqrt{E(Y^2)} = \sqrt{\frac{k_B T}{m}} = O(\varepsilon^{-3/2})\,, \qquad m = O(a^3) = O(\varepsilon^3)$$

We select the scaling for *Y* to make it O(1)

$$\hat{Y} = \varepsilon^{3/2} Y, \qquad \Rightarrow \qquad \sqrt{E(\hat{Y}^2)} = O(1)$$

The full Langevin equation in new variables

$$m dY = -bY\,dt + F(X,t)dt + b\sqrt{2D}\,dW$$

$$\Longrightarrow \quad \frac{m}{b} dY = -Y\,dt + \frac{F(X,t)}{b} dt + \sqrt{2D}\,dW$$

$$\Longrightarrow \quad \varepsilon^2(\varepsilon^{-3/2} d\hat{Y}) = -(\varepsilon^{-3/2}\hat{Y})dt + \frac{\varepsilon^{1/2} b_0 f(\hat{X},t)}{\varepsilon b_0} dt + \sqrt{2(\varepsilon^{-1} D_0)}\,dW$$

$$\Longrightarrow \quad \varepsilon^{1/2} d\hat{Y} = -\varepsilon^{-3/2}\hat{Y}\,dt + \varepsilon^{-1/2} f(\hat{X},t)dt + \varepsilon^{-1/2}\sqrt{2D_0}\,dW$$

$$\Longrightarrow \quad d\hat{Y} = \frac{-1}{\varepsilon^2}\hat{Y}\,dt + \frac{1}{\varepsilon} f(\hat{X},t)dt + \frac{1}{\varepsilon}\sqrt{2D_0}\,dW$$

The over-damped Langevin equation in new variables

$$dX = \frac{F(X,t)}{b} dt + \sqrt{2D}\,dW$$

$$\Longrightarrow \quad \varepsilon^{-1/2} d\hat{X} = \frac{\varepsilon^{1/2} b_0 f(\hat{X},t)}{\varepsilon b_0} dt + \sqrt{2(\varepsilon^{-1} D_0)}\,dW$$

$$\Longrightarrow \quad d\hat{X} = f(\hat{X},t)dt + \sqrt{2D_0}\,dW$$

Note that after scaling, the over-damped Langevin equation is free of ε.

After scaling, we revert back to the simple notation of (*X*, *Y*, *t*) to write out the starting SDE and the end SDE of the S-K approximation.

The starting SDE of S-K approximation

$$\begin{aligned} dX &= \frac{1}{\varepsilon} Y\,dt \\ dY &= \frac{-1}{\varepsilon^2} Y\,dt + \frac{1}{\varepsilon} f(X,t)dt + \frac{1}{\varepsilon}\sqrt{2D_0}\,dW \end{aligned} \qquad \text{(S-1)}$$

The end SDE of S-K approximation

$$dX = f(X,t)dt + \sqrt{2D_0}\,dW \qquad \text{(S-2)}$$

Backward equation for the starting SDE

For the starting SDE (S-1), we consider function

$$u(x,y,t) \equiv \Pr\left\{ X \text{ having crossed } x_B \text{ by time } t \,\middle|\, X(0)=x, Y(0)=y \right\}$$

We derive the governing equation of $u(x, y, t)$. For dt small enough, we have

$$u(x,y,t) = E\Big(u(x+dX, y+dY, t-dt)\Big) + o(dt) \qquad \text{(E-1)}$$

Note that in the SDE, ε = fixed and $dt \to 0$. We have $dX = O(dt)$ and $dY = O(\sqrt{dt})$.

Specifically we have the moments of dX and dY from the SDE.

$$\Big(dX \,\Big|\, X(0)=x, Y(0)=y\Big) = \frac{y}{\varepsilon}dt \qquad \text{not a random variable}$$

$$E\Big(dY \,\Big|\, X(0)=x, Y(0)=y\Big) = \left(\frac{-1}{\varepsilon^2}y + \frac{1}{\varepsilon}f(x,t)\right)dt$$

$$E\Big((dY)^2 \,\Big|\, X(0)=x, Y(0)=y\Big) = \frac{1}{\varepsilon^2}2D_0 dt + o(dt)$$

$$E\Big((dY)^n \,\Big|\, X(0)=x, Y(0)=y\Big) = o(dt) \quad \text{for } n \geq 3$$

We expand (E-1)

$$u(x,y,t) = E\Big(u(x+dX, y+dY, t-dt)\Big) + o(dt)$$

$$= E\left(u(x,y,t) + u_x dX + u_y dY + \frac{1}{2}u_{yy}(dY)^2 - u_t dt \right) + o(dt)$$

$$= u + u_x \frac{y}{\varepsilon}dt + u_y\left(\frac{-1}{\varepsilon^2}y + \frac{1}{\varepsilon}f(x,t)\right)dt + \frac{1}{2}u_{yy}\frac{1}{\varepsilon^2}2D_0 dt - u_t dt + o(dt)$$

...

The governing equation for $u(x, y, t)$ is the backward equation of (S-1).

$$\varepsilon^2 u_t = -y u_y + D_0 u_{yy} + \varepsilon\Big(y u_x + f(x,t)u_y\Big) \qquad \text{(BE-1)}$$

Backward equation for the end SDE

For the end SDE (S-2), we consider function

$$w(x,t) \equiv \Pr\left\{ X \text{ having crossed } x_B \text{ by time } t \,\middle|\, X(0)=x \right\}$$

The governing equation for $w(x, t)$ is the backward equation of (S-2).

$$\frac{\partial w}{\partial t} = D_0 \frac{\partial^2 w}{\partial x} + f(x,t)\frac{\partial w}{\partial x} \qquad \text{(BE-2)}$$

Convergence of $u(x, y, t)$ to $w(x, t)$

For backward equation (BE-1), we seek solutions of the form

$$u(x,y,t)=u_0(x,y,t)+\varepsilon u_1(x,y,t)+\varepsilon^2 u_2(x,y,t)+\cdots$$

We substitute the expansion into (BE-1) and keep terms up to $O(\varepsilon^2)$.

$$\varepsilon^2\frac{\partial u_0}{\partial t}=-y\left(\frac{\partial u_0}{\partial y}+\varepsilon\frac{\partial u_1}{\partial y}+\varepsilon^2\frac{\partial u_2}{\partial y}\right)+D_0\left(\frac{\partial^2 u_0}{\partial y^2}+\varepsilon\frac{\partial^2 u_1}{\partial y^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial y^2}\right)$$
$$+\varepsilon\left(y\left(\frac{\partial u_0}{\partial x}+\varepsilon\frac{\partial u_1}{\partial x}\right)+f(x,t)\left(\frac{\partial u_0}{\partial y}+\varepsilon\frac{\partial u_1}{\partial y}\right)\right)+o(\varepsilon^2)$$

$$\Longrightarrow\quad 0=\underbrace{\left(-y\frac{\partial u_0}{\partial y}+D_0\frac{\partial^2 u_0}{\partial y^2}\right)}_{O(1)\text{ terms}}+\underbrace{\varepsilon\left(-y\frac{\partial u_1}{\partial y}+D_0\frac{\partial^2 u_1}{\partial y^2}+y\frac{\partial u_0}{\partial x}+f(x,t)\frac{\partial u_0}{\partial y}\right)}_{O(\varepsilon)\text{ terms}}$$
$$\underbrace{+\varepsilon^2\left(-y\frac{\partial u_2}{\partial y}+D_0\frac{\partial^2 u_2}{\partial y^2}+y\frac{\partial u_1}{\partial x}+f(x,t)\frac{\partial u_1}{\partial y}-\frac{\partial u_0}{\partial t}\right)}_{O(\varepsilon^2)\text{ terms}}+o(\varepsilon^2)$$

Below, we show that

- $u_0(x, y, t) = u_0(x, t)$, independent of y.
- $u_0(x, t)$ satisfies backward equation (BE-2).

Step A: $u_0(x, y, t)$ is independent of y.

We balance the O(1) terms.

$$-y\frac{\partial u_0}{\partial y}+D_0\frac{\partial^2 u_0}{\partial y^2}=0$$

Multiplying by the integrating factor yields

$$\exp\left(\frac{-y^2}{2D_0}\right)\left(\frac{-y}{D_0}\right)\frac{\partial u_0}{\partial y}+\exp\left(\frac{-y^2}{2D_0}\right)\frac{\partial^2 u_0}{\partial y^2}=0$$

$$\Longrightarrow\quad \frac{\partial}{\partial y}\left[\exp\left(\frac{-y^2}{2D_0}\right)\frac{\partial u_0}{\partial y}\right]=0$$

$$\Longrightarrow\quad \exp\left(\frac{-y^2}{2D_0}\right)\frac{\partial u_0}{\partial y}=c_1(x,t)\quad\text{independent of } y$$

==> $\dfrac{\partial u_0}{\partial y}=c_1(x,t)\exp\left(\dfrac{y^2}{2D_0}\right)$

==> $u_0=c_0(x,t)+c_1(x,t)\displaystyle\int_0^y \exp\left(\frac{z^2}{2D_0}\right)dz$

Recall that $u(x, y, t)$ = probability.

==> $u(x, y, t)$ is bounded as $y \to \infty$.

==> $u_0(x, y, t)$, as the leading term, is bounded as $y \to \infty$

==> $c_1(x,t)=0$

==> $u_0=c_0(x,t)$ independent of y

Step B: $u_0(x, t)$ satisfies backward equation (BE-2)

Step B1: We balance the $O(\varepsilon)$ terms

$$-y\frac{\partial u_1}{\partial y}+D_0\frac{\partial^2 u_1}{\partial y^2}+y\frac{\partial u_0}{\partial x}+f(x,t)\frac{\partial u_0}{\partial y}=0$$

We write it as an equation for u_1 and multiply by the integrating factor

$$-y\frac{\partial u_1}{\partial y}+D_0\frac{\partial^2 u_1}{\partial y^2}=-y\frac{\partial u_0}{\partial x}$$

==> $\exp\left(\dfrac{-y^2}{2D_0}\right)\left(\dfrac{-y}{D_0}\right)\dfrac{\partial u_1}{\partial y}+\exp\left(\dfrac{-y^2}{2D_0}\right)\dfrac{\partial^2 u_1}{\partial y^2}=\exp\left(\dfrac{-y^2}{2D_0}\right)\left(\dfrac{-y}{D_0}\right)\dfrac{\partial u_0}{\partial x}$

==> $\dfrac{\partial}{\partial y}\left(\exp\left(\dfrac{-y^2}{2D_0}\right)\dfrac{\partial u_1}{\partial y}\right)=\dfrac{\partial}{\partial y}\left(\exp\left(\dfrac{-y^2}{2D_0}\right)\right)\dfrac{\partial u_0}{\partial x}$

Note that $\partial u_0/\partial x$ is independent of y.

$$\exp\left(\frac{-y^2}{2D_0}\right)\frac{\partial u_1}{\partial y}=\exp\left(\frac{-y^2}{2D_0}\right)\frac{\partial u_0}{\partial x}+c_3(x,t)$$

==> $\dfrac{\partial u_1}{\partial y}=\dfrac{\partial u_0}{\partial x}+c_3(x,t)\exp\left(\dfrac{y^2}{2D_0}\right)$

==> $u_1=y\dfrac{\partial u_0}{\partial x}+c_2(x,t)+c_3(x,t)\displaystyle\int_0^y \exp\left(\frac{z^2}{2D_0}\right)dz$

u_1 is solvable without imposing any constraint on u_0.

To find a constraint on u_0, we need to examine $O(\varepsilon^2)$ terms.

Step B2: We balance the $O(\varepsilon^2)$ terms

$$-y\frac{\partial u_2}{\partial y}+D_0\frac{\partial^2 u_2}{\partial y^2}+y\frac{\partial u_1}{\partial x}+f(x,t)\frac{\partial u_1}{\partial y}-\frac{\partial u_0}{\partial t}=0$$

==> $$-y\frac{\partial u_2}{\partial y}+D_0\frac{\partial^2 u_2}{\partial y^2}=\frac{\partial u_0}{\partial t}-y\frac{\partial u_1}{\partial x}-f(x,t)\frac{\partial u_1}{\partial y}$$

==> $$L_1[u_2]=\frac{\partial u_0}{\partial t}+L_2[u_1]$$

where operators L_1 and L_2 are defined as

$$L_1[\cdot]\equiv -y\frac{\partial \cdot}{\partial y}+D_0\frac{\partial^2 \cdot}{\partial y^2}$$

$$L_2[\cdot]\equiv -y\frac{\partial \cdot}{\partial x}-f(x,t)\frac{\partial \cdot}{\partial y}$$

Theorem (solvability condition)

Equation $L[u] = g$ is solvable if and only if $\langle g, v\rangle = 0$ for all v satisfying $L^*[v] = 0$ (and satisfying $v(y) \to 0$ rapidly as $|y| \to \infty$)

Illustration:

Let A be an $m\times n$ matrix .

$A\,u = b$ is solvable if and only if $b \in \mathrm{Col}(A)$

if and only if $\langle b, v\rangle$ for all $v \in \mathrm{Col}(A)^{\perp} = \mathrm{Nul}(A^{T})$.

Important note:

When operator L_1 is in variable y, we view $v(y)$ as a function of y and the inner-product $\langle g, v\rangle$ is an integral with respect to y.

Step B3: Solvability of $L_1[u_2]=\frac{\partial u_0}{\partial t}+L_2[u_1]$.

u_2 is solvable if and only if $\left\langle \frac{\partial u_0}{\partial t}+L_2[u_1], v\right\rangle=0$ for all v satisfying $L_1^*[v] = 0$.

<u>Step B4:</u> Solution of $L_1^*[v] = 0$

$$L_1[\cdot] = -y\frac{\partial \cdot}{\partial y} + D_0\frac{\partial^2 \cdot}{\partial y^2} \quad ==> \quad L_1^*[\cdot] = \frac{\partial (y\,\cdot)}{\partial y} + D_0\frac{\partial^2 \cdot}{\partial y^2}$$

$L_1^*[v] = 0$ yields

$$\frac{\partial (y\,v)}{\partial y} + D_0\frac{\partial^2 v}{\partial y^2} = 0$$

$$==> \quad \frac{y}{D_0}v + \frac{\partial v}{\partial y} = d_1$$

$$==> \quad \exp\left(\frac{y^2}{2D_0}\right)\frac{y}{D_0}v + \exp\left(\frac{y^2}{2D_0}\right)\frac{\partial v}{\partial y} = d_1\exp\left(\frac{y^2}{2D_0}\right)$$

$$==> \quad \frac{\partial}{\partial y}\left[\exp\left(\frac{y^2}{2D_0}\right)v\right] = d_1\exp\left(\frac{y^2}{2D_0}\right)$$

$$==> \quad \exp\left(\frac{y^2}{2D_0}\right)v(y) = d_1\int_0^y \exp\left(\frac{z^2}{2D_0}\right)dz + d_0$$

$$==> \quad v(y) = \underbrace{d_1\exp\left(\frac{-y^2}{2D_0}\right)\int_0^y \exp\left(\frac{z^2}{2D_0}\right)dz}_{\text{Not decaying to zero as } y\to\infty} + d_0\exp\left(\frac{-y^2}{2D_0}\right)$$

$$==> \quad v(y) = d_0\exp\left(\frac{-y^2}{2D_0}\right)$$

<u>Step B5:</u> Back to solvability of $L_1[u_2] = \dfrac{\partial u_0}{\partial t} + L_2[u_1]$.

We select d_0 to make $v(y)$ a Gaussian density.

$$v(y) = \frac{1}{\sqrt{2\pi D_0}}\exp\left(\frac{-y^2}{2D_0}\right) = \rho_{N(0,D_0)}(y)$$

u_2 is solvable if and only if $\displaystyle\int_{-\infty}^{+\infty}\left(\frac{\partial u_0}{\partial t} + L_2[u_1]\right)\rho_{N(0,D_0)}(y)dy = 0$.

Notice that $\partial u_0/\partial t$ is independent of y. The solvability condition gives us

$$\frac{\partial u_0}{\partial t}+\int_{-\infty}^{+\infty} L_2[u_1]\rho_{N(0,D_0)}(y)dy=0 \qquad \text{(Cond-1)}$$

<u>Step B6:</u> Calculation of $\int_{-\infty}^{+\infty} L_2[u_1]\rho_{N(0,D_0)}(y)dy$

Recall that $u_1=y\frac{\partial u_0}{\partial x}+c_2(x,t)+c_3(x,t)\int_0^y \exp\left(\frac{z^2}{2D_0}\right)dz$. We calculate $L_2[u_1]$.

$$L_2[u_1]=-y\frac{\partial u_1}{\partial x}-f(x,t)\frac{\partial u_1}{\partial y}=\underbrace{-y^2\frac{\partial^2 u_0}{\partial x^2}-f(x,t)\frac{\partial u_0}{\partial x}}_{\text{effect of } u_0}\underbrace{-y\frac{\partial c_2}{\partial x}}_{\text{effect of } c_2}$$
$$\underbrace{-y\frac{\partial c_3}{\partial x}\int_0^y \exp\left(\frac{z^2}{2D_0}\right)dz-f(x,t)c_3\exp\left(\frac{y^2}{2D_0}\right)}_{\text{effect of } c_3}$$

We examine the inner product of each part with $\rho_{N(0,D)}(y)$.

$u_0(x,t)$ is independent of y.

$$\int_{-\infty}^{+\infty}\underbrace{\left[-y^2\frac{\partial^2 u_0}{\partial x^2}-f(x,t)\frac{\partial u_0}{\partial x}\right]}_{\text{effect of } u_0}\rho_{N(0,D_0)}(y)dy=-D_0\frac{\partial^2 u_0}{\partial x^2}-f(x,t)\frac{\partial u_0}{\partial x}$$

$c_2(x,t)$ is independent of y.

$$\int_{-\infty}^{+\infty}\underbrace{-y\frac{\partial c_2}{\partial x}}_{\text{effect of } c_2}\rho_{N(0,D_0)}(y)dy=0$$

$c_3(x,t)$ is independent of y.

$$\int_{-\infty}^{+\infty}\underbrace{\left[-y\frac{\partial c_3}{\partial x}\int_0^y \exp\left(\frac{z^2}{2D_0}\right)dz-f(x,t)c_3\exp\left(\frac{y^2}{2D_0}\right)\right]}_{\text{effect of } c_3}\rho_{N(0,D_0)}(y)dy=\infty \quad \text{if } c_3\neq 0.$$

To make $\int_{-\infty}^{+\infty} L_2[u_1]\rho_{N(0,D_0)}(y)dy$ finite, we must have $c_3(x,t)\equiv 0$.

Thus, we obtain the expression

$$\int_{-\infty}^{+\infty} L_2[u_1]\rho_{N(0,D_0)}(y)dy=-D_0\frac{\partial^2 u_0}{\partial x^2}-f(x,t)\frac{\partial u_0}{\partial x} \qquad \text{(Res-1)}$$

Step B7: Governing equation for u_0

Substitute result (Res-1) into solvability condition (Cond-1), we conclude

$$\frac{\partial u_0}{\partial t} = D_0 \frac{\partial^2 u_0}{\partial x^2} + f(x,t)\frac{\partial u_0}{\partial x}$$

which is backward equation (BE-2).

Remark:

The key component of the derivation is the solvability condition.

Below we illustrate the use of solvability condition in two simple examples.

Example: (the solvability condition)

Consider solving a 2×2 linear system $Ax = b$ where

$$A = \begin{pmatrix} 1 & 1+\varepsilon \\ 1+\varepsilon & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1-\varepsilon \\ 1+\varepsilon \end{pmatrix}$$

The exact solution is

$$x_{\text{exa}} = \begin{pmatrix} \dfrac{3+\varepsilon}{2+\varepsilon} \\ \dfrac{-(1+\varepsilon)}{2+\varepsilon} \end{pmatrix} \approx \begin{pmatrix} 1.5 \\ -0.5 \end{pmatrix}$$

Here we use asymptotic analysis to solve this example.

The goal is to see the application of solvability condition in a simple setting.

We seek solutions of the form

$$x = x^{(0)} + \varepsilon x^{(1)} + \cdots$$

We write matrix A and vector b as

$$A = A^{(0)} + \varepsilon A^{(1)}, \qquad A^{(0)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$b = b^{(0)} + \varepsilon b^{(1)}, \qquad b^{(0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad b^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

We expand linear system $Ax - b = 0$ into $O(1)$ terms, $O(\varepsilon)$ terms, …

$$\left(A^{(0)} + \varepsilon A^{(1)}\right)\left(x^{(0)} + \varepsilon x^{(1)} + \cdots\right) - \left(b^{(0)} + \varepsilon b^{(1)}\right) = 0$$

==> $\left(A^{(0)}x^{(0)}-b^{(0)}\right)+\varepsilon\left(A^{(0)}x^{(1)}+A^{(1)}x^{(0)}-b^{(1)}\right)+\cdots=0$

First we look at the O(1) terms.

$$A^{(0)}x^{(0)}=b^{(0)}$$

This linear system is underdetermined.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

==> $x_1^{(0)}+x_2^{(0)}=1$ (CNS-1)

From the O(1) terms, we obtain only one constraint for two unknowns.

Next we look at the O(ε) terms.

$$A^{(0)}x^{(1)}=-A^{(1)}x^{(0)}+b^{(1)} \qquad \text{(L-1)}$$

Matrix $A^{(0)}$ is symmetric. The null space of $A^{(0)}$ is

$$\mathrm{Nul}\left(A^{(0)}\right)=\left\{v_0\right\}, \qquad v_0=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Linear system (L-1) is solvable if and only if

$$\left\langle -A^{(1)}x^{(0)}+b^{(1)},\ v_0\right\rangle=0$$

==> $x_1^{(0)}-x_2^{(0)}-2=0$ (CNS-2)

Combining constraints (CNS-1) and (CNS-2), we conclude

$$x^{(0)}=\begin{pmatrix} 1.5 \\ -0.5 \end{pmatrix}$$

Example: (the solvability condition)

Consider the BVP of 2nd order linear ODE

$$\begin{cases} L[u]+\varepsilon x u=3\sin(2x), \qquad L[u]\equiv u''+u \\ u(0)=0, \qquad u(\pi)=0 \end{cases}$$

Here we use asymptotic analysis to solve this example.

We seek solutions of the form

$$u(x)=u^{(0)}(x)+\varepsilon u^{(1)}(x)+\cdots$$

We expand the ODE into O(1) terms, O(ε) terms, …

$$L\left[u^{(0)\prime\prime}(x)+\varepsilon u^{(1)\prime\prime}(x)\right]+\varepsilon x u^{(0)}(x)-3\sin(2x)=0$$

==> $$\left(L\left[u^{(0)}(x)\right]-3\sin(2x)\right)+\varepsilon\left(L\left[u^{(1)}(x)\right]+x u^{(0)}(x)\right)+\cdots=0$$

First we look at the O(1) terms.

$$\begin{cases} L\left[u^{(0)}\right]=3\sin(2x) \\ u^{(0)}(0)=0, \qquad u^{(0)}(\pi)=0 \end{cases}$$

This BVP is <u>underdetermined</u>.

$u^{(0)}=-\sin(2x)+c\sin(x)$ is a solution for any c. (CNS-3)

Next we look at the O(ε) terms.

$$\begin{cases} L\left[u^{(1)}\right]=-x u^{(0)}(x) \\ u^{(1)}(0)=0, \qquad u^{(1)}(\pi)=0 \end{cases} \qquad \text{(BVP-1)}$$

Operator L with zero-BCs is self-adjoint (symmetric).

The null space of L with zero-BCs is

$$\text{Nul}\left(L\right)=\left\{v_0(x)\right\}, \qquad v_0(x)=\sin(x)$$

(BVP-1) is solvable if and only if

$$\left\langle -x u^{(0)}(x),\ v_0\right\rangle=0$$

==> $$\int_0^{\pi}\left(x\sin(2x)-c\,x\sin(x)\right)\sin(x)dx=0$$

==> $$-\frac{8}{9}-c\frac{\pi^2}{4}=0 \qquad \text{==>} \qquad c=\frac{-32}{9\pi^2} \qquad \text{(CNS-4)}$$

Combining constraints (CNS-3) and (CNS-4), we conclude

$$u^{(0)}=-\sin(2x)-\frac{32}{9\pi^2}\sin(x)$$