AM216 Homework #8

Problem 1:

Definition of martingale:

A stochastic process $X(t)$ is called a martingale if

1) $E\left(|X(t)|\right)<\infty$

2) $E\left(X(t)\middle| X(t_i)=a_i,\ i=1,2,\ldots,n,\ t_1<t_2<\cdots<t_n<t\right)=a_n$

Part 1:

Show that $X(t)=\exp\left(\alpha W(t)-\frac{\alpha^2 t}{2}\right)$ is a martingale.

Part 2:

Show that $X(t)=\exp\left(W(t)\right)$ is NOT a martingale.

Problem 2:

Let $X(t)$ = C $\exp(W(t)-t/2)$.

Part 1:

We work with the increment directly

$dX = X(t+dt) - X(t)$ = C $\exp(W(t)+dW-(t+dt)/2)$ – C $\exp(W(t) - t/2)$

Expand it to show that $X(t)$ satisfies the Ito interpretation of

$$dX=\sqrt{X(t)^2}dW \qquad \text{(SDE-1)}$$

Part 2:

Use this approach to find a general solution of

$$dX=b\,X(t)+\sqrt{X(t)^2}dW \qquad \text{(SDE-2)}$$

Hint: Try solution of the form $X(t)$ = C $\exp(W(t)-\beta t)$ and find β.

Problem 3:

Continue with (SDE-1) in Problem 2.

Suppose we start with $X(0) > 0$. The SDE does not directly tell us $X(t) > 0$.

We like to convince ourselves that $X(t)$ will not hit 0.

We discretize the SDE using the Ito interpretation.

$$X(t+\Delta t)=X(t)+\sqrt{dt}\sqrt{X(t)^2}\,N(0,1) \qquad \text{(DIS-1)}$$

<u>Part 1:</u>

Show that in each Δt time step starting with $X(t) > 0$, we have

$$\Pr\left(X(t+\Delta t)\le 0\right)=\Pr\left(N(0,1)\le\frac{-1}{\sqrt{\Delta t}}\right)=\frac{1}{2}\operatorname{erfc}\left(\frac{1}{\sqrt{2\Delta t}}\right)$$

where erfc(·) is the complementary error function

$$\operatorname{erfc}(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty \exp(-s^2)ds$$

<u>Part 2:</u>

Show that for $z > 0$, we have $\operatorname{erfc}(z)\le\frac{1}{z\sqrt{\pi}}\exp(-z^2)$.

<u>Hint:</u> Use change of variables $s = z + w$.

<u>Part 3:</u>

Consider discretization (DIS-1) on the grid

$$\Delta t=\frac{t}{N}, \qquad t_j=j\Delta t, \quad j=0,1,\ldots,N$$

Show that for $X(0) > 0$, we have

$$\Pr\left(\min_{1\le j\le N}X(t_j)\le 0\right)\le\frac{\sqrt{Nt}}{\sqrt{2\pi}}\exp\left(\frac{-N}{2t}\right)$$

which converges to zero exponentially as numerical resolution N is refined.

<u>Remark:</u>

Consider SDE $dX=b(X)dt+\sqrt{a(X)^2}\,dW$. Suppose $\sqrt{a(X)^2}\le|X|$ for small $|X|$.

The result above shows that

$$\Pr\left(\min X(t)\le 0\,\middle|\,X(0)>0\right)=0$$

Problem 4:

One approach for solving an SDE exactly is to use the λ-chain rule.

Suppose $X(t) = H(t, W(t))$. The λ-chain rule gives

$$dH(t,W(t)) = \left(H_t + (\tfrac{1}{2}-\lambda)H_{ww}\right)dt + H_w dW, \qquad \lambda = 0 \text{ for Ito}$$

We study the IVP of SDE below:

$$dX = \frac{4X^3}{(X^2+1)^3}dt + \frac{2X^2}{X^2+1}dW, \qquad X(0)=2 \qquad \text{(SDE-3)}$$

We notice that $\sqrt{a(X)^2} = \dfrac{2X^2}{X^2+1} \le |X|$ for small $|X|$, which implies $X(t) > 0$.

Use the method of λ-chain rule to solve (SDE-3).

<u>Hint:</u>

Identify H_w and view it as an ODE of $H(w)$ …

<u>Remark:</u>

This approach works only when X has the form $H(t, \mathrm{W}(t))$.

Problem 5:

Another approach for solving an SDE exactly is the integrating factor method.

This approach works when the SDE is linear.

We use the integrating factor method to solve the IVP of SDE below:

$$dX = \frac{X}{1+t}dt + dW, \qquad X(0)=1 \qquad \text{(SDE-4)}$$

Write $X(t)$ as an integral of dW.

Show that $X(t)$ has the normal distribution. Find $E(X(t))$ and $\mathrm{var}(X(t))$.