INTEGRABILITY OF SIEGEL TRANSFORMS AND AN APPLICATION

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Abstract. We give an algebraic characterization of the L^p-integrability, for $p = 1, 2, \infty$, of a natural generalization of the Siegel transform to the setting of rational representations of semisimple algebraic $\mathbb{Q}$-groups, extending Siegel's analytic work in the geometry of numbers.

As an application, on the space of real points of a generalized flag variety of rank one, we establish an effective asymptotic formula for the number of rational approximations at the Diophantine exponent to almost every point with respect to the Riemannian measure. The proof relies on the effective single and double equidistribution property of expanding orbits of maximal compact subgroups.

1. Introduction

The Siegel transform, introduced in 1945 by *Siegel* [33], maps a function of sufficient decay on the Euclidean space $\mathbb{R}^n$ to a function on the moduli space of unimodular lattices $\Omega = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, equipped with the unique $\mathrm{SL}_n(\mathbb{R})$-invariant probability measure μ_Ω, by averaging over the lattice. Let $\mathcal{P}(\mathbb{Z}^n)$ denote the set of primitive elements of $\mathbb{Z}^n$ and let $B_c^\infty(\mathbb{R}^n)$ be the space of Borel-measurable bounded compactly supported functions $f : \mathbb{R}^n \to \mathbb{C}$. For every $f \in B_c^\infty(\mathbb{R}^n)$, the *primitive* Siegel transform $Sf : \Omega \to \mathbb{C}$ of f is defined by

$$\forall\, g \in \mathrm{SL}_n(\mathbb{R}), \qquad Sf(g\mathbb{Z}^n) = \sum_{\boldsymbol{v}\in\mathcal{P}(\mathbb{Z}^n)} f(g\boldsymbol{v}). \tag{1.1}$$

Let ζ be the Riemann zeta function and let $\lambda_{\mathbb{R}^n}$ be the usual Lebesgue measure on $\mathbb{R}^n$. Siegel's mean value theorem [33] expresses the average of Sf in terms of the average of f:

$$\int_\Omega Sf \,\mathrm{d}\mu_\Omega = \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} f \,\mathrm{d}\lambda_{\mathbb{R}^n}. \tag{1.2}$$

Later, generalizing Siegel's result, *Rogers* [29] proved a k-th moment formula for the Siegel transform for k up to $n-1$. A remarkable application of the second moment formula was given by *Schmidt* [31], who derived asymptotic formulas for counting lattice points in an increasing family of sets in Euclidean space from the variance bound

$$\int_\Omega \left| Sf - \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} f \,\mathrm{d}\lambda_{\mathbb{R}^n} \right|^2 \mathrm{d}\mu_\Omega \ll \int_{\mathbb{R}^n} |f|^2 \,\mathrm{d}\lambda_{\mathbb{R}^n}. \tag{1.3}$$

In other words, the *centered* Siegel transform $\overline{S}f = Sf - \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} f \,\mathrm{d}\lambda_{\mathbb{R}^n}$ extends to a bounded operator $\overline{S} : L^2(\mathbb{R}^n) \to L^2(\Omega)$.

In this paper, we study the L^p-integrability, for $p = 1, 2, \infty$, of a natural generalization of the primitive Siegel transform (1.1). In particular, our results establish a framework for the development of higher moment formulas in the spirit of *Rogers*

[29] and have applications to metric Diophantine approximation on generalized flag varieties (see Theorem D below).

1.1. **Main results.** Let $\mathbf{G}$ be a connected simply-connected almost $\mathbb{Q}$-simple $\mathbb{Q}$-group and let $\mathbf{P}$ be a proper parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$. We denote algebraic varieties defined over $\mathbb{Q}$ by bold letters and their sets of real points by ordinary letters. For instance, we write $G = \mathbf{G}(\mathbb{R})$ to denote the group of real points of $\mathbf{G}$. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup of G. Let $\pi_\chi : \mathbf{G} \to \mathrm{GL}(\mathbf{V}_\chi)$ be an irreducible representation defined over $\mathbb{Q}$ which is generated by a rational line $\mathbf{D}_\chi$ of highest weight χ such that $\mathbf{P} = \mathrm{Stab}_{\mathbf{G}}\, \mathbf{D}_\chi$. We fix a highest weight vector $\boldsymbol{e}_\chi \in \mathbf{D}_\chi(\mathbb{Q})$ and define $\widetilde{X}$ to be the orbital set $\widetilde{X} = G\,\boldsymbol{e}_\chi \subset V_\chi$. Fix a Γ-stable lattice $\mathbf{V}_\chi(\mathbb{Z}) \subset \mathbf{V}_\chi(\mathbb{Q})$ of V_χ and denote by $\mathcal{P}_\chi$ the set of primitive elements of $\mathbf{V}_\chi(\mathbb{Z}) \cap \widetilde{X}$. Let $B_c^\infty(\widetilde{X})$ be the space of Borel measurable bounded compactly supported functions $f : \widetilde{X} \to \mathbb{C}$.

Definition 1.1 (Siegel transform)**.** For every $f \in B_c^\infty(\widetilde{X})$, we define the *Siegel transform* $S_\chi f : \Omega \to \mathbb{C}$ of f by

$$\forall\, g \in G, \quad S_\chi f(g\Gamma) = \sum_{\boldsymbol{v} \in \mathcal{P}_\chi} f(g\boldsymbol{v}).$$

Let μ_Ω be the unique G-invariant Borel probability measure on the homogeneous space $\Omega = G/\Gamma$. Given $p = 1, 2, \infty$, we will answer the question: *What are necessary and sufficient conditions for S_χ to map $B_c^\infty(\widetilde{X})$ into $L^p(\Omega)$?*

In our first result, the equivalences (1) - (4) are likely known to experts; the formula (1.4) below is simply a consequence of a general integration formula due to *Weil* [17, Theorem 2.51]. Let $\mathbf{L} = \mathrm{Stab}_{\mathbf{G}}\, \boldsymbol{e}_\chi \subset \mathbf{P}$.

Theorem A (L^1-integrability)**.** *The following assertions are equivalent.*

(1) The Siegel transform S_χ maps $B_c^\infty(\widetilde{X})$ into $L^1(\Omega)$.
(2) There exists a unique (up to scaling) G-invariant Radon measure $\lambda_{\widetilde{X}}$ on $\widetilde{X}$, the Siegel transform S_χ extends to a bounded operator $S_\chi : L^1(\widetilde{X}) \to L^1(\Omega)$, and $\lambda_{\widetilde{X}}$ can be normalized so that we have a convergent mean value formula:

$$\forall\, f \in L^1(\widetilde{X}), \quad \int_\Omega S_\chi f \; \mathrm{d}\mu_\Omega = \int_{\widetilde{X}} f \; \mathrm{d}\lambda_{\widetilde{X}}. \tag{1.4}$$

(3) The Lie group $L = \mathbf{L}(\mathbb{R})$ is unimodular and $\Gamma_L = \Gamma \cap L$ is a lattice in L.
(4) The parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ of $\mathbf{G}$ is maximal.
(5) There exists $\varepsilon > 0$ such that S_χ maps $B_c^\infty(\widetilde{X})$ into $L^{1+\varepsilon}(\Omega)$.

As for higher integrability, we derive the following algebraic and surprisingly easy-to-verify criteria. Let $\mathbf{P}_0$ be a minimal parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ contained in $\mathbf{P}$ and let $\mathbf{T}$ be a maximal $\mathbb{Q}$-split torus of $\mathbf{G}$ contained in $\mathbf{P}_0$. Let Δ be the associated set of simple roots of $\mathbf{G}$ relative to $\mathbf{T}$. For each subset θ of Δ, write $\mathbf{P}_\theta$ for the associated standard parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$.

Theorem B (L^2-integrability)**.** *If the Siegel transform S_χ maps $B_c^\infty(\widetilde{X})$ into $L^2(\Omega)$, then the parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ of $\mathbf{G}$ is maximal and the simple root $\alpha \in \Delta$, satisfying that $\mathbf{P} = \mathbf{P}_{\Delta \smallsetminus \{\alpha\}}$, has at most one neighbor in the associated Dynkin diagram.*

We were unable to determine whether the converse to Theorem B is true: *Assuming that the parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ of $\mathbf{G}$ is maximal and the simple root $\alpha \in \Delta$,* *satisfying that* $\mathbf{P} = \mathbf{P}_{\Delta\smallsetminus\{\alpha\}}$*, has at most one neighbor in the associated Dynkin diagram, does the Siegel transform* S_χ *map* $B_c^\infty(\widetilde{X})$ *into* $L^2(\Omega)$*?*

Theorem C (L^∞-integrability)**.** *The following assertions are equivalent.*

(1) The Siegel transform S_χ *maps* $B_c^\infty(\widetilde{X})$ *into* $L^\infty(\Omega)$*.*
(2) The $\mathbb{Q}$*-rank of* $\mathbf{G}$ *is* 1*.*
(3) The discrete group Γ_L *is a cocompact lattice in* L*.*

For every $f \in B_c^\infty(\widetilde{X})$, define the *centered Siegel transform* $\overline{S}_\chi f : \Omega \to \mathbb{C}$ of f by

$$\forall\, g \in G, \qquad \overline{S}_\chi f(g\Gamma) = S_\chi f(g\Gamma) - \int_{\widetilde{X}} f \;\mathrm{d}\lambda_{\widetilde{X}}.$$

Beyond the case $p = q = 1$, we were not able to determine for which pairs $p, q \in [1, +\infty]$ the Siegel transform, or its centered counterpart, extends to a bounded operator from $L^p(\widetilde{X})$ to $L^q(\Omega)$. More specifically, we would like to mention a question raised by *Saxcé*, suggesting a fractional version of the variance bound (1.3) that also takes into account point (5) in Theorem A, and Theorem B: *Assuming that the parabolic* $\mathbb{Q}$*-subgroup* $\mathbf{P}$ *is maximal, does there exist* $\varepsilon > 0$ *such that the centered Siegel transform* $\overline{S}_\chi$ *extends to a bounded operator*

$$\overline{S}_\chi : L^{1+\varepsilon}(\widetilde{X}) \to L^{1+\varepsilon}(\Omega)\,?$$

1.2. **Application to Diophantine approximation on flag varieties.** Many classical results in Diophantine approximation on the real line $\mathbb{R}$ or in Euclidean space $\mathbb{R}^n$ admit a dynamical reinterpretation in terms of properties of certain diagonal orbits in the space of lattices $\Omega = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$; this is known as Dani's correspondence [8]. Via this dynamical reinterpretation and building on influential work of *Margulis*, *Kleinbock* and others [22, 23, 24, 16], *Saxcé* [9] extended analogues of these classical results to generalized flag varieties, that is, varieties that can be written as the quotient $\mathbf{X} = \mathbf{G}/\mathbf{P}$ of a connected semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$ by a parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ of $\mathbf{G}$. First examples of such varieties include projective n-space $\mathbb{P}^n(\mathbb{R})$, the Grassmann variety $\mathrm{Gr}_{\ell,n}(\mathbb{R})$ of ℓ-dimensional subspaces of $\mathbb{R}^n$, projective quadric hypersurfaces (that is, the solution set in $\mathbb{P}^n(\mathbb{R})$ of a non-degenerate rational quadratic form in $n+1$ variables), and more general flag varieties, parametrizing flags of subspaces of a Euclidean space.

Let $\psi : \mathbb{N} \to (0, +\infty)$ be a non-increasing function. Khintchine's theorem [21] asserts that the inequality

$$0 \le qx - p < \psi(q)$$

admits infinitely (resp. finitely) many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$ for almost every $x \in \mathbb{R}$, if the series $\sum_{q=1}^{\infty} \psi(q)$ diverges (resp. converges). In the case the series is divergent, Schmidt [30] strengthened Khintchine's theorem. More precisely, for every $x \in \mathbb{R}$ and $T \ge 1$, he considered the counting function

$$\mathcal{N}_\psi(x, T) = \#\left\{(p, q) \in \mathbb{Z} \times \mathbb{N} : 0 \le qx - p < \psi(q),\ 1 \le q < T\right\} \tag{1.5}$$

and showed that for almost every $x \in \mathbb{R}$, $\mathcal{N}_\psi(x, T)$ is asymptotically equal to $\sum_{1 \le q < T} \psi(q)$ as T goes to infinity, with an explicit error term. In fact, Schmidt's result holds not only for the real line, but also for the Euclidean space $\mathbb{R}^n$ of any dimension $n \ge 1$.

Our goal is to prove a version of this theorem, where the Euclidean space $\mathbb{R}^n$ is replaced by the set of real points $X = \mathbf{X}(\mathbb{R})$ of a generalized flag variety $\mathbf{X}$ defined over $\mathbb{Q}$. To state our theorem, we need to introduce some more notation.

We assume that $\mathbf{P}$ is a maximal parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ with abelian unipotent radical. In particular, $\mathbf{X}$ has $\mathbb{Q}$-rank 1. We may, without loss of generality, assume that the group $\mathbf{G}$ is simply connected and almost $\mathbb{Q}$-simple. Let K be a maximal compact subgroup of G. Let σ_X be the unique K-invariant probability measure on X. We equip X with a K-invariant Riemannian distance $d(\cdot,\cdot)$ and the set of rational points $\mathbf{X}(\mathbb{Q})$ with a height function H_χ associated to an irreducible rational representation $\pi_\chi : \mathbf{G} \to \mathrm{GL}(\mathbf{V}_\chi)$ which is generated by a unique rational line $\mathbf{D}_\chi$ of highest weight χ such that $\mathrm{Stab}_{\mathbf{G}}\, \mathbf{D}_\chi = \mathbf{P}$ (see Section 2.2). By [9, Théorèmes 2.4.5 et 3.2.1], there exists a rational number $\beta_\chi > 0$ such that, for every $c > 0$ and for almost every $x \in X$, the inequality

$$d(x, v) < c\, H_\chi(v)^{-\tau} \tag{1.6}$$

admits infinitely (resp. finitely) many solutions $v \in \mathbf{X}(\mathbb{Q})$, if $\tau \le \beta_\chi$ (resp. $\tau > \beta_\chi$). We refer to β_χ as the *Diophantine exponent* of X with respect to χ.

In analogy to (1.5), for every constant $c > 0$, exponent $\tau \in [0, \beta_\chi]$, element $x \in X$, and parameter $T \ge 1$, we define

$$\mathcal{N}_{c,\tau}(x, T) = \#\left\{v \in \mathbf{X}(\mathbb{Q}) : d(x, v) < c\, H(v)^{-\tau},\, 1 \le H(v) < T\right\}.$$

In [28], we provided an almost-sure asymptotic formula for $\mathcal{N}_{c,\tau}(x, T)$ as $T \to +\infty$ – with an explicit error term in the case where $\tau \in [0, \beta_\chi)$. Our method did not yield an effective estimate when counting *at the Diophantine exponent*, that is, when $\tau = \beta_\chi$. In our application, we upgrade our previous result to an effective asymptotic estimate. Our approach is inspired by a recent effective counting result due to *Ouaggag* [27, Theorem 1.2] for spheres, and our result may be viewed as a substantial generalization thereof. We were unable to handle the case of a general decreasing approximation function ψ.

Theorem D (Effective counting at the Diophantine exponent)**.** *Let $\mathbf{G}$ be a connected simply connected almost $\mathbb{Q}$-simple $\mathbb{Q}$-group, $\mathbf{P}$ be a maximal parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ with abelian unipotent radical and $\mathbf{X} = \mathbf{G}/\mathbf{P}$. Let $d = \dim X$ be the dimension of X. Let H_χ be the height function on $\mathbf{X}(\mathbb{Q})$ associated to a dominant $\mathbb{Q}$-weight χ of $\mathbf{G}$ and let $c > 0$. Then there exists an explicit constant $\varkappa > 0$ and $\varepsilon > 0$ such that for σ_X-almost every $x \in X$, as $T \to +\infty$,*

$$\mathcal{N}_{c,\beta_\chi}(x, T) = \varkappa\, c^d \ln(T)\left(1 + O_x(\ln(T)^{-\varepsilon})\right). \tag{1.7}$$

In the course of establishing Theorem D, we also prove an effective equidistribution theorem for expanding translates of maximal compact subgroup orbits, derived from an analogous result for expanding translates of horospherical orbits due to *Björklund* and *Gorodnik* (see [4, Theorem 1.2]). Let $\alpha \in \Delta$ be the simple root corresponding to $\mathbf{P}$, that is, $\mathbf{P} = \mathbf{P}_{\Delta\smallsetminus\{\alpha\}}$, and let Y_α be the unique element in the Lie algebra of $\mathbf{T}(\mathbb{R})^\circ$ such that $\alpha(Y_\alpha) = -1$ and $\beta(Y_\alpha) = 0$ for all other simple roots $\beta \ne \alpha$. For all $y \in \mathbb{R}_+^\times$, put $a(y) = \exp(\ln(y) Y_\alpha)$. Then the one-parameter diagonal subgroup $A = \{a(y) : y \in \mathbb{R}_+^\times\}$ is the connected component of the center of the standard Levi subgroup of P. Let μ_K be the unique Haar probability measure on K.

Theorem E (Effective single and double equidistribution)**.** *There exist constants $c > 0$, $C > 0$ and an integer $r \ge 1$ such that for all $\phi \in C_c^\infty(\Omega)$ and $y \ge 1$, we have*

$$\left| \int_K \phi\big(a(y)k\big)\, d\mu_K(k) - \mu_\Omega(\phi) \right| \le C\, y^{-c} \mathcal{S}_r(\phi), \tag{1.8}$$

and, for all $\phi_1, \phi_2 \in C_c^\infty(\Omega)$ and $y_2 \geq y_1 \geq 1$, we have

(1.9)
$$\left| \int_K \phi_1\big(a(y_1)k\big)\phi_2\big(y(y_2)k\big)\,d\mu_K(k) - \mu_\Omega(\phi_1)\mu_\Omega(\phi_2) \right| \leq C\min\{y_1, y_2/y_1\}^{-c}\mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2).$$

where $\mathcal{S}_r$ denotes a certain degree r Sobolev norm on $C_c^\infty(\Omega)$.

1.3. **Proof sketch of Theorem D.** We take this opportunity to illustrate Theorem D in the special case of the Grassmann variety $X = \mathrm{Gr}_{\ell,n}(\mathbb{R})$ ($\ell, n \in \mathbb{N}$, $1 \leq \ell < n$), parametrizing ℓ-dimensional subspaces of the Euclidean space $\mathbb{R}^n$; this theorem also applies to projective quadric hypersurfaces and we refer the reader to [28, Sections 1.2 and 8] and the references therein. We sketch the proof of the argument, which involves introducing the Siegel transform in this specific setting, studying its analytic properties, and establishing equidistribution of expanding translates of orbits of maximal compact subgroups. This should help the reader become familiar with the definition of the Siegel transform and make the proof of Theorem D in the general case more accessible.

Let $\mathbf{G} = \mathrm{SL}_n$, let $\mathbf{T} \leq \mathbf{G}$ be the maximal $\mathbb{Q}$-split $\mathbb{Q}$-torus given by the subgroup of $\mathbf{G}$ consisting of all diagonal matrices, and let $\mathbf{P}_0$ be the Borel subgroup of $\mathbf{G}$ consisting of all upper-triangular matrices. Let $\Phi(\mathbf{G}, \mathbf{T})$ be the associated root system with ordering induced by $\mathbf{P}_0$, $\Delta = \{\alpha_1, \dots, \alpha_{n-1}\}$ the set of simple roots, and $\{\lambda_1, \dots, \lambda_{n-1}\}$ the set of fundamental $\mathbb{Q}$-weights. Fix $\alpha_\ell \in \Delta$ and let $\chi = \lambda_\ell$ be the associated fundamental $\mathbb{Q}$-weight. Recall that for all $a = \mathrm{diag}(a_1, \dots, a_n) \in \mathbf{T}$, we have $\chi(a) = a_1 \cdots a_\ell$. Let $\mathbf{P} = \mathbf{P}_{\Delta \smallsetminus \{\alpha_\ell\}}$ be the corresponding standard parabolic $\mathbb{Q}$-subgroup. Then $\mathbf{P}$ is the stabilizer in $\mathbf{G}$ of the rational line spanned by the pure tensor $\boldsymbol{e}_\chi = \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_\ell$ in the ℓ-th exterior power of the standard representation of $\mathbf{G}$. The Siegel transform in this case is defined as follows. Let $\widetilde{X} = G\,\boldsymbol{e}_\chi \subset \bigwedge^\ell \mathbb{R}^n$: this is the set of all non-zero pure tensors of $\bigwedge^\ell \mathbb{R}^n$. Let $\mathbf{L} = \mathrm{Stab}_{\mathbf{G}}\,\boldsymbol{e}_\chi$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ and let $\mathcal{P}_\chi$ be the set of all primitive elements of $\bigwedge^\ell \mathbb{Z}^n$ that are contained in $\widetilde{X}$. The group Γ acts transitively on $\mathcal{P}_\chi$: $\mathcal{P}_\chi = \Gamma\,\boldsymbol{e}_\chi \cong \Gamma/\Gamma_L$. Therefore, for every $f \in B_c^\infty(\widetilde{X})$, the Siegel transform $S_\chi f : G/\Gamma \to \mathbb{C}$ is given by

$$\forall\, g \in G, \qquad S_\chi f(g\Gamma) = \sum_{\boldsymbol{v} \in \mathcal{P}_\chi} f(g\boldsymbol{v}) = \sum_{\gamma \in \Gamma/\Gamma_L} f(g\gamma\boldsymbol{e}_\chi).$$

Then $X = G/P$, viewed as a subvariety of $\mathbb{P}(\bigwedge^\ell \mathbb{R}^n)$ via the embedding $gP \mapsto g[\boldsymbol{e}_\chi]$ (here $[\boldsymbol{e}_\chi]$ denotes the projectivization of $\boldsymbol{e}_\chi$), is the Grassmann variety $\mathrm{Gr}_{\ell,n}(\mathbb{R})$ of ℓ-dimensional subspaces of $\mathbb{R}^n$. This is in accordance with Schmidt's paper [32], where he used the Plücker embedding to define the height $H(v)$ of a rational subspace v of $\mathbb{R}^n$: for $v \in \mathrm{Gr}_{\ell,n}(\mathbb{Q})$ pick $\boldsymbol{v} \in \mathcal{P}_\chi$ with $v = [\boldsymbol{v}]$ and set $H(v) = \|\boldsymbol{v}\|$, where $\|\cdot\|$ denotes the $\mathrm{SO}_n(\mathbb{R})$-invariant norm on $\bigwedge^\ell \mathbb{R}^n$ induced from the standard Euclidean norm on $\mathbb{R}^n$. The distance used on X is the usual Riemannian distance and we equip X with the unique probability measure σ_X invariant under the action of the maximal compact subgroup $K = \mathrm{SO}_n(\mathbb{R}) \leq G$. We study the approximation of a real subspace chosen randomly according to σ_X by rational subspaces. Write d for the dimension $\dim_{\mathbb{R}} X = \ell(n-\ell)$. The Diophantine exponent of $X = \mathrm{Gr}_{\ell,n}(\mathbb{R})$ with respect to $\chi = \lambda_\ell$ is given by $\beta_\chi = \frac{n}{\ell(n-\ell)}$ (see [10, Théorème 1]). We wish to determine the asymptotic behavior of the counting function

$$\mathcal{N}_{c,\beta_\chi}(x, T) = \#\left\{v \in \mathrm{Gr}_{\ell,n}(\mathbb{Q}) : d(x, v) < c\,H(v)^{-\beta_\chi},\ H(v) < T\right\} \tag{1.10}$$

as $T \to +\infty$, for σ_X-almost every $x \in X$. In fact, Theorem D takes the following form in this special case.

Corollary 1.2. *Fix integers $1 \leq \ell < n$ and let $X = \mathrm{Gr}_{\ell,n}(\mathbb{R})$ be the Grassmann variety of ℓ-dimensional subspaces in $\mathbb{R}^n$. Then there exists an explicit constant $\varkappa > 0$ and $\varepsilon > 0$ such that for σ_X-almost every $x \in X$, as $T \to +\infty$,*

$$\mathcal{N}_{c,\beta}(x,T) = \varkappa\, c^d \ln(T)\left(1 + O_x(\ln(T)^{-\varepsilon})\right). \tag{1.11}$$

Let us now go through the main steps of the argument. For simplicity, we assume that $c = 1$ and we write $\mathcal{N}_{\beta_\chi}(x,T) = \mathcal{N}_{1,\beta_\chi}(x,T)$. The first observation is that the quantity $\mathcal{N}_{\beta_\chi}(x,T)$ can be understood as the Siegel transform of the indicator function of a certain subset $\mathcal{E}_{\beta_\chi}(T) \subset \widetilde{X}$ evaluated at a certain point in $\Omega = G/\Gamma$: we can associate to each $x \in X$ a rotation $k_x \in K$ such that

$$\mathcal{N}_{\beta_\chi}(x,T) = \#(\mathcal{P}_\chi \cap k_x\, \mathcal{E}_{\beta_\chi}(T)) = S_\chi \mathbb{1}_{\mathcal{E}_{\beta_\chi}(T)}(k_x^{-1}\Gamma).$$

By Theorem A, since $\mathbf{P} = \mathbf{P}_{\Delta \smallsetminus \{\alpha_\ell\}}$ is maximal, the group $L = \mathbf{L}(\mathbb{R})$ is unimodular, $\widetilde{X} = G/L$ admits a unique up to scaling Radon measure $\lambda_{\widetilde{X}}$ and the expected value of $S_\chi \mathbb{1}_{\mathcal{E}_\beta(T)}$, viewed as a random variable on Ω, is given by

$$\int_\Omega S_\chi \mathbb{1}_{\mathcal{E}_{\beta_\chi}(T)}\, \mathrm{d}\mu_\Omega = \int_{\widetilde{X}} \mathbb{1}_{\mathcal{E}_{\beta_\chi}(T)}\, \mathrm{d}\lambda_{\widetilde{X}} = \lambda_{\widetilde{X}}(\mathcal{E}_{\beta_\chi}(T)).$$

The hope is that, for σ_X-almost every $x \in X$, the quantity $\mathcal{N}_{\beta_\chi}(x,T)$ is asymptotically equal to the volume $\lambda_{\widetilde{X}}(\mathcal{E}_{\beta_\chi}(T))$, as $T \to +\infty$, and this is what we will show. In fact, the main term on the right-hand side in (1.11) is just the explicit value of the (main term of the) volume $\lambda_{\widetilde{X}}(\mathcal{E}_{\beta_\chi}(T))$. In order to prove the desired asymptotic estimate, we will exploit the special geometry of the set $\mathcal{E}_{\beta_\chi}(T)$. In fact, this set can be approximated by a set $\mathcal{E}_{\beta_\chi}(T)^+$ that admits a simple decomposition under the action of the diagonal subgroup

$$\forall\, y \in \mathbb{R}_+^\times, \quad a(y) = \mathrm{diag}\,\big(\underbrace{y^{-(n-\ell)/n}, \ldots, y^{-(n-\ell)/n}}_{\ell \text{ times}}, \underbrace{y^{\ell/n}, \ldots, y^{\ell/n}}_{n-\ell \text{ times}}\big).$$

Indeed, there exists a subset $\mathcal{F} \subset \widetilde{X}$ such that for all integers $N \geq 1$

$$\mathcal{E}_{\beta_\chi}(e^N)^+ = \bigsqcup_{i=0}^{N-1} a(e^{\beta_\chi})^{-i}\mathcal{F}. \tag{1.12}$$

On the level of the Siegel transform this yields the sum decomposition

$$S_\chi \mathbb{1}_{\mathcal{E}_{\beta_\chi}(T)^+}(k_x^{-1}\Gamma) = \sum_{i=0}^{N-1} \mathbb{1}_{\mathcal{F}}(a(e^{\beta_\chi})^i k_x^{-1}\Gamma).$$

From now on, we simply view $S_\chi \mathbb{1}_{\mathcal{E}_{\beta_\chi}(T)^+}(k\Gamma)$ as a random variable on the probability space (K, μ_K), where μ_K is the Haar probability measure of K. Up to dividing the right-hand side by N, it is a Birkhoff sum, but we will not take this viewpoint. Instead, we shall try to bound a quantity related to the variance of $S_\chi \mathbb{1}_{\mathcal{E}_{\beta_\chi}(T)^+}$ and then conclude by a Borel-Cantelli argument. More specifically, we shall bound a $(1+\varepsilon)$-moment, for some $\varepsilon > 0$, of the centered Siegel transform $S_\chi \mathbb{1}_{\mathcal{E}_{\beta_\chi}(T)^+} - \lambda_{\widetilde{X}}(\mathcal{E}_{\beta_\chi}(T)^+)$, viewed as a random variable on (K, μ_K): if we can show that for some $\varepsilon > 0$ and all $N \geq 1$,

$$\int_K \left| S_\chi \mathbb{1}_{\mathcal{E}_{\beta_\chi}(e^N)^+}(k\Gamma) - \lambda_{\widetilde{X}}(\mathcal{E}_{\beta_\chi}(e^N)^+)\right|^{1+\varepsilon} \mathrm{d}\mu_K(k) \ll N, \tag{1.13}$$

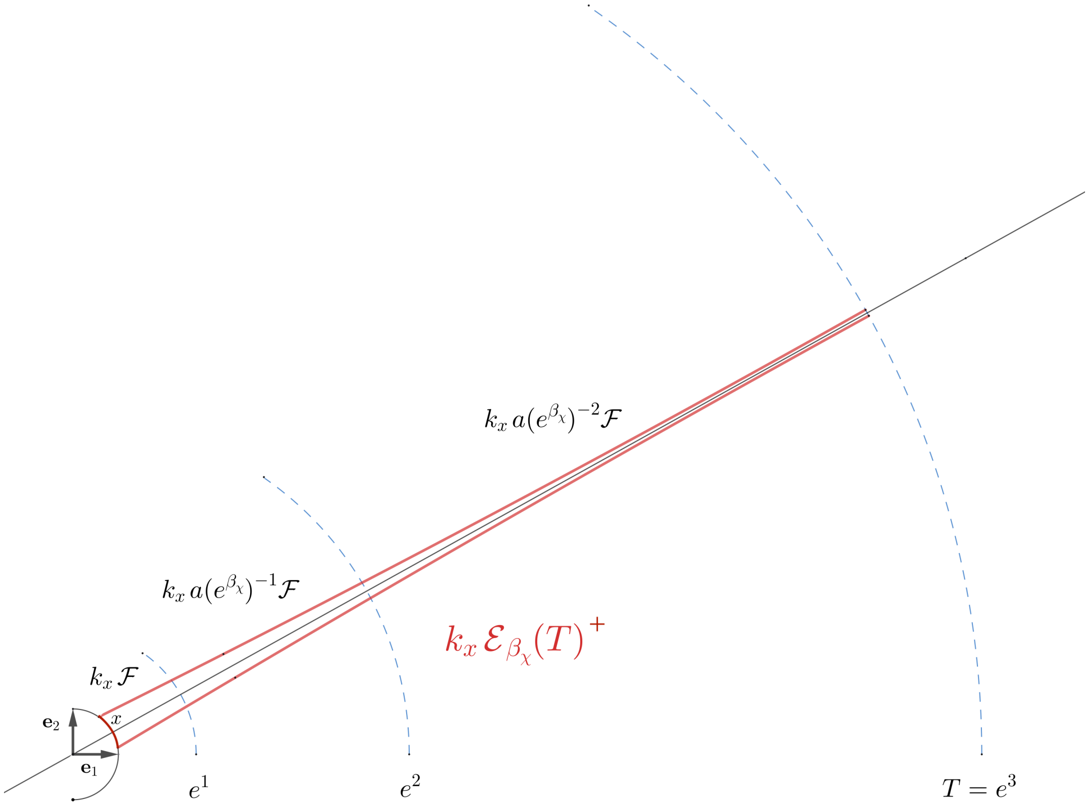

Figure 1. The set $\mathcal{E}_{\beta_\chi}(T)^+$ for the group $G = \mathrm{SL}_2(\mathbb{R})$, the flag variety the real projective line $X = \mathbb{P}^1(\mathbb{R}) = \mathrm{Gr}_{1,2}(\mathbb{R})$, the punctured affine cone $\widetilde{X} = \mathbb{R}^2 \setminus \{0\}$ above X, and the set $\mathcal{P}_\chi = \mathcal{P}(\mathbb{Z}^2)$ of primitive elements of $\mathbb{Z}^2$. Rational approximations to a point $x \in X$ of height bounded by T correspond to primitive lattice points of $\mathbb{Z}^2$ in the red region $k_x\mathcal{E}_{\beta_\chi}(T)$, where $k_x \in \mathrm{SO}_2(\mathbb{R})$ is a rotation such that $x = k_x[\boldsymbol{e}_1]$. The action of $a(y) = \mathrm{diag}(y^{-1/2}, y^{1/2})$ with $y > 1$ on $\widetilde{X} = \mathbb{R}^2 \setminus \{0\}$ contracts the line through $\boldsymbol{e}_\chi = \boldsymbol{e}_1$ and expands the line through $\boldsymbol{e}_2$. The domain $\mathcal{E}_{\beta_\chi}(T)^+$ can be decomposed into translates of the elementary domain $\mathcal{F}$ under the action of $a(y)$. The hope is that for x chosen randomly according to the Lebesgue measure on X the number of primitive lattice points in the red region $k_x\mathcal{E}_{\beta_\chi}(T)$, that is, the quantity $\#(\mathcal{P}_\chi \cap k_x\, \mathcal{E}_{\beta_\chi}(T)$, which is the classical primitive Siegel transform of the indicator function $\mathbb{1}_{\mathcal{E}_{\beta_\chi}(T)}$ evaluated at the rotated lattice $k_x^{-1}\mathbb{Z}^2$, is approximately given (up to a scalar) by the volume of $\mathcal{E}_{\beta_\chi}(T)$.

then there exists $c > 0$ and $\nu(\varepsilon) \in (0, 1)$ such that for μ_K-almost every $k \in K$,

$$S_\chi \mathbb{1}_{\mathcal{E}_{\beta_\chi}(e^N)^+}(k\Gamma) = c\, N\, \left(1 + O_x(N^{-\nu(\varepsilon)})\right),$$

as required. Due to integrability issues of the Siegel transform at this level of generality (see Theorems A and B), we are forced to work with $1 + \varepsilon$ for some small $\varepsilon > 0$ instead of 2, which would represent the usual variance. Using the decomposition (1.12), we express the argument in the integral of (1.13) as

$$S_\chi \mathbb{1}_{\mathcal{E}_{\beta_\chi}(e^N)^+}(k\Gamma) - \lambda_{\widetilde{X}}(\mathcal{E}_{\beta_\chi}(e^N)^+) = \sum_{i=0}^{N-1} \left(S_\chi \mathbb{1}_{\mathcal{F}}(a(e^{\beta_\chi})k_x^{-1}\Gamma) - \lambda_{\widetilde{X}}(\mathcal{F}) \right). \tag{1.14}$$

and obtain the bound in (1.13) using the effective single and double equidistribution property of expanding translates of K-orbits. In particular, we will need to work with smooth compactly supported functions that, on translated K-orbits, approximate the Siegel transform $S_\chi \mathbb{1}_{\mathcal{F}}$, which typically is neither smooth nor compactly supported.

1.4. **Notation and conventions.** We use the Landau notation $O(\cdot)$ and the Vinogradov symbol $\ll$. Given $A, B > 0$, we use the notation $A \gg B$ for $B \ll A$, and $A \asymp B$ for $A \ll B \ll A$. We use subscripts to indicate the dependence of the constant on parameters. For simplicity of exposition, we will work with the set of complex points of an algebraic variety defined over $\mathbb{Q}$, and refer to it simply as the variety itself when no confusion arises. For instance, we write $G = \mathbf{G}(\mathbb{R})$ and $\mathbf{G} = \mathbf{G}(\mathbb{C})$ to denote the groups of real and complex points of $\mathbf{G}$, respectively. Given a discrete subgroup $\Gamma \leq G$ and a closed subgroup $H \leq G$, we write Γ_H for $\Gamma \cap H$. Discrete groups are always equipped with the counting measure.

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2. Notation and preliminary results

Unless stated otherwise, we will always denote by $\mathbf{G}$ a connected semisimple $\mathbb{Q}$-group and by $\mathbf{P}$ a proper parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$. Let $\mathbf{P}_0$ be a minimal parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ contained in $\mathbf{P}$ and let $\mathbf{T}$ be a maximal $\mathbb{Q}$-split $\mathbb{Q}$-torus of $\mathbf{G}$ contained in $\mathbf{P}_0$. Let Φ, Δ and $\{\lambda_\alpha\}_{\alpha\in\Delta}$ be the set of roots of $\mathbf{G}$ relative to $\mathbf{T}$, with the ordering associated to $\mathbf{P}_0$, the set of simple roots and the set of relative fundamental $\mathbb{Q}$-weights (see [7, Section 12]), respectively. We let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup of G. We normalize the Haar measure μ_G on G so that the induced G-invariant measure μ_Ω on the quotient $\Omega = G/\Gamma$ is a probability measure.

2.1. **Structure of parabolic $\mathbb{Q}$-subgroups.** Let us record some facts concerning the structure of standard parabolic $\mathbb{Q}$-subgroups of $\mathbf{G}$ (see [5, Section 11.7]). Let $\mathbf{U}_0$ be the unipotent radical of $\mathbf{P}_0$. For each subset θ of Δ, we define the $\mathbb{Q}$-subtorus $\mathbf{T}_\theta$ of $\mathbf{T}$ to be the connected component of the intersection of the kernels of the $\alpha \in \theta$ and the parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_\theta$, containing $\mathbf{P}_0$, to be the product of the centralizer $\mathcal{Z}(\mathbf{T}_\theta)$ of $\mathbf{T}_\theta$ in $\mathbf{G}$ and $\mathbf{U}_0$. In fact, this group is a semi-direct product $\mathbf{P}_\theta = \mathcal{Z}(\mathbf{T}_\theta)\mathbf{U}_\theta$ of $\mathcal{Z}(\mathbf{T}_\theta)$ and its unipotent radical $\mathbf{U}_\theta$. Let $[\theta]$ be the set of $\mathbb{Q}$-roots that are linear combinations of elements of θ. Let $\mathbf{Q}$ be the largest connected $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of $\mathcal{Z}(\mathbf{T})$. There exist a connected semisimple $\mathbb{Q}$-subgroup $\mathbf{H}_\theta$ of $\mathcal{Z}(\mathbf{T}_\theta)$ and a connected $\mathbb{Q}$-subgroup $\mathbf{Q}_\theta$ of $\mathbf{Q}$ such that:

- the $\mathbb{Q}$-rank of $\mathbf{H}_\theta$ is equal to the number of elements of θ;
- $\mathbf{S}_\theta = (\mathbf{H}_\theta \cap \mathbf{T})^\circ$ is a maximal $\mathbb{Q}$-split torus of $\mathbf{H}_\theta$;
- $[\theta]$ is the system of $\mathbb{Q}$-roots of $\mathbf{H}_\theta$;

- $\mathcal{Z}(\mathbf{T}_\theta)$ is the almost direct product of $\mathbf{Q}_\theta$, $\mathbf{H}_\theta$, $\mathbf{T}_\theta$.

Let $\mathbf{M}_\theta$ be the identity component of the intersection of the kernels of the $\mathbb{Q}$-characters of $\mathcal{Z}(\mathbf{T}_\theta)$. By [5, Proposition 10.7, (b)], we have $X^*(\mathbf{M}_\theta)_\mathbb{Q} = \{1\}$, the $\mathbb{Q}$-character group of $\mathbf{M}_\theta$. Then $\mathbf{M}_\theta = \mathbf{Q}_\theta \mathbf{H}_\theta$ and $\mathbf{P}_\theta$ is the almost direct product:

$$\mathbf{P}_\theta = \mathbf{M}_\theta \, \mathbf{T}_\theta \, \mathbf{U}_\theta. \tag{2.1}$$

2.2. **Representations and height functions.** Assume that $\pi_\chi : \mathbf{G} \to \mathrm{GL}(\mathbf{V}_\chi)$ is an irreducible rational representation which is generated by a rational line $\mathbf{D}_\chi$ of highest $\mathbb{Q}$-weight $\chi \in X^*(\mathbf{T})$ such that $\mathbf{P} = \mathrm{Stab}_\mathbf{G} \, \mathbf{D}_\chi$. Such representations are referred to as strongly rational over $\mathbb{Q}$ and we refer the reader to [7, Section 12] for the details. We fix a highest weight vector $\boldsymbol{e}_\chi \in \mathbf{D}_\chi(\mathbb{Q})$ and denote by $x_0 = [\boldsymbol{e}_\chi] \in \mathbb{P}(V_\chi)$ the corresponding point in projective space. In particular, the space of real points $X = \mathbf{X}(\mathbb{R})$ of the generalized flag variety $\mathbf{X} = \mathbf{G}/\mathbf{P}$ can be identified with the orbital set $G\,[\boldsymbol{e}_\chi]$ via the map $gP \mapsto gx_0$. We define $\widetilde{X}$ to be the orbital set $\widetilde{X} = G\,\boldsymbol{e}_\chi$ in V_χ. By abuse of notation, we shall refer to $\widetilde{X}$ as the cone over X. Fix a Γ-stable lattice $\mathbf{V}_\chi(\mathbb{Z}) \subset \mathbf{V}_\chi(\mathbb{Q})$ of V_χ and denote by $\mathcal{P}_\chi$ the set of primitive elements of $\mathbf{V}_\chi(\mathbb{Z})$ that are contained in $\widetilde{X}$. Let K be a maximal compact subgroup of G whose Lie algebra is orthogonal that of $T = \mathbf{T}(\mathbb{R})$ (with respect to the Killing form on the Lie algebra $\mathfrak{g}$ of G). We equip V_χ with a Euclidean inner product $\langle \cdot, \cdot \rangle$ for which $\pi_\chi(g)$ is unitary (resp. self-adjoint) whenever $g \in K$ (resp. $g \in T$). We denote the induced norm by $\| \cdot \|$. We assume that $\|\boldsymbol{e}_\chi\| = 1$ and that $\boldsymbol{e}_\chi \in \mathcal{P}_\chi$. First, we define a height function H on $\mathbb{P}(\mathbf{V}_\chi)(\mathbb{Q})$ by $H([\boldsymbol{v}]) = \|\boldsymbol{v}\|$, where $\boldsymbol{v}$ is a primitive vector in the lattice $\mathbf{V}_\chi(\mathbb{Z})$ representing $[\boldsymbol{v}]$. Then, using the embedding ι_χ, we obtain a height function H_χ on $\mathbf{X}(\mathbb{Q})$, which is given by

$$\forall\, v \in \mathbf{X}(\mathbb{Q}), \quad H_\chi(v) = H(\iota_\chi(v)).$$

2.3. **Measures and coordinates.** In this subsection, we assume that the parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ of $\mathbf{G}$ is maximal. In particular, there exists a unique simple root $\alpha \in \Delta$ such that $\mathbf{P} = \mathbf{P}_{\Delta \smallsetminus \{\alpha\}}$ is the standard parabolic $\mathbb{Q}$-subgroup associated with the subset $\Delta \smallsetminus \{\alpha\}$ of Δ. In this case, we shall denote the almost direct product decomposition in (2.1) simply by $\mathbf{P} = \mathbf{M}\,\mathbf{A}\,\mathbf{U}$, where $\mathbf{M} = \mathbf{M}_{\Delta \smallsetminus \{\alpha\}}$, $\mathbf{A} = \mathbf{T}_{\Delta \smallsetminus \{\alpha\}}$, and $\mathbf{U} = \mathbf{U}_{\Delta \smallsetminus \{\alpha\}}$. We note that $\mathbf{L} = \mathrm{Stab}_\mathbf{G}\, \boldsymbol{e}_\chi$ satisfies $\mathbf{L}^\circ = \mathbf{M}\,\mathbf{U}$ and hence $\mathbf{L}^\circ$ does not admit any non-trivial $\mathbb{Q}$-characters. Hence, by a theorem of *Borel* and *Harish-Chandra* (see [6, Theorem 9.4]), the group $L = \mathbf{L}(\mathbb{R})$ is unimodular, the discrete subgroup $\Gamma_L = \Gamma \cap L$ is a lattice in L, and the quotient G/L, that we identify with the cone $\widetilde{X}$ over X via the orbital map $gL \mapsto g\boldsymbol{e}_\chi$, admits a unique (up to scaling) G-invariant Radon measure $\mu_{G/L} = \lambda_{\widetilde{X}}$. We let μ_L be the Haar measure on L, normalized so that the induced L-invariant measure on the quotient L/Γ_L is a probability measure and we normalize $\mu_{G/L}$ so that (see [17, Theorem 2.51]),

$$\forall\, f \in C_c(G), \quad \int_G f(g)\, \mathrm{d}\mu_G(g) = \int_{G/L} \int_L f(gl)\, \mathrm{d}\mu_L(l)\, \mathrm{d}\mu_{G/L}(gL). \tag{2.2}$$

By a slight abuse of notation, we denote by A the connected component with respect to the real topology of $\mathbf{A}(\mathbb{R})$. Let us parametrize A as follows. Let $\mathfrak{t}$ be the Lie algebra of $T = \mathbf{T}(\mathbb{R})$. There exists a unique element $Y_\alpha \in \mathfrak{t}$ such that $\alpha(Y_\alpha) = -1$ and $\beta(Y_\alpha) = 0$ for all other simple roots $\beta \in \Delta \smallsetminus \{\alpha\}$. We let

$$\forall\, y \in \mathbb{R}_+^\times, \quad a(y) = \exp(\log(y) Y_\alpha). \tag{2.3}$$

Then $A = \{a(y) : y \in \mathbb{R}_+^\times\}$. Let $d = \dim X$ be the dimension of X. Let μ_K be the Haar probability measure on K and $d\mu_A(a(y)) = y^{-1}\,\mathrm{d}y$ the push-forward to A of the Haar measure on $\mathbb{R}_+^\times$ via the map $y \mapsto a(y)$. By [28, Section 2.7], the group G admits an almost direct product decomposition $G = K\,A\,L$ and there exists a normalizing constant $\omega_0 > 0$ such that the Haar measure μ_G of G is given by

$$\mathrm{d}\mu_G = \omega_0\, y^{-(d+1)}\, \mathrm{d}\mu_K\ \mathrm{d}y\ \mathrm{d}\mu_L. \tag{2.4}$$

Moreover, let $K_L = K \cap L$ and let σ be the pushforward of the measure μ_K on K to K_L via the map $k \mapsto k\,K_L$. The map of $(K/K_L) \times A$ to $G/L = \widetilde{X}$ given by $(k\,K_L, a(y)) \mapsto ka(y)\boldsymbol{e}_\chi$ is a homeomorphism. In these coordinates, the measure $\mu_{G/L} = \lambda_{\widetilde{X}}$ is given by

$$\mathrm{d}\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi) = \omega_0\, y^{-(d+1)}\ \mathrm{d}\sigma(k)\ \mathrm{d}y. \tag{2.5}$$

2.4. **Distance on X.** For our application we also need to specify a probability measure σ_X and a distance $d(\cdot,\cdot)$ on X. By the Iwasawa decomposition $G = KP$, the group K acts transitively by left multiplication on X and we let σ_X be the unique K-invariant probability measure on X. Let $\mathbb{S} = \{\boldsymbol{x} \in V_\chi : \|\boldsymbol{x}\| = 1\}$ be the unit sphere in V_χ, viewed as a Riemannian submanifold of V_χ. The K-equivariant projection map $\mathbb{S} \to \mathbb{P}(V_\chi)$, $\boldsymbol{v} \mapsto [\boldsymbol{v}]$, induces a K-invariant Riemannian metric on $\mathbb{P}(V_\chi)$, and by restriction also on X. The associated Riemannian measure equals $\mathrm{vol}_{\mathrm{R}}(X)\,\sigma_X$, where $\mathrm{vol}_{\mathrm{R}}(X)$ is the total Riemannian volume of X. We denote the induced distance on X by $d(\cdot,\cdot)$. Let $\mathfrak{u}^-$ be the Lie algebra of the unipotent subgroup U^- opposite to P. Let $\phi : U^- \to X$ be the map given by $\phi(u) = ux_0$. By [28, Section 2.5], we have that $D_1\phi : \mathfrak{u}^- \to T_{x_0}X$ is a linear isomorphism. We equip $\mathfrak{u}^-$ with a Euclidean structure for which this isomorphism is an isometry and we denote the implied norm on $\mathfrak{u}^-$ by $\|\cdot\|_{\mathfrak{u}^-}$. Then, for all $u \in \mathfrak{u}^-$, we have that

$$d(x_0, \exp(u)x_0) = \|u\|_{\mathfrak{u}^-} + O\big(\|u\|_{\mathfrak{u}^-}^2\big), \tag{2.6}$$

where $\exp : \mathfrak{u}^- \to U^-$ is the exponential map (see [28, Lemma 2.1]).

3. Integrability of Siegel transforms

Let us recall the definition of the Siegel transform S_χ (see Definition 1.1): for every $f \in B_c^\infty(\widetilde{X})$, we defined $S_\chi f : \Omega \to \mathbb{C}$ by

$$\forall\, g \in G, \quad S_\chi f(g\Gamma) = \sum_{\boldsymbol{v} \in \mathcal{P}_\chi} f(g\boldsymbol{v}). \tag{3.1}$$

Since f is compactly supported, the sum on the right-hand side is finite, and hence converges absolutely. Also, since the subset $\mathcal{P}_\chi \subset \widetilde{X}$ is Γ-stable, the Siegel transform $S_\chi f$ is a well-defined function on $\Omega = G/\Gamma$.

3.1. **Some preliminary observations.** The action of Γ on the discrete set $\mathcal{P}_\chi$ is not transitive in general. However, as we will now show, $\mathcal{P}_\chi$ can always be expressed as a finite union of Γ-orbits. By a theorem of *Borel* and *Harish-Chandra* [5, Proposition 15.6], the set of double cosets $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$ is finite. Moreover, according to [5, Theorem 11.8], we have $(\mathbf{G}/\mathbf{P})(\mathbb{Q}) = \mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$. Thus, the set of rational points $(\mathbf{G}/\mathbf{P})(\mathbb{Q})$ of the generalized flag variety $\mathbf{G}/\mathbf{P}$ is a finite union of Γ-orbits. The orbit map $\mathbf{G} \to \mathbb{P}(\mathbf{V}_\chi)$ given by $g \mapsto gx_0$ induces an isomorphism $\mathbf{G}/\mathbf{P} \to \mathbf{G}\,x_0$ defined over $\mathbb{Q}$. We identify $\mathbf{G}/\mathbf{P}$ with $\mathbf{G}\,x_0$ via this isomorphism. Next, we note that there is a one-to-one correspondence between

$(\mathbf{G}/\mathbf{P})(\mathbb{Q})$ and lines passing through elements of $\mathcal{P}_\chi$. Hence there exist finitely many representatives $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathcal{P}_\chi$, with $\boldsymbol{v}_1 = \boldsymbol{e}_\chi$, such that

$$\mathcal{P}_\chi = \bigsqcup_{i=1}^{m} \Gamma\, \boldsymbol{v}_i. \tag{3.2}$$

Let us define, for every $f \in B_c^\infty(\widetilde{X})$, arithmetic subgroup $\Gamma' \subset \mathbf{G}(\mathbb{Q})$ of G and $\boldsymbol{v} \in \widetilde{X} \cap \mathbf{V}_\chi(\mathbb{Q})$, the *incomplete Eisenstein series* $E_{\chi,\Gamma',\boldsymbol{v}} f : G/\Gamma' \to \mathbb{C}$ by

$$\forall\, g \in G, \quad E_{\chi,\Gamma',\boldsymbol{v}} f(g\Gamma') = \sum_{\gamma \in \Gamma'/(\Gamma' \cap L_{\boldsymbol{v}})} f(g\gamma\boldsymbol{v}), \tag{3.3}$$

where $L_v = \mathrm{Stab}_G\, \boldsymbol{v}$. Letting $g_{\boldsymbol{v}} \in \mathbf{G}(\mathbb{Q})$ be such that $[\boldsymbol{v}] = g_{\boldsymbol{v}}[\boldsymbol{e}_\chi]$, we have $L_{\boldsymbol{v}} = g_{\boldsymbol{v}} L g_{\boldsymbol{v}}^{-1}$. For simplicity, we write $E_\chi = E_{\chi,\Gamma,\boldsymbol{e}_\chi}$. Hence, by (3.2), the Siegel transform $S_\chi f$ is a finite sum of incomplete Eisenstein series:

$$\forall\, g \in G, \quad S_\chi f(g\Gamma) = \sum_{i=1}^{m} E_{\chi,\Gamma,\boldsymbol{v}_i} f(g\Gamma)$$

The following lemma shows in particular that the fact that S_χ maps $B_c^\infty(\widetilde{X})$ into $L^p(\Omega)$ is independent of the choices of the arithmetic subgroup Γ and the Γ-stable lattice $\mathbf{V}_\chi(\mathbb{Z})$; The proof is in the appendix.

Lemma 3.1. *Let $p \in [1, +\infty]$. The following assertions hold.*

1. *The fact that $E_{\chi,\Gamma',\boldsymbol{v}}$ maps $B_c^\infty(\widetilde{X})$ into $L^p(G/\Gamma')$ is independent of the choice of the arithmetic subgroup Γ' and the rational element $\boldsymbol{v} \in \widetilde{X} \cap \mathbf{V}_\chi(\mathbb{Q})$.*
2. *The Siegel transform S_χ maps $B_c^\infty(\widetilde{X})$ into $L^p(\Omega)$ if and only if E_χ does so.*

Thus, when studying the integrability properties of S_χ, we may always work with E_χ instead. To prove item (1), we use that, for a fixed $\mathbb{Q}$-structure on $\mathbf{G}$, any two arithmetic subgroups $\Gamma, \Gamma' \subset \mathbf{G}(\mathbb{Q})$ of G are commensurable.

3.2. **L^1-integrability.** In this section, we prove Theorem A. We define

$$\forall\, g \in G, \quad \lambda_\chi(g\Gamma) = \min_{\boldsymbol{v} \in \mathbf{V}_\chi(\mathbb{Z}) \smallsetminus \{\mathbf{0}\}} \|g\boldsymbol{v}\|$$

to be the length of the shortest non-zero vector of $g\mathbf{V}_\chi(\mathbb{Z})$. For every $\mathbb{Q}$-weight $\mu \in X^*(\mathbf{T})$ of the representation π_χ let

$$\mathbf{V}^\mu = \{\boldsymbol{v} \in \mathbf{V}_\chi : \forall\, t \in \mathbf{T},\ \pi_\chi(t)\boldsymbol{v} = \mu(t)\boldsymbol{v}\}.$$

This is a $\mathbb{Q}$-subspace of $\mathbf{V}_\chi$. It is known that $\mathbf{V}^\chi$ is one-dimensional, that $\mathbf{V}_\chi$ is the direct sum of the linear subspaces $\mathbf{V}^\mu$,

$$\mathbf{V}_\chi = \bigoplus_{\mu} \mathbf{V}^\mu, \tag{3.4}$$

and that every $\mathbb{Q}$-weight of π_χ has the form

$$\mu = \chi - \sum_{\alpha \in \Delta} c_\alpha(\mu)\alpha \quad \text{with } c_\alpha(\mu) \in \mathbb{N}. \tag{3.5}$$

We may assume that $\mathbf{V}_\chi(\mathbb{Z})$ is the $\mathbb{Z}$-span of an orthonormal basis consisting of weight vectors for the action of $T = \mathbf{T}(\mathbb{R})$. Moreover, by Lemma 3.1, we may assume that Γ is given by the stabilizer in G of the lattice $\mathbf{V}_\chi(\mathbb{Z})$. In the following lemma, we bound the Siegel transform of a function $f \in B_c^\infty(\widetilde{X})$ in terms of λ_χ.

Lemma 3.2. *Suppose that the parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ is maximal. Then, for every $f \in B_c^\infty(\widetilde{X})$, we have*

$$\forall\, g \in G, \quad |S_\chi f(g\Gamma)| \ll_{\operatorname{supp}(f)} \|f\|_\infty\, \lambda_\chi(g\Gamma)^{-\beta_\chi d}. \tag{3.6}$$

Although this bound is not optimal, it is sufficient for our purposes.

Proof. We shall need the following consequence of the proof of [28, Theorem C]. For every $T \geq 1$, consider the function

$$\mathcal{N}(T) = \#\{v \in \mathbf{X}(\mathbb{Q}) : H_\chi(v) < T\}$$

counting rational points in X of height $< T$. Let $\beta_\chi \in \mathbb{Q}_{>0}$ be the Diophantine exponent of X with respect to χ (see [9, Définition 2.4.1 et Théorème 2.4.5]) and let $d = \dim X$ be the dimension of X. Then, as $T \to +\infty$, we have $\mathcal{N}(T) \sim T^{\beta_\chi d}$. Since there is a one-to-one correspondence between points in $\mathbf{X}(\mathbb{Q})$ and lines passing through $\mathcal{P}_\chi$, by the definition of the height function H_χ, we also have that, as $T \to +\infty$

$$\#\{\boldsymbol{v} \in \mathcal{P}_\chi : \|\boldsymbol{v}\| < T\} \asymp T^{\beta_\chi d}. \tag{3.7}$$

Fix $f \in B_c^\infty(\widetilde{X})$ and pick $r = r(\operatorname{supp}(f)) \geq 1$ such that $\operatorname{supp}(f)$ is contained in $B_{\widetilde{X}}(r) = \{\boldsymbol{v} \in \widetilde{X} : \|\boldsymbol{v}\| < r\}$. The proof now proceeds using reduction theory as presented, for instance, in [5, Section 12, Theorem 13.1]. By a slight abuse of notation, we let $\mathfrak{a}$ be the Lie algebra of T° and, for every $\tau \geq 0$, let $\mathfrak{a}_\tau = \{Y \in \mathfrak{a} : \forall\, \beta \in \Delta,\ \beta(Y) \leq \tau\}$. We set $A_\tau = \exp \mathfrak{a}_\tau$ and an note that $\mathfrak{a}^- = \mathfrak{a}_0$ is the negative Weyl chamber of $\mathfrak{a}$ with respect to Δ. Let $\mathbf{M}_0$ be the largest $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of the centralizer $\mathscr{Z}_{\mathbf{G}}(\mathbf{T})^\circ$ in $\mathbf{G}$ of $\mathbf{T}$ and let $\mathbf{U}_0$ be the unipotent radical of the minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_0$. There exist $\tau > 0$, a compact subset $\boldsymbol{\omega}$ of M_0U_0, and a finite subset $C \subset \mathbf{G}(\mathbb{Q})$ such that the Siegel set $\mathfrak{S} = K\, A_\tau\, \boldsymbol{\omega}$ satisfies

$$G = \mathfrak{S}\, C\, \Gamma.$$

In particular, we can express, though not uniquely, each $g \in G$ as $g = kanc\gamma$ with $k \in K$, $a \in A_\tau$, $n \in \boldsymbol{\omega}$, $c \in C$, and $\gamma \in \Gamma$. Fix any norm $\|\cdot\|_{\mathfrak{a}}$ on $\mathfrak{a}$ and, for $r_0 > 0$, let $B_{\mathfrak{a}}(r_0)$ denote the corresponding ball centered at the origin with radius r_0. Let $r_0 > 0$ be such that $\mathfrak{a}_\tau$ is contained in $\mathfrak{a}^- + B_{\mathfrak{a}}(r_0)$. Let $k \in K$, $n \in \boldsymbol{\omega}$, $a \in A_\tau$, and $\gamma \in \Gamma$. We express $a = a^- \exp(O(1))$ with $a^- \in \exp(\mathfrak{a}^-)$. Using that λ_χ is right Γ-invariant, that K is compact and that $\bigcup_{a \in A_\tau} a\boldsymbol{\omega}a^{-1}$ is relatively compact (see [5, Lemma 12.2]), we have

$$\lambda_\chi(kan\gamma\Gamma) \asymp \lambda_\chi(a^-\Gamma). \tag{3.8}$$

By the description of the $\mathbb{Q}$-weights of the representation π_χ in (3.5), for every $\mathbb{Q}$-weight μ of π_χ, we have

$$\chi(a^-) \leq \mu(a^-).$$

Hence, since we assumed $\mathbf{V}_\chi(\mathbb{Z})$ to be spanned over $\mathbb{Z}$ by an orthonormal basis consisting of weight vectors for the action of T, we have $\lambda_\chi(a^-) = \chi(a^-)$. Thus, for every $\boldsymbol{v} \in V_\chi$, we have $\lambda_\chi(a^-)\|\boldsymbol{v}\| \leq \|a^-\boldsymbol{v}\|$. Using that the norm $\|\cdot\|$ on V_χ is K-invariant, that $\bigcup_{a \in A_\tau} a\boldsymbol{\omega}a^{-1}$ is relatively compact, there exists a constant $C_0 \geq 1$, independent of f, such that, for every $g \in G$ with Siegel decomposition

$g = kan\gamma$ (and writing $a = a^- \exp(O(1))$ as above), we have

$$\begin{aligned}|S_\chi f(g\Gamma)| &\le \|f\|_\infty \#\{\boldsymbol{v} \in \mathcal{P}_\chi : \|g\boldsymbol{v}\| < r\} \\ &\le \|f\|_\infty \#\{\boldsymbol{v} \in \mathcal{P}_\chi : \|a^-\boldsymbol{v}\| < C_0\, r\} \\ &\le \|f\|_\infty \#\{\boldsymbol{v} \in \mathcal{P}_\chi : \|\boldsymbol{v}\| < C_0\, \lambda_\chi(a^-)^{-1}\, r\}.\end{aligned}$$

By the estimate in (3.7), we further have

$$\#\{\boldsymbol{v} \in \mathcal{P}_\chi : \|\boldsymbol{v}\| < C_0\, \lambda_\chi(a^-)^{-1}\, r\} \ll_{\mathrm{supp}(f)} \lambda_\chi(a^-)^{-\beta_\chi d}.$$

This together with (3.8) now implies that

$$|S_\chi f(g\Gamma)| \ll_{\mathrm{supp}(f)} \|f\|_\infty\, \lambda_\chi(g\Gamma)^{-\beta_\chi d},$$

finishing the proof of the lemma. □

Proof of Theorem A. We first show that (1) ⇒ (2). In view of the Riesz-Markov-Kakutani representation theorem, since $\Lambda(f) = \int_\Omega S_\chi f \, \mathrm{d}\mu$ defines a positive G-invariant linear functional on $B_c^\infty(\widetilde{X})$ by assumption, there exists a unique G-invariant Radon measure $\lambda_{\widetilde{X}}$ on $\widetilde{X}$ such that for all $f \in B_c^\infty(\widetilde{X})$,

$$\int_\Omega S_\chi f \, \mathrm{d}\mu_\Omega = \int_{\widetilde{X}} f \, d\lambda_{\widetilde{X}}. \tag{3.9}$$

In particular, we have $\|S_\chi f\|_{L^1(\Omega)} \le \|f\|_{L^1(\widetilde{X})}$ and S_χ extends to a bounded operator $S_\chi : L^1(\widetilde{X}) \to L^1(\Omega)$; the formula (3.9) continues to hold for all $f \in L^1(\widetilde{X})$.

To see the implication (2) ⇒ (3), we first note that since $\widetilde{X} = G/L$ carries a positive G-invariant Radon measure $\lambda_{\widetilde{X}}$, L must be a unimodular subgroup of G (see, for instance, [17, Theorem 2.51]). Fix a Haar measure μ_L on L and the counting measure on the discrete group $\Gamma_L = \Gamma \cap L$. By assumption, for every non-negative $f \in B_c^\infty(\widetilde{X})$, we have

$$\int_{G/\Gamma} \sum_{\gamma \in \Gamma/\Gamma_L} f(g\gamma \boldsymbol{e}_\chi) \, \mathrm{d}\mu_\Omega(g\Gamma) \le \int_\Omega S_\chi f \, \mathrm{d}\mu_\Omega < +\infty.$$

Using a standard folding/unfolding argument, there exists (unique up to scaling) G and L-invariant measures $\mu_{G/L}$ and μ_{L/Γ_L} on G/L and L/Γ_L, respectively, that can be normalized such that for every non-negative $f \in B_c^\infty(\widetilde{X})$,

$$\int_{G/\Gamma} \sum_{\gamma \in \Gamma/\Gamma_L} f(g\gamma \boldsymbol{e}_\chi) \, \mathrm{d}\mu_\Omega(g\Gamma) = \int_{G/L} \int_{L/\Gamma_L} f(gl\boldsymbol{e}_\chi) \, \mathrm{d}\mu_{L/\Gamma_L}(l\Gamma_L) \, \mathrm{d}\mu_{G/L}(gL).$$

Using that L stabilizes the vector $\boldsymbol{e}_\chi$, we further have

$$\int_{G/L} \int_{L/\Gamma_L} f(gl\boldsymbol{e}_\chi) \, \mathrm{d}\mu_{L/\Gamma_L}(l\Gamma_L) \, \mathrm{d}\mu_{G/L}(gL) = \mu_{L/\Gamma_L}(L/\Gamma_L) \int_{G/L} f(g\boldsymbol{e}_\chi) \, \mathrm{d}\mu_{G/L}(gL).$$

Choosing $f \in B_c^\infty(\widetilde{X})$ non-negative so that $\int_{G/L} f(g\boldsymbol{e}_\chi) \, \mathrm{d}\mu_{G/L}(g) > 0$, we have

$$\mu_{L/\Gamma_L}(L/\Gamma_L) \le \frac{\int_\Omega S_\chi f \, \mathrm{d}\mu}{\int_{G/L} f(g\boldsymbol{e}_\chi) \, \mathrm{d}\mu_{G/L}(g)} < +\infty,$$

which shows that Γ_L is a lattice in L, as required.

Next we show that (3) ⇔ (4). As recorded in Section 2.1, the group $\mathbf{P} = \mathbf{P}_\theta$ is an almost direct product $\mathbf{P} = \mathbf{M}_\theta \mathbf{T}_\theta \mathbf{U}_\theta$. The stabilizer $\mathbf{L}$ in $\mathbf{G}$ of $\boldsymbol{e}_\chi$ then satisfies $\mathbf{L}^\circ = \mathbf{M}_\theta(\mathbf{T}_\theta \cap \ker(\chi))^\circ \mathbf{U}_\theta$. Now, by a theorem of *Borel* and *Harish-Chandra* [6, Theorem 9.4], the Lie group L is unimodular and Γ_L is a lattice in L if and only if $\mathbf{L}^\circ$ does not admit any non-trivial $\mathbb{Q}$-characters. The group of $\mathbb{Q}$-characters $X^*(\mathbf{L}^\circ)_\mathbb{Q}$ can be identified, by restriction, with $X^*(\mathbf{M}_\theta(\mathbf{T}_\theta \cap \ker(\chi))^\circ)_\mathbb{Q}$. We claim that the latter is trivial if and only if the central $\mathbb{Q}$-split torus $(\mathbf{T}_\theta \cap \ker(\chi))^\circ$ is trivial. In fact, if $(\mathbf{T}_\theta \cap \ker(\chi))^\circ$ is trivial, then $\mathbf{M}_\theta$ does not admit any non-trivial $\mathbb{Q}$-characters since it is an almost direct product of a $\mathbb{Q}$-anisotropic and a semisimple $\mathbb{Q}$-subgroup (see Section 2.1). Conversely, suppose that the $\mathbb{Q}$-split torus $(\mathbf{T}_\theta \cap \ker(\chi))^\circ$ is non-trivial. Since $\mathbf{M}_\theta$ is a normal $\mathbb{Q}$-subgroup of $\mathcal{Z}(\mathbf{T}_\theta)$, the quotient map $q : \mathcal{Z}(\mathbf{T}_\theta) \to \mathcal{Z}(\mathbf{T}_\theta)/\mathbf{M}_\theta$ is a $\mathbb{Q}$-morphism of algebraic $\mathbb{Q}$-groups. Since its restriction to $\mathbf{T}_\theta$ is still surjective, we see that $\mathcal{Z}(\mathbf{T}_\theta)/\mathbf{M}_\theta$ is a non-trivial $\mathbb{Q}$-split torus (see, for instance, [5, Corollary10.4]). In particular, the composition of q with any non-trivial $\mathbb{Q}$-character of $\mathcal{Z}(\mathbf{T}_\theta)/\mathbf{M}_\theta$ gives a non-trivial $\mathbb{Q}$-character of $\mathcal{Z}(\mathbf{T}_\theta)$, as required. The torus $\mathbf{T}_\theta$ acts non-trivially on the line through $\boldsymbol{e}_\chi$ via the character χ. In particular, the quotient $\mathbf{T}_\theta/(\mathbf{T}_\theta \cap \ker(\chi))^\circ$ is one-dimensional. Thus $(\mathbf{T}_\theta \cap \ker(\chi))^\circ$ is trivial if and only if $\mathbf{T}_\theta$ is one-dimensional if and only if $\mathbf{P}$ is maximal, as claimed.

Next we show that (4) $\Rightarrow$ (5). By assumption, $\mathbf{P}$ is a maximal parabolic $\mathbb{Q}$-subgroup. We may assume that the action of Γ on $\mathcal{P}_\chi$ is transitive. Using a folding/unfolding argument similar to that in the proof of the implication (2) $\Rightarrow$ (3), for every $\varepsilon > 0$ and non-negative $f \in B_c^\infty(\widetilde{X})$, we have

$$\begin{aligned}\int_\Omega |S_\chi f|^{1+\varepsilon}\ \mathrm{d}\mu_\Omega &= \int_\Omega \sum_{\gamma\in\Gamma/\Gamma_L} \left(f(g\gamma\boldsymbol{e}_\chi)\left(S_\chi f(g\Gamma)\right)^\varepsilon\right)\ \mathrm{d}\mu_\Omega(g\Gamma)\\ &= \int_{G/L}\int_{L/\Gamma_L} f(gl\boldsymbol{e}_\chi)\left(S_\chi f(gl\Gamma)\right)^\varepsilon\ \mathrm{d}\mu_{L/\Gamma_L}(l\Gamma_L)\ \mathrm{d}\mu_{G/L}(gL)\\ &= \int_{G/L} f(g\boldsymbol{e}_\chi)\int_{L/\Gamma_L}\left(S_\chi f(gl\Gamma)\right)^\varepsilon\ \mathrm{d}\mu_{L/\Gamma_L}(l\Gamma_L)\ \mathrm{d}\mu_{G/L}(gL).\end{aligned}$$

Moreover, using the identification $\widetilde{X} = G/L = K/K_L \times A$, this further equals

$$\int_{\widetilde{X}} f(ka(y)\boldsymbol{e}_\chi)\int_{L/\Gamma_L}\left(S_\chi f(ka(y)l\Gamma)\right)^\varepsilon\ \mathrm{d}\mu_{L/\Gamma_L}(l\Gamma_L)\, d\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi).$$

By Lemma 3.2, we can estimate $|S_\chi f(g\Gamma)| \ll_{\mathrm{supp}(f)} \|f\|_\infty \lambda_\chi(g\Gamma)^{-\beta_\chi d}$. Putting everything together, we obtain, with $\varepsilon' = \varepsilon\beta_\chi d$,

(3.10)
$$\begin{aligned}&\int_\Omega |S_\chi f|^{1+\varepsilon}\ \mathrm{d}\mu_\Omega\\ &\ll_{\mathrm{supp}\, f} \|f\|_\infty^\varepsilon \int_{\widetilde{X}} f(ka(y)\boldsymbol{e}_\chi)\int_{L/\Gamma_L}\lambda_\chi(ka(y)l\Gamma)^{-\varepsilon'}\ \mathrm{d}\mu_{L/\Gamma_L}(l\Gamma_L)\,\mathrm{d}\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi).\end{aligned}$$

Since f is continuous and compactly supported and $\lambda_\chi(ka(y)l)^{-\varepsilon'} \ll \lambda_\chi(l)^{-\varepsilon'}$ for all $l \in L$ and all $(kK_L, a(y)) \in K/K_L \times A$ such that $ka(y)\boldsymbol{e}_\chi \in \mathrm{supp}\, f$, it suffices to show that, for $\varepsilon' > 0$ small enough,

$$\int_{L/\Gamma_L}\lambda_\chi(l\Gamma)^{-\varepsilon'}\ \mathrm{d}\mu_{L/\Gamma_L}(l\Gamma_L) < +\infty.$$

By the maximality of $\mathbf{P}$, we know from Section 2.1 (on the structure of parabolic subgroups) that $\mathbf{L}^\circ = \mathbf{M}\,\mathbf{U}$ is the semidirect product of the unipotent radical of $\mathbf{P}$ and a connected reductive $\mathbb{Q}$-subgroup $\mathbf{M}$ of $\mathbf{G}$, that, in turn, is an almost direct product $\mathbf{M} = \mathbf{Q}\,\mathbf{H}$ of a connected $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup $\mathbf{Q}$ and a semisimple $\mathbb{Q}$-subgroup $\mathbf{H}$ satisfying the following conditions:

- the $\mathbb{Q}$-rank of $\mathbf{H}$ equals $\operatorname{rank}_{\mathbb{Q}} \mathbf{G} - 1$;
- $\mathbf{T}_{\mathbf{H}} = (\mathbf{H} \cap \mathbf{T})^\circ$ is a maximal $\mathbb{Q}$-split torus of $\mathbf{H}$;
- $\mathbf{P}_{0,\mathbf{H}} = (\mathbf{H} \cap \mathbf{P}_0)^\circ$ is a minimal parabolic $\mathbb{Q}$-subgroup of $\mathbf{H}$;
- $K_H = K \cap H$ is a maximal compact subgroup of H;
- $\Delta_H = \Delta \smallsetminus \{\alpha\}$ is the set of simple roots with ordering associated to $\mathbf{P}_0'$;
- $[\Delta_H]$ is the system of $\mathbb{Q}$-roots of $\mathbf{H}$.

For any subgroup N of G, let us denote by Γ_N the intersection $\Gamma \cap N$. The subgroup $\Gamma_Q\,\Gamma_H\,\Gamma_U$ has finite index in Γ_L (see [6, Corollary 6.4]) and $Q\,H\,U$ has finite index in L. Without loss, we may assume that $\Gamma_L = \Gamma_Q\,\Gamma_H\,\Gamma_U$ and that $L = Q\,H\,U$. If Ω_Q, Ω_H, and Ω_U are fundamental sets (in the sense of [5, Definition 9.6]) for Γ_Q in Q, Γ_H in H, and Γ_U in U, respectively, then $\Omega_Q\,\Omega_H\,\Omega_U$ is a fundamental set for Γ_L in L. Moreover, we can choose the sets Ω_Q and Ω_U to be compact (since $\mathbf{Q}$ is $\mathbb{Q}$-anisotropic and $\mathbf{U}$ unipotent). Denoting μ_Q, μ_H, and μ_U the corresponding Haar measures, it suffices to show the finiteness of

$$\int_{\Omega_Q}\int_{\Omega_H}\int_{\Omega_U} \lambda_\chi(qhu\Gamma)^{-\varepsilon'}\,\mathrm{d}\mu_Q(q)\,\mathrm{d}\mu_H(h)\,\mathrm{d}\mu_U(u).$$

Since Ω_Q is compact and $\lambda_\chi(qg\Gamma) \asymp_{\Omega_Q} \lambda_\chi(qg\Gamma)$ for all $q \in \Omega_Q$ and $g \in G$, we further reduce to show that

$$\int_{\Omega_H}\int_{\Omega_U} \lambda_\chi(hu\Gamma)^{-\varepsilon'}\,\mathrm{d}\mu_H(h)\,\mathrm{d}\mu_U(u) \;<\; +\infty.$$

The argument now relies again on reduction theory (see [5, Section 12, Theorem 13.1]), but this time applied to Γ_H and H. We let $\mathfrak{a}_H$ be the Lie algebra of T_H° and, for every $\tau \geq 0$, let $\mathfrak{a}_{\tau,H} = \{Y \in \mathfrak{a}_H : \forall\,\beta \in \Delta_H,\ \beta(Y) \leq \tau\}$. We set $A_{\tau,H} = \exp\,\mathfrak{a}_{\tau,H}$ and an note that $\mathfrak{a}_H^- = \mathfrak{a}_{0,H}$ is the negative Weyl chamber of $\mathfrak{a}_H$ with respect to Δ_H. Let $\mathbf{M}_{0,\mathbf{H}}$ be the largest $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of the centralizer $\mathscr{Z}_{\mathbf{H}}(\mathbf{T}_H)^\circ$ in $\mathbf{H}$ of $\mathbf{T}_H$ and let $\mathbf{U}_{0,\mathbf{H}}$ be the unipotent radical of the minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_{0,\mathbf{H}}$. There exist $\tau > 0$, a compact subset $\boldsymbol{\omega}_H$ of $M_{0,H}U_{0,H}$, and a finite subset $C_H \subset \mathbf{H}(\mathbb{Q})$ such that the Siegel set $\mathfrak{S}_H = K_H\,A_{\tau,H}\,\boldsymbol{\omega}_H$ satisfies

$$H = \mathfrak{S}_H\,C_H\,\Gamma_H.$$

We let $\Omega_H = \mathfrak{S}_H\,C_H$. We claim that $\lambda_\chi(a\Gamma) \ll \lambda_\chi(hu\Gamma)$ for all $h = kanc$ with $k \in K_H$, $a \in A_{\tau,H}$, $n \in \boldsymbol{\omega}_H$, $c \in C_H$, and $u \in \Omega_U$. Indeed, using the fact that $\bigcup_{a\in A_{\tau,H}} a\boldsymbol{\omega}_H a^{-1}$ is relatively compact (see [5, Lemma 12.2]) and since $\min_{\boldsymbol{v}\in\mathbf{V}_\chi(\mathbb{Z})\smallsetminus\{\mathbf{0}\},\,c\in C_H,\,u\in\Omega_U}\|cu\boldsymbol{v}\| > 0$ (since C_H is finite, Ω_U is compact, and $\mathbf{V}_\chi(\mathbb{Z}) \smallsetminus \{\mathbf{0}\}$ is discrete), we have

$$\lambda_\chi(hu\Gamma) = \min_{\boldsymbol{v}\in\mathbf{V}_\chi(\mathbb{Z})\smallsetminus\{0\}} \|kan(a)^{-1}acu\boldsymbol{v}\| \gg \lambda_\chi(a) \min_{\boldsymbol{v}\in\mathbf{V}_\chi(\mathbb{Z})\smallsetminus\{0\},\,c\in C_H,\,u\in\Omega_U} \|cu\boldsymbol{v}\|,$$

which yields the claim. Let $\rho : \mathfrak{a}_H \to (0,+\infty)$ be the sum of the positive roots of H relative to T_H with multiplicities counted; it can be written as $\rho = \sum_{\beta\in\Delta_H} n_\beta\beta$ where the n_β are strictly positive integers. Let $\log : T_H^\circ \to \mathfrak{a}_H$ be the logarithm map. According to [25, Proposition 8.44], since we have a Cartan decomposition $H = K_H P_{0,H}^\circ$ with $P_{0,H}^\circ = M_{0,H}^\circ\,T_H^\circ\,U_{0,H}$, the corresponding Haar measures μ_{K_H}, $\mu_{M_{0,H}^\circ}$, $\mu_{T_H^\circ}$, and $\mu_{U_{0,H}}$ can be normalized so that

$$\mathrm{d}\mu_H(h) = e^{\rho(\log(a))}\;\mathrm{d}\mu_{K_H}(k)\;\mathrm{d}\mu_{M_{0,H}^\circ}(m)\;\mathrm{d}\mu_{T_H^\circ}(a)\;\mathrm{d}\mu_{U_{0,H}}(u).$$

Putting everything together and using the structure of the Siegel set $\mathfrak{S}_H$,

$$\int_{\Omega_H}\int_{\Omega_U} \lambda_\chi(hu\Gamma)^{-\varepsilon'} \, \mathrm{d}\mu_H(h)\,\mathrm{d}\mu_U(u) \ll \int_{A_{\tau,H}} \lambda_\chi(a)^{-\varepsilon'} e^{\rho(\log(a))} \, \mathrm{d}\mu_{T_H^\circ}(a).$$

The exponential map $\exp : \mathfrak{a}_H \to T_H^\circ$ is an isomorphism that carries a Lebesgue measure to a Haar measure. Hence for a suitable normalization of the Lebesgue measure $\mathrm{d}Y$ on $\mathfrak{a}_H$, we have

$$\int_{A_{\tau,H}} \lambda_\chi(a)^{-\varepsilon'} e^{\rho(\log(a))} \, \mathrm{d}\mu_{T_H^\circ}(a) = \int_{\mathfrak{a}_{\tau,H}} \lambda_\chi(\exp(Y))^{-\varepsilon'} e^{\rho(Y)} \, \mathrm{d}Y.$$

Since the group $\mathbf{H}$ is semisimple, the set of differentials associated to the elements $\beta \in \Delta_H$, which we continue to denote by $\beta : \mathfrak{a}_H \to \mathbb{R}$, forms a basis of the dual $\mathfrak{a}_H^*$. Via the identification of $\mathfrak{a}_H^*$ with $\mathfrak{a}_H$ (through the choice of an admissible inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}_H$), this gives a basis $(\beta^*)_{\beta\in\Delta_H}$ of $\mathfrak{a}_H$. Let $\mathcal{I}$ denote the set of $\mathbb{Q}$-weights on V_χ for the action of T. Since $\mathbf{V}_\chi(\mathbb{Z})$ is spanned by an orthonormal basis consisting of corresponding weight vectors, we observe that, for every $a \in T^\circ$ (in particular, for every $a \in A_{\tau,H}$), $\lambda_\chi(a) = \min_{\widehat{\chi}\in\mathcal{I}} \widehat{\chi}(a)$ and therefore

$$\lambda_\chi(a)^{-\varepsilon'} = \left(\min_{\widehat{\chi}\in I} \widehat{\chi}(a)\right)^{-\varepsilon'} = \max_{\widehat{\chi}\in I} \left(\widehat{\chi}(a)^{-\varepsilon'}\right) \le \sum_{\widehat{\chi}\in I} \widehat{\chi}(a)^{-\varepsilon'}.$$

Given a character $\widehat{\chi} \in \mathcal{I}$, let us also denote by $\widehat{\chi} : \mathfrak{a}_H \to \mathbb{R}$ the corresponding differential and let us express $\widehat{\chi} = \sum_{\beta\in\Delta_H} c_\beta(\widehat{\chi})\,\beta$ ($c_\beta(\widehat{\chi}) \in \mathbb{R}$). The claim now amounts to show that for every $\widehat{\chi} \in I$ the integral

$$\begin{aligned}\int_{\mathfrak{a}_{\tau,H}} \widehat{\chi}(\exp(Y))^{-\varepsilon'} e^{\rho(Y)} \, \mathrm{d}Y &= \int_{\mathfrak{a}_{\tau,H}} e^{\langle \rho - \varepsilon' \widehat{\chi}, Y\rangle} \, \mathrm{d}Y \\ &= \prod_{\beta\in\Delta_H} \int_{-\infty}^{\ln\tau} e^{\langle \rho-\varepsilon'\widehat{\chi}, x_{\beta^*}\beta^*\rangle} \, \mathrm{d}x_{\beta^*} \\ &= \prod_{\beta\in\Delta_H} \int_{-\infty}^{\ln\tau} e^{(n_\beta - \varepsilon' c_\beta(\widehat{\chi}))x_{\beta^*}} \, \mathrm{d}x_{\beta^*}\end{aligned}$$

converges. By choosing $\varepsilon' > 0$ sufficiently small, the implication (4) $\Rightarrow$ (5) follows.

Since (5) $\Rightarrow$ (1) is immediate, this concludes the proof of Theorem A. $\square$

3.3. **L^2-integrability.** In this section, we prove Theorem B. The following lemma turns the L^2-condition into an L^1-condition.

Lemma 3.3. *The Siegel transform S_χ maps $B_c^\infty(\widetilde{X})$ into $L^2(\Omega)$ if and only if*

$$\forall f \in B_c^\infty(\widetilde{X}) \qquad \int_{L/\Gamma_L} |S_\chi f| \, \mathrm{d}\mu_{L/\Gamma_L} < +\infty.$$

For any function $f : \widetilde{X} \to \mathbb{R}$ and $g \in G$, we let $g \cdot f : \widetilde{X} \to \mathbb{R}$ denote the function

$$\forall \boldsymbol{v} \in V_\chi, \quad (g\cdot f)(\boldsymbol{v}) = f(g^{-1}\boldsymbol{v}).$$

Proof. Suppose first that the Siegel transform S_χ maps $B_c^\infty(\widetilde{X})$ into $L^2(\Omega)$ (thus, in particular, into $L^1(\Omega)$). By Theorem A, the parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ must be maximal. Fix $f \in B_c^\infty(\widetilde{X})$; we need to show that $\int_{L/\Gamma_L} |S_\chi f| \, \mathrm{d}\mu_{L/\Gamma_L}$ converges.

Let $\rho \in B_c^\infty(\widetilde{X})$ be a non-negative K-invariant function such that $\rho(\boldsymbol{e}_\chi) > 0$ and such that for every $k \in K$ and $y \in [1/2, 3/2]$, $|f| \le (ka(y))^{-1} \cdot \rho$. By assumption,

$$\int_\Omega |S_\chi \rho|^2 \ \mathrm{d}\mu_\Omega < +\infty.$$

On the other hand, using the identification $\widetilde{X} = G/L = K/K_L \times A$ and the corresponding measure description of $\lambda_{\widetilde{X}}$ in Equation (2.5), by applying again a standard folding/unfolding argument, we have

$$\begin{aligned}
\int_\Omega |S_\chi \rho|^2 \ \mathrm{d}\mu_\Omega &= \int_{G/\Gamma} \sum_{\gamma \in \Gamma/\Gamma_L} \left(\rho(g\gamma \boldsymbol{e}_\chi)(S_\chi\rho)(g\Gamma)\right) \mathrm{d}\mu_\Omega(g\Gamma) \\
&= \int_{G/\Gamma_L} \rho(g\gamma \boldsymbol{e}_\chi)(S_\chi\rho)(g\Gamma) \ \mathrm{d}\mu_{G/\Gamma_L}(g\Gamma_L) \\
&= \int_{\widetilde{X}} \rho(ka(y)\boldsymbol{e}_\chi) \left(\int_{L/\Gamma_L} S_\chi\rho(ka(y)l\Gamma) \ \mathrm{d}\mu_{L/\Gamma_L}(l\Gamma_L) \right) d\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi).
\end{aligned}$$

Since $\rho(\boldsymbol{e}_\chi) > 0$ and ρ is K-invariant, by continuity there exists $\varepsilon \in (0, 1/2)$ such that $\rho(ka(y)\boldsymbol{e}_\chi) > 0$ for every $k \in K$ and $y \in [1-\varepsilon, 1+\varepsilon]$. Since the above integral converges, for $\lambda_{\widetilde{X}}$-almost every $ka(y)\boldsymbol{e}_\chi$ with $k \in K$ and $y \in [1-\varepsilon, 1+\varepsilon]$, we have

$$\int_{L/\Gamma_L} S_\chi\rho(ka(y)l) \ \mathrm{d}\mu_{L/\Gamma_L}(l) < +\infty.$$

Fix such an element $ka(y)\boldsymbol{e}_\chi$. By construction of ρ, since $|f| \le (ka(y))^{-1} \cdot \rho$,

$$\begin{aligned}
\int_{L/\Gamma_L} |S_\chi f| \ \mathrm{d}\mu_{L/\Gamma_L} &\le \int_{L/\Gamma_L} S_\chi((ka(y))^{-1} \cdot \rho) \ \mathrm{d}\mu_{L/\Gamma_L} \\
&= \int_{L/\Gamma_L} (S_\chi\rho)(ka(y)l\Gamma) \ \mathrm{d}\mu_{L/\Gamma_L}(l) < +\infty.
\end{aligned}$$

Let us now prove the other implication. Fix $f \in B_c^\infty(\widetilde{X})$. Since $\int_\Omega |S_\chi f|^2 \ \mathrm{d}\mu_\Omega \le \int_\Omega (S_\chi|f|)^2 \ \mathrm{d}\mu_\Omega$, we may without loss assume that f is non-negative. Let $\rho \in B_c^\infty(\widetilde{X})$ be a non-negative function with $\rho(\boldsymbol{e}_\chi) = 1$. Then, for every $l \in L$, we have $S_\chi\rho(l\Gamma) = \sum_{\gamma \in \Gamma/\Gamma_L} \rho(l\gamma\boldsymbol{e}_\chi) \ge 1$ and thus

$$\mu_{L/\Gamma_L}(L/\Gamma_L) \le \int_{L/\Gamma_L} S_\chi\rho(l\Gamma)\, \mu_{L/\Gamma_L}(l\Gamma_L) \ < \ +\infty,$$

showing that Γ_L is a lattice in L and hence, by Theorem A, that the parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ is maximal. In particular, the measure description of $\lambda_{\widetilde{X}}$ in Equation (2.5) applies again. Using that f is non-negative, by the same argument as before,

$$\int_\Omega |S_\chi f|^2 \ \mathrm{d}\mu_\Omega = \int_{\widetilde{X}} f(ka(y)\boldsymbol{e}_\chi) \left(\int_{L/\Gamma_L} S_\chi f(ka(y)l\Gamma) \ \mathrm{d}\mu_{L/\Gamma_L}(l\Gamma_L) \right) d\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi).$$

By assumption, the assignment $(kK_L, a(y)) \mapsto \int_{L/\Gamma_L} S_\chi f(ka(y)l\Gamma) \ \mathrm{d}\mu_{L/\Gamma_L}(l\Gamma_L)$ defines a function $K/K_L \times A \to \mathbb{R}$. Since the support of f is compact, in order to conclude, it suffices to establish the continuity of this function, which is in fact a consequence of Lebesgue's dominated converges theorem. $\square$

Let W be the Weyl group, obtained as the quotient of the normalizer in $\mathbf{G}$ of $\mathbf{T}$ by its centralizer. We also fix an admissible (in particular, W-invariant) inner product $\langle \cdot, \cdot \rangle$ on $X^*(\mathbf{T}) \otimes \mathbb{R}$ and recall that two simple roots $\beta_1, \beta_2 \in \Delta$ with $\beta_1 \neq \beta_2$

are said to be *neighbors* if $\langle \beta_1, \beta_2 \rangle \neq 0$. We identify $X^*(\mathbf{T}) \otimes \mathbb{R}$ with the dual $\mathfrak{a}^*$ of the Lie algebra $\mathfrak{a}$ of T.

Proof of Theorem B. So suppose that the Siegel transform S_χ maps $B_c^\infty(\widetilde{X})$ into $L^2(\Omega)$. Since $L^2(\Omega)$ is contained in $L^1(\Omega)$, Theorem A implies that $\mathbf{P}$ is a maximal parabolic $\mathbb{Q}$-subgroup. Hence there exists a unique simple root $\alpha \in \Delta$, such that $\mathbf{P} = \mathbf{P}_{\Delta \smallsetminus \{\alpha\}}$. We need to show that the root α has at most one neighbor in the Dynkin diagram associated to the root system Φ of $\mathbf{G}$ relative to $\mathbf{T}$.

We first show that, for every $w \in W$, the group $(w\mathbf{L}w^{-1} \cap \mathbf{L})^\circ$ does not admit any non-trivial $\mathbb{Q}$-characters. Let $w \in W$. Observe that the orbit of $\boldsymbol{e}_\chi$ under W is contained in $\mathcal{P}_\chi$. Consider the element $w\boldsymbol{e}_\chi \in \mathcal{P}_\chi$. Its stabilizer in L is given by $L_w = wLw^{-1} \cap L$. Using Lemma 3.3 and the fact that $\Gamma_L \cdot w\, \boldsymbol{e}_\chi \subseteq \mathcal{P}_\chi$, for every non-negative function $f \in B_c^\infty(\widetilde{X})$,

$$\int_{L/\Gamma_L} \sum_{\gamma \in \Gamma_L/(\Gamma \cap L_w)} f(l\gamma w\boldsymbol{e}_\chi)\, \mathrm{d}\mu_{L/\Gamma_L} \leq \int_{L/\Gamma_L} S_\chi f\, \mathrm{d}\mu_{L/\Gamma_L} < +\infty.$$

Therefore,

$$\Lambda_w(f) = \int_{L/\Gamma_L} \sum_{\gamma \in \Gamma_L/(\Gamma \cap L_w)} f(l\gamma w\boldsymbol{e}_\chi)\, \mathrm{d}\mu_{L/\Gamma_L}$$

defines a positive L-invariant linear functional on $B_c^\infty(\widetilde{X})$, inducing a unique L-invariant Radon measure λ_w on $\widetilde{X}$, whose support is contained in the closure of the orbit $L \cdot w\boldsymbol{e}_\chi$. Therefore $L \cdot w\boldsymbol{e}_\chi = L/L_w$ carries an L-invariant Radon measure and since L is unimodular, this implies that L_w is also unimodular. Hence, applying again a standard folding/unfolding argument, there exists an L-invariant Radon measure μ_{L/L_w} on L/L_w that can be normalized such that, for every non-negative $f \in B_c^\infty(\widetilde{X})$,

$$\begin{aligned}\Lambda_w(f) &= \int_{L/L_w} \int_{L_w/(\Gamma \cap L_w)} f(ll'w\boldsymbol{e}_\chi)\, \mathrm{d}\mu_{L_w/(\Gamma\cap L_w)}(l'(\Gamma \cap L_w))\, \mathrm{d}\mu_{L/L_w}(lL_w) \\ &= \mu_{L_w/(\Gamma \cap L_w)}(L_w/(\Gamma \cap L_w)) \int_{L/L_w} f(lw\boldsymbol{e}_\chi)\, \mathrm{d}\mu_{L/L_w}(lL_w).\end{aligned}$$

It follows that $\Gamma \cap L_w$ is a lattice in L_w. Thus, by a theorem of *Borel* and *Harish-Chandra* [6, Theorem 9.4], $(w\mathbf{L}w^{-1} \cap \mathbf{L})^\circ$ does not admit any non-trivial $\mathbb{Q}$-characters, as was claimed.

Denote the set of neighbors of α by $V(\alpha)$ and set $B(\alpha) = V(\alpha) \cup \{\alpha\}$. Let $s_\alpha \in W$ be the reflection across the hyperplane defined by α. We claim that

$$\mathbf{L}_\alpha = \left(s_\alpha \mathbf{L} s_\alpha^{-1} \cap \mathbf{L}\right)^\circ \subseteq \mathbf{P}_{\Delta \smallsetminus B(\alpha)}.$$

Since $\mathbf{L}_\alpha$ is normalized by $\mathbf{T}$, its Lie algebra decomposes as a direct sum of a maximal toral subalgebra and root spaces. By the description of $\mathbf{P}_{\Delta \smallsetminus B(\alpha)}$ in terms of the simple roots $\Delta \smallsetminus B(\alpha)$, it suffices to check that every negative root $\lambda = \sum_{\beta \in \Delta} n_\beta \beta$ arising in the adjoint representation of $\mathbf{T}$ on $\mathbf{L}_\alpha$ has $n_\beta = 0$ for all $\beta \in B(\alpha)$. Now $\mathbf{L}_\alpha$ is s_α-invariant, hence $s_\alpha(\lambda)$ is again a root of $\mathbf{L}_\alpha$. Moreover $s_\alpha(\lambda)$ cannot have $-\alpha$ in its support (since $\mathbf{L}_\alpha \subseteq \mathbf{L} \subseteq \mathbf{P}_{\Delta \setminus \alpha}$). Applying s_α yields

$$s_\alpha(\lambda) = \sum_{\beta \in \Delta \smallsetminus \{\alpha\}} n_\beta s_\alpha(\beta) = \left(\sum_{\beta \in \Delta \smallsetminus \{\alpha\}} n_\beta \beta\right) - \left(\sum_{\beta \in V(\alpha)} \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} n_\beta\right) \alpha.$$

Thus $n_\beta = 0$ for all $\beta \in V(\alpha)$, proving that $\mathbf{L}_\alpha \subseteq \mathbf{P}_{\Delta\setminus B(\alpha)}$. Moreover, the $\mathbb{Q}$-rank of $\mathbf{L}_\alpha$ is at least $\operatorname{rank}\mathbb{Q}\,\mathbf{G}-2$, since it contains $\mathbf{T}\cap\ker\chi\cap\ker(s_\alpha\cdot\chi)$. On the other hand, the $\mathbb{Z}$-rank of $X^*(\mathbf{P}_{\Delta\setminus B(\alpha)})_{\mathbb{Q}}$ equals $|B(\alpha)|$. If $|V(\alpha)|>1$ (so $|B(\alpha)|>2$), then $\mathbf{L}_\alpha$ would admit a nontrivial $\mathbb{Q}$-character. Hence $|V(\alpha)|\le 1$, which establishes the implication (1) $\Rightarrow$ (2) of Theorem B. $\square$

3.4. **L^∞-integrability.** In this section, we prove Theorem C.

Proof of Theorem C. We first show that (1) $\Leftrightarrow$ (2). Suppose that the $\mathbb{Q}$-rank of $\mathbf{G}$ is 1. In particular, the set of simple roots Δ consists of a single element, say α, and $\mathbf{P}$ is the standard parabolic $\mathbb{Q}$-subgroup associated to the empty set $\Delta\smallsetminus\{\alpha\}=\varnothing$, and hence equals the minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_0$. We need to show that S_χ maps $B_c^\infty(\widetilde{X})$ into $L^\infty(\Omega)$. Since any bounded compactly supported function f on $\widetilde{X}$ can be bounded by $\|f\|_\infty \mathbb{1}_{B_{\widetilde{X}}(r)}$ for some large radius $r>0$ depending only on the support of f, it suffices to show that $S_\chi \mathbb{1}_{B_{\widetilde{X}}(r)} \in L^\infty(\Omega)$. We note that in this case $T=\mathbf{T}(\mathbb{R})^\circ$ is one-dimensional and equals the subgroup A given as the identity component of $\mathbf{T}_{\Delta\smallsetminus\{\alpha\}}(\mathbb{R})$ (see Section 2.1). By Lemma **??**, we may assume that Γ acts transitively on $\mathcal{P}_\chi$. Using the reduction theory for Γ in G, as detailed in Section **??**, we have $G=\mathfrak{S}\,C\,\Gamma$, where $\mathfrak{S}=KA_\tau\boldsymbol{\omega}$ is a Siegel set and the set A_τ can be described as $A_\tau=\{a(y)\in A:\alpha(a(y))=y^{-1}\le e^\tau\}$ for some $\tau>0$. Since Γ acts transitively on $\mathcal{P}_\chi$, it also acts transitively on the set of rational points of $\mathbf{X}=\mathbf{G}/\mathbf{P}$. Hence, by [5, Proposition 15.6], we have $C=\{1\}$. This allows us to express g as $g=ka(y)n\gamma$, where $ka(y)n\in\mathfrak{S}=KA_\tau\boldsymbol{\omega}$, and $\gamma\in\Gamma$. Using that $B_{\widetilde{X}}(r)$ is K-invariant and that $\bigcup_{a(y)\in A_\tau} a(y)\boldsymbol{\omega}a(y)^{-1}$ is relatively compact, there exists $R\ge r$, depending on r, the choice of the Siegel set $\mathfrak{S}$ and the finite set C, such that

$$S_\chi \mathbb{1}_{B_{\widetilde{X}}(r)}(g)=\#(B_{\widetilde{X}}(r)\cap g\mathcal{P}_\chi)\le \#(B_{\widetilde{X}}(R)\cap a(y)\mathcal{P}_\chi).$$

Since the $\mathbb{Q}$-rank of $\mathbf{G}$ is 1, the Weyl group $W\cong\mathbb{Z}/2\mathbb{Z}$ consists of 2 elements and we let $w\in N_G(\mathbf{T})(\mathbb{Q})$ be a representative of the non-trivial element in W. According to [7, Théorème 5.15], one has the Bruhat decomposition $G=P\sqcup PwP$. Since $w\cdot\chi=-\chi$, the element $a(y)$ acts on $\boldsymbol{e}_{-\chi}=w\boldsymbol{e}_\chi$ by

$$a(y)w\boldsymbol{e}_\chi = ww^{-1}a(y)w\boldsymbol{e}_\chi = w\chi(w^{-1}a(y)w)\boldsymbol{e}_\chi = w(w\cdot\chi)(a(y))\boldsymbol{e}_\chi = y^{-\chi(Y_\alpha)}\boldsymbol{e}_{-\chi},$$

and since $\chi(Y_\alpha)<0$ this action expands $\boldsymbol{e}_{-\chi}$ as $y\to+\infty$. We may assume that $\boldsymbol{e}_{-\chi}$ is in the basis $\mathcal{B}$ for V_χ. We first show that any $\boldsymbol{v}\in\mathcal{P}_\chi$ with $\boldsymbol{v}\ne\pm\boldsymbol{e}_\chi$ satisfies $\|a(y)\boldsymbol{v}\|\to+\infty$ as $y\to+\infty$. This follows if we can show that for any such $\boldsymbol{v}$ one has $|\langle\boldsymbol{v},\boldsymbol{e}_\chi\rangle|\ge 1$. Using the Bruhat decomposition $G=P\sqcup PwP$, there exist $p_1,p_2\in P$ such that $\boldsymbol{v}=p_1wp_2\boldsymbol{e}_\chi$. Now, applying $a(y)$, one gets

$$a(y)\boldsymbol{v}=a(y)p_1a(y)^{-1}ww^{-1}a(y)w\chi(p_2)\boldsymbol{e}_\chi = a(y)p_1a(y)^{-1}wy^{-\chi(Y_\alpha)}\chi(p_2)\boldsymbol{e}_\chi.$$

Noting that $a(y)p_1a(y)^{-1}w$ converges to some element $p_\infty w$ with $p_\infty\in P$ as y tends to infinity, we deduce that $\|a(y)\boldsymbol{v}\|$ grows at the highest possible rate: $y^{-\chi(Y_\alpha)}$. Hence using that $\langle\boldsymbol{v},\boldsymbol{e}_{-\chi}\rangle\in\mathbb{Z}$, we must have $|\langle\boldsymbol{v},\boldsymbol{e}_{-\chi}\rangle|\ge1$. Therefore, choosing y so large that $y^{-\chi(Y_\alpha)}>R$, we have for all $\boldsymbol{v}\in\mathcal{P}_\chi\smallsetminus\{\pm\boldsymbol{e}_\chi\}$,

$$\|a(y)\boldsymbol{v}\|\ge\|a(y)\boldsymbol{e}_{-\chi}\|=y^{-\chi(Y_\alpha)}>R.$$

Hence for all y large enough, $B_{\widetilde{X}}(R)\cap a(y)\mathcal{P}_\chi=\{\pm\boldsymbol{e}_\chi\}$. Since $f\in B_c^\infty(\widetilde{X})$ was arbitrary, this shows that the Siegel transform S_χ maps $B_c^\infty(\widetilde{X})$ into $L^\infty(\Omega)$.

Conversely, suppose that we have $\operatorname{rank}_{\mathbb{Q}} \mathbf{G} \geq 2$. Let $\mathfrak{a}$ be the Lie algebra of $\mathbf{T}(\mathbb{R})^\circ$. The negative Weyl chamber is defined by

$$\mathfrak{a}^- = \{X \in \mathfrak{a} : \forall \beta \in \Delta,\ \beta(X) < 0\}.$$

Fix a K-invariant non-negative function $\rho \in B_c^\infty(\widetilde{X})$ such that $\rho \geq \mathbb{1}_{B_{\widetilde{X}}(1)}$. To show that the Siegel transform S_χ does not map $B_c^\infty(\widetilde{X})$ into $L^\infty(\Omega)$, it suffices to show that $S_\chi \rho$ is unbounded on Ω. By continuity of ρ (and hence of $S_\chi \rho$), it further suffices to show that $S_\chi \rho$ is unbounded on $A^- = \exp \mathfrak{a}^-$. Roughly speaking, we would like to find $a \in A^-$ such that $a\mathcal{P}_\chi$ has many short vectors. Let $\mathbf{U}_0$ be the unipotent radical of the minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_0$. Since the orbit of $\boldsymbol{e}_\chi$ under W is contained in $\mathcal{P}_\chi$, it suffices to find $w \in W$ and $Y \in \mathfrak{a}^-$ such that $\Gamma_{U_0} w^{-1} \boldsymbol{e}_\chi \subseteq \mathcal{P}_\chi$ is infinite and gets contracted by $\widehat{a}(y) = \exp(\log(y)Y) \in A^-$ as $y \to +\infty$. For any $Y \in \mathfrak{a}^-$, $u \in U_0$, $w \in W$, we have

$$\begin{aligned}\|\widehat{a}(y)uw^{-1}\boldsymbol{e}_\chi\| &= \|\widehat{a}(y)u\widehat{a}(y)^{-1}w^{-1}w\widehat{a}(y)w^{-1}\boldsymbol{e}_\chi\| \\ &\asymp_{u,w} \|\exp(\log(y)\operatorname{Ad}_w Y)\boldsymbol{e}_\chi\| = y^{(w^{-1}\cdot\chi)(Y)}\end{aligned}$$

Hence it is enough to pick $w \in W$ and $Y \in \mathfrak{a}^-$ such that

$$(w^{-1} \cdot \chi)(Y) = \langle \chi, \operatorname{Ad}_w Y \rangle < 0$$

and $\Gamma_{U_0} w^{-1} \boldsymbol{e}_\chi$ is infinite.

Fact: The group $\Gamma_{U_0} w^{-1} \boldsymbol{e}_\chi$ is infinite, unless w is the identity element.

Proof of fact: Indeed, the stabilizer of $w^{-1}\boldsymbol{e}_\chi$ in Γ_{U_0} is given by

$$\Gamma_{U_0} \cap w^{-1}\Gamma_{U_0} w = U_0 \cap w^{-1}U_0 w \cap \Gamma \cap w^{-1}\Gamma w.$$

Since Γ and $w^{-1}\Gamma w$ are commensurable, the intersection $\Gamma \cap w^{-1}\Gamma w$ has finite index in Γ. Setting $U_{0,w} = U_0 \cap w^{-1}U_0 w$, it suffices to show that $\Gamma_{U_0}/\Gamma_{U_{0,w}}$ is infinite. Note that the Lie algebra of $U_{0,w}$ is the sum of all the root spaces associated to positive roots $\beta > 0$ such that $w \cdot \beta > 0$. Moreover, define $U_{w,0}^- = U_0 \cap w^{-1}U_0^- w$, where U_0^- is the unipotent subgroup opposite to U_0. Then its Lie algebra is the sum of all the root spaces associated to positive roots $\beta > 0$ such that $w \cdot \beta < 0$. Since the Weyl groups acts simply transitively on the set of bases of Φ, for each non-trivial w there exists a positive root $\beta > 0$ such that $w \cdot \beta < 0$. In particular, $U_{0,w}^-$ is a non-trivial subgroup of U_0 and the Lie algebra of U_0 is the direct sum of the Lie algebras of $U_{0,w}$ and $U_{0,w}^-$. In particular, the group $\Gamma_{U_{w,0}^-}$ is infinite (as it is a lattice in $U_{w,0}^-$) and the inclusion map followed by the quotient map

$$\Gamma_{U_{w,0}^-} \to \Gamma_{U_0} \to \Gamma_{U_0}/\Gamma_{U_{0,w}}$$

is injective. Thus for every non-trivial $w \in W$, the orbit $\Gamma_{U_0} w^{-1} \boldsymbol{e}_\chi$ is infinite, as required.

Let w_0 be the unique element in the Weyl group W that takes the positive Weyl chamber to the negative one. Our goal is to find $w \in W \setminus \{1, w_0\}$ and $Y \in \mathfrak{a}^-$ such that $\langle \chi, \operatorname{Ad}_w Y \rangle < 0$.

We first claim that there exists $\varepsilon > 0$ and Y_0 in the negative Weyl chamber $\mathfrak{a}^-$ such that the ε-ball $B_{\mathfrak{a}}(Y_0, \varepsilon)$ with center Y_0 is still contained in the negative Weyl chamber and for all $Y \in B_{\mathfrak{a}}(Y_0, \varepsilon)$,

$$\langle \chi, \operatorname{Ad}_{w_0} Y \rangle + \langle \chi, Y \rangle \geq 0. \tag{3.11}$$

To show this claim, we first suppose that w_0 acts on χ by $w_0 \cdot \chi = -\chi$. Then (3.11) holds for all $Y \in \mathfrak{a}^-$, and we are done. Now suppose that $w_0 \cdot \chi \neq -\chi$. Let

Y_χ be the unique element in the closure of the positive Weyl chamber with norm $\|Y_\chi\| = \|\chi\|$ such that $\chi(Y_\chi) = \|\chi\|^2$. Set $Y_0 = \mathrm{Ad}_{w_0^{-1}} Y_\chi$. Then

$$\langle \chi, \mathrm{Ad}_{w_0} Y_0 \rangle = \langle \chi, Y_\chi \rangle = \|\chi\|^2,$$

and expressing $w_0 \cdot \chi = a\chi + b\chi^\perp$ as an orthogonal sum with $\chi^\perp$ in the orthogonal complement of $\mathbb{R}\chi$ with $a > -1$, we have

$$\langle \chi, Y_0 \rangle = \langle w_0 \cdot \chi, Y_\chi \rangle = \langle a\chi + b\chi^\perp, Y_\chi \rangle = a\|\chi\|^2.$$

Hence $\langle \chi, \mathrm{Ad}_{w_0} Y_0 \rangle + \langle \chi, Y_0 \rangle = \|\chi\|^2 + a\|\chi\|^2 > 0$. If Y_0 lies in a wall of the negative Weyl chamber, we may replace Y_0 with a closeby element Y_0' in the interior of $\mathfrak{a}^-$ and still have that $\langle \chi, \mathrm{Ad}_{w_0} Y_0' \rangle + \langle \chi, Y_0' \rangle > 0$. The claim (3.11) follows now by continuity of the inner product.

Observe that for all $Y \in \mathfrak{a}$, we have $\sum_{w \in W} \mathrm{Ad}_w Y = 0$, since the only element of $\mathfrak{a}$ invariant under the Weyl group is 0. In particular, for all $Y \in B_{\mathfrak{a}}(Y_0, \varepsilon)$, using (3.11), one has

$$\sum_{w \in W \smallsetminus \{1, w_0\}} \langle \chi, \mathrm{Ad}_w Y \rangle \leq 0.$$

Since the $\mathbb{Q}$-rank of G is at least 2, the set $W \smallsetminus \{1, w_0\}$ is non-empty. Fix $Y \in B_{\mathfrak{a}}(Y_0, \varepsilon)$. If there is $w \in W \smallsetminus \{1, w_0\}$ with $\langle \chi, \mathrm{Ad}_w Y \rangle < 0$, then we are done. Otherwise, we have for all such w that $\langle \chi, \mathrm{Ad}_w Y \rangle = 0$. Since the ball $B_{\mathfrak{a}}(Y_0, \varepsilon)$ is not contained in a proper linear subspace of $\mathfrak{a}$, we can pick $Y' \in B_{\mathfrak{a}}(Y_0, \varepsilon)$ closeby such that $\langle \chi, \mathrm{Ad}_w Y' \rangle < 0$, as required.

Finally, we show that the $\mathbb{Q}$-rank of $\mathbf{G}$ is 1 if and only if Γ_L is a cocompact (or uniform) lattice in L. To this end, we recall the structure of the stabilizer $\mathbf{L}$ in $\mathbf{G}$ of $\boldsymbol{e}_\chi$, as described in the proof of Theorem A. Let $\theta \subset \Delta$ be the proper subset of simple roots such that $\mathbf{P}$, the stabilizer in $\mathbf{G}$ of the line through $\boldsymbol{e}_\chi$, is the associated standard parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_\theta$. Then $\mathbf{L}$ is the semi-direct product of the reductive $\mathbb{Q}$-group $\widetilde{\mathbf{M}_\theta} = \mathbf{Q}_\theta \mathbf{M}_\theta (\mathbf{T}_\theta \cap \ker(\chi))$ and $\mathbf{U}_\theta$ (the unipotent radical of $\mathbf{P} = \mathbf{P}_\theta$), where $\mathbf{Q}_\theta$ is a connected $\mathbb{Q}$-subgroup of the largest $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of $\mathcal{Z}(\mathbf{T})$, $\mathbf{M}_\theta$ is a connected semisimple $\mathbb{Q}$-subgroup and $\mathbf{T}_\theta = \left(\bigcap_{\beta \in \theta} \ker \beta\right)^\circ$.

Now, if the $\mathbb{Q}$-rank of $\mathbf{G}$ is 1, then $\theta = \varnothing$ is the empty set and $\mathbf{P} = \mathbf{P}_\varnothing$ is the minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_0$ and $\mathbf{U} = \mathbf{U}_\varnothing$, the unipotent radical of $\mathbf{P}$, is the unipotent radical $\mathbf{U}_0$ of $\mathbf{P}_0$. In particular, $\mathbf{T}_\varnothing$ is the maximal $\mathbb{Q}$-split $\mathbb{Q}$-torus $\mathbf{T}$. By [5, Section 11.7], the centralizer $\mathcal{Z}(\mathbf{T})$ is an almost direct product of $\mathbf{T}$ and the largest connected $\mathbb{Q}$-anisotropic subgroup $\mathbf{M}$ of $\mathcal{Z}(\mathbf{T})$. Since $\mathrm{char}(\mathbb{Q}) = 0$, we have $X^*(\mathbf{M})_{\mathbb{Q}} = \{1\}$ and $\mathbf{M}(\mathbb{Q})$ consists of semisimple elements (see just below the [5, Definition 10.5]). Let $M = \mathbf{M}(\mathbb{R})$. By [5, Theorem 8.4], $\Gamma_M = \Gamma \cap M$ is thus a cocompact lattice in M. Also, $\Gamma_U = \Gamma \cap U$ is a cocompact lattice in U. Let $\widetilde{K} = K \cap M$. Hence, we can find a compact fundamental set Ω_1 for Γ_M in M such that $\widetilde{K}\Omega_1 = \Omega_1$, and a compact fundamental set for Ω_2 for Γ_U in U (see [5, Definition 9.6]). By [5, Remark 9.9], the compact set $\Omega_1\Omega_2$ is a fundamental set for $\Gamma_M\Gamma_U$ in L. Since by [6, Corollary 6.4] the group $\Gamma_M\Gamma_U$ has finite index in Γ_L, we deduce that Γ_L is a uniform lattice in L, as desired.

Conversely, suppose that Γ_L is a uniform lattice in L. Without loss of generality, we may assume that $\Gamma_L = \Gamma_M \Gamma_{U_\theta}$. As follows from [5, Theorem 8.7], $X^*(\mathbf{L}^\circ)_{\mathbb{Q}} = \{1\}$ and every unipotent element of $\mathbf{L}(\mathbb{Q})$ is contained in the unipotent radical of $\mathbf{L}$. Since $\mathbf{L} = \widetilde{\mathbf{M}_\theta}\mathbf{U}_\theta$ is the semi-direct product of a reductive $\mathbb{Q}$-subgroup and the connected unipotent $\mathbb{Q}$-subgroup $\mathbf{U}_\theta$, it follows from [5, Section 7.15] that

$R_u(\mathbf{L}) = \mathbf{U}_\theta$. In view of Theorem A, since $X^*(\mathbf{L}^\circ)_{\mathbb{Q}} = \{1\}$, we must have that $\mathbf{P} = \mathbf{P}_\theta$ is maximal, and hence $\Delta \smallsetminus \theta = \{\alpha\}$ consists of a single element. To conclude, it is enough to show that $\theta = \varnothing$ is the empty set. Since $X^*(\mathbf{L}^\circ)_{\mathbb{Q}} = \{1\}$ and every unipotent element of $\mathbf{L}(\mathbb{Q})$ is contained in the unipotent radical of $\mathbf{L}$, we must have that $X^*(\widetilde{\mathbf{M}}_\theta^\circ)_{\mathbb{Q}} = \{1\}$ and every unipotent element of $\widetilde{\mathbf{M}}_\theta(\mathbb{Q})$ is trivial. This implies that $\widetilde{\mathbf{M}}_\theta$ does not contain any non-trivial $\mathbb{Q}$-split torus. In particular, the semisimple $\mathbb{Q}$-subgroup $\mathbf{M}_\theta$ of $\widetilde{\mathbf{M}}_\theta$ does not contain any non-trivial $\mathbb{Q}$-split torus. However, by [5, Section 11.7], the number of elements in θ is equal to the $\mathbb{Q}$-rank of $\mathbf{M}_\theta$, and hence zero. Thus, we have shown that the $\mathbb{Q}$-rank of $\mathbf{G}$ is 1. The proof of Theorem C is complete. □

4. Application to Diophantine approximation on flag varieties

Let us briefly recall the notation and standing assumptions. Let $\mathbf{X}$ be a generalized flag variety defined over $\mathbb{Q}$, obtained as the quotient $\mathbf{X} = \mathbf{G}/\mathbf{P}$ of a connected semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$ by a proper parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ with abelian unipotent radical. In particular, the $\mathbb{Q}$-rank of $\mathbf{X}$ is 1, or equivalently, $\mathbf{P}$ is a maximal parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$. By [28, Section 2.1], we may assume that $\mathbf{G}$ is simply connected and almost $\mathbb{Q}$-simple. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup of G and let K be the maximal compact subgroup of G as before. We refer the reader to the introductory Section 2 for the specification of the height function H_χ on $\mathbf{X}(\mathbb{Q})$, the K-invariant probability measure σ_X and the K-invariant Riemannian distance $d(\cdot,\cdot)$ on X. We recall that by [9, Théorèmes 2.4.5 et 3.2.1] there exists $\beta_\chi \in \mathbb{Q}_{>0}$ such that, for every $c > 0$ and for almost every $x \in X$, the inequality

$$d(x, v) < c\, H_\chi(v)^{-\tau}$$

admits infinitely (resp. at most finitely) many solutions $v \in \mathbf{X}(\mathbb{Q})$, if $\tau \leq \beta_\chi$ (resp. $\tau > \beta_\chi$). We refer to β_χ as the *Diophantine exponent* of X with respect to χ. Our goal is to determine, for every constant $c > 0$, the asymptotic behavior of counting function

$$\mathcal{N}_{c,\beta_\chi}(x, T) = \#\{v \in \mathbf{X}(\mathbb{Q}) : d(x, v) < c\, H(v)^{-\beta_\chi},\, 1 \leq H(v) < T\}$$

as $T \to +\infty$ for σ_X-almost every $x \in X$. In the proof of Theorem D, we work for simplicity with $c = 1$ and we write $\mathcal{N}_{\beta_\chi}(x, T)$ for $\mathcal{N}_{1,\beta_\chi}(x, T)$. Moreover, it suffices to study the asymptotic behavious of

$$\#\{\gamma \in \Gamma/\Gamma \cap P : d(x, \gamma x_0) < c\,\|\gamma \boldsymbol{e}_\chi\|^{-\beta_\chi},\, 1 \leq \|\gamma \boldsymbol{e}_\chi\| < T\}$$

4.1. **Diophantine approximation and counting lattice points.** In this section, we translate the problem of counting rational approximations of bounded height in X to the problem of counting primitive lattice points in a certain family of growing sets in the Euclidean space V_χ.

For every $T \geq 1$, we define the set

$$\mathcal{E}_{\beta_\chi}(T) = \big\{\boldsymbol{v} \in \widetilde{X} : d(x_0, [\boldsymbol{v}]) < \|\boldsymbol{v}\|^{-\beta_\chi}, 1 \leq \|\boldsymbol{v}\| < T\big\}. \tag{4.1}$$

Fix a section $\mathfrak{s} : X \to K$ of the orbital map $K \to X$ sending k to kx_0. Given $x \in X$, we shall write $k_x = \mathfrak{s}(x)$. As the following lemma shows, estimating the counting function $\mathcal{N}_{\beta_\chi}(x, T)$ amounts to counting lattice points in the increasing family $\{\mathcal{E}_T\}_{T\geq 1}$. Let $[K \cap P : K \cap L] \in \{1, 2\}$ be the index of $K \cap L$ in $K \cap P$.

Lemma 4.1. *For every $x \in X$ and $T \geq 1$, we have*

$$\mathcal{N}_{\beta_\chi}(x, T) = [K \cap P : K \cap L]^{-1}\, \#\left(k_x^{-1}\mathcal{P}_\chi \cap \mathcal{E}_{\beta_\chi}(T)\right). \tag{4.2}$$

Proof. It suffices to show that $\mathcal{N}_{\beta_\chi}(x,T) = \#\left(\mathcal{P}_\chi \cap k_x\mathcal{E}_{\beta_\chi}(T)\right)$. We first note that

$$k_x\mathcal{E}_{\beta_\chi}(T) = \left\{\boldsymbol{v} \in \widetilde{X} : d(x,[\boldsymbol{v}]) < \|\boldsymbol{v}\|^{-\beta_\chi}, 1 \le \|\boldsymbol{v}\| < T\right\}.$$

Now a rational point $v = g[\boldsymbol{e}_\chi] \in \mathbf{X}(\mathbb{Q})$ satisfies $d(x,v) < H_\chi(v)^{-\beta_\chi}$ and $1 \le H_\chi(v) < T$ if and only if any of the primitive vectors $\boldsymbol{v} \in \mathcal{P}_\chi$ representing v satisfies $d(x,[\boldsymbol{v}]) < \|\boldsymbol{v}\|^{-\beta_\chi}$ and $1 \le \|\boldsymbol{v}\| < T$. This finishes the proof of the lemma. □

4.2. **Approximation of $\mathcal{E}_{\beta_\chi}(T)$.** In this section, we approximate the region $\mathcal{E}_{\beta_\chi}(T)$ from inside and from outside by regions that admit a decomposition with respect to the action of the one-parameter diagonal subgroup A.

Recall that the map $\mathfrak{u}^- \to X$ sending $u \mapsto \exp(u)x_0$ restricts to a diffeomorphism from a neighborhood of $1 \in \mathfrak{u}^-$ to a neighborhood of $x_0 \in X$. In particular, any $\boldsymbol{v} \in \widetilde{X}$, so that $[\boldsymbol{v}]$ is close to x_0, defines an element $u^-_{\boldsymbol{v}}$ in the Lie algebra $\mathfrak{u}^-$ by $[\boldsymbol{v}] = \exp(u^-_{\boldsymbol{v}})x_0$. The adjoint action of $a(y) \in A$ on $\mathfrak{u}^- = T_{x_0}X$ acts by scalar multiplication $\mathrm{Ad}(a(y))u^- = y\,u^-$ by y. Observe that

$$[a(y)\boldsymbol{v}] = a(y)[\boldsymbol{v}] = a(y)\exp(u^-_{\boldsymbol{v}})a(y)a(y)^{-1}x_0 = \exp(\mathrm{Ad}(a(y))u^-_{\boldsymbol{v}})x_0.$$

But we also have $[a(y)\boldsymbol{v}] = \exp(u^-_{a(y)\boldsymbol{v}})x_0$. By uniqueness, this gives the relation

$$u^-_{a(y)\boldsymbol{v}} = y\,u^-_{\boldsymbol{v}}. \tag{4.3}$$

Moreover, by the distance estimate (2.6), there exists a constant $C_0 > 0$ such that $d(x_0,[\boldsymbol{v}]) \le \|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-}(1 + C_0\|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-})$. Let $\pi^+ : V_\chi \to V_\chi$ be the orthogonal projection onto $\mathbb{R}\boldsymbol{e}_\chi$ and we abbreviate $\pi^+(\boldsymbol{v})$ by $\boldsymbol{v}^+$.

For every $T \ge 1$ and $c > 0$ close to 1, we will work with regions

$$\mathcal{E}^+_{T,c} = \{\boldsymbol{v} \in \widetilde{X} : \|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-} < c\,\|\boldsymbol{v}^+\|^{-\beta_\chi}, 1 \le \|\boldsymbol{v}^+\| < c\,T\}.$$

By enlarging C_0 if necessary, we can assume that $\|\boldsymbol{v}^+\| \ge C_0^{-1}\|\boldsymbol{v}\|$ as soon as $d(x_0,[\boldsymbol{v}]) < 1$. For every integer $\ell \ge 1$, we let

$$Q_\ell = \{\boldsymbol{v} \in \widetilde{X} : \|\boldsymbol{v}\| \le C_0\,\ell\}$$

and we define

$$\widehat{c}_\ell = \left(1 + C_0\,\ell^{-\beta_\chi}\right)^{-(1+\beta_\chi)} \in (0,1).$$

In particular, we have $\widehat{c}_\ell \nearrow 1$ as $\ell \to +\infty$.

The sets $\mathcal{E}^+_{T,c}$ have the following nice properties. For every $c > 0$, let

$$\mathcal{F}_c = \{\boldsymbol{v} \in \widetilde{X} : \|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-} < c\|\boldsymbol{v}^+\|^{-\beta_\chi}, 1 \le \|\boldsymbol{v}^+\| < e\}. \tag{4.4}$$

Lemma 4.2. *(1) (Approximation) For all large enough $\ell \ge 1$ and $T \ge 1$,*

$$\mathcal{E}^+_{T,\widehat{c}_\ell} \smallsetminus Q_{2\ell} \subseteq \mathcal{E}_{\beta_\chi}(T) \smallsetminus Q_\ell \subseteq \mathcal{E}^+_{T,\widehat{c}_\ell^{-1}}. \tag{4.5}$$

(2) (Tessellation) For every $c > 0$ and $T \ge 1$ such that $cT = e^N$ for some $N \in \mathbb{N}$, we have

$$\mathcal{E}^+_{T,c} = \bigsqcup_{i=0}^{N-1} a(y_i)^{-1}\,\mathcal{F}_c, \qquad \text{with } y_j = e^{\beta_\chi j} \text{ for } j \in \mathbb{N}. \tag{4.6}$$

Proof. Let $\boldsymbol{v} \in \mathcal{E}^+_{T,\widehat{c}_\ell} \smallsetminus Q_{2\ell}$. Let us first prove that for all sufficiently large ℓ and T, we have

$$d(x_0,[\boldsymbol{v}]) < \|\boldsymbol{v}\|^{-\beta_\chi},\ C_0\,\ell < \|\boldsymbol{v}\| < T.$$

Let $\boldsymbol{v}^{\perp} = \boldsymbol{v} - \boldsymbol{v}^{+}$. Observe that, for large enough ℓ, we have

$$\frac{\|\boldsymbol{v}^{\perp}\|}{\|\boldsymbol{v}^{+}\|} \asymp d(x_0, [\boldsymbol{v}]) < \|\boldsymbol{v}\|^{-\beta_\chi}.$$

Therefore, we have

$$\frac{\|\boldsymbol{v}\|^2}{\|\boldsymbol{v}^{+}\|^2} = 1 + \frac{\|\boldsymbol{v}^{\perp}\|^2}{\|\boldsymbol{v}^{+}\|^2} \leq 1 + C_0 \,\|\boldsymbol{v}^{+}\|^{-2\beta_\chi},$$

and hence

$$\frac{\|\boldsymbol{v}\|^{\beta_\chi}}{\|\boldsymbol{v}^{+}\|^{\beta_\chi}} \leq \left(1 + C_0 \,\|\boldsymbol{v}^{+}\|^{-2\beta_\chi}\right)^{\beta_\chi/2}.$$

Since $\boldsymbol{v}$ does not lie in $Q_{2\ell}$, we have $\|\boldsymbol{v}^{+}\| \geq C_0^{-1}\|\boldsymbol{v}\| \geq 2\ell$. Thus, using the definition of $\widehat{c}_\ell$, we have (by enlarging C_0 where necessary)

$$\begin{aligned}
d(x_0, [\boldsymbol{v}]) &\leq \|u_{\boldsymbol{v}}^{-}\|_{\mathfrak{u}^-} \left(1 + C_0 \,\|u_{\boldsymbol{v}}^{-}\|_{\mathfrak{u}^-}\right) \\
&\leq \widehat{c}_\ell \|\boldsymbol{v}^{+}\|^{-\beta_\chi} \left(1 + C_0 \,\|\boldsymbol{v}^{+}\|^{-\beta_\chi}\right) \\
&= \|\boldsymbol{v}\|^{-\beta_\chi} \left(\widehat{c}_\ell \frac{\|\boldsymbol{v}\|^{\beta_\chi}}{\|\boldsymbol{v}^{+}\|^{\beta_\chi}} (1 + C_0 \,\|\boldsymbol{v}^{+}\|^{-\beta_\chi})\right) \\
&\leq \|\boldsymbol{v}\|^{-\beta_\chi} \left(\widehat{c}_\ell \left(1 + C_0 \,\ell^{-2\beta_\chi}\right)^{\beta_\chi/2} (1 + C_0 \,\ell^{-\beta_\chi})\right) \\
&\leq \|\boldsymbol{v}\|^{-\beta_\chi}
\end{aligned}$$

Since $\boldsymbol{v}$ does not lie in $Q_{2\ell}$, it does, in particular, not lie in Q_ℓ. Moreover, we have $\|\boldsymbol{v}\| = \|\boldsymbol{v}^{+}\| \frac{\|\boldsymbol{v}\|}{\|\boldsymbol{v}^{+}\|} \leq \widehat{c}_\ell \frac{\|\boldsymbol{v}\|}{\|\boldsymbol{v}^{+}\|} T < T$. This shows the left inclusion in Equation (4.5). The other inclusion is proved similarly.

To see the last claim in the lemma, let us recall that a_y acts on $\boldsymbol{v}^{+}$ by $a_y \boldsymbol{v}^{+} = y^{-\frac{1}{\beta_\chi}} \boldsymbol{v}^{+}$. Then, using (4.3) and observing that

$$\widehat{a}(t_i)^{-1} \mathcal{F}_c = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^{-}\|_{\mathfrak{u}^-} < c \,\|\boldsymbol{v}^{+}\|^{-\beta_\chi}, e^j \leq \|\boldsymbol{v}^{+}\| < e^{j+1}\}$$

yields the desired tessellation. $\square$

Intersecting (4.5) with the discrete set $k_x^{-1}\mathcal{P}_\chi$ and using that the number of lattice points in the set Q_ℓ is bounded by an absolute constant times $\ell^{\beta_\chi d}$ (see Corollary **??**), we get, by enlarging C_0 if necessary,

(4.7)
$$\#\left(k_x^{-1}\mathcal{P}_\chi \cap \mathcal{E}_{T,\widehat{c}_\ell}^{+}\right) - C_0 \ell^{\beta_\chi d} \leq \#\left(k_x^{-1}\mathcal{P}_\chi \cap \mathcal{E}_T\right) \leq \#\left(k_x^{-1}\mathcal{P}_\chi \cap \mathcal{E}_{T,\widehat{c}_\ell^{-1}}^{+}\right) + C_0 \ell^{\beta_\chi d}.$$

Using the tessellation of $\mathcal{E}_{T,c}^{+}$ as in (4.6) with $T = \frac{1}{c} e^N$ for every integer $N \geq 1$, we have

$$\begin{aligned}
\#(k_x^{-1}\mathcal{P}_\chi \cap \mathcal{E}_{T,c}^{+}) &= \#(\mathcal{P}_\chi \cap k_x \mathcal{E}_{T,c}^{+}) \\
&= \sum_{i=0}^{N-1} S_\chi \mathbb{1}_{k_x \widehat{a}(t_i)^{-1}\mathcal{F}_c}(\Gamma) = \sum_{i=0}^{N-1} S_\chi \mathbb{1}_{\mathcal{F}_c}(\widehat{a}(t_i) k_x^{-1}\Gamma).
\end{aligned} \tag{4.8}$$

Plugging this back into (4.7), we get the lower and upper bounds, for every $T' \geq 1$ and large enough ℓ,

$$\sum_{i=0}^{\lfloor T'+\ln \widehat{c}_\ell \rfloor -1} S_\chi \mathbb{1}_{\mathcal{F}_{c_\ell}}(\widehat{a}(t_i)k_x^{-1}\Gamma) - C_0\, \ell^{\beta_\chi d} \leq \# \left(k_x^{-1}\mathcal{P}_\chi \cap \mathcal{E}_{\beta_\chi}(e^{T'})\right)$$
$$\leq \sum_{i=0}^{\lceil T'-\ln \widehat{c}_\ell \rceil -1} S_\chi \mathbb{1}_{\mathcal{F}_{\widehat{c}_\ell^{-1}}}(\widehat{a}(t_i)k_x^{-1}\Gamma) + C_0\, \ell^{\beta_\chi d}. \tag{4.9}$$

The proof of Theorem D consists of effectively estimating the left- and right-hand sides of (4.9). To this end, we will now develop the necessary tools and ingredients for these estimates.

5. A uniform upper bound for K-averages

A crucial estimate used in this approximation is given in the following lemma. Using Theorem A, we fix $\varepsilon > 0$ such that the Siegel transform S_χ maps $B_c^\infty(\widetilde{X})$ into $L^{1+\varepsilon}(\Omega)$.

Lemma 5.1. *Let $s \in (0, 1+\varepsilon)$. For every $c > 0$ and $g \in G$, we have*

$$\sup_{t\geq 0} \int_K \left|S_\chi \mathbb{1}_{\mathcal{F}_c}(\widehat{a}_t k g)\right|^s \, \mathrm{d}\mu_K(k) < +\infty.$$

The upper bound is uniform as g and c vary in compact sets.

Let $d_G(\cdot,\cdot)$ be a left-invariant Riemannian metric on G; this induces left-invariant metrics on the parabolic subgroup P and on the quotient $\Omega = G/\Gamma$, and we denote them by $d_P(\cdot,\cdot)$ and $d_\Omega(\cdot,\cdot)$, respectively. The key observation used in the proof is that the translated sets $a(y)^{-1}\mathcal{F}_c$ are almost invariant under a small neighborhood of the identity in P.

Proof. We fix $s \in (0, 1+\varepsilon)$. We show that for $c > 0$ and g_0 in a compact subset Q of G, we have

$$\sup_{t\geq 0} \int_K \left(S_\chi \mathbb{1}_{\mathcal{F}_c}(\widehat{a}_t k g_0)\right)^s \, \mathrm{d}\mu_K(k) \ll_{s,Q} \int_{\widetilde{X}} \mathbb{1}_{\mathcal{F}_c}\, d\lambda_{\widetilde{X}}.$$

For every $\delta \in (0, \frac{1}{2})$, we define a δ-neighborhood of $\mathcal{F}_c$ by

$$\mathcal{F}_c(\delta) = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} < (1+\delta)^{1+\beta_\chi}\, c\, \|\boldsymbol{v}^+\|^{-\beta_\chi}, (1+\delta)^{-1} \leq \|\boldsymbol{v}^+\| < (1+\delta)e\}.$$

One can show that $\lambda(\mathcal{F}_c(\delta) \smallsetminus \mathcal{F}_c) \ll_c \delta$. For every $r > 0$, let $B_P(r)$ denote the metric open ball in P with radius $r > 0$ and center $1 \in P$. We claim that there exists some small constant $\widetilde{c} > 0$, independent of δ, such that for all $0 < \delta < \frac{1}{2}$, $p \in B_P(\widetilde{c}\delta)$, and $t \geq 0$, we have

$$pa(y)^{-1}\mathcal{F}_c \subseteq a(y)^{-1}\mathcal{F}_c(\delta).$$

First, using the relationship (4.3) and the fact that, for every $t \in \mathbb{R}$, $a(y)$ acts on $\boldsymbol{e}_\chi$ via $a(y)\,\boldsymbol{e}_\chi = e^{-\frac{t}{\beta_\chi}}\,\boldsymbol{e}_\chi$, we have

$$a(y)^{-1}\mathcal{F}_c = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} < c\,\|\boldsymbol{v}^+\|^{-\beta_\chi}, e^{\frac{t}{\beta_\chi}} \leq \|\boldsymbol{v}^+\| < e^{\frac{t}{\beta_\chi}+1}\},$$

and likewise

$$a(y)^{-1}\mathcal{F}_c(\delta) = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} < (1+\delta)^{1+\beta_\chi}\, c\,\|\boldsymbol{v}^+\|^{-\beta_\chi},$$
$$(1+\delta)^{-1}e^{\frac{t}{\beta_\chi}} \leq \|\boldsymbol{v}^+\| < (1+\delta)e^{\frac{t}{\beta_\chi}+1}\}.$$

So we need to show that for some small $\widetilde{c} > 0$, for all $0 < \delta < \frac{1}{2}$, $p \in B_P(\widetilde{c}\delta)$, $t \geq 0$, and $\boldsymbol{v} \in a(y)^{-1}\mathcal{F}_c$, we have

(1) $\|u^-_{p\boldsymbol{v}}\|_{\mathfrak{u}^-} < (1+\delta)^{1+\beta_\chi} c \,\|(p\boldsymbol{v})^+\|^{-\beta_\chi}$, and

(2) $(1+\delta)^{-1} e^{\frac{t}{\beta_\chi}} \leq \|(p\boldsymbol{v})^+\| < (1+\delta)\, e^{\frac{t}{\beta_\chi}+1}$.

Let us start with condition (2). Since $\boldsymbol{v}$ satisfies $e^{\frac{t}{\beta_\chi}} \leq \|\boldsymbol{v}^+\| < e^{\frac{t}{\beta_\chi}+1}$, it is enough to show that for some small $\widetilde{c} > 0$, for all $0 < \delta < \frac{1}{2}$, $p \in B_P(\widetilde{c}\delta)$, $t \geq 0$, and $\boldsymbol{v} \in a(y)^{-1}\mathcal{F}_c$, we have

$$(1+\delta)^{-1}\|\boldsymbol{v}^+\| \leq \|(p\boldsymbol{v})^+\| < (1+\delta)\|\boldsymbol{v}^+\|. \tag{5.1}$$

However, applying the triangle inequality, we already see that

$$\|(p\boldsymbol{v})^+\| - \|\boldsymbol{v}^+\| \leq \|(p\boldsymbol{v})^+ - \boldsymbol{v}^+\| \leq \|p\boldsymbol{v} - \boldsymbol{v}\| \ll \widetilde{c}\delta\|\boldsymbol{v}\| \ll \widetilde{c}\delta\|\boldsymbol{v}^+\|.$$

By shrinking $\widetilde{c} > 0$ if necessary, this gives $\|(p\boldsymbol{v})^+\| < (1+\varepsilon)\|\boldsymbol{v}^+\|$, as required. The other inequality is shown similarly and we omit the details.

So it remains to show the first condition. Recall that $u_{p\boldsymbol{v}} \in \mathfrak{u}^-$ is defined by the relation $p[\boldsymbol{v}] = \exp(u^-_{p\boldsymbol{v}})x_0$. On the other hand, since x_0 is fixed by p, we have $p[\boldsymbol{v}] = \exp(\operatorname{Ad}(p)u^-_{\boldsymbol{v}})x_0$. Consider the decomposition $\mathfrak{g} = \mathfrak{u}^-\oplus$, where denotes the Lie algebra of P. The map $\mathfrak{u}^-\oplus \to G$ sending (u^-, p') to $\exp(u^-)\exp(p')$ is a local diffeomorphism. In particular, there exists a unique $(u^-, p') \in \mathfrak{u}^-\oplus$ such that $\exp(\operatorname{Ad}(p)u^-_{\boldsymbol{v}}) = \exp(u^-)\exp(p')$. When applied to x_0, we have

$$p[\boldsymbol{v}] = \exp(\operatorname{Ad}(p)u^-_{\boldsymbol{v}})x_0 = \exp(u^-)x_0 \quad \text{and} \quad p[\boldsymbol{v}] = \exp(u^-_{p\boldsymbol{v}})x_0.$$

This implies that $u^- = u^-_{p\boldsymbol{v}}$. The induced map $\mathfrak{g} \to \mathfrak{u}^-$, sending $Y \in \mathfrak{g}$ to u^-_Y such that $\exp(Y) = \exp(u^-_Y)\exp(p_Y)$ with $p_Y \in$, is Lipschitz on a bounded neighborhood of the origin in $\mathfrak{g}$. Thus

$$\|u^-_{p\boldsymbol{v}}\|_{\mathfrak{u}^-} = \|u^-\|_{\mathfrak{u}^-} \ll \|\operatorname{Ad}(p)u^-_{\boldsymbol{v}}\|_{\mathfrak{g}^-} \leq \|\operatorname{Ad}(p)\|\|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-},$$

where $\|\operatorname{Ad}(p)\|$ denotes the operator norm as an operator on $\mathfrak{g}$. By shrinking $c > 0$ if necessary, one has $p \in B_P(\widetilde{c}\varepsilon)$ implies that $\|\operatorname{Ad}(p)\|_{\mathrm{op}} < 1+\varepsilon$.

Since $\boldsymbol{v} \in a(y)^{-1}\mathcal{F}_c$, we get from the first inequality in (5.1) applied to p^{-1}

$$\begin{aligned}\|u^-_{p\boldsymbol{v}}\|_{\mathfrak{u}^-} \leq (1+\varepsilon)\|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-} &\leq (1+\varepsilon)\, c\,\|\boldsymbol{v}^+\|^{-\beta_\chi} = (1+\varepsilon)c\,\|(p^{-1}p\boldsymbol{v})^+\|^{-\beta_\chi} \\ &\leq (1+\varepsilon)^{1+\beta_\chi}\, c\,\|(p\boldsymbol{v})^+\|^{-\beta_\chi},\end{aligned}$$

as desired. This completes the proof of the claim.

We can now prove Lemma 5.1. Letting $g_0 \in Q$, we see that

$$\begin{aligned}\int_K \left|S_\chi \mathbb{1}_{\mathcal{F}_c}(a_t k g_0)\right|^s \,\mathrm{d}\mu_K(k) &= \int_K \left|S_\chi \mathbb{1}_{a(y)^{-1}\mathcal{F}_c}(k g_0)\right|^s \,\mathrm{d}\mu_K(k) \\ &\leq \frac{1}{\mu_P(B_P(\widetilde{c}\delta))} \int_{B_P(\widetilde{c}\delta)} \int_K \left|S_\chi \mathbb{1}_{a(y)^{-1}\mathcal{F}_{c(\varepsilon)}}(pkg_0)\right|^s \,\mathrm{d}\mu_K(k)\,\mathrm{d}\mu_P(p) \\ &= \frac{1}{\mu_P(B_P(\widetilde{c}\delta))} \int_{B_P(\widetilde{c}\delta)\,K} \left|S_\chi \mathbb{1}_{a(y)^{-1}\mathcal{F}_{c(\varepsilon)}}(gg_0)\right|^s d\mu_G(g) \\ &= \frac{1}{\mu_P(B_P(\widetilde{c}\delta))} \int_{(B_P(\widetilde{c}\delta)\,K)g_0} \left|S_\chi \mathbb{1}_{a(y)^{-1}\mathcal{F}_{c(\varepsilon)}}(g)\right|^s d\mu_G(g).\end{aligned}$$

Similarly as in the proof of Theorem A, let $\mathfrak{S}$ be a Siegel set and $C \subset \mathbf{G}(\mathbb{Q})$ a finite subset such that $G = \mathfrak{S}\,C\,\Gamma$. Fix a fundamental domain $F \subseteq \mathfrak{S}\,C$ with nonempty interior for the right action of Γ on G. There exist a finite subset I of G such that the union $\bigcup_{h\in I} hF$ covers the closure of $B_P(\widetilde{c}\delta)\,K\,Q$ in G, which is compact. Together with the G-invariance of μ_Ω, we have

$$\int_K \left|S_\chi \mathbb{1}_{\mathcal{F}_c}(\widehat{a}_t kg)\right|^s \,\mathrm{d}\mu_K(k) \ll_\varepsilon \sum_{h\in I}\int_{hF}\left|S_\chi \mathbb{1}_{a(y)^{-1}\mathcal{F}_c(\varepsilon)}(g)\right|^s d\mu_G(g)$$
$$\ll_\varepsilon \sum_{h\in I}\int_F \left|S_\chi \mathbb{1}_{a(y)^{-1}\mathcal{F}_c(\varepsilon)}(hg)\right|^s \,\mathrm{d}\mu_G(g) \ll_{\varepsilon,F} \int_\Omega \left|S_\chi \mathbb{1}_{\mathcal{F}_c(\varepsilon)}(g)\right|^s d\mu_\Omega(g\Gamma).$$

To conclude, let us recall a few facts. First, applying the inequality (3.10) in the proof of Theorem A to $f = \mathbb{1}_{\mathcal{F}_c(\varepsilon)}$ and using that $\|f\|_\infty^\delta = 1$, we have

$$\int_\Omega |S_\chi f|^s \,\mathrm{d}\mu_\Omega \ll \int_{\widetilde{X}} f(a_y \boldsymbol{e}_\chi)\int_{L/L(\mathbb{Z})}\lambda_\chi(a_y l)^{-(s-1)a_\chi}\,\mathrm{d}\mu_{L/L(\mathbb{Z})}(l) d\lambda_{\widetilde{X}}(k a_y \boldsymbol{e}_\chi).$$

where the implicit constant depends only on the support of f. Since this support is compact, and consequently $a_y \in A$ varies in a compact set, we can estimate: $\forall\, l \in L$, $\lambda_\chi(a(y)l)^{-(s-1)a_\chi} \ll_{\operatorname{supp} f} \lambda_\chi(l)^{-(s-1)a_\chi}$, and hence

$$\int_\Omega |S_\chi f|^s \,\mathrm{d}\mu_\Omega \ll \int_{\widetilde{X}} f(a_y \boldsymbol{e}_\chi)\, d\lambda_{\widetilde{X}}(k a_y \boldsymbol{e}_\chi)\int_{L/L(\mathbb{Z})}\lambda_\chi(l)^{-(s-1)a_\chi}\,\mathrm{d}\mu_{L/L(\mathbb{Z})}(l)$$

and the integral $\int_{L/L(\mathbb{Z})}\lambda_\chi(l)^{-(s-1)a_\chi}\,\mathrm{d}\mu_{L/L(\mathbb{Z})}(l)$ converges by the choice of s and the arguments in the proof of Theorem A. Putting everything together, we have

$$\int_K \left(S_\chi \mathbb{1}_{\mathcal{F}_c}(\widehat{a}_t k\Delta)\right)^s \,\mathrm{d}\mu_K(k) \ll_{\varepsilon,F,c,s} \int_{\widetilde{X}} \mathbb{1}_{\mathcal{F}_c(\varepsilon)}\, d\lambda_{\widetilde{X}} \ll_{\varepsilon,F,c,s} \int_{\widetilde{X}} \mathbb{1}_{\mathcal{F}_c}\, d\lambda_{\widetilde{X}},$$

as desired. Since the sets $\mathcal{F}_c(\varepsilon)$, for c and ε varying in fixed compact sets, are all contained in a fixed compact subset of $\widetilde{X}$, one sees that the implicit constant only depends on s and the compact set Q, completing the proof. □

6. Equidistribution of compact-orbit translates

In this section, we prove an effective double equidistribution result for expanding translates of K-orbits. The proof is based on the one of Ouaggag's result [?, Proposition 4.1], which is derived from the effective equidistribution of unstable horospherical orbits established in [4].

In order to state the result, we shall need to introduce certain Sobolev norms on $C_c^\infty(\Omega)$ and $C^\infty(K)$. Each element Z in the Lie algebra of G defines a first order differential operator $\mathcal{D}_Z$ on $C_c^\infty(\Omega)$ by

$$\forall\,\phi \in C_c^\infty(\Omega),\ \forall\, x\in\Omega,\quad \mathcal{D}_Z\phi(x) = \frac{d}{dt}\Big|_{t=0}\phi(\exp(tZ)x).$$

Let D be the dimension of and let $\mathcal{B} = (Z_i)_{1\le i\le D}$ be a basis of the real vector space . Then each monomial

$$\mathcal{D}_Z = \mathcal{D}_{Z_1}^{j_1}\circ\cdots\circ\mathcal{D}_{Z_D}^{j_D} \tag{6.1}$$

with $(j_1,\ldots,j_D)\in\mathbb{N}^D$ defines a differential operator of degree $\deg(\mathcal{D}_Z) = j_1+\cdots+j_D$. For all $r\ge 1$ and $\phi\in C_c^\infty(\Omega)$, we define the *degree r Sobolev norm* of ϕ by

$$\mathcal{S}_r(\phi) = \sum_{\deg(\mathcal{D})\le r}\|\mathcal{D}\phi\|_\infty\,, \tag{6.2}$$

where $\mathcal{D}$ ranges over all monomials of elements in $\mathcal{B}$ of degree $\leq r$. Similarly, one can define a degree r Sobolev norm on $C^\infty(K)$, which, by abuse of notation, we also denote by $\mathcal{S}_r$.

For every bounded continuous function $\phi : \Omega \to \mathbb{R}$ and time $t \geq 0$, we introduce

$$\Delta_\phi(t) = \left| \int_K \phi\big(a(y)k\big)\, d\mu_K(k) - \mu_\Omega(\phi_1)\mu_\Omega(\phi_2) \right|. \tag{6.3}$$

Moreover, for all $t_2 \geq t_1 > 0$ we shall write $m(t_1, t_2) = \min\{t_1, t_2 - t_1\}$ and given bounded continuous functions $\phi_1, \phi_2 : \Omega \to \mathbb{R}$, and times $t_2 \geq t_1 \geq 0$, we introduce

$$\Delta_{\phi_1,\phi_2}(t_1, t_2) = \left| \int_K \phi_1\big(\widehat{a}(t_1)k\big)\phi_2\big(\widehat{a}(t_2)k\big) d\mu_K(k) - \mu_\Omega(\phi_1)\mu_\Omega(\phi_2) \right|. \tag{6.4}$$

Proposition 6.1 (Effective single and double equidistribution)**.** *There exist constants $c > 0$, $C > 0$ and an integer $r \geq 1$ such that for all $\phi \in C_c^\infty(\Omega)$ and $t \geq 0$, we have*

$$\Delta_\phi(t) \;\leq\; C\, e^{-c\,t} \mathcal{S}_r(\phi). \tag{6.5}$$

and, for all $\phi_1, \phi_2 \in C_c^\infty(\Omega)$ and $t_2 \geq t_1 \geq 0$, we have

$$\Delta_{\phi_1,\phi_2}(t_1, t_2) \;\leq\; C\, e^{-c\, m(t_1,t_2)} \mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2). \tag{6.6}$$

Observe that U^-, the unipotent subgroup opposite to the unipotent radical U of P, is the expanding horospherical subgroup with respect to the one-parameter diagonal subgroup $A = \{a(y) : t \in \mathbb{R}\}$:

$$U^- = \left\{ g \in G : \lim_{t \to -\infty} a(y)\, g\, \widehat{a}(-t) = 1 \right\}.$$

Let μ_{U^-} be a Haar measure on U^-. The idea of proof of Proposition 6.1 is to deduce the effective single and double equidistribution of expanding compact-orbit translates from the corresponding effective single and double equidistribution of expanding horospherical translates, as proved by Björklund and Gorodnik (see [4, Theorem 1.2]).

Theorem 6.2. *There exists a constants $c' > 0$ and an integer $r \geq 1$ such that the following hold. Let Q be a compact subset of G. Then, for all $f \in C_c^\infty(U)$, $\phi \in C_c^\infty(\Omega)$, $g \in Q$, and $t \geq 0$, we have*

$$\left| \int_{U^-} f(u)\phi\big(\widehat{a}(t_1)u^- g\big) d\mu_{U^-}(u^-) - \mu_{U^-}(f)\mu_\Omega(\phi) \right| \ll_Q \; e^{-c'\,t} \mathcal{S}_r(f)\mathcal{S}_r(\phi).$$

and, for all $f \in C_c^\infty(U)$, $\phi_1, \phi_2 \in C_c^\infty(\Omega)$, $g \in Q$, and $t_2 \geq t_1 \geq 0$, we have

$$\begin{aligned} &\left| \int_{U^-} f(u)\phi_1\big(\widehat{a}(t_1)u^- g\big)\phi_2\big(\widehat{a}(t_2)u^- g\big) d\mu_{U^-}(u^-) - \mu_{U^-}(f)\mu_\Omega(\phi_1)\mu_\Omega(\phi_2) \right| \\ &\qquad\qquad \ll_Q \; e^{-c'\, m(t_1,t_2)} \mathcal{S}_r(f)\mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2). \end{aligned} \tag{6.7}$$

Proof of Proposition 6.1. Let us prove the effective double equidistribution statement in (6.6). The proof of the corresponding effective single equidistribution statement in (6.5) follows along the same line of argument and we omit the details.

Let $\mathfrak{k}_P$ be the Lie algebra of the stabilizer $K_P = K \cap P$ in K of $x_0 \in X$. Let us define to be the orthogonal complement inside the Lie algebra $\mathfrak{k}$ of K of the Lie algebra $\mathfrak{k}_P$ with respect to the Killing form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$ of G. In particular, we have $\mathfrak{k} = \mathfrak{s} \oplus \mathfrak{k}_P$. Let V be a neighborhood of the origin in $\mathfrak{g}$ such that the exponential map $\exp : \mathfrak{g} \to G$ restricts to a diffeomorphism

$\exp|_V : V \to \exp(V)$. Consider the embedded submanifold $S = \exp(V \cap \mathfrak{s})$ of $\exp(V)$. In order to relate the double equidistribution of translated K-orbits to that of translated horospherical orbits, we first construct a local diffeomorphism from S to U^-.

Note that the product map $S \times K_P \to K$ restricts to a diffeomorphism in a neighborhood of the identity, giving a decomposition $k = s(k)k_P(k)$ with $s(k) \in S$ and $k_P(k) \in K_P$ for every $k \in K$ in this neighborhood. We recall that $^-$ and $\mathfrak{p}$ denote the Lie algebras of U^- and P respectively and that the Lie algebra of G decomposes as the direct sum $\mathfrak{g} =^- \oplus \mathfrak{p}$. In particular, every element $g \in G$ close to the identity $1 \in G$ can be uniquely decomposed as $g = u^-(g)p(g)$ with $u^-(g) \in U^-$ and $p(g) \in P$.

We claim that there exists a neighborhood V_S of the identity $1 \in S$ such that the map $h : V_S \to U^-$ given by sending $s \in S$ to $u^-(s)$ is well-defined and defines a diffeomorphism onto a neighborhood of the identity $1 \in U^-$. Observe first that the dimensions of S and U^- agree:

$$\dim S = \dim K - \dim K_P = \dim X = \dim U^-.$$

To show the claim, by the inverse function theorem, it suffices to show that the derivative $D_1 h$ of the map h at $1 \in S$ is injective. This however follows from the fact that $\ker D_1 h$ is contained in $\mathfrak{s} \cap \mathfrak{p} = \{0\}$.

Let μ_{K_P} be the Haar probability measure on K_P. Let us now equip S with a measure μ_S such that, for all integrable functions $f : K \to \mathbb{R}$ supported in a sufficiently small neighborhood of $1 \in K$, we have

$$\int_K f(k)\, d\mu_K(k) = \int_S \int_{K_P} f(sk_P)\, d\mu_S(s) d\mu_{K_P}(k_P).$$

Let μ_{K/K_P} be the pushforward of μ_K along the projection map $K \to K/K_P$. By [17, Theorem 2.51], for all integrable functions $f : K \to \mathbb{R}$ supported in a sufficiently small neighborhood of $1 \in K$, we have

$$\int_K f(k)\, d\mu_K(k) = \int_{K/K_P} \int_{K_P} f(xk_P)\, d\mu_{K/K_P}(x) d\mu_{K_P}(k_P).$$

Since the kernel of the derivative at $1 \in K$ of the projection map $K \to K/K_P$ is $\mathfrak{k}_P$, the restriction to S of this projection map defines a local diffeomorphism at 1. We let μ_S be the pushforward of the restriction of μ_{K/K_P} to a suitable neighborhood along the inverse of this diffeomorphism.

Let $r \geq 1$ be as in Theorem 6.2. There exist constants $c_1 > 0$ and $c_2 > 0$ such that the following holds. For every small $r_0 > 0$, there exists $N \in \mathbb{N}$ with $N \ll r_0^{-c_1}$, non-negative functions $\kappa_i \in C_c^\infty(K)$ with $1 \leq i \leq N$, all supported in $B_K(r_0)$ and satisfying $\|\kappa_i\|_r \ll r_0^{-c_2}$, and elements $k_i \in K$ with $1 \leq i \leq N$ such that we have a partition of unity: for every $k \in K$, we have $1 = \sum_{i=1}^N \kappa_i(kk_i^{-1})$.

For all $\phi_1, \phi_2 \in C_c^\infty(\Omega)$ and $t_2 \geq t_1 \geq 0$, we define

$$I_{\phi_1,\phi_2}(t_1, t_2) = \int_K \phi_1\big(\widehat{a}(t_1)k\big)\phi_2\big(\widehat{a}(t_2)k\big) d\mu_K(k).$$

Let $r_0 > 0$ be small, to be fixed later. Using the direct sum decompositions

$$\mathfrak{k} = \mathfrak{k}_P \oplus \mathfrak{s} \quad \text{and} \quad \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}^-,$$

for every k close the identity in K, we have

$$k = k_P(k)s(k) = k_P(k)p(s(k))u^-(s(k)).$$

Observe that K_P is the centralizer $\mathcal{Z}_K(A)$ in K of A. Putting everything together and letting, for $j = 1, 2$,

$$g(k, t_j) = k_P(k)\widehat{a}(t_j)p(s(k))\widehat{a}(-t_j),$$

we have

$$\begin{aligned}I_{\phi_1,\phi_2}(t_1, t_2) &= \sum_{i=1}^{N}\int_K \kappa_i(k)\phi_1(\widehat{a}(t_1)kk_i)\phi_2(\widehat{a}(t_2)kk_i)d\mu_K(k)\\ &= \sum_{i=1}^{N}\int_K \kappa_i(k)\phi_1(g(k,t_1)\widehat{a}(t_1)u^-(s(k))k_i)\phi_2(g(k,t_2)\widehat{a}(t_2)u^-(s(k))k_i)\, d\mu_K(k).\end{aligned}$$

We recall that P is the semi-direct product of the centralizer $\mathcal{Z}_G(A)$ in G of A and the unipotent radical U of P. This unipotent radical is also the contracting horospherical subgroup with respect to the one-parameter diagonal subgroup $A = \{a(y) : t \in \mathbb{R}\}$:

$$U = \left\{g \in G : \lim_{t\to+\infty} a(y)\, g\, \widehat{a}(-t) = 1\right\}.$$

In particular, for every $t \geq 0$, elements $p \in P$ do not get expanded by the conjugation action of $a(y)$. By Lipschitz continuity of the coordinate maps $k \mapsto k_P(k)$, $k \mapsto p(s(k))$ on $B_K(r_0)$ with $r_0 > 0$ small enough, there exists a constant $C_1 > 0$, independent of t_1 and t_2, such that for every $k \in B_K(r_0)$, we have

$$k_P(k),\, \widehat{a}(t_j)p(s(k))\widehat{a}(-t_j) \in B_G(C_1 r_0).$$

By the Lipschitz continuity of ϕ_1 and ϕ_2, we have

$$\left| I_{\phi_1,\phi_2}(t_1,t_2) - \sum_{i=1}^{N}\int_K \kappa_i(k)\phi_1(\widehat{a}(t_1)u^-(s(k))k_i)\phi_2(\widehat{a}(t_2)u^-(s(k))k_i)\, d\mu_K(k)\right| \\ \ll_r\ r_0\, \mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2).$$

Recall that there exists a neighborhood V_S of the identity $1 \in S$ such that the map $h : V_S \to U^-$ given by sending $s \in S$ to $u^-(s)$ is well-defined and defines a diffeomorphism onto a neighborhood of the identity $1 \in U^-$. Hence, denoting by $u^- \mapsto s(u^-)$ the local inverse of this diffeomorphism, there exists a smooth density ρ_0 defined in a neighborhood of $1 \in U^-$ such that for all sufficiently small $r_0 > 0$ and all $f \in C_c(S)$ supported in $B_S(r_0)$, we have

$$\int_S f(s)\, d\mu_S(s) = \int_{U^-} f(s(u^-))\rho_0(u^-)\, d\mu_{U^-}(u^-).$$

Using the local decomposition of the measure μ_K as a product of the measures μ_{K_P} and μ_S, we have

$$\sum_{i=1}^{N}\int_K \kappa_i(k)\phi_1(\widehat{a}(t_1)u^-(s(k))k_i)\phi_2(\widehat{a}(t_2)u^-(s(k))k_i)\,d\mu_K(k)$$
$$= \sum_{i=1}^{N}\int_{K_P}\int_S \kappa_i(k_P s)\phi_1(\widehat{a}(t_1)u^-(s)k_i)\phi_2(\widehat{a}(t_2)u^-(s)k_i)\,d\mu_S(s)d\mu_{K_P}(k_P)$$
(6.8)
$$= \int_{K_P}\sum_{i=1}^{N}\left(\int_{U^-}\kappa_i(k_P s(u^-))\phi_1(\widehat{a}(t_1)u^-k_i)\phi_2(\widehat{a}(t_2)u^-k_i)\rho_0(u^-)\,d\mu_{U^-}(u^-)\right)d\mu_{K_P}(k_P).$$

By Theorem 6.2, applied to ϕ_1, ϕ_2 and the functions $f_{k_P,i}$, for $1\le i\le N$, defined on a neighborhood of $1\in U^-$ by

$$u^-\mapsto f_{k_P,i}(u^-)=\kappa_i(k_P s(u^-))\rho_0(u^-),$$

there exist constants $c'>0$, $C'>0$, independent of $f_{k_P,i}$, ϕ_1, ϕ_2, such that we have

$$\left|\int_{U^-} f_{k_P,i}(u^-)\phi_1(\widehat{a}(t_1)u^-k_i)\phi_2(\widehat{a}(t_2)u^-k_i)\,d\mu_{U^-}(u^-)-\mu_{U^-}(f_{k_P,i})\mu_\Omega(\phi_1)\mu_\Omega(\phi_2)\right|$$
$$\le C'\,e^{-c'\,m(t_1,t_2)}\mathcal{S}_r(f_{k_P,i})\mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2).$$

Observing that $\mathcal{S}_r(f_{k_P,i})\ll\mathcal{S}_r(\kappa_i)\mathcal{S}_r(\rho_0)$, that $N\le r_0^{-c_1}$, that $\mathcal{S}_r(\kappa_i)\ll r_0^{-c_2}$, that $\mathcal{S}_r(\rho_0|_{B_{U^-}(r_0)})\ll 1$, and that

$$\int_{K_P}\sum_{i=1}^{N}\mu_{U^-}(f_{k_P,i})d\mu_{K_P}(k_P)=\int_{K_P}\int_S\sum_{i=1}^{N}\kappa_i(k_P s)\,d\mu_S(s)d\mu_{K_P}(k_P)=1,$$

the integral (6.8) is equal to

$$\int_{K_P}\sum_{i=1}^{N}\left(\mu_{U^-}(f_{k_P,i})\mu_\Omega(\phi_1)\mu_\Omega(\phi_2)+O\left(e^{-c'\,m(t_1,t_2)}\,r_0^{-c_2}\,\mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2)\right)\right)d\mu_{K_P}(k_P)$$
$$=\mu_\Omega(\phi_1)\mu_\Omega(\phi_2)+O\left(e^{-c'\,m(t_1,t_2)}\,r_0^{-c_1-c_2}\,\mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2)\right)$$

Hence, putting everything together, we have

$$I_{\phi_1,\phi_2}(t_1,t_2)=\mu_\Omega(\phi_1)\mu_\Omega(\phi_2)+O\left((e^{-c'\,m(t_1,t_2)}\,r_0^{-c_1-c_2}+r_0)\,\mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2)\right).$$

Setting $c=\frac{c'}{1+c_1+c_2}$ and $r_0=e^{-c\,m(t_1,t_2)}$ now completes the proof of Proposition 6.1. ☐

7. Non-escape of mass

We equip the K-orbit $Y=K\Gamma/\Gamma$ of the identity coset $\Gamma\in\Omega$ with the push-forward measure ν_Y, defined via the orbital map $k\mapsto k\Gamma$, of the Haar probability measure μ_K on K. We will be interested in establishing upper bounds for the Siegel transform of bounded compactly supported functions $f:\widetilde{X}\to\mathbb{R}$ when evaluated at points of the translated K-orbit $a(y)Y$. To this end, similarly as in [**?**, Proposition 4.5], we begin by establishing a non-escape of mass property for $a(y)Y$.

We recall that, for every $g \in G$, we defined

$$\lambda_\chi(g\Gamma) = \min_{\boldsymbol{v} \in \mathbf{V}_\chi(\mathbb{Z}) \smallsetminus \{\mathbf{0}\}} \|g\boldsymbol{v}\|$$

to be the length of the shortest non-zero vector in $g\mathbf{V}_\chi(\mathbb{Z})$. Let us also recall that d stands for the dimension of X and β_χ is the Diophantine exponent of X with respect to χ. For each $0 < \delta < 1$, we define an open cusp neighborhood in Ω by

$$\Omega_\delta = \{g\Gamma \in \Omega : \lambda_\chi(g\Gamma) < \delta\}.$$

There exists a constant $\kappa > 0$ such that for all $\delta \in (0,1)$ and $t \in [\kappa \ln \delta^{-1}, +\infty)$, we have

$$\nu_Y\big(\{y \in Y : \lambda_\chi(a(y)y) < \delta\}\big) \ll \delta^{\beta_\chi d}.$$

Proof. Let us denote by $\mathbb{1}_{\Omega_\delta^c}$ the indicator function of the complementary subset $\Omega_\delta^c = \Omega \smallsetminus \Omega_\delta$. By Mahler's compactness criterion, the support of $\mathbb{1}_{\Omega_\delta^c}$ is compact. We fix once and for all a non-negative function $\rho_1 \in C_c^\infty(G)$ with $\int_G \rho_1 \, d\mu_G = 1$ and define $\chi_\delta = \rho_1 * \mathbb{1}_{\Omega_\delta^c} : \Omega \to [0, +\infty)$. Since μ_Ω is G-invariant, we have

$$\int_\Omega \chi_\delta \, d\mu_\Omega = \int_\Omega \mathbb{1}_{\Omega_\delta^c} \, d\mu_\Omega = \mu_\Omega(\{\lambda_\chi \geq \delta\}).$$

Moreover, by the G-invariance of μ_Ω again, for any differential operator $\mathcal{D}$, we have $\mathcal{D}\chi_\delta = \mathcal{D}(\rho) * \mathbb{1}_{\Omega_\delta^c}$. In particular, we have $\chi_\delta \in C_c^\infty(\Omega)$ and, letting $r \geq 1$ be the integer from Proposition 6.1, also $\mathcal{S}_r(\chi_\delta) \ll \mathcal{S}_r(\rho_1) \ll 1$. Moreover, there exists $\xi = \xi(\rho_1) > 1$ such that, for every $g \in \operatorname{supp}(\rho_1)$ and $x \in \Omega$, we have $\lambda_\chi(gx) \leq \xi \lambda_\chi(x)$. Therefore, for every $g \in \operatorname{supp}(\rho_1)$, we have

$$\{x \in \Omega : \lambda_\chi(gx) \geq \delta\} \subseteq \{x \in \Omega : \lambda_\chi(x) \geq \xi^{-1}\delta\}$$

and hence $\chi_\delta \leq \mathbb{1}_{\Omega^c_{\xi^{-1}\delta}}$. Thus, for every $t \geq 0$, we have

$$\begin{aligned}
\nu_Y(\{y \in Y : \lambda_\chi(a(y)y) \geq \xi^{-1}\delta\}) &= \int_Y \mathbb{1}_{\Omega^c_{\xi^{-1}\delta}}(a(y)y) \, d\nu_Y(y) \\
&\geq \int_Y \chi_\delta(a(y)y) \, d\nu_Y(y).
\end{aligned}$$

By Proposition 6.1, since ν_Y is the pushforward to Y of the Haar probability measure μ_K on K, there exists $c > 0$ such that, for every $t \geq 0$, we have

$$\begin{aligned}
\int_Y \chi_\delta(a(y)y) \, d\nu_Y(y) &= \int_K \chi_\delta(a(y)k\Gamma) \, d\mu_K(k) \\
&= \int_\Omega \chi_\delta \, d\mu_\Omega + O\left(e^{-ct}\mathcal{S}_r(\chi_\delta)\right) \\
&= \mu_\Omega(\{\lambda_\chi \geq \delta\}) + O(e^{-ct}).
\end{aligned}$$

By [?, Proposition 3.1.1], we have the measure estimate

$$\mu_\Omega(\Omega_\delta) \asymp \delta^{\beta_\chi d}. \tag{7.1}$$

Putting everything together, for every $t \geq 1$, this yields

$$\nu_Y(\{y \in Y : \lambda_\chi(a(y)y) \geq \xi^{-1}\delta\}) \geq \mu_\Omega(\{\lambda_\chi \geq \delta\}) + O\left(e^{-ct}\right) = 1 + O\left(\delta^{\beta_\chi d} + e^{-ct}\right).$$

Therefore, since ρ_1 is fixed and ξ only depends on ρ_1, for every $t \geq 1$, we have

$$\nu_Y(\{y \in Y : \lambda_\chi(a(y)y) < \delta\}) \ll (\xi\delta)^{\beta_\chi d} + e^{-ct} \ll \delta^{\beta_\chi d} + e^{-ct}.$$

Letting $\kappa = \frac{\beta_\chi d}{c}$, the claim holds for all $\delta \in (0,1)$ and $t \in [\kappa \ln \delta^{-1}, +\infty)$. $\square$

8. Approximation by smooth compactly supported functions

The Siegel transform of the indicator function of the set $\mathcal{F}_c$ appearing in the tessellation (4.6) is neither smooth nor bounded. In order to apply effective equidistribution results, we approximate $S_\chi \mathbb{1}_{\mathcal{F}_c}$ by smooth compactly supported functions. We again fix the integer $r \geq 1$ as in Proposition 6.1.

Lemma 8.1. *For every $\xi > 1$, there exists a family of functions $(D_\delta)_{\delta \in (0,1)}$ in $C_c^\infty(\Omega)$ satisfying*

$$0 \leq D_\delta \leq 1, \quad D_\delta = 1 \text{ on } \{\lambda_\chi \geq \xi\delta\}, \quad D_\delta = 0 \text{ on } \{\lambda_\chi < \xi^{-1}\delta\}, \quad \mathcal{S}_r(D_\delta) \ll 1.$$

The proof is essentially analogous to that of [?, Lemma 4.11] and we omit the details. We refer to the family $(D_\delta)_{\delta \in (0,1)}$ as a family of smooth cut-off functions on Ω and, fixing once and for all a $\xi > 1$ in the above lemma, we will omit ξ from the notation. For every $\delta \in (0,1)$ and measurable bounded compactly supported function $f : \widetilde{X} \to \mathbb{R}$, we define the *$\delta$-truncated Siegel transform* $S_\chi^{(\delta)} f : \Omega \to \mathbb{R}$ of f by

$$\forall\, g \in G, \quad S_\chi^{(\delta)} f(g) = D_\delta(g)\, S_\chi f(g\Gamma). \tag{8.1}$$

Next, we approximate $\mathbb{1}_{\mathcal{F}_c}$ for c arbitrarily close to 1 by a family of nonnegative smooth compactly supported functions.

For every $\varepsilon \in (0,1)$, we recall the definition of the ε-neighborhood $\mathcal{F}_c(\varepsilon)$ of $\mathcal{F}_c$ given by

$$\mathcal{F}_c(\varepsilon) = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_- < (1+\varepsilon)^{1+\beta_\chi}\, c\, \|\boldsymbol{v}^+\|^{-\beta_\chi}, (1+\varepsilon)^{-1} \leq \|\boldsymbol{v}^+\| < (1+\varepsilon)e\}.$$

There exists a family $(f_{\varepsilon,c})_{\varepsilon \in (0,1),\, c \in [1/2,3/2]} \subset C_c^\infty(\widetilde{X})$ and $\mathrm{supp}(f_{\varepsilon,c}) \subset \mathcal{F}_c(\varepsilon)$ satisfying the following properties:

$$\forall\, \varepsilon \in (0,1), \quad \mathbb{1}_{\mathcal{F}_c} \leq f_{\varepsilon,c} \leq 1, \quad \|f_{\varepsilon,c} - \mathbb{1}_{\mathcal{F}_c}\|_{L^1(\widetilde{X})} \ll \varepsilon, \quad \mathcal{S}_r(f_{\varepsilon,c}) \ll \varepsilon^{-r}, \tag{8.2}$$

and the implicit constants are uniform in $c \in [1/2, 3/2]$.

There exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0,1)$ and $t \in [-\frac{\beta_\chi}{\varepsilon_1} \ln(\beta_\chi \varepsilon), +\infty)$, we have

$$\int_Y |S_\chi f_{\varepsilon,c}(a(y)y) - S_\chi \mathbb{1}_{\mathcal{F}_c}(a(y)y)|\, d\nu_Y(y) \ll \varepsilon.$$

The implicit constant is uniform in $c \in [1/2, 3/2]$, but depends on β_χ and d.

Proof. Since $\mathrm{supp}(f_{\varepsilon,c}) \subset \mathcal{F}_c(\varepsilon)$, we have $S_\chi \mathbb{1}_{\mathcal{F}_c(\varepsilon)} - S_\chi \mathbb{1}_{\mathcal{F}_c} \geq S_\chi f_{\varepsilon,c} - S_\chi \mathbb{1}_{\mathcal{F}_c}$. The difference $\mathbb{1}_{\mathcal{F}_c(\varepsilon)} - \mathbb{1}_{\mathcal{F}_c}$ is bounded by the sum $\mathbb{1}_{\mathcal{R}_1(\varepsilon)} + \mathbb{1}_{\mathcal{R}_2(\varepsilon)} + \mathbb{1}_{\mathcal{R}_3(\varepsilon)}$ of indicator functions of the sets

$$\mathcal{R}_1(\varepsilon) = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_- < (1+\varepsilon)^{1+\beta_\chi}\, c\, \|\boldsymbol{v}^+\|^{-\beta_\chi}, (1+\varepsilon)^{-1} \leq \|\boldsymbol{v}^+\| < 1\},$$
$$\mathcal{R}_2(\varepsilon) = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_- < (1+\varepsilon)^{1+\beta_\chi}\, c\, \|\boldsymbol{v}^+\|^{-\beta_\chi}, e \leq \|\boldsymbol{v}^+\| < (1+\varepsilon)e\},$$
$$\mathcal{R}_3(\varepsilon) = \{\boldsymbol{v} \in \widetilde{X} : c\, \|\boldsymbol{v}^+\|^{-\beta_\chi} \leq \|u_{\boldsymbol{v}}^-\|_- < (1+\varepsilon)^{1+\beta_\chi}\, c\, \|\boldsymbol{v}^+\|^{-\beta_\chi}, 1 \leq \|\boldsymbol{v}^+\| < e\}.$$

In particular, we have $S_\chi f_{\varepsilon,c} - S_\chi \mathbb{1}_{\mathcal{F}_c} \leq S_\chi \mathbb{1}_{\mathcal{R}_1(\varepsilon)} + S_\chi \mathbb{1}_{\mathcal{R}_2(\varepsilon)} + S_\chi \mathbb{1}_{\mathcal{R}_3(\varepsilon)}$, and it is enough to show that for all $t \geq 0$ sufficiently large in terms of ε, we have, for every $i \in \{1, 2, 3\}$,

$$J_i(t) = \int_Y S_\chi \mathbb{1}_{\mathcal{R}_i(\varepsilon)}(a(y)y)\, d\nu_Y(y) = \sum_{\boldsymbol{v} \in \mathcal{P}_\chi} \int_K \mathbb{1}_{\mathcal{R}_i(\varepsilon)}(a(y)k\boldsymbol{v})\, d\mu_K(k) \ll \varepsilon.$$

We start with $J_1(t)$. Using polar coordinates on $\widetilde{X} \smallsetminus \{\mathbf{0}\}$ (see Section **??**), for every $\boldsymbol{v} \in \mathcal{P}_\chi$ there exist $k_{\boldsymbol{v}} \in K$ and $t(\boldsymbol{v}) \in \mathbb{R}$ such that $\boldsymbol{v} = k_{\boldsymbol{v}} \widehat{a}(t(\boldsymbol{v}))\boldsymbol{e}_\chi$. Taking norms of both sides and recalling that $\widehat{a}(t(\boldsymbol{v}))$ acts through the character χ on $\boldsymbol{e}_\chi$, we have $\boldsymbol{v} = \|\boldsymbol{v}\| k_{\boldsymbol{v}} \boldsymbol{e}_\chi$. Now, the right K-invariance of μ_K gives, for every $t \in \mathbb{R}$,

$$J_1(t) = \sum_{\boldsymbol{v}\in\mathcal{P}_\chi} \int_K \mathbb{1}_{\mathcal{R}_1(\varepsilon)}(a(y)\|\boldsymbol{v}\|k\boldsymbol{e}_\chi)\, d\mu_K(k).$$

Let $\boldsymbol{v} \in \mathcal{P}_\chi$ and let us define the two intervals $I_1(\varepsilon) = [(1+\varepsilon)^{-1}, 1)$ and $I_2(\varepsilon) = [0, (1+\varepsilon)^{1+\beta_\chi} c)$. We observe that, for every $k \in K$, we have $\mathbb{1}_{\mathcal{R}_1(\varepsilon)}(a(y)\|\boldsymbol{v}\|k\boldsymbol{e}_\chi) = 1$ if and only if

$$\mathbb{1}_{I_1(\varepsilon)}\left(\|(a(y)\|\boldsymbol{v}\|k\boldsymbol{e}_\chi)^+\|\right) \mathbb{1}_{I_2(\varepsilon)}\left(\|u^-_{a(y)k\boldsymbol{e}_\chi}\|_- \, \|(a(y)\|\boldsymbol{v}\|k\boldsymbol{e}_\chi)^+\|^{\beta_\chi}\right) = 1. \tag{8.3}$$

For every $\boldsymbol{v} \in \widetilde{X} \smallsetminus \{\mathbf{0}\}$ and $k \in K$, we have $(a(y)\|\boldsymbol{v}\|k\boldsymbol{e}_\chi)^+ = e^{-\frac{t}{\beta_\chi}}(\|\boldsymbol{v}\|k\boldsymbol{e}_\chi)^+$ and $\|u^-_{a(y)k\boldsymbol{e}_\chi}\|_- = e^t\|u^-_{k\boldsymbol{e}_\chi}\|_-$ (see Equation (4.3)). Moreover, we have $\|(\|\boldsymbol{v}\|k\boldsymbol{e}_\chi)^+\| = \|\boldsymbol{v}\|\,|\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|$, since $\boldsymbol{e}_\chi$ is unitary. Hence, (8.3) holds if and only if

$$\mathbb{1}_{I_1(\varepsilon)}\left(e^{-\frac{t}{\beta_\chi}}\|\boldsymbol{v}\|\,|\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|\right) \mathbb{1}_{I_2(\varepsilon)}\left(\|u^-_{k\boldsymbol{e}_\chi}\|_- \, (\|\boldsymbol{v}\|\,|\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|)^{\beta_\chi}\right) = 1.$$

By the definition of $\mathcal{R}_1(\varepsilon)$, there exists an absolute constant $\widehat{C} > 1$ such that for every $\boldsymbol{v} \in \mathcal{R}_1(\varepsilon)$, we have $\|u^-_{\boldsymbol{v}}\| < \widehat{C}$. In particular, if $[\mathcal{R}_1(\varepsilon)]$ denotes the corresponding set in X, then $[\mathcal{R}_1(\varepsilon)] \subset B_-(\widehat{C})\, x_0$. This implies that there exists a small constant $\widehat{c} > 0$ such that the region $\mathcal{R}_1(\varepsilon)$ is contained in the set $\mathcal{C} = \{\boldsymbol{v} \in \widetilde{X} \smallsetminus \{\mathbf{0}\} : \|\boldsymbol{v}^+\| \geq \widehat{c}\,\|\boldsymbol{v}\|\}$. For every $t \geq 0$, the set $\mathcal{C}$ is stable under the action of $a(y)$ and for every $\boldsymbol{v} \in \mathcal{C}$, written as $\boldsymbol{v} = \|\boldsymbol{v}\|k\boldsymbol{e}_\chi$ as above, we have $|\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle| \geq \widehat{c}$. Therefore, letting $K(\widehat{c}) = \{k \in K : |\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle| \geq \widehat{c}\}$, we have that $J_1(t)$ is given by

$$\sum_{\boldsymbol{v}\in\mathcal{P}_\chi} \int_{K(\widehat{c})} \mathbb{1}_{I_1(\varepsilon)}\left(e^{-\frac{t}{\beta_\chi}}\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|\right) \mathbb{1}_{I_2(\varepsilon)}\left(\|u^-_{k\boldsymbol{e}_\chi}\|_-(\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|)^{\beta_\chi}\right) d\mu_K(k).$$

Let us write $J_1(t) = B_1(t) + B_2(t)$, where $B_1(t)$ is given by

$$\sum_{\boldsymbol{v}\in\mathcal{P}_\chi} \int_{K((1+\varepsilon)^{-1})} \mathbb{1}_{I_1(\varepsilon)}\left(e^{-\frac{t}{\beta_\chi}}\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|\right) \mathbb{1}_{I_2(\varepsilon)}\left(\|u^-_{k\boldsymbol{e}_\chi}\|_-(\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|)^{\beta_\chi}\right) d\mu_K(k).$$

and $B_2(t) = J_1(t) - B_1(t)$. Consequently, for every $k \in K$ satisfying that $|\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle| \geq (1+\varepsilon)^{-1}$, if $\mathbb{1}_{I_1(\varepsilon)}\left(e^{-\frac{t}{\beta_\chi}}\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|\right) = 1$, then

$$(1+\varepsilon)^{-1} e^{\frac{t}{\beta_\chi}} \leq \|\boldsymbol{v}\| \leq (1+\varepsilon) e^{\frac{t}{\beta_\chi}}$$

Hence, if also $\mathbb{1}_{I_2(\varepsilon)}\left(\|u^-_{k\boldsymbol{e}_\chi}\|_-(\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|)^{\beta_\chi}\right) = 1$, then

$$\|u^-_{k\boldsymbol{e}_\chi}\|_- \ll e^{-t}.$$

The latter implies that we have $d(x_0, kx_0) \ll e^{-t}$. Together this gives,

$$\begin{aligned} B_1(t) &\ll \sum_{\boldsymbol{v}\in\mathcal{P}_\chi,\, \|\boldsymbol{v}\| = e^{\frac{t}{\beta_\chi}} + O(\varepsilon e^{\frac{t}{\beta_\chi}})} \int_X \mathbb{1}_{B_X(e^{-t})}(x)\, d\sigma_X(x) \\ &\ll \sum_{\boldsymbol{v}\in\mathcal{P}_\chi,\, \|\boldsymbol{v}\| = e^{\frac{t}{\beta_\chi}} + O(\varepsilon e^{\frac{t}{\beta_\chi}})} e^{-dt}. \end{aligned}$$

Using Theorem **??**, there exist constants $\varkappa_1 > 0$ and $\varepsilon_1 > 0$ such that

$$\#\{\boldsymbol{v} \in \mathcal{P}_\chi : \|\boldsymbol{v}\| \leq T\} = \varkappa_1 \, T^{\beta_\chi d}(1 + O(T^{-\varepsilon_1})),$$

and consequently

$$\#\{\boldsymbol{v} \in \mathcal{P}_\chi : (1+\varepsilon)^{-1} e^{\frac{t}{\beta_\chi}} \leq \|\boldsymbol{v}\| \leq (1+\varepsilon) e^{\frac{t}{\beta_\chi}}\} = 2\,\beta_\chi\, d\, \varkappa_1\, \varepsilon\, e^{dt} + O(\varepsilon^2 e^{dt} + e^{-\frac{\varepsilon_1 t}{\beta_\chi}} e^{dt}).$$

Hence we get $B_1(t) \ll \beta_\chi\, \varepsilon$ for all $t \geq -\frac{\beta_\chi}{\varepsilon_1} \ln(\beta_\chi\, \varepsilon)$.

Let us now bound the term $B_2(t)$, where we integrate over all $k \in K$ such that $\widehat{c} \leq |\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle| \leq (1+\varepsilon)^{-1}$. The vector $(k\boldsymbol{e}_\chi)^\perp = k\boldsymbol{e}_\chi - \langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle$ satisfies

$$\|u_{k\boldsymbol{e}_\chi}\|_- \asymp d(kx_0, x_0) \asymp \|(k\boldsymbol{e}_\chi)^\perp\| \geq 1 - |\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle| \gg \varepsilon.$$

In particular, there exists $\delta_0 > 0$ such that $\|u_{k\boldsymbol{e}_\chi}\|_- |\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|^{\beta_\chi} \geq \delta_0 \varepsilon$ is bounded away from zero. Thus, if $\mathbb{1}_{I_2(\varepsilon)}\left(\|u^-_{k\boldsymbol{e}_\chi}\|_- (\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|)^{\beta_\chi}\right) = 1$, then $\mathbb{1}_{I_2(\varepsilon)}\left(\delta_0\varepsilon\|\boldsymbol{v}\|^{\beta_\chi}\right) = 1$ and hence

$$\|\boldsymbol{v}\|^{\beta_\chi} < (1+\varepsilon)^{1+\beta_\chi} c(\delta_0\varepsilon)^{-1}. \tag{8.4}$$

Therefore, for every $t \geq 0$, the non-negative term $B_2(t)$ is bounded from above by

$$\sum_{\substack{\boldsymbol{v} \in \mathcal{P}_\chi \\ \|\boldsymbol{v}\|^{\beta_\chi} < (1+\varepsilon)^{1+\beta_\chi} c(\delta_0\varepsilon)^{-1}}} \int_{K((1+\varepsilon)^{-1}) \smallsetminus K(\widehat{c})} \mathbb{1}_{I_1(\varepsilon)}\left(e^{-\frac{t}{\beta_\chi}}\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|\right) d\mu_K(k). \tag{8.5}$$

We claim that, for all $t \geq \ln\left((1+\varepsilon)^{1+\beta_\chi} c(\delta_0\varepsilon)^{-1}\right)$, the expression (8.5) is zero. In fact, by (8.4) and since $\widehat{c} \leq |\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle| \leq (1+\varepsilon)^{-1}$, we have

$$e^{-\frac{t}{\beta_\chi}}\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle| \leq e^{-\frac{t}{\beta_\chi}}\left((1+\varepsilon)^{1+\beta_\chi} c(\delta_0\varepsilon)^{-1}\right)^{\frac{1}{\beta_\chi}} (1+\varepsilon)^{-1}$$

and, for all $t \geq \ln\left((1+\varepsilon)^{1+\beta_\chi} c(\delta_0\varepsilon)^{-1}\right)$, the right-hand side is smaller than $(1+\varepsilon)^{-1}$ (and hence $\mathbb{1}_{I_1(\varepsilon)}\left(e^{-\frac{t}{\beta_\chi}}\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|\right) = 0$).

The calculations for $J_2(t)$ and $J_3(t)$ are essentially analogous to that for $J_1(t)$ and we omit the details. □

9. Proof of Theorem D

We first generalize [**?**, Lemma 1.4], a useful tool in the theory of metric Diophantine approximation for deriving effective counting statements from an L^2-bound, to obtain such effective statements from an L^p-bound for arbitrary $p \in (1, 2]$. The idea of the proof goes back to the work of H. Weyl [**?**] on the equidistribution of numbers modulo one. Moreover, W. Schmidt strengthened [**?**, Lemma 1.4] in [**?**], obtaining the optimal error term, though we shall content ourselves with the following weaker version. Let $(\widehat{Y}, \nu_{\widehat{Y}})$ be a probability space and let $(\phi_{i,\ell} : \widehat{Y} \to \mathbb{R})_{i,\ell \in \mathbb{N}^*}$ be a family of non-negative random variables. Let $C_1 > 1$ be a constant and $(\overline{\phi}_{i,\ell})_{i,\ell \in \mathbb{N}^*}$ and $(\overline{\phi}_i)_{i \in \mathbb{N}^*}$ be families of real numbers satisfying, for all $i, \ell \in \mathbb{N}^*$, $0 \leq \overline{\phi}_{i,\ell} \leq \overline{\phi}_i \leq C_1$, and put $Z_{i,\ell} = \phi_{i,\ell} - \overline{\phi}_{i,\ell}$. Assume that $\sum_{i=1}^\infty \overline{\phi}_i = +\infty$ and that for some $p \in (1, 2]$ and $C_2 > 0$, we have

$$\forall\, N \in \mathbb{N}^*, \forall\, \ell \in \mathbb{N}^*, \quad \int_{\widehat{Y}} \Big|\sum_{i=1}^N Z_{i,\ell}(y)\Big|^p d\nu_{\widehat{Y}}(y) \leq C_2 \sum_{i=1}^N \overline{\phi}_i. \tag{9.1}$$

Let $\varepsilon > 0$ and let $(\ell_N)_{N\geq 1}$ be a sequence of positive integers. Then there is a constant $C_3 > 0$ such that almost surely

$$\forall\, N \in \mathbb{N}^*, \qquad \left|\sum_{i=1}^{N} Z_{i,\ell_N}\right| \leq C_3 \left(\sum_{i=1}^{N} \overline{\phi}_i\right)^{\frac{2}{p+1}+\varepsilon} \tag{9.2}$$

Proof. Let $\varepsilon > 0$ and let $(\ell_N)_{N\geq 1}$ be a sequence of positive integers. For every $N \in \mathbb{N}^*$ and $y \in \widehat{Y}$, we define

$$\Psi(N, y) = \sum_{i=1}^{N} \phi_{i,\ell_N}(y), \quad \Psi(N) = \sum_{i=1}^{N} \overline{\phi}_{i,\ell_N}, \quad \Phi(N) = \sum_{i=1}^{N} \overline{\phi}_i,$$

$$E(N, y) = \Psi(N, y) - \Psi(N) = \sum_{i=1}^{N} Z_{i,\ell_N}(y).$$

Let $x > 1$. For every $k \in \mathbb{N}^*$, let $N_k \in \mathbb{N}^*$ be the smallest integer with

$$\Phi(N_k) > k^{px-1}. \tag{9.3}$$

We remark that the sequence $(N_k)_{k\in\mathbb{N}^*}$ is increasing. Next, for every $k \in \mathbb{N}^*$, we define the subset A_k of $\widehat{Y}$ by

$$A_k = \{y \in \widehat{Y} : |E(N_k, y)| > k^{x+\varepsilon}\}.$$

By Chebyshev's inequality and our assumption (9.1), we have

$$\nu_{\widehat{Y}}(A_k) \leq \frac{1}{k^{px+p\varepsilon}} \int_{\widehat{Y}} |E(N_k, y)|^p \, d\nu_Y(y) \leq \frac{C_2 k^{px-1} + C_1}{k^{px+p\varepsilon}} \ll k^{-1-p\varepsilon}.$$

Hence $\sum_{k=1}^{\infty} \nu_{\widehat{Y}}(A_k)$ converges. By the Borel-Cantelli lemma, we have, for almost every $y \in Y$, that there exists an integer $k(y) \in \mathbb{N}^*$ such that for all $k \geq k(y)$, we have

$$|E(N_k, y)| \leq k^{x+\varepsilon}$$

and also, by (9.3),

$$k^{x+\varepsilon} \ll_x (k-1)^{x+\varepsilon} \ll \Phi(N_{k-1})^{\frac{x+\varepsilon}{px-1}}.$$

Now, for an arbitrary $N \in \mathbb{N}^*$ there exists $k \in \mathbb{N}^*$ such that $N_{k-1} \leq N < N_k$ and

$$\Psi(N_{k-1}, y) \leq \Psi(N, y) \leq \Psi(N_k, y).$$

Thus, for almost every $y \in Y$, there exists an integer $k(y) \in \mathbb{N}^*$ such that for all $k \geq k(y)$, we have

$$\Psi(N, y) = \Psi(N) + O\left(\Psi(N_k) - \Psi(N_{k-1}) + \Phi(N_{k-1})^{\frac{x+\varepsilon}{px-1}}\right), \tag{9.4}$$

Next, we note that, for all $k \in \mathbb{N}^*$, we have

$$\Psi(N_k) - \Psi(N_{k-1}) \leq \Phi(N_k) - \Phi(N_{k-1}) \ll k^{px-2} \ll \Phi(N_{k-1})^{\frac{px-2}{px-1}}. \tag{9.5}$$

We put $x = \frac{2+\varepsilon}{p-1}$, so that the exponents in (9.4) and (9.5) match. Plugging this back into $\frac{x+\varepsilon}{px-1}$, we find that

$$\frac{\frac{2+\varepsilon}{p-1} + \varepsilon}{p\frac{2+\varepsilon}{p-1} - 1} = \frac{2 + \varepsilon(2-p)}{p + 1 + p\varepsilon} \leq \frac{2 + \varepsilon(2-p)}{p+1} \leq \frac{2}{p+1} + \varepsilon.$$

The proof of Lemma 9 is complete. □

9.1. **Upper bound estimate.** We recall that, for all $\ell \in \mathbb{N}^*$, we defined

$$\widehat{c}_\ell = \Big(1 + C_0\,\ell^{-\beta_\chi}\Big)^{-(1+\beta_\chi)} \in (0,1)$$

and we let, for every $c \in [1/2, 3/2]$,

$$\mathcal{F}_c = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_- < c\|\boldsymbol{v}^+\|^{-\beta_\chi}, 1 \le \|\boldsymbol{v}^+\| < e\}.$$

We also defined, for every $i \in \mathbb{N}$, the times $t_i = \beta_\chi i$ and recall from Section 4.2 that there exists a constant $C_0 > 0$ such that for every $T' \geq 1$ and $x \in X$, we have the following lower and upper bounds on the lattice point counting function $\#\Big(k_x^{-1}\mathcal{P}_\chi \cap \mathcal{E}_{\beta_\chi}(e^{T'})\Big)$: for all large enough $\ell \in \mathbb{N}^*$, we have

$$\sum_{i=0}^{\lfloor T'+\ln\widehat{c}_\ell\rfloor-1} S_\chi \mathbb{1}_{\mathcal{F}_{c_\ell}}(\widehat{a}(t_i)k_x^{-1}\Gamma) - C_0\,\ell^{\beta_\chi d} \le \#\Big(k_x^{-1}\mathcal{P}_\chi \cap \mathcal{E}_{\beta_\chi}(e^{T'})\Big)$$

$$\le \sum_{i=0}^{\lceil T'-\ln\widehat{c}_\ell\rceil-1} S_\chi \mathbb{1}_{\mathcal{F}_{\widehat{c}_\ell^{-1}}}(\widehat{a}(t_i)k_x^{-1}\Gamma) + C_0\,\ell^{\beta_\chi d}. \tag{9.6}$$

The proof of Theorem D consists of effectively estimating the left- and right-hand sides of (9.6) and we start with the latter. There exist $\delta' \in (0,1)$ and $\varepsilon \in (0,1)$ such that the following holds. Let $\varkappa_1 = \lambda_{\widetilde{X}}(\mathcal{F})$ be the volume of $\mathcal{F} = \mathcal{F}_1$ and, for every $N \in \mathbb{N}^*$, let $\ell_N = \lfloor N^{\delta'}\rfloor$. Then for almost every $x \in X$ and for all large enough $N \in \mathbb{N}^*$, we have

$$\sum_{i=0}^{N-1} S_\chi \mathbb{1}_{\mathcal{F}_{\widehat{c}_{\ell_N}^{-1}}}(\widehat{a}(t_i)k_x^{-1}\Gamma) + C_0\,\ell_N^{\beta_\chi d} = \varkappa_1\,N + O_x(N^{1-\varepsilon}). \tag{9.7}$$

Using the inequality (9.6), by Lemma 4.1 and Proposition 9.1, this prove the desired upper bound in (1.7). The lower bound in (1.7) is shown analogously and we omit the details.

Proof of Proposition 9.1. We recall that Y denotes the K-orbit of the identity coset $\Gamma \in \Omega$ and ν_Y is the pushforward to Y under the orbit map $k \mapsto k\Gamma$ of the Haar probability measure μ_K on K. We will apply Lemma 9 with $(\widehat{Y}, \nu_{\widehat{Y}}) = (Y, \nu_Y)$. Let δ' and $(\ell_N)_{N\in\mathbb{N}^*}$ be as in the statement of Proposition 9.1. Let us define, for all $i \in \mathbb{N}^*$ and $\ell \in \mathbb{N}$, the function $\phi_{i,\ell} : Y \to \mathbb{R}$ and the number $\overline{\phi}_{i,\ell}$ by

$$\forall\, y \in Y, \quad \phi_{i,\ell}(y) = S_\chi \mathbb{1}_{\mathcal{F}_{c_\ell^{-1}}}(\widehat{a}(t_i)y), \quad \overline{\phi}_{i,\ell} = \int_\Omega S_\chi \mathbb{1}_{\mathcal{F}_{c_\ell^{-1}}}\, d\mu_\Omega.$$

Moreover, we define the function $Z_{i,\ell} : Y \to \mathbb{R}$ by $Z_{i,\ell} = \phi_{i,\ell} - \overline{\phi}_{i,\ell}$. Let us also put $\overline{\phi}_i = \overline{\phi}_{i,1}$ (so that, $i \in \mathbb{N}^*$ and $\ell \in \mathbb{N}$, $\overline{\phi}_i \geq \overline{\phi}_{i,\ell}$). By Lemma 9, we need to show that there exists $p \in (1,2]$, such that

$$\forall\, N \in \mathbb{N}^*, \forall\, \ell \in \mathbb{N}^*, \qquad \int_Y \left|\sum_{i=1}^N Z_{i,\ell}(y)\right|^p d\nu_Y(y) \ll N. \tag{9.8}$$

Fix $\ell \in \mathbb{N}^*$ and $\varepsilon_0 \in (0,1]$ such that the Siegel transform S_χ maps $B_c^\infty(\widetilde{X})$ into $L^{1+\varepsilon_0}(\Omega)$ (see Theorem A). Let us simply write $\mathbb{1}_\ell$ for $\mathbb{1}_{\mathcal{F}_{c_\ell^{-1}}}$.

First, we approximate the Siegel transform $S_\chi \mathbb{1}_\ell$ by the δ-truncated Siegel transform $S_\chi^{(\delta)}\mathbb{1}_\ell = D_\delta\, S_\chi \mathbb{1}_\ell$ (with $(D_\delta)_{\delta\in(0,1)}$ as in Lemma 8.1) with $\delta \in (0,1)$ to be determined later. That is, we would like to give an upper bound in terms of the truncation parameter δ for

$$\left\| \left(S_\chi \mathbb{1}_\ell \circ a(y) - \int_\Omega S_\chi \mathbb{1}_\ell \right) d\mu_\Omega - \left(S_\chi^{(\delta)} \mathbb{1}_\ell \circ a(y) - \int_\Omega S_\chi^{(\delta)} \mathbb{1}_\ell \, d\mu_\Omega \right) \right\|_{L^p(Y)}.$$

Note that, using Theorem A and Hölder's inequality with $p' = 1 + \varepsilon_0 > 1$ and $q' = (1 - \frac{1}{p'})^{-1}$, we have

$$\begin{aligned} \int_\Omega \left| S_\chi \mathbb{1}_\ell - S_\chi^{(\delta)} \mathbb{1}_\ell \right| d\mu_\Omega &= \int_\Omega \left| (S_\chi \mathbb{1}_\ell)(1 - D_\delta) \right| d\mu_\Omega \\ &\leq \left(\int_\Omega (S_\chi \mathbb{1}_\ell)^{p'} \, d\mu_\Omega \right)^{\frac{1}{p'}} \mu_\Omega \left(\{ \lambda_\chi \leq \xi^{-1} \delta \} \right)^{\frac{1}{q'}} \\ &\ll_{\mathrm{supp}(\mathbb{1}_\ell), p'} \delta^{\beta_\chi d (1 - \frac{1}{1+\varepsilon_0})}. \end{aligned}$$

Using Hölder's inequality again with $p \in (1, 1 + \varepsilon_0/2)$, $s = 1 + \varepsilon_0/2$, and $q = (\frac{1}{p} - \frac{1}{s})^{-1}$, and Lemma 7 with $t \geq \kappa \ln(\delta^{-1})$, we thus have

$$\begin{aligned} &\left\| \left(S_\chi \mathbb{1}_\ell \circ a(y) - \int_\Omega S_\chi \mathbb{1}_\ell \, d\mu_\Omega \right) - \left(S_\chi^{(\delta)} \mathbb{1}_\ell \circ a(y) - \int_\Omega S_\chi^{(\delta)} \mathbb{1}_\ell \, d\mu_\Omega \right) \right\|_{L^p(Y)} \\ &\quad \leq \left\| S_\chi \mathbb{1}_\ell \circ a(y) - S_\chi^{(\delta)} \mathbb{1}_\ell \circ a(y) \right\|_{L^p(Y)} + \int_\Omega \left| S_\chi \mathbb{1}_\ell - S_\chi^{(\delta)} \mathbb{1}_\ell \right| d\mu_\Omega \\ &\quad \ll \left\| S_\chi \mathbb{1}_\ell \circ a(y) \right\|_{L^s_{\nu_Y}(Y)} \nu_Y \left(\{ y \in Y : \lambda_\chi(a(y)y) < \delta \} \right)^{1/q} + \delta^{\beta_\chi d (1 - \frac{1}{1+\varepsilon_0})} \\ &\quad \ll \delta^{\beta_\chi d (\frac{1}{p} - \frac{1}{1+\varepsilon_0/2})}. \end{aligned} \tag{9.9}$$

Next, we approximate the δ-truncated Siegel transform $S_\chi^{(\delta)} \mathbb{1}_\ell$ by the Siegel transform of the smooth compactly supported approximation function $f_{\varepsilon,\ell} = f_{\varepsilon,c_\ell}$ with $\varepsilon \in (0,1)$, as constructed in (8.2). That is, we would like to give an upper bound in terms of the truncation parameter δ and in terms of the approximation parameter ε for

$$\left\| \left(S_\chi^{(\delta)} \mathbb{1}_\ell \circ a(y) - \int_\Omega S_\chi^{(\delta)} \mathbb{1}_\ell \right) d\mu_\Omega - \left(S_\chi^{(\delta)} f_{\varepsilon,\ell} \circ a(y) - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell} \, d\mu_\Omega \right) \right\|_{L^p(Y)}.$$

By the mean value formula in Theorem A and by the approximation properties of $f_{\varepsilon,\ell}$ (see Equation (8.2)), we have

$$\int_\Omega \left| S_\chi^{(\delta)} f_{\varepsilon,\ell} - S_\chi^{(\delta)} \mathbb{1}_\ell \right| d\mu_\Omega \leq \int_\Omega \left| S_\chi f_{\varepsilon,\ell} - S_\chi \mathbb{1}_\ell \right| d\mu_\Omega = \int_{\widetilde{X}} (f_{\varepsilon,\ell} - \mathbb{1}_\ell) \, d\lambda_{\widetilde{X}} \ll \varepsilon.$$

By the estimate in (3.6) applied with $\mathbb{1}_\ell - f_{\varepsilon,\ell}$, we have

$$\forall g \in G, \quad |S_\chi(\mathbb{1}_\ell - f_{\varepsilon,\ell})(g)| \ll_{\mathrm{supp}(\mathbb{1}_\ell - f_{\varepsilon,\ell})} \lambda_\chi(g\Gamma)^{-\beta_\chi d}.$$

This together with the fact that $\mathrm{supp}(D_\delta) \subseteq \{ x \in \Omega : \lambda_\chi(x) \geq \xi^{-1} \delta \}$, gives

$$\left\| S_\chi^{(\delta)} (\mathbb{1}_\ell - f_{\varepsilon,\ell}) \circ a(y) \right\|_{L^\infty(Y)}^{\frac{p-1}{p}} \ll \delta^{-\beta_\chi d \frac{p-1}{p}}.$$

Putting everything together, by Proposition 8, there exists $\varepsilon_1 > 0$ such that for all $t \in [-\frac{\beta_\chi}{\varepsilon_1}\ln(\beta_\chi \varepsilon), +\infty)$, we have

$$\begin{aligned}
&\Big\|\Big(S_\chi^{(\delta)}\mathbb{1}_\ell \circ a(y) - \int_\Omega S_\chi^{(\delta)}\mathbb{1}_\ell \, d\mu_\Omega\Big) - \Big(S_\chi^{(\delta)} f_{\varepsilon,\ell} \circ a(y) - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell}\, d\mu_\Omega\Big)\Big\|_{L^p(Y)} \\
&\quad \le \Big\| S_\chi^{(\delta)}\mathbb{1}_\ell \circ a(y) - S_\chi^{(\delta)} f_{\varepsilon,\ell} \circ a(y)\Big\|_{L^p(Y)} + \int_\Omega \Big| S_\chi^{(\delta)}\mathbb{1}_\ell - S_\chi^{(\delta)} f_{\varepsilon,\ell}\Big|\, d\mu_\Omega \\
&\quad \le \Big\| (S_\chi^{(\delta)}(\mathbb{1}_\ell - f_{\varepsilon,\ell})) \circ a(y)\Big\|_{L^\infty(Y)}^{\frac{p-1}{p}} \Big\| (S_\chi^{(\delta)}(\mathbb{1}_\ell - f_{\varepsilon,\ell}) \circ a(y)\Big\|_{L^1(Y)}^{\frac{1}{p}} + \varepsilon \\
&\quad \ll \delta^{-\beta_\chi d \frac{p-1}{p}} \varepsilon^{\frac{1}{p}} + \varepsilon \ll \delta^{-\beta_\chi d (1-\frac{1}{p})} \varepsilon^{\frac{1}{p}}.
\end{aligned} \tag{9.10}$$

Next, using the effective equidistribution of compact-orbit translates (see Proposition 6.1), we would like to give an upper bound in terms of the rate of equidistribution on

$$\int_Y \Big| \sum_{i=1}^N \Big(S_\chi^{(\delta)} f_{\varepsilon,\ell} - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell}\, d\mu_\Omega\Big)\Big|^p d\nu_Y = \Big\| \sum_{t=1}^N \Big(S_\chi^{(\delta)} f_{\varepsilon,\ell} - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell}\, d\mu_\Omega\Big) \circ a(y)\Big\|_{L^p(Y)}^p$$

Note that, for every $f \in C_c^\infty(\widetilde{X})$, we have $S_\chi^{(\delta)} f \in C_c^\infty(\Omega)$. Let us now show that, for every $f \in C_c^\infty(\widetilde{X})$, we have

$$\forall\, r \in \mathbb{N}^*, \quad \mathcal{S}_r(S_\chi^{(\delta)} f) \ll_{\mathrm{supp}(f)} \delta^{-\beta_\chi d} \mathcal{S}_r(f). \tag{9.11}$$

First, for every $f \in C_c^\infty(\widetilde{X})$ and every differential operator $\mathcal{D}$ as in (6.1), we note that $\mathcal{D}(S_\chi f) = S_\chi(\mathcal{D} f)$. Then, by the point-wise upper bound estimate (3.6) for the Siegel transform applied with $\mathcal{D} f$, we have

$$\forall\, g \in G, \quad |S_\chi(\mathcal{D} f)(g)| \ll_{\mathrm{supp}(f)} \mathcal{S}_r(f)\, \lambda_\chi(g\Gamma)^{-\beta_\chi d}.$$

Since $\mathrm{supp}(D_\delta) \subseteq \{x \in \Omega : \lambda_\chi(x) \ge \xi^{-1}\delta\}$ and $\mathcal{S}_r(D_\delta) \ll 1$, we deduce (9.11), as desired.

We recall that for every $i \in \mathbb{N}$, we defined $t_i = \beta_\chi i$. For every $i, j \ge 0$ we shall write $m(t_i, t_j) = \min\{t_i, t_j, |t_i - t_j|\}$. By Proposition 6.1, there are constants $c > 0$ and $C > 0$ such that, using also the estimate for the Sobolev norm (9.11) and the inequality $\mathcal{S}_r(f_{\varepsilon,\ell}) \ll \varepsilon^{-r}$ (see (8.2)), we have

$$\begin{aligned}
&\left\| \sum_{i=1}^{N} \left(S_\chi^{(\delta)} f_{\varepsilon,\ell} - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell} \right) \circ \widehat{a}(t_i) \right\|_{L^p(Y)} \leq \left\| \sum_{i=1}^{N} \left(S_\chi^{(\delta)} f_{\varepsilon,\ell} - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell} \right) \circ \widehat{a}(t_i) \right\|_{L^2_{\nu_Y}(Y)} \\
&= \left(\sum_{i,j=1}^{N} \int_Y \left(S_\chi^{(\delta)} f_{\varepsilon,\ell} - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell} \right) \circ \widehat{a}(t_i) \left(S_\chi^{(\delta)} f_{\varepsilon,\ell} - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell} \right) \circ \widehat{a}(t_j)\, d\nu_Y \right)^{1/2} \\
&\ll \left(\sum_{i,j=1}^{N} C\, \mathcal{S}_r(S_\chi^{(\delta)} f_{\varepsilon,\ell})^2\, e^{-c\,m(t_i,t_j)} \right)^{1/2} \\
&\ll \left(\sum_{i,j=1}^{N} C\, \delta^{-2\beta_\chi d}\, \mathcal{S}_r(f_{\varepsilon,\ell})^2\, e^{-c\,m(t_i,t_j)} \right)^{1/2} \\
&\ll \delta^{-\beta_\chi d}\, \varepsilon^{-r} \left(\sum_{i,j=1}^{N} e^{-c\,m(t_i,t_j)} \right)^{1/2}.
\end{aligned}$$

Next, we would like to show that

$$\left(\sum_{i,j=1}^{N} e^{-c\,m(t_i,t_j)} \right)^{1/2} \ll N^{1/2}. \tag{9.12}$$

We have

$$\sum_{\substack{i,j=1 \\ t_1 \leq \min\{t_j, |t_i - t_j|\}}}^{N} e^{-c\,m(t_i,t_j)} \leq \sum_{i,j=1}^{N} e^{-c\,t_i} = N \sum_{i=1}^{N} e^{-c\,t_i} \ll N.$$

By symmetry, we also have $\sum_{\substack{i,j=1 \\ t_j \leq \min\{t_i, |t_i - t_j|\}}}^{N} e^{-c\,m(t_i,t_j)} \ll N$, and, by letting $n = i - j$ and noting that $t_i - t_j = t_{i-j}$ if $i, j \geq 0$, we have

$$\sum_{\substack{i,j=1 \\ |t_i - t_j| \leq \min\{t_i, t_j\}}}^{N} e^{-c\,m(t_i,t_j)} \leq \sum_{n=-N}^{N} \sum_{\substack{i=1 \\ i \in [1+n, N+n]}}^{N} e^{-c\,|t_n|} \leq \sum_{n=-N}^{N} (N - |n|) e^{-c\,|t_{|n|}|} \ll N.$$

as desired. Hence, putting everything together, we have shown that

$$\left\| \sum_{i=1}^{N} \left(S_\chi^{(\delta)} f_{\varepsilon,\ell} - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell} \right) \circ \widehat{a}(t_i) \right\|_{L^p(Y)} \ll \delta^{-\beta_\chi d}\, \varepsilon^{-r}\, N^{1/2}. \tag{9.13}$$

Let κ be as in Proposition 7 and ε_1 as in Proposition 8. Later we will choose $\lambda_1, \lambda_2 > 0$ and let $\delta = N^{-\lambda_1}$ and $\varepsilon = N^{-\lambda_2}$. Let us note that for every such choices of λ_1, λ_2 and for all large $N \in \mathbb{N}^*$, we have

$$\frac{|\{i \in \{1, \ldots, N\} : t_i \geq \max\{\kappa\lambda_1 \ln N, \frac{\beta_\chi}{\varepsilon_1}\gamma_2 \ln(\beta_\chi^{1/\gamma_2} N)\}\}|}{N} = O\left(\frac{\ln N}{N} \right),$$

Hence the proportion of indices $i \in \{1, \ldots, N\}$ for which Lemma 7 and Proposition 8 do not apply to the time t_i is asymptotically negligible. Since we are free to choose $\lambda_1, \lambda_2 > 0$, let us determine them by a heuristic argument. So suppose for a moment that Propositions 7 and 8 hold for all $t \geq 0$. Then using Minkowski's inequality and combining (9.9), (9.10), and (9.13), we would have

$$\left\| \sum_{i=1}^{N} \left(S_\chi \mathbb{1}_\ell - \int_\Omega S_\chi \mathbb{1}_\ell \, d\mu_\Omega \right) \circ \widehat{a}(t_i) \right\|_{L^p(Y)} \ll N(\delta^{\beta_\chi d(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2})} + \delta^{-\beta_\chi d(1-\frac{1}{p})} \varepsilon^{\frac{1}{p}}) + N^{1/2} \delta^{-\beta_\chi d} \varepsilon^{-r}. \tag{9.14}$$

By setting

$$\delta = N^{-\frac{1}{2\beta_\chi d(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2} + 1 + rp(1 - \frac{1}{1+\varepsilon_0/2}))}} \quad \text{and} \quad \varepsilon = \delta^{\beta_\chi dp(1 - \frac{1}{1+\varepsilon_0/2})} \tag{9.15}$$

the exponents of the three terms on the right-hand side in (9.14) match. Hence we let

$$\lambda_1 = \frac{1}{2\beta_\chi d(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2} + 1 + rp(1 - \frac{1}{1+\varepsilon_0/2}))}$$

and

$$\lambda_2 = \frac{p(1 - \frac{1}{1+\varepsilon_0/2})}{2(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2} + 1 + rp(1 - \frac{1}{1+\varepsilon_0/2}))}.$$

Now having fixed λ_1 and λ_2, let us show that (up to a multiplicative constant) (9.14) holds true. We recall that, by Lemma 5.1, there exists an absolute constant $C \geq 1$ such that, for every $c \in [1/2, 2/3]$, we have

$$\sup_{t \geq 0} \int_Y \left| S_\chi \mathbb{1}_{\mathcal{F}_c}(\widehat{a}_t y) \right|^p d\nu_Y(y) \leq C.$$

Applying this estimate in the case where $t \in \mathbb{R}_+$ satisfies

$$t < \max\{\kappa \lambda_1 \ln N, \tfrac{\beta_\chi}{\varepsilon_1} \gamma_2 \ln(\beta_\chi^{1/\gamma_2} N)\},$$

using Minkowski's inequality and combining (9.9), (9.10), and (9.13), we have

$$\begin{aligned}
&\left\| \sum_{i=1}^{N} \left(S_\chi \mathbb{1}_\ell - \int_\Omega S_\chi \mathbb{1}_\ell \right) \circ \widehat{a}(t_i) \right\|_{L^p(Y)} \ll \left(\sum_{\substack{i=1 \\ t_i \geq \kappa \lambda_1 \ln(N)}}^{N} \delta^{\beta_\chi d(1-\frac{1}{p}} \right) + \left(\sum_{\substack{i=1 \\ t_i \leq \kappa \lambda_1 \ln(N)}}^{N} C \right) \\
&+ \left(\sum_{\substack{i=1 \\ t_i \geq \frac{\beta_\chi}{\varepsilon_1} \gamma_2 \ln(\beta_\chi^{1/\gamma_2} N)}}^{N} \delta^{\beta_\chi d(1-\frac{1}{p})} \right) + \left(\sum_{\substack{i=1 \\ t_i < \frac{\beta_\chi}{\varepsilon_1} \gamma_2 \ln(\beta_\chi^{1/\gamma_2} N)}}^{N} C \right) + \delta^{-\beta_\chi d} \varepsilon^{-r} N^{1/2} \\
&\ll N(\delta^{\beta_\chi d(1-\frac{1}{p})} + \delta^{-\beta_\chi d(1-\frac{1}{p})} \varepsilon^{\frac{1}{p}}) + N^{1/2} \delta^{-\beta_\chi d} \varepsilon^{-r}.
\end{aligned}$$

Plugging in the formulas for δ and ε, we have

$$\left\| \sum_{i=1}^{N} \left(S_\chi \mathbb{1}_\ell - \int_\Omega S_\chi \mathbb{1}_\ell \right) \circ \widehat{a}(t_i) \right\|_{L^p(Y)} \ll N^{1 - \frac{(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2})}{2(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2} + 1 + rp(1 - \frac{1}{1+\varepsilon_0/2}))}}. \tag{9.16}$$

Consider the function $h : (1, 1 + \varepsilon_0/2) \to \mathbb{R}$ given by

$$h(p) = 1 - \frac{(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2})}{2(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2} + 1 + rp(1 - \frac{1}{1+\varepsilon_0/2}))}.$$

This function satisfies $\lim_{p\to 1} h(p) < 1$. Let $p \in (1, 1+\varepsilon_0/2)$ be maximal with the property that $h(p) \leq \frac{1}{p}$. Then we have

$$\left\| \sum_{i=1}^{N} \left(S_\chi \mathbb{1}_\ell - \int_\Omega S_\chi \mathbb{1}_\ell \right) \circ \widehat{a}(t_i) \right\|_{L^p(Y)}^p \ll N.$$

Let $\delta' > 0$ and, for every $N \in \mathbb{N}^*$, let $\ell_N = \lfloor N^{\delta'} \rfloor$. By Lemma 9, for every $\varepsilon' > 0$ and for almost every $y \in Y$, we have

$$\forall\, N \in \mathbb{N}^*, \quad \sum_{i=0}^{N-1} S_\chi \mathbb{1}_{\mathcal{F}_{\widehat{c}_\ell^{-1}}} (\widehat{a}(t_i) k_x^{-1} \Gamma) = \int_\Omega S_\chi \mathbb{1}_{\mathcal{F}_{c_\ell^{-1}}} \, d\mu_\Omega \, N + O\left(N^{\frac{2}{p+1}+\varepsilon'} \right)$$

By the mean value formula in Theorem A, we have

$$\int_\Omega S_\chi \mathbb{1}_{\mathcal{F}_{c_\ell^{-1}}} \, d\mu_\Omega = \lambda_{\widetilde{X}} \left(\mathcal{F}_{c_\ell^{-1}} \right).$$

There exists a constant $\kappa' > 0$ such that $\lambda_{\widetilde{X}} \left(\mathcal{F}_{c_\ell^{-1}} \right) = \lambda_{\widetilde{X}} (\mathcal{F}_1) + O(\ell^{-\kappa'})$. Plugging this into the inequality (9.6), we have, for all $N \in \mathbb{N}^*$,

$$\begin{aligned} &\sum_{i=0}^{N-1} S_\chi \mathbb{1}_{\mathcal{F}_{\widehat{c}_\ell^{-1}}} (\widehat{a}(t_i) k_x^{-1} \Gamma) + C_0 \, \ell_N^{\beta_\chi d} \\ &\qquad = \lambda_{\widetilde{X}} (\mathcal{F}_1) \, N + O(\lfloor N^{\delta'} \rfloor^{-\kappa'}) + O\left(N^{\frac{2}{p+1}+\varepsilon'} \right) + O\left(\lfloor N^{\delta'} \rfloor^{\beta_\chi d} \right). \end{aligned}$$

This completes the proof of Proposition 9.1 with

$$\delta' = \frac{1}{\beta_\chi d} \frac{2}{p+1}, \quad \varepsilon = 1 - \frac{2}{p+1}, \quad \text{and} \quad \varkappa_1 = \lambda_{\widetilde{X}}(\mathcal{F}).$$

□

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