Fast BEM-Solution of Laplace Problems With $\mathcal{H}$-Matrices and ACA

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The main problems for applying *boundary element methods* (BEM) in computational electromagnetism are related to the large memory requirements of the matrices and the convergence of the iterative solver. In this paper, we solve a Laplace problem with mixed boundary conditions by making use of a variational symmetric direct boundary integral equation. The Galerkin discretization results in densely populated matrices that are here compressed by *adaptive cross approximation*. This leads to an approximation of the underlying BEM-operator by means of so-called *hierarchical matrices* ($\mathcal{H}$-Matrices). These matrices are then used to construct an effective preconditioner for the iterative solver. Numerical experiments demonstrate the application of the method.

***Index Terms*—Algebra, boundary element methods (BEM), boundary integral equations, data compression, stability, symbol manipulation.**

I. Introduction

ONE OF the most often appearing operators in scientific computing of physical problems is the Laplace operator. It occurs on bounded or unbounded domains with different types of boundary conditions, depending on the problem. When talking about electromagetism, one may think about electrostatics, magnetostatics, or dc conduction. A general, stable, and precise numerical solution procedure would, therefore, be an ultimate goal for multiple applications. We will present here such a solution, that is applicable on Laplace problems on homogeneous domains with mixed boundary conditions or pure Dirichlet boundary conditions. The method can be generalized to multiple homogeneous domains and for pure Neumann problems [4]. But these generalizations are not part of this paper. Here we will restrict ourselves on a model problem in order to make the basic ideas of symmetric variational formulation, discretization, compression, and preconditioning clearly visible.

The solution procedure is based on the *boundary element method* (BEM), since it has to treat not only bounded, but also unbounded domains. We are faced with the standard problem of densely populated BEM-matrices and we will apply the *adaptive cross approximation* (ACA) [1], [2] in order to overcome this problem. Unfortunately, this is not the only challenge when talking about industrial applications. Due to the often very complicated underlying geometry, we find the matrices to be ill conditioned. Thus, the iterative solution poses a serious problem. An effective (i.e., storage saving and fast) preconditioner is required, that is very well adapted to the geometry and the operator. The clustering of the boundary integral operator in near-field and far-field blocks, together with the ACA matrix approximations, reflect in a sense the geometry as well as the operator. These ACA matrix approximations have a special structure, they can be regarded as *hierarchical matrices* ($\mathcal{H}$-matrices) [9], [12], [13]. Hackbusch *et al.* first introduced arithmetical operations on the set of these $\mathcal{H}$-matrices. The arithmetics can be used for the construction of an approximative LU-decomposition $M \approx L \cdot U$ of the BEM stiffness matrix M by a lower triangular matrix L and an upper triangular matrix U. This yields the desired highly effective preconditioner [3].

An alternative to our approach is to make use of the multipole method for compression and to precondition the matrix with an operator of inverse order [7], [15], [16], [18], [19]. An advantage of our approach is that ACA and $\mathcal{H}$-matrix preconditioning are purely algebraic manipulations acting on the original matrix entries. Thus they can also be used for all other types of asymptotically smooth operators.

II. Model Problem

The model problem that we will attack is an ohmic conductor with a homogeneous linear conductivity σ that occupies the region Ω. Constant voltages of $\pm U$ are applied at the ends of the conductor $\partial\Omega^{\pm U}$, as shown in Fig. 1. We compute the resulting current density $\mathbf{j}$ in the conductor. Ampère's law for the electric field $\mathbf{curl}\,\mathbf{E} = 0$ allows the use of a scalar electric potential φ for its description $\mathbf{E} = -\mathbf{grad}\,\varphi$. The continuity equation of the current density $\mathrm{div}\,\mathbf{j} = 0$ together with Ohm's law $\mathbf{j} = \sigma \cdot \mathbf{E}$ leads to a Laplace problem with mixed boundary conditions

$$\triangle\varphi(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \Omega, \tag{1}$$

$$\partial_{\mathbf{n}}(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \Gamma_N := \partial\Omega \backslash \partial\Omega^{\pm U} \tag{2}$$

$$\varphi(\mathbf{x}) = \pm U, \quad \forall \mathbf{x} \in \Gamma_D := \partial\Omega^{\pm U}. \tag{3}$$

III. Symmetric Formulation and Discretization

The potential in the interior of the conductor can be calculated by making use of Green's representation formula [14]

$$\varphi(\mathbf{x}) = V[t] - K[\varphi], \quad \forall \mathbf{x} \in \Omega \tag{4}$$

where V represents the single-layer boundary integral operator

$$V(\mathbf{x}) := \int_{\Gamma} \frac{t(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} dS(\mathbf{y})$$

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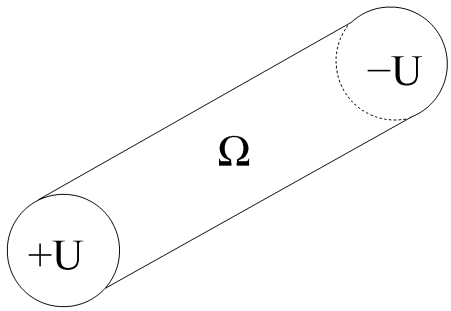

Fig. 1. Cylindrical conductor contacted at its ends.

acting on the Neumann-data $t := \partial_{\mathbf{n}}\varphi$ of the boundary $\Gamma := \partial\Omega$. K is the double-layer boundary integral operator

$$K(\mathbf{x}) := \int_C \varphi(\mathbf{y}) \cdot \partial_{\mathbf{n_y}} \left(\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \right) dS(\mathbf{y})$$

acting on the Dirichlet-data φ. Next, the Calderón projector (i.e., the Dirichlet-trace γ_0 and the Neumann-trace γ_1) is applied to the representation formula (4). This operation leads to a system of two boundary integral equations

$$\begin{bmatrix} \gamma_0\varphi(\mathbf{x}) \\ \gamma_1\varphi(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K^* \end{bmatrix} \begin{bmatrix} \gamma_0\varphi \\ \gamma_1\varphi \end{bmatrix}, \quad \forall \mathbf{x} \in \partial\Omega. \tag{5}$$

Herein, I is the identity, D is the hypersingular operator, and K^* is the adjoint of the double-layer operator. Testing these equations, i.e., multiplying them with a testfunction and integrating them over the boundary (abbreviation $<,>$) yields a variational formulation of the problem [8], [17].

A boundary triangulation enables the use of a discretization $\varphi_h = \sum_{i=1}^{n_N} \varphi_i h_i$ and $t_h = \sum_{j=1}^{n_D} t_j c_j$. Herein, h_i are linear nodal basis functions, $1 \le i \le n_N$, for the unknown potential φ on the n_N nodes of the Neumann boundary Γ_N. For the unknown normal derivative t of the potential on the n_D triangles of the Dirichlet-boundary Γ_D, we make use of constant basis functions $c_j, 1 \le j \le n_D$. This leads to the following linear system of equations:

$$\begin{bmatrix} -V & K \\ K^* & D \end{bmatrix} \begin{bmatrix} t \\ \varphi \end{bmatrix} = \begin{bmatrix} f_D \\ f_N \end{bmatrix} \tag{6}$$

with

$$\begin{aligned} V_{ij} &:= \langle c_i, V[c_j] \rangle, \quad i,j = 1,\ldots,n_D \\ K_{ij} &:= \langle c_i, K[h_j] \rangle, \quad i = 1,\ldots,n_D,\ j = 1,\ldots,n_N \\ D_{ij} &:= \langle h_i, D[h_j] \rangle, \quad i,j = 1,\ldots,n_N \end{aligned}$$

that has to be solved. The right-hand sides

$$\begin{aligned} (f_D)_i &:= -\frac{1}{2}\langle c_i, g_D \rangle - \langle c_i, K[g_D] \rangle, \quad i = 1,\ldots,n_D \\ (f_N)_j &:= -\langle h_j, D[g_D] \rangle, \quad j = 1,\ldots,n_N \end{aligned}$$

result from the given voltage $g_D := \pm U$ on the Dirichlet boundary. The stiffness matrix is symmetric and the problem is uniquely solvable for nonvanishing Dirichlet-boundaries Γ_D. Edges or corners of the workpiece pose no problem for this Galerkin-type of discretization, which is a great advantage for practical simulations.

IV. ACA-Compression

The resulting BEM-matrix is densely populated and has to be compressed in order to compute problems of realistic problem sizes that result in the analysis of complicated three-dimensional (3-D)-industrial applications. A general method for the compression of boundary integral operators with asymptotically smooth kernels, like all the BEM-operators that are involved in our symmetric formulation of (6), is the so-called adaptive cross approximation [1], [2]. In this method, one first divides the boundary triangulation of Γ into pairwise disjoint clusters Γ_τ of contiguous triangles. Diameter, center, and distance of such clusters can be defined easily. The partition of Γ automatically also divides the BEM-stiffness matrix M corresponding to the boundary-integral over $\Gamma \times \Gamma$ into submatrices $M_{\Gamma_\tau \times \Gamma_\sigma}$ corresponding to the clusters $\Gamma_\tau \times \Gamma_\sigma$. A submatrix belongs to the far-field if the diameters of the clusters are small compared to their distance, meaning that they fulfill the admissibility condition

$$\min\{\operatorname{diam}\Gamma_\tau, \operatorname{diam}\Gamma_\sigma\} \le \eta \cdot \operatorname{dist}(\Gamma_\tau, \Gamma_\sigma)$$

with the admissibility parameter η, which is typically chosen between (0.5, 1.5). If the admissibility condition is not fulfilled, then the submatrix belongs to the near-field. The submatrices of the near-field remain uncompressed, whereas the normally dense submatrices of the far-field are approximated using only a few rows and columns, i.e., crosses of the submatrix. Other matrix entries, except for these crosses, do not need to be calculated. One applies this approximation iteratively, starting from only one cross, adding more and more crosses and stopping the sequence of approximations if the difference between two consecutive cross-approximations is small enough. Thus one finds adaptively a low-rank approximation of the formerly dense submatrices of the far-field. This is the way in which ACA compresses the BEM-stiffness-matrix M to an approximation $\tilde{M}$. Thereby, the matrix-vector multiplication speeds up significantly. A picture of the resulting ACA-matrix-approximation with the ranks of the submatrices is shown in Fig. 2. Full rank submatrices have a dark color. The ACA-method uses the matrix entries only and is, therefore, kernel-independent in contrast to other compression methods like fast multipole [10], panel clustering [11], or the $\mathcal{H}^2$-matrix approximation [5], [6] that are based on expansions or interpolations of the kernel-function.

V. $\mathcal{H}$-Matrix Preconditioning

The BEM-stiffness-matrix is often badly conditioned in real life applications, due to the complicated underlying geometry. Thus a good preconditioner is required if an iterative Krylov-type solver is to converge sufficiently quickly. Such a preconditioner can be constructed by making use of the special submatrix-structure of the ACA matrix approximation $\tilde{M}$, since the submatrices can be regarded as hierarchical matrices

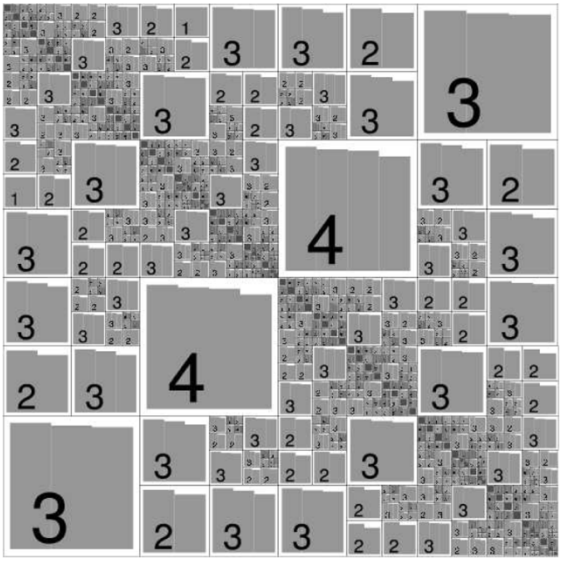

Fig. 2. An ACA-matrix-approximation with its rank distribution.

[9], [12], [13]. The set of $\mathcal{H}$-matrices $\mathcal{H}(\Gamma \times \Gamma, k)$ of rank k on $\Gamma \times \Gamma$ is defined as

$$\mathcal{H}(\Gamma \times \Gamma, k) := \{M_{\Gamma_\tau \times \Gamma_\sigma} \mid \operatorname{rank}(M_{\Gamma_\tau \times \Gamma_\sigma}) < k, \text{ for all admissible } \Gamma_\tau, \Gamma_\sigma\}.$$

When trying to introduce arithmetical operations on the set of these $\mathcal{H}$-matrices, one is faced with the problem, that the space $\mathcal{H}(\Gamma \times \Gamma, k)$ is not closed with respect to these operations. It holds for example, that the rank of the sum of two rank-k matrices can exceed k. Therefore, one has to define special, so-called formatted additions $\oplus$ and multiplications $\odot$, that give results within the space $\mathcal{H}(\Gamma \times \Gamma, k)$, i.e.,

$$A \oplus B \in \mathcal{H}(\Gamma \times \Gamma, k), \quad A \odot B \in \mathcal{H}(\Gamma \times \Gamma, k), \quad \forall A, B \in \mathcal{H}(\Gamma \times \Gamma, k).$$

For the formatted addition of two $\mathcal{H}$-matrices $A, B \in \mathcal{H}(\Gamma \times \Gamma, k)$, this is done by the following procedure. First, perform a singular value decomposition of $C := A + B$. Next, order the singular values $s_1 \geq s_2 \geq, \ldots, \geq s_{2k}$ values with respect to their magnitude and cut of all s_i with $i > k$. Fig. 3 illustrates the procedure. The result is an optimal approximation $\tilde{C} \in \mathcal{H}(\Gamma \times \Gamma, k)$ of C, in the sense that $||\tilde{C} - C||$ is minimal in $\mathcal{H}(\Gamma \times \Gamma, k)$ in the Frobenius or the spectral norm. The formatted multiplication can be defined in a similar fashion. These formatted arithmetics are now used in a recursive procedure for the construction of an approximative LU-decomposition of the ACA-compressed stiffness matrix $\tilde{M}$, see Fig. 4. The idea of $\mathcal{H}$-matrix preconditioning is to choose a reasonable small $k' < k$ and perform the LU-decomposition with this small rank. The result is the desired efficient preconditioner, which uses much less memory than $\tilde{M}$ due to the smaller rank.

The described LU-decomposition can be accomplished regardless of the underlying integral equations, since the $\mathcal{H}$-matrix arithmetics as well as the ACA-compression are purely algebraic manipulations that are concentrated on the matrix level. For further details on the topic we refer to [3] and [9].

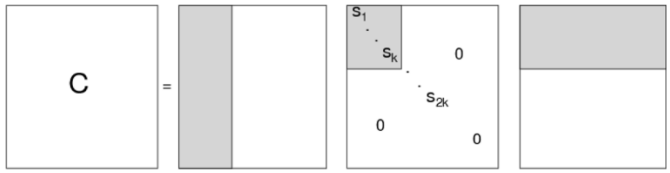

Fig. 3. Formatted addition by singular value decomposition.

$$\begin{aligned} &\text{Step 1:} && L_{11} \odot U_{11} \approx \tilde{M}_{11} \\ &\text{Step 2:} && L_{11} \odot U_{12} \approx \tilde{M}_{12} \\ &\text{Step 3:} && L_{21} \odot U_{11} \approx \tilde{M}_{21} \\ &\text{Step 4:} && L_{22} \odot U_{22} \oplus L_{21} \odot U_{12} \approx \tilde{M}_{22} \end{aligned}$$

$$\Longrightarrow$$

$$\begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} \approx \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix}$$

Fig. 4. Approximate LU-decomposition.

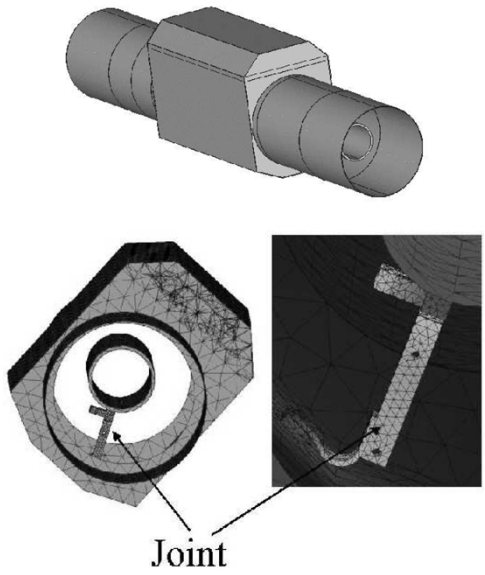

Fig. 5. Two conducting parts connected by a thin joint.

VI. Numerical Experiments

The geometry of Fig. 5, which is a part of an ABB generator circuit breaker, is used to perform some numerical experiments with the described method. It consists of two cylindrical parts (enclosure and breaking chamber) that are connected via a thin joint (earthing knife). The enclosure and the breaking chamber are both connected at one end to the power supply, i.e., the mentioned voltage $\pm U$. The biggest potential-drop takes place in the earthing knife. The current density in the conductor can easily be calculated by $\mathbf{j} = -\sigma \cdot \mathbf{grad}\, \varphi$. The small geometrical detail of the thin joint, which is quite tiny compared to the whole geometry, causes the convergence problems. It must be resolved properly for a precise result, since the potential drops there. Additionally the convergence gets worse if the Dirichlet boundary Γ_D, i.e., the contacts are small compared with the workpiece. Both must be cured by the preconditioner.

The BEM-stiffness matrix is symmetric but not necessarily positive definite. Therefore, we apply an iterative biconjugate

TABLE I
EXPERIMENTS ON COMPRESSION AND PRECONDITIONING

N	Co.	Matrix	Pre	No Pre	LU Pre
	%	MB / sec	MB / sec		Steps / sec
6901	40	73 / 118	25 / 12	0.009	11 / 2
11371	29	143 / 235	44 / 24	0.023	15 / 6
23291	18	364 / 609	101 / 66	0.002	15 / 13
36873	15	797 / 1382	173 / 149	0.029	22 / 41

gradient stabilized method (BiCGStab). We either use the described $\mathcal{H}$-matrix-based LU-decomposition for preconditioning, or we do not use any preconditioner at all. Table I presents the results of some numerical experiments that are computed on a 2.8-GHz Intel Xeon CPU with 2.0 GB RAM for several different discretizations of the geometry.

The first column of Table I shows the number of unknowns. The second column shows the amount of memory that is required to store the ACA-compression matrix. It is expressed as a percentage of the memory required by an uncompressed matrix. The third column shows the amount of storage for the compressed matrix in MB and the time to fill it in seconds. The fourth column shows the amount of storage for the preconditioner in MB and the time to construct it in seconds. Since the unpreconditioned BiCGStab did not converge for this example, we show in the fifth column the residual of the unpreconditioned BiCGStab after 1000 steps. The last column shows the number of BiCGStab steps and the time in seconds that is required to reach a relative residual smaller than 10^{-10} if the LU-preconditioner is applied.

We want to remark, that the implementation of the symmetric formulation is rather trivial, if one has ACA and the $\mathcal{H}$-matrix arithmetics as library modules on-hand. Due to their kernel independency, ACA-compression and $\mathcal{H}$-matrix preconditioning can be utilized for quick solutions of other systems of equations.

VII. CONCLUSION

The numerical experiments in Table I show that the memory and convergence problems are removed by the applied method and that Laplace problems with more than 36 000 BEM unknowns on complex 3-D industrial geometries can be solved on standard 32-bit desktop computers. The method can handle parts with edges and corners and is also applicable on unbounded domains. Major parts of the program, namely ACA-compression and $\mathcal{H}$-matrices preconditioning, are purely algebraic based algorithms. Thus, these code-modules can easily be reused for the compression and preconditioning of other BEM-operators satisfying the required smoothness conditions.

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