ST3233: Tutorial 4

September 2016

1 MA(1): size of the first autocorrelation coefficient

Prove that for any value of $\alpha \in \mathbb{R}$, the first autocorrelation coefficient $\rho(1)$ of the MA(1) process with dynamics $X_k = W_k + \alpha\, W_{k-1}$ is such that $|\rho(1)| < 1/2$.

2 AR(1): accuracy of the Yule-Walker estimates

Consider a AR(1) model of the type $X_k = \alpha\, X_{k-1} + W_k$ for a Gaussian white noise process $\{W_k\}_{k \geq 0}$. Suppose that we collect data $\{x_1, \ldots, x_T\}$ and would like to fit such an AR(1) model to it.

1. Show that the Yule-Walker estimate for α coincides with the estimate of the first autocorrelation coefficient: $\widehat{\alpha} = \widehat{\rho}(1)$.
2. Generate a time series of length $T = 50$ following an AR(1) model with $\alpha = 1/2$ and compute the estimate $\widehat{\alpha}$.
3. Do the same experiement 10^3 times and plot a histogram of the estimates $\widehat{\alpha}$. This is called a bootstrap estimate of the ditribution of $\widehat{\alpha}$. What accuracy can one expect from using the Yule-Walker estimate on a time series of length $T = 50$?
4. Do the same as above, but with time series of length $T = 500$.

3 Partial Autocorrelation (slightly difficult but instructive)

Consider a stationary time series $\{y_k\}_{k \geq 0}$ with mean zero. For any $h \geq 0$, consider the linear regression of y_k on $(y_{k-1}, y_{k-2}, \ldots, y_{k-h})$; in other words, we are looking for the coefficients $(c_{h,1}, c_{h,2}, \ldots, c_{h,h})$ that minimizes

$$\mathbb{E}\left[\left(y_k - [c_{h,1}\, y_{k-1} + c_2\, y_{k-2} + \ldots + c_{h,h}\, y_{k-h}]\right)^2\right]$$

The coefficient $\phi(h) = c_{h,h}$ is called the partial autocorrelation of order h.

1. Express $\phi(1)$ in terms of the usual autocorrelation coefficient $\rho(1)$. To do so, minimize the function $F(\alpha) = \mathbb{E}\left[\left(y_k - \alpha\, y_{k-1}\right)^2\right]$.
2. Express $\phi(2)$ in terms of the usual autocorrelation coefficients $\rho(1)$ and $\rho(2)$. To do so, minimize the function $G(\alpha, \beta) = \mathbb{E}\left[\left(y_k - [\alpha\, y_{k-1} + \beta\, y_{k-2}]\right)^2\right]$.