Tricks and Techniques

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I knew a lot of things. Over time, I forgot many things. Now, I only know a few things. Therefore I took the wise decision of noting stuff down. To start, I try to summarize the most elementary things I knew, and the hope is that I'll keep on adding new things that I learn. For now, it's almost all the undergrad stuff which I am trying to summarize as quickly as possible.

Contents

Chapter 1

Feel of the scale : Read twice everyday

I always disliked remembering numerical values. I was satisfied with the qualitative nature of things. Fortunately, I found a very old book (Newtonian Mechanics by AP French) which made me realise the kind of grave mistake I was doing. If you don't wish to read him, the summary is : A physicist must know the rough scale of things.

Gravitational acceleration (g)	$10\ \mathrm{m/sec^2}$
Densities of solids and liquids	10^3–$10^4\ \mathrm{kg/m^3}$
Density of air at sea level	$1\ \mathrm{kg/m^3}$ (approx.)
Length of day	10^5 sec (approx.)
Length of year	3.16×10^7 sec $\approx 10^{7.5}$ sec
Earth's radius	6400 km
Angle subtended by finger thickness at arm's length	1° (approx.)
Thickness of paper	0.1 mm (approx.)
Mass of a paperclip	0.5 g (approx.)
Highest mountains, deepest oceans	10 km (approx.)
Earth–moon separation	3.8×10^5 km
Earth–sun separation	1.5×10^8 km
Atmospheric pressure	Equivalent to weight of $1\ \mathrm{kg/cm^2}$ or a 10-m column of water
Avogadro's number	6.0×10^{23}
Atomic masses	1.6×10^{-27} kg to 4×10^{-25} kg
Linear dimensions of atoms	10^{-10} m (approx.)
Molecules/$\mathrm{cm^3}$ in gas at STP	2.7×10^{19}
Atoms/$\mathrm{cm^3}$ in solids	10^{23} (approx.)
Elementary charge (e)	1.6×10^{-19} C
Electron mass	10^{-30} kg (approx.)
Speed of light	3×10^8 m/sec
Wavelength of light	6×10^{-7} m (approx.)

Chapter 2

Magic of Infinitesimals

2.1 Continuity

2.1.1 Any mid value

Statement: If function $f(x)$ is continuous in the interval $[a, b]$ with $f(a) = A$ and $f(b) = B$, then every value in the interval $[A, B]$ is achieved. Namely, $\forall N \in [A, B], \exists \xi \in [a, b]$ such that $f(\xi) = N$

Proof: Not required.

2.1.2 Max and Min value

Statement: If the function is continuous in a closed interval than it has a largest and a smallest value for atleast one x in that interval. (Other way of saying that it doesn't become infinite).

Use: In rolle's theorem, for the existence of highest value.

2.2 Differentiability

2.2.1 Rolle's Theorem

Statement : If the function is differentiable in the interval $[a, b]$ and $f(a) = f(b)$, $\exists\ \xi \in (a, b)$ such that $f'(\xi) = 0$.

Proof : Use sign change/existence argument at maximum.

Use : Finding remainder term in taylor series. Proving L'Hopital rule and various other places where we construct a new function on which we can apply Rolle to prove certain thing about a given function.

2.2.2 Mean Value theorem

Statemnt : For a differentiable function in a given interval $[a, b]$, $\exists\ \xi \in (a, b)$ such that $f'(\xi) = \frac{f(b)-f(a)}{b-a}$

Proof : Using rolle's theorem.

Use : $f(b) = f(a) + (b-a)f'(a + \theta(b-a))$, $\theta \in [0, 1]$.

2.2.3 Taylor series

A function which can be differentiated arbitary number of times can always be expanded as

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^n}{n!}f^{(n)}(a) + R_n \quad (2.1)$$

The remainder term can be beautifully calculated using rolle's theorem as

$$R_n = \frac{(b-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi)$$

Clearly, this tends to zero as n tends to infinity if the derivative is bounded (which is, because we assumed it exists). A clever trick for finding out taylor series for indefinite integrals is to find the taylor series of integrand and integrate them.

2.3 Infinitesimals

Summing up (infinite times), or dividing is same as summing up (infinite times) or divding only the principal part.

Three properties are important for geometric visualization of calculus:

Property 1 : Any general arc differes from its chord by an infinitestimal of a higher order. Using this, we can imainge arcs as chords (and same extension in higher dimensions, curved surface as planes). We can find the curve length by ignorigng this higher order infinitesimal differnce between arc and chord because infinite sum of infinitesimals is same as infinite sum of their principal parts.

Property 2 : Perpendicular distance from one end of the infintestimal arc to the tangent at the other end is an infitesimal of higher order than arc, however the length of the tangent from the foot of prependicular to the point of tangency is of the same order. You can prove this by using maclaurine series in

transformed coordinates or by simple geometric construction.

Property 3 : Same trigonometric laws are obeyed. This is because the chord becomes tangent to the arc as their length approaches zero. The rules of chord extend to the arcs.

2.4 Differentials

Note that for an independent variable $d^2x = 0$, not otherwise. For independent variable x, differential dx is same as infinitesimal increment Δx to x. For dependent variable y, the differential dy is the principal part of the infinitesimal change Δy in y. The rules for differentials can easily be derived.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$$

$$\Delta y = f'(x)\Delta x + \epsilon \Delta x$$

$$dy = f'(x)dx$$

Knowing usual rules for differnetials, we can use them as algebraic quantities, no need to think of differential operators.

Note the difference between the context of the symbol $\frac{d^2y}{dx^2}$. Which has context independent meaning only when x is independent variable.

Implicit Function Theorem : If $F(x, y) = 0$ and $|F_x| < \infty$ with $F_y \neq 0$, then $y(x)$ exists and is differentiable. Can be easily extended to higher dimensions and/or involving more than one function using jacobians.

Chapter 3

Sequences

I list all the theorems which you can prove in order and digest.

Defn : A sequence (x_n) is said to convere to x if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|x_n - x| < \epsilon, \forall n > N$.

Theorem 1 : If sequence (x_n) converges, it has a unique limit point.

Thorem 2 : If sequence (x_n) converges, it is bounded. (Bounded : $\exists M \in \mathbb{R}$ such that $\forall n, |x_n| < M$).

Theorem 3 : If a **monotone** sequence (x_n) is bounded, then it converges. The limit is $sup\{x_n : n \in \mathbb{N}\}$

Theorem 4 : Every subsequence of a convergent sequence converges to the limit of sequence.

Theorem 5 : Every sequence has a monotone subsequence.

Theorem 6 : (Bolzano) Every bounded sequence has a convergent subsequence. (Just a consequence of prev theorem)

Defn : Limsup/ Liminf of sequence (a_n) are defined by

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} sup(\{a_k : k >= n\})$$
$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} inf(\{a_k : k >= n\})$$

Thoerem 7 : limsup/liminf of a bounded sequence always exists.

Theorem 7 : Let (x_n) be a bounded sequence. Then $\exists$ subsequences (x_{n_k}) and (x_{m_k}) s.t.

$$\lim_{n\to\infty} x_{n_k} = \limsup x_n$$

$$\lim_{n\to\infty} x_{n_k} = \liminf x_n$$

Theorem 8 : If every convergent subsequence of (x_n) has the same limit x, then (x_n) converges to x.

Cauchy Sequence : A sequence (x_n) is cauchy sequence, if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon, \forall m, n > N$

Theorem 9 : A sequence (x_n) converges iff it's a Cauchy sequence.

This theorem is great, because we don't need to know x beforehand to check if the sequence converges to x. All we care about is the behaviour of the sequence. Namely, terms get closer and closer. Also useful for deriving various tricks to check convergence of a series.

Chapter 4

Series

Same presentation as chapter 2.

Defn : If (a_n) is a a sequence of real numbers then the series $\sum_{n=1}^{\infty} a_n$ converges to $S \in \mathbb{R}$ if the seuqnece (S_n) of partial sum converges to S.

Trick : For geometric and telescopic series, you can find explicit formula for n^{th} term of the sequence (s_n).

Theorem 1 : A series Σa_n of positive term converges iff its partial sums are bounded.

Theorem 2 : (Cauchy Condition) The series Σa_n converges iff $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|S_n - S_m| < \epsilon, \forall n > m > N$

Defn : Series Σa_n converges absolutely if $\Sigma |a_n|$ converges. It converges conditionally if Σa_n converges but $\Sigma |a_n|$ diverges.

Theorem 4 : Absolutely convergent series Σa_n converges. Moreover, Σa_n converges absolutely, iff series Σa_n^+ and Σa_n^- converges.

Follwoing tests can all be proved simply, using cauchy criterion.

Test for divergence : If series Σa_n converges, then $\lim_{n\to\infty} a_n = 0$

Comparision test : If $b_n \geq 0, \Sigma b_n$ converges, then Σa_n converges absolutely if $|a_n| \leq b_n$

Ratio test : If (a_n) is a sequence of nonzero real numbers, let

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Series Σa_n converges absolutely if $0 \leq r < 1$, diverges if $1 < r \leq \infty$

Root test : For (a_n)

$$r = \lim_{n \to \infty} sup|a_n|^{\frac{1}{n}}$$

The series converges absolutely if $0 \leq r < 1$ and diverges if $1 < r < \infty$.

Alternating series test : If (a_n) is a decreasing sequence of nonnegative real numbers, such that $lim_{n \to \infty} a_n = 0$, then the alternating series $\Sigma(-1)^{n+1} a_n$ converges.

Limit Comparisiont test : If Σa_n and Σb_n are two seires and

$$r := \lim_{n \to \infty} \left(a_n / b_n \right)$$

If $r = 0$: If Σb_n converges, then Σa_n converges.
If $r = \infty$: If Σb_n diverges, then Σa_n diverges.
IF r is finite, either both converge or both diverge.

Theorem 4 : Every rearrangement of an absolutely convergent series converges to the same sum.

4.1 Power series :

There's only one power series for a given funcion (Coefficients are obviously uniquely defined by differentiation).

- If a power series converges for some $x = x_1$, it converges absolutely and uniformly for all $|x_2| < |x_1|$. This can be proven using comparision test.

  Follwowing are the conseqences of uniform convergence :

  1. The function defined by power series is continuous inside the region of convergence.
  2. The power series can be integrated term by term in the region of convergence.
  3. The power series can be differentiated term by term. This is a more restrictive condition : the continuity of all individual derivatives as well as the uniform convergence of the series formed by derviative is required.

- On the boundary of region of convergence, there is atleast one singularity.

Chapter 5

Complex Variables

- Cauchy reimann conditions for analyticity are sufficient when first order derivatives are also continuous. Analyticity is defined as differentiability in a region.
- As an example $|z|^2$ is differentiable at $z = 0$ but not analytic anywhere.
- The corresponding real functions of analytic function are harmonic and the conjugate functions form an orthogonal family. Moving along one would give maximum change in the other. The mapping is conformal at points when $f'(z) \neq 0$ or $f'(z) \neq \infty$.

5.1 Cauchy's Theorem and Integral Formula

If $f(z)$ is analytic in a simply connected region and C is a closed contour in that region then

$$\oint_C f(z)dz = 0 \tag{5.1}$$

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz = f(z_0) \tag{5.2}$$

1. Taylor and laurent series are simple consequence of this.
2. Residue is very important for performing various integrals and summing series(as that is the only term that will remain after integration).
3. Apart from poles and essential singularity, branch points are also a type of singularity.
4. for $z^{\frac{1}{2}}$, the branch point is at $z = 0$, order is defined as the number of paths around that point to return to same value. In this case it's 2.

5.2 Analytic Continuation

In real analysis, a function defined on a given range can be smoothly extended to outer regions in many ways. There's only a unique way to do so for analytic functions. This great property makes complex analysis almost GODLY.

If two analytic functions coincide in any region, or on any finite line segment, they are the same function.

Reminiscent of the fact that if a function is is analytic once, it can be differentiated arbitary number of times, hence we have all coefficients of the taylor series. For functions of real variables, the coefficients of taylor series can be chosen arbitarily.

Chapter 6

Differential Equations

6.1 A notational devil

You have seen the notation

$$\frac{\partial f}{\partial y^i} = \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial y^i}$$

Here, f is a function which can have different form as expressed in y coordinates. The notation $\frac{\partial f}{\partial y^i}$ really means that we are differentiating the function f expressed in y coordinates with respect to the i^{th} entry. In a more precise form it is seen as

$$\frac{\partial f}{\partial y^i} := \partial_i (f \circ y^{-1})$$

I won't even mention ODE's with constant coefficients.

6.2 First Order ODE

- One can always find integrating factor for first order ODE in principle. In practice, we only know direct formula for **linear** first order ODE. This also means that every differential constraint (in two dimensions) is holonomic.
- List of famous non-linear first order equations that can be solved

  1. Bernoulli equation

  $$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

  2. Clairaut's equation

  $$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$$

3. D'Alembert's equation

$$y = xf\left(\frac{dy}{dx}\right) + g\left(\frac{dy}{dx}\right)$$

6.3 Second Order Linear ODE

Most important differential equations arising in physics are almost always second order and linear (ordinary or partial). Let's consider only the homogenous part right now.

$$y'' + P(x)y' + Q(x) = 0$$

Ordinary Point : x_0 for which $P(x_0)$ and $Q(x_0)$ are finite.

Regular Singular Point : Singular x_0 for which $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ remain finite.

Irregular Singular Point : Singular x_0 for which one of $(x - x_0)P(x)$ or $(x - x_0)^2Q(x)$ diverge.

- The only trick we know is Frobenius! Always check if the final solution converges. You have to expand about a clever point. Expanding about essential singularity won't work.

- Expressing ODE as $L(x)y(x) = 0$. If $L(x) = L(-x)$, then if $y(x)$ is a solution, so is $y(-x)$. So you can express general solution as a linear combination of odd and even.

Wronskian : To test for linear independence of functions, we obtain n equations by differentiating the relation

$$\sum_{i=0}^{n} a_i\phi_i(x) = 0$$

This is the origin of Wronskian! Also used in some ninja-technical way to find particular solution for inhomogenous linear equation.

6.3.1 Sturm-Liouville Theory

In physics, we don't really require general solutions. We require solutions with specific boundary conditions. Specific boundary conditions impose conditions on the parameters of the equation. For instance, for a string clamped at $x = 0$ and $x = l$, in the differential equation $\frac{d^2\psi}{dx^2} + k^2\psi(x) = 0$, k will have certain restrictions as you know. Here, k^2 is the eigenvalue.

Charecterization of general features of eigenproblems arising **from linear second-order differential equations** is known as Strum-Liouville theory.

Consider $\mathcal{L}\psi(x) = \lambda\psi(x)$, where

$$\mathcal{L}(x) = p_0(x)\frac{d^2\psi}{dx^2} + p_1(x)\frac{d\psi}{dx} + p_2(x)$$

Now $\mathcal{L}$ is known as self adjoint if $p_0'(x) = p_1(x)$, which enables us to write

$$\mathcal{L}(x) = \frac{d}{dx}\left[p_0(x)\frac{d}{dx}\right] + p_2(x)$$

We can show that

$$\int_a^b v^*(x)\mathcal{L}u(x)\,dx = [v^*p_0u' - (v^*)'p_0u]_a^b + \int_a^b (\mathcal{L}v)^*u\,dx$$

- Dirichlet Boundary : u and v both vanish at endpoints.
- Neumann Boundary : u' and v' both vanish at endpoints.
- Any second order linear operator can be turned into Sturm-Liouville form by changing the scalar product to include weight factor.

If u and v are eigenfunctions of $\mathcal{L}$ with respective eigenvalues λ_u and λ_v, then we can see that they are orthogonal. **This is great beacause we have just proven orthogonality of Trigonometric, Bessel, Legendre, Hermite, Laguerre, Chebyshev. Further, completeness of eigenfunctions is the origin of fourier series!**

Namely, if $f(x)$ satisfy boundary conditions of Sturm-Liouville and is continous and piecewise differentiable on $[a, b]$ then eignefunction expansion of f converges uniformly to f on $[a, b]$. If it's piecewise differentiable, then it converges to $[f(x_+) + f(x_-)]/2$ on $[a, b]$.

To solve, inhomogenous strum lioville equation, we solve the eigenequation of the given operator with given boundary condition, which allows us to find the eigenfunction expansion of the inhomogenous equation. There is a theorem which states that there will be a unique solution (modulo homegenous solution) for non-homogenous strum-liouville boundary problem for a given boundary equations.

6.3.2 Special Functions and Polynomials

A summary of results about special functions can be found Here

6.3.3 Variational Method

Suppose that the spectrum of the hamiltonian H has a finite greatest lower bound. In such cases, we have a simple yet very powerful method to guess the ground state energy and wavefunction.

$$\langle \Psi | H | \Psi \rangle \geq E_0$$

This allows us to guess a paramaterized wavefunction and choose the parameter which minimizes the expectation value. The result can only get better because the ground state has lowest energy among all such ψ

6.4 Partial Differential Equations

Second order (mostly linear, but even non-linear) partial differential equations are everywhere. It's all of physics.

6.4.1 Charecteristic

Understanding boundary conditions is important. Consider for example a simple first order PDE

$$\mathcal{L}\phi = a\frac{\partial \phi}{\partial x} + b\frac{\partial \phi}{\partial y} = 0$$

We generally try to change variables in such a manner that the equation reduces to something which only contains derivative w.r.t one term. Here $s = ax+br, t = bx - ay$ does the job. Reducing it to

$$\frac{\partial \phi}{\partial s} = 0 \quad \implies \phi(x, y) = f(bx - ay)$$

Observe that the solution is constant along constant t. It is known as the charecterisic curve. It's not necessary that solution remains constant on charecteristic cuve, but it can be solved using ODE methods on the charecteristic curve. If $\phi(x, y)$ is specified on a curve segment, one can deduce its value on all the charecteristics that intersects it.

$$\phi(x, y) = \phi(x_0, y_0) + \frac{\partial \phi(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial \phi(x_0, y_0)}{\partial y}(y - y_0) + ...$$

We can obtain the values of partial derivatives using the equations determining the curve and the given partial differential equation (Arfken p.405).

If the boundary curve is along a charecteristic, then there may be inconsistencies. IF the boundary intersects same charecteristic on more than one point, then it may lead to inconsistencies as well.

Exact general analysis of boundary conditions is complicated. Look up when you need it. For reference of three simple yet important PDE's

Table 9.1 Relation between PDE and Boundary Conditions

Boundary Conditions	Class of Partial Differential Equation		
	Elliptic	Hyperbolic	Parabolic
	Laplace, Poisson in (x, y)	Wave equation in (x, t)	Diffusion equation in (x, t)
Cauchy			
Open surface	Unphysical results (instability)	**Unique, stable solution**	Too restrictive
Closed surface	Too restrictive	Too restrictive	Too restrictive
Dirichlet			
Open surface	Insufficient	Insufficient	**Unique, stable solution** in one direction
Closed surface	**Unique, stable solution**	Solution not unique	Too restrictive
Neumann			
Open surface	Insufficient	Insufficient	**Unique, stable solution** in one direction
Closed surface	**Unique, stable solution**	Solution not unique	Too restrictive

6.5 Integral Transforms

Due to integration by parts, we can change derivatives to algebraic equations by integral transforms. Additionally, we have shifting property to receive answers modulo exponential.

6.5.1 Fourier Transform

6.5.2 Laplace Transform

6.5.3 Legendre Transformation

A function $f(x, y)$ can be seen as a function g of u and y by certain transformation. (Note : $u = \frac{\partial f}{\partial x}$)

$$g = f - ux$$

This is key to going from lagrangian formalism to hamiltonian.

6.6 Dirac Delta

Any sequence of function, whose integral over all space is unity and it vanishes for all points except at zero as n goes to ∞ will converge to dirac delta function. Extremely useful in quantum field theory.

1. $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$

6.7 Green's Functions

Linear differential equations have the property of superposition. Let's solve for a δ function source. If we call the solution G, the problem with arbitary source can be reduced to quadrature (to solving integrals), if we know G.

$$\hat{L}(x)G(x,x') = \delta(x-x') \tag{6.1}$$

$$\hat{L}(x)u(x) = f(x) \tag{6.2}$$

$$u(x) = \int f(x')G(x,x')dx'$$

Here, $hatL$ is a linear operator.

Chapter 7

Algebra

A beautiful summary can be found in Weyl's book on Classical groups.

- It is useful to think of two different vector spaces, one consisting of matrices and the other of operators. One consisting of row/column vectors and the other consisting of abstract vectors. The matrices and row/column are then just representation of these abstract stuff.
- Check Dirac's quantum book for relation between minimal polynomial and the eigenvalues.
- Same operator is represented by different matrices in different basis. All this matrices are connected by Similarity transformation ($S^{-1}A\,S$). Since the properties of the operator are not dependent on the basis chosen, all these matrices will share these properties. Note again that these matrices are different stuff in the vector space consisting of matrices, but they are same in a sense since they describe the same abstract operator. Therefore, they have same determinant, trace and eigenvalues (as can be easily checked). They form an equivalence class in the vector space of matrices.
- Inner product is an additional structure over vector space which has 3 defining properties.

  1. Conjugate Symmerty : $< x, y >= < y, \bar{x} >$
  2. Linearity in first argument : $< ax + by, z >= a < x, z > +b < y, z >$
  3. $< x, x >> 0$

  Without this structure, it's not possible to talk to about angles between vectors and norm of the vectors. Inner product is just a metric. Note however, that linear transformations are represented by matrices, just in the same way, inner products (metric) is represented by matrix (with elements $g(e_i, e_j)$). However, inner product and linear transformations are fundamentally different things. Linear transformations are operators

which take a vector to another vector, that is $T : V \to V$ whereas metric (or inner product) is $g : V \times V \to \mathbb{F}$. In a particular basis the inner product gives $\bar{x}^\intercal M x$, where x is just a column vector containinig components. Firstly, M can always be taken to be a symmetric (hermitian) matrix by the axioms of inner product. We know that under a different basis, this column vector will change like Sx where S is an invertible matrix. Since we want the value to remain same, we see that the representation of M under change of basis changes like $S^\intercal M S$. For linear operator, the representation would change like $S^{-1} T S$. Note that any symmetric matrix can be diagonalized using an orthogonal (unitary) matrix. What really defines the inner product is than the number of positive, negative and zero numbers in the diagonal form. This is sylvester's law of inertia which reveals the fundamental form of metric. There can only be as many inner product in n dimensional vector space as there are combinations of $(+, -, 0)$. Note that because of different transformation properties of linear operator and of the metric, we cannot bring the linear operator in $(+1, -1, 0)$ form but we can always bring the metric in $(+1, -1, 0)$ form. Obviously we cannot have n fixed eigenvectors for an operator acting on n dimensional vector space unless it's the identity operator, which has same matrix in every basis. Further, others which have fixed eigenvectors(upto inversion) are also special operators, cannot be done for every operator. To summarize, $(1, 1)$ tensors have eigenvalues and $(0, 2)$ tesnors have signature.

- An algbera is a vector space with product between the vectors. It is defined by 3 additional properties
  1. Right distributivity : $(x + y)z = xz + yz$
  2. Left distributivity : $z(x + y) = zx + zy$
  3. Scalar compatibility : $(ax)(by) = (ab)(xy)$
- Eigenvalues of Hermitian operator are real and eigenvectors with different eigenvalues are orthogonal. Eigenvectors are complete. (Similarly for symmetric operator).
- Eigenvalue of $MM\dagger$ is **positive**. Prove by inserting the identity operator!
- Hermitian operators can always be diagonalized by a unitary matrix. A quick way to see this is to realize that eigenvectors are orthogonal, so we just need to do rotation of the axes to align them in the directiof eigenvectors. Therefore the change of basis matrix is a rotation(orthogonal, unitary in complex case) matrix.
- In differential geometry, we have vector space at each point (made up by differentials).

7.1 Invariant Subspace

The concept of invariance is widespread in mathematics. We know what is an invariant subgroup, invariant subalgebra. Similarly, for a given linear operator, the invariant subspace is defined. It is useful in characterizing reducible and irreducible representations.A representation is reducible if it contains a nontrivial invariant subspace. Note that the representation is just a linear operator. Invariant subspace W of operator $T : V \to V$ is a subspace W of V that is preserved. That is $T(W) \subseteq W$. Given an invariant subspace W of T, V can be decomoposed into the direct sum $V = W \oplus W'$.

Matrix representation of T in an appropriately chosen basis (Most natural way ofc, the basis of invariant subspace and add the remaining) is upper triangular. Note that off diagonal terms of a matrix serve the role of mixing two subspaces. Now, when V is broken into $W \oplus W'$, the top right off-diagonal term brings components of W' subspace into W. There's no restriction there. But, the bottom left off-diagonal part brings components W into W', which cannot happen since W is an invariant subspace of T.

Chapter 8

Asymptotics and Perturbation Methods

$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1 := f(x) \sim g(x) \quad \text{as} \quad x \to x_0$

$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0 := f(x) << g(x) \quad \text{as} \quad x \to x_0$

Given an asymptotic sequence(each successive term having lower order) $\{\phi_i\}$
The coefficients of the assymptotic expansion

$$f(x) \sim a_1\phi_1(x) + a_2\phi_2(x) + ...$$

are uniquely determined.

If the remainder term is of lower order than the last term of the expansion for all n, the exapnsion is called asymptotic.

Caution1 : Two given functions differing by transcendentally small terms (exponentials as compared to powers) can have same asymptotic expansions. Hence, these terms are ignored in asymptotic expnsions (in powers of x). TST are related to essential singularities.

Caution2 : Need to keep higher order terms in exponentials (including sin,cos,sinh,cosh etc). Not doing so, can give wrong prefactor.

Caution3 : Differentiating can cause trouble too. Look at "Tauberian Theorems" (BO - p.127)

8.1 For Integrals

8.1.1 Integration by Parts

In general, if we have

$$I(x) = \int_a^b f(t)e^{x\phi(t)}dt \quad \text{as} \quad x \to \infty$$

$$I(x) = \int_a^b \frac{f(t)}{x\phi'(t)} d(e^{x\phi(t)})$$

$$I(x) = \frac{1}{x}\left[\frac{f(t)}{\phi(t)}e^{x\phi'(t)}\right]_{t=a}^{t=b} - \frac{1}{x}\int_a^b \frac{d}{dt}\left[\frac{f}{\phi'}\right]e^{x\phi(t)}dt$$

We hope that the second integral has lower order than the first term and that $\phi'(t)$ is not zero anywhere in the interval. This will fail when asymptotic expansion involves *log* or fractional powers of x.

8.1.2 Laplace's Method

For "sharply-peaked" integrands, where the dominant contribution comes from the nbd of a single point, we can expand the function involved in exponent around that point.

Example1 :

$$I(x) = \int_{-10}^{10} e^{-xt^2}dt \quad \text{as} \quad x \to \infty$$

Integration by parts won't work because $\phi'(0) = 0$. Dominant contribution comes for $t = 0$.

$$I(x) = \int_{-\infty}^{\infty} e^{-xt^2}dt - 2\int_{10}^{\infty} e^{-xt^2}dt$$

First part is simple gaussian integral, we can estimate the second integral by replacing t^2 bt $10t$. Which will show that the second term is transcendentally smaller than the first term.

$$I(x) \sim \sqrt{\frac{\pi}{x}}$$

Stirling's Approximation :

$$\Gamma(x+1) = \int_0^{\infty} t^x e^{-t}dt$$

Where's the peak of this integrand ?

$$\Gamma(x+1) = \int_0^\infty e^{x\log(t)-t}dt$$

Maximum occurs when the exponent is maximized, which gives $x = t$. Maximum moves with x. We change coordinates where it doesn't change. Define $s = \frac{t}{x}$

$$\Gamma(x+1) = x^{x+1}\int_0^\infty e^{x\log(s)-s}ds$$

Expanding log about $s = 1$.

$$\Gamma(x+1) \sim x^{x+1}\int_0^\infty e^{-x\left(1+\frac{(1-s)^2}{2}\right)}ds$$

$$\Gamma(x+1) \sim x^{x+1}e^{-x}\int_0^\infty e^{-x\frac{(1-s)^2}{2}}ds$$

We can't do this integral. But if we add TST, we have usual gaussian integral!

$$\Gamma(x+1) \sim x^{x+1}e^{-x}\sqrt{\frac{2\pi}{x}}$$

8.1.3 Stationary Phase

For an integral of the function $f(t)$ multiplied with an oscillating function, we get cancellations. If the frequency of oscillation is sufficiently big, we expect the integral to vanish. Rigorously speaking :

$$I(x) = \int_a^b f(t)e^{ix\psi(t)}dt$$

As $x \to \infty$, we try integration by parts to get

$$I(x) = \left[\frac{f(t)e^{ix\psi(t)}}{ix\psi'(t)}\right]_{t=a}^{t=b} - \frac{1}{ix}\int_a^b \frac{d}{dt}\left[\frac{f}{\psi'}\right]e^{ix\psi(t)}$$

Again, just hoping for usual conditions on $\psi'(t)$ and the second integral being of lower order, we see that

$$I(x) \sim O\left(\frac{1}{x}\right)$$

At points when $\psi'(t) = 0$, the first order oscillation of ψ vanishes, meaning it oscillates much more slowely (in second order of t) near this point. It gives a contribution of higher order than $O(\frac{1}{x})$. Therefore, we only integrate near neighbourhood of that point of stationary phase. After expanding around that point, a standard trick would be then to extend this region to infinity, if corresponding integral is easy to do. Since this extension would contribute to higher order terms only.

Expanding around the stationary phase ($\psi'(c) = 0$), and using Fresnel's integral(really just use Gamma function formula), we can show that if

$$\text{if} \quad f^{(i)}(c) = 0 \quad \forall i < n \quad \text{then} \quad I(x) \sim O\left(\frac{1}{x^{\frac{1}{n}}}\right)$$

A useful way to think about both the laplace and stationary phase method in a coherent manner is to see that the function multiplying $f(t)$ has an exponential order. Meaning they change much faster than $f(t)$ as long as $f(t)$ doesn't have exponential order. Note that complex representation of sin and cosine make this evident for this oscillatory functions as well. In some sense then, the integral is wholly dominated by this exponential ordered terms (including sine,cosine). Therefore we only care about the regions where this exponential ordered terms are slowly varying (given that they decay eventually in case of laplace method). In laplace method, it was the peak. In stationary phase method, it is the stationary phase.

Bessel function

$$J_0(x) = \frac{1}{\pi}\int_0^{\infty} \cos(x\sin(t))dt$$

$\psi'(t) = 0$ at $t = \frac{\pi}{2}$. Expanding $\sin(t)$ about $t = \frac{pi}{2}$

$$\sin(t) = \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)\left(t - \frac{\pi}{2}\right) - \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\left(t - \frac{\pi}{2}\right)^2 + ...$$

$$\text{As} \quad x \to \infty$$

$$\frac{1}{\pi}\int_0^{\pi} e^{ix\sin(t)}dt \sim \frac{1}{\pi}\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} e^{ix\left[1-\frac{1}{2}\left(t-\frac{\pi}{2}\right)^2+...\right]}dt$$

As done again and again, expanding the limits of integration because it only affects higher order terms, we get

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right) \quad \text{as} \quad x \to \infty$$

Group velocity

In quantum mechanics, we consider a superposition of waves.

$$\Psi(x,t) = \int \phi(k)e^{i(kx-\omega(k)t)}dk$$

What part gives the dominant contribution to the integral? The part where the phase is stationary. Giving us

$$x = \frac{d\omega}{dk} t$$

Hence, The part which gives dominant contribution to the overall integral moves with the velocity $\frac{d\omega}{dk}$.

8.1.4 Steepest Descent

This method makes previous two methods a consequence. Consider the integral

$$I(x) = \int_C h(t)\, e^{x\phi(t)}\, e^{ix\psi(t)}$$

If we choose a contour such that the $\psi(t)$ is constant on this contour, we can bring it out of the integral and use laplace's method. This will enable us to find higher order terms that we missed in stationary phase method. Please check the video by Professor Strogatz for example.

Chapter 9

Groups

9.1 Finite Groups

Once and Only Once Rule: In the multiplication table, each row or each column contains each element of the group exactly once. Try constructing all posible finite groups of order n.

Lagrange's Theorem: Order of every subgroup divides the order of original group. (Using disjointness of cosets)

Centre: The set defined by elements z_i such that z_i commutes with all elements of the group is an abelian subgroup of G. Not only it's an abelian subgroup, but it's also an invariant subgroup of G.

Permutations: A permutation can be decomposed into product of two cycles. It's even or odd depending on number of two cycles present. (Imagine the ball hiding guy arranging the glasses. He achieves all sort of permutation by doing exchanges again and again). Odd and even because matrix representation corresponds to elementary operations which have determinants 1 or -1.

Cayley's Theorem: Any finite group G with n elements is isomorphic to a subgroup of S_n (Permutation Group). (Associate each g_i with corresponding permutation by checking the permutation offered by observing the multiplication table.)

Conjugation: Two elements g and g' are similar if there exists an element f such that $f^{-1}gf = g'$. Conjugation is an equivalence relation, therefore we can form equivalence classes. (Doing something, doing main thing, undoing something leads to a similar element in some sense. Which is what defines this relation). Equivalence classes provide disjoint partitions of the set (Simple proof). Equivalence relations provide a zoom out lens of a given set.

Invariant Subgroup: Say H is a group. Conjugate H with some g not in H. It will give another group. Conjugation will lend to similar elements but not same. If the conjugated group is same as the original group **for all g**. Then that group is special, and rightly deserves the name Invariant subgroup. All elements equivalent to elements in the subgroup H are also in H. It's a sort of an equivalence relation that preserves the group structure.

The terminology "Normal", "Invariant", "Ideal" are used analogously for all algebraic structures. Similarly, the term "Simple" and "Semi-Simple" are used analogously, namely algebraic structures having no non-trivial Invariant substructures.

Quotient Group: If G has an invariant subgroup H, we can form another group consisting of left cosets g_aH as elements called the Quotient group G/H. We can zoom out sets using equivalence relation, in the same way we need invariant subgroups to zoom out of the group if we want to preserve the group strucutre. It's a somewhat stronger condition than just using equivalence class to zoom out in order to keep group structure.

Automorphism Group An isomorphism between G to G is called an automorphism. Different automorphisms form a group under composition of automorphisms. It is denoted by $Aut(G)$ We can check that ϕ_a defined by conjugation with a ($\phi_a(g) = aga^{-1}$) is an automorphism. Automorphisms of this type are called inner automorphisms. **Inner automorphisms form a normal subgroup of** $Aut(G)$.

9.2 Rotations and Lie Algebra

Calculus is mankind's greatest invention. We analyze behaviours of functions in infinitesimal neighbourhood and then form differential equations to find the right relationship (from infinitesimals to reals). Taylor series is, in a sense $f(x+\Delta x) = exp(\Delta x \frac{d}{dx}) f(x)$. For groups, the higher derivatives $\frac{d^n}{dx^n}$ are simple matrix multiplication of the generator $\frac{d}{dx}$. Rotation is defined by

$$R^T R = I \quad det(R) = 1$$

Check that generators are antisymmetric matrices. We stick in $-i$ to make the generators hermitian.

9.2.1 Lie Algebra

Instead of looking at the actual elements of the group, we are now concerned with generators only. We can go back to the actual group using exponention. For instance, $SO(n)$ generators are antisymmetric matrices. A general antisymmetric matrix can be seen as linear combination of $\frac{n(n-1)}{2}$ antisymmetric

matrices. Now you know why lorentz group is 6 parameters group. It is a weird kind of rotation in 4 dimenstion.

The structure of the group is determined by the commutation relation of the generator. Commutation relation of the generators form an algebra which is called lie algebra of the cooresponding lie group.

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

We can do something like this for all sorts of lie groups. Where ϵ are replaced by some other constants known as the structure cosntatnts.

To see how, consider the product of two group elements near identity $A(\alpha)A(\beta)$, the product itself is a group element with different parameter γ. Compare $A(\gamma)$ with with the result of previous product, which will give you a closed relation between the commutators. When you read theory of lie, you will understand that infinitesimal generator corresponding to each parameter is a vector. We have now identified a way to multiply these vectors (with three specific rules required for a vector space to be an algebra). The structure constants will be different for different choice of basis.

9.3 Representation Theory

It's useful to represent the group elements g as matrices which preserve the multiplication structure of the group. Namely,

$$D(g_1 g_2) = D(g_1)D(g_2)$$

Obviously, $D(g) = 1$ (one dimensional identity matrix) is a perfectly good representation. There's a loss of information in some sense. D is homomorphism as is written, but if it is an isomorphism, we say that the representation is faithful (useful). Since every finite group is a subgroup of S_n, and S_n has obvious representation of dimension n, we have faithful representations of every finite group. (A bit complicated for continuous groups, but we're physicists)

9.3.1 Character is a function of class

$$\chi^{(r)}(g) = Tr(D^{(r)}(g))$$

There may be many representations of the same group. We denote a particular representation r by $D^{(r)}$. Character of a particular element under a given representation is the trace of $D^{(r)}(g)$, and is denoted by $\chi^{(r)}(g)$. It's easy to prove that **character of an entire equivalence class is same.** (Using cyclic property of trace).

9.3.2 Equivalent represntation

There may be more than one representation of a given group. Clearly, We are only concerned with reducible representations. Even among the reducible representations, given a representation $D(g)$, we can construct another representation $D'(g) = S^{-1}D(g)S$ (check that D' is really a presentation). These representations are esentially the same connected by similarity transformation. If we are given two represenations, are they same? One thing you can check is that the character is invariant under the representations connected by similarity transformation. Therefore, if each matrix was conjugated (similarly transformed) and we were to be presented with weird looking set of matrices as representation, they may just be realted to some simple looking set of matrices via similarity transformation.

Other way to look at this is that a representation is really a map from group to a linear operator. This linear operator is represented by all matrices connected by similarity transformation (just a change of basis). So, matrices related by similarity transformation represent the group element being mapped to just a single linear operator. This dichotomy shows up everywhere again and again.

9.3.3 Irreducible or Reducible

The dimension of the matrix is said to be the dimension of repersentation. You can make any dimensional representation just by direct summing 1 dimensional trivial representation. (Note that the direct sum of two operators acts independently on the vectors from the direct summed vector spaces, in matrix language, it will be a block diagonal matrix and the vector will be an extended row). Although, by similarity transformation in higher dimension, we could make this block diagonal matrices uglier, and it will be messy to see if the representation is irreducible. In any case, ideally we say that the representation is irreducible no subspace of that representation is also a representation. Goal is to figure out the number of irreducible representations.

Chapter 10

Theory of Lie

All functions considered are smooth etc. x without subscript inside the function is to be seen as something which involves all x_i. Imagine a flat space and cartesian coordinates of dimension n. Now consider, a set of transformations (Called point transformation)

$$x_i' = f_i(x_1, x_2, x_3, ...) \quad \forall i \in 1, 2, ..n$$

We can imagine x moving to x' in the same system or a region in the domain of x system being mapped to a correesponding region of x' system.

We wish to see this transformation as a continous group. What do we need? We want it to be invertible. Which demands usual condition of Jacobian being non-zero under the region of consideration (Chapter 1 final theorem). Further, if we were to transform twice using different parameters, we should end up in the same family. Let's introduce a notation for finite parameter continuous group to make this more clear.

A continuous transformation involving finite parameters will be represented as $x' = f(x; a)$. For this family to form a group, we demand that $x'' = f(x'; b) = f(x; c)$. Now if we have a r parameter transformation

$$x_i' = f_i(x_1, x_2, ...; a_1, a_2, ..., a_r)$$

We can eliminate the constants to form a set of differential equations for x_i'. This differential equations will have an amazing property. Namely, if we find two different set of families of solution $x_i' = g_i(x; b_1, b_2, ..., b_r)$ and $x_i' = h_i(x; a)$. Then the composition $y_i' = g_i(h_1(x, a), h_2(x, a), ...; b_1, b_2, ..., b_r)$ is also a solution by definition of the group property of initial equation.

A function f in x system may look very different in x' system. If it looks the same, we say that the trasformation which takes x to x' is a symmetry of f. The magic in theory of continuous groups is that we can always analyze upto

the first order near identity, namely, we are dealing with generators only. An explicit example is considered in the (Section of Killing Vectors).

More generally, let x be transformed to y and $u_i(x)$ be transformed to different functions under the group action, which in y coordinates, have the form $v(y)$, then the function $P(x, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, ...)$ is an invariant of the group if

$$P\left(x, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, ...\right) = P\left(y, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial y^2}, ...\right)$$

10.1 Infinitesimal Transformation and Vector field generating the Transformation

Consider the n parameter transformation, let's think of infinitesimal increment ϵ of the parameterk k, all the rest of parameters are chosen such that it corresponds to identity transformation.

$$x_i' = f_i(x; e_1, ..., e_k + \epsilon, ..., e_n) = x_i + \frac{\partial f_i}{\partial a_k}(x; e)\epsilon + ... \quad (i = 1...n) \tag{10.1}$$

If we introduce the vector field

$$X^e{}_k = \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_k}(x; e)\frac{\partial}{\partial x_i} := \sum_{i=1}^{n} \xi_{ki}(x)\frac{\partial}{\partial x_i}$$

We have $x_i' = x_i + \epsilon\xi_{ki} + ... \quad (i = 1...n)$

More generally, arbitary increments from identity to all parameters $(e_1+\epsilon\lambda_1, ..., e_k+\epsilon\lambda_k, ...e_r + \epsilon\lambda_r)$ will give

$$x' = x + \epsilon X + ... \tag{10.2}$$
$$X := \lambda_1 X_1^e + \lambda_1 X_2^e + ... \tag{10.3}$$

Clearly, the infinitesimal generators are vectors at each point of the manifold (Since dx is a vector always).

10.2 A brief note on invariance

There's a deep but trivial connection between the coordinate transformation that preserve the form of the function and the symmetry of the function. As stressed many times, Transformation we will consider are infinitesimal (Finite can be generated easily from them).

Say we have a new coordinate system $x' = x + \epsilon\,\eta(x)$. Any function $f(x)$ will have a totally different form $f'(x')$. Here, x and x' refer to the same point on the manifold, but in different coordinate systems. Hence, the numerical value of two expression is same $f'(x') = f(x)$. However, if we want the form of the function itself to be the same in new coordinate system, we need $f'(x') = f(x')$. But if the transformation $x' = x'(x)$ is such that $f(x') = f(x)$ then we clearly have $f'(x') = f(x')$. Such transformation for which $f(x') = f(x)$ are called infinitesimal symmetry of the function. Therefore, the infinitesimal transformations which doesn't change the value of the function are the corresponding coordinate transformations which preserve the form of the function under coordinate transformations. It's trivial, but took me long to realise.

In Noether's theorem, think of coordinate tranformations as active transformations (meaning that we bring the field backwards, instead of moving the coordinates forward. So both, coordinate and field transformations correspond to the field transformation effectively. The reason I want you to think in terms of active transformation is because I don't want you to get confused with the definition of Scalars (Especially the notation). In our notation, $f'(x') = f(x)$ is always true where $'$ denotes f in new coordinates. In many books though, Notation $'$ is used to describe the new (varied) field. Varied field is a different scalar field altogether, and in their notation $'$ is not describing the same field in new coordinates but the different field in same coordinates.

10.2.1 Killing vectors

Consider finding out symmetries of the metric in relativity. We consider an infinitesimal transformation at all coordinates

$$x'^{\mu} = x^{\mu} + \epsilon\,\eta^{\mu}(x)$$

Here, $\eta^{\mu}(x)$ is like dx^{μ}, hence it can be treated as vector. Namely, its indices can be lowered and uppered. Note also, that the inverse transformation can be given by (since first order change inside the function η will give second order overall change)

$$x^{\mu} = x'^{\mu} - \epsilon\,\eta^{\mu}(x')$$

The metric transforms under general coordinate transformation as

$$g'_{\alpha\beta}(x') = g_{\mu\nu}(x)\frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}}$$

We want the metric to keep the same form under such coordinate transformation, meaning that we want $g'_{\alpha\beta}(x') = g_{\alpha\beta}(x')$. This connects the x system and x' system in a sense. Equation says that the point x(system 1) in the manifold which is also the point x'(system 2) in different system is related to the point x'(system 1) of the original system in this fashion.

$$g_{\alpha\beta}(x) + \epsilon\,\partial_\sigma g_{\alpha\beta}\,\eta^\sigma(x) = g_{\mu\nu}(x)\left[\delta^\mu{}_\alpha - \epsilon\,\partial_\alpha\eta^\mu\right]\left[\delta^\nu{}_\beta - \epsilon\,\partial_\beta\eta^\nu\right]$$

Here, we have ignored ϵ^2 terms arising from partial derivatives of η evaluated at infinitesimally different point. Which will give us a coordinate independent equation for vector η^μ

$$\nabla_\mu\eta_\nu + \nabla_\nu\eta_\mu = 0 \tag{10.4}$$

The vector field η is then the generator of the continuous symmetry of the metric.

10.3 Lie Algebra

It can be shown that the infinitesimal generators form the relation

$$[X_\rho, X_\sigma] = c^\tau{}_{\rho\sigma}X_\tau$$

With $c^\tau{}_{\rho\sigma} = -c^\tau{}_{\sigma\rho}$. Further, they satisfy the Jacobi identity

$$[X_\rho,[X_\sigma,X_\nu]] + [X_\sigma,[X_\nu,X_\rho]] + [X_\nu,[X_\rho,X_\sigma]] = 0$$

Such a vector space with product defined as above form Lie algebra (Tangent space at the identity element of the manifold). Note that we can also realize matrices to represent the same thing, but the generators are vectors in reality. Matrices are just reprenerntations. Formally, the lie algebra has following three properties

$$[\alpha X + \beta Y, Z] = \alpha[X,Z] + \beta[Y,Z] \tag{10.5}$$
$$[X,Y] + [Y,X] = 0 \tag{10.6}$$
$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \tag{10.7}$$

Under a transformation of basis (using matrix $a^\nu{}_\rho$), the structure constants change as

$$c'^\tau{}_{\rho\sigma} = a^\nu{}_\rho a^\lambda{}_\sigma c^\kappa{}_{\nu\lambda}(a^\kappa{}_\tau)^{-1}$$

Different Lie groups may have the same structure constants and as a result the same Lie algebra, but be radically different groups globally. Groups possessing common lie algebra are said to be locally isomorpihc, that is, isomorphic in the neighbourhood of the identity. The Lie algebra $su(2)$ and $so(3)$ are isomorphic while the Lie groups $SU(2)$ and $SO(3)$ are only locally isomorphic. A radical example is $SO(2)$ and $T(1)$ have single selfcommuting generator but the groups are obviously not isomorphic globally.

10.3.1 Subalgebras, Ideals and Proper Ideals

If a subspace Z (as a vector subspace) of lie algebra A is closed, it's a subalgebra.

$$[X, Y] \in Z \quad \text{for any} \quad (X, Y \in Z)$$

A subset Z is ideal (or invariant subalgebra) of A if it is closed under product with another element in A. This is analogous to the invariant subgroup which remains as it is under conjugation by any element.

$$[X, Y] \in Z \quad \text{for any} \quad (X \in Z, Y \in A)$$

that is,

$$[X_\rho, X_\sigma] = c^\tau{}_{\rho\sigma} X_\tau \quad (\rho, \tau \in Z, \sigma \in A)$$

If the algeba has members which are not in the ideal, the ideal is called proper ideal. Basically, it is not the whole agebra itself.

We had a concept of abelian group. In close analogy, a subalgebra Z is abelian if commutator $[X, Y] = 0$ for any $(X, Y \in Z)$. Similarly, we had a concept of center (the set of elements which commute with all other elemnts of the group). In close analogy, the set of all elements X_σ of algebra A is said to form the maximal ideal or center of the algebra if $[X_\rho, X_\sigma] = 0 \quad (\rho \in Z, \sigma \in A)$.

Obviously, the ideal is a subalgebra itself. Even more so, it is a special type of subalgbera, it's an invariant subalgebra. We will see that lie algebras which have no non-trivial ideals are simplest in some sense. Those which can be seen as direct product of these are semisimple.

Basically, we tried to see the product of vectors as some group theoretic multiplication and tried to identify corresponding things here.

10.3.2 Adjoint representation

We really have vectors and a product of vectors when we talk about algebras. However, we represent elements of the algebra (vectors) by matrices which follow the same commutation relation. Which we call a representation of the algebra. Just like we have a representation of group by matrices. Now, whenever we represent stuff by matrices, we should note that we're really talking about a linear operator in $GL(n, \mathbb{C})$. Which can be represented by a lot of matrices connected by similarity transformation. The closed lie algebra provides a cannonical such linear operator (just like inner product provides a natural isomorphism between dual space). Representation using this cannonical linear operator provided by the algebra is called adjoint representation. We need to show that this operator respects the commutator algebra.

$$ad(X) : Z \to [X, Z] \quad \text{for any} \, (Z \in A)$$

$$[ad(Y), ad(Z)]K = ad(Y)ad(Z)K - ad(Z)ad(Y)K \tag{10.8}$$
$$= ad(Y)[Z, K] - ad(Z)[Y, K] \tag{10.9}$$
$$= [Y, [Z, K]] - [Z, [Y, K]] \tag{10.10}$$
$$= ad([Y, Z])K \tag{10.11}$$

In this sense, we already have a faithful representation of the **simple** lie algebra. The main purpose is not to find faithful representation of the lie algebra, but to find all representations (upto equivalency(by similarity transformation) ofc).

10.3.3 Simple and Semisimple Lie Algebra

A lie algebra is simple if it has no proper ideals (rightly named). It's semisimple if it has no maximal (abelian) ideals. Alternate definition of semisimple lie algebra is if

$$A = A_1 \oplus \cdots \oplus A_N$$

where each A_i(are ideals of A) form a simple lie algebra.

Semi-simplicity is a widespread concept in linear-algebra, abstract algebra and representation theory. Semi-simple object is one which can be decomposed into direct sum of simple objects and simple objects are those which do not contain non-trivial proper(invariant) sub-objects.

10.4 Killing Form and Cartan's Criterion for Semi-Simple Lie Algebras

Using the structure constants of a lie group, we can define a symmetric tensor (ofc it will depend on the basis)

$$g_{\sigma\lambda} = g_{\lambda\sigma} = c^{\tau}{}_{\sigma\rho}c^{\rho}{}_{\lambda\tau}$$

Theorem : Lie algebra is semisimple if and only if $det|g_{\sigma\lambda}| \neq 0$
Wow! Great! We just need to know the structure constants of a Lie algebra. From that, we need to find the determinant of killing form to check if the lie algebra is semisimple.

Chapter 11

Classical Mechanics

11.1 Lagrange Multipliers

How would you find the max/min of a function $f(x, y)$? We get two equations by setting both partial derivatives equal to zero. We can simply solve for x and y. Now if there's a constraint of the form $g(x, y) = c$, it may happen that the solution (x_0, y_0) doesn't lie on g. What can we do? We introduce additional parameter λ to adjust itself so as to make x and y fit on g. We find max/min for $f'(x, y) = f(x, y) + \lambda g(x, y)$, this maximum will depend on the parameter λ. We can then impose $g(x, y) = c$ to solve for x, y, λ.

Geometric intuition for lagrange multipliers is that the change in f is maximum along the gradient of f. If, along the curve g, there's a point where gradient of f is parallel to gradient of g, then moving along g will give min/max change in f at that point (it's like f has min/max at that point as seen by g, the contour lines just touch).

Consider the action

$$S = \int L(q, \dot{q}, t)dt$$

If we have n constraint realtion of the form $\sum_k a_{ik}dq_k = 0$, we can introduce n additional parameters λ_i and add all these terms in the variation of orginal δL.

11.2 General remarks

- If q_k does not appear in the lagrangian, we have conserved quantity(constant of motion) corresponding to it. However, We cannot directly substitute this conserved quantity back into the lagrangian and then use euler lagrange equations for other coordinates. Because, (firstly, the form of new lagrangian is clearly different from original, so there's no reason to believe

it would work anyway) the euler-lagrange equations were derived based on the assumption that the coordinates q_i's are independent(David Tong Ch2, p.29). In that sense, even though q_k is a ignorable in lagrangian formalism, it cannot really be ignored when we wish to derive equations of motion. In Hamiltonian formalism, if q_k is ignorable, then it is truly ignorable in the sense that p_k can be substituted for a constant in the hamiltonian, and problem becomes $n-1$ dimensional one.

$$L = L(q_1, q_2, ...q_{n-1}; \dot{q1}, ..., \dot{q_n}; t) \tag{11.1}$$
$$H = H(q_1, q_2, ...q_{n-1}; p_1, ..., p_{n-1}; \alpha; t) \tag{11.2}$$

The motion of nth coordinate can simply be given by

$$\dot{q_n} = \frac{\partial H}{\partial \alpha}$$

- A beautiful geometric interpretation of noether theorem can be found on Wikipedia. In summary, if the action doesn't change under some transformation, since first order changes around equation of motion doesn't change action, a 'slanting term' would be conserved.

11.3 Small Oscillations and Stability

Consider $\ddot{x} = f(x)$. Point x_0 is called stable if $f(x_0) = 0$. For if initial conditions were $x = x_0$ and $\dot{x} = 0$, the system would stay at x_0. Consider small oscillations $x = x_0 + \eta(t)$ about this point. Upto the first order,

$$\ddot{\eta} = f'(x_0)\eta$$

In case $f'(x_0) < 0$, the solution is oscillatory, otherwise exponential. It's just about linear second order system now. Let's see this in more dimensions

$$\ddot{q_i} = f_i(q_1, q_2, ...)$$

Equilibrium point q_i^0 satisfies $f_i(q_1^0, ..., q_n^0)$ for all $i = 1...n$. Expanding about this point, we get again

$$\ddot{\boldsymbol{\eta}} = F\boldsymbol{\eta}$$

Here F is seen to be a matrix $\frac{\partial f_i}{\partial q_j}$ evaluated at the equilibrium point. In one dimension we only cared about the sign of this matrix to check for stability. In higher dimension, the analysis depends on the nature of this matrix. If we assume that the matrix is symmetric, then it has complete set of orthogonal eigenvectors with real eigenvalues. If $\boldsymbol{\mu_a}$ are the eignevectors of F with eigenvalue λ_a^2, that is $F\boldsymbol{\mu_a} = \lambda_a^2\boldsymbol{\mu_a}$. As you may have guessed, the solution depends on the eigenvectors like

$$\eta(t) = \sum_a \boldsymbol{\mu_a} \left[A_a e^{\lambda_a t} + B_a e^{-\lambda_a t}\right]$$

The system will be stable only if all eigenvalues are negative (since the solution depends on the square root of eigenvalue which turns exponential into trigonometric functions). Further, the special vectors $\boldsymbol{\mu_a}$ that have arisen out of beautiful combination of linear algebra and differential equation are called normal modes of the system. The most general solution has the form

$$\boldsymbol{\eta} = \sum_{a,\lambda_a^2>0} \boldsymbol{\mu_a} \left[A_a e^{\lambda_a t} + B_a e^{-\lambda_a t}\right] + \sum_{a,\lambda_a^2<0} \boldsymbol{\mu_a} A_a cos(\omega_a(t - t_a))$$

11.4 A comparision of Hamiltonian and Lagrangian Formalism

We have n second order equations in lagrangian formalism. Unfortunately, the lagrangian is considered to be a function of $2n$ coordinates, where last n coordinates are NOT REALLY on an independent footing from first n coordinates. By legendre transformation, we arrive at a formalism where we can have $2n$ truly independent coordinates and the equation of motion becomes $2n$ coupled first order. It can be shown that euler-lagrange equation do not change their form under point transformation. The analogy of point transformations in hamiltonian formalism are cannonical transformations. In hamiltonian formalism, we have $2n$ independent coordinates which can all be pointwise transformed, unlinke only n coordinates in the lagrangian formalism. However, not all such transformation preserve the form of the equation. Those which do, are called cannonical transformations.

11.4.1 Cannonical Transformation

In phase space, we can associate a $2n$ dimensional vector $\boldsymbol{x}$ corresponding to the coordinates $(q_1, ..., q_n, p_1, ..., p_n)$. The hamilton's equation are then simply

$$\dot{\boldsymbol{x}} = J\frac{\partial H}{\partial \boldsymbol{x}}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Let's make a point transformation $x \to y(x)$ in phase space and check if the equation of motion have the same form.

$$\dot{\boldsymbol{y}} = \mathbb{J} J \mathbb{J}^T \frac{\partial H}{\partial \boldsymbol{y}}$$

Here $\mathbb{J}_{ij} = \frac{\partial y^i}{\partial x^j}$. To have the same form in y

$$\mathbb{J}J\mathbb{J}^T = J$$

Transformations whose jacobian have these form are called canonical. Rings bell from group theory? Already thinking about generators and infinitesimal transformations? Further, we can show that poisson brackets remain invariant under cannonical transformation in a sense that you can calculate them in any particular coordinate system (connected by cannonical transformation) you like. Don't they give you a feeling of dot product (quadratic form so to speak)? It's kind of a scalar on some manifold? Like in general relativity $\partial_\mu\phi\partial^\mu\phi$ is a scalar. (Tong p.101 last chapter).
The associated symmetry group (for cannonical transformation) is called symplectic group. (Check Classical Groups wyborne P.14).
Converse is also true. That is, if transformations are such that they preserve the cannonical commutation relations ($\{Q_i, Q_j\} = 0, \{Q_i, P_j\} = \delta_{ij}$), these transformations are cannonical.

Now, Lie's theory is everywhere. Until now, we thought of cannonical transformation as such coordinate transformations which preserve the form of the hamilton's equation. Just like how lorentz invariant metric preserves the form of equations. Infinitestimal cannonical transformation can also be thought of as active (acting on the same coordinate system). They take (q_i, p_i) to a different point $(q_i + \epsilon E_i, p_i + \epsilon F_i)$. For this transformation to be cannonical, we can show that it has to be in the form $(q_i + \epsilon\frac{\partial G}{\partial p_i}, p_i - \epsilon\frac{\partial G}{\partial q_i})$. $G(q,p)$ is called the generating function. For example, if $G = p_k$, $q_i \to q_i + \epsilon\delta_{ik}$ and $p_i \to p_i$. We say that momentum is a generator of translation.

11.5 Constrained Hamiltonian Systems

If the hessian $W_{ij} = \frac{\partial^2 L}{\partial\dot{q}_i\partial\dot{q}_j}$ is singular then we cannot express all $\dot{q}_i$ as functions of q and p. If rank of W is $n - m$ where n is the number of coordinates. The legendre transformation maps us onto $2n-m$ dimensional subspace of the phase space. then we would have m constraint realtions $\phi_a(q,p) = 0$ restricting us onto $2n - m$ dimensional subspace of the phase space. As an example, think of the dissipative lagrangian (Do the stuff I described, if you wish, you can check here)

$$L = \frac{1}{2}\dot{q_i}^2 + c\dot{q_2}q_1 - \frac{m}{2}(q_1^2 + q_2^2 + q_3^2)$$

Chapter 12

Quantum Mechanics

- A state of quantum system is defined by a ket $|\psi\rangle$ or the corresponding bra $\langle\psi|$ (For every vector space, we can construct a dual space (of linear functionals)). There is a one-one correspondence between this space.

  Pedantry : If V is a finite dimensional vector space, its dual V^* has the same dimension. Therefore, they're isomorphic. (Note that finite dimensional vector spaces with same dimensions are obviously isomorphic.) By establishing the dual basis, we establish this isomorphism between V and V^*. But this isomorphism is not natural. However, if we have an inner product space, we can have almost natural isomorphism between a vector $|\alpha\rangle$ and its dual, $\langle\alpha|$ is naturally defined by fixing the first element of inner product as $|\alpha\rangle$.

- Inner product is positive definite and has conjugate symmetry by defintion. Important for probabilistic interpreation :

$$\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^* \quad \text{and} \quad \langle\alpha|\alpha\rangle \geq 0$$

- The eigenvalues of a Hermitian operator are real and Eigenkets corresponding to different eigenvalues are orthogonal. Every observable corresponds to a Hermitian operator. Converse, I am not aware of yet.

  Postulate: The eigenkets of an observable are complete. This is a physical requirement, not a mathematical truth. As a consequence, we start by identifying the observables, and since they're hermitian, we choose an orthonormal basis consisting of eigenkets of this observable, which will span the entire hilbert space.

- If $[A, B] = 0$, every eigenvector of A is an eigenvector of B and vice versa.

- Given two sets of orthonormal base kets $\left|a^{(i)}\right\rangle$ and $\left|b^{(i)}\right\rangle$, there is a unitary operator

$$U = \sum_k \left|b^{(k)}\right\rangle \left\langle a^{(k)}\right|$$

such that $\left|b^{(i)}\right\rangle = U \left|a^{(i)}\right\rangle$
Check $U^\dagger U = I$

- Matrix representation of the same operator in different basis are connected by similarity transformations $X' = U^{-1}XU$. LHS is the matrix of operator in new basis, RHS are matrices in old basis. If both basis are orthonormal, the matrix U is unitary. Trace is a property of the operator, not of the basis (Although we are only talking about transformations connecting orthonormal basis here, it's true for general transformation because of the cyclic property of trace). **Every hermitian matrix is unitary diagonalizable.**

- Similarity transformation are much more general. As we know, all finite vector spaces of a given dimension are isomorphic to each other. Similarity transformation connects two matrices A and B by an invertible matrix P(change of basis matrix) such that $B = P^{-1}AP$. In general linear group $GL(n, \mathbb{C})$, this is same as conjugation that we discussed in group theory. Character is same for an entire equivalence class means that trace is same for all matrices connected by similarity transformation. In linear algebra language, they represent same operator in different basis.

- There is a minor contextual linguistic difficulty here. When seen purely as a group $GL(n)$, the matrices connected by similarity transformation form an equivalence class. When this group is seen to be acting on an abstract vector space, all the elements of equivalence class represent the same operator(acting on this abstract vector space) in different basis. We can naturally ask, is there a relation between operators connected by unitary transformation (Note that it is different from the representation of same operator in different basis). There is a theorem which says that operators connected by similarity transformation are physically the same (Sakurai pp. 39).

- In summary, it is useful to associate the matrix representation to elements of $GL(n)$, which I call pure matrices. Think of operators as abstract quantities then, which are represented by an entire equivalence class of $GL(n)$.

12.1 Difference between direct sum(product) and tensor product

Direct sum of two vector spaces A and B is such that (a, b) with $a \in A, b \in B$ and (ca, b) is considered diferent from (a, cb) (Imagine $\mathbb{R} \bigoplus \mathbb{R} = \mathbb{R}^2$, both are different vectors in $\mathbb{R}^2$). We don't need to explain anything, they act as independent vector spaces and it's simple. For Direct Product, we need to restrict the structure by defining properties for the product. The dimension of Direct Sum of vector spaces add, they multiply in case of tensor product.

Fundamental property defining tensor product of vectors is following (which is not true for direct sum(product))

$$\begin{aligned}(\alpha v) \otimes w &= \alpha(v \otimes w) = v \otimes (\alpha w) \\ (v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w \end{aligned} \tag{12.1}$$

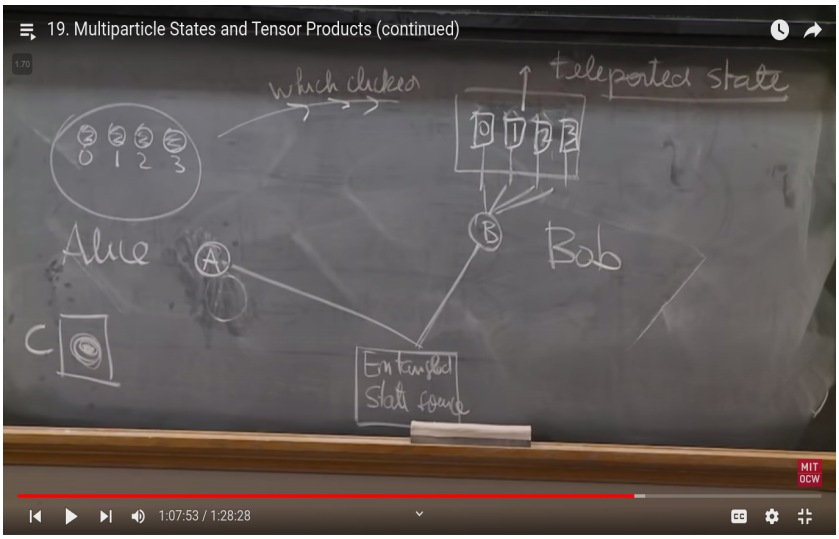
If V has basis $(e_1, e_2, ..)$ and W has basis $(f_1, f_2, ...)$ then $(e_i \otimes f_j)$ is basis for $V \bigotimes W$. Tensor produt of oprtators is defined in the natural way.

Inner product is also defined most naturally.

$$\left\langle \sum_{ij} a_{ij} v_i \otimes w_j \middle| \sum_{pq} b_{pq} v_p \otimes w_q \right\rangle = \sum_{ij} \sum_{pq} a^*_{ij} b_{pq} \langle v_i | v_p \rangle_V \langle w_j | w_q \rangle_W$$

12.2 Quantum Teleportation

We note that the pauli matrices are both hermitian unitary, and can be realized physically as hamiltonians by setting up appropriate magnetic field. We cannot have a unitary operator which clones the state (No cloning theorem).



Say Alice wants to teleport the state $(\alpha |+\rangle_C + \beta |-\rangle_C)$. We use an auxiliary entangled state (we use $|\Phi_i\rangle$ to denote the bell basis)

$$|\Phi_0\rangle = \frac{1}{\sqrt{2}}(|+\rangle_A |+\rangle_B + |-\rangle_A |-\rangle_B)$$

The total state $|\Psi\rangle_T = |\Phi_0\rangle_{AB} \otimes (\alpha |+\rangle_C + \beta |-\rangle_C)$ is (by trivial algebra)

$$\begin{aligned} |\Psi_T\rangle = &\frac{1}{2} |\Phi_0\rangle_{AC} (\alpha |+\rangle_B + \beta |-\rangle_B) \\ &+ \frac{1}{2} |\Phi_1\rangle_{AC} (\beta |+\rangle_B + \alpha |-\rangle_B) \\ &+ \frac{1}{2} |\Phi_2\rangle_{AC} (i\alpha |-\rangle_B + i\beta |+\rangle_B) \\ &+ \frac{1}{2} |\Phi_3\rangle_{AC} (\alpha |+\rangle_B - \beta |-\rangle_B) \end{aligned} \tag{12.2}$$

Alice measures the bell state. She sends this information to bob classically. Corresponding to each bell state that Alice measures, Bob can set up appropriate magnetic field to get the original state.

12.3 Baker-Hausdorff Lemma

In Heisenberg picture, the operators (and therefore, their eigenkets) evolve with time. The state ket remains what it was at $t = 0$. In schrodinger picture, the operators are what they are, the state ket evolves according to the hamiltonian. Now, how do we evaluate the operator after finite time t in Heisenberg picture?

$$A(t) = exp\left(\frac{iHt}{\hbar}\right) A(0)\, exp\left(\frac{-iHt}{\hbar}\right)$$

Prove the following :

$$exp(iG\lambda)\, A\, exp(-iG\lambda) = A + i\lambda[G,A] + \left(\frac{i^2\lambda^2}{2!}\right)[G,[G,A]] \tag{12.3}$$
$$+ ... + \left(\frac{i^n\lambda^n}{n!}\right)[G,[G,[G,...[G,A]]]...] + ...$$

Since we have an algbra, every commutator will be some combination of the elements of the algebra. Therefore, we only need to know commutators of the algebra for finite transformations (elements of the actual group). This is just theory of lie.

12.4 Angular Momentum

Momentum is the generator of translation in space, Hamiltonian is the generator of translation in time, Angular momentum is the generator of rotations.

Tranlsations in different directions commute, therefore $[p_i, p_j] = 0$ (please derive). Rotation in different directions don't commute, so we have a different algebra for their generators. This algebra defines everything.

12.4.1 Operator Vector Analysis

If $\vec{a}$ and $\vec{b}$ are vector operators, we have

$$\vec{a}\cdot\vec{b} = a_i b_i \quad \text{(definition)} \tag{12.4}$$
$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} + [a_i, b_i] \tag{12.5}$$
$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a} + \epsilon_{ijk}[a_j, b_k] \tag{12.6}$$

Of course, not any pair of operators can be considered vector. An operator V is a vector operator if its expectation values transform as real vectors under rotation (Turn it into mathematical definition (Sakurai p.246)). We simply state the definition

$$[L_i, U_j] = i\hbar\epsilon_{ijk}U_k$$

We can show that if $\vec{u}$ and $\vec{v}$ are vector operators under rotation, $\vec{u}\cdot\vec{v}$ is a scalar(commutator with L_i is zero) and $\vec{u}\times\vec{v}$ is a vector. For practice, show that $\vec{r}$ and $\vec{p}$ are vector under rotation.

12.5 Addition of Angular Momentum

If $J^{(1)}$ is angular momentum in space V_1 and J_2 is angular momentum in space V_2 then $J = J^{(1)} \bigotimes \mathbb{1} + \mathbb{1} \bigotimes J^{(2)}$ is angular momentum in space $V_1 \bigotimes V_2$ (Check algebra).

12.5.1 Spin Orbit Coupling

(Check out thomas precession. It's beautiful) Consider hydrogen atom. Since electron has intrinsic spin, due to the motion of proton around electron, there will be a magnetic field which changes the energy by $\Delta H = \vec{\mu} \cdot \vec{B}$. $\vec{B}$ turns out to be proportional to $\vec{L}$.

The new hamiltonian $H_{TOT} = H_0 + \Delta H$ contains an additional $\vec{L} \cdot \vec{S}$ term.

$$\vec{J}^2 = J_i J_i = \vec{L}^2 \otimes \mathbb{1} + \mathbb{1} \otimes \vec{S}^2 + 2 \sum_i L_i \otimes S_i$$

$$\vec{L} \cdot \vec{S} = \frac{1}{2}(J^2 - L^2 - S^2) \tag{12.7}$$

Due to this, the CSCO is no longer $(H_0, L^2, S^2, L_z, S_z)$ but $(H_{TOT}, L^2, S^2, J^2, J_z)$. For example, consider the subspace having angular momentum 1 and spin 1/2. This is a tensor product of two spaces having the dimension $3 \times 2 = 6$. It breaks into the direct sum of two vector spaces with total angular momentum 3/2 or 1/2.

$$1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$$

Chapter 13

General Relativity

No better summary than Dirac's 60 page book. Roughly speaking, ∂_μ can be treated as a vector direction if one imagines y^n to be accompanied with it. Also explains how it changes if surface is not flat or the coordinates are curved.

- An observer is a worldine where the basis vector he chooses at each point of the worldline reduces the metric to η_{ab}. In effect, he's someone who uses flat metric at all points of the worldline. (Schuller Lect 13. 41:26)
- All momentum components p^μ are not independent due to reparameterization invariance of the lagrangian. In classical mechanics, time was a thing which was used to define action. (David Tong p.22 GR).
- In 2 dimesnions, gaussian curvature is sufficient to describe inner properties of the space. In higher dimensions, we need more.
- Jacobi proved that any quadratic form can be put into the diagonal form by a change of basis.

  $$\lambda_1 \bar{x}_1^2 + \lambda_2 \bar{x}_2^2 + ... + \lambda_n \bar{x}_n^2$$

  Namely, if the quadratic form q has matrix A in a given basis then

  $$q(v) = x^T A\, x \tag{13.1}$$
  $$A \to B = S^T A\, S \quad \text{Basis change by S} \tag{13.2}$$

  Then B can be put into a diagonal form. This is reminiscent of the gaussian normal coordinates. Sylvester law of inertia states that the number of positive, negative and zero λ's are a characteristic property of the quadratic form independent of the basis. In a manifold, we have vector spaces at each point. However, metric tensor g is such that it's signature remains the same when carry this out at tangent space of any point. Note

that since A is symmetric, it can be unitary diagonalized. That's why S^T can diagonalize it, we don't need inverse.

1. Metric can be chosen at a point such that $g_{\mu\nu}(P) = \eta_{\mu\nu}$ and $\Gamma^{\alpha}{}_{\beta\gamma}(P) = 0$ (Poisson p.27)
2. Suppose $P_{\mu\nu\lambda}$ is such that $A^\mu P_{\mu\nu\lambda}$ is a tensor for any vector A^μ. Then $P_{\mu\nu\lambda}$ is a tensor. Useful for quickly proving something is a tensor.
3. $\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda} \left[\partial_\gamma g_{\lambda\beta} + \partial_\beta g_{\lambda\gamma} - \partial_\lambda g_{\beta\gamma} \right]$
4. $\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^{\nu}{}_{\lambda\mu} A^\lambda$
5. $\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^{\lambda}{}_{\nu\mu} A_\lambda$
   For general tensors, use leibniz rule. $g_{\mu\nu}$ can be treated as a constant under covariant derivative.
6. $\nabla_\rho \nabla_\sigma A_\nu - \nabla_\sigma \nabla_\rho A_\nu = A_\lambda R^{\lambda}{}_{\nu\rho\sigma}$
7. $R^{\beta}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\beta}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\beta}{}_{\nu\rho} + \Gamma^{\alpha}{}_{\nu\sigma} \Gamma^{\beta}{}_{\alpha\rho} - \Gamma^{\alpha}{}_{\nu\rho} \Gamma^{\beta}{}_{\alpha\sigma}$

Chapter 14

Gibberish that can be made precise

- In theories with torsion, the translations don't commute. What's the effect on the commutator of momenum in different directions.
- A uniform interpretation of wavefunction among all theories?