Bayesian games

Stanisław Bitner, Aleksander Wojsz

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Bayesian games - intuitive definition

So far we have assumed that each player knows the payoff of other players. Ex. Prisoners dilemma. We didn't know whether our colleague will cooperate with police, but we knew payoffs for all possibilities.

Closer to real life are games with incomplete information (uncertainty about payoffs) called Bayesian Games.

Bayesian games - intuitive definition

In games with incomplete information we have

- Players
  - One player can have multiple types each having different payoffs
  - Player has probability assigned for each type of his opponent
  - Player 2 knows that player 1 has beliefs.
    Player 1 knows that player 2 knows that player 1 has beliefs.
    Player 2 knows that player 1 knows that player 2 knows that player 1 has beliefs.
    ... and so on
- Actions
- Payoffs

So in short, the difference is these types and beliefs about these types.

Bayesian games - intuitive definition

Bayesian game eliminates infinite loops in situations where players try to predict each other's thoughts. For instance, a player might think, "If I expect *player B* to take a certain action, then *player B* will predict that I expect this action, so I need to predict *player B's* prediction", and so on. Bayesian games simplify this by assigning probability weights to each outcome.

Example on Battle of the Sexes

Complete information

	Star Wars	Titanic
Star Wars	(2, 1)	(0, 0)
Titanic	(0, 0)	(2, 1)

Incomplete information

p=0.75

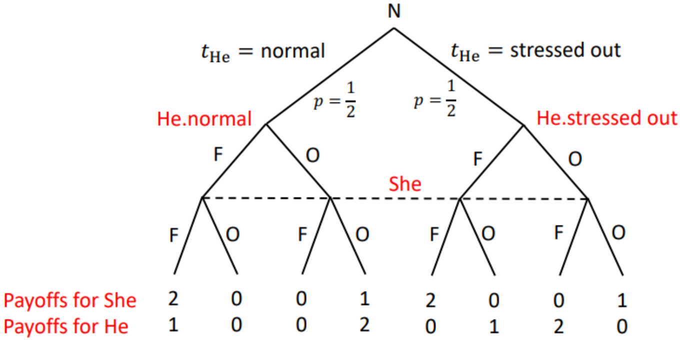
	Star Wars	Titanic
Star Wars	(2, 1)	(0, 0)
Titanic	(0, 0)	(2, 1)

Woman wishes to meet with man

p=0.25

	Star Wars	Titanic
Star Wars	(2, 0)	(0, 2)
Titanic	(0, 1)	(1, 0)

Woman wishes to avoid man



Watch out! Here she prefers football and he prefers opera.

Suspect is criminal with prob p

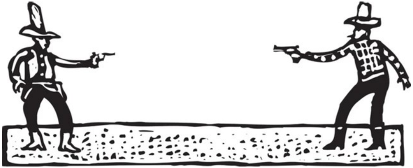
Sheriff's action (columns); *Suspect's action* (rows)

	Shoot	Not
Shoot	(0, 0)	(2, -2)
Not	(-2, -1)	(-1, 1)

Suspect is civilian with prob 1-p

Sheriff's action (columns); *Suspect's action* (rows)

	Shoot	Not
Shoot	(-3, -1)	(-1, -2)
Not	(-2, -1)	(0, 0)



Example on Sheriff's dilemma

Suspect is criminal with prob p

Sheriff's action

Suspect's action

	Shoot	Not
Shoot	(0, 0)	(2, -2)
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Suspect is civilian with prob 1-p

Sheriff's action

Suspect's action

	Shoot	Not
Shoot	(-3, -1)	(-1, -2)
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Dominant strategy for suspect (who knows his type)

p

Sheriff's action

Suspect's action

	Shoot	Not
Shoot	(0, 0)	(2, -2)

1-p

Sheriff's action

Suspect's action

	Shoot	Not
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Example on Sheriff's dilemma

p

Sheriff's action

Suspect's action

	Shoot	Not
Shoot	(0, 0)	(2, -2)

1-p

Sheriff's action

Suspect's action

	Shoot	Not
Not	(-2, -1)	(0, 0)

Expected payoff for sheriff if he shoots:

$$0 \cdot p + (-1)(1 - p) = p - 1$$

and if he does not shoot:

$$-2 \cdot p + 0 \cdot (1 - p) = -2p$$

Therefore, sheriff should shoot only when

$$p - 1 > -2p \qquad\qquad p > \frac{1}{3}$$

Mathematical definition

A Bayesian game is defined by (N, A, T, p, u), where:

- N - **Set of players**
- $a_i \in A$ - **Actions:** The set of actions available to Player i.
- $t_i \in T$ - **Types:** The set of types for player i. Captures the private information a player can have.
- u - **Payoff functions:** Assign a payoff to a player given their type and the action profile.
- p - **Types probabilities:** Where $p(t) = p(t_1, \ldots, t_N)$ is the probability that Player 1 has type t_1 and Player N has type t_N.

Mathematical definition example

- $N = \{\text{Suspect, Sheriff}\}$
- $A_{\text{Suspect}} = \{\text{Shoot, Not}\}$, $A_{\text{Sheriff}} = \{\text{Shoot, Not}\}$
- $T_{\text{Suspect}} = \{\text{Criminal, Civilian}\}$, $T_{\text{Sheriff}} = \{\text{Default}\}$
- $p_{\text{Criminal}} = p$, $p_{\text{Civilian}} = (1 - p)$
- Payoffs u are the tables we have seen before

Extensions

- She and He have an arbitrary number of types each.
- There are more than two players.
- Players choose sequentially. The player playing second can observe the action, but not the type of the player playing first.

Impact of Asymmetric Information on the Market

- There are two cars:
  - A high-quality car worth \$100,000, sold for \$100,000.
  - A defective car with hidden flaws worth \$50,000, also sold for \$100,000.
- The buyer does not know which car has hidden flaws, so they take this into account and negotiate a price in the middle: \$75,000.
- Since sellers of high-quality cars cannot sell them for lower prices than the cars are worth, they leave the market. Only low-quality cars can be sold for lower prices.
- As a result, the average quality and price in the market decrease.
- This cycle repeats until buyers only want cars for free.

Impact of Asymmetric Information on the Market

In Poland, nearly twice as many used cars are sold as new ones, so there are, of course, ways to deal with this problem, such as warranties.

Bayesian Nash Equilibrium

Definition

A BNE is a set of strategies, one for each type of player, such that no type has incentive to change his or her strategy given the beliefs about the types and what the other types are doing.

Bayesian Nash Equilibrium

Definition

A BNE is a set of strategies, one for each type of player, such that no type has incentive to change his or her strategy given the beliefs about the types and what the other types are doing.

Consider a game with 2 players.
Player 1 may be either type a or type b.
Player 2 is always of one type.

BNE – Example

a type $= p$

a type $= 1 - p$

	Left	**Right**
Up	$3, 4$	$1, 0$
Down	$4, 3$	$2, 0$

BNE – Example

a type $= p$

	Left	Right
Up	$3, 4$	$1, 0$
Down	$4, 3$	$2, 0$

a type $= 1 - p$

	Left	Right
Up	$6, 2$	$0, 4$
Down	$5, 1$	$-1, 4$

a type $= p$

	Left	Right
Up	$3, 4$	$1, 0$
Down	$4, 3$	$2, 0$

BNE – Example

a type $= p$

	Left	**Right**
Up	$3, 4$	$1, 0$
Down	$\boxed{4}, 3$	$\boxed{2}, 0$

a type $= 1 - p$

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Utility

BNE – Example

a type $= p$

	Left	Right
Up	3, 4	1, 0
Down	$\boxed{4}$, 3	$\boxed{2}$, 0

a type $= 1 - p$

	Left	Right
Up	$\boxed{6}$, 2	$\boxed{0}$, 4
Down	5, 1	-1, 4

Utility

$$u(\text{left}) = p \cdot 3 + (1 - p) \cdot 2$$

BNE – Example

a type $= p$

	Left	Right
Up	$3, 4$	$1, 0$
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a type $= 1 - p$

	Left	Right
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Utility

$$u(\text{left}) = p \cdot 3 + (1 - p) \cdot 2$$
$$u(\text{right}) = p \cdot 0 + (1 - p) \cdot 4$$

a type $= p$

	Left	**Right**
Up	3, 4	1, 0
Down	$\boxed{4}$, 3	$\boxed{2}$, 0

a type $= 1 - p$

	Left	**Right**
Up	$\boxed{6}$, 2	$\boxed{0}$, 4
Down	5, 1	-1, 4

Utility

$$u(\text{left}) = p \cdot 3 + (1 - p) \cdot 2$$
$$u(\text{right}) = p \cdot 0 + (1 - p) \cdot 4$$
$$p \cdot 3 + (1 - p) \cdot 2 > p \cdot 0 + (1 - p) \cdot 4$$

BNE – Example

a type $= p$

	Left	Right
Up	3, 4	1, 0
Down	$\boxed{4}$, 3	$\boxed{2}$, 0

a type $= 1 - p$

	Left	Right
Up	$\boxed{6}$, 2	$\boxed{0}$, 4
Down	5, 1	-1, 4

Utility

$$u(\text{left}) = p \cdot 3 + (1-p) \cdot 2$$
$$u(\text{right}) = p \cdot 0 + (1-p) \cdot 4$$
$$p \cdot 3 + (1-p) \cdot 2 > p \cdot 0 + (1-p) \cdot 4$$
$$3p + 2 - 2p > 4 - 4p$$
$$p > 2/5$$

BNE – Example

- Player 1, a type chooses down;
- Player 1, b type chooses up;
- Player 2 chooses:
    - if $p > 2/5$ – left;
    - if $p = 2/5$ – left or right;
    - if $p < 2/5$ – right.

Rules

- All bidders simultaneously submit bids so that no bidder knows the bid of any other participant.
- The highest bidder pays the price that was submitted and wins the auction.

First-Price Auctions example

- The auction house has a book for sale.
- You can quote prices that are a multiple of 10 greater than 0. In the event of a tie, the winner is drawn by lot.
- Julek values the book at $28, and Janek at $24.

What is the optimal bid value for each of them?

Obviously, it is not profitable for Julek and Janek to bid more than \$28 and \$24 respectively

Pay-offs:

	10\$	**20\$**
10\$		
20\$		

First-Price Auctions example

Pay-offs:

	10\$	**20\$**
10\$		\$0, \$4
20\$	\$8, \$0	

Pay-offs:

	10\$	**20\$**
10\$	\$9, \$7	\$0, \$4
20\$	\$8, \$0	

First-Price Auctions example

Pay-offs:

	10\$	**20\$**
10\$	\$9, \$7	\$0, \$4
20\$	\$8, \$0	\$4, \$2

So Julek should bid as much as he expects Janek to bid and vice versa.

Second-Price Auctions

Rules

The same as for first-price auctions, but the winner pays the **second** highest bid.

Example

If three participants A, B and C submit bids of 100, 80 and 120, C wins, but pays 100 (the second highest bid).

Resources

- https://en.wikipedia.org/wiki/Bayesian_game
- https://www.ehu.eus/iaguirre/Chapter%201.%20Bayesian%20Games%20in%20Normal%20Form.pdf