SUPPLEMENTARY MATERIALS: POLICY GRADIENT METHODS FOR THE NOISY LINEAR QUADRATIC REGULATOR OVER A FINITE HORIZON

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SM1. Market Simulator for Linear Price Dynamics. We estimate the parameters for the LQR model using NASDAQ ITCH data taken from Lobster[1].

Permanent Price Impact and Volatility. The model in (2.7) implies that prices changes are proportional to the *market-order flow imbalances* (MFI). We adopt the framework from [1], namely that the price change ΔS is given by

$$\Delta S = \gamma \,\mathrm{MFI} + \sigma\, \epsilon, \tag{SM1.1}$$

with $\mathrm{MFI} = M^b - M^s$ where M^s and M^b are the volumes of market sell orders and market buy orders respectively during a time interval $\Delta T = 5$mins and $\epsilon \sim \mathcal{N}(0,1)$. We then estimate γ and σ from the data.

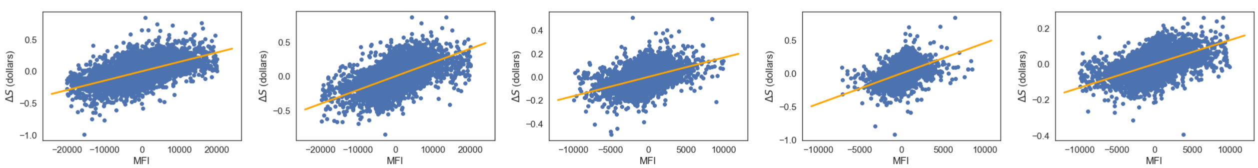

Fig. SM1: Relationship between MFI and ΔS. (Example (from left to right): AAP, FB, JPM, IBM and AAL, 10:00AM-11:00AM 01/01/2019-08/31/2019, $\Delta T = 1$min)

Temporary Price Impact. We assume the LOB has a flat shape with constant queue length l for the first few levels. Figure SM2 shows the average queue lengths for the first 5 levels so that our assumption is not too unreasonable. Therefore the following equation, on the amount received when we liquidate u shares with best bid price S, holds

$$u(S - \beta u) = \int_{S - \frac{u\,\Delta}{l}}^{S} l v dv.$$

Therefore we have $\beta = \frac{\Delta}{2l}$, where Δ is the tick size and l is the average queue length.

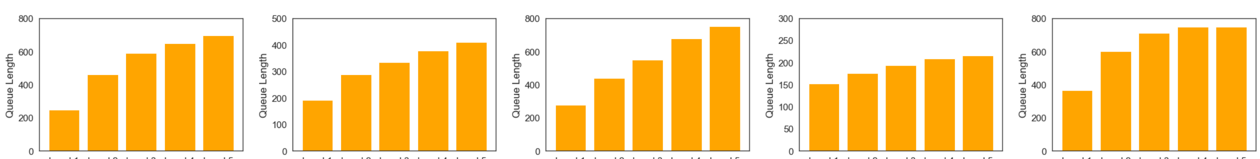

Fig. SM2: Average queue length (volume) of the first five levels on the limit buy side (Example (from left to right): AAP, FB, JPM, IBM and AAL, 10:00AM-11:00AM 01/01/2019-08/31/2019 with 5000 samples uniformly sampled with natural time clock in each trading day.)

Parameter Estimation. See the estimates for AAPL, FB, IBM, JPM, and AAL in Table SM1.

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[1]https://lobsterdata.com/

Paramters/Stock	AAPL	FB	IBM	JPM	AAL
β	1.03×10^{-5}	1.30×10^{-5}	$2.65 \times 10{*}{*}{-5}$	9.28×10^{-6}	3.27×10^{-5}
γ	7.27×10^{-6}	1.40×10^{-5}	4.60×10^{-5}	1.65×10^{-5}	1.3310×10^{-5}
σ	0.107	0.115	0.082	0.059	0.042

Table SM1: Parameter estimation from NASDAQ ITCH Data (10:00AM-11:00PM 01/01/2019-08/31/2019).

SM2. Proofs of Technical Results.

We now give the proofs that were omitted in the text.

SM2.1. Proofs in Section 3.1.

Proof of Lemma 3.2. Denote by $\{x_t\}_{t=0}^T$ the state trajectory induced by an arbitrary control $\boldsymbol{K}$. By Assumption 3.1 the matrix $\mathbb{E}[x_0 x_0^\top]$ is positive definite. For $t \geq 1$, we have

$$\mathbb{E}[x_t x_t^\top] = (A - BK_{t-1})\mathbb{E}[x_{t-1}x_{t-1}^\top](A - BK_{t-1})^\top + \mathbb{E}[w_{t-1}w_{t-1}^\top].$$

Now $(A-BK_{t-1})\mathbb{E}[x_{t-1}x_{t-1}^\top](A-BK_{t-1})^\top$ is positive semi-definite and $\mathbb{E}[w_{t-1}w_{t-1}^\top]$ is positive definite. Hence $\mathbb{E}[x_t x_t^\top]$ is positive definite and as a result $\underline{\sigma}_{\boldsymbol{X}} > 0$. In this case, we can simply take $\underline{\sigma}_{\boldsymbol{X}} = \min(\mathbb{E}[x_0 x_0^\top], \sigma_{\min}(W))$. ∎

Proof of Proposition 3.4. This can be proved by backward induction. For $t = T$, $P_T^{\boldsymbol{K}} = Q_T$ is positive definite since Q_T is positive definite. Assume $P_{t+1}^{\boldsymbol{K}}$ is positive definite for some $t+1$, then take any $z \in \mathbb{R}^d$ such that $z \neq 0$,

$$z^\top P_t^{\boldsymbol{K}} z = z^\top Q_t\, z + z^\top K_t^\top R_t K_t z + z^\top \left(A - BK_t\right)^\top P_{t+1}^{\boldsymbol{K}} \left(A - BK_t\right) z > 0.$$

The last inequality holds since $z^\top Q_t\, z > 0$, $z^\top K_t^\top R_t K_t z \geq 0$ and $z^\top \left(A - BK_t\right)^\top P_{t+1}^{\boldsymbol{K}} \left(A - BK_t\right) z \geq 0$. By backward induction, we have $P_t^{\boldsymbol{K}}$ positive definite, $\forall\, t = 0, 1, \cdots, T$. ∎

To prove Lemma 3.6, let us start with a useful result for the value function. Define the value function $V_{\boldsymbol{K}}(x, \tau)$ for $\tau = 0, 1, \cdots, T-1$, as

$$V_{\boldsymbol{K}}(x,\tau) = \mathbb{E}_{\boldsymbol{w}}\left[\sum_{t=\tau}^{T-1}(x_t^\top Q_t x_t + u_t^\top R_t u_t) + x_T^\top Q_T x_T \,\middle|\, x_\tau = x\right] = x^\top P_\tau x + L_\tau,$$

with terminal condition

$$V_{\boldsymbol{K}}(x, T) = x^\top Q_T x,$$

where L_τ is defined in (3.10). We then define the Q function, $Q_{\boldsymbol{K}}(x, u, \tau)$ for $\tau = 0, 1, \cdots, T-1$ as

$$Q_{\boldsymbol{K}}(x,u,\tau) = x^\top Q_\tau x + u^\top R_\tau u + \mathbb{E}_{w_\tau}\left[V_{\boldsymbol{K}}(Ax + Bu + w_\tau, \tau+1)\right],$$

and the advantage function

$$A_{\boldsymbol{K}}(x,u,\tau) = Q_{\boldsymbol{K}}(x,u,\tau) - V_{\boldsymbol{K}}(x,\tau).$$

Note that $C(\boldsymbol{K}) = \mathbb{E}_{x_0 \sim \mathcal{D}}[V(x_0, 0)]$. Then we can write the difference of value functions between $\boldsymbol{K}$ and $\boldsymbol{K}'$ in terms of advantage functions.

Lemma SM2.1. *Assume $\boldsymbol{K}$ and $\boldsymbol{K}'$ have finite costs. Denote $\{x_t'\}_{t=0}^T$ and $\{u_t'\}_{t=0}^{T-1}$ as the state and control sequences of a single trajectory generated by $\boldsymbol{K}'$ starting from $x_0' = x_0 = x$, then*

$$V_{\boldsymbol{K}'}(x,0) - V_{\boldsymbol{K}}(x,0) = \mathbb{E}_{\boldsymbol{w}}\left[\sum_{t=0}^{T-1} A_{\boldsymbol{K}}(x_t', u_t', t)\right], \quad \text{(SM2.1)}$$

and $A_{\boldsymbol{K}}(x, -K_\tau' x, \tau) \;= 2x^\top (K_\tau' - K_\tau)^\top E_\tau x + x^\top (K_\tau' - K_\tau)^\top (R_\tau + B^\top P_{\tau+1} B)(K_\tau' - K_\tau) x$, *where E_τ is defined in* (3.11).

Proof. Denote by $c_t'(x)$ the cost generated by $\boldsymbol{K}'$ with a single trajectory starting from $x_0' = x_0 = x$. That is, $c_t'(x) = (x_t')^\top Q_t x_t' + (u_t')^\top R_t u_t',\ t = 0, 1, \cdots, T-1$, and $c_T'(x) = (x_T')^\top Q_T x_T'$, with $u_t' = -K_t' x_t',\ \ x_{t+1}' = Ax_t' + Bu_t' + w_t,\ \ x_0' = x$.
Therefore,

$$
\begin{aligned}
V_{\boldsymbol{K}'}(x,0) - V_{\boldsymbol{K}}(x,0) &= \mathbb{E}_{\boldsymbol{w}}\left[\sum_{t=0}^{T} c_t'(x)\right] - V_{\boldsymbol{K}}(x,0) = \mathbb{E}_{\boldsymbol{w}}\left[\sum_{t=0}^{T} \left(c_t'(x) + V_{\boldsymbol{K}}(x_t',t) - V_{\boldsymbol{K}}(x_t',t)\right)\right] - V_{\boldsymbol{K}}(x,0)\\
&= \mathbb{E}_{\boldsymbol{w}}\left[\sum_{t=0}^{T-1} \left(c_t'(x) + V_{\boldsymbol{K}}(x_{t+1}',t+1) - V_{\boldsymbol{K}}(x_t',t)\right)\right]\\
&= \mathbb{E}_{\boldsymbol{w}}\left[\sum_{t=0}^{T-1} \left(Q_{\boldsymbol{K}}(x_t',u_t',t) - V_{\boldsymbol{K}}(x_t',t)\right)\middle|\, x_0 = x\right] = \mathbb{E}_{\boldsymbol{w}}\left[\sum_{t=0}^{T-1} A_{\boldsymbol{K}}(x_t',u_t',t)\,\middle|\, x_0 = x\right],
\end{aligned}
$$

where the third equality holds since $c_T'(x) = V_{\boldsymbol{K}}(x_T', T)$ with the same single trajectory. For $u = -K_\tau' x$,

$$
\begin{aligned}
A_{\boldsymbol{K}}(x,-K_\tau'x,\tau) &= Q_{\boldsymbol{K}}(x,-K_\tau'x,\tau) - V_{\boldsymbol{K}}(x,\tau)\\
&= x^\top(Q_\tau + (K_\tau')^\top R_\tau K_\tau')x + \mathbb{E}_{w_\tau}\left[V_{\boldsymbol{K}}((A-BK_\tau')x + w_\tau, \tau+1)\right] - V_{\boldsymbol{K}}(x,\tau)\\
&= x^\top(Q_\tau + (K_\tau')^\top R_\tau K_\tau')x + \left(x^\top(A-BK_\tau')^\top P_{\tau+1}(A-BK_\tau')x + \mathrm{Tr}(WP_{\tau+1}) + L_{\tau+1}\right)\\
&\quad - \left(x^\top P_\tau x + L_\tau\right)\\
&= x^\top(Q_\tau + (K_\tau' - K_\tau + K_\tau)^\top R_\tau(K_\tau' - K_\tau + K_\tau))x\\
&\quad + x^\top(A - BK_\tau - B(K_\tau' - K_\tau))^\top P_{\tau+1}(A - BK_\tau - B(K_\tau' - K_\tau))x\\
&\quad - x^\top(Q_\tau + K_\tau^\top R_\tau K_\tau + (A-BK_\tau)^\top P_{\tau+1}(A-BK_\tau))x\\
&= 2x^\top(K_\tau' - K_\tau)^\top((R_\tau + B^\top P_{\tau+1}B)K_\tau - B^\top P_{\tau+1}A)x\\
&\quad + x^\top(K_\tau' - K_\tau)^\top(R_\tau + B^\top P_{\tau+1}B)(K_\tau' - K_\tau)x.
\end{aligned}
\tag{SM2.2}
$$

$\square$

Proof of Lemma 3.6. First for any K_τ', from (SM2.2),

$$
\begin{aligned}
A_{\boldsymbol{K}}(x,-K_\tau'x,\tau) &= Q_{\boldsymbol{K}}(x,-K_\tau'x,\tau) - V_{\boldsymbol{K}}(x,\tau)\\
&= 2\,\mathrm{Tr}(xx^\top(K_\tau' - K_\tau)^\top E_\tau) + \mathrm{Tr}(xx^\top(K_\tau' - K_\tau)^\top(R_\tau + B^\top P_{\tau+1}B)(K_\tau' - K_\tau))\\
&= \mathrm{Tr}\left(xx^\top(K_\tau' - K_\tau + (R_\tau + B^\top P_{\tau+1}B)^{-1}E_\tau)^\top(R_\tau + B^\top P_{\tau+1}B)\right.\\
&\quad \left.(K_\tau' - K_\tau + (R_\tau + B^\top P_{\tau+1}B)^{-1}E_\tau)\right) - \mathrm{Tr}(xx^\top E_\tau^\top(R_\tau + B^\top P_{\tau+1}B)^{-1}E_\tau)\\
&\geq -\mathrm{Tr}(xx^\top E_\tau^\top(R_\tau + B^\top P_{\tau+1}B)^{-1}E_\tau),
\end{aligned}
\tag{SM2.3}
$$

with equality holds when $K_\tau' = K_\tau - (R_\tau + B^\top P_{\tau+1}B)^{-1}E_\tau$. Then,

$$
\begin{aligned}
C(\boldsymbol{K}) - C(\boldsymbol{K}^*) &= -\mathbb{E}\sum_{t=0}^{T-1} A_{\boldsymbol{K}}(x_t^*,u_t^*,t) \leq \mathbb{E}\sum_{t=0}^{T-1}\mathrm{Tr}\left(x_t^*(x_t^*)^\top E_t^\top(R_t + B^\top P_{t+1}B)^{-1}E_t\right)\\
&\leq \|\Sigma_{\boldsymbol{K}^*}\|\sum_{t=0}^{T-1}\mathrm{Tr}(E_t^\top(R_t + B^\top P_{t+1}B)^{-1}E_t) \leq \frac{\|\Sigma_{\boldsymbol{K}^*}\|}{\underline{\sigma}_{\boldsymbol{R}}}\sum_{t=0}^{T-1}\mathrm{Tr}(E_t^\top E_t)\\
&\leq \frac{\|\Sigma_{\boldsymbol{K}^*}\|}{4\,\underline{\sigma}_{\boldsymbol{X}}^2\,\underline{\sigma}_{\boldsymbol{R}}}\sum_{t=0}^{T-1}\mathrm{Tr}(\nabla_t C(\boldsymbol{K})^\top\nabla_t C(\boldsymbol{K})),
\end{aligned}
$$

where $\underline{\sigma}_{\boldsymbol{X}}$ is defined in (3.3) and $\underline{\sigma}_{\boldsymbol{R}}$ is defined in (3.4). For the lower bound, consider $K_t' = K_t - (R_t +$

$B^\top P_{t+1}B)^{-1}E_t$ where the equality holds in (SM2.3). Using $C(\boldsymbol{K}^*) \le C(\boldsymbol{K}')$

$$
\begin{aligned}
C(\boldsymbol{K}) - C(\boldsymbol{K}^*) \ge C(\boldsymbol{K}) - C(\boldsymbol{K}') &= -\mathbb{E}\sum_{t=0}^{T-1} A_{\boldsymbol{K}}(x_t', u_t', t) = \mathbb{E}\sum_{t=0}^{T-1} \mathrm{Tr}(x_t'(x_t')^\top E_t^\top (R_t + B^\top P_{t+1}B)^{-1}E_t) \\
&\ge \underline{\sigma}_{\boldsymbol{X}} \sum_{t=0}^{T-1} \frac{1}{\|R_t + B^\top P_{t+1}B\|} \mathrm{Tr}(E_t^\top E_t)
\end{aligned} \tag{SM2.4}
$$

$\square$

Proof of Lemma 3.7. By lemma SM2.1 we have

$$
\begin{aligned}
C(\boldsymbol{K}') - C(\boldsymbol{K}) &= \mathbb{E}\left[\sum_{t=0}^{T-1} A_{\boldsymbol{K}}(x_t', -K_t'x_t', t)\right] \\
&= \sum_{t=0}^{T-1} \left(2\,\mathrm{Tr}(\Sigma_t'(K_t' - K_t)^\top E_t) + \mathrm{Tr}(\Sigma_t'(K_t' - K_t)^\top (R_t + B^\top P_{t+1}B)(K_t' - K_t))\right).
\end{aligned}
$$

$\square$

Proof of Lemma 3.8. For $t = 0, 1, \cdots, T$,

$$
C(\boldsymbol{K}) \ge \mathbb{E}[x_t^\top P_t x_t] \ge \|P_t\|\sigma_{\min}(\mathbb{E}[x_t x_t^\top]) \ge \underline{\sigma}_{\boldsymbol{X}}\,\|P_t\|,
$$

$$
C(\boldsymbol{K}) = \sum_{t=0}^{T-1} \mathrm{Tr}(\mathbb{E}[x_t x_t^\top](Q_t + K_t^\top R_t K_t)) + \mathrm{Tr}(\mathbb{E}[x_T x_T^\top]Q_T) \ge \underline{\sigma}_{\boldsymbol{Q}}\,\mathrm{Tr}(\Sigma_{\boldsymbol{K}}) \ge \underline{\sigma}_{\boldsymbol{Q}}\,\|\Sigma_{\boldsymbol{K}}\|.
$$

Therefore the statement in Lemma 3.8 follows provided that $\underline{\sigma}_{\boldsymbol{X}} > 0$ and Assumption 2.1 holds. $\square$

Proof of Proposition 3.9 . Recall that $\Sigma_t = \mathbb{E}\left[x_t x_t^\top\right]$. Note that

$$
\begin{aligned}
\Sigma_1 = \mathbb{E}\left[x_1 x_1^\top\right] &= \mathbb{E}\left[\left((A - B\,K_0)x_0 + w_0\right)\left((A - B\,K_0)x_0 + w_0\right)^\top\right] \\
&= (A - B\,K_0)\Sigma_0\left(A - B\,K_0\right)^\top + W = \mathcal{G}_0(\Sigma_0) + W.
\end{aligned}
$$

Now we first prove that

$$
\Sigma_t = \mathcal{G}_{t-1}(\Sigma_0) + \sum_{s=1}^{t-1} D_{t-1,s} W D_{t-1,s}^\top + W,\ \forall\, t = 2, 3, \cdots, T. \tag{SM2.5}
$$

When $t = 2$,

$$
\begin{aligned}
\Sigma_2 = \mathbb{E}\left[x_2 x_2^\top\right] &= \mathbb{E}\left[\left((A - B\,K_1)x_1 + w_1\right)\left((A - B\,K_1)x_1 + w_1\right)^\top\right] \\
&= (A - B\,K_1)\Sigma_1\left(A - B\,K_1\right)^\top + W = \mathcal{G}_1(\Sigma_0) + (A - BK_1)W(A - BK_1)^\top + W,
\end{aligned}
$$

which satisfies (SM2.5). Assume (SM2.5) holds for $t \le k$. Then for $t = k+1$,

$$
\begin{aligned}
\mathbb{E}\left[x_{t+1}x_{t+1}^\top\right] &= \mathbb{E}\left[\left((A - B\,K_t)x_t + w_t\right)\left((A - B\,K_t)x_t + w_t\right)^\top\right] \\
&= (A - B\,K_t)\Sigma_t\left(A - B\,K_t\right)^\top + W = \mathcal{G}_t(\Sigma_0) + \sum_{s=1}^{t} D_{t,s} W D_{t,s}^\top + W.
\end{aligned}
$$

Therefore (SM2.5) holds, $\forall\, t = 1, 2, \cdots, T$. Finally,

$$
\Sigma_{\boldsymbol{K}} = \sum_{t=0}^{T} \Sigma_t = \Sigma_0 + \sum_{t=1}^{T-1} \mathcal{G}_t(\Sigma_0) + \sum_{t=1}^{T-1}\sum_{s=1}^{t} D_{t,s} W D_{t,s}^\top + TW = \mathcal{T}_{\boldsymbol{K}}(\Sigma_0) + \Delta(\boldsymbol{K}, W).
$$

$\square$

SM2.2. Proofs in Section 3.2.

Proof of Lemma 3.13. By direct calculation,

$$\|\mathcal{G}_t\| \le \rho^{2(t+1)}, \quad \text{and} \quad \|\mathcal{G}'_t\| \le \rho^{2(t+1)}. \tag{SM2.6}$$

Denote $\mathcal{F}_t = \mathcal{F}_{K_t}$ and $\mathcal{F}'_t = \mathcal{F}_{K'_t}$ to ease the exposition. Then for any symmetric matrix $\Sigma \in \mathbb{R}^{d\times d}$ and $t \ge 0$,

$$\begin{aligned}
\|(\mathcal{G}'_{t+1} - \mathcal{G}_{t+1})(\Sigma)\| &= \|\mathcal{F}'_{t+1} \circ \mathcal{G}'_t(\Sigma) - \mathcal{F}_{t+1} \circ \mathcal{G}_t(\Sigma)\| \\
&= \|\mathcal{F}'_{t+1} \circ \mathcal{G}'_t(\Sigma) - \mathcal{F}'_{t+1} \circ \mathcal{G}_t(\Sigma) + \mathcal{F}'_{t+1} \circ \mathcal{G}_t(\Sigma) - \mathcal{F}_{t+1} \circ \mathcal{G}_t(\Sigma)\| \\
&\le \|\mathcal{F}'_{t+1} \circ \mathcal{G}'_t(\Sigma) - \mathcal{F}'_{t+1} \circ \mathcal{G}_t(\Sigma)\| + \|\mathcal{F}'_{t+1} \circ \mathcal{G}_t(\Sigma) - \mathcal{F}_{t+1} \circ \mathcal{G}_t(\Sigma)\| \\
&= \|\mathcal{F}'_{t+1} \circ (\mathcal{G}'_t - \mathcal{G}_t)(\Sigma)\| + \|(\mathcal{F}'_{t+1} - \mathcal{F}_{t+1}) \circ \mathcal{G}_t(\Sigma)\| \\
&\le \|\mathcal{F}'_{t+1}\| \, \|(\mathcal{G}'_t - \mathcal{G}_t)(\Sigma)\| + \|\mathcal{G}_t\| \, \|\mathcal{F}'_{t+1} - \mathcal{F}_{t+1}\| \, \|\Sigma\| \\
&\le \rho^2 \|(\mathcal{G}'_t - \mathcal{G}_t)(\Sigma)\| + \rho^{2(t+1)} \|\mathcal{F}'_{t+1} - \mathcal{F}_{t+1}\| \|\Sigma\|.
\end{aligned}$$

Therefore,

$$\|(\mathcal{G}'_{t+1} - \mathcal{G}_{t+1})(\Sigma)\| \le \rho^2 \|(\mathcal{G}'_t - \mathcal{G}_t)(\Sigma)\| + \rho^{2(t+1)} \|\mathcal{F}'_{t+1} - \mathcal{F}_{t+1}\| \|\Sigma\|. \tag{SM2.7}$$

Summing (SM2.7) up for $t \in \{1, 2, \cdots, T-2\}$ with $\|\mathcal{G}'_0 - \mathcal{G}_0\| = \|\mathcal{F}'_0 - \mathcal{F}_0\|$, we have

$$\sum_{t=0}^{T-1} \left\| (\mathcal{G}_t - \mathcal{G}'_t)(\Sigma) \right\| \le \frac{\rho^{2T} - 1}{\rho^2 - 1} \Big(\sum_{t=0}^{T-1} \|\mathcal{F}_t - \mathcal{F}'_t\| \Big) \|\Sigma\|.$$

□

SM2.3. Proofs in Section 3.3.

Proof of Lemma 3.15. Given (3.22) and condition (3.23), we have $\|K'_t - K_t\| = \eta \|\nabla_t C(\boldsymbol{K})\| \le \frac{\sigma_{\boldsymbol{Q}} \, \sigma_{\boldsymbol{X}}}{2C(\boldsymbol{K})\|B\|}$. Therefore,

$$\|B\| \|K'_t - K_t\| \le \frac{\sigma_{\boldsymbol{Q}} \, \sigma_{\boldsymbol{X}}}{2C(\boldsymbol{K})} \le \frac{1}{2}.$$

The last inequality holds since $\sigma_{\boldsymbol{X}} \le \frac{C(\boldsymbol{K})}{\sigma_{\boldsymbol{Q}}}$ given by Lemma 3.8. Therefore, by Lemma 3.12,

$$\sum_{t=0}^{T-1} \|\mathcal{F}_{K_t} - \mathcal{F}_{K'_t}\| \le (2\rho + 1)\|B\| \left(\sum_{t=0}^{T-1} \|K_t - K'_t\| \right). \tag{SM2.8}$$

By Lemmas 3.5 and 3.7,

(SM2.9)

$$\begin{aligned}
C(\boldsymbol{K}') - C(\boldsymbol{K}) &= \sum_{t=0}^{T-1} \Big[2\,\mathrm{Tr}\Big(\Sigma'_t (K'_t - K_t)^\top E_t\Big) + \mathrm{Tr}\Big(\Sigma'_t (K'_t - K_t)^\top (R_t + B^\top P_{t+1} B)(K'_t - K_t)\Big)\Big] \\
&= \sum_{t=0}^{T-1} \Big[-4\eta\,\mathrm{Tr}\Big(\Sigma'_t \Sigma_t E_t^\top E_t\Big) + 4\eta^2\,\mathrm{Tr}\Big(\Sigma'_t \Sigma_t E_t^\top (R_t + B^\top P_{t+1} B) E_t \Sigma_t\Big)\Big] \\
&= \sum_{t=0}^{T-1} \Big[-4\eta\,\mathrm{Tr}\Big((\Sigma'_t - \Sigma_t + \Sigma_t) \Sigma_t E_t^\top E_t\Big) + 4\eta^2\,\mathrm{Tr}\Big(\Sigma'_t \Sigma_t E_t^\top (R_t + B^\top P_{t+1} B) E_t \Sigma_t\Big)\Big] \\
&\le \sum_{t=0}^{T-1} \Big[-4\eta\,\mathrm{Tr}\Big(\Sigma_t E_t^\top E_t \Sigma_t\Big) + 4\eta\,\mathrm{Tr}((\Sigma'_t - \Sigma_t)\Sigma_t E_t^\top E_t \Sigma_t \Sigma_t^{-1}) \\
&\qquad + 4\eta^2\,\mathrm{Tr}\Big(\Sigma'_t \Sigma_t E_t^\top (R_t + B^\top P_{t+1} B) E_t \Sigma_t\Big)\Big] \\
&\le \sum_{t=0}^{T-1} \Big[-4\eta\,\mathrm{Tr}\Big(\Sigma_t E_t^\top E_t \Sigma_t\Big) + 4\eta \frac{\|\Sigma'_t - \Sigma_t\|}{\sigma_{\min}(\Sigma_t)}\,\mathrm{Tr}\Big(\Sigma_t E_t^\top E_t \Sigma_t\Big) \\
&\qquad + 4\eta^2 \|\Sigma'_t (R_t + B^\top P_{t+1} B)\|\,\mathrm{Tr}\Big(\Sigma_t E_t^\top E_t \Sigma_t\Big)\Big] \\
&\le -\eta \Big(1 - \frac{\sum_{t=0}^{T-1} \|\Sigma'_t - \Sigma_t\|}{\underline{\sigma}_{\boldsymbol{X}}} - \eta \|\Sigma_{\boldsymbol{K}'}\| \sum_{t=0}^{T-1} \|R_t + B^\top P_{t+1} B\| \Big) \sum_{t=0}^{T-1} \Big[\mathrm{Tr}(\nabla_t C(\boldsymbol{K})^\top \nabla_t C(\boldsymbol{K})) \Big].
\end{aligned}$$

By Lemma 3.6, we have

(SM2.10)
$$C(\boldsymbol{K'}) - C(\boldsymbol{K}) \leq -\eta\Big(1 - \frac{\sum_{t=0}^{T-1}\|\Sigma_t' - \Sigma_t\|}{\underline{\sigma}_{\boldsymbol{X}}} - \eta\|\Sigma_{\boldsymbol{K'}}\|\sum_{t=0}^{T-1}\|R_t + B^\top P_{t+1}B\|\Big)\Big(\frac{4\,\underline{\sigma}_{\boldsymbol{X}}{}^2\,\underline{\sigma}_{\boldsymbol{R}}}{\|\Sigma_{\boldsymbol{K}^*}\|}\Big)\Big(C(\boldsymbol{K}) - C(\boldsymbol{K}^*)\Big)$$

provided that

$$1 - \frac{\sum_{t=0}^{T-1}\|\Sigma_t' - \Sigma_t\|}{\underline{\sigma}_{\boldsymbol{X}}} - \eta\|\Sigma_{\boldsymbol{K'}}\|\sum_{t=0}^{T-1}\|R_t + B^\top P_{t+1}B\| > 0. \tag{SM2.11}$$

By (3.21), (3.22), and (SM2.8),

$$\sum_{t=0}^{T-1}\|\Sigma_t' - \Sigma_t\| \leq \frac{\rho^{2T}-1}{\rho^2-1}\left(\frac{C(\boldsymbol{K})}{\underline{\sigma}_{\boldsymbol{Q}}} + T\|W\|\right)\left(\eta(2\rho+1)\|B\|\sum_{t=0}^{T-1}\|\nabla_t C(\boldsymbol{K})\|\right).$$

Given the step size condition in (3.23), we have
(SM2.12)
$$\eta(2\rho+1)\|B\|\sum_{t=0}^{T-1}\|\nabla_t C(\boldsymbol{K})\| \leq \eta(2\rho+1)\|B\|\Big(T\cdot\max_t\{\|\nabla_t C(\boldsymbol{K})\|\}\Big) \leq \frac{(\rho^2-1)\,\underline{\sigma}_{\boldsymbol{Q}}\,\underline{\sigma}_{\boldsymbol{X}}}{2(\rho^{2T}-1)(C(\boldsymbol{K}) + \underline{\sigma}_{\boldsymbol{Q}}\,T\|W\|)}.$$

Then, by Corollary 3.14 and (SM2.8),

$$\begin{aligned}\frac{\|\Sigma_{\boldsymbol{K'}} - \Sigma_{\boldsymbol{K}}\|}{\underline{\sigma}_{\boldsymbol{X}}} &\leq \frac{\rho^{2T}-1}{\rho^2-1}\Big(\sum_{t=0}^{T-1}\|\mathcal{F}_{K_t} - \mathcal{F}_{K_t'}\|\Big)\frac{\|\Sigma_0\| + T\|W\|}{\underline{\sigma}_{\boldsymbol{X}}}\\ &\leq \frac{\rho^{2T}-1}{\rho^2-1}(2\rho+1)\|B\|\left(\sum_{t=0}^{T-1}\eta\|\nabla_t C(\boldsymbol{K}\|\right)\frac{C(\boldsymbol{K}) + \underline{\sigma}_{\boldsymbol{Q}}\,T\|W\|}{\underline{\sigma}_{\boldsymbol{Q}}\,\underline{\sigma}_{\boldsymbol{X}}} \leq \frac{1}{2},\end{aligned}$$

where the last step holds by (SM2.12). Therefore, the bound of $\|\Sigma_{\boldsymbol{K'}}\|$ in (SM2.11) is given by

$$\|\Sigma_{\boldsymbol{K'}}\| \leq \|\Sigma_{\boldsymbol{K'}} - \Sigma_{\boldsymbol{K}}\| + \|\Sigma_{\boldsymbol{K}}\| \leq \frac{1}{2}\,\underline{\sigma}_{\boldsymbol{X}} + \frac{C(\boldsymbol{K})}{\underline{\sigma}_{\boldsymbol{Q}}} \leq \frac{1}{2}\|\Sigma_{\boldsymbol{K'}}\| + \frac{C(\boldsymbol{K})}{\underline{\sigma}_{\boldsymbol{Q}}}, \tag{SM2.13}$$

which indicates that $\|\Sigma_{\boldsymbol{K'}}\| \leq \frac{2C(\boldsymbol{K})}{\underline{\sigma}_{\boldsymbol{Q}}}$. Therefore, (SM2.11) gives

$$\begin{aligned}&1 - \frac{\sum_{t=0}^{T-1}\|\Sigma_t' - \Sigma_t\|}{\underline{\sigma}_{\boldsymbol{X}}} - \eta\|\Sigma_{\boldsymbol{K'}}\|\sum_{t=0}^{T-1}\|R_t + B^\top P_{t+1}B\|\\ &\geq 1 - \frac{(\rho^{2T}-1)}{(\rho^2-1)\,\underline{\sigma}_{\boldsymbol{X}}}\left(\frac{C(\boldsymbol{K})}{\underline{\sigma}_{\boldsymbol{Q}}} + T\|W\|\right)\left(\eta(2\rho+1)\|B\|\sum_{t=0}^{T-1}\|\nabla_t C(\boldsymbol{K})\|\right) - \eta\frac{2C(\boldsymbol{K})}{\underline{\sigma}_{\boldsymbol{Q}}}\sum_{t=0}^{T-1}\|R_t + B^\top P_{t+1}B\|\\ &= 1 - C_1\eta,\end{aligned}$$

where C_1 is defined in (3.24). So if $\eta \leq \frac{1}{2C_1}$, then,

$$1 - \frac{\sum_{t=0}^{T-1}\|\Sigma_t' - \Sigma_t\|}{\underline{\sigma}_{\boldsymbol{X}}} - \eta\|\Sigma_{\boldsymbol{K'}}\|\sum_{t=0}^{T-1}\|R_t + B^\top P_{t+1}B\| \geq 1 - C_1\eta \geq \frac{1}{2} > 0.$$

Hence,$C(\boldsymbol{K'}) - C(\boldsymbol{K}) \leq -\frac{\eta}{2}\Big(\frac{4\,\underline{\sigma}_{\boldsymbol{X}}{}^2\,\underline{\sigma}_{\boldsymbol{R}}}{\|\Sigma_{\boldsymbol{K}^*}\|}\Big)\Big(C(\boldsymbol{K}) - C(\boldsymbol{K}^*)\Big)$, and

$$C(\boldsymbol{K'}) - C(\boldsymbol{K}^*) = (C(\boldsymbol{K'}) - C(\boldsymbol{K})) + (C(\boldsymbol{K}) - C(\boldsymbol{K}^*)) \leq \Big(1 - 2\eta\frac{\underline{\sigma}_{\boldsymbol{X}}{}^2\,\underline{\sigma}_{\boldsymbol{R}}}{\|\Sigma_{\boldsymbol{K}^*}\|}\Big)\Big(C(\boldsymbol{K}) - C(\boldsymbol{K}^*)\Big).$$

$\square$

SM2.4. Proofs in Section 4.

Proof of Lemma 4.6. Under Assumption 4.3, we have $\mathbb{E}\left[x_0x_0^\top\right] = \widetilde{W}_0\mathbb{E}\left[z_0z_0^\top\right]\widetilde{W}_0^\top$, and $\left\|\mathbb{E}\left[x_0x_0^\top\right]\right\| \leq \sigma_0^2\|\widetilde{W}_0\|^2$. With the sub-Gaussian distributed noise, $W = \mathbb{E}\left[w_tw_t^\top\right] = \widetilde{W}\mathbb{E}\left[v_tv_t^\top\right]\widetilde{W}^\top$, then we have $\|W\| \leq \sigma_w^2\left\|\widetilde{W}^2\right\|$.

Denote $S_t = Q_t + K_t^TR_tK_t$, $\forall\, t = 1,\cdots,T-1$. Thus, for $t = 0,1,\cdots,T-2$,

$$\begin{aligned}\mathbb{E}[x_{t+1}^\top Q_{t+1}x_{t+1} + u_{t+1}^\top R_{t+1}u_{t+1}] &= \mathbb{E}[x_{t+1}^\top S_{t+1}x_{t+1}] = \mathrm{Tr}(\mathbb{E}[x_{t+1}^\top S_{t+1}x_{t+1}]) = \mathrm{Tr}(\mathbb{E}[x_{t+1}x_{t+1}^\top]S_{t+1})\\ &= \mathrm{Tr}\left(\mathcal{G}_t(\Sigma_0)S_{t+1} + \sum_{s=1}^{t} D_{t,s}WD_{t,s}^\top S_{t+1} + WS_{t+1}\right).\end{aligned}$$

The last equality holds by (SM2.5). Therefore,

$$\begin{aligned}C(\boldsymbol{K}') - C(\boldsymbol{K}) = &\underbrace{\mathbb{E}[x_0^\top(K_0')^\top R_0K_0'x_0 - x_0^\top K_0^\top R_0K_0x_0]}_{(I)} + \underbrace{\sum_{t=0}^{T-2}\mathrm{Tr}\left(\mathcal{G}_t'(\Sigma_0)S_{t+1}' - \mathcal{G}_t(\Sigma_0)S_{t+1}\right)}_{(II)}\\ &+ \underbrace{\sum_{t=0}^{T-2}\mathrm{Tr}\left(\sum_{s=1}^{t}\left(D_{t,s}'W(D_{t,s}')^\top S_{t+1}' - D_{t,s}WD_{t,s}^\top S_{t+1}\right) + W(S_{t+1}' - S_{t+1})\right)}_{(III)}\\ &+ \underbrace{\mathrm{Tr}\left(\mathcal{G}_{T-1}(\Sigma_0)Q_T - \mathcal{G}_{T-1}'(\Sigma_0)Q_T + \sum_{s=1}^{T-1}\left(D_{T-1,s}'W(D_{T-1,s}')^\top Q_T - D_{T-1,s}WD_{T-1,s}^\top Q_T\right)\right)}_{(IV)}.\end{aligned}$$

For the first term, $(I) \leq \mathrm{Tr}(\mathbb{E}[x_0x_0^\top])\|(K_0')^\top R_0K_0' - K_0^\top R_0K_0\|$. For the second term (II), since

$$\sum_{t=0}^{T-2}\left(\mathrm{Tr}\left(\mathcal{G}_t(\Sigma_0)S_{t+1}\right)\right) = \mathbb{E}\left[\sum_{t=0}^{T-2}\left(\mathrm{Tr}\left(\Pi_{i=0}^t(A-BK_i)x_0x_0^\top\Pi_{i=0}^t(A-BK_{t-i})^\top S_{t+1}\right)\right)\right] \leq \mathrm{Tr}\left(\mathbb{E}\left[x_0x_0^\top\right]\right)\left\|\sum_{t=0}^{T-2}\mathcal{G}_t(S_{t+1})\right\|,$$

we have, $(II) \leq \mathrm{Tr}\left(\mathbb{E}\left[x_0x_0^\top\right]\right)\left\|\sum_{t=0}^{T-2}\left(\mathcal{G}_t'\left(S_{t+1}'\right) - \mathcal{G}_t\left(S_{t+1}\right)\right)\right\|$.

We denote $\mathcal{G}_d := \sum_{t=0}^{T-2}\left(\mathcal{G}_t'\left(S_{t+1}'\right) - \mathcal{G}_t\left(S_{t+1}\right)\right)$, then

$$\begin{aligned}\|\mathcal{G}_d\| \leq &\sum_{t=0}^{T-2}\Big\|\mathcal{G}_t'\left(Q_{t+1} + (K_{t+1}')^\top R_{t+1}K_{t+1}'\right) - \mathcal{G}_t\left(Q_{t+1} + (K_{t+1}')^\top R_{t+1}K_{t+1}'\right) -\\ &\mathcal{G}_t\circ\left(K_{t+1}^\top R_{t+1}K_{t+1} - (K_{t+1}')^\top R_{t+1}K_{t+1}'\right)\Big\|\\ \leq &\frac{\rho^{2T}-1}{\rho^2-1}\left((2\rho+1)\|B\|\sum_{t=0}^{T-2}\|K_t - K_t'\|\right)\left(\sum_{t=1}^{T-1}\|Q_t + (K_t')^\top R_tK_t'\|\right)\\ &+\sum_{t=0}^{T-2}\|\mathcal{G}_t\|\left\|(K_{t+1}')^\top R_{t+1}K_{t+1}' - K_{t+1}^\top R_{t+1}K_{t+1}\right\|\\ \leq &\frac{\rho^{2T}-1}{\rho^2-1}\left((2\rho+1)\|B\|\sum_{t=0}^{T-2}\|K_t - K_t'\|\right)\left(\sum_{t=1}^{T-1}\|Q_t + (K_t')^\top R_tK_t' - K_t^\top R_tK_t + K_t^\top R_tK_t\|\right)\\ &+\frac{\rho^2(\rho^{2(T-1)}-1)}{\rho^2-1}\sum_{t=1}^{T-1}\left\|(K_t')^\top R_tK_t' - K_t^\top R_tK_t\right\|\\ \leq &\frac{\rho^{2T}-1}{\rho^2-1}(2\rho+1)\|B\|\,\|\!|\boldsymbol{K}' - \boldsymbol{K}|\!\|\left(\|\!|\boldsymbol{Q}|\!\| + \|\!|\boldsymbol{K}|\!\|^2\,\|\!|\boldsymbol{R}|\!\|\right)\\ &+\left(\frac{\rho^{2T}-1}{\rho^2-1}(2\rho+1)\|B\|\,\|\!|\boldsymbol{K}' - \boldsymbol{K}|\!\| + \frac{\rho^2(\rho^{2(T-1)}-1)}{\rho^2-1}\right)\sum_{t=1}^{T-1}\left\|(K_t')^\top R_tK_t' - K_t^\top R_tK_t\right\|.\end{aligned} \tag{SM2.14}$$

where the second inequality holds by Lemma 3.13 and (SM2.8), and the third inequality holds by (SM2.6). For the first term in (III), we have

$$
\begin{aligned}
&\sum_{t=0}^{T-2} \mathrm{Tr}\left(\sum_{s=1}^{t} D'_{t,s} W (D'_{t,s})^\top S'_{t+1} - D_{t,s} W D_{t,s}^\top S_{t+1}\right) \\
&= \sum_{t=0}^{T-2} \mathrm{Tr}\left(\sum_{s=1}^{t} D'_{t,s} W (D'_{t,s})^\top (S'_{t+1} - S_{t+1}) + (D'_{t,s} W (D'_{t,s})^\top - D_{t,s} W D_{t,s}^\top) S_{t+1}\right) \\
&\leq \Big(\sum_{t=0}^{T-2}\sum_{s=1}^{t} \mathrm{Tr}(W) \|D'_{t,s}\|^2\Big) \left\|\sum_{t=1}^{T-1} (K'_t)^\top R_t K'_t - K_t^\top R_t K_t\right\| \\
&\quad + \sum_{t=0}^{T-2} \left\|\sum_{s=1}^{t} D'_{t,s} W (D'_{t,s})^\top - D_{t,s} W D_{t,s}^\top\right\| \Big(\sum_{t=1}^{T-1} \mathrm{Tr}(Q_t) + \|K_t\|^2 \mathrm{Tr}(R_t)\Big) \\
&\leq \mathrm{Tr}(W) \frac{(T-1)(\rho^{2(T-1)}-1)}{\rho^2-1} \left\|\sum_{t=1}^{T-1} (K'_t)^\top R_t K'_t - K_t^\top R_t K_t\right\| \\
&\quad + T\frac{(\rho^{2T}-1)}{\rho^2-1}(2\rho+1)\|B\|\,\|W\|\,|||\boldsymbol{K}'-\boldsymbol{K}|||\left(\mathrm{Tr}\left(\sum_{t=1}^{T-1} Q_t\right) + |||\boldsymbol{K}|||^2 \mathrm{Tr}\left(\sum_{t=1}^{T-1} R_t\right)\right),
\end{aligned}
$$

where the last step holds by (3.20). The second term in (III) is bounded by

$$
\sum_{t=0}^{T-2} \mathrm{Tr}\Big(W(S'_{t+1} - S_{t+1})\Big) \leq \mathrm{Tr}(W) \sum_{t=1}^{T-1} \left\|(K'_t)^\top R_t K'_t - K_t^\top R_t K_t\right\|.
$$

Similarly, by (3.20) and (SM2.8), (IV) is bounded by

$$
\begin{aligned}
(IV) &\leq \mathrm{Tr}(\mathbb{E}[x_0 x_0^\top]) \sum_{t=0}^{T-1} \left\|(\mathcal{G}'_t - \mathcal{G}_t)(Q_T)\right\| + \mathrm{Tr}\left(\sum_{s=1}^{T-1} D'_{T-1,s} W (D'_{T-1,s})^\top Q_T - D_{T-1,s} W D_{T-1,s}^\top Q_T\right) \\
&\leq \mathrm{Tr}(\mathbb{E}[x_0 x_0^\top]) \frac{\rho^{2T}-1}{\rho^2-1}(2\rho+1)\|B\|\|Q_T\|\,|||\boldsymbol{K}'-\boldsymbol{K}||| + \mathrm{Tr}(Q_T)\frac{\rho^{2T}-1}{\rho^2-1}(2\rho+1)\|B\|\,\|W\|\,|||\boldsymbol{K}'-\boldsymbol{K}|||.
\end{aligned}
$$

Now we bound the term $\sum_{t=1}^{T-1} \left\|(K'_t)^\top R_t K'_t - K_t^\top R_t K_t\right\|$, which appears several times in previous inequalities:

$$
\begin{aligned}
\sum_{t=1}^{T-1} \left\|(K'_t)^\top R_t K'_t - K_t^\top R_t K_t\right\| &= \sum_{t=1}^{T-1} \left\|(K'_t - K_t + K_t)^\top R_t (K'_t - K_t + K_t) - K_t^\top R_t K_t\right\| \\
&\leq \sum_{t=1}^{T-1} \|K'_t - K_t\|^2 \|R_t\| + 2\|K_t\|\|R_t\|\|K'_t - K_t\| \leq 3|||\boldsymbol{K}|||\,|||\boldsymbol{R}|||\,|||\boldsymbol{K}'-\boldsymbol{K}|||.
\end{aligned}
$$

The last step holds since $\|K'_t - K_t\| \leq \|K_t\|$ by assumption.

Therefore,

$$\begin{aligned}|C(\boldsymbol{K}')-C(\boldsymbol{K})| \le \mathrm{Tr}(\mathbb{E}[x_0x_0^\top])\Big\{&3|||\boldsymbol{K}||| \|R_0\| |||\boldsymbol{K}'-\boldsymbol{K}||| + \frac{\rho^{2T}-1}{\rho^2-1}(2\rho+1)\|B\|\|Q_T\|\, |||\boldsymbol{K}'-\boldsymbol{K}||| \\ &+ \frac{\rho^{2T}-1}{\rho^2-1}(2\rho+1)\|B\|\, |||\boldsymbol{K}'-\boldsymbol{K}|||\left(|||\boldsymbol{Q}||| + |||\boldsymbol{K}|||^2\, |||\boldsymbol{R}|||\right) \\ &+ \left(\frac{\rho^{2T}-1}{\rho^2-1}(2\rho+1)\|B\|\, |||\boldsymbol{K}'-\boldsymbol{K}||| + \frac{\rho^2(1-\rho^{2(T-1)})}{\rho^2-1}\right) 3|||\boldsymbol{K}|||\, |||\boldsymbol{R}|||\, |||\boldsymbol{K}'-\boldsymbol{K}|||\Big\} \\ &+ 3\mathrm{Tr}(W)\Big(\frac{(T-1)(\rho^{2(T-1)}-1)}{\rho^2-1}+1\Big)|||\boldsymbol{K}|||\, |||\boldsymbol{R}|||\, |||\boldsymbol{K}'-\boldsymbol{K}||| \\ &+ \left(T\frac{(\rho^{2T}-1)}{\rho^2-1}(2\rho+1)\|B\|\,\|W\|\, |||\boldsymbol{K}'-\boldsymbol{K}|||\right)\left(\mathrm{Tr}\left(\sum_{t=1}^{T-1}Q_t\right) + |||\boldsymbol{K}|||^2\,\mathrm{Tr}\left(\sum_{t=1}^{T-1}R_t\right)\right) \\ &+ \mathrm{Tr}(Q_T)\frac{\rho^{2T}-1}{\rho^2-1}(2\rho+1)\|B\|\,\|W\|\, |||\boldsymbol{K}'-\boldsymbol{K}|||.\end{aligned}$$

By (3.27), Lemma 3.8, and Lemma 3.16, ρ is bounded above by polynomials in $\|A\|$, $\|B\|$, $|||\boldsymbol{R}|||$, $\frac{1}{\underline{\sigma}_{\boldsymbol{X}}}$, $\frac{1}{\underline{\sigma}_{\boldsymbol{R}}}$ and $C(\boldsymbol{K})$, or a constant $1+\xi$. Therefore, we rewrite the above inequality by

$$|C(\boldsymbol{K}')-C(\boldsymbol{K})| \le h_{CK}|||\boldsymbol{K}'-\boldsymbol{K}||| + h'_{CK}|||\boldsymbol{K}'-\boldsymbol{K}|||^2, \tag{SM2.15}$$

where $h_{CK}\in\mathcal{H}(\mathrm{C}(\boldsymbol{K}))$ and $h'_{CK}\in\mathcal{H}(\mathrm{C}(\boldsymbol{K}))$ are polynomials in $C(\boldsymbol{K})$ and model parameters. Given assumption (4.5), we have $|||\boldsymbol{K}'-\boldsymbol{K}|||\le 1$ and hence

$$|||\boldsymbol{K}'-\boldsymbol{K}||| \ge |||\boldsymbol{K}'-\boldsymbol{K}|||^2.$$

Define $h_{cost}=h_{CK}+h'_{CK}$, then (SM2.15) gives

$$|C(\boldsymbol{K}')-C(\boldsymbol{K})| \le h_{cost}|||\boldsymbol{K}'-\boldsymbol{K}|||,$$

with $h_{cost}\in\mathcal{H}(\mathrm{C}(\boldsymbol{K}))$. ∎

Proof of Lemma 4.7. Recall $\nabla_t C(\boldsymbol{K}) = 2E_t\Sigma_t$ and $W=\mathbb{E}\left[w_tw_t^\top\right]=\widetilde{W}\mathbb{E}\left[v_tv_t^\top\right]\widetilde{W}^\top$. We have,

$$\|\nabla_t C(\boldsymbol{K}')-\nabla_t C(\boldsymbol{K})\| = \|2E'_t\Sigma'_t - 2E_t\Sigma_t\| \le 2\|E'_t-E_t\|\|\Sigma'_t\| + 2\|E_t\|\|\Sigma'_t-\Sigma_t\|, \tag{SM2.16}$$

For the second term, by Lemma 3.6 and Cauchy-Schwarz inequality,

$$\|E_t\| \le \sum_{t=0}^{T-1}\|E_t\| \le \sum_{t=0}^{T-1}\sqrt{\mathrm{Tr}(E_t^\top E_t)} \le \sqrt{T\cdot\frac{\max_t\|R_t+B^\top P_{t+1}B\|}{\underline{\sigma}_{\boldsymbol{X}}}(C(\boldsymbol{K})-C(\boldsymbol{K}^*))}. \tag{SM2.17}$$

By (SM2.7) and direct calculation, we have

$$\|(\mathcal{G}'_{t+1}-\mathcal{G}_{t+1})(\Sigma_0)\| \le \rho^{2(t+1)}\left(\sum_{s=0}^{t+1}\|\mathcal{F}_{K'_s}-\mathcal{F}_{K_s}\|\,\|\Sigma_0\|\right).$$

By (SM2.8) and (3.20), for $t=1,2,\cdots,T-1$,

$$\begin{aligned}\|\Sigma'_t-\Sigma_t\| &\le \|(\mathcal{G}'_t-\mathcal{G}_t)(\Sigma_0)\| + \left\|\sum_{s=0}^{t-1}D_{t-1,s}WD_{t-1,s}^\top - D'_{t-1,s}W(D'_{t-1,s})^\top\right\| \\ &\le \rho^{2t}(2\rho+1)\|B\|\,\|\Sigma_0\|\, |||\boldsymbol{K}'-\boldsymbol{K}||| + \frac{(\rho^{2T}-1)}{\rho^2-1}(2\rho+1)\|B\|\,\|W\|\, |||\boldsymbol{K}'-\boldsymbol{K}|||.\end{aligned} \tag{SM2.18}$$

Therefore the second term in (SM2.16) is bounded by the product of (SM2.17) and (SM2.18).

Next we bound the first term in (SM2.16). Similar to (SM2.13), $\|\Sigma_t'\| \le \|\sum_{t=0}^{T}\Sigma_t'\| = \|\Sigma_{\boldsymbol{K}'}\| \le \|\Sigma_{\boldsymbol{K}}' - \Sigma_{\boldsymbol{K}}\| + \|\Sigma_{\boldsymbol{K}}\| \le \frac{C(\boldsymbol{K})}{\underline{\sigma}_{\boldsymbol{Q}}} + \|\Sigma_{\boldsymbol{K}}\|$. For $\|E_t' - E_t\|$, we first need a bound on $\|P_t' - P_t\|$. Since $P_0 = S_0 + \sum_{t=0}^{T-2}\mathcal{G}_t(S_{t+1}) + \mathcal{G}_{T-1}(Q_T)$, by (SM2.14), we have

(SM2.19)

$$\begin{aligned}
\|P_t' - P_t\| \le \|P_0' - P_0\| \le 3\|K_0\|\|R_0\|\|K_0' - K_0\| + \|\mathcal{G}_d\| + \frac{\rho^{2T}-1}{\rho^2-1}(2\rho+1)\|B\|\|Q_T\|\left(\sum_{t=0}^{T-1}\|K_t - K_t'\|\right) \\
\le \frac{\rho^{2T}-1}{\rho^2-1}(2\rho+1)\|B\|\,|||\boldsymbol{K}'-\boldsymbol{K}|||\left(|||\boldsymbol{Q}||| + |||\boldsymbol{K}|||^2|||\boldsymbol{R}|||\right) \\
+ 3\left(1 + \frac{\rho^{2T}-1}{\rho^2-1}(2\rho+1)\|B\|\,|||\boldsymbol{K}'-\boldsymbol{K}||| + \frac{\rho^2(1-\rho^{2(T-1)})}{\rho^2-1}\right)\cdot |||\boldsymbol{K}|||\,|||\boldsymbol{R}|||\,|||\boldsymbol{K}'-\boldsymbol{K}||| \\
+ \frac{\rho^{2T}-1}{\rho^2-1}(2\rho+1)\|B\|\|Q_T\||||\boldsymbol{K}'-\boldsymbol{K}|||.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|E_t' - E_t\| &= \left\|R_t(K_t'-K_t) - B^\top(P_{t+1}'-P_{t+1})A + B^\top(P_{t+1}'-P_{t+1})BK_t' + B^\top P_{t+1}B(K_t'-K_t)\right\| \\
&\le \left(\|R_t\| + \|B\|^2\|P_0\|\right)|||\boldsymbol{K}'-\boldsymbol{K}||| + \|B\|\,\|P_0'-P_0\|\,\|A\| + 2\|B\|^2\|P_0'-P_0\||||\boldsymbol{K}|||.
\end{aligned}$$

Given the bound on $|||\boldsymbol{K}||| = \sum_{t=0}^{T-1}\|K_t\|$ in Lemma 3.16 and the bound on $\|P_t\|$ in Lemma 3.8, all the terms in (SM2.16) can be bounded by polynomials of related parameters multiplied by $|||\boldsymbol{K}'-\boldsymbol{K}|||$ and $|||\boldsymbol{K}'-\boldsymbol{K}|||^2$. Similarly to the proof of Lemma 4.6, we have $|||\boldsymbol{K}'-\boldsymbol{K}||| \le 1$ and

$$\|\nabla_t C(\boldsymbol{K}') - \nabla_t C(\boldsymbol{K})\| \le h_{grad}|||\boldsymbol{K}'-\boldsymbol{K}|||, \qquad \square$$

for some polynomial $h_{grad} \in \mathcal{H}(\mathrm{C}(\boldsymbol{K}))$.

SM2.5. Proofs in Section 5.

Proof of Proposition 5.2. Denote $H_t := \begin{pmatrix} 1+\gamma k_t^1 & \gamma k_t^2 \\ k_t^1 & 1+k_t^2 \end{pmatrix}$. Since H_t has two eigenvalues 1 and $\gamma k_t^1 + k_t^2 + 1$, H_t is positive definite when $\gamma k_t^1 + k_t^2 > -1$ $(0 \le t \le T-1)$.

Then let us show the first claim by induction. Assume $\mathbb{E}[x_s x_s^\top]$ is positive definite for all $s \le t$, then

$$\begin{aligned}
\mathbb{E}[x_{t+1}x_{t+1}^\top] &= \mathbb{E}[((A-BK_t)x_t + w_t)\left((A-BK_t)x_t + w_t\right)^\top] = \mathbb{E}[(H_t x_t + w_t)\left(H_t x_t + w_t\right)^\top] \\
&= \mathbb{E}[H_t x_t x_t^\top H_t^\top + w_t w_t^\top + w_t w_t^\top + 2H_t x_t w_t^\top] = H_t\mathbb{E}[x_t x_t^\top]H_t^\top + \begin{pmatrix}\sigma & 0\\ 0 & 0\end{pmatrix}.
\end{aligned}$$

Hence $\mathbb{E}[x_{t+1}x_{t+1}^\top]$ is positive definite since $\mathbb{E}[x_t x_t^\top]$ is positive definite and H_t is positive definite. Therefore $\underline{\sigma}_{\boldsymbol{X}} > 0$.

The second claim can be proved by backward induction. For $t = T$, $P_T^{\boldsymbol{K}} = Q_T$ is positive definite since Q_T is positive definite. Assume $P_{t+1}^{\boldsymbol{K}}$ is positive definite for some $t+1$, then take any $z \in \mathbb{R}^d$ such that $z \neq 0$,

$$z^\top P_t^K z = z^\top Q_t\, z + z^\top K_t^\top R_t K_t z + z^\top H_t^\top P_{t+1}^{\boldsymbol{K}} H_t z > 0.$$

Note that H_t is positive definite when $\gamma k_t^1 + k_t^2 > -1$ and $1 + \gamma k_t^1 > 0$. The last inequality holds since Q_t and $H_t^\top P_{t+1}^{\boldsymbol{K}} H_t$ are positive definite, and $K_t^\top R_t K_t$ is positive semi-definite. Hence we have $P_t^{\boldsymbol{K}}$ positive definite for all $t = 0, 1, 2, \cdots, T$. $\square$

References.

[1] R. Cont, A. Kukanov, and S. Stoikov, *The price impact of order book events*, Journal of Financial Econometrics, 12 (2014), pp. 47–88.