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LQR Control for Stabilizing Triple Link Inverted Pendulum System

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***Abstract*--This paper presents the model of car triple inverted pendulum system using Lagrange equation in detail. This model is then linearized to design an LQR controller to maintain the triple inverted pendulum on a cart around its unstable equilibrium position using single control input. The controllability of the triple inverted pendulum system is investigated and the choice of weights in LQR is also discussed. The system is simulated in MATLAB environment and the simulation results establish the satisfactory performance of the LQR controller in stabilizing the system.**

***Index Terms*--Linear Quadratic Regulator, state feedback, Triple Inverted Pendulum System.**

I. Introduction

IN control theory, the inverted pendulum system is an important benchmark to verify new theories and methods for solving the control problem of general complex system because of the simplicity of the structure [1]. Triple link inverted pendulum is a highly non-linear, multi-variable, high-order, unstable system that resembles many features found in, for instance, walking robots, flexible space structures, an automatic aircraft landing system, and many other industrial applications.

A vast range of contributions exist for the stabilization of Triple Inverted Pendulum System (TIPS), [2], [3]. A robust servo controller is designed for a triple inverted pendulum with two motors providing two control inputs in [2]. This system does not have a movable cart. Instead the bottom link is hinged to the ground and horizontal bars are fixed to increase their inertia and facilitate the control. The system described in [3] is more complicated, it stabilizes a triple inverted pendulum on a cart by using two DC motors, one motor is moving the cart and the other is attached to the third arm of the pendulum system.

In the engineering applications, simple controller structures e.g. linear controller, are often required [4]. So it is significant to research the control strategy with simple structures and good performances. In this paper, a continuous Linear Quadratic Regulator (LQR) is used for controlling the linearized model of the triple inverted pendulum in simulation. The aim of the work is to improve the overall performance of the cart TIPS in the steady state. Some abstract concepts such as stability and controllability are also discussed.

The rest of this paper is organized as follows. Section 2 represents the mathematical model and equations of motion. Section 3 is devoted to the design of LQR controller. Simulation experiments and results are presented in section 4 followed by conclusion in section 5.

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II. The Mathematical Model

The mathematical model of the triple inverted pendulum is established by the Lagrange method [5], [6] and its schematic representation is shown in Fig. 1.

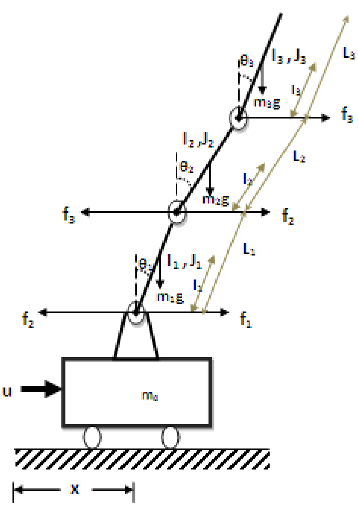

Fig.1 Configuration of cart triple inverted pendulum.

The cart TIPS considered here consists of a straight line rail, a cart moving on the rail, a lower pendulum, which is hinged on the car, middle and an upper pendulum. In the same vertical plane with the rail, the lower pendulum can rotate around the pivot, the middle and the upper pendulum can rotate around the linkage in the same vertical plane. This passage is on the stipulation that the turn angle of clockwise and force moment are positive, u is external action; x is displacement of cart; θ_1, θ_2, θ_3 are the angles of the lower, middle, and upper pendulum bars respectively with respect to the vertical line; m_0 is the mass of the cart; m_1, m_2, m_3 are the masses of the lower, middle, upper pendulum bar respectively; f_0 is the friction factor of cart and track; J_1, J_2, J_3 are the rotary inertia of lower, middle and upper pendulum bar respectively, K_s is the overall system's input conversion gain.

The differential equations set of cart triple inverted

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pendulum are:

$$H_1(\vec{z})\ddot{\vec{z}} = H_2(\vec{z},\dot{\vec{z}})\dot{\vec{z}} + h_3\vec{z} + h_0 u \quad (1)$$

$$\vec{z} = (x, \theta_1, \theta_2, \theta_3)^T \quad (2)$$

$$h_0 = (1\,0\,0\,0)^T \quad (3)$$

where:

$$H_1(\vec{z}) = \begin{bmatrix} a_0 & a_1\cos\theta_1 & a_2\cos\theta_2 & m_3 l_3\cos\theta_3 \\ a_1\cos\theta_1 & b_1 & a_2 L_1\cos(\theta_2-\theta_1) & m_3 L_1 l_3\cos(\theta_3-\theta_1) \\ a_2\cos\theta_2 & a_2 L_1\cos(\theta_2-\theta_1) & b_2 & m_3 L_2 l_3\cos(\theta_3-\theta_2) \\ m_3 l_3\cos\theta_3 & m_3 L_1 l_3\cos(\theta_3-\theta_1) & m_3 L_2 l_3\cos(\theta_3-\theta_2) & J_3 + m_3 l_3^2 \end{bmatrix} \quad (4)$$

where the coefficients are given by:

$$\left.\begin{aligned} a_0 &= m_0 + m_1 + m_2 + m_3 \\ a_1 &= m_1 l_1 + m_2 L_1 + m_3 L_1 \\ a_2 &= m_2 l_2 + m_3 L_2 \\ b_1 &= J_1 + m_1 l_1^2 + m_2 L_1^2 + m_3 L_1^2 \\ b_2 &= J_2 + m_2 l_2^2 + m_3 L_2^2 \end{aligned}\right\} \quad (5)$$

$$H_2(\vec{z},\dot{\vec{z}}) = \begin{bmatrix} -f_0 & a_1\sin\theta_1.\dot{\theta}_1 & a_2\sin\theta_2.\dot{\theta}_2 & m_3 l_3\sin\theta_3.\dot{\theta}_3 \\ 0 & -f_1-f_2 & a_2 L_1\sin\theta_2.\dot{\theta}_2 & m_3 L_1 l_3\sin(\theta_3-\theta_1).\dot{\theta}_3 \\ 0 & -a_2 L_1\sin(\theta_2-\theta_1).\dot{\theta}_1 + f_2 & -f_2-f_3 & m_3 L_2 l_3\sin(\theta_3-\theta_2).\dot{\theta}_3 + f_3 \\ 0 & -m_3 L_1 l_3\sin(\theta_3-\theta_1).\dot{\theta}_1 & -m_3 L_2 l_3\sin(\theta_3-\theta_2).\dot{\theta}_2 + f_3 & -f_3 \end{bmatrix} \quad (6)$$

$$h_3(\vec{z}) = \begin{bmatrix} 0 & a_1 g\sin\theta_1 & a_2 g\sin\theta_2 & m_3 g l_3\sin\theta_3 \end{bmatrix}^T \quad (7)$$

The non-linear model, described in (1), is linearized about the upright position with zero input.

The state vector is defined by:

$$X = \begin{bmatrix} x & \theta_1 & \theta_2 & \theta_3 & \dot{x} & \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 \end{bmatrix}^T$$

The linearized model is best represented in state-space form as follows:

$$\dot{X} = \begin{bmatrix} 0 & I_{4X4} \\ E^{-1}H & E^{-1}G \end{bmatrix} X + \begin{bmatrix} 0 \\ E^{-1}h_0 \end{bmatrix} U = AX + BU \quad (8)$$

$$Y = CX$$

where:

$$E = \begin{bmatrix} a_0 & a_1 & a_2 & m_3 l_3 \\ a_1 & b_1 & a_2 L_1 & m_3 L_1 l_3 \\ a_2 & a_2 L_1 & b_2 & m_3 L_2 l_3 \\ m_3 l_3 & m_3 L_1 l_3 & m_3 L_2 l_3 & J_3 + m_3 l_3^2 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_1 g & 0 & 0 \\ 0 & 0 & a_2 g & 0 \\ 0 & 0 & 0 & m_3 l_3 g \end{bmatrix},$$

$$G = \begin{bmatrix} -f_0 & 0 & 0 & 0 \\ 0 & -f_1-f_2 & f_2 & 0 \\ 0 & f_2 & -f_2-f_3 & f_3 \\ 0 & 0 & f_3 & -f_3 \end{bmatrix} \text{ and}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The co-efficient matrix of state equation (8) of TIPS is as follows:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -12.4928 & -2.0824 & 2.2956 & -5.1127 & 0.0075 & 0.0024 & -0.0053 \\ 0 & 67.1071 & 65.2564 & -71.9704 & 14.0176 & 0.0039 & -0.1948 & 0.1659 \\ 0 & 144.5482 & -394.2536 & 272.1049 & 5.2021 & -0.4334 & 1.1287 & -0.7492 \\ 0 & -300.4564 & 512.8310 & -258.9198 & -10.8077 & 0.6476 & -1.3621 & 0.826 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 3.651 & -10.012 & -3.716 & 7.720 \end{bmatrix}^T$$

III. Linear Quadratic Regulator: A Brief Overview

Extensive research in the control field has shown on multiple occasions that an LQR is well suited for inverted pendulum stabilization [7], [8]. It is a multivariable controller [9] as it can control displacement of the cart and the angles of the triple inverted pendulum at the same time.

For the controlled system described by (8), the quadratic performance index is given by:

$$J = \frac{1}{2}\int_0^{\infty} [x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]\,dt \qquad (9)$$

where, *u(t)* is not constrained, *Q(t)* is required to be symmetric, positive semi definite matrix. *R(t)* is required to be symmetric positive definite matrix. The principle of LQR is as shown Fig.2.

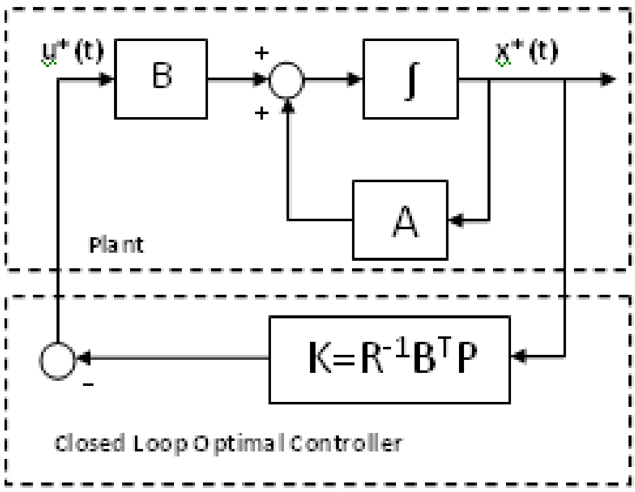

Fig.2 Implementation of Closed Loop Optimal Controller

The optimal control is obtained as:

$$u^*(t) = -Kx^*(t); K = -R^{-1}(t)B^T(t)P(t) \qquad (10)$$

where, K is Kalman gain, P(t) is positive definite matrix is the solution of matrix differential riccati equation (DRE)

$$\begin{aligned}\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) - Q(t) + \\ P(t)B(t)R^{-1}(t)B^T(t)P(t)\end{aligned} \qquad (11)$$

The optimal state is the solution of:

$$\dot{x}^*(t) = \left[A(t) - B(t)R^{-1}(t)B^T(t)P(t)\right]x^*(t) \qquad (12)$$

such that the optimal control u*(t) given by (10) is linear in the optimal state x*(t).

IV. Simulation Experiment And Results

The simulation parameters used for the MATLAB/SIMULINK are given in the appendix.

The eigen values of the system matrix A for the system described by (8) are given in Table I. The system is unstable as it has positive eigen values. The model is simulated in MATLAB/SIMULINK and the step response of the open loop TIPS is as shown in Fig.3.

TABLE I
EIGEN VALUES OF THE SYSTEM

S.No.	Eigen Values
1	0
2	1.0503+27.213i
3	1.0503-27.213i
4	11.0342
5	-12.5674
6	4.2312
7	-1.9243
8	-6.0251

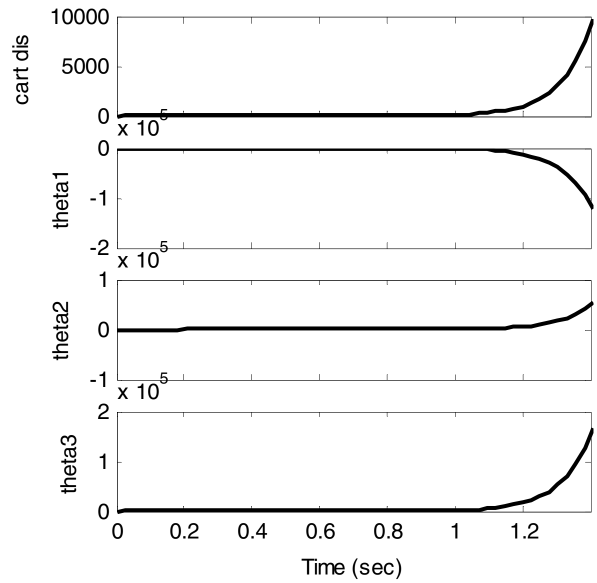

Fig.3 Step Response of Triple Inverted Pendulum System without controller.

The controllability matrix of the system is

$$CO = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B & A^6B & A^7B \end{bmatrix}$$

The rank of the CO matrix using MATLAB command, rank(CO), comes out to be 8 i.e. each of the 8 states is reachable giving the system proper input through u(t). So the system is completely controllable.

The observability matrix of the system is

$$OB = \begin{bmatrix} C & CA & CA^2 & CA^3 & CA^4 & CA^5 & CA^6 & CA^7 \end{bmatrix}^T$$

The rank of the OB matrix using MATLAB command, rank(OB), comes out to be 8 i.e. each of the 8 states is viewable through linear combinations of output variables y(t). So the system is completely observable.

From the above analysis it can be concluded that the linearized model of the system given in (8) is controllable and observable so, the LQR algorithm can be implemented to control the system.

In the design of the controller, one of the key problems is to select weight matrix Q and R in the quadratic performance indexes functional given in (9). During the operation process, the main controlled variables of triple inverted pendulum system are outputs of the system, which in this case are: x, θ_1, θ_2, θ_3.

Therefore Q and R are selected as diagonal matrices [10]:

$$Q = \begin{bmatrix} Q_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, R = 1$$

For the TIPS, several different weighting matrices were tried and tested. The elements of the final Q matrix are larger than the elements of the R matrix. This selection translates into a controller that is more sensitive to the states of the system than the control input. The logic behind this choice is that since the main design criterion is stability, therefore the system states should dictate stability. The elements of the *Q* and *R* matrices of the LQR selected for the system under consideration are:

$$Q_{11} = 700, Q_{22} = 3000, Q_{33} = 3000, Q_{44} = 3000, R = 1$$

It is clear from the above values that the angular control weights are comparatively larger than displacement control weight which clearly establishes the dominance of the angular control weight as compared to the displacement control weight. The optimal control problem is solved and the state feedback control parameters K and P are obtained by using LQR function in MATLAB.

$$\left[K, P, E\right] = lqr(A, B, Q, R)$$

where E is the open loop eigen value.

The optimal feedback gain matrix is

K = [26.4575 -89.2095 88.9158 187.7696 31.0493 10.9616 29.0913 21.2290]

With a full-state feedback controller all the states are fedback. The steady-state value of the states are first computed and then multiplied by a chosen gain K to provide a new value as the reference for computing the input. The step response of the inverted pendulum system on using these new values of states is as shown in Fig.4. It can be seen from the figure that the settling time for θ_1, θ_2, θ_3 is around 3 seconds.

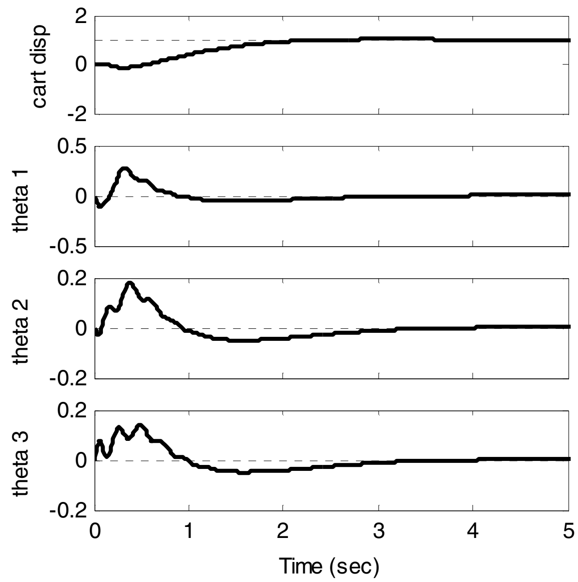

Fig.4 Step Response of Triple Inverted Pendulum System with controller.

V. Conclusion

This paper proposed an LQR controller to stabilize the TIPS about its vertical position. The performance of the proposed LQR controller is found to be good and the settling time is also small. Simulation results clearly establish the effectiveness of the proposed controller as the system performance and stability are satisfactory.

VI. Appendix

m_0 = 2.4 Kg

m_1=1.323 Kg	m_2=1.389 Kg	m_3 = 0.8655 Kg
L_1=0.402 m	L_2=0.332 m	L_3=0.72 m
l_1=0.2449 m	l_2=0.193 m	l_3=0.3405 m
J_1=.0119 Kgm^2	J_2=.0069e^{-3} Kgm^2	J_3=.0291 Kgm^2
f_0=13.611 Nsm^{-1}	f_1=.0045 Nsm	f_2=0.0045 Nsm
f_3=0.0045 Nsm	K_s = 9.722 NV	g=9.81ms^{-2}

VII. References

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