IEEE International Conference on Recent Advances and Innovations in Engineering (ICRAIE-2014),May 09-11, 2014, Jaipur India

Stabilization of Triple Link Inverted Pendulum system based on LQR control Technique

Mukul K. Gupta
Assistant Professor Electrical Engineering University of Petroleum and Energy Studies, Dehradun, INDIA
mukulvjti@gmail.com,

Kamal Bansal
Professor Electrical Engineering, University of Petroleum and Energy Studies, Dehradun, INDIA
kamalbansal@ddn.upes.ac.in,

Arun Kumar Singh
Assistant Professor, Mechanical Engineering, Visvasvaraya National Institute of Technology Nagpur, INDIA,
aksinghb@gmail.com

***Abstract*- This paper first discusses the mathematical modeling of a dynamical system consists of triple link inverted pendulum (TLIP) on a moving cart using Lagrangian equation. This model is then used to design a controller based on the LQR method to maintain the TLIP vertically on the moving cart. The stability and controllability of the triple link inverted pendulum system are investigated and the choice of weights in LQR is also discussed. The system is simulated in MATLAB environment and the simulation results establish the satisfactory performance of the LQR controller in stabilizing the system.**

Keywords-Triple link inverted Pendulum; Linear Quadratic Regulator; Lagrangian system.

I. INTRODUCTION

Inverted pendulums are inherently unstable and nonlinear systems thus present a challenging control problem [1]. As such, they provide an excellent test bench for the evaluation and comparison of different control strategies. The stabilization of single and double inverted pendulums has been the subject of numerous papers [1-4].

The study of a triple link system is, in contrast, a highly non-linear, multi-variable, higher order, unstable system. This can be used for the development of walking robots, flexible space structures, automatic aircraft landing system, biped locomotive machines since it can be considered a simplified model of the human standing on one leg[4-6]. In this paper, a TLIP mounted on a cart that can move horizontally is controllable and can be stabilized in the upward position with a single control input [3,7-8]. In this paper, a continuous time Linear Quadratic Regulator (LQR) is used for controlling the Triple link inverted pendulum in simulation.
The aim of the work is to improve the overall performance of the TLIP in the steady state. Some concepts such as stability and controllability are also discussed in this paper. The rest of this paper is organized as follows. Section 2 describes the mathematical model and equations of system. Section 3 is related to the design of the LQR controller. Simulation results are presented in section 4 followed by the conclusions in section 5.

II. MATHEMATICAL MODELING

The mathematical model of the TLIP is established by the Lagrange method [3] and its schematic representation is shown in Fig. 1. The pendulum's system consists of three steel links of various lengths mounted on a cart. The cart is driven on a rail track by a permanent magnet dc servo motor through a cable pully assembly. The mathematical model for the pendulum is constructed using the Lagrange method [3-4] under the assumptions similar to those Furuta *et al.* (1980) [1, 5].

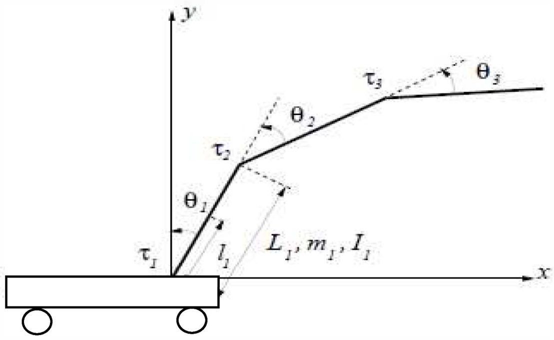

Fig.1. Triple inverted pendulum system on cart

The TLIP in Fig.1 shows that M is Mass of the cart, m_i - Mass of the *i*th link, l_i - distance from the lower position sensor to the center of gravity of the *i*th link, L_i - Total length of the *i*th link, I_i - Mass moment of inertia of the *i*th link about its center of gravity, r - Cart's position from the middle of the rail track, θ_i - Angle of the *i*th link from the vertical position, C_c Dynamic friction coefficient between the cart and the track, C_i - Dynamic friction coefficient for the *i*th link, μ_j - Coulomb friction coefficient for the *j*th link.

The kinetic, potential and friction dissipation energies are given as follows: the kinetic energy is

$$T = 0.5\sum_{i=1}^{3} I_i \dot{\theta}_i^2 + m_i \left\{ \left[\frac{d}{dt}(r + \sum_{k=i-2}^{i-1} L_k \sin(\theta_k) + l_i \sin(\theta_i)) \right]^2 + \left[\frac{d}{dt}(r + \sum_{k=i-2}^{i-1} L_k \cos(\theta_k) + l_i \cos(\theta_i)) \right]^2 \right\} + 0.5 M \dot{r}^2$$

(1)

The potential energy is

$$V = \sum_{i=1}^{3} m_i g (l_i \cos(\theta_i) + \sum_{k=i-2}^{i-1} L_k \cos(\theta_k)) \qquad (2)$$

And the dissipation energy is

$$D = 0.5\sum_{i=1}^{3} C_1 (\dot{\theta}_i^2 - \dot{\theta}_{i-1}^2) + 0.5 C_c \dot{r}^2 \qquad (3)$$

The system's nonlinear dynamics equations can then be given in the following form

$$F(q)\ddot{q} = -G(q,\dot{q})\dot{q} - H(q) + L(q,u) \qquad (4)$$

Where

$$F(q) = \begin{bmatrix} A_1 & A_2\cos(\theta_1) & A_3\cos(\theta_2) & A_4\cos(\theta_3) \\ A_9\cos(\theta_1) & A_{10} & A_{11}\cos(\theta_1-\theta_2) & A_{12}\cos(\theta_1-\theta_3) \\ A_{18}\cos(\theta_2) & A_{19}\cos(\theta_1-\theta_2) & A_{20} & A_{21}\cos(\theta_2-\theta_3) \\ A_{28}\cos(\theta_3) & A_{29}\cos(\theta_1-\theta_3) & A_{30}\cos(\theta_2-\theta_3) & A_{31} \end{bmatrix}$$

$$G(q,\dot{q}) = \begin{bmatrix} A_5 & A_6\sin(\theta_1)\theta_1 & A_7\sin(\theta_2)\dot{\theta}_2 & A_8\sin(\theta_3)\theta_3 \\ 0 & A_{13} & A_{14}\sin(\theta_1-\theta_2)\dot{\theta}_2 + A_{15} & A_{16}\sin(\theta_1-\theta_3)\dot{\theta}_3 \\ 0 & A_{22}\sin(\theta_1-\theta_2)\theta_1 + A_{23} & A_{24} & A_{25}\sin(\theta_2-\theta_3)\dot{\theta}_3 + A_{26} \\ 0 & A_{33}\sin(\theta_1-\theta_3)\theta_1 & A_{35}\sin(\theta_2-\theta_3)\dot{\theta}_2 + A_{36} & A_{32} \end{bmatrix}$$

$$q = \begin{bmatrix} r \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}, \quad H(q) = \begin{bmatrix} 0 \\ A_{17}\sin(\theta_1) \\ A_{27}\sin(\theta_2) \\ A_{34}\sin(\theta_3) \end{bmatrix},$$

$$L(q,u) = \begin{bmatrix} K_s u - sign(r)\mu_r N_r \\ -sign(\theta_1)\mu_{\theta 1} N_{\theta 1} \\ -sign(\theta_1)\mu_{\theta 2} N_{\theta 2} \\ -sign(\theta_1)\mu_{\theta 3} N_{\theta 3} \end{bmatrix}$$

Where u is the control input and K_s is the overall systems input. The system parameters, as given in the appendix Matrix A, are determined either by the direct measurements or from the experimental data.

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -7.6 & -0.156 & -0.0005 & -4.9 & 0.0005 & -0.0005 & 0 \\ 0 & 38.9 & -23.9 & -0.07 & 11.11 & -0.0046 & 0.0087 & -0.0037 \\ 0 & -37.03 & 82.75 & -2.01 & -10.55 & 0.0087 & -0.0234 & 0.0253 \\ 0 & -1.7 & -52.8 & 71.99 & -0.49 & -0.0037 & 0.0253 & -0.4 \end{bmatrix} \quad (5)$$

$$B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.903 \\ -2.02 \\ 1.91 \\ 0.09 \end{bmatrix} \quad (6)$$

The A_i is the systems constants given in the appendix.

III. LINEAR QUADRATIC REGULATOR

The plant is linear time invariant and the state space equation is

$$\dot{X} = AX + Bu \quad (7)$$

It can make the performance index J achieve the minimum value

$$J = \int_0^{\infty} (X^T QX + u^T Ru)dt \quad (8)$$

Where Q is a semi definite matrix, R is a positive definite matrix, Q and R are the weighting matrixes of state variable and input variables respectively [2] [8].For the smallest performance index function. At first Hamilton function is constructed, derivation of this was obtained and makes it equal to zero, which can determined the optimal control rate:

$$u(t) = -Kx(t) = R^{-1}B^T Px(t) \quad (9)$$

Where P is the only positive definite symmetric solution which meets the Riccati equation

$$\mathrm{PA} + \mathrm{A}^T\mathrm{P} - \mathrm{PBR}^{-1} B^T P + Q = 0 \quad (10)$$

In the design of the controller, one of the key problems is to select weight matrix Q and R in the quadratic performance indexes. For the Triple link inverted pendulum system, several different weighting matrices were tried and tested. The elements of the final Q matrix are larger than the elements of the R matrix. This selection translates into a controller that is more sensitive to the states of the system than the control input. The logic behind this choice is that since the main design criterion is stability, therefore the system states should dictate stability. The elements of the Q and R matrices of the LQR selected for the system under consideration are:

Q=diag([400 6000 700 0 0 140 0 0.1]);
R=1

Therefore the optimal feedback gain matrix is

$K=$

1.0e+003 *

[-0.0200 -0.3296 0.8522 -3.5432 -0.0458 -0.1442 -0.0777 -0.4105]

The Eigen value of system Matrix A and closed loop system matrix Ac is given below in the following table 1.

TABLE I. EIGEN VALUES OF MATRIX A AND AC

Sino	Eigen value A	Eigen value Ac
1	0	-25.7123
2	9.8691	-9.6201
3	8.1493	-8.8233
4	4.3265	-8.0303
5	-10.4277	-6.1828 + 1.0041i
6	-8.6358	-6.1828 - 1.0041i
7	-6.3749	-1.2164 + 0.9812i
8	-2.2345	-1.2164 - 0.9812i

From the Eigen value in Table 1, it is clear that for the system matrix A, it is unstable and after adding gain K and Matrix B, the new Eigen value of Ac are stable in nature.

IV. SIMULATION RESULTS

Based on the simulations, the plot in Figs.2a & 2b shows position and velocity of the cart respectively. It may be seen in Figs.2a & 2b that the cart stabilizes after going through initial transition. Similarly, Figs.3a&3b, Figs.4a &4b, and Figs. 5a&5b present the plots related with angular displacement and angular velocity of the bottom link, middle link and bottom link of the

TLIP. These plots also confirm that like cart, the links also stabilize with time. At the same time, it is obvious that settling time is very small. It is less than 4 seconds in almost all the cases which is a very good sign the efficacy of the LQR controller. In all simulations the systems was stable, conforming the LQR properties of the control system.

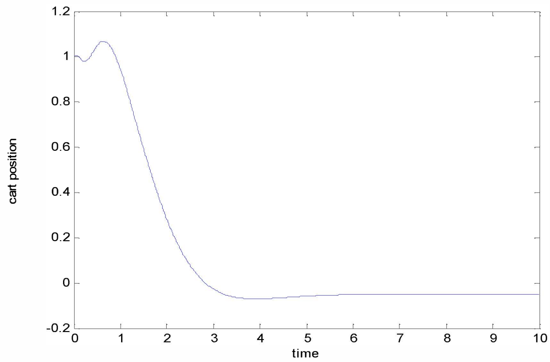

Fig. 2a. Cart Position

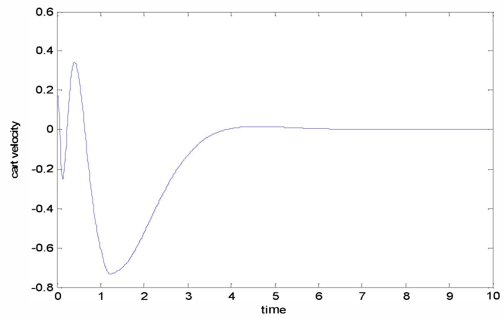

Fig. 2b. Cart Velociy.

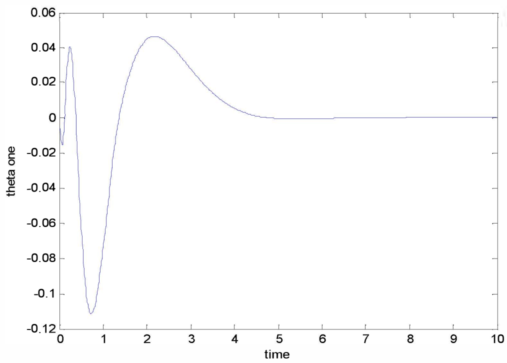

Fig. 3a. First link: Angular displacement

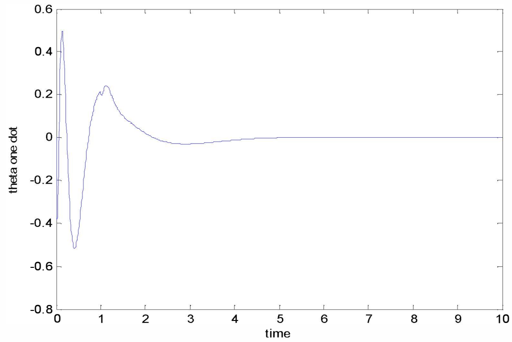

Fig. 3b. First link : Angular Velocity

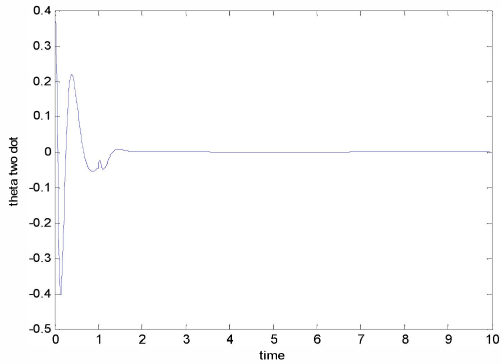

Fig. 4a. (Second link: Angular displacement)

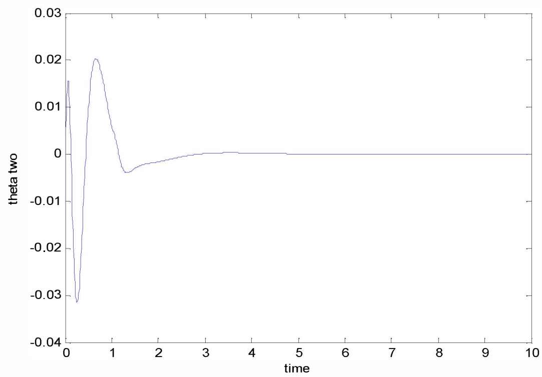

Fig. 4b. Second link: Angular velocity.

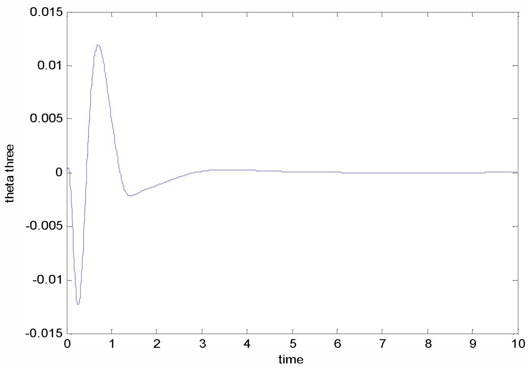

Fig. 5a. Third link: Angular displacement.

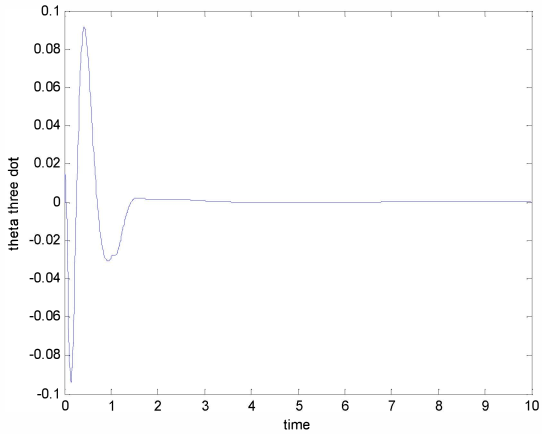

Fig. 5b. Third Link: Angular velocity.

V. CONCLUSION

This paper presents the use of the LQR controller to stabilize a TLIP on a moving cart. An optimal control system was designed to balance the same designed successfully. Simulation results clearly show the effectiveness of the proposed controller. Many other research problem are left for future work such as one can used intelligent control technique for the TLIP and other higher degree of freedom system.

Appendix

Constants Value	Constants Value
$A_1 = M + m_1 + m_2 + m_3$	$A_{19} = (m_2 l_2 + m_3 L_2) L_1$
$A_2 = m_1 l_1 + (m_2 + m_3) L_1$	$A_{20} = I_2 + m_3 L_2^2 + m_2 l_2^2$
$A_3 = m_2 l_2 + m_3 L_2$	$A_{21} = m_3 l_3 L_2$
$A_4 = m_3 l_3$	$A_{22} = -(m_2 l_2 + m_3 L_2) L_1$
$A_5 = C_c$	$A_{23} = -C_2$
$A_6 = -m_1 l_1 - (m_2 + m_3) L_1$	$A_{24} = C_2 + C_3$
$A_7 = -(m_2 l_2 + m_3 L_2)$	$A_{25} = m_3 l_3 L_2$
$A_8 = -m_3 l_3$	$A_{26} = -C_3$
$A_9 = [m_1 l_1 + (m_2 + m_3)] L_1$	$A_{27} = -g(m_2 l_2 + m_3 L_2)$
$A_{10} = I_1 + m_2 l_1^2 + (m_2 + m_3) L_1^2$	$A_{28} = m_3 l_3$
$A_{11} = (m_2 l_2 + m_3 L_2) L_1$	$A_{29} = m_3 l_3 L_1$
$A_{12} = m_3 l_3 L_1$	$A_{30} = m_3 l_3 L_2$
$A_{13} = C_1 + C_2$	$A_{31} = I_3 + m_3 l_3^2$
$A_{14} = (m_2 l_2 + m_3 L_2) L_1$	$A_{32} = C_3$
$A_{15} = -C_2$	$A_{33} = -m_3 l_3 L_1$
$A_{16} = m_3 l_3 L_1$	$A_{34} = -g m_3 l_3$
$A_{17} = -g(m_1 l_1 + m_2 L_1 + m_3 L_1)$	$A_{35} = -m_3 l_3 L_2$
$A_{18} = m_2 l_2 + m_3 L_2$	$A_{36} = -C_3$

REFERENCES

[1] K.Furtura, T.Ochiai, and N.Ono, "Attitude Control of a triple inverted pendulum," International Journal of Control, vol.39, no.6, pp. 1351- 1365, 1984.

[2] Desineni Subbaram Naidu, "Optimal Control Systems," Idaho State University, 2002, p. 129.

[3] Eltohamy, K. G., and Kuo, C. Y., 1997, "Real-time stabilization of a triple link inverted pendulum using single control inpu,". Proceedings of the Institution of Electrical Engineers, Pt D, **144**, 498- 504.

[4] Elthomy K G and Kuo C.Y, 1999, "Nonlinear optimal control of a triple link inverted pendulum with single control input," INT. J. CONTROL, Taylor and Francis 1998, VOL. 69, NO. 2, 239-256

[5] Lijuan Zhang, Yaqing Tu, "Research of Car Inverted Pendulum Model Based on Lagrange Equation," Proceedings of the 6th World Congress on Intelligent Control and Automation, IEEE Press, June 2006, pp. 820-824.

[6] L. Y. Hu, G. P. Liu, X. P. Liu, et al., "Linear quadratic optimal algorithm for inverted pendulum system simulation and real-time control," Mechanical Design and Manufacturing, vol.47, no.l, pp.89-91, 2010.

[7] I. fatoni and R Lozano," Non linear control for under actuated system," London Springer 2002

[8] Medrano-Cerda, G. A., Eldukhri, E. E.,and Cetin, M., 1995, "Balancing and attitude control of double and triple inverted pendulums," Transactions of the Institute of Measurement and Control, **17**, 143- 154.

[9] G. A. Medrano, "Cerda Robust stabilization of a triple inverted pendulum cart," International Journal of Control, Taylor and Francis, 08 Nov 2010, 849-866

[10] Sponge, M. W., 1995, "the swing up control problem for the acrobat," IEEE Control Systems Magazine, **15**, 49-55.