Small World, Power Law

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Outline

1. Small world graphs
2. Clustering coefficients
3. Power law graphs
4. Barabasi-Albert and other models

Small world

Real graph do not typically look like ER graphs, or CL graphs with uniform degree distribution;

Some observed characteristics include:

- relatively short paths between nodes (small diameter)
- local density behaviour (triangles, communities)
- large number of low degree nodes
- presence of high degree nodes (hubs, authorities)

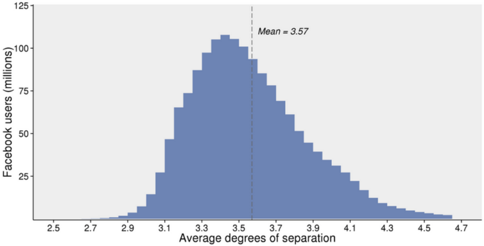
Definition: A graph exhibits *small world* behaviour if the characteristic path length (average distance between nodes) grows proportionally to the logarithm of the number of nodes n.

Small world

Famous ”6 degrees of separation” experiment: chain letters in the USA (Milgram, 1963)

The ”6 degree” refer to the characteristic path length

Social networks today typically have lower values:



REF: Facebook Research (2016)

Clustering

Most networks, in particular social networks, exhibit *homophily*;

Friends share common friends with higher than random probability;

We can measure this by looking at triangles in a graph (graph transitivity) or density of edges between neighbours of each node (clustering coefficient);

We assume undirected graphs $G = (V, E)$.

Clustering

A *triad* is a subgraph of 3 nodes forming a tree; let $n_\wedge$ the number of triads in a graph G;

A *triangle* is a fully-connected subgraph with 3 nodes; let n_Δ be the number of triangles in graph G;

Definition: The *graph transitivity* of G is defined as

$$T(G) = \frac{3n_\Delta}{n_\wedge}.$$

Clustering

For a given node $i \in V$ of degree d_i, let $n_i(e)$ be the number of edges between neighbours of node i;

Node i's (local) clustering coefficient is defined as:

$$c_i = \frac{n_i(e)}{\binom{d_i}{2}}.$$

It is also called the local transitivity, as is corresponds to T for the ego-net of node i.

Clustering

The clustering coefficient of graph G is defined as:

$$CC(G) = \frac{1}{n} \sum_i c_i$$

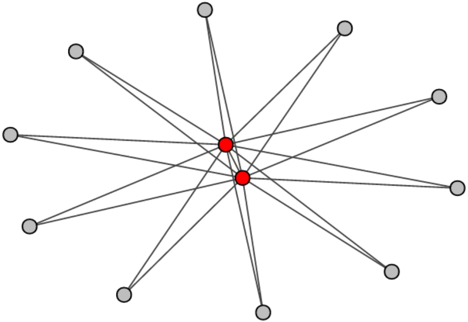
the average over all local clustering coefficients.

It is also called the average *local* transitivity.

Clustering

Quantities $T(G)$ and $CC(G)$ are often similar, but this can be misleading;

Consider the following graph with $n+2$ nodes (n grey nodes):



Clustering

In this graph, $n_\Delta = n$ and $n_\wedge = 3n + n(n-1) = n^2 + 2n$, so

$$T(G) = \frac{3}{n+2} \to 0$$

with the limit as $n \to \infty$.

For the 2 red nodes, we get $c_i = 2/(n+1)$ while for the grey nodes, $c_i = 1$. The clustering coeffient is therefore:

$$CC(G) = \frac{n^2 + n + 4}{n^2 + 3n + 2} \to 1.$$

Clustering

For an ER graph, the probability of ”closing” any given wedge is given by p, the probability of each node pair forming an edge;

Thus, the (expected) clustering coefficient and transitivity is p; examples with $n = 500$:

	p	T	CC
0	0.1	0.099344	0.099291
1	0.2	0.200384	0.200429
2	0.3	0.299115	0.299171
3	0.4	0.398637	0.398633
4	0.5	0.498615	0.498613
5	0.6	0.601092	0.601100

Power law

In a *power law* function, one quantity varies as a power of the other;

This is often observed in real graphs, in particular for the degree distribution;

Let p_k the proportion of nodes with degree k, then

$$p_k = ck^{-\gamma}$$

is an example of a power-law distribution, where c is the normalizing constant.

For degree distribution in social networks, the values typically observed are $2 \leq \gamma \leq 3$.

Power law

A famous power law distribution is known as **Zipf's law**;

In a written language, rank the words in decreasing order of appearance starting from rank $r = 1$

With Zipf's law, the proportion of times word at rank r appears is proportional to $1/r$, in other terms:

$$p_r \propto r^{-1}.$$

Power law

A power law function is *scale-free* or *scale-invariant*

With the function for p_k, we get:

$$p_{ak} = c(a) \cdot p_k$$

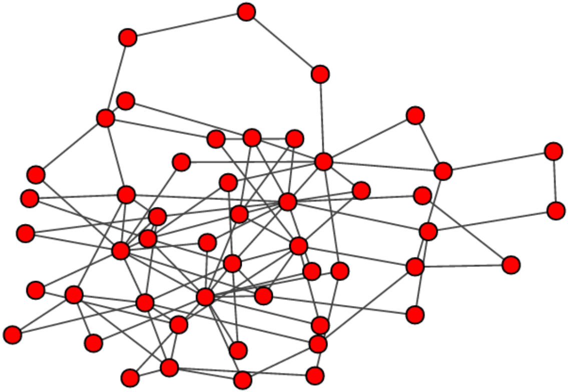
where only the constant changes as a function of the multiplier a.

In practice, we can draw a sequence of degrees from a power law distribution with parameter γ, in some interval $[d_{min}, d_{max}]$.

We then generate a graph via the configuration or CL model.

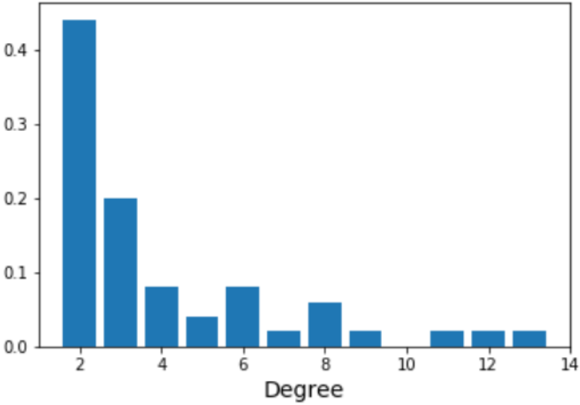
Power law

Here is an example of a graph with power-law degree distribution where $d_i \in [2, 20]$ and $\gamma = 2$, built with the configuration model:



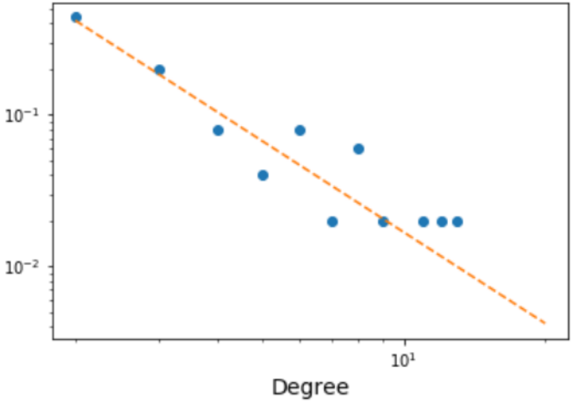
Power law

it's degree distribution:



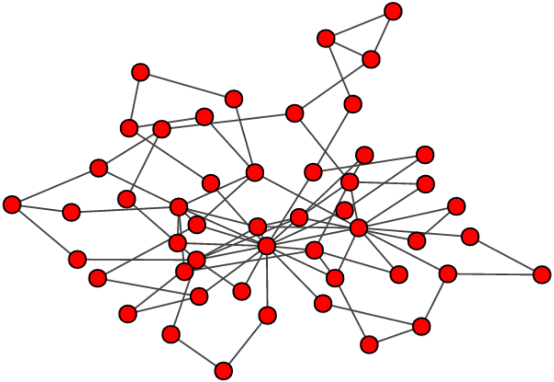
Power law

On a log-log plot, we see a roughly linear relation (with log binning of degrees):



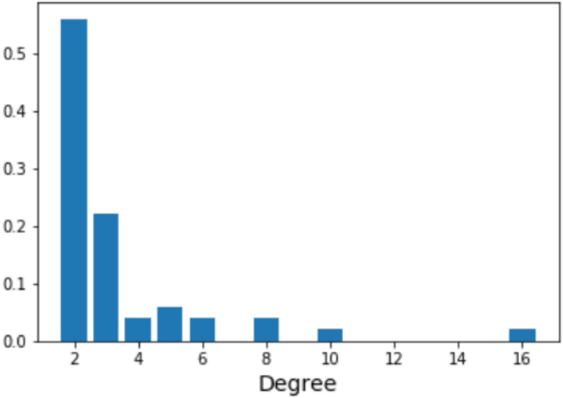
Power law

Here is a graph with $\gamma = 3$:



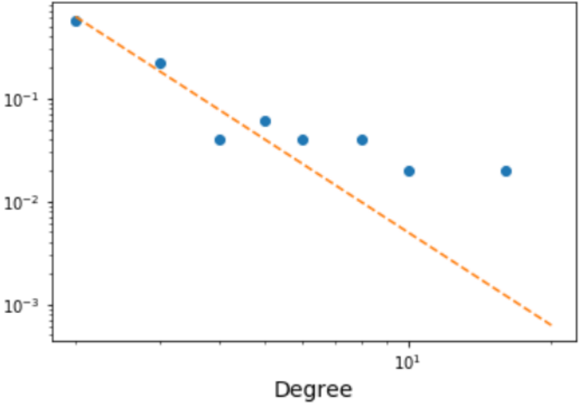
Power law

We see a higher proportion of low degree nodes:



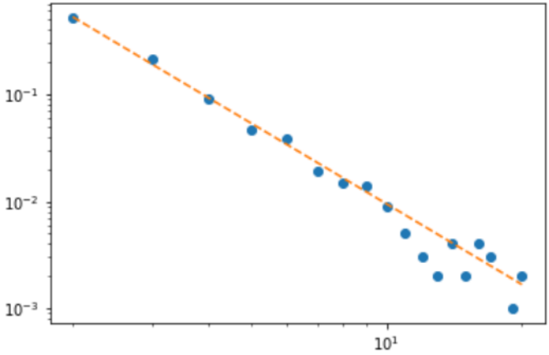
Power law

log-log plot:



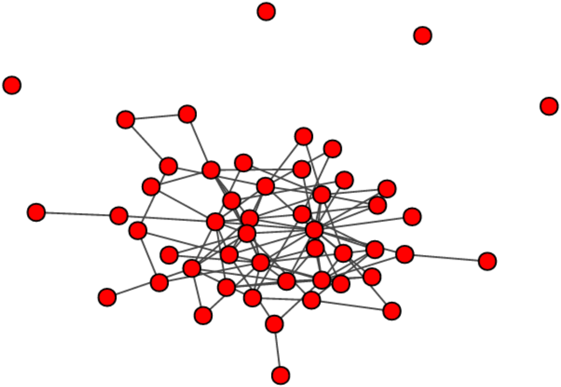
Power law

Here is another log-log plot for a much larger graph with $\gamma = 2.5$:



Power law

We can also use CL, but isolates may appear given the large number of low (expected) degree nodes:



Power law

Therefore, power-law graphs exhibit a degree distribution with:

- a fairly large number of small degree nodes, and
- a ”long tail” which amounts to the presence of high degree nodes.

Such high degree nodes are called hubs (hubs and authorities for directed graphs).

Barabasi-Albert

This is an example of a preferential attachment model, also referred to as *the rich gets richer*;

A graph is generated node by node:

- start from some initial graph (ex: empty graph, small clique) at step $t = 0$
- at step $t > 0$, add a new node with (up to) m edges to existing nodes
- the probability that the new node has an edge with (existing) node i is proportional to $d_i(t)$, the degree on node i before adding this new node.
- continue until n nodes are generated.

Barabasi-Albert

The model favours attaching to high degree nodes, thus forming hubs;

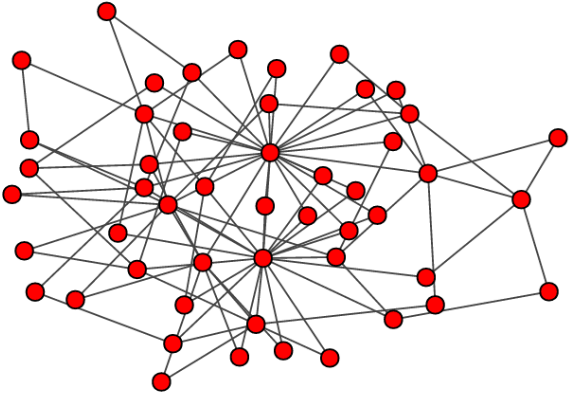
Asympotically, the proportion of degree k nodes is:

$$p_k \approx 2m(m+1)k^{-3}$$

There are several variations on that theme.

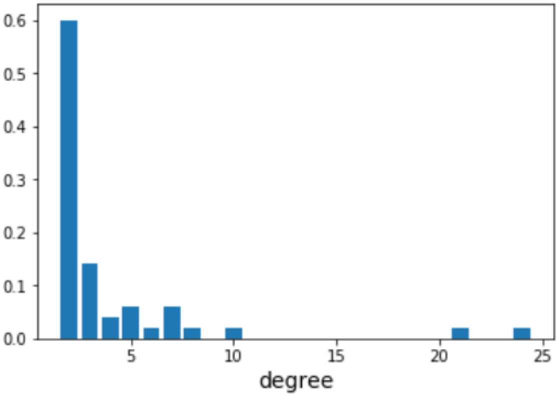
Barabasi-Albert

BA graph with $n = 50$ and $m = 2$:



Barabasi-Albert

The degree distribution:



Barabasi-Albert

Binned log-log plot of the degree distribution:

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fitted gamma: -2.3930958076836606
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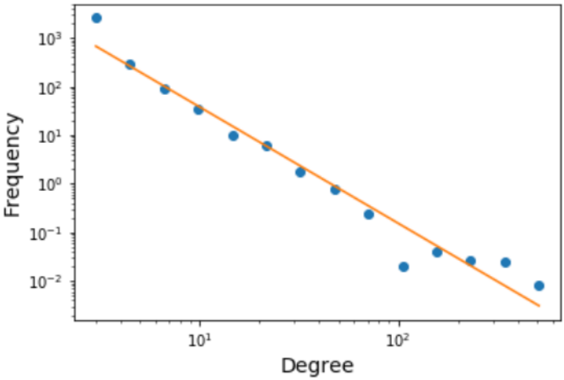
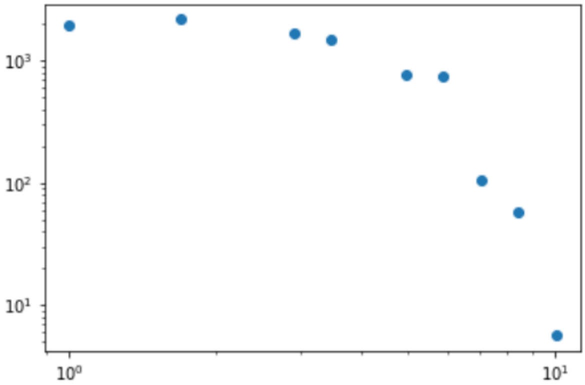


Barabasi-Albert

The same log-log plot for an ER graph with same number of nodes and edges; clearly no power law;



Random Graphs

Notebook #3