Chapter 3

Computer Simulation of Brownian Motion and Ito Processes

3.1 Introduction

In Chapter 2, we have introduced Brownian motion (the Wiener process) as a stationary, continuous stochastic process with independent increments. This process is a unique one to model the irregular noise such as Gaussian white noise in systems, and once such a process is incorporated in differential equations, the process of obtaining solutions involve stochastic calculus. Only a limited number of stochastic differential equations have analytical solutions and some of these equations are given by Kloeden and Platen (1992). In many instances we have to resort to numerical methods. The objective of this chapter is to illustrate the behavior of the Wiener process and Ito processes through computer simulations so that reader can appreciate the variable nature of individual realizations. The routines are written in Mathematica® (1999). Some of these routines may be useful in constructing numerical solutions of stochastic differential equations later in this book .

3.2 A Standard Wiener Process Simulation

For the numerical implementation, it is most convenient to use the variance specification of the Wiener process $B(t)$ in the form of equation (2.21). The time span of the simulation, [0,1] is discretized into small equal time increments *delt* , and the corresponding independent Wiener increments selected randomly from a normal distribution with zero mean and standard deviation equal to $\sqrt{delt}$. This is implemented by the following simple Mathematica program:

```
<< Statistics`ContinuousDistributions`

wienerProcess[delt_, Ni_] := Module[{std, ii = 1, kk = 1},
(*Ni---number of intervals; delt--- interval *)
std = N[Sqrt[delt], 32];
incrementList = Table[N[Random[NormalDistribution[0.0, std]], 32], {Ni}];
incrementList = PrependTo[incrementList, 0.0];
standardWiener = N[Apply[Plus, Table[Table[incrementList[[ii]],
                {ii, 1, kk}], {kk, 1, Ni + 1}], {1}], 32];];
wienerProcess[0.001, 1000] // Timing
ListPlot[standardWiener, PlotJoined -> True, PlotRange -> All] // Timing
```

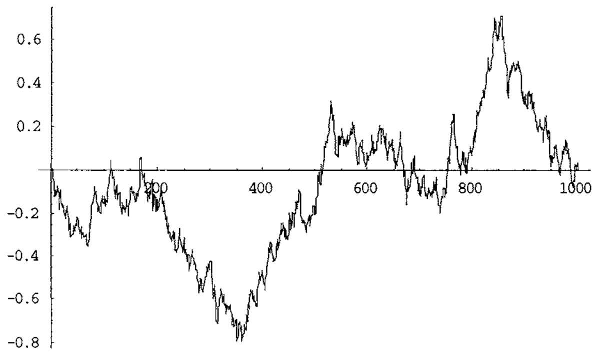

Figure 3.1 Mathematica program for the standard Wiener process and a sample realization.

In the example given above we have used the definition of Ito integral by evaluating the increment during the interval [i - 1, i] at (i -1) and computing the Wiener process by adding the increment to the value of Brownian motion at [i - 1]. In this example the time interval [0, 1] is divided into very small equal divisions, and the graph is shown in terms of the number of time intervals instead of time t. We have generated 1000 Brownian motion increments that are Gaussian random variables, and Figure 3.2 shows these increments as a single stochastic process. Since Gaussian white noise is the derivative of Brownian motion and as the time interval is a constant, Figure 3. 2 depicts a realization of a white noise process.

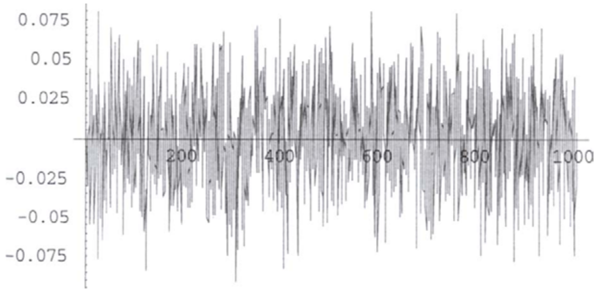

Figure 3.2 A realization of Brownian increments.

In Figure 3.1, the realization shown tend to come back to original position, but Figure 3.3 shows a significant diversion from the origin.

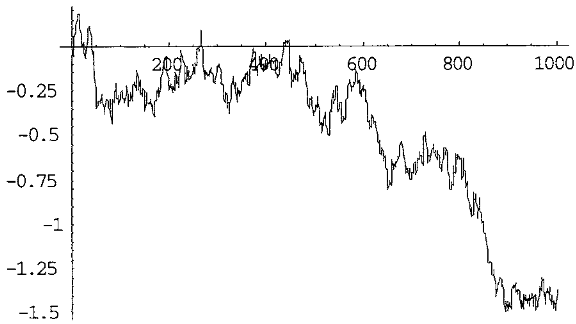

Figure 3.3 Another realization of the standard Brownian motion.

These two realizations have quite different directions of movement, even though expected value of Brownian motion at a given time is zero. To investigate the behavior further we have produced 10 realizations in Figure 3.4.

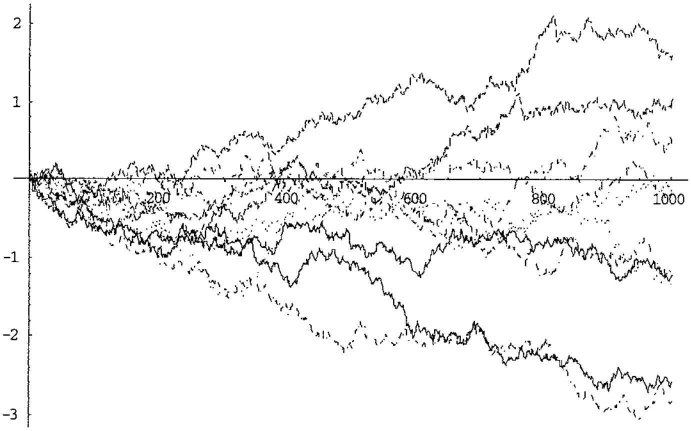

Figure 3.4 Ten different realizations of Brownian motion.

It is seen that the motion is very irregular, and the only discernible pattern is that as time progresses, the position tends to wander away from the starting position at the origin. In other words, if the statistical variance over realizations for a fixed time is evaluated, this increases gradually – a property referred to as time varying variance.

The use of the process in a modeling situation to represent the noise in the system should be carefully thought through. If the noise can be represented as white noise, then Brownian motion enters into the equation because of the relationship between the white noise and Brownian motion as a Gaussian white noise process can be approximated as the derivative of Brownian motion.

3.3 Simulation of Ito Integral and Ito Processes

It is important to realize that the Ito integral is a stochastic process dependent on the Wiener process. This is analogous to integration in standard calculus because an indefinite integral is a function of the independent, deterministic variable. Given the Brownian motion realization depicted in Figure 5, we will compute the Ito integral of Brownian motion:

$$\int_0^t B(t,\omega)dB.$$

As we have previously seen, this integral can be evaluated by using the following stochastic relationship converging in probability:

$$\int_0^t B(s,\omega)dB(s,\omega) = \frac{1}{2}B^2(t,\omega) - \frac{1}{2}t. \tag{3.1}$$

We have computed this Ito integral using the following Mathematica statement taking the time interval as 0.001 which was the value used for the generation of the Brownian realization in Figure 3.5:

```
itoIntegralBdB = Table[0.5 (standardWiener[[i]])^2 + 0.5 0.001 (i - 1), {i, 1, 1000}];
```

The corresponding realization is given in Figure 3.6.

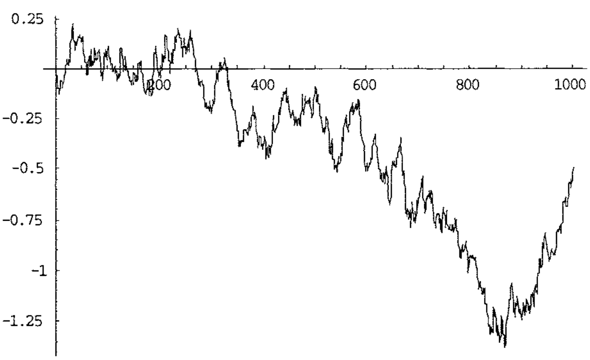

Figure 3.5 The realization of the Wiener process used in the calculation of the Ito Integral depicted in Figure 3.6.

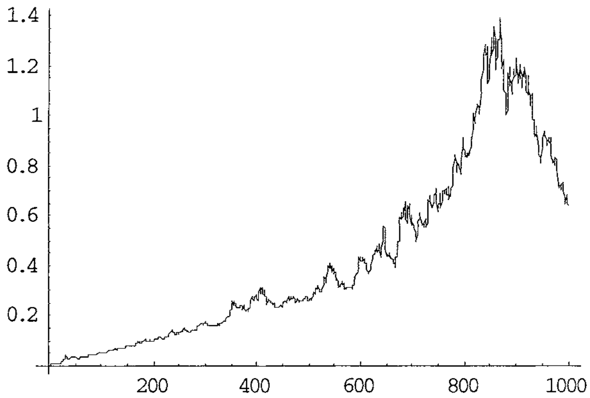

Figure 3.6 A realization of $\int_0^t B(t,\omega)dB$.

As seen in Figures 3.5 and 3.6, the realization of the Ito integral depends strongly on the square of Brownian motion as time increases and this tendency weakens as time advances beyond 1.

Next we will compute another realization of Brownian motion (Figure 3.7) and corresponding Ito integral $\int_0^t B(t,\omega)dB$ (Figure 3.8).

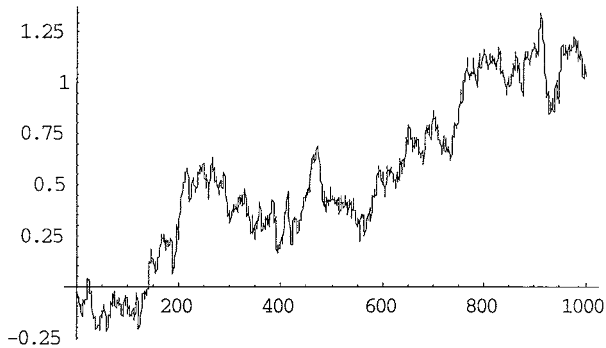

Figure 3.7 Another realization of Brownian motion.

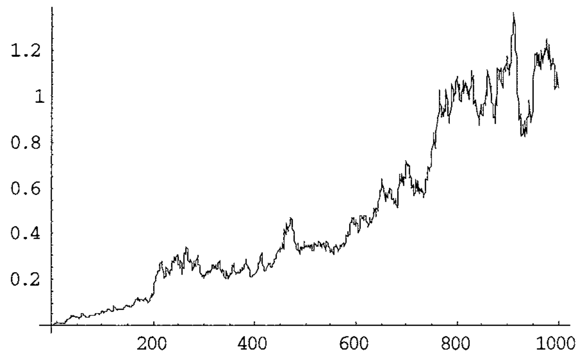

Figure 3.8 Ito integral $\int_0^t B(t,\omega)dB$ corresponding to the Brownian realization in Figure 3.7.

Let us consider the following Ito process which we have derived in Chapter 2. In differential notation,

$$d(B^4(t))=6B^2(t)dt+4B^3(t)dB(t),$$

which means,

$B^4(t)=B^4(0)+\int_0^t 6B^2(t)dt+\int_0^t 4B^3(t)dB(t)$, and

$$B^4(t)=\int_0^t 6B^2(t)dt+\int_0^t 4B^3(t)dB(t) \ . \tag{3.2}$$

This Ito process has a drift term as well as a diffusion term, and the process can be evaluated by using the following Mathematica code given that we have evaluated a new standard Wiener realization:

$$\text{itoProcessB4} = \text{Table}\left[\sum_{j=1}^{i} (6\,(\text{standardWiener[[j]]})^2\,0.001 + 4\,(\text{standardWiener[[j]]})^3\,\text{incrementList[[j]]}),\ \{i,\ 1,\ 999\}\right];$$

The Ito process given in equation (3.2) is simulated in Figure 3.10 for the Wiener realization depicted in Figure 3.9.

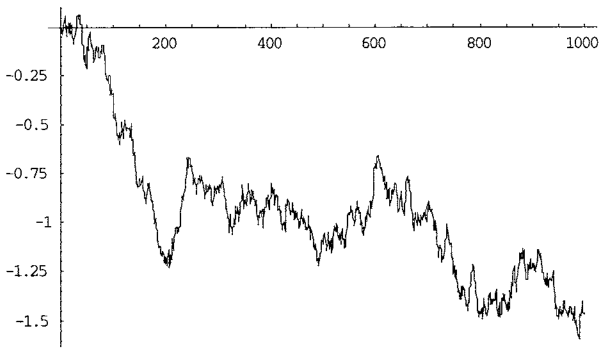

Figure 3.9 Wiener realization used in evaluating the Ito process $B^4(t)$ as seen in Figure 3.10.

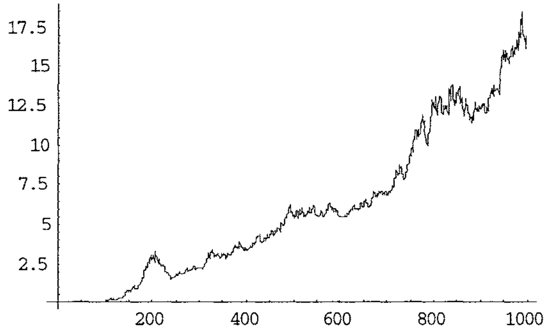

Figure 3.10 Ito process $B^4(t)=\int_0^t 6B^2(t)dt+\int_0^t 4B^3(t)dB(t)$.

Even for a decreasing and erratic Brownian motion, the Ito process $\left\{\int_0^t 6B^2(t)dt+\int_0^t 4B^3(t)dB(t)\right\}$ in general has a smoother realization which has an overall growth in positive direction. The effect of Ito integration tends to smother the erratic behavior of Brownian motion. We have evaluated the above Ito process for 3 different realizations of the standard Wiener process, and they are shown in Figure 3.11.

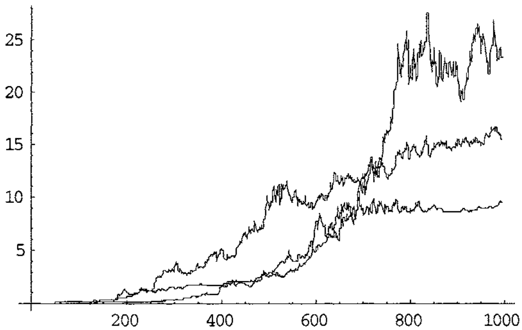

Figure 3.11 Three realizations of $\left\{\int_0^t 6B^2(t)dt+\int_0^t 4B^3(t)dB(t)\right\}$.

As seen in Figure 3.11, individual realizations of the Ito process $\left\{\int_0^t 6B^2(t)dt + \int_0^t 4B^3(t)dB(t)\right\}$ are distinct from each other and therefore shows the complexity in stochastic integration as opposed to integration in the standard calculus, and this illustrates that it is important to simulate the stochastic processes in applications to better understand the variability of observations. Among a large number of realizations, one may observe extreme events which can not be mathematically obtained. This leads us to discuss a specific stochastic model and we will go back to our stochastic population dynamic model.

3.4 Simulation of Stochastic Population Growth

We consider equation (2.5), which is the solution to the population growth model with a variable coefficient. The Mathematica® code for the solution of equation (2.5) is given below for r = *1.5* and σ = *1.0*. Note that for these particular values, the coefficients of t and B_t in the exponents both reduce to a value of 1. Figure 3.13 shows a sample of realizations (or sample paths) of the solution and the horizontal axis gives the number of time intervals.

```
(* initial value of the population=1.0 *)

Do[
        Module[{t=0,i=1,delt= 2^-9}, B[0]=0.0;
                s=delt ^0.5;
        x[0]=1.0;      xd[0]=1.0;
        While[t< 1.0,
                B[i]=B[i-1]+ Random[NormalDistribution[0.0,s]];
                x[i]=x[0] Exp[t + B[i-1]];
                xd[i]=xd[0] Exp[t];
                t=t+delt;   i++
        ];
        Print[i-1];
        ListPlot[Table[x[j],{j,1,i-1}],PlotJoined->True]
        ],
 {n,1,10}]
```

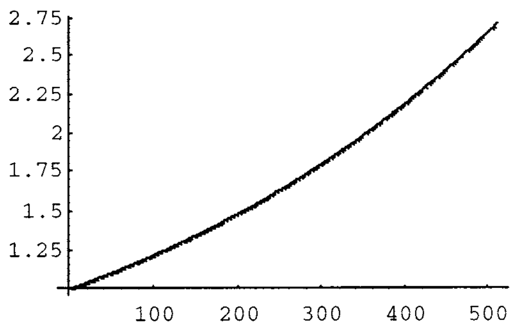

Figure 3.12 Deterministic solution of population growth equation.

Figures 3.12 and 3.13 show the deterministic solution and a sample of different realizations of the stochastic solution. The most striking aspect is how little of the behavior of the deterministic exponential growth curve remains recognizable in the stochastic realizations. In fact, it would be difficult to distinguish by inspection between the realizations of the simple Brownian motion and those of exponential growth in Figure 3.13.

To an observer, any one of these realizations in Figure 3.13 can be seen as an outcome of the process. A limited number of samples will obscure the fact that these realizations result from a mechanistic relationship, but with noisy coefficients with irregular behavior. This also shows that the variability in parameters can significantly change the outcome of the process.

These observations apply, of course, to the cases where the amplitude of the noise term is comparable to the growth rate. Figure 3.14 shows realizations with $\sigma = 0.5$ and $r = 1.5$ and comparing with Figure 3.13, the underlying exponential growth can now be recognized.

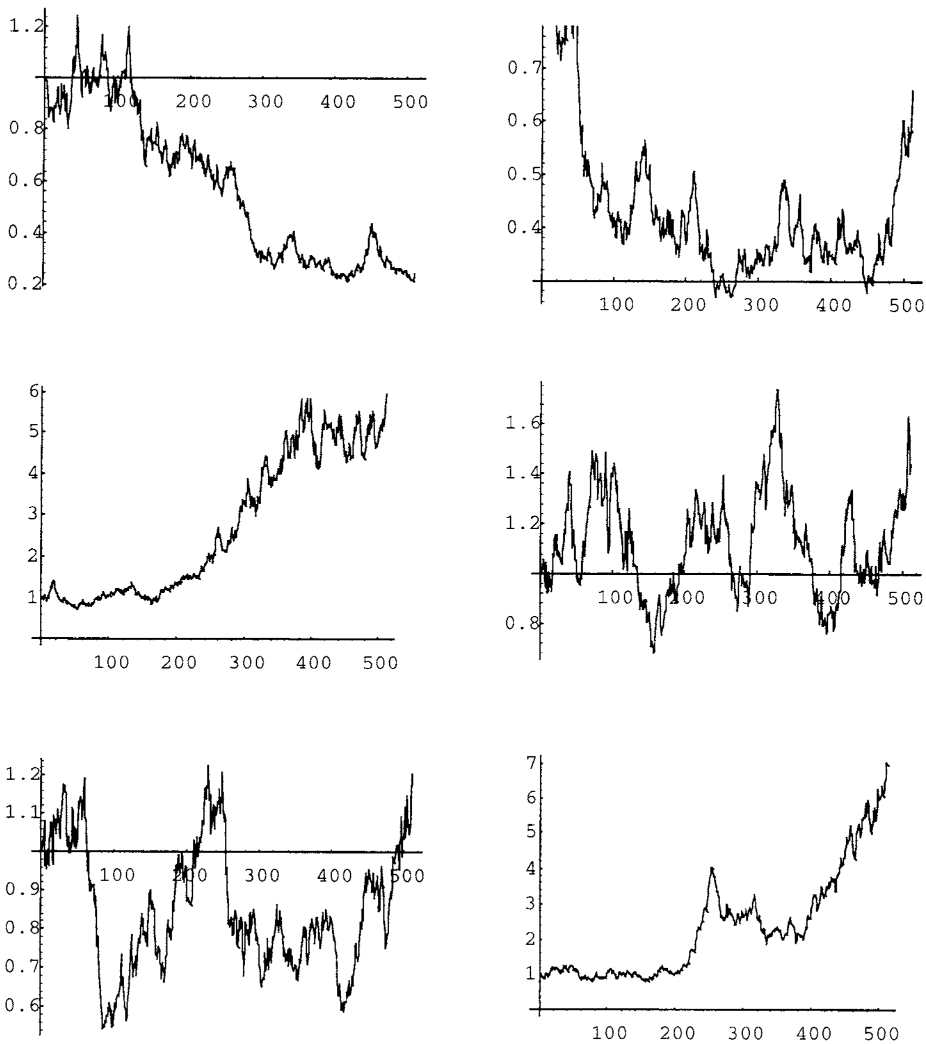

Figure 3.13 A sample of different realizations of equation (2.5).

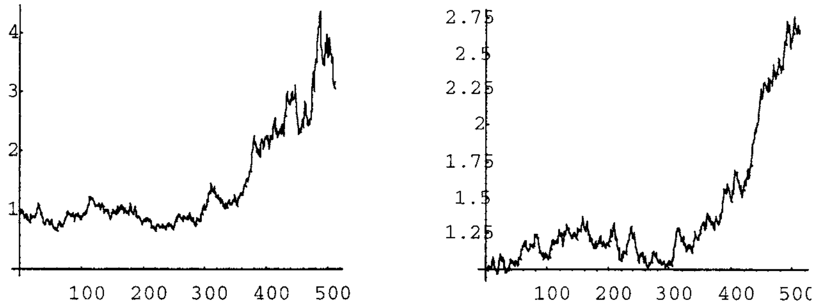

Figure 3.14 Realizations of equation (2.5) with $\sigma = 0.5$ and $r = 1.5$.

If the deterministic solution is not known, the statistical moments such as the mean and the variance could be calculated from a sufficiently large sample of realizations at a given time, *t*, to discuss the behavior of the system. It seems reasonable to expect that such mean values will approximate the behavior of the deterministic solution and allow determination of the growth rate, provided that σ can also be determined. However, the variance depends on both σ and the time at which the sample is taken. Clearly, the extraction of model parameters from a limited set of individual realizations is not straightforward.

Moreover, as will be shown in Chapter 5, the intuitive expectation about the representativeness of a finite set of samples is not fully justified. While the mean over all realizations indeed follows the deterministic behavior, it turns out that the majority of realizations do not reach the deterministic population value at a given time. Hence a finite sample is likely to underestimate the underlying growth rate. To fully explore this phenomenon requires considerable additional theoretical background and that is the subject of Chapter 5.

Leaving these issues aside, we note that Figures 3.13 and 3.14 show the effects of σ on the variance, and this provides us with a way of constructing confidence intervals for the results of experiments. Confidence intervals along with the moments should be used to validate the model given by equation (2.1) with "field" data from experiments. Therefore, when models are constructed using stochastic differential equations with Brownian motion to represent the noise in the system, they can be used to conduct computer experiments to understand and predict the behavior of the systems under study. The results of the experiments should be analyzed and interpreted using appropriate statistical methods. For simple models such as the one given in equation (2.5), solutions can be found and the need for extensive statistical analysis of results is not necessary. But for complex systems, incorporation of noise can result in mathematically unpredictable behavior; therefore, computer experimentation with the system models is the only way of examining the randomness affecting different parameters in the model. The only validation that can be done is to compare results from computer experiments with data from the actual system. (See Brown and Kulasiri (1996) for a discussion of validation of complex, stochastic, biological systems.)