Chapter 4

Solving Stochastic Differential Equations

4.1 Introduction

In Chapter 2, we discussed the elementary concepts in stochastic calculus and showed in a limited number of situations how it differs from the standard calculus. We have defined Ito integrals and introduced Ito processes along with some of the tools that could be useful in working with stochastic calculus. In this chapter, we intend to review stochastic differential equations (SDEs) briefly and the ways of solving them analytically. The main aim of this chapter is to present a very limited number of solution methods which are useful within the context of the scope of this book but more importantly to encourage reader to pursue this subject using more rigorous treatments available.

4.2 General Form of Stochastic Differential Equations

Let us consider an ordinary differential equation which relates the derivative of the dependent variable ($y(t)$) to the independent variable (t) through a function, $\phi(y(t),t)$, with the initial condition $y(0) = y_0$:

$$\frac{dy}{dt} = \phi(y,t), \tag{4.1}$$

and

$$dy = \phi(y,t)dt.$$

In many natural systems, this rate of change can be influenced by random noise caused by a combination of factors, which could be difficult to model. As a model of this random fluctuations, white noise ($\xi(t)$) is a suitable candidate. Therefore we can write, in general, the increments of the noise process as $\sigma(y,t)\,\xi(t)$ where σ is an amplitude function modifying the white noise.

Hence,

$$\frac{dy}{dt} = \phi(y,t) + \sigma(y,t)\,\xi(t). \tag{4.2}$$

As we have see from Chapter 2,

$$\sigma(y,t)\,\xi(t) = \sigma(y,t)\frac{dB}{dt} \tag{4.3}$$

where, $B(t)$ = Brownian motion.

Therefore,

$$\frac{dy}{dt} = \phi(y,t) + \sigma(y,t)\frac{dB}{dt}, \tag{4.4}$$

$$dy = \phi(y,t)d\phi + \sigma(y,t)dB. \tag{4.5}$$

In general, $\phi(y,t)$ and $\sigma(y,t)$, could be stochastic processes. This equation is called a stochastic differential equation (SDE) driven by Brownian motion. Once Brownian motion enters into equation (4.3), y becomes a stochastic process, $Y(t,\omega)$, and in the differential notation SDE is written as

$$dY(t) = \phi(Y(t),t)dt + \sigma(Y(t),t)\,dB(t). \tag{4.6}$$

This actually means,

$$Y(t) = Y(0) + \int_o^t \phi(Y(t),t)\,dt + \int_o^t \sigma(Y(t),t)\,dB(t). \tag{4.7}$$

If we can find a function of Brownian motion in the form of an Ito process that satisfies the above integral equation (4.7), we call that function a strong solution of SDE.

Strong solutions do not depend on individual realizations of Brownian motion. In other words, all possible realizations of an Ito process, which is a strong solution of a SDE, satisfy the SDE under consideration. Not all the SDEs have strong solutions. The other class of solutions are called weak solutions where solution to each individual realization is different from each other. In this chapter we will focus only on strong solutions. In many situations, finding analytical solutions to SDEs is impossible and therefore we will review a minimum number of SDEs and their solutions so that reader can embark on learning this challenging area of applied mathematics.

4.3 A Useful Result

If $X(t)$ is a stochastic process and another stochastic process $Y(t)$ is related to $X(t)$ through the stochastic differential,

$$dY(t) = Y(t)\, dX(t), \tag{4.8}$$

with $Y(0) = 1$.

Thus $Y(t)$ is called the stochastic exponential of $X(t)$. If $X(t)$ is a stochastic process of finite variation, then the solution to equation (4.8) is,

$$Y(t) = e^{X(t)}, \tag{4.9}$$

and, for any process $X(t)$,

$Y(t) = e^{\xi(t)}$ satisfies the stochastic differential given above when

$$\xi(t) = X(t) - X(0) - \frac{1}{2}[X,X](t). \tag{4.10}$$

$[X, X](t)$ is quadratic variation of $X(t)$ and for a continuous function with finite variation $[X,X](t) = 0$.

For example, consider the following stochastic differential equation in differential form,

$$dX(t) = X(t)\, dB(t). \tag{4.11}$$

This SDE does not have a drift term and the diffusion term is an Ito integral. We know, $[B, B](t) = t$.

Therefore from the above result,

$$\begin{aligned} \xi(t) &= B(t) - B(0) - \frac{1}{2}t, \\ &= B(t) - \frac{1}{2}(t). \end{aligned} \tag{4.12}$$

Then the solution to the SDE is

$$X(t) = e^{B(t) - \frac{1}{2}t}. \tag{4.13}$$

Now let us consider a similar SDE with a drift term:

$$dX(t) = \alpha\, X(t)\, dt + \beta\, X(t)\, dB(t), \tag{4.14}$$

where α and β are constants.

Dividing it by $X(t)$,

$$\frac{dX(t)}{X(t)} = \alpha\ dt + \beta\ dB(t). \tag{4.15}$$

This differential represents,

$$\begin{aligned} \int_0^t \frac{dX(t)}{X(t)} &= \int_0^t \alpha\, dt + \int_0^t \beta\, dB(t), \\ &= \alpha\, t + \beta(B(t) - B(0)), \\ &= \alpha\, t + \beta\, B(t). \end{aligned} \tag{4.16}$$

The second term on the right hand side comes from Ito integration.

Now let us assume

$$\phi(t) = \alpha t + \beta B(t).$$

Then the SDE becomes,

$$\int_0^t \frac{dX(t)}{X(t)} = \phi(t). \tag{4.17}$$

$X(t)$ is a stochastic exponential of $\phi(t)$ with corresponding $\xi(t)$:

$$\xi(t) = \phi(t) - \phi(0) - \frac{1}{2}[\phi,\phi](t). \tag{4.18}$$

$$\begin{aligned}[\phi,\phi](t) &= [(\alpha t + \beta B(t)), (\alpha t + \beta B(t))](t), \\ &= [\alpha t, \alpha t](t) + 2\alpha \beta [t, B(t)](t) + \beta^2 [B,B](t), \\ &= 0 + 0 + \beta^2 t.\end{aligned} \tag{4.19}$$

Therefore,

$$\xi(t) = \alpha t + \beta B(t) - 0 - \frac{1}{2}\beta^2 t.$$

The solution to the SDE is

$$X(t) = \exp((\alpha - \frac{1}{2}\beta^2)t + \beta B(t)). \tag{4.20}$$

Let us examine whether the stochastic process

$$X(t) = \exp((\alpha - \frac{1}{2}\beta^2)t + \beta B(t))$$

is a strong solution to the differential equation

$$dX(t) = \alpha X(t)dt + \beta X(t)\, dB(t). \tag{4.21}$$

We will define a function,

$$f(x,t) = \exp((\alpha - \frac{1}{2}\beta^2)t + \beta\, x).$$

Then

$$\begin{aligned} X(t) &= f(B(t),t), \\ &= \exp((\alpha - \frac{1}{2}\beta^2)t + \beta\, B(t)). \end{aligned} \tag{4.22}$$

We need to apply Ito formula for the two Ito processes $X_1(t)$ and $X_2(t)$ (equation (2.114)).

$X_1(t) = B(t)$; $X_2(t) = t$ (a continuous function with finite variation);

$dX_1.dX_2(t) = d[X_1, X_2] = 0$; $(dX_1)^2 = dt$; $(dX_2)^2 = 0$.

Differentiating the function f with respect to x,

$$\frac{\partial f}{\partial x} = \beta \exp((\alpha - \frac{1}{2}\beta^2)t + \beta x),$$

and differentiating again w.r.t. x,

$$\frac{\partial^2 f}{\partial x^2} = \beta^2 \exp((\alpha - \frac{1}{2}\beta^2)t + \beta x).$$

Differentiating f with respect to t,

$$\frac{\partial f}{\partial t} = (\alpha - \frac{1}{2}\beta^2)\exp((\alpha - \frac{1}{2}\beta^2)t + \beta x).$$

From Ito formula (equation 2.114),

$$\begin{aligned} d(f(X_1, X_2)) &= \frac{\partial f}{\partial x} dB(t) + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(0) + \frac{1}{2}\frac{\partial^2 f}{\partial x\, \partial t}(0), \\ &= \beta \exp((\alpha - \frac{1}{2}\beta^2)t + \beta\, B(t)) + (\alpha - \frac{1}{2}\beta^2)\exp(\alpha - \frac{1}{2}\beta^2)dt \\ &\quad + \frac{1}{2}\beta^2 \exp(\alpha - \frac{1}{2}\beta^2)dt. \end{aligned} \tag{4.23}$$

$$
\begin{aligned}
d(X(t)) &= d(f(B(t),t)), \\
&= \alpha \exp((\alpha - \frac{1}{2}\beta^2)t + \beta\, B(t))dt \\
&\quad + \beta \exp((\alpha - \frac{1}{2}\beta^2)t + \beta\, B(t))dB(t), \\
&= \alpha X(t)dt + \beta X(t)dB(t).
\end{aligned} \tag{4.24}
$$

This proves that $X(t) = f(B(t),t)$ is a strong solution of the SDE given by equation (4.22).

We can see that if we can find a function $f(x,t)$, and for a given Brownian motion $B(t)$, $X(t) = f(B(t),t)$ is a solution to the SDE of the form

$$dX(t) = \mu(X(t),t)dt + \sigma(X(t),t)\, dB(t).$$

$X(t)$ should also satisfy,

$$X(t) = X(0) + \int_0^t \mu(X(s),s) + \int_0^t \sigma\, dB(s), \tag{4.25}$$

provided that $\int_o^t \mu\, ds$ and $\int_o^t \sigma dB(s)$ exist.

4.4 Solution to the General Linear SDE

Solution to the general linear SDE of the form,

$$dX(t) = (\alpha(t) + \beta(t)X(t))dt + (\gamma(t) + \delta(t)X(t))\, dB(t), \tag{4.26}$$

where α, β, γ and δ are given adapted processes and continuous functions of t, can be quite useful in applications.

The solution can be expresses as a product of two Ito processes (Klebaner, 1998)

$$X(t) = u(t)\, v(t), \tag{4.27}$$

where

$$du(t) = \beta\, u(t)dt + \delta u(t)\, dB(t), \tag{4.28}$$

and

$$dv(t) = a\, dt + b\, dB(t). \tag{4.29}$$

$u(t)$ can be solved by using a stochastic exponential as shown above and once we have a solution, we can obtain $a(t)$, $b(t)$ by solving the following two equations:

$$b(t)\, u(t) = \gamma(t), \text{ and} \tag{4.30}$$

$$a(t)\, u(t) = \alpha(t) - \delta(t)\, \gamma(t). \tag{4.31}$$

Then the solution to the general linear SDE is given by (Klebaner, 1998):

$$X(t) = u(t)\left(X(0) + \int_0^t \frac{\alpha(s) - \delta(s)\, \gamma(s)}{u(s)} ds + \int_0^t \frac{\gamma(s)}{u(s)} dB(s) \right). \tag{4.32}$$

As an example let us solve the following linear SDE:

$$dx(t) = a\, X(t)\, dt + dB(t), \tag{4.33}$$

where a is a constant.

Here $\beta(t) = a,\ \gamma(t) = 1,\ \alpha(t) = 0$, and $\delta(t) = 0$.

Using the general solution with

$$\begin{aligned} du(t) &= a\ u(t)\ dt + (0)\ dB(t), \\ &= a(t)\ u(t)\ dt. \end{aligned}$$

From stochastic exponential, $u(t) = \exp(at)$.

Therefore,

$$X(t) = \exp(at)(X(o) + \int_o^t \exp(-a\ s)\ \ dB(s)). \tag{4.35}$$

This is a strong solution of the SDE given by equation (4.35).

The integral in the solution given above is an Ito integral and should be calculated according Ito integration. For nonlinear stochastic differential equations, some times appropriate substitutions can be found to reduce them to linear ones. In some situations, we can simplify a nonlinear SDE to a linear one to study the approximate behavior of the system. In these situations analytical results provide insightful information about the system behavior.