# 1 The model

For a time horizon $T > 0$ and a parameter $\alpha > 0$ and $(\theta_t)_{t\in[0,T]}$ a positive deterministic function, let us consider the model given by

$$dX_t = \big(\dot{p}_t - \theta_t(X_t - p_t)\big)dt + \sqrt{2\alpha\theta_t X_t(1-X_t)}dW_t \quad t \in [0,T] \tag{1}$$
$$X_0 = x_0 \in [0,1],$$

where $(p_t)_{t\in[0,T]}$ denotes the prediction function that satisfies $0 \le p_t \le 1$ for all $t \in [0,T]$. This prediction function is assumed to be a smooth function of the time so that

$$\sup_{t\in[0,T]} \big(|p_s| + |\dot{p}_s|\big) < +\infty.$$

The following proofs are based on standard arguments for stochastic processes that can be found e.g. in Alfonsi [1] and Karatzas and Shreve [2] that we adapted to the setting of our model (1).

**Theorem 1.1.** Assume that

$$\forall t \in [0,T],\ \ 0 \le \dot{p}_t + \theta_t p_t \le \theta_t, \quad \text{and} \quad \sup_{t\in[0,T]} |\theta_t| < +\infty. \tag{A}$$

Then, there is a unique strong solution to (1) s.t. for all $t \in [0,T]$, $X_t \in [0,1]$ a.s.

*Proof.* Let us first consider the following SDE for $t \in [0,T]$

$$X_t = x_0 + \int_0^t \big(\dot{p}_s - \theta_s(X_s - p_s)\big)ds + \int_0^t \sqrt{2\alpha\theta_s|X_s(1-X_s)|}dW_s, \quad x_0 > 0. \tag{2}$$

According to Proposition 2.13, p. 291 of [2], under assumption (A) there is a unique strong solution $X$ to (2). Moreover, as the diffusion coefficient is of linear growth, we have for all $p > 0$

$$\mathbb{E}[\sup_{t\in[0,T]} |X_t|^p] < \infty. \tag{3}$$

Then, it remains to show that for all $t \in [0,T]$, $X_t \in [0,1]$ a.s. For this aim, we need to use the so-called Yamada function $\psi_n$ that is a $\mathcal{C}^2$ function that satisfies a bench of useful properties:

$$|\psi_n(x)| \underset{n\to+\infty}{\to} |x|, \ \ x{\psi'}_n(x) \underset{n\to+\infty}{\to} |x|, \ \ |\psi_n(x)| \wedge |x{\psi'}_n(x)| \le |x|$$
$${\psi'}_n(x) \le 1, \quad \text{and } {\psi''}_n(x) = g_n(|x|) \ge 0 \quad \text{with} \quad g_n(x)x \le \frac{2}{n} \quad \text{for all } x \in \mathbb{R}.$$

See the proof of Proposition 2.13, p. 291 of [2] for the construction of such fucntion. Applying Itô's formula we get

$$\psi_n(X_t) = \psi_n(x_0) + \int_0^t \psi'_n(X_s)\big(\dot p_s + \theta_s p_s - \theta_s X_s\big)ds + \int_0^t \psi'_n(X_s)\sqrt{2\alpha\theta_s|X_s(1-X_s)|}dW_s$$
$$+ \alpha\int_0^t \theta_s g_n(|X_s|)|X_s(1-X_s)|ds.$$

Now, thanks to (A), (3) and to the above properties of $\psi_n$ and $g_n$, we get

$$\mathbb{E}[\psi_n(X_t)] \le \psi_n(x_0) + \int_0^t \big(\dot p_s + \theta_s p_s - \theta_s\mathbb{E}[\psi'_n(X_s)X_s]\big)ds + \frac{2\alpha}{n}\int_0^t \theta_s\mathbb{E}|1-X_s|ds.$$

Therefore, letting $n$ tends to infinity we use Lebesgue's theorem to get

$$\mathbb{E}[|X_t|] \le x_0 + \int_0^t \big(\dot p_s + \theta_s p_s - \theta_s\mathbb{E}|X_s|\big)ds.$$

Besides, taking the expectation of (2), we get

$$\mathbb{E}X_t = x_0 + \int_0^t \big(\dot p_s + \theta_s p_s - \theta_s\mathbb{E}X_s\big)ds$$

and thus we have

$$\mathbb{E}[|X_t| - X_t] \le \int_0^t \theta_s\mathbb{E}[|X_s| - X_s]ds.$$

Then, Gronwall's lemma gives us $\mathbb{E}[|X_t|] = \mathbb{E}X_t$ and thus for any $t\in[0,T]$ $X_t \ge 0$ a.s. The same arguments work to prove that for any $t\in[0,T]$ $Y_t := 1 - X_t \ge 0$ a.s. since the process $(Y_t)_{t\in[0,T]}$ is solution to

$$dY_t = \big(\theta_t(1-p_t) - \dot p_t - \theta_t Y_t\big)dt - \sqrt{2\alpha\theta_t Y_t(1-Y_t)}dW_t,$$

Then similarly, we need to assume that $\dot p_t + \theta_t p_t \ge 0$. This completes the proof. □

**Theorem 1.2.** Assume that assumptions of Theorem 1.1 hold with $x_0 \in ]0,1[$. Let $\tau_0 := \inf\{t\in[0,T],\ X_t = 0\}$ and $\tau_1 := \inf\{t\in[0,T],\ X_t = 1\}$ with the convention that $\inf\emptyset = +\infty$. Assume in addition that

$$0 < \alpha < \frac{1}{2} \quad\text{and}\quad \forall t\in[0,T],\ \ \alpha\theta_t \le \dot p_t + \theta_t p_t \le (1-\alpha)\theta_t. \tag{B}$$

Then, $\tau_0 = \tau_1 = +\infty$ a.s.

*Proof.* For $t \in [0, \tau_0[$, we have

$$\frac{dX_t}{X_t} = \frac{\dot{p}_t + \theta_t p_t}{X_t} - \theta_t dt + \sqrt{\frac{2\alpha\theta_t(1 - X_t)}{X_t}} dW_t$$

so that

$$X_t = x_0 \exp\Big( \int_0^t \frac{\dot{p}_s + \theta_s(p_s - \alpha)}{X_s} ds - (1 - \alpha) \int_0^t \theta_s ds + M_t \Big),$$

where $M_t = \int_0^t \sqrt{\frac{2\alpha\theta_s(1-X_s)}{X_s}} dW_s$ is a continuous martingale. Then as for all $t \in [0, T]$, we have $\dot{p}_t + \theta_t(p_t - \alpha) \geq 0$, we deduce that

$$X_t \geq x_0 \exp\Big( - (1 - \alpha) \int_0^t \theta_s ds + M_t \Big).$$

By way of contradiction let us assume that $\{\tau_0 < \infty\}$, then letting $t \to \tau_0$ we deduce that $\lim_{t\to\infty} \mathbf{1}_{\{\tau_0<\infty\}} M_{t\wedge\tau_0} = -1_{\{\tau_0<\infty\}}\infty$ a.s. This leads to a contradiction since we know that continuous martingales likewise the Brownian motion cannot converge almost surely to $+\infty$ or $-\infty$. It follows that $\tau_0 = \infty$ almost surely. Next, recalling that the process $(Y_t)_{t\geq 0}$ given by $Y_t = 1 - X_t$ is solution to

$$dY_t = \big(\theta_t(1 - p_t) - \dot{p}_t - \theta_t Y_t\big) dt - \sqrt{2\alpha\theta_t Y_t(1 - Y_t)} dW_t$$

we deduce using similar arguments as above $\tau_1 = \infty$ a.s. provided that $\theta_t(1-p_t)-\dot{p}_t-\alpha\theta_t \geq 0$. $\square$

When using Lamperti, the form

## References


[1] Alfonsi, A. (2015) Affine diffusions and related processes: simulation, theory and applications. *Springer, Cham; Bocconi University Press, Milan, 2015.*

[2] Karatzas, I. and Shreve, S. E. (1991), Brownian motion and stochastic calculus *Springer-Verlag, New York.*