Lamperti Transform for the processes X and V

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New model for the SDE: $\theta_t = \theta_0$ in the diffusion

$$X_t\colon\ \mathrm{d}X_t = \left(\dot{p}_t - \theta_t(X_t - p_t)\right)\mathrm{d}t + \sqrt{2\theta_0\alpha X_t(1-X_t)}\,\mathrm{d}W_t$$

$$V_t\colon\ \mathrm{d}V_t = -\theta_t V_t\,\mathrm{d}t + \sqrt{2\theta_0\alpha(V_t+p_t)(1-V_t-p_t)}\,\mathrm{d}W_t$$

Lamperti transform for V_t:

$$\psi(V_t,t) = \int \frac{1}{\sqrt{2\theta_0\alpha(u+p_t)(1-u-p_t)}}\,\mathrm{d}u\bigg|_{u=V_t} = -\sqrt{\frac{2}{\alpha\theta_0}}\arcsin\left(\sqrt{1-V_t-p_t}\right),$$

$$= -\sqrt{\frac{2}{\alpha\theta_0}}\arcsin\left(\sqrt{1-X_t}\right).$$

We can see that for every $t=t^*$, the primitive function of $\frac{1}{\sigma(v,t^*)}$ is well defined for all $v\in\left[-p(t^*),1-p(t^*)\right]\subset[-1,1]$ (recall $v=x-p_t$, and $x\in[0,1]$; then, when $x=0$ and $x=1$, we have that $v=-p$ and $x=1-p$, respectively).

Identities for the Lamperti transform of V_t:

- $\psi(V_t, t) = -\sqrt{\frac{2}{\alpha\theta_0}} \arcsin(\sqrt{1 - V_t - p_t})$.
- $\psi_v(V_t, t) = \frac{1}{\sigma(V_t,t)} = \frac{1}{\sqrt{2\alpha\theta_0(V_t+p_t)(1-V_t-p_t)}}$.
- $\psi_{vv}(V_t, t) = \frac{\mathrm{d}}{\mathrm{d}v}\left[\frac{1}{\sigma(V_t,t)}\right] = -\frac{\sigma_v(V_t,t)}{\sigma^2(V_t,t)} = -\frac{1}{\sigma^2(V_t,t)} \cdot \sqrt{\frac{\alpha\theta_0}{2}} \frac{1-2V_t-2p_t}{\sqrt{(V_t+p_t)(1-V_t-p_t)}}$.
- $\psi_t(V_t, t) = \frac{\dot{p}_t}{\sqrt{2\alpha\theta_0(V_t+p_t)(1-V_t-p_t)}}$.

Recall $V_t = X_t - p_t$. Then ψ_v, ψ_{vv}, and ψ_t are not defined when $X_t = 0$ or $X_t = 1$. However, this only happens in the boundary of the domain $(0, 1)$.

Can we apply Itô's lemma to ψ? Maybe as the singularities are in the boundary, it is possible despite that we are not strictly in the hypothesis of the lemma.

SDE for $Z_t = \psi(V_t, t)$: (Verified with Mathematica)

By Itô's lemma, if $\psi(v,t)$ is $C^2([-p_t, 1-p_t])$ for v and $C^1([0,T])$ for t, then:

$$dZ_t = \left(\psi_t + \psi_v \cdot f + \frac{1}{2}\psi_{vv} \cdot \sigma^2 \right) dt + \psi_v \cdot \sigma \, dW_t.$$

If we substitute the terms related with $\psi(V_t, t)$ from slide (3), we have

$$\begin{aligned} dZ_t = \Bigg[& \frac{\dot{p}_t}{\sqrt{2\alpha\theta_0 (V_t + p_t)(1 - V_t - p_t)}} \\ & - \frac{\theta_t V_t}{\sqrt{2\alpha\theta_0 (V_t + p_t)(1 - V_t - p_t)}} - \frac{1}{2}\sqrt{\frac{\alpha\theta_0}{2}} \frac{1 - 2V_t - 2p_t}{\sqrt{(V_t + p_t)(1 - V_t - p_t)}} \Bigg] dt + 1 \cdot dW_t. \end{aligned}$$

Recall $Z_t = -\sqrt{\frac{2}{\alpha\theta_t}} \arcsin\left(\sqrt{1 - V_t - p_t}\right)$, where $Z_t \in \left[-\frac{\pi}{\sqrt{2\alpha\theta_t}}, 0 \right]$.

SDE for $Z_t = \psi(V_t, t)$: (Computed with Mathematica)

$$\mathrm{d}Z_t = \underbrace{\left[\frac{\alpha\theta_0 \cos(Z_t\sqrt{2\alpha\theta_0}) - \theta_t \cos(Z_t\sqrt{2\alpha\theta_0}) + 2\theta_t p_t + 2\dot{p}_t - \theta_t}{\sqrt{\alpha\theta_0}\sqrt{1-\cos(2Z_t\sqrt{2\alpha\theta_0})}}\right]}_{f(Z_t,t)} \mathrm{d}t + 1\cdot \mathrm{d}W_t.$$

$$\lim_{z\to 0^-} f(z,t) = \infty \times \left[\frac{\mathrm{sign}\,(2\theta_t p_t + 2\dot{p}_t + \alpha\theta_0 - 2\theta_t)}{\mathrm{sign}(\alpha)\,\mathrm{sign}(\theta_0)}\right].$$

$$\lim_{z\to \left[\frac{-\pi}{\sqrt{2\alpha\theta_0}}\right]^+} f(z,t) = \infty \times \left[\frac{\mathrm{sign}\,(2\theta_t p_t + 2\dot{p}_t - \alpha\theta_0)}{\mathrm{sign}(\alpha)\,\mathrm{sign}(\theta_0)}\right].$$

We want to find the correct conditions for θ_t.

To simplify the SDE, Mathematica has used:
$\sin^2(x) - \sin^4(x) = \sin^2(x)\cos^2(x) = \frac{1}{4}\sin^2(2x) = \frac{1}{8}(1-\cos(4x))$.

Limit when $z \to 0^-$:

Recall we have a bijective mapping $Z_t([0,1]) = \left[\frac{-\pi}{\sqrt{2\alpha\theta_0}}, 0\right]$. This helps the intuition, as when $X_t = 1$, we expect the diffusion to be negative, and $z \to 0^-$ is equivalent to $x \to 1^-$.

We want $\lim_{z\to 0^-} f(z,t)$ to be $-\infty$ or zero, so we do not escape from $z = 0$ to $z > 0$ ($x = 1$ to $x > 1$). Then, we need $\alpha\theta_0 - 2\theta_t + 2\theta_t p_t + 2\dot{p}_t \leq 0$. Then:

- If $p_t < 1$, we have that $\theta_t \geq \frac{\alpha\theta_0 + 2\dot{p}_t}{2(1-p_t)}$.

Limit when $z \to \left[\frac{-\pi}{\sqrt{2\alpha\theta_0}}\right]^+$:

Recall we have a bijective mapping $Z_t([0,1]) = \left[\frac{-\pi}{\sqrt{2\alpha\theta_0}}, 0\right]$. This helps the intuition, as when $X_t = 0$, we expect the diffusion to be positive, and $z \to \frac{-\pi}{\sqrt{2\alpha\theta_0}}^+$ is equivalent to $x \to 0^+$.

We want $\lim_{z \to \left[\frac{-\pi}{\sqrt{2\alpha\theta_0}}\right]^+} f(z,t)$ to be $+\infty$ or zero, so we do not escape from $z = \frac{-\pi}{\sqrt{2\alpha\theta_t}}$ to $z < \frac{-\pi}{\sqrt{2\alpha\theta_t}}$ ($x = 0$ to $x < 0$). Then, we need $2\theta_t p_t + 2\dot{p}_t - \alpha\theta_0 \geq 0$. Then:

- If $p_t > 0$, we have that $\theta_t \geq \frac{\alpha\theta_0 - 2\dot{p}_t}{2p_t}$.

Controlled drift:

From both orange conditions in slides (6) and (7), we create a more restrictive condition:

$$\max\left(\frac{\alpha\theta_0 + 2\dot{p}_t}{2(1-p_t)}, \frac{\alpha\theta_0 - 2\dot{p}_t}{2p_t}\right) \leq \frac{\alpha\theta_0 + |2\dot{p}_t|}{2\min(1-p_t, p_t)}.$$

Then, we choose

$$\theta_t = \max\left(\theta_0, \frac{\alpha\theta_0 + |2\dot{p}_t|}{2\min(1-p_t, p_t)}\right). \tag{1}$$

Recall that in the paper, we start by choosing $\theta_t = \max\left(\theta_0, \frac{|\dot{p}_t|}{\min(1-p_t, p_t)}\right)$. Our new condition (1) is slightly more restrictive.

Limits when $p_t \approx 0$ and $p_t \approx 1$:

It $p_t \approx 0$, then $\theta_t = \frac{\alpha\theta_0 + |2\dot{p}_t|}{2p_t}$, and we have the limit:

$$\lim_{z \to \left[\frac{-\pi}{\sqrt{2\alpha\theta_0}}\right]^+} f(z,t) = \infty \times \mathrm{sign}(|\dot{p}_t| + \dot{p}_t).$$

As $p_t \approx 0$, it is reasonable to assume $\dot{p}_t \geq 0$. Then, the limit is $+\infty$.

It $p_t \approx 1$, then $\theta_t = \frac{\alpha\theta_0 + |2\dot{p}_t|}{2(1-p_t)}$, and we have the limit:

$$\lim_{z \to 0^-} f(z,t) = \infty \times \mathrm{sign}(\dot{p}_t - |\dot{p}_t|).$$

As $p_t \approx 1$, it is reasonable to assume $\dot{p}_t \leq 0$. Then, the limit is $-\infty$.

Conditions summary:

- First model: $\theta_t^{first} = \max\left(\theta_0, \frac{|\dot{p}_t|}{\min(1-p_t, p_t)}\right)$.
- New condition from Lamperti: $\theta_t^{lamperti} = \max\left(\theta_0, \frac{\alpha\theta_0 + |2\dot{p}_t|}{2\min(1-p_t, p_t)}\right)$. Notice $\theta_t^{first} \leq \theta_t^{lamperti}$.
- Professor Kebaier's condition: $0 < \alpha < 1/2$, $\alpha < p_t < 1-\alpha$, and $\theta_t^{kebaier} = \max\left(\theta_0, \frac{|\dot{p}|}{\min(p-\alpha, 1-p-\alpha)}\right)$.

Now, given $p_t \approx \alpha$, we have that $\theta_t^{lamperti} = \theta_t^{kebaier}$ if $\theta_0 = \frac{|2\dot{p}_t|}{p_t - \alpha} > 0$. Then, which is more restrictive depends on $|\dot{p}_t|$ and p_t.

Now, given $p_t \approx 1-\alpha$, we have that $\theta_t^{lamperti} = \theta_t^{kebaier}$ if $\theta_0 = \frac{|2\dot{p}_t|}{1-p_t-\alpha} > 0$. Then, which is more restrictive depends on $|\dot{p}_t|$ and p_t.

We can see that, we can have $\theta_t^{lamperti} > \theta_t^{kebaier}$ or $\theta_t^{lamperti} < \theta_t^{kebaier}$, depending on the values of p_t and $\dot{p}_t$.

Particular questions:

- In slide (3), we can see that the Lamperti transform $\psi(v,t)$ has undefined partial derivatives when $x=0$, or $x=1$. This is a consequence of the singularities of $\frac{1}{\sigma(v,t)}$ when $x=0$, or $x=1$. What can we say about the SDE of $Z_t=\psi(V_t,t)$ in the sense of existence and unicity? Can we use Itô's lemma considering that the singularities are in the boundary of the domain?
- In slide (5), the limits for the drift when Z_t touch the boundaries of its domain depend on α. Then, the condition for Z_t to stay always in $\left[\frac{-\pi}{\sqrt{2\alpha\theta_0}},0\right]$ also depends on α. This is not intuitive because the condition for X_t to be in $[0,1]$, and there is a bijective mapping between X_t and Z_t, so they both should require the same condition.