Asymptotic properties of maximum likelihood estimator for the growth rate of jump-type CIR processes

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Outline of The Talk

1. Introduction
   - The jump-type CIR process
   - Statistical inference of the model
2. Stochastic Analysis of the Model
   - Existence and uniqueness
   - Stationarity and ergodicity of the model
   - Joint Laplace transform of Y_t and $\int_0^t Y_s\,ds$
3. Statistical Inference of the growth rate
   - The maximum likelihood estimator (MLE) of b
   - Asymptotics of MLE
4. The Alpha-CIR process
   - The model
   - Maximum likelihood estimator
5. Statistical Inference of the Wishart process
   - ergodic case and non ergodic cases

Overview of the main contributions

- In Ben Alaya and K. (12 & 13) :
  - ➢ we prove original limit theorems on the drift parameters MLE of the continuously observed CIR process including the critical and subcritical cases .
  - ➢ We provide sufficient conditions so that these limit theorems can be easily carried out for the discretely observed process.
- In Alfonsi, K. and Rey (16):
  - ➢ we extend the above results to the setting of matrix Wishart processes
  - ➢ We provide asymptotic behavior and local asymptotic properties of the associated drift parameters, in the ergodic and several non-ergodic cases.
- In Barczy et al (17 &18+) : we study the drift parameters MLE properties for jump-type CIR models.

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Diffusion-type CIR process

- Cox-Ingersoll-Ross (CIR) process (Feller (1951) and Cox, Ingersoll and Ross (1985)):

$$\boxed{\mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}W_t, \qquad t \geq 0,}$$

where Y_0 is a non-negative initial value, $a \geq 0$, $b \in \mathbb{R}$, $\sigma > 0$, and $(W_t)_{t\geq 0}$ is a standard Wiener process, independent of Y_0.

- Y is also called a square root process or a Feller process.
  - ➢ The existence of a pathwise unique non-negative strong solution can be found in Ikeda and Watanabe (1981).
  - ➢ If $a \geq \frac{\sigma^2}{2}$, then $\mathbb{P}(Y_t > 0$ for all $t > 0) = 1$.

A jump-type CIR process driven by a subordinator

- We consider the SDE

$$\boxed{\mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}W_t + \mathrm{d}J_t, \qquad t \geq 0,}$$

where Y_0 is an a.s. non-negative initial value, $a \geq 0$, $b \in \mathbb{R}$, $\sigma > 0$, $(W_t)_{t\geq 0}$ is a standard Wiener process, and $(J_t)_{t\geq 0}$ is a subordinator.

- The Lévy measure m concentrating on $(0, \infty)$ satisfies

$$\int_0^\infty z\, m(\mathrm{d}z) \in [0, \infty), \qquad \text{(A1)}$$

that is, for $t \geq 0$ and $u \in \mathbb{C}$ with $\Re(u) \leq 0$,

$$\mathbb{E}(\mathrm{e}^{uJ_t}) = \exp\left\{ t \int_0^\infty (\mathrm{e}^{uz} - 1)\, m(\mathrm{d}z) \right\}$$

- We suppose that Y_0, $(W_t)_{t\geq 0}$ and $(J_t)_{t\geq 0}$ are independent.

A special case: Basic Affine Jump-Diffusion (BAJD)

It was introduced by Duffie and Gârleanu (2001):

- $a = \kappa\theta$ and $b = \kappa$, where $\kappa > 0$ and $\theta \geq 0$, i.e., the drift takes form $\kappa(\theta - Y_t)$ (only subcritical case),
- the Lévy measure m takes form

$$m(\mathrm{d}z) = c\lambda \mathrm{e}^{-\lambda z} \mathbf{1}_{(\mathbf{0},\infty)}(\mathbf{z})\, \mathrm{d}z$$

with some constants $c \geq 0$ and $\lambda > 0$.

Then J is a compound Poisson process,
its first jump time $\sim Exp(c)$ and its jump size $\sim Exp(\lambda)$.

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Aim

- study the asymptotic properties of the MLE of $b \in \mathbb{R}$ under the conditions:
  - $a \geq 0$, $\sigma > 0$ and the Lévy measure m are known,
  - based on continuous time observations $(Y_t)_{t\in[0,T]}$ with $T \in (0, \infty)$,
  - known non-random initial value $y_0 \geq 0$: $\mathbb{P}(Y_0 = y_0) = 1$,
  - sample size tends to ∞, i.e., $T \to \infty$.
- It will turn out that for the calculation of the MLE of b, one does not need to know σ and m.
- At the moment, we can not handle the MLE of a supposing that b is known or the joint MLE of (a, b).
  Reason: limit behavior of $\int_0^t \frac{1}{Y_s} ds$ as $t \to \infty$, is not known to us.

On the parameter σ

We do not estimate the parameter σ, since it is a measurable function (statistic) of $(Y_t)_{t\in[0,T]}$ for any $T>0$, following from

$$\frac{1}{\frac{1}{n}\sum_{i=1}^{\lfloor nT\rfloor} Y_{\frac{i-1}{n}}}\left[\sum_{i=1}^{\lfloor nT\rfloor}\left(Y_{\frac{i}{n}}-Y_{\frac{i-1}{n}}\right)^2-\sum_{u\in[0,T]}(\Delta Y_u)^2\right]\xrightarrow{\mathbb{P}}\frac{\langle Y^{\mathrm{cont}}\rangle_T}{\int_0^T Y_u\,\mathrm{d}u}=\sigma^2$$

as $n\to\infty$, where

- $\Delta Y_u := Y_u - Y_{u-}$, $u>0$, and $\Delta Y_0 := 0$,
- $Y_t^{\mathrm{cont}} = \sigma\int_0^t \sqrt{Y_u}\,\mathrm{d}W_u$, $t\geq 0$, denotes the continuous martingale part of Y,
- the convergence holds almost surely as well along a suitable subsequence.

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Existence and uniqueness of a strong solution

Recall that

$$\boxed{\mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}W_t + \mathrm{d}J_t, \qquad t \geq 0.}$$

Proposition. Let η_0 be a random vector independent of $(W_t)_{t\geq 0}$ and $(J_t)_{t\geq 0}$ satisfying $\mathbb{P}(\eta_0 \geq 0) = 1$ and $\mathbb{E}(\eta_0) < \infty$. Then for all $a \geq 0$, $b \in \mathbb{R}$, $\sigma > 0$ and Lévy measure m on $(0, \infty)$ satisfying (A1),

- there is a pathwise unique strong solution $(Y_t)_{t\geq 0}$ such that $\mathbb{P}(Y_0 = \eta_0) = 1$ and $\mathbb{P}(Y_t \geq 0 \text{ for all } t \geq 0) = 1$. (It is a consequence of Dawson and Li (2006).)
- if $\mathbb{P}(\eta_0 > 0) = 1$ or $a > 0$, then $\mathbb{P}\left(\int_0^t Y_s\,\mathrm{d}s > 0\right) = 1$, $t > 0$.

Remark

- The infinitesimal generator of Y takes the form

$$(\mathcal{A}f)(y) = (a - by)f'(y) + \frac{\sigma^2}{2} y f''(y) + \int_0^\infty (f(y+z) - f(y))\, m(\mathrm{d}z),$$

where $y \geq 0$, $f \in C_c^2(\mathbb{R}_+, \mathbb{R})$, and f' and f'' denote the first and second order derivatives of f.

- Y is a CBI process having a branching mechanism

$$R(u) = \frac{\sigma^2}{2} u^2 - bu, \qquad u \in \mathbb{C} \text{ with } \Re(u) \leq 0,$$

and an immigration mechanism

$$F(u) = au + \int_0^\infty (\mathrm{e}^{uz} - 1)\, m(\mathrm{d}z), \qquad u \in \mathbb{C} \text{ with } \Re(u) \leq 0.$$

The jump part has effects only on the immigration mechanism.

- Then, the Laplace transform of Y_t takes the form

$$\mathbb{E}(e^{-\lambda Y_t} \mid Y_0 = y_0) = \exp\left\{-y_0 v_t(\lambda) - \int_0^t F(v_s(\lambda)) \, ds\right\} \tag{1}$$

$$\frac{\partial}{\partial t} v_t(\lambda) = -R(v_t(\lambda)), \qquad v_0(\lambda) = \lambda. \tag{2}$$

then we have

$$\mathbb{E}(e^{-\lambda Y_t} \mid Y_0 = y_0) = \exp\left\{-y_0 v_t(\lambda) + \int_\lambda^{v_t(\lambda)} \frac{F(z)}{R(z)} \, dz\right\}. \tag{3}$$

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Stationarity and ergodicity in the subcritical case, I

Theorem. Let $a \geq 0$, $b > 0$, $\sigma > 0$, and let m be a Lévy measure on $(0, \infty)$ satisfying (A1). Let $(Y_t)_{t \geq 0}$ be the unique strong solution satisfying $\mathbb{P}(Y_0 \geq 0) = 1$ and $\mathbb{E}(Y_0) < \infty$.

(i) Then $(Y_t)_{t \geq 0}$ converges in law to its unique stationary distribution π having Laplace transform

$$\int_0^\infty e^{uy}\,\pi(dy) = \exp\left\{ \int_u^0 \frac{av + \int_0^\infty (e^{vz} - 1)\,m(dz)}{\frac{\sigma^2}{2}\,v^2 - bv}\,dv \right\}, \qquad u \leq 0.$$

In the special case $m = 0$ (diffusion-type CIR process),

$$\int_0^\infty e^{uy}\,\pi(dy) = \left(1 - \frac{\sigma^2}{2b} u\right)^{-\frac{2a}{\sigma^2}}, \qquad u \leq 0,$$

i.e., π has Gamma distribution with parameters $\frac{2a}{\sigma^2}$ and $\frac{2b}{\sigma^2}$.

Stationarity and ergodicity in the subcritical case, II

(ii) If, in addition, $a > 0$ and the extra moment condition

$$\int_0^1 z \log\left(\frac{1}{z}\right) m(\mathrm{d}z) < \infty$$

holds, then the process $(Y_t)_{t \geq 0}$ is exponentially ergodic, namely, there exist constants $\beta \in (0,1)$, $\gamma > 0$ and $C > 0$ such that

$$\|\mathbb{P}_{Y_t | Y_0 = y_0} - \pi\|_{\mathrm{TV}} \leq C(y_0 + 1)\beta^t, \qquad t, y_0 \geq 0.$$

Moreover, for all Borel measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ with $\int_0^\infty |f(y)| \, \pi(\mathrm{d}y) < \infty$, we have

$$\frac{1}{T} \int_0^T f(Y_s) \, \mathrm{d}s \xrightarrow{\text{a.s.}} \int_0^\infty f(y) \, \pi(\mathrm{d}y) \qquad \text{as} \quad T \to \infty.$$

References for stationarity and ergodicity

For the existence of a unique stationary distribution, see Keller-Ressel and Steiner (2008), Li (2011) and Keller-Ressel and Mijatović (2012).

For the exponential ergodicity, see Jin, Rüdiger and Trabelsi (2016).

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Theorem. Let $a \geq 0$, $b \in \mathbb{R}$, $\sigma > 0$, and let m be a Lévy measure on $(0, \infty)$ satisfying (A1). Let $(Y_t)_{t \geq 0}$ be the unique strong solution satisfying $\mathbb{P}(Y_0 = y_0) = 1$ with some $y_0 \geq 0$. Then for all $u, v \leq 0$,

$$\mathbb{E}\left[\exp\left\{uY_t + v\int_0^t Y_s\,\mathrm{d}s\right\}\right]$$
$$= \exp\left\{\psi_{u,v}(t)y_0 + \int_0^t \left(a\psi_{u,v}(s) + \int_0^\infty \left(\mathrm{e}^{z\psi_{u,v}(s)} - 1\right) m(\mathrm{d}z)\right)\mathrm{d}s\right\}$$

for $t \geq 0$, where $\psi_{u,v} : [0, \infty) \to (-\infty, 0]$ takes the form

$$\psi_{u,v}(t) = \begin{cases} \dfrac{u\gamma_v \cosh\left(\frac{\gamma_v t}{2}\right) + (-ub+2v)\sinh\left(\frac{\gamma_v t}{2}\right)}{\gamma_v \cosh\left(\frac{\gamma_v t}{2}\right) + (-\sigma^2 u + b)\sinh\left(\frac{\gamma_v t}{2}\right)} & \text{if } v < 0 \text{ or } b \neq 0, \\ \dfrac{u}{1 - \frac{\sigma^2 u}{2} t} & \text{if } v = 0 \text{ and } b = 0, \end{cases}$$

where $\gamma_v := \sqrt{b^2 - 2\sigma^2 v}$.

Remark

- The proof is based on the fact that $(Y_t, \int_0^t Y_s)$, $t \geq 0$, is a 2-dimensional CBI process yielding that

$$\mathbb{E}\left[\exp\left\{uY_t + v\int_0^t Y_s\,\mathrm{d}s\right\}\right] = \exp\left\{\psi_{u,v}(t)y_0 + \int_0^t \left(a\psi_{u,v}(s) + \int_0^\infty (\mathrm{e}^{z\psi_{u,v}(s)} - 1)\,m(\mathrm{d}z)\right)\mathrm{d}s\right\}$$

for $t \geq 0$, $u, v \leq 0$, where $\psi_{u,v} : [0,\infty) \to (-\infty, 0]$ is the unique locally bounded solution to the Riccati DE

$$\psi'_{u,v}(t) = \frac{\sigma^2}{2}\psi_{u,v}(t)^2 - b\psi_{u,v}(t) + v, \qquad t \geq 0, \qquad \psi_{u,v}(0) = u.$$

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The likelihood ratio: Jacod and Shiryaev (2003)

Assume, we have a semimartingale representation of a càdlàg process η under $\mathbb{P}_\psi$ for a given truncation function h

$$\eta_t = \boldsymbol{x}_0 + B_t^{(\psi)} + (\eta^{\mathrm{cont}})_t^{(\psi)} + \int_0^t \int_{\mathbb{R}^d} h(\boldsymbol{x})\,(\mu^\eta - \nu^{(\psi)})(\mathrm{d}s, \mathrm{d}\boldsymbol{x}) + \int_0^t \int_{\mathbb{R}^d} (\boldsymbol{x} - h(\boldsymbol{x}))\,\mu^\eta(\mathrm{d}s, \mathrm{d}\boldsymbol{x}), \qquad t \in \mathbb{R}_+.$$

Assume that there exists a nondecreasing, continuous, adapted process $(F_t^{(\psi)})_{t\in\mathbb{R}_+}$ with $F_0^{(\psi)} = 0$ and a predictable process $(c_t^{(\psi)})_{t\in\mathbb{R}_+}$ with values in the set of all symmetric positive semidefinite $d \times d$ matrices such that

$$(\eta^{\mathrm{cont}})_t^{(\psi)} = \int_0^t c_s^{(\psi)}\,\mathrm{d}F_s^{(\psi)}$$

Assume also that there exist a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function $V^{(\widetilde{\psi},\psi)} : D(\mathbb{R}_+, \mathbb{R}^d) \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_{++}$ and a predictable $\mathbb{R}^d$-valued process $\beta^{(\widetilde{\psi},\psi)}$ satisfying

$$\nu^{(\psi)}(\mathrm{d}t, \mathrm{d}\boldsymbol{x}) = V^{(\widetilde{\psi},\psi)}(t, \boldsymbol{x})\nu^{(\widetilde{\psi})}(\mathrm{d}t, \mathrm{d}\boldsymbol{x}), \tag{4}$$

$$\text{with} \quad \int_0^t \int_{\mathbb{R}^d} \left(\sqrt{V^{(\widetilde{\psi},\psi)}(s, \boldsymbol{x})} - 1\right)^2 \nu^{(\widetilde{\psi})}(\mathrm{d}s, \mathrm{d}\boldsymbol{x}) < \infty, \tag{5}$$

$$\begin{aligned} B_t^{(\psi)} = B_t^{(\widetilde{\psi})} &+ \int_0^t c_s^{(\psi)} \beta_s^{(\widetilde{\psi},\psi)} \, \mathrm{d}F_s^{(\psi)} \\ &+ \int_0^t \int_{\mathbb{R}^d} (V^{(\widetilde{\psi},\psi)}(s, \boldsymbol{x}) - 1) h(\boldsymbol{x}) \, \nu^{(\widetilde{\psi})}(\mathrm{d}s, \mathrm{d}\boldsymbol{x}), \end{aligned} \tag{6}$$

$$\text{with} \quad \int_0^t (\beta_s^{(\widetilde{\psi},\psi)})^\top c_s^{(\psi)} \beta_s^{(\widetilde{\psi},\psi)} \, \mathrm{d}F_s^{(\psi)} < \infty, \tag{7}$$

$\mathbb{P}_\psi$-almost sure for every $t \in \mathbb{R}_+$.

If local uniqueness holds for the martingale problem on the canonical space corresponding to the triplet $(B^{(\psi)}, (\eta^{\mathrm{cont}})^{(\psi)}, \nu^{(\psi)})$ with the given initial value $\boldsymbol{x}_0$ with $\mathbb{P}_\psi$ as its unique solution. Then for each $T \in \mathbb{R}_+$, $\mathbb{P}_{\psi,T}$ is absolutely continuous with respect to $\mathbb{P}_{\widetilde{\psi},T}$,

$$\begin{aligned}
\log \frac{\mathrm{d}\mathbb{P}_{\psi,T}}{\mathrm{d}\mathbb{P}_{\widetilde{\psi},T}}(\eta) &= \int_0^T (\beta_s^{(\widetilde{\psi},\psi)})^\top \,\mathrm{d}(\eta^{\mathrm{cont}})_s^{(\widetilde{\psi})} - \frac{1}{2}\int_0^T (\beta_s^{(\widetilde{\psi},\psi)})^\top c_s^{(\psi)} \beta_s^{(\widetilde{\psi},\psi)} \,\mathrm{d}F_s^{(\psi)} \\
&+ \int_0^T \int_{\mathbb{R}^d} (V^{(\widetilde{\psi},\psi)}(s,\boldsymbol{x}) - 1)\,(\mu^\eta - \nu^{(\widetilde{\psi})})(\mathrm{d}s,\mathrm{d}\boldsymbol{x}) \\
&+ \int_0^T \int_{\mathbb{R}^d} (\log(V^{(\widetilde{\psi},\psi)}(s,\boldsymbol{x})) - V^{(\widetilde{\psi},\psi)}(s,\boldsymbol{x}) + 1)\,\mu^\eta(\mathrm{d}s,\mathrm{d}\boldsymbol{x})
\end{aligned} \tag{8}$$

Existence and uniqueness of MLE

Proposition. Let $b, \widetilde{b} \in \mathbb{R}$. Then, for all $T > 0$, the probability measures $\mathbb{P}_{b,T}$ and $\mathbb{P}_{\widetilde{b},T}$ are absolutely continuous with respect to each other, and, under $\mathbb{P}$,

$$\log\left(\frac{\mathrm{d}\mathbb{P}_{b,T}}{\mathrm{d}\mathbb{P}_{\widetilde{b},T}}(\widetilde{Y})\right) = -\frac{b-\widetilde{b}}{\sigma^2}(\widetilde{Y}_T - y_0 - aT - J_T) - \frac{b^2-\widetilde{b}^2}{2\sigma^2}\int_0^T \widetilde{Y}_s\,\mathrm{d}s,$$

where $\widetilde{Y}$ is the process corresponding to the parameter $\widetilde{b}$.

Then, for each $T > 0$, there exists a unique MLE $\widehat{b}_T$ of b a.s. having the form

$$\widehat{b}_T = -\frac{Y_T - y_0 - aT - J_T}{\int_0^T Y_s\,\mathrm{d}s},$$

provided that $\int_0^T Y_s\,\mathrm{d}s > 0$ (Valid if $a > 0$ or $y_0 > 0$).

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Asymptotics of MLE: subcritical case $(b > 0)$

Theorem. Let $a > 0$, $b > 0$, $\sigma > 0$, m be a Lévy measure on $(0, \infty)$ satisfying (A1), and $\mathbb{P}(Y_0 = y_0) = 1$ with some $y_0 \geq 0$. Then the MLE $\widehat{b}_T$ of b is asymptotically normal, i.e.,

$$\sqrt{T}(\widehat{b}_T - b) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\sigma^2 b}{a + \int_0^\infty z\, m(\mathrm{d}z)}\right) \qquad \text{as} \quad T \to \infty.$$

Especially, $\widehat{b}_T$ is weakly consistent, i.e., $\widehat{b}_T \xrightarrow{\mathbb{P}} b$ as $T \to \infty$.

With a random scaling,

$$\frac{1}{\sigma}\left(\int_0^T Y_s \,\mathrm{d}s\right)^{1/2}(\widehat{b}_T - b) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1) \qquad \text{as} \quad T \to \infty.$$

Under the additional moment condition $\int_0^1 z \log\left(\frac{1}{z}\right) m(\mathrm{d}z) < \infty$, $\widehat{b}_T$ is strongly consistent, i.e., $\widehat{b}_T \xrightarrow{\text{a.s.}} b$ as $T \to \infty$.

A proof is based on

- the decomposition

$$\sqrt{T}(\widehat{b}_T - b) = -\sigma \frac{\frac{1}{\sqrt{T}} \int_0^T \sqrt{Y_s}\, \mathrm{d}W_s}{\frac{1}{T} \int_0^T Y_s\, \mathrm{d}s}, \qquad T > 0.$$

- by the explicit form of the Laplace transform of $\int_0^T Y_s\, \mathrm{d}s$,

$$\frac{1}{T}\int_0^T Y_s\, \mathrm{d}s \xrightarrow{\mathbb{P}} \frac{1}{b}\left(a + \int_0^\infty z\, m(\mathrm{d}z)\right) = \int_0^\infty y\, \pi(\mathrm{d}y) \qquad \text{as} \quad T \to \infty,$$

- a limit theorem for continuous local martingales.
- under the moment assumption $\int_0^1 z \log\left(\frac{1}{z}\right) m(\mathrm{d}z) < \infty$, we have $\frac{1}{T}\int_0^t Y_s\, \mathrm{d}s \xrightarrow{\text{a.s.}} \int_0^\infty y\, \pi(\mathrm{d}y)$ as $T \to \infty$, yielding $\int_0^T Y_s\, \mathrm{d}s \xrightarrow{\text{a.s.}} \infty$ as $T \to \infty$, and then one can use a SLLN for continuous local martingales.

Theorem (van Zanten (2000)) for continuous local martingales

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space satisfying the usual conditions. Let $(\boldsymbol{M}_t)_{t\geq 0}$ be a d-dimensional square-integrable continuous local martingale w.r.t the filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathbb{P}(\boldsymbol{M}_0 = \boldsymbol{0}) = 1$. Suppose that

- there exists a function $\boldsymbol{Q} : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ such that $\boldsymbol{Q}(t)$ is an invertible (non-random) matrix for all $t \geq 0$,
- $\lim_{t\to\infty} \|\boldsymbol{Q}(t)\| = 0$,
- $\boldsymbol{Q}(t)\langle \boldsymbol{M}\rangle_t \, \boldsymbol{Q}(t)^\top \xrightarrow{\mathbb{P}} \boldsymbol{\eta}\boldsymbol{\eta}^\top$ as $t \to \infty$, where $\boldsymbol{\eta}$ is a $d \times d$ (possibly) random matrix.

Then $\boldsymbol{Q}(t)\boldsymbol{M}_t \xrightarrow{\mathcal{L}} \boldsymbol{\eta}\boldsymbol{Z}$ as $t \to \infty$, where $\boldsymbol{Z}$ is a d-dimensional standard normally distributed random vector independent of $\boldsymbol{\eta}$. That is, $\boldsymbol{Q}(t)\boldsymbol{M}_t$ has a mixed normal limit distribution as $t \to \infty$.

Asymptotics of MLE: critical case $(b = 0)$

Theorem. Let $a \geq 0$, $b = 0$, $\sigma > 0$, m be a Lévy measure on $(0, \infty)$ satisfying (A1), and $\mathbb{P}(Y_0 = y_0) = 1$ with some $y_0 \geq 0$. Suppose that $a > 0$ or $y_0 > 0$. Then

$$T(\widehat{b}_T - b) = T\widehat{b}_T \xrightarrow{\mathcal{L}} \frac{a + \int_0^\infty z\, m(\mathrm{d}z) - \mathcal{Y}_1}{\int_0^1 \mathcal{Y}_s \, \mathrm{d}s} \qquad \text{as } T \to \infty,$$

where $(\mathcal{Y}_t)_{t \geq 0}$ is the critical (diffusion type) CIR process

$$\mathrm{d}\mathcal{Y}_t = \left(a + \int_0^\infty z\, m(\mathrm{d}z) \right) \mathrm{d}t + \sigma \sqrt{\mathcal{Y}_t}\, \mathrm{d}\mathcal{W}_t, \quad t \geq 0, \quad \text{with } \mathcal{Y}_0 = 0,$$

where $(\mathcal{W}_t)_{t \geq 0}$ is a standard Wiener process. As a consequence, $\widehat{b}_T$ is weakly consistent. With a random scaling,

$$\frac{1}{\sigma}\left(\int_0^T \mathcal{Y}_s \, \mathrm{d}s \right)^{1/2} (\widehat{b}_T - b) \xrightarrow{\mathcal{L}} \frac{a + \int_0^\infty z\, m(\mathrm{d}z) - \mathcal{Y}_1}{\sigma \left(\int_0^1 \mathcal{Y}_s \, \mathrm{d}s \right)^{1/2}} \qquad \text{as } T \to \infty.$$

A proof is based on

- the decomposition

$$T\widehat{b}_T = -\frac{\frac{Y_T}{T} - \frac{y_0}{T} - a - \frac{J_T}{T}}{\frac{1}{T^2}\int_0^T Y_s\,\mathrm{d}s}, \qquad T > 0.$$

- by SLLN for Lévy processes, $\frac{J_T}{T} \xrightarrow{\text{a.s.}} \mathbb{E}(J_1) = \int_0^\infty z\, m(\mathrm{d}z)$ as $T \to \infty$.
- $\left(\frac{1}{T}Y_T, \frac{1}{T^2}\int_0^T Y_s\,\mathrm{d}s\right) \xrightarrow{\mathcal{L}} \left(\mathcal{Y}_1, \int_0^1 \mathcal{Y}_s\,\mathrm{d}s\right) \qquad$ as $T \to \infty$,

where the Laplace transform of the limit law takes the form

$$\mathbb{E}\left(\mathrm{e}^{u\mathcal{Y}_1 + v\int_0^1 \mathcal{Y}_s\,\mathrm{d}s}\right) = \begin{cases} \left(\cosh\left(\frac{\gamma_v}{2}\right) - \frac{\sigma^2 u}{\gamma_v}\sinh\left(\frac{\gamma_v}{2}\right)\right)^{-\frac{2}{\sigma^2}\left(a + \int_0^\infty z\, m(\mathrm{d}z)\right)} & \text{if } u \le 0,\ v < 0, \\ \left(1 - \frac{\sigma^2 u}{2}\right)^{-\frac{2}{\sigma^2}\left(a + \int_0^\infty z\, m(\mathrm{d}z)\right)} & \text{if } u \le 0,\ v = 0, \end{cases}$$

where $\gamma_v = \sqrt{-2\sigma^2 v}$, $v \le 0$.

Asymptotics of MLE: supercritical case $(b < 0)$

Theorem. Let $a > 0$, $b < 0$, $\sigma > 0$, m be a Lévy measure on $(0, \infty)$ satisfying (A1), and $\mathbb{P}(Y_0 = y_0) = 1$ with some $y_0 \geq 0$. Then $\widehat{b}_T$ is strongly consistent, and asymptotically mixed normal, namely

$$e^{-bT/2}(\widehat{b}_T - b) \xrightarrow{\mathcal{L}} \sigma Z \left(-\frac{V}{b}\right)^{-1/2} \qquad \text{as} \quad T \to \infty,$$

where V is a positive r. v. having Laplace transform

$$\mathbb{E}(e^{uV}) = \exp\left\{\frac{uy_0}{1 + \frac{\sigma^2 u}{2b}}\right\} \left(1 + \frac{\sigma^2 u}{2b}\right)^{-\frac{2a}{\sigma^2}} \exp\left\{\int_0^\infty \left(\int_0^\infty \left(\exp\left\{\frac{zue^{by}}{1 + \frac{\sigma^2 u}{2b} e^{by}}\right\} - 1\right) m(\mathrm{d}z)\right) \mathrm{d}y\right\}$$

for all $u \leq 0$, and Z is a standard normally distributed r. v., independent of V. With a random scaling, we have

$$\frac{1}{\sigma}\left(\int_0^T Y_s \, \mathrm{d}s\right)^{1/2} (\widehat{b}_T - b) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1) \qquad \text{as} \quad T \to \infty.$$

A stochastic representation $V \stackrel{\mathcal{L}}{=} \widetilde{\mathcal{V}} + \widetilde{\widetilde{\mathcal{V}}}$

- $\widetilde{\mathcal{V}}$ and $\widetilde{\widetilde{\mathcal{V}}}$ are independent random variables,
- $\mathrm{e}^{bt}\widetilde{\mathcal{Y}}_t \xrightarrow{\text{a.s.}} \widetilde{\mathcal{V}}$ as $t \to \infty$, where $(\widetilde{\mathcal{Y}}_t)_{t\geq 0}$ is the (diffusion-type) supercritical CIR process

$$\mathrm{d}\widetilde{\mathcal{Y}}_t = (a - b\widetilde{\mathcal{Y}}_t)\,\mathrm{d}t + \sigma\sqrt{\widetilde{\mathcal{Y}}_t}\,\mathrm{d}\widetilde{\mathcal{W}}_t, \qquad t \geq 0, \qquad \text{with} \quad \widetilde{\mathcal{Y}}_0 = y_0,$$

  where $(\widetilde{\mathcal{W}}_t)_{t\geq 0}$ is a standard Wiener process,
- $\mathrm{e}^{bt}\widetilde{\widetilde{\mathcal{Y}}}_t \xrightarrow{\text{a.s.}} \widetilde{\widetilde{\mathcal{V}}}$ as $t \to \infty$, where $(\widetilde{\widetilde{\mathcal{Y}}}_t)_{t\geq 0}$ is the jump-type supercritical CIR process

$$\mathrm{d}\widetilde{\widetilde{\mathcal{Y}}}_t = -b\widetilde{\widetilde{\mathcal{Y}}}_t\,\mathrm{d}t + \sigma\sqrt{\widetilde{\widetilde{\mathcal{Y}}}_t}\,\mathrm{d}\widetilde{\widetilde{\mathcal{W}}}_t + \mathrm{d}J_t, \qquad t \geq 0, \qquad \text{with} \quad \widetilde{\widetilde{\mathcal{Y}}}_0 = 0,$$

  where $(\widetilde{\widetilde{\mathcal{W}}}_t)_{t\geq 0}$ is a standard Wiener process indep. of $\widetilde{\mathcal{W}}$.
- $\widetilde{\mathcal{V}} \stackrel{\mathcal{L}}{=} \mathcal{Z}_{-\frac{1}{b}}$, where $\mathrm{d}\mathcal{Z}_t = a\,\mathrm{d}t + \sigma\sqrt{\mathcal{Z}_t}\,\mathrm{d}\mathcal{W}_t$, $t \geq 0$ with $\mathcal{Z}_0 = y_0$.

A proof is based on

- the decomposition

$$e^{-bT/2}(\widehat{b}_T - b) = -\sigma \frac{e^{bT/2}\int_0^T \sqrt{Y_s}\, dW_s}{e^{bT}\int_0^T Y_s\, ds}, \qquad T > 0.$$

- there exists a non-negative random variable V such that

$$e^{bT} Y_T \xrightarrow{\text{a.s.}} V \qquad \text{and} \qquad e^{bT}\int_0^T Y_u\, du \xrightarrow{\text{a.s.}} -\frac{V}{b} \qquad \text{as} \quad T \to \infty,$$

  following from submartingale convergence theorem applied to $(e^{bT} Y_T)_{t\geq 0}$, and from integral Kronecker lemma.

- positivity of V following from the absolute continuity of $\widetilde{\mathcal{V}}$ due to $\widetilde{\mathcal{V}} \stackrel{\mathcal{L}}{=} \mathcal{Z}_{-\frac{1}{b}}$.
- van Zanten's theorem for continuous local martingales.
- SLLN for Lévy processes: $\frac{J_T}{T} \xrightarrow{\text{a.s.}} \mathbb{E}(J_1) = \int_0^\infty z\, m(dz)$ as $T \to \infty$.

Remarks on the limit theorems

(i) In the subcritical case, the limit distribution of $\sqrt{T}(\widehat{b}_T - b)$, and in the critical case, the limit distribution of $T(\widehat{b}_T - b)$, does not depend on the intial value y_0.

But, in the supercritical case, the limit law of $e^{-bT/2}(\widehat{b}_T - b)$ does depend on the initial value y_0.

(ii) Unified theory: a common (random) normalization for the MLE $\widehat{b}_T$ to have a non-trivial limit in all cases.

Namely, for all $b \in \mathbb{R}$,

$$\frac{1}{\sigma}\left(\int_0^T Y_s \, ds\right)^{1/2}(\widehat{b}_T - b) \quad \text{converges in distribution as} \quad T \to \infty,$$

and the limit distribution is standard normal for the non-critical cases, while it is non-normal for the critical case.

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An α-stable CIR process

$$\boxed{\mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}W_t + \delta\sqrt[\alpha]{Y_{t-}}\,\mathrm{d}L_t, \qquad t \geq 0,}$$

where Y_0 is an a.s. non-negative initial value, $a \geq 0$, $b \in \mathbb{R}$, $\sigma \geq 0$, $\delta > 0$, $\alpha \in (1, 2)$, $(W_t)_{t\geq 0}$ is a standard Wiener process, and $(L_t)_{t\geq 0}$ is a spectrally positive α-stable Lévy process with Lévy–Khintchine representation:

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta L_1}) = \exp\left\{\int_0^\infty (\mathrm{e}^{\mathrm{i}\theta z} - 1 - \mathrm{i}\theta z) C_\alpha z^{-1-\alpha}\,\mathrm{d}z\right\}, \qquad \theta \in \mathbb{R},$$

where $C_\alpha := (\alpha\Gamma(-\alpha))^{-1}$ and Γ is the Gamma function.

We suppose that Y_0, $(W_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$ are independent.

Note that the Lévy measure of L_1 is $C_\alpha z^{-1-\alpha}\mathbf{1}_{(\mathbf{0},\infty)}(\mathbf{z})\,\mathrm{d}\mathbf{z}$.

It will turn out that b can be interpreted as a growth rate.

Some recent work on α-stable CIR processes

Carr and Wu (2004): considered a stochastic process admitting the same infinitesimal generator in case of $\sigma = 0$.

Fu and Li (2010): existence of a pathwise unique non-negative strong solution (for more details, see later on).

Li and Ma (2015):

- ergodicity provided that $a > 0$ and $b > 0$.
- asymptotic behaviour of the conditional least squares estimator of the drift parameters (a, b) based on discrete time observations in case of $b > 0$ and $\sigma = 0$.

Jiao, Ma and Scotti (2017): applications for interest rate modelling and pricing.

Peng (2016): α-stable CIR process with restart $\rightsquigarrow$ *internet congestion*.

Aim

Recall that $\mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}W_t + \delta\sqrt[\alpha]{Y_{t-}}\,\mathrm{d}L_t, \quad t \geq 0.$

To study the asymptotic properties of the MLE of $b \in \mathbb{R}$ under the conditions:

- $a \geq 0, \quad \sigma > 0, \quad \delta > 0$ and $\alpha \in (1, 2)$ are known,
- known non-random initial value $y_0 \geq 0$: $\mathbb{P}(Y_0 = y_0) = 1$,
- based on continuous time observations $(Y_t)_{t \in [0,T]}$ with $T \in (0, \infty)$,
- sample size tends to ∞, i.e., $T \to \infty$.

Some properties of α-stable CIR processes

Proposition. Let η_0 be a random variable independent of $(W_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$ satisfying $\mathbb{P}(\eta_0 \geq 0) = 1$ and $\mathbb{E}(\eta_0) < \infty$. Let $a \geq 0$, $b \in \mathbb{R}$, $\sigma \geq 0$ and $\delta > 0$. Then

- there is a pathwise unique strong solution $(Y_t)_{t\geq 0}$ such that $\mathbb{P}(Y_0 = \eta_0) = 1$ and $\mathbb{P}(Y_t \geq 0 \text{ for all } t \geq 0) = 1$.
- if, in addition, $\mathbb{P}(\eta_0 > 0) = 1$ or $a > 0$, then $\mathbb{P}\big(\int_0^t Y_s \, ds > 0\big) = 1$, $t > 0$.
- If, in addition, $\sigma \in \mathbb{R}_{++}$ and $a \geq \frac{\sigma^2}{2}$, then $\mathbb{P}\big(Y_t \in \mathbb{R}_{++} \text{ for all } t \in \mathbb{R}_{++}\big) = 1$.
- If, in addition, $\mathbb{P}(\eta_0 \in \mathbb{R}_{++}) = 1$, $a = 0$ and $b \in \mathbb{R}_+$, then $\mathbb{P}(\tau_0 < \infty) = 1$, where $\tau_0 := \inf\{s \in \mathbb{R}_+ : Y_s = 0\}$, and $\mathbb{P}(Y_t = 0 \text{ for all } t \geq \tau_0) = 1$.

- The process $(Y_t)_{t\in\mathbb{R}_+}$ is a CBI process having branching and immigration mechanisms

$$R(z) = \frac{\sigma^2}{2}z^2 + \frac{\delta^\alpha}{\alpha}z^\alpha + bz, \qquad F(z) = az, \quad z \in \mathbb{R}_+.$$

- For all $t \in \mathbb{R}_+$ and $y_0 \in \mathbb{R}_+$, the Laplace transform of Y_t takes the form

$$\mathbb{E}(\mathrm{e}^{-\lambda Y_t} \mid Y_0 = y_0) = \exp\left\{-y_0 v_t(\lambda) - \int_0^t F(v_s(\lambda))\,\mathrm{d}s\right\} \tag{9}$$

$$\frac{\partial}{\partial t}v_t(\lambda) = -R(v_t(\lambda)), \qquad v_0(\lambda) = \lambda. \tag{10}$$

then we have

$$\mathbb{E}(\mathrm{e}^{-\lambda Y_t} \mid Y_0 = y_0) = \exp\left\{-y_0 v_t(\lambda) + \int_\lambda^{v_t(\lambda)} \frac{F(z)}{R(z)}\,\mathrm{d}z\right\}. \tag{11}$$

Joint Laplace transform of Y_t and $\int_0^t Y_s \, ds$

Theorem. Let $a \geq 0$, $b \in \mathbb{R}$, $\sigma \geq 0$, and $\delta > 0$. Let $(Y_t)_{t \geq 0}$ be the unique strong solution satisfying $\mathbb{P}(Y_0 = y_0) = 1$ with some $y_0 \geq 0$. Then for all $u, v \leq 0$,

$$\mathbb{E}\left[\exp\left\{uY_t + v\int_0^t Y_s \, ds\right\}\right] = \exp\left\{\psi_{u,v}(t)y_0 + a\int_0^t \psi_{u,v}(s) \, ds\right\}$$

for $t \geq 0$, where $\psi_{u,v} : [0, \infty) \to (-\infty, 0]$ is the unique locally bounded solution to the Riccati type differential equation

$$\psi'_{u,v}(t) = \frac{\sigma^2}{2}\psi_{u,v}(t)^2 + \frac{\delta^\alpha}{\alpha}(-\psi_{u,v}(t))^\alpha - b\psi_{u,v}(t) + v, \quad t \geq 0$$

with $\psi_{u,v}(0) = u$.

Note that this Laplace transform is an exponentially affine function of the initial value $(y_0, 0)$.

Proof

By Theorem 4.10 in Keller-Ressel (2008):
$\left(Y_t, \int_0^t Y_s \, \mathrm{d}s\right)_{t\in\mathbb{R}_+}$ is a 2-dimensional CBI process with branching mechanism $\widetilde{R}(z_1, z_2) = (\widetilde{R}_1(z_1, z_2), \widetilde{R}_2(z_1, z_2))$, $z_1, z_2 \in \mathbb{R}_+$, with

$$\widetilde{R}_1(z_1, z_2) = R(z_1) - z_2, \qquad \widetilde{R}_2(z_1, z_2) = 0, \qquad z_1, z_2 \in \mathbb{R}_+,$$

and with immigration mechanism $\widetilde{F}(z_1, z_2) = F(z_1)$, $z_1, z_2 \in \mathbb{R}_+$, where R and F have explicit expression. Then,

$$\begin{aligned}\mathbb{E}\left[\exp\left\{uY_t + v\int_0^t Y_s \, \mathrm{d}s\right\}\right] &= \exp\left\{y_0\psi_{u,v}(t)\right. \\ &\qquad \left.- \int_0^t \widetilde{F}(-\psi_{u,v}(s), -\varphi_{u,v}(s)) \, \mathrm{d}s\right\} \\ &= \exp\left\{y_0\psi_{u,v}(t) + a\int_0^t \psi_{u,v}(s) \, \mathrm{d}s\right\}.\end{aligned}$$

Stationarity and ergodicity

Theorem. Let $a \in \mathbb{R}_+$, $b \in \mathbb{R}_+$, $\sigma \in \mathbb{R}_+$, and $\delta \in \mathbb{R}_{++}$.

1. Then $(Y_t)_{t \in \mathbb{R}_+}$ converges in law to its unique stationary distribution π having Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda y}\,\pi(\mathrm{d}y) = \exp\left\{ -\int_0^\lambda \frac{F(x)}{R(x)}\,\mathrm{d}x \right\} = \exp\left\{ -\int_0^\lambda \frac{ax}{\frac{\sigma^2}{2}x^2 + \frac{\delta^\alpha}{\alpha}x^\alpha + bx}\,\mathrm{d}x \right\}$$

2. If, in addition, $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}_{++}$, then the process $(Y_t)_{t \in \mathbb{R}_+}$ is exponentially ergodic, i.e., there exist constants $C \in \mathbb{R}_{++}$ and $D \in \mathbb{R}_{++}$ such that

$$\|\mathbb{P}_{Y_t \mid Y_0 = y} - \pi\|_{\mathrm{TV}} \leq C(y+1)\mathrm{e}^{-Dt}, \qquad t \in \mathbb{R}_+, \qquad y \in \mathbb{R}_+.$$

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Grigelionis representation

$$Y_t = Y_0 + \int_0^t (a - bY_u)\,\mathrm{d}u + \int_0^t \sigma\sqrt{Y_u}\,\mathrm{d}W_u + \gamma\delta\int_0^t \sqrt[\alpha]{Y_{u-}}\,\mathrm{d}u$$
$$+ \delta\int_0^t\int_{\mathbb{R}} \sqrt[\alpha]{Y_{u-}}h(z)\,\widetilde{\mu}^L(\mathrm{d}u,\mathrm{d}z) + \delta\int_0^t\int_{\mathbb{R}} \sqrt[\alpha]{Y_{u-}}(z - h(z))\,\mu^L(\mathrm{d}u,\mathrm{d}z)$$

The Grigelionis form of $(Y_t)_{t\in\mathbb{R}_+}$ takes the form

$$Y_t = Y_0 + \int_0^t (a - bY_u + \gamma\delta\sqrt[\alpha]{Y_u})\mathrm{d}u + \int_0^t \Big(\int_{\mathbb{R}} (h(z\delta\sqrt[\alpha]{Y_u}) - \delta\sqrt[\alpha]{Y_u}h(z))m(\mathrm{c}$$
$$+ \sigma\int_0^t \sqrt{Y_u}\,\mathrm{d}W_u + \int_0^t\int_{\mathbb{R}} h(z\delta\sqrt[\alpha]{Y_{u-}})\,\widetilde{\mu}^L(\mathrm{d}u,\mathrm{d}z)$$
$$+ \int_0^t\int_{\mathbb{R}} (z\delta\sqrt[\alpha]{Y_{u-}} - h(z\delta\sqrt[\alpha]{Y_{u-}}))\,\mu^L(\mathrm{d}u,\mathrm{d}z) \tag{12}$$

for $t\in\mathbb{R}_+$, where $h:\mathbb{R}\to[-1,1]$, $h(z) := z\mathbf{1}_{[-1,1]}(\mathbf{z})$, $z\in\mathbb{R}$.

Characteristic triplet

Consequently, under the probability measure $\mathbb{P}_b$ the canonical process $(\eta_t)_{t\in\mathbb{R}_+}$ is a semimartingale with characteristics $(B^{(b)}, C, \nu)$, where

$$B_t^{(b)} = \int_0^t \left(a - b\eta_u + \gamma\delta\sqrt[\alpha]{\eta_u} + \int_{\mathbb{R}} (h(z\delta\sqrt[\alpha]{\eta_u}) - h(z)\delta\sqrt[\alpha]{\eta_u})\, m(\mathrm{d}z) \right) \mathrm{d}u$$

$$C_t = \int_0^t (\sigma\sqrt{\eta_u})^2\, \mathrm{d}u = \sigma^2 \int_0^t \eta_u\, \mathrm{d}u, \qquad t \in \mathbb{R}_+,$$

$$\nu(\mathrm{d}t, \mathrm{d}y) = K(\eta_t, \mathrm{d}y)\, \mathrm{d}t$$

with the Borel transition kernel K from $\mathbb{R}_+ \times \mathbb{R}$ into $\mathbb{R}$ given by

$$K(y, R) := \int_{\mathbb{R}} \mathbf{1}_{\mathbf{R}\setminus\{\mathbf{0}\}}(\mathbf{z}\delta\sqrt[\alpha]{y})\, \mathbf{m}(\mathrm{d}\mathbf{z}) \qquad \text{for} \quad y \in \mathbb{R}_+ \quad \text{and} \quad R \in \mathcal{B}(\mathbb{R})$$

with $m(\mathrm{d}z) = C_\alpha z^{-1-\alpha}\mathbf{1}_{(\mathbf{0},\infty)}(\mathbf{z})\, \mathrm{d}\mathbf{z}$.

Likelihhod ratio

Consequently, for all $b, \widetilde{b} \in \mathbb{R}$,

$$B_t^{(b)} - B_t^{(\widetilde{b})} = -(b - \widetilde{b}) \int_0^t \eta_u \, \mathrm{d}u = \int_0^t c_u \beta_u^{(\widetilde{b}, b)} \, \mathrm{d}F_u .$$

Recall that by Jacod and Shiryaev (2003), we have

$$\frac{\mathrm{d}\mathbb{P}_{b,T}}{\mathrm{d}\mathbb{P}_{\widetilde{b},T}}(\eta) = \exp\left\{ \int_0^T \beta_u^{(\widetilde{b},b)} \, \mathrm{d}(\eta^{\mathrm{cont}})_u^{(\widetilde{b})} - \frac{1}{2} \int_0^T \left(\beta_u^{(\widetilde{b},b)}\right)^2 c_u \, \mathrm{d}u \right\},$$

Hence,

$$\log\left(\frac{\mathrm{d}\mathbb{P}_{b,T}}{\mathrm{d}\mathbb{P}_{\widetilde{b},T}}(\widetilde{Y})\right) = -\frac{b - \widetilde{b}}{\sigma^2} \int_0^T (\mathrm{d}\widetilde{Y}_u - \delta \sqrt[\alpha]{\widetilde{Y}_{u-}} \mathrm{d}L_u) + \frac{b - \widetilde{b}}{\sigma^2} \int_0^T a \, \mathrm{d}u$$

$$- \frac{b^2 - \widetilde{b}^2}{2\sigma^2} \int_0^T \widetilde{Y}_u \, \mathrm{d}u,$$

Existence and uniqueness of MLE

Proposition. Let $a \geq 0$, $b \in \mathbb{R}$, $\sigma > 0$, $\delta > 0$, and $y_0 \geq 0$. If $a > 0$ or $y_0 > 0$, then for each $T > 0$, there exists a unique MLE $\widehat{b}_T$ of b a.s. having the form

$$\widehat{b}_T = -\frac{Y_T - y_0 - aT - \delta \int_0^T \sqrt[\alpha]{Y_{u-}}\, \mathrm{d}L_u}{\int_0^T Y_s\, \mathrm{d}s},$$

provided that $\int_0^T Y_s\, \mathrm{d}s > 0$ (which holds a.s.).

Task: using the explicit forms above, let us describe the asymptotics of $\widehat{b}_T$ as $T \to \infty$.

Asymptotics of MLE: subcritical case $(b > 0)$

Theorem. Let $a > 0$, $b > 0$, $\sigma > 0$, $\delta > 0$, and $\mathbb{P}(Y_0 = y_0) = 1$ with some $y_0 \geq 0$. Then the MLE $\widehat{b}_T$ of b is strongly consistent and asymptotically normal, i.e.,
$\widehat{b}_T \xrightarrow{\text{a.s.}} b$ as $T \to \infty$, and

$$\sqrt{T}(\widehat{b}_T - b) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\sigma^2 b}{a}\right) \qquad \text{as} \quad T \to \infty.$$

With a random scaling,

$$\frac{1}{\sigma}\left(\int_0^T Y_s \, \mathrm{d}s\right)^{1/2}(\widehat{b}_T - b) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \qquad \text{as} \quad T \to \infty.$$

Asymptotics of MLE: supercritical case $(b < 0)$

Theorem. Let $a > 0$, $b < 0$, $\sigma > 0$, $\delta > 0$ and $\mathbb{P}(Y_0 = y_0) = 1$ with some $y_0 \geq 0$. Then $\widehat{b}_T$ is strongly consistent, and asymptotically mixed normal, namely

$$e^{-bT/2}(\widehat{b}_T - b) \xrightarrow{\mathcal{L}} \sigma Z \left(-\frac{V}{b}\right)^{-1/2} \qquad \text{as} \quad T \to \infty,$$

where V is a positive r. v. described by its Laplace transform and Z is a standard normally distributed r. v., independent of V.

With a random scaling, we have

$$\frac{1}{\sigma}\left(\int_0^T Y_s \, ds\right)^{1/2}(\widehat{b}_T - b) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \qquad \text{as} \quad T \to \infty.$$

Laplace transform of V

For all $u \leq 0$,

$$\mathbb{E}(e^{uV}) = \exp\left\{ y_0 \psi_u^* + \int_0^{-\psi_u^*} \frac{F(z)}{R(z)}\, dz \right\},$$

where

- $F(z) = az, \quad z \geq 0,$ and

$$R(z) = \frac{\sigma^2}{2} z^2 + \frac{\delta^\alpha}{\alpha} z^\alpha + bz, \qquad z \geq 0,$$

- $\psi_u^* := \lim_{t\to\infty} \psi_{ue^{bt},0}(t)$. Especially, $\psi_0^* = 0$.

Possible future research questions for this model

For the α-stable CIR process

$$\boxed{\mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}W_t + \delta\sqrt[\alpha]{Y_{t-}}\,\mathrm{d}L_t, \qquad t \geq 0,}$$

one could investigate

- the asymptotics of the MLE $\widehat{b}_T - b$ as $T \to \infty$ in the critical case $(b = 0)$. Open problem.
- the MLE of a supposing that b is known based on continuous time observations. For this, e.g., we should find the limit behavior of $\int_0^t \frac{1}{Y_s}\,\mathrm{d}s$ as $t \to \infty$.
- the MLE of (a, b) based on continuous time observations.
- estimation of α (some work has already been started by Jiao, Ma and Scotti).

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Introduction to Wishart processes

- Wishart processes have been first introduced by Bru [91].
- We consider the affine diffusion on $\mathcal{S}_d^+$ solution to

$$\begin{cases} dX_t = \left[\alpha a^\top a + bX_t + X_t b^\top\right] dt + \sqrt{X}_t dW_t a + a^\top dW_t^\top \sqrt{X}_t, t > 0 \\ X_0 = x \in \mathcal{S}_d^+, \end{cases} \quad ($$

where $\alpha \geqslant d-1$, $a \in \mathcal{M}_d$, $b \in \mathcal{M}_d$.

- It has a unique strong solution when $\alpha \geqslant d+1$ and a unique weak solution when $\alpha \geqslant d-1$.
- When $d=1$, Wishart processes are known as CIR processes.
- We denote by $WIS_d(x, \alpha, b, a)$ the law of $(X_t, t \geqslant 0)$.

$$WIS_d(x, \alpha, b, a) \underset{law}{=} WIS_d(x, \alpha, b, \sqrt{a^\top a}),$$

How to get rid of the parameter $a^\top a$?

- We follow the theory developed in the books by Liptser and Shiryaev [74] and Kutoyants [04] and assume that we observe the full path $(X_t, t \in [0, T])$ up to time $T > 0$.

- It is worth noticing that once we observe the path $(X_t, t \in [0, T])$, the parameter $a^\top a$ is known.

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- We follow the theory developed in the books by Liptser and Shiryaev [74] and Kutoyants [04] and assume that we observe the full path $(X_t, t \in [0, T])$ up to time $T > 0$.
- It is worth noticing that once we observe the path $(X_t, t \in [0, T])$, the parameter $a^\top a$ is known.
- For $i, j, k, l \in \{1, \cdots, d\}$, $\langle X_{i,j}, X_{k,l} \rangle_T$ is equal to

$$\int_0^T (a^\top a)_{j,l}(X_s)_{i,k} + (a^\top a)_{j,k}(X_s)_{i,l} + (a^\top a)_{i,l}(X_s)_{j,k} + (a^\top a)_{i,k}(X_s)_{j,l} ds.$$

How to get rid of the parameter $a^\top a$?

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$$\begin{cases} (a^\top a)_{i,i} = & \frac{1}{4}\langle X_{i,i}\rangle_T \left(\int_0^T (X_s)_{i,i} ds \right)^{-1}, \\ (a^\top a)_{i,j} = & \left(\frac{1}{2}\langle X_{i,j}, X_{i,i}\rangle_T - (a^\top a)_{i,i} \int_0^T (X_s)_{i,j} ds \right) \left(\int_0^T (X_s)_{i,i} ds \right)^{-1}, \end{cases}$$

How to get rid of the parameter $a^\top a$?

- Let us assume $a^\top a \in \mathcal{S}_d^{+,*}$.
- Then, according to Ahdida and Alfonsi [13] we have

$$Y_t := (a^\top)^{-1} X_t \, a^{-1} \underset{law}{=} WIS_d\left((a^\top)^{-1} x a^{-1}, \alpha, (a^\top)^{-1} b a^\top, I_d\right)$$

- It is sufficient to focus on the estimation of the parameter $\theta = (\alpha, b)$ with $a = I_d$.
- We denote by $\mathbb{P}_\theta$ the original probability measure under which X satisfies

$$dX_t = \left[\alpha I_d + bX_t + X_t b^\top\right] dt + \sqrt{X}_t dW_t + dW_t^\top \sqrt{X}_t.$$

- We will assume for the joint estimation of α and b that

$$\alpha \geqslant d+1 \text{ and } x \in \mathcal{S}_d^{+,*}.$$

Change of probability measure

- For $\theta_0 = (\alpha_0, 0)$ with $\alpha_0 \geqslant d+1$, we have according to Mayerhofer [12]

$$\frac{d\mathbb{P}_{\theta_0,T}}{d\mathbb{P}_{\theta,T}} := \exp\left(\int_0^T \mathrm{Tr}[H_s dW_s] - \frac{1}{2}\int_0^T \mathrm{Tr}[H_s H_s^\top] ds\right), \text{ with}$$

$$H_t = \frac{\alpha_0 - \alpha}{2}(\sqrt{X_t})^{-1} - b\sqrt{X_t}$$

with $\tilde{W}_t = W_t - \int_0^t H_s^\top ds$ is a $d \times d$-B. m. under $\mathbb{P}_{\theta_0,T}$.

- Then, X follows a Wishart process with parameter θ_0 under $\mathbb{P}_{\theta_0}$.

$$dX_t = \alpha_0 I_d dt + \sqrt{X}_t d\tilde{W}_t + d\tilde{W}_t^\top \sqrt{X}_t,$$

Likelihood Ratio

Theorem 1

We have $$\frac{d\mathbb{P}_{\theta,T}}{d\mathbb{P}_{\theta_0,T}} \in \mathcal{F}_T^X \iff b \in \mathcal{S}_d$$

and we have in this case $L_T^{\theta,\theta_0} = \frac{d\mathbb{P}_{\theta,T}}{d\mathbb{P}_{\theta_0,T}}$ *where*

$$\begin{aligned} L_T^{\theta,\theta_0} = \exp\Big(& \frac{\alpha-\alpha_0}{4}\log\left(\frac{\det[X_T]}{\det[x]}\right) + \frac{\mathrm{Tr}[bX_T]-\mathrm{Tr}[bx]}{2} \\ & - \frac{\alpha-\alpha_0}{4}\big(\frac{\alpha+\alpha_0}{2} - 1 - d\big)\int_0^T \mathrm{Tr}[X_s^{-1}]ds \\ & - \frac{\alpha T}{2}\mathrm{Tr}[b] - \frac{1}{2}\int_0^T \mathrm{Tr}[b^2 X_s]ds\Big). \end{aligned}$$

Likelihood Ratio

- For $b \notin \mathcal{S}_d$, the likelihood is then defined by (see Lipster and Shiryaev [01])

$$L_T^{\theta,\theta_0} = \frac{1}{\mathbb{E}\left[\exp\left(\int_0^T \mathrm{Tr}[H_s dW_s] - \frac{1}{2}\int_0^T \mathrm{Tr}[H_s H_s^\top]ds\right) \Big| \mathcal{F}_T^X\right]}, \quad (14)$$

where $(\mathcal{F}_t^X)_{t \geqslant 0}$ denote the filtration generated by the process X.

Likelihood Ratio

Theorem 2

For $X \in \mathcal{S}_d^{+,*}$, let $\mathcal{L}_X : \mathcal{S}_d \to \mathcal{S}_d$ defined by $\mathcal{L}_X(Y) = XY + YX$. It is invertible, and the likelihood of ratio is given by

$$
\begin{aligned}
L_T^{\theta,\theta_0} = \exp\Bigg\{ & \frac{\alpha-\alpha_0}{4}\log\left(\frac{\det[X_T]}{\det[x]}\right) - \frac{\alpha T}{2}\mathrm{Tr}[b] \\
& - \frac{\alpha-\alpha_0}{4}\Big(\frac{\alpha+\alpha_0}{2} - 1 - d\Big)\int_0^T \mathrm{Tr}[X_s^{-1}]ds \\
& + \frac{1}{2}\int_0^T \mathrm{Tr}\left[\mathcal{L}_{X_t}^{-1}\left(bX_t + X_t b^\top\right) dX_t\right] \\
& - \frac{1}{4}\int_0^T \mathrm{Tr}\left[\mathcal{L}_{X_t}^{-1}\left(bX_t + X_t b^\top\right)(bX_t + X_t b^\top)\right] dt\Bigg\}.
\end{aligned}
$$

Notations

- For simplicity, we assume $b \in \mathcal{S}_d$.
- We introduce the following shorthand notation

$$R_T := \int_0^T X_s ds, \, Q_T := \left(\int_0^T \mathrm{Tr}[X_s^{-1}] ds \right)^{-1}, Z_T := \log \left(\frac{\det[X_T]}{\det[x]} \right),$$

  - Note that Q_T and Z_T are defined only for $\alpha \geqslant d+1$ while R_T is defined for $\alpha \geqslant d-1$ and belongs almost surely to $\mathcal{S}_d^{+,*}$.
  - For $X \in \mathcal{S}_d$ and $a \in \mathbb{R}$, we define the linear applications

$$\begin{aligned} \mathcal{L}_X : \ & \mathcal{S}_d \to \mathcal{S}_d \\ & Y \mapsto YX + XY \end{aligned} \quad \text{and } \begin{aligned} \mathcal{L}_{X,a} : \ & \mathcal{S}_d \to \mathcal{S}_d \\ & Y \mapsto YX + XY - 2a\mathrm{Tr}[Y] I_d. \end{aligned}$$

Maximum Likelihood Estimator

The MLE $\hat{\theta}_T = (\hat{\alpha}_T, \hat{b}_T)$ is then characterized by the following equations:

$$\begin{cases} \hat{\alpha}_T & = 1 + d \\ & +Q_T\Big(Z_T - 2T\mathrm{Tr}\big[\mathcal{L}^{-1}_{R_T,T^2Q_T}\left(X_T - x - T\left[Q_T Z_T + 1 + d\right] I_d\right)\big]\Big) \\ \hat{b}_T & = \mathcal{L}^{-1}_{R_T,T^2Q_T}\left(X_T - x - T\left[Q_T Z_T + 1 + d\right] I_d\right). \end{cases}$$

Ergodicity

- If $-b \in \mathcal{S}_d^{+,*}$, then $X_t \overset{law}{\underset{t\to+\infty}{\Longrightarrow}} X_\infty \sim WIS_d(0, \alpha, 0, \sqrt{b^{-1}}; 1/2)$, $\forall x \in \mathcal{S}_d^+$
- This is the unique stationary law which is thus extremal, and we know by Stroock ('93) that it is then ergodic. That is

$$\frac{R_T}{T} \xrightarrow{a.s.} \overline{R}_\infty := \mathbb{E}_\theta(X_\infty) = -\frac{\alpha}{2} b^{-1} \in \mathcal{S}_d^{+,*}, \quad \text{as } T \to +\infty.$$

and when $\alpha \geqslant d+1$,

$$TQ_T \xrightarrow{a.s.} \overline{Q}_\infty = \frac{1}{\mathbb{E}_\theta(\mathrm{Tr}[X_\infty^{-1}])} = \frac{\alpha - (1+d)}{2\mathrm{Tr}[-b]}, \quad \text{as } T \to +\infty.$$

Subcritical case $-b \in S_d^{+,*}$, for $\alpha > d+1$

Theorem 3

Assume that $-b \in S_d^{+,*}$ *and* $\alpha > d+1$. *Under* $\mathbb{P}_\theta$,

$$\left(\sqrt{T}(\hat{b}_T - b, \hat{\alpha}_T - \alpha)\right) \underset{T \to +\infty}{\overset{law}{\Longrightarrow}} (\mathbf{G}, H) \in \mathcal{S}_d \times \mathbb{R},$$

where for $c, \lambda \in \mathcal{S}_d \times \mathbb{R}$,

$$\mathbb{E}_\theta \left[\exp\left(\mathrm{Tr}[c\mathbf{G}] + \lambda H\right)\right] = \exp\left(\frac{2\overline{Q}_\infty \lambda^2}{1-\overline{Q}_\infty \mathrm{Tr}[\overline{R}_\infty^{-1}]} - \frac{2\overline{Q}_\infty \lambda}{1-\overline{Q}_\infty \mathrm{Tr}[\overline{R}_\infty^{-1}]} \mathrm{Tr}[c\overline{R}_\infty^{-1}] + \mathrm{Tr}[c\mathcal{L}^{-1}_{\overline{R}_\infty, \overline{Q}_\infty}(c)]\right)$$

Subcritical case $-b \in S_d^{+,*}$, for $\alpha = d + 1$

Theorem 4

Assume $-b \in S_d^{+,*}$ *and* $\alpha = d + 1$. *Then, under* $\mathbb{P}_\theta$,

$$\left(\sqrt{T}(\hat{b}_T - b), T(\hat{\alpha}_T - \alpha)\right) \underset{T \to +\infty}{\overset{law}{\Longrightarrow}} \left(\mathbf{G}, -2\tau^{-1}_{-\mathrm{Tr}[b]} \mathrm{Tr}[b + \mathbf{G}]\right),$$

where $\tau_a = \inf\{t \geq 0,\ B_t = a\}$ *with* $(B_t)_{t \geq 0}$ *a given one-dimensional standard Brownian motion and* $\mathbf{G}$ *is a Gaussian vector independent of* B *such that*
$\mathbb{E}_\theta\left[\exp\left(\mathrm{Tr}[c\mathbf{G}]\right)\right] = \exp\left(\mathrm{Tr}[c\mathcal{L}^{-1}_{R_\infty}(c)]\right)$, $c \in \mathcal{S}_d$.

Critical case $b = 0$, for $\alpha > d + 1$

Theorem 5

$$\left(T(\hat{b}_T - b), \sqrt{\log(T)}(\hat{\alpha}_T - \alpha)\right) \underset{T \to +\infty}{\overset{law}{\Longrightarrow}} \left(\mathcal{L}^{-1}_{R_1^0}\left(X_1^0 - \alpha I_d\right), 2\sqrt{\frac{\alpha - (d+1)}{d}} G\right)$$

where $X_t^0 = \alpha t I_d + \int_0^t \sqrt{X_s^0} dW_s + dW_s^\top \sqrt{X_s^0}$, $R_t^0 = \int_0^t X_s^0 ds$ *and* $G \sim \mathcal{N}(0,1)$ *is an independent standard Normal variable.*

Critical case $b = 0$, for $\alpha = d + 1$

Theorem 6

Assume that $b = 0$ and $\alpha = d + 1$. Then, under $\mathbb{P}_\theta$

$$\left(T(\hat{b}_T - b), \log(T)(\hat{\alpha}_T - \alpha)\right) \underset{T\to+\infty}{\overset{law}{\Longrightarrow}} \left(\mathcal{L}_{R_1^0}^{-1}\left(X_1^0 - \alpha I_d\right), \frac{4}{d\tau_1}\right),$$

where $X_t^0 = \alpha t I_d + \int_0^t \sqrt{X_s^0}dW_s + dW_s^\top \sqrt{X_s^0}$, $R_t^0 = \int_0^t X_s^0 ds$ and $\tau_1 = \inf\{t \geqslant 0, B_t = 1\}$ where B is a standard Brownian motion independent from W.

MLE of $\theta = (\alpha, b)$ when $b = b_0 I_d$, $b_0 > 0$ and $\alpha \geqslant d - 1$

Theorem 7

$$\exp(b_0 T)(\hat{b}_T - b) \underset{T \to +\infty}{\overset{law}{\Longrightarrow}} \mathcal{L}_X^{-1}\left(\sqrt{X}\tilde{\mathbf{G}} + \tilde{\mathbf{G}}\sqrt{X}\right)$$

where $X \sim WIS_d\left(\frac{x}{2b_0}, \alpha, 0, I_d; \frac{1}{4b_0^2}\right)$ and $\tilde{\mathbf{G}}$ is an independent d-square matrix whose elements are independent standard Normal variables.

- When $b = b_0 I_d$ with $b_0 \geqslant 0$ the convergence of all the matrix terms occurs at the same speed, namely $1/\sqrt{T}$ for the ergodic case, $1/T$ for $b = 0$ and $e^{-b_0 T}$ when $b_0 > 0$.
- In the other cases, there is no such a simple scalar rescaling because of the different matrix products.

The Laplace transform of (X_T, R_T)

Theorem 8

Let $\alpha \geqslant d-1$, $x \in \mathcal{S}_d^+$, $b \in \mathcal{S}_d$ *and* $X \sim WIS_d(x, \alpha, b, I_d)$. *Let* $v, w \in \mathcal{S}_d$ *be such that* $\exists m \in \mathcal{S}_d,\ \frac{v}{2} - mb - bm - 2m^2 \in \mathcal{S}_d^+$ *and* $\frac{w}{2} + m \in \mathcal{S}_d^+$. *Then,*

$$\mathbb{E}\Big[\exp\Big(-\frac{1}{2}\mathrm{Tr}[wX_t] - \frac{1}{2}\mathrm{Tr}[vR_t]\Big)\Big] = \frac{\exp\big(-\frac{\alpha}{2}\mathrm{Tr}[b]t\big)}{\det[V_{v,w}(t)]^{\frac{\alpha}{2}}} \exp\Big(-\frac{1}{2}\mathrm{Tr}\big[(V'_{v,w}(t)V_{v,w}(t)^{-1} + b)x\big]\Big),$$

where $V_{v,w}(t) = (\sqrt{\tilde{v}})^{-1}\sinh(\sqrt{\tilde{v}}t)\tilde{w} + \cosh(\sqrt{\tilde{v}}t)$ *with* $\tilde{v} = v + b^2 \in \mathcal{S}_d^{+,*}$, *we have*

This talk is based on:

- ALFONSI, A., KEBAIER, A., REY, C.,
  Maximum likelihood estimation for Wishart processes
  Stochastic Process. Appl. (2016)
- BARCZY, M., BEN ALAYA, M., KEBAIER, A., PAP, G.,
  Asymptotic properties of maximum likelihood estimator for the growth rate for a jump-type CIR process based on continuous time observations. **Stochastic Process. Appl.** (2018).
- BARCZY, M., BEN ALAYA, M., KEBAIER, A., PAP, G.,
  Asymptotic properties of maximum likelihood estimator for the growth rate of a stable CIR process based on continuous time observations. **Statistics**, to appear (2019).

Thank you for your attention!