First Estimates on the Parameter based on LSE and Ergodicity

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1 Notes and calculations

Recall our model is given as follows,

$$\begin{aligned} dV_t &= -\theta_t V_t \; dt + \sqrt{2\theta_t \alpha (V_t + p_t)(1 - V_t - p_t)} \; dW_t \\ V_0 &= v_0 \end{aligned} \tag{1}$$

where $V_t = X_t - p_t$ and $\theta_t = \max\left(\theta_0, \frac{|\dot{p}_t|}{\min(p_t, 1-p_t)}\right)$. We write Euler's scheme for the model,

this an approx, Euler Maruyama
unless dt between obs is small. We also have model error

$$\begin{aligned} V_{j,i+1} - V_{j,i} &= -\theta_{j,i} V_{j,i} \; dt + \sqrt{2\theta_{j,i} \alpha (V_{j,i} + p_{j,i})(1 - V_{j,i} - p_{j,i})} \; dW_{j,i} \\ V_0 &= v_0 \end{aligned} \tag{2}$$

\Delta W_ji

Note that in this document j represents the path index and i represents the transition index inside that path.

Re-arranging,

\Delta W_ji

\Delta t

$$\begin{aligned} dW_{j,i} &= \frac{V_{j,i+1} - V_{j,i} + -\theta_{j,i} V_{j,i} \; dt}{\sqrt{2\theta_{j,i} \alpha (V_{j,i} + p_{j,i})(1 - V_{j,i} - p_{j,i})}} \\ &\cancel{V_0 = v_0} \end{aligned} \tag{3}$$

and variance \Delta t

We know that $dW_{j,i}$ is normally distributed with mean zero. Thus, we can obtain the following variance, which is also called the contrast function.

$$\rho(\theta_{j,i}, \alpha | \{V\}_{i,j}) = \sum_{j,i} \left(\frac{V_{j,i+1} - V_{j,i} + -\theta_{j,i} V_{j,i} \; dt}{\sqrt{2\theta_{j,i} \alpha (V_{j,i} + p_{j,i})(1 - V_{j,i} - p_{j,i})}} \right)^2 \; -1 \tag{4}$$

and normalize

overlap between paths
for longer T prediction power, check this relation

is there asymptotic results for when dt

We can find a first estimate of θ_0 by choosing the one that minimizes the above contrast function,

$$\min_{\theta_0} \rho(\theta_t, \alpha | \{V\}_{i,j}) \tag{5}$$

which we can obtain by setting $\frac{\partial \rho}{\partial \theta_0} \equiv 0$. We proceed to find that,

$$\begin{aligned} \frac{\partial \rho}{\partial \theta_{j,i}} &= \frac{\Delta t V_{j,i}(\Delta t \theta_t V_{j,i} - V_{j,i} + V_{j,i+1})}{\alpha \theta_t(-p - V_{j,i} + 1)(p + V_{j,i})} - \frac{(\Delta t \theta_t V_{j,i} - V_{j,i} + V_{j,i+1})^2}{2\alpha \theta_t^2(-p - V_{j,i} + 1)(p + V_{j,i})} \\ &= -\frac{(\Delta t \theta_{j,i} V_{j,i} + V_{j,i} - V_{j,i+1})(V_{j,i}(\Delta t \theta_{j,i} - 1) + V_{j,i+1})}{2\alpha \theta_{j,i}^2 (p + V_{j,i} - 1)(p + V_{j,i})} \end{aligned} \tag{6}$$

Recall that $\theta_t = \max\left(\theta_0, \frac{|\dot{p}_t|}{\min(p_t, 1-p_t)}\right)$, then

$$\frac{\partial \theta_{j,i}}{\partial \theta_0} = \begin{cases} 1 & \text{if} \quad \theta_0 \geq \left(\frac{|\dot{p}_t|}{\min(p_t, 1-p_t)}\right) \\ 0 & \text{if} \quad \theta_0 < \left(\frac{|\dot{p}_t|}{\min(p_t, 1-p_t)}\right) \end{cases} \tag{7}$$

And we have that,

$$\frac{\partial \rho}{\partial \theta_0} = \sum_{j,i} \frac{\partial \rho}{\partial \theta_{j,i}} \frac{\partial \theta_{j,i}}{\partial \theta_0} \tag{8}$$

Setting $\frac{\partial \rho}{\partial \theta_0} \equiv 0$, we have

$$\sum_{j,i} \frac{\partial \rho}{\partial \theta_{j,i}} \frac{\partial \theta_{j,i}}{\partial \theta_0} \equiv 0 \tag{9}$$

The result is an equation in θ_0 to be solved algebriaclly and we pick the positive root as an estimate for θ_0. However, note that the expression requires us to know $\alpha\theta_0$ a priori in the denominator of (6). Next we show how to find an estimate of the product $\theta_{j,i}\alpha$. Assuming that we are in the ~~ergodic~~ regime, we have the following expression for one path,

$$\lim_{n\to\infty} \sum_{i=1}^{2^n} \left(V_{j,(i+1)t2^{-n}} - V_{j,it2^{-n}}\right)^2 = \int_0^t 2\theta_s \alpha (V_s + p_s)(1 - V_s - p_s)\, ds \tag{10}$$

Discritizing the above according to our discrete observations,

$$\sum_i (V_{j,i+1} - V_{j,i})^2 = \sum_i 2\theta_{j,i}\alpha(V_{j,i} + p_{j,i})(1 - V_{j,i} - p_{j,i})\Delta t \tag{11}$$

which can be solved for $\alpha\theta_0$ algebraically. Recall that $\theta_{j,i} = \max\left(\theta_0, \frac{|\dot{p}_{j,i}|}{\min(p_{j,i}, 1-p_{j,i})}\right)$. Then, we obtain the product $\alpha\theta_{j,i}$ which we use in estimating θ_0 by solving (9). Then, we trivially have $\alpha = \frac{\alpha\theta_0}{\theta_0}$

In conclusion, this seems to be a way to obtain first estimates on θ_0 and α in this two stage procedure by estimating the product $\theta_0\alpha$ then finding θ_0 knowing an estimate of the product $\theta_0\alpha$.