Lamperti Transform for the processes X and V

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SDE for X_t:

$$\mathrm{d}X_t = \left(\dot{p}_t - \theta_t(X_t - p_t)\right)\mathrm{d}t + \sqrt{2\theta_t\alpha X_t(1-X_t)}\,\mathrm{d}W_t \implies$$

$$\psi(X_t, t) = \int \frac{1}{\sqrt{2\theta_t\alpha u(1-u)}}\,\mathrm{d}u\bigg|_{u=X_t} = -\sqrt{\frac{2}{\alpha\theta_t}}\arcsin\left(\sqrt{1-X_t}\right).$$

SDE for V_t:

$$\mathrm{d}V_t = \underbrace{-\theta_t V_t}_{f(V_t,t)}\mathrm{d}t + \underbrace{\sqrt{2\theta_t\alpha(V_t+p_t)(1-V_t-p_t)}}_{\sigma(V_t,t)}\mathrm{d}W_t \implies$$

$$\psi(V_t,t) = \int \frac{1}{\sqrt{2\theta_t\alpha(u+p_t)(1-u-p_t)}}\,\mathrm{d}u\Bigg|_{u=V_t} = -\sqrt{\frac{2}{\alpha\theta_t}}\arcsin\left(\sqrt{1-V_t-p_t}\right).$$

We can see that for every $t=t^*$, the primitive function of $\frac{1}{\sigma(v,t^*)}$ is well defined for all $v\in\left[-p(t^*),1-p(t^*)\right]\subset[-1,1]$ (recall $V_t = X_t - p_t$, and $x\in[0,1]$).

SDE for V_t:

We define $\Psi(V_t,t) = \int_\varepsilon^u \frac{1}{\sigma(u,t)}\,\mathrm{d}u\Big|_{u=V_t}$ where ε is in the space of V_t (which depends on p_t).

We have that $V_t = X_t - p_t \iff V_t + p_t = X_t \implies 0 \le V_t + p_t \le 1 \iff -p_t \le V_t \le 1 - p_t$. Then, for each $t = t^*$, the space of the process is $[-p_{t^*}, 1 - p_{t^*}]$. Now, notice that for all $p_{t^*} \in [0,1]$, we have that $q \in [-p_{t^*}, 1 - p_{t^*}]$ if and only if $q = 0$.

We conclude that if we want a fix value for ε, necessarily we need to choose $\varepsilon = 0$.

We have that $\psi_v(V_t,t) = \Psi_v(V_t,t)$. However, $\psi_t(V_t,t) = \Psi_t(V_t,t)$ is in general no true, and $\Psi_t(V_t,t)$ has a more complicate expression.

If we choose $\varepsilon = 0$, we have that $\Psi_t(V_t,t) = \psi_t(V_t,t) + \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\mathrm{d}u}{\sigma(u,t)}\Big|_{u=\varepsilon=0}$.

SDE for V_t:

- $\Psi(V_t,t) = \psi(V_t,t) - \psi(\varepsilon,t)\Big|_{\varepsilon=0} =$
$\sqrt{\frac{2}{\alpha\theta_t}}\left[\arcsin(\sqrt{1-\varepsilon-p_t}) - \arcsin(\sqrt{1-V_t-p_t})\right]\Big|_{\varepsilon=0} =$
$\sqrt{\frac{2}{\alpha\theta_t}}\left[\arcsin(\sqrt{1-p_t}) - \arcsin(\sqrt{1-V_t-p_t})\right].$
- $\Psi_v(V_t,t) = \psi_v(V_t,t) = \frac{1}{\sigma(V_t,t)}.$
- $\Psi_{vv}(V_t,t) = \psi_{vv}(V_t,t) = \frac{\mathrm{d}}{\mathrm{d}v}\left[\frac{1}{\sigma(V_t,t)}\right] = -\frac{\sigma_v(V_t,t)}{\sigma^2(V_t,t)} = -\frac{1}{\sigma^2(V_t,t)}\cdot\sqrt{\frac{\alpha\theta_t}{2}}\frac{1-2V_t-2p_t}{\sqrt{(V_t+p_t)(1-V_t-p_t)}}.$
- $\psi_t(V_t,t) = \frac{\dot{p}_t}{\sqrt{2\alpha\theta_t}(V_t+p_t)(1-V_t-p_t)} + \frac{\alpha\dot{\theta}_t\arcsin(\sqrt{1-V_t-p_t})}{\sqrt{2}(\alpha\theta_t)^{3/2}}.$
- $\Psi_t(V_t,t) = \psi_t(V_t,t) - \psi_t(\varepsilon,t)\Big|_{\varepsilon=0} =$
$\frac{\dot{p}_t}{\sqrt{2\alpha\theta_t(V_t+p_t)(1-V_t-p_t)}} + \frac{\alpha\dot{\theta}_t\arcsin(\sqrt{1-V_t-p_t})}{\sqrt{2}(\alpha\theta_t)^{3/2}} - \frac{\dot{p}_t}{\sqrt{2\alpha\theta_t(0+p_t)(1-0-p_t)}} - \frac{\alpha\dot{\theta}_t\arcsin(\sqrt{1-0-p_t})}{\sqrt{2}(\alpha\theta_t)^{3/2}} =$
$\frac{1}{\sqrt{2\alpha\theta_t}}\left(\frac{\dot{p}_t}{\sqrt{(V_t+p_t)(1-V_t-p_t)}} + \frac{\alpha\dot{\theta}_t\arcsin(\sqrt{1-V_t-p_t})}{\alpha\theta_t} - \frac{\dot{p}_t}{\sqrt{(0+p_t)(1-0-p_t)}} - \frac{\alpha\dot{\theta}_t\arcsin(\sqrt{1-0-p_t})}{\alpha\theta_t}\right).$

SDE for $Z_t = \Psi(V_t, t)$: (Verified with Mathematica)

By Itô's lemma:

$$dZ_t = \left(\Psi_t + \Psi_v \cdot f + \frac{1}{2}\Psi_{vv} \cdot \sigma^2\right) dt + \Psi_v \cdot \sigma \, dW_t.$$

If we substitute the terms related with $\Psi(V_t, t)$:

$$dZ_t = \left[\frac{1}{\sqrt{2\alpha\theta_t}}\left(\frac{\dot{p}_t}{\sqrt{(V_t+p_t)(1-V_t-p_t)}} + \frac{\alpha\dot{\theta}_t \arcsin(\sqrt{1-V_t-p_t})}{\alpha\theta_t} - \frac{\dot{p}_t}{\sqrt{(p_t)(1-p_t)}} - \frac{\alpha\dot{\theta}_t \arcsin(\sqrt{1-p_t})}{\alpha\theta_t}\right)\right.$$
$$\left. - \frac{\theta_t V_t}{\sqrt{2\alpha\theta_t(V_t+p_t)(1-V_t-p_t)}} - \frac{1}{2}\sqrt{\frac{\alpha\theta_t}{2}}\frac{1-2V_t-2p_t}{\sqrt{(V_t+p_t)(1-V_t-p_t)}}\right] dt + 1 \cdot dW_t.$$

SDE for $Z_t = \psi(V_t, t)$: (Verified with Mathematica)

By Itô's lemma, if $\psi(v,t)$ is $C^2([-p_t, 1-p_t])$ for v and $C^1([0,T])$ for t, then:

$$dZ_t = \left(\psi_t + \psi_v \cdot f + \frac{1}{2}\psi_{vv} \cdot \sigma^2\right) dt + \psi_v \cdot \sigma \, dW_t.$$

If we substitute the terms related with $\psi(V_t, t)$:

$$dZ_t = \left[\frac{1}{\sqrt{2\alpha\theta_t}}\left(\frac{\dot{p}_t}{\sqrt{(V_t+p_t)(1-V_t-p_t)}} + \frac{\alpha\dot{\theta}_t \arcsin(\sqrt{1-V_t-p_t})}{\alpha\theta_t}\right)\right.$$
$$\left. - \frac{\theta_t V_t}{\sqrt{2\alpha\theta_t(V_t+p_t)(1-V_t-p_t)}} - \frac{1}{2}\sqrt{\frac{\alpha\theta_t}{2}}\frac{1-2V_t-2p_t}{\sqrt{(V_t+p_t)(1-V_t-p_t)}}\right] dt + 1 \cdot dW_t.$$

Recall $Z_t = -\sqrt{\frac{2}{\alpha\theta_t}} \arcsin\left(\sqrt{1-V_t-p_t}\right)$. Then, we have the next identities: **1)** $\sqrt{1-V_t-p_t} = \sin\left(-Z_t\sqrt{\frac{\alpha\theta_t}{2}}\right)$, **2)** $1-V_t-p_t = \sin^2\left(-Z_t\sqrt{\frac{\alpha\theta_t}{2}}\right)$, and **3)** $V_t+p_t = 1-\sin^2\left(-Z_t\sqrt{\frac{\alpha\theta_t}{2}}\right)$.

SDE for $Z_t = \psi(V_t, t)$:

$$
\begin{aligned}
dZ_t = \Bigg[& \frac{1}{\sqrt{2\alpha\theta_t}} \left(\frac{\dot{p}_t}{\sqrt{\left(1-\sin^2\left(-Z_t\sqrt{\frac{\alpha\theta_t}{2}}\right)\right)\left(\sin^2\left(-Z_t\sqrt{\frac{\alpha\theta_t}{2}}\right)\right)}} + \frac{\dot{\theta}_t\left(-Z_t\sqrt{\frac{\alpha\theta_t}{2}}\right)}{\theta_t} \right) \\
& - \frac{\theta_t\left(1-p_t-\sin^2\left(-Z_t\sqrt{\frac{\alpha\theta_t}{2}}\right)\right)}{\sqrt{2\alpha\theta_t\left(1-\sin^2\left(-Z_t\sqrt{\frac{\alpha\theta_t}{2}}\right)\right)\left(\sin^2\left(-Z_t\sqrt{\frac{\alpha\theta_t}{2}}\right)\right)}} \\
& - \frac{1}{2}\sqrt{\frac{\alpha\theta_t}{2}} \frac{1-2\left(1-\sin^2\left(-Z_t\sqrt{\frac{\alpha\theta_t}{2}}\right)\right)}{\sqrt{\left(1-\sin^2\left(-Z_t\sqrt{\frac{\alpha\theta_t}{2}}\right)\right)\left(\sin^2\left(-Z_t\sqrt{\frac{\alpha\theta_t}{2}}\right)\right)}} \Bigg] dt + 1 \cdot dW_t.
\end{aligned}
$$

Range for $Z_t = \psi(V_t, t)$:

We have that $Z_t = -\sqrt{\frac{2}{\alpha\theta_t}} \arcsin\left(\sqrt{1 - V_t - p_t}\right) = -\sqrt{\frac{2}{\alpha\theta_t}} \arcsin\left(\sqrt{1 - X_t}\right)$, where $X_t \in [0,1]$ almost sure. Then, we have that $Z_t \in \left[-\sqrt{\frac{2}{\alpha\theta_t}} \frac{\pi}{2}, 0\right] = \left[-\frac{\pi}{\sqrt{2\alpha\theta_t}}, 0\right]$, because $\arcsin\left([0,1]\right) = \left[0, \frac{\pi}{2}\right]$.

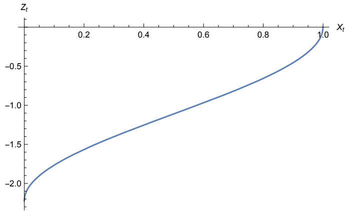


Figure 1: Plot of Z_t as a function of X_t from 0 to 1. We choose $\alpha = 1/2$, and $\theta_t = 1$. Notice that $Z_t(0) = -\frac{\pi}{\sqrt{2}}$, and $Z_t(1) = 0$.

SDE for $Z_t = \psi(V_t, t)$ with the c.o.v. $Y_t = Z_t\sqrt{\frac{\alpha\theta_t}{2}}$:

We have that $Z_t \in \left[-\frac{\pi}{\sqrt{2\alpha\theta_t}}, 0\right] \iff Y_t \in \left[-\frac{\pi}{2}, 0\right]$.

$$
\begin{aligned}
\mathrm{d}Z_t = \Bigg[& \frac{1}{\sqrt{2\alpha\theta_t}} \left(\frac{\dot{p}_t}{\sqrt{\left(1-\sin^2(Y_t)\right)\left(\sin^2(Y_t)\right)}} - \frac{\dot{\theta}_t Y_t}{\theta_t} \right) \\
& - \frac{\theta_t\left(1-p_t-\sin^2(Y_t)\right)}{\sqrt{2\alpha\theta_t\left(1-\sin^2(Y_t)\right)\left(\sin^2(Y_t)\right)}} \\
& - \frac{1}{2}\sqrt{\frac{\alpha\theta_t}{2}} \frac{1-2\left(1-\sin^2(Y_t)\right)}{\sqrt{\left(1-\sin^2(Y_t)\right)\left(\sin^2(Y_t)\right)}} \Bigg] \mathrm{d}t + 1\cdot \mathrm{d}W_t.
\end{aligned}
$$

SDE for $Z_t = \psi(V_t, t)$: Singularity in $Z_t = -\frac{\pi}{\sqrt{2\alpha\theta_t}}$, and $Z_t = 0$

We call $f_Z(Z_t, t)$ to the drift of Z_t. We define $Y_t = Z_t\sqrt{\frac{\alpha\theta_t}{2}}$, notice that $Y_t(Z_t = 0) = 0$, and $Y_t\left(Z_t = -\frac{\pi}{\sqrt{2\alpha\theta_t}}\right) = -\frac{\pi}{2}$. Now, we define $\hat{f}_Z$ a simplified version of f_Z which has the same limits in the spatial boundaries.

We have that:

$$\hat{f}_Z(y,t) = \left[\frac{\dot{p}_t}{\sqrt{2\alpha\theta_t}}\right]\frac{1}{(1-\sin^2(y))\sin^2(y)}.$$

We have the next limits for some fixed t:

$$\lim_{y\to 0^-}\hat{f}_Z(y,t) = \lim_{y\to -\frac{\pi}{2}^+}\hat{f}_Z(y,t) = \infty \times \operatorname{sign}(\dot{p}_t).$$

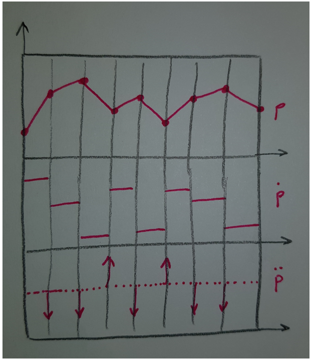
This result is a bit scaring because the sing depends on $\dot{p}_t$ when the process Y_t touch the boundaries, not for the forecast p_t. LIMITS MAY BE WRONG! CHECK WELL!

What about the forecast $p(t)$?

We do not have control about the input $p(t)$. However, as it is discrete, we can choose how to make it continuous.

If we have a measurement every Δt-intervals, and we realize linear interpolation, then $|\dot{p}| \leq \frac{1}{\Delta t}$ as $p(t) \in [0, 1]$. In our concrete case we have that $\frac{1}{\Delta t} = 144$.
Theoretically, $\dot{p}(t)$ is not defined at the measurement times, and it is constant in the time between consecutive measurements.

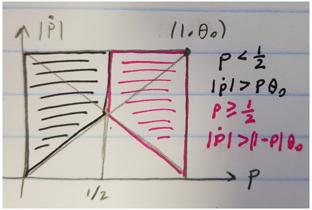
In the case of $\ddot{p}(t)$, theoretically it is zero a.e., and it has Dirac's deltas at the measurement times.



What about $\dot{\theta}_t$?

$$\dot{\theta}_t = \begin{cases} \frac{\operatorname{sign}(\dot{p}_t)\ddot{p}_t p_t - |\dot{p}_t|\dot{p}_t}{p_t^2} & \text{if} \quad |\dot{p}_t| > \theta_0 p_t \textbf{ and } p_t < \frac{1}{2} \\ \frac{\operatorname{sign}(\dot{p}_t)\ddot{p}_t(1-p_t) + |\dot{p}_t|\dot{p}_t}{(1-p_t)^2} & \text{if} \quad |\dot{p}_t| > \theta_0(1-p_t) \textbf{ and } p_t \geq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

We need to take care of the extremes $p_t \approx 0$ and $1 - p_t \approx 0$.
We will assume that always all the term involving $\ddot{p}_t$ is very small.



What about $\dot{\theta}_t$?

We assume that p_t is near the extremes, and then we do **not** have $\theta_t = \theta_0$. Also, we assume $\ddot{p}_t \approx 0$ (because I do not know how to model it).

- $p_t < \frac{1}{2}$ and $|\dot{p}_t| > \theta_0 p_t$: $\frac{\dot{\theta}_t}{\theta_t} \approx -\frac{|\dot{p}_t|\dot{p}_t}{p_t^2}\frac{p_t}{|\dot{p}_t|} = -\frac{\dot{p}_t}{p_t}$. Then, we have $\lim_{p_t \to 0^+}\left[\frac{\dot{\theta}_t}{\theta_t}\right] \approx \lim_{p_t \to 0^+}\left[-\frac{\dot{p}_t}{p_t}\right] = -\infty \times \text{sign}(\dot{p}_t)$. In this situation, the drift for Z_t tends to $-\infty \times \text{sign}(\dot{p}_t)$.
- $p_t \geq \frac{1}{2}$ and $|\dot{p}_t| > \theta_0(1-p_t)$: $\frac{\dot{\theta}_t}{\theta_t} \approx \frac{|\dot{p}_t|\dot{p}_t}{(1-p_t)^2}\frac{(1-p_t)}{|\dot{p}_t|} = \frac{\dot{p}_t}{(1-p_t)}$. Then, we have $\lim_{p_t \to 1^-}\left[\frac{\dot{\theta}_t}{\theta_t}\right] \approx \lim_{p_t \to 1^-}\left[\frac{\dot{p}_t}{(1-p_t)}\right] = \infty \times \text{sign}(\dot{p}_t)$. In this situation, the drift for Z_t tends to $\infty \times \text{sign}(\dot{p}_t)$.

Maybe we can use a model where $\theta_t = \theta_0$ only in the diffusion. The SDE would remain between 0 and 1, and the Lamperti transform would be simpler.

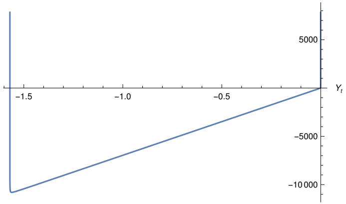
X_t, V_t, Z_t, and Y_t:

It is essential to understand what happens with the domain of each process to have a correct intuition about the SDEs.

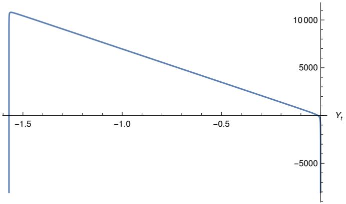
1. $X_t \in [0, 1]$.
2. $V_t \in [-p_t, 1 - p_t]$.
3. $Z_t \in \left[-\frac{\pi}{\sqrt{2\alpha\theta_t}}, 0\right]$.
4. $Y_t \in \left[-\frac{\pi}{2}, 0\right]$.

Notice that none of the changes of variables inverters the domain (red corresponds to red, and blue to blue).

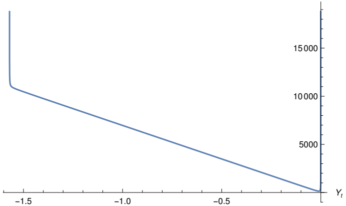
Plotting the drift for Z_t: (θ_t and $\dot{\theta}_t$ with real definitions, and $\ddot{p}_t = 0$)



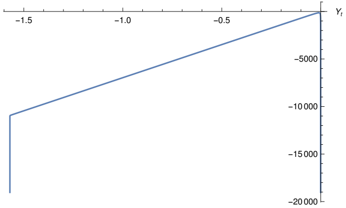
$p_t = 0.01, \alpha = 0.1, \dot{p}_t = 0.1$ (common case).



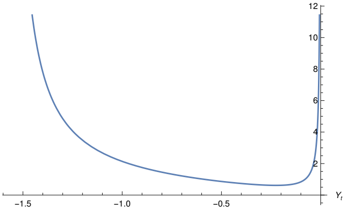
$p_t = 0.01, \alpha = 0.1, \dot{p}_t = -0.1$ (uncommon case).



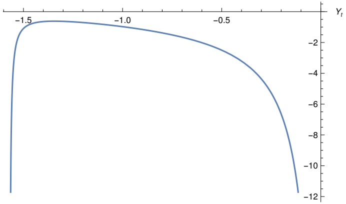
$p_t = 0.99, \alpha = 0.1, \dot{p}_t = 0.1$ (uncommon case).



$p_t = 0.99, \alpha = 0.1, \dot{p}_t = -0.1$ (common case).

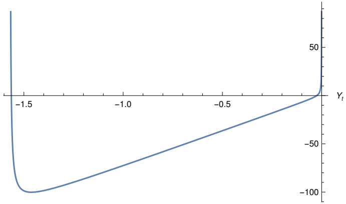


$p_t = 1/2, \alpha = 0.1, \dot{p}_t = 0.2$ (common case).

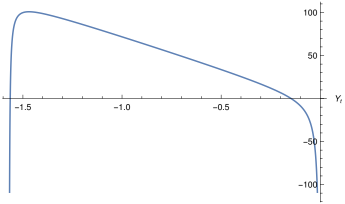


$p_t = 1/2, \alpha = 0.1, \dot{p}_t = -0.2$ (common case).

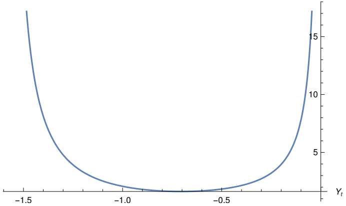
Plotting the drift for Z_t: (θ_t and $\dot{\theta}_t$ with real definitions, and $\ddot{p}_t = 0$)



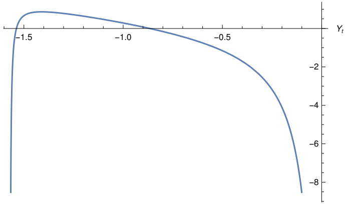
$p_t = 0.1, \alpha = 0.1, \dot{p}_t = 0.15$ (common case).



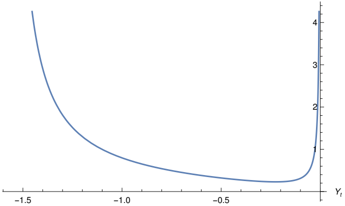
$p_t = 0.1, \alpha = 0.1, \dot{p}_t = -0.15$ (uncommon case).



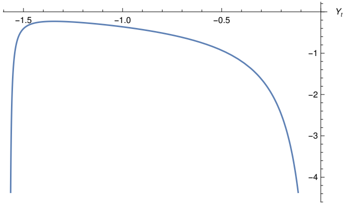
$p_t = 0.9, \alpha = 0.1, \dot{p}_t = 0.15$ (uncommon case).



$p_t = 0.9, \alpha = 0.1, \dot{p}_t = -0.15$ (common case).



$p_t = 0.1, \alpha = 0.1, \dot{p}_t = 0.05$ (common case).



$p_t = 0.9, \alpha = 0.1, \dot{p}_t = -0.05$ (common case).

Professor Kebaier's condition: (In this slide, we remove the time subindex)

We assume that **1)** $0 < \alpha < \frac{1}{2}$, and **2)** $\alpha\theta \leq \dot{p} + \theta p \leq (1-\alpha)\theta$.

From **2)**: $\frac{\alpha\theta - \dot{p}}{p} \leq \theta \leq \frac{\theta - \alpha\theta - \dot{p}}{p}$ (recall $p \in [0,1]$, but in this step we assume $p \neq 0$). We well check the two inequalities $\frac{\alpha\theta - \dot{p}}{p} \leq \theta$ and $\theta \leq \frac{\theta - \alpha\theta - \dot{p}}{p}$.

$\theta(1-p-\alpha) \geq \dot{p}$ is implied by the more restrictive inequality $\theta(1-p-\alpha) \geq |\dot{p}|$. If we assume that $1-p-\alpha > 0$ (or equivalently $p < 1-\alpha$), we have the more restrictive inequality $\theta \geq \frac{|\dot{p}|}{(1-p-\alpha)}$ which implies the blue one.

$\theta(\alpha - p) \leq \dot{p}$ is equivalent to $\theta \geq \frac{\dot{p}}{\alpha - p}$ assuming that $\alpha - p < 0$ (or $p > \alpha$). Now, violet is implied by the more restrictive inequality $\theta \geq \left|\frac{\dot{p}}{\alpha - p}\right| = \frac{|\dot{p}|}{p-\alpha}$.

Combining the two assumptions, we have that **2)** is true if $\alpha < p < 1-\alpha$, and $\theta = \max\left(\theta_0, \frac{|\dot{p}|}{\min(p-\alpha, 1-p-\alpha)}\right)$. For very small α, we recuperate our classical condition.

Professor Kebaier's Lamperti SDE:

If we fix $\theta_t = \theta_0$ in the diffusion in both SDEs (our original one and Kebaier's one), then assuming the new condition $\alpha < p < 1 - \alpha$, we have that both SDEs share the same Lamperti transform (up to the definition of θ_t in the drift).

New Model

New model for the SDE: $\theta_t = \theta_0$ in the diffusion

X_t: $\mathrm{d}X_t = \left(\dot{p}_t - \theta_t(X_t - p_t)\right)\mathrm{d}t + \sqrt{2\theta_0\alpha X_t(1-X_t)}\,\mathrm{d}W_t$

V_t: $\mathrm{d}V_t = -\theta_t V_t\,\mathrm{d}t + \sqrt{2\theta_0\alpha(V_t+p_t)(1-V_t-p_t)}\,\mathrm{d}W_t$

Lamperti transform for V_t:

$$\psi(V_t,t) = \int \frac{1}{\sqrt{2\theta_0\alpha(u+p_t)(1-u-p_t)}}\,\mathrm{d}u\Bigg|_{u=V_t} = -\sqrt{\frac{2}{\alpha\theta_0}}\arcsin\left(\sqrt{1-V_t-p_t}\right),$$

$$= -\sqrt{\frac{2}{\alpha\theta_0}}\arcsin\left(\sqrt{1-X_t}\right).$$

We can see that for every $t = t^*$, the primitive function of $\frac{1}{\sigma(v,t^*)}$ is well defined for all $v \in \left[-p(t^*), 1-p(t^*)\right] \subset [-1,1]$ (recall $v = x - p_t$, and $x \in [0,1]$).

Identities for the Lamperti transform of V_t:

- $\psi(V_t, t) = -\sqrt{\frac{2}{\alpha\theta_0}} \arcsin(\sqrt{1 - V_t - p_t})$.
- $\psi_v(V_t, t) = \frac{1}{\sigma(V_t,t)}$.
- $\psi_{vv}(V_t, t) = \frac{\mathrm{d}}{\mathrm{d}v}\left[\frac{1}{\sigma(V_t,t)}\right] = -\frac{\sigma_v(V_t,t)}{\sigma^2(V_t,t)} = -\frac{1}{\sigma^2(V_t,t)} \cdot \sqrt{\frac{\alpha\theta_0}{2}} \frac{1-2V_t-2p_t}{\sqrt{(V_t+p_t)(1-V_t-p_t)}}$.
- $\psi_t(V_t, t) = \frac{\dot{p}_t}{\sqrt{2\alpha\theta_0(V_t+p_t)(1-V_t-p_t)}}$.

SDE for $Z_t = \psi(V_t, t)$: (Verified with Mathematica)

By Itô's lemma, if $\psi(v,t)$ is $C^2([-p_t, 1-p_t])$ for v and $C^1([0,T])$ for t, then:

$$\mathrm{d}Z_t = \left(\psi_t + \psi_v \cdot f + \frac{1}{2}\psi_{vv} \cdot \sigma^2\right)\mathrm{d}t + \psi_v \cdot \sigma \,\mathrm{d}W_t.$$

If we substitute the terms related with $\psi(V_t, t)$:

$$\mathrm{d}Z_t = \left[\frac{\dot{p}_t}{\sqrt{2\alpha\theta_0(V_t+p_t)(1-V_t-p_t)}}\right.$$
$$\left. - \frac{\theta_t V_t}{\sqrt{2\alpha\theta_0(V_t+p_t)(1-V_t-p_t)}} - \frac{1}{2}\sqrt{\frac{\alpha\theta_0}{2}}\frac{1-2V_t-2p_t}{\sqrt{(V_t+p_t)(1-V_t-p_t)}}\right]\mathrm{d}t + 1\cdot\mathrm{d}W_t.$$

Recall $Z_t = -\sqrt{\frac{2}{\alpha\theta_t}}\arcsin\left(\sqrt{1-V_t-p_t}\right)$, where $Z_t \in \left[-\frac{\pi}{\sqrt{2\alpha\theta_t}}, 0\right]$.

SDE for $Z_t = \psi(V_t, t)$: (Computed with Mathematica)

$$\mathrm{d}Z_t = \underbrace{\left[\frac{\alpha\theta_0\cos(Z_t\sqrt{2\alpha\theta_0}) - \theta_t\cos(Z_t\sqrt{2\alpha\theta_0}) + 2\theta_t p_t + 2\dot{p}_t - \theta_t}{\sqrt{\alpha\theta_0}\sqrt{1-\cos(2Z_t\sqrt{2\alpha\theta_0})}}\right]}_{f(Z_t,t)}\mathrm{d}t + 1\cdot\mathrm{d}W_t.$$

$$\lim_{z\to 0^-} f(z,t) = \infty \times \left[\frac{\operatorname{sign}(2\theta_t p_t + 2\dot{p}_t + \alpha\theta_0 - 2\theta_t)}{\operatorname{sign}(\alpha)\operatorname{sign}(\theta_0)}\right].$$

$$\lim_{z\to\left[\frac{-\pi}{\sqrt{2\alpha\theta_0}}\right]^+} f(z,t) = \infty \times \left[\frac{\operatorname{sign}(2\theta_t p_t + 2\dot{p}_t - \alpha\theta_0)}{\operatorname{sign}(\alpha)\operatorname{sign}(\theta_0)}\right].$$

We want to find the correct conditions for θ_t.

To simplify the SDE, Mathematica has used:
$\sin^2(x) - \sin^4(x) = \sin^2(x)\cos^2(x) = \frac{1}{4}\sin^2(2x) = \frac{1}{8}(1-\cos(4x))$.

Limit when $z \to 0^-$:

Recall we have a bijective mapping $Z_t([0,1]) = \left[\frac{-\pi}{\sqrt{2\alpha\theta_0}}, 0\right]$.

We want $\alpha\theta_0 - 2\theta_t + 2\theta_t p_t + 2\dot{p}_t \leq 0$ so we do not scape from $x = 1$ to $x > 1$. Then:

- If $p_t < 1$, we have that $\theta_t \geq \frac{\alpha\theta_0 + 2\dot{p}_t}{2(1-p_t)}$.
- If $p_t = 1$, $\lim_{z\to 0^-} f(z,t) = \alpha\theta_0 + 2\dot{p}_t$.
  Then, if $\alpha\theta_0 > |2\dot{p}_t|$, we scape from $x = 1$ to $x > 1$.

Limit when $z \to \left[\frac{-\pi}{\sqrt{2\alpha\theta_0}}\right]^+$:

Recall we have a bijective mapping $Z_t([0,1]) = \left[\frac{-\pi}{\sqrt{2\alpha\theta_0}}, 0\right]$.

We want $2\theta_t p_t + 2\dot{p}_t - \alpha\theta_0 \geq 0$ so we do not scape from $x = 0$ to $x < 0$. Then:

- If $p_t > 0$, we have that $\theta_t \geq \frac{\alpha\theta_0 - 2\dot{p}_t}{2p_t}$.
- If $p_t = 0$, $\lim_{z \to \left[\frac{-\pi}{\sqrt{2\alpha\theta_0}}\right]^+} f(z,t) = 2\dot{p}_t - \alpha\theta_0$.

  Then, if $\alpha\theta_0 > |2\dot{p}_t|$, we scape from $x = 0$ to $x < 0$.

Controlled drift:

From both orange conditions in slides (25) and (26), we create a more restrictive condition:

$$\max\left(\frac{\alpha\theta_0 + 2\dot{p}_t}{2(1-p_t)}, \frac{\alpha\theta_0 - 2\dot{p}_t}{2p_t}\right) \leq \frac{\alpha\theta_0 + |2\dot{p}_t|}{2\min(1-p_t, p_t)}.$$

Then, we choose

$$\theta_t = \max\left(\theta_0, \frac{\alpha\theta_0 + |2\dot{p}_t|}{2\min(1-p_t, p_t)}\right). \tag{1}$$

Notice that we still need $\alpha\theta_0 > |2\dot{p}_t|$ when $\{x = 0, p = 0\}$ or $\{x = 1, p = 1\}$. As we no control over this condition, it is enough to ensure that we never reach these pairs $\{x, p_t\}$.

Recall that in the paper, we start by choosing $\theta_t = \max\left(\theta_0, \frac{|\dot{p}_t|}{\min(1-p_t, p_t)}\right)$. Our new condition (1) is mroe restrictive.

Conditions summary:

- Initial condition: $\theta_t^{initial} = \max\left(\theta_0, \frac{|\dot{p}_t|}{\min(1-p_t, p_t)}\right)$
- New condition from Lamperti: $\theta_t^{lamperti} = \max\left(\theta_0, \frac{\alpha\theta_0 + |2\dot{p}_t|}{2\min(1-p_t, p_t)}\right)$. Also, we need that $\alpha\theta_0 > |2\dot{p}_t|$ when $\{x = 0, p = 0\}$ or $\{x = 1, p = 1\}$. Notice $\theta_t^{initial} \leq \theta_t^{lamperti}$.
- Professor Kebaier's condition: $0 < \alpha < 1/2$, $\alpha < p_t < 1 - \alpha$, and $\theta_t^{kebaier} = \max\left(\theta_0, \frac{|\dot{p}|}{\min(p-\alpha, 1-p-\alpha)}\right)$.

Now, given $p_t < 1/2$, we have that $\theta_t^{lamperti} = \theta_t^{kebaier}$ if $\theta_0 = \frac{|2\dot{p}_t|}{p_t - \alpha}$ (assuming that in both cases the maximum is not θ_0). Then, how much each one is restrictive depends on $|\dot{p}_t|$ and p_t.

Conditions $0 < \alpha < 1/2$ and $\alpha < p_t < 1 - \alpha$ implies that we never have $\{x = 0, p = 0\}$ or $\{x = 1, p = 1\}$