Chapter 6

Stochastic Modeling of the Velocity

6.1 Introduction

The purpose of developing the theory of stochastic processes and stochastic differential equations (SDE's) in Chapters 2-4, was to apply it to the problem of random variations of the flow velocity on a microscopic scale, for a fluid flowing through a porous medium. The macroscopic flow is adequately represented by Darcy's equation; an obvious idea is to represent the flow velocity by augmenting this with random fluctuations such as represented by a white noise term:

$$\underline{V}(\underline{x},t) = -\frac{K(\underline{x})}{\varphi(\underline{x})}\nabla\phi(\underline{x},t) + \underline{W}(\underline{x},t)\,. \tag{6.1}$$

Here, as before, K is the hydraulic conductivity, φ the porosity, and ϕ the hydraulic pressure, also called the piezometric head. This equation shows the interrelatedness of the mean velocity and the noise component, and the Darcian term on the right hand side can be replaced by a known function for a particular physical situation.

Equation (6.1) is not a differential equation, but it becomes one if the velocity is expressed as the derivative of the position vector of a fluid element. Equation (6.1) then assumes a form very similar to that of equation (2.4). One can envisage that in a similar way as in chapter 2, the requirement that the displacement of the fluid element must be continuous even if subject to random increments, will eventually lead to an SDE similar to equation (2.4), in which the random variations are represented by a Wiener process.

However, to actually do this requires some extensions of the theory as developed so far. Firstly, equation (6.1) is a vector equation in which the velocity, position and the noise term are all spatial vectors in 1,2 or 3

dimensions. This is not a major concern, as by writing it in components the single equation becomes a set of equations all of the same form and although the examples discussed in Chapters 2-4 all contained only a single equation, the theoretical expressions do apply to sets of linear equations as was briefly discussed after equation (6.4).

More importantly, the random term is shown in equation (6.1) as depending on both space and time coordinates and consideration has to be given to the nature of the random dependence on more than one variable. This is further explored in the next section. Note that it is, indeed, physically plausible to expect random variation associated with both position and time. In an experiment such as that by Rashidi et al. (1996), which was discussed in section 1.2.3, if a snapshot is taken of the velocities at different positions, these vary randomly about the average drift velocity. Conversely, if one focuses on one position, and registers the velocity as a function of time, random variation is generally observed indicating that on the microscopic scale the flow is not laminar. From another perspective, if the velocity is interpreted as that of a particular fluid element, its instantaneous velocity will undergo random fluctuations over time as it progresses along its trajectory, because of the pore structure that it encounters along the way.

One may also reflect on the physical interpretation of the random variation introduced here, in the light of the fact previously stressed that an SDE arises when the driving coefficients in a differential equation varies stochastically. In Darcy's equation, expressed as a differential equation for the displacement, the driving coefficient is derived from the medium properties K and φ, and an external system property ϕ. Is it reasonable to use random variations in any of these to produce random microscopic fluid displacements? An argument is most easily made out for the hydraulic conductivity K. As formulated above, K is shown as a scalar quantity that in principle indicates a displacement of the fluid element in the direction of the pressure gradient. However, more generally K is a tensor and its off-diagonal elements represent displacements in other directions than that of the pressure gradient that drives them. This is exactly the effect that the pore structure of the medium has on the microscopic velocity: a fluid element that strikes a grain wall, is deflected away from the direction of the external pressure gradient, and in turn will also affect the directions in which neighboring fluid elements travel. Thus it is entirely reasonable to imagine replacing the physical obstacles that the porous medium offers to the flow, by fluctuations in the conductivity tensor. Conceptually then, one may contemplate using a full tensor conductivity, and put random variations into each of its components; but as the net result will simply be to produce random displacements away from that produced by the

scalar macroscopic tensor, the same effect is reached by directly introducing the random displacements themselves. That is essentially what is done when using equation (6.1) in the stated form.

6.2 Spectral Expansion of Wiener Processes in Time and in Space

If we replace the white noise term in equation (6.1) by a Wiener process that is random with respect to both space and time, this will have severe consequences for numerical modeling. As illustrated in Chapter 3, a discrete grid needs to be introduced for each independent variable; if we choose N grid points for each, this will give N^2 grid points for a one dimensional problem, or N^4 grid points for a problem in 3 dimensions. At each grid point, a random Gaussian value must be calculated.

However, it may be questioned if this amount of randomness is meaningful. The goal to make a fluid element follow a random path through the porous medium is equally well achieved by adding a Wiener process that only varies stochastically with either time or space alone. Suppose that we add a time dependent Wiener process. As the fluid element proceeds along its path, a random displacement will be added to its instantaneous motion at every point along the path even if there is no explicit spatial randomness. Also, a subsequent fluid element that starts from an identical initial position and velocity, will follow a different random path. This is a plausible representation of the non-laminar aspect of the microscopic flow. The only aspect that would not be plausible with a purely time-dependent Wiener term, is that if we could take a snapshot of the random displacements of all fluid elements at one instant, these would all be the same. In other words, the displacements would be perfectly correlated in space over the entire extent of the medium. This is not physically plausible; however, a spatial correlation over small distances would not only be acceptable but would in fact add to the realism of the stochastic model. For example, all the fluid elements within the volume of a single pore can clearly not move independently and their motions should be correlated.

The conclusion is that if we add a stochastic term that is a Wiener process with respect to the time variable, but is spatially correlated over a finite range (the correlation length, b), this will give a model that is both physically reasonable and numerically tractable. The remaining question is how to formulate such a stochastic term mathematically. To answer that, it is first

necessary to explore some properties of the particular kind of stochastic process defined below. The treatment presented here, is based largely on that discussed by Ghanem and Spanos (1991).

Consider an arbitrary set of functions $\{f_n(x)\}$, that is orthonormal over some domain D; i.e:

$$\int_D f_n(x) f_m(x) dx = \delta_{nm} . \tag{6.2}$$

We also select a set of random variables, $\xi_n(\omega)$ where as in Chapter 2, ω is a label that identifies a particular value of the n-th random variable. The ξ_n are restricted to have a zero mean over all values of ω, for each fixed index n. Furthermore, they are statistically independent so that they satisfy

$$E[\xi_n(\omega)\xi_m(\omega)] = \delta_{nm}. \tag{6.3}$$

Then the sum

$$\alpha(x,\omega) = \sum_{n=0}^{\infty} \sqrt{\lambda_n} f_n(x) \xi_n(\omega) \tag{6.4}$$

constitutes a stochastic process; e.g., for each value of ω, the values taken by $\alpha(x,\omega)$ is a realization of the process α. The λ_n are numerical coefficients, whose values will be specified below. By taking the mean of equation (6.4), it follows that α is a process with zero mean value. A process defined in this way has an implied correlation between values at different values of its independent argument, as is shown by calculating the covariance. Because it is a zero-mean process, the covariance function $C(x_1,x_2)$ reduces to

$$\begin{aligned} C(x_1,x_2) &= E[\alpha(x_1,\omega)\alpha(x_2,\omega)], \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \sqrt{\lambda_n \lambda_m} f_n(x_1) f_m(x_2) E[\xi_n(\omega)\xi_m(\omega)]. \end{aligned} \tag{6.5}$$

Applying equation (6.3) removes one summation in the second line of equation (6.5):

$$C(x_1,x_2) = \sum_{m=0}^{\infty} \lambda_m f_m(x_1) f_m(x_2) \tag{6.6}$$

which will not, in general, be zero, hence proving the existence of correlation. Multiplying this equation through by $f_k(x)$ and integrating over D leads, by use of the orthonormality condition equation , to the equation

$$\int_D C(x_1, x_2) f_k(x_2) dx_2 = \lambda_k f_k(x_1) . \tag{6.7}$$

Equation (6.7) has the form of an eigenvalue equation, where λ_k is now recognized as the eigenvalue and f_k as the eigenfunction of the integral kernel function $C(x_1,x_2)$.

To make the derivation consistent, it is necessary that equation (6.7) must have solutions forming an orthonormal set. It is well known from the theory of integral eigenvalue equations, that this is guaranteed provided that the kernel $C(x_1,x_2)$ is bounded, symmetric and positive definite. These properties are indeed valid, because $C(x_1,x_2)$ was defined in equation (6.5) as an autocovariance function. As a result, it has a complete orthonormal set of eigenfunctions and its eigenvalues are all positive real numbers.

Having derived the covariance function for a stochastic process defined by an expansion of the form of equation (6.4), we can now turn the argument around. Suppose that we are given a a functional expression for a covariance function $C(x_1,x_2)$. Then we can construct a stochastic process that is guaranteed to have the specified covariance, by performing the expansion in equation (6.4). In order to do this, the integral eigenvalue equation (6.7) must be soluble (and be solved) for the given $C(x_1,x_2)$.

Equation (6.4) plays a central role in this approach, and is known as a Karhunen-Loeve expansion or spectral expansion of the stochastic process $\alpha(x,\omega)$. The salient features of the Karhunen-Loeve expansion are that:

- the random behavior, and the functional dependence on the independent variable, are separated into factors in each term,

- this functional dependence can be considered known, being carried by a precisely specified set of orthnormal functions,

- one possibility is that a known set of orthogonal functions are chosen, in which case there is an implied covariance given by equation (6.6), and

- alternatively, the covariance function is predetermined, and the orthogonal functions calculated by solving an integral eigenvalue equation using the chosen covariance as the kernel of the equation.

The same idea is used in the flow problem to separate a deterministic spatial dependence and a stochastic time dependence by writing:

$$B(x,t,\omega) = \sum_{n=0}^{\infty} \sqrt{\lambda_n} f_n(x) b_n(t,\omega), \tag{6.8}$$

where now $b_n(t,\omega)$ is a set of independent Wiener processes in time, of the type discussed extensively in Chapter 2. At a fixed time t, the Wiener process $b_n(t,\omega)$ reduces to a random variable with zero mean, such as ξ above. The independence of the b_n allows us to replace equation (6.3) by

$$E[b_n(t_1,\omega) b_m(t_2,\omega)] = \min(t_1,t_2)\,\delta_{nm}, \tag{6.9}$$

where use was made of one of the defining properties of a standard Wiener process. Using this relation in calculating the combined space and time covariance function $\mathcal{C}(x_1,t_1,x_2,t_2)$ in an analogous way to that above, the time dependent part factors out of the sum and we get

$$\mathcal{C}(x_1,t_1,x_2,t_2) = \min(t_1,t_2)\, C(x_1,x_2), \tag{6.10}$$

where $C(x_1,x_2)$ is given by equation (6.6) as before and now represents the spatial covariance part. This factorization of the combined covariance only relied on the fact that the set of b_n all shared the same Wiener process time correlation, so equation (6.10) is easily generalized to the case where b_n are taken to be generalized Wiener processes defined by equation (2.45).

6.3 Solving the Covariance Eigenvalue Equation

The Karhunen-Loeve expansion enables us to construct a Wiener process with a predetermined spatial correlation, but it still remains to make the choice and solve for the corresponding orthonormal basis functions. The functional form of the covariance would be expected to depend on details of the flow within a pore and may depend on the properties and structure of the porous medium. However, for a general discussion it suffices to choose a simple form that encapsulates only a single property, that of a finite correlation length, and that will allow easy solution of the resulting eigenvalue equation. We assume only a single spatial dimension and choose a covariance kernel that decreases exponentially with separation:

$$C(x_1, x_2) = \sigma^2 e^{-\frac{|x_1 - x_2|}{b}} . \tag{6.11}$$

Notice that the correlation length b appears explicitly in this expression, serving as a scale constant that regulates the distance over which correlation between stochastic processes at neighboring positions is extinguished. σ^2 is the variance of the covariance kernel which can be thought of as an amplitude factor. Equation (6.11) is to be substituted into equation (6.7) and the resulting integral equation solved for the set of λ_n and $f_n(x)$.

We take D to be a finite interval of interest, e.g. the left and right boundaries of the porous medium for which flow is modeled; D = [$-a,a$]. The equation is of the type classified as a homogeneous Fredholm equation of the second kind (See Morse and Feshbach 1953). It is most easily solved by conversion to a differential equation. The integration interval is split at the value $x_2 = x_1$, in order to avoid the modulus sign in the exponent:

$$\lambda f(x_1) = \sigma^2 \int_{-a}^{x_1} e^{-\frac{x_1 - x_2}{b}} f(x_2) dx_2 + \sigma^2 \int_{x_1}^{a} e^{+\frac{x_1 - x_2}{b}} f(x_2) dx_2 . \tag{6.12}$$

Dividing this by σ^2 and differentiating with respect to x_1 gives

$$\bar{\lambda}\ f(x_1) = f(x_1) + \left(\frac{-1}{b}\right) \int_{-a}^{x_1} e^{-\frac{x_1 - x_2}{b}} f(x_2)\ dx_2 \\ -f(x_1) + \left(\frac{1}{b}\right) \int_{x_1}^{a} e^{\frac{x_1 - x_2}{b}}\ f(x_2)\ dx_2, \tag{6.13}$$

where $\bar{\lambda} = \lambda / \sigma^2$.

The first and third terms on the right hand side come from differentiation of the integral limits, and cancel. Differentiating the equation once more and resubstituting equation (6.12) leads to

$$\bar{\lambda} f''(x_1) = -2(\tfrac{1}{b}) f(x_1) + (\tfrac{1}{b})^2 \bar{\lambda} f(x_1). \tag{6.14}$$

This is a simple harmonic differential equation, and has the general solution

$$f(x) = A\cos(kx) + B\sin(kx) \tag{6.15}$$

where A and B are constant coefficients determined by the boundary conditions, and k is a wave number given by

$$k^2 = \frac{2b - \bar{\lambda}}{\bar{\lambda} b^2}. \tag{6.16}$$

The boundary conditions are derived from the integral equation for f, by evaluating equations (6.12) and (6.13) at the points $x_1 = \pm a$. This yields the pair of equations

$$\begin{aligned} f(a) + b f'(a) &= 0, \\ f(-a) - b f'(a) &= 0, \end{aligned} \tag{6.17}$$

and by substituting equation (6.15) into these, a pair of homogeneous equations for A and B are obtained. The condition for the existence of non-trivial solutions to the homogeneous system is that the determinant of coefficients must be zero, and this is met only by the discrete set of k-values that satisfy either of the following pair of transcendental equations:

$$\begin{aligned} 1 - bk\tan(ka) &= 0, \\ bk + \tan(ka) &= 0. \end{aligned} \tag{6.18}$$

Given numerical values of the correlation length b and the value a that specifies the extent of the spatial region of interest, each of the equations in (6.18) can be solved numerically for k. The two sets of discrete solutions are interleaved, and may be collected together by assigning a single numbering index n of which even values refer to solutions of the first, and odd values to the second equation. Through equation (6.16) the discretization of k implies that of the $\bar{\lambda}$ values, so that we end up with the discrete eigenvalues of integral equation (6.7) (for the exponential kernel) given by

$$\bar{\lambda}_n = \frac{2b}{1+b^2 k_n{}^2}. \tag{6.19}$$

Finally, substituting the k_n into the homogeneous equations and solving for A and B, the corresponding eigenfunctions of the integral equation can be written as

$$f_n(x) = \begin{cases} \dfrac{\cos(k_n x)}{\sqrt{a + \frac{\sin(2k_n a)}{2k_n}}} & ;\text{n even} \\ \dfrac{\sin(k_n x)}{\sqrt{a - \frac{\sin(2k_n a)}{2k_n}}} & ;\text{n odd} \end{cases} \tag{6.20}$$

The square root factors in equation (6.20) ensure that the eigenfunctions are normalized over the x-interval $[-a,a]$. At this point the determination of the orthonormal basis functions is complete, and by substituting equations (6.19) and (6.20) into equation (6.4) the Karhunen-Loeve spectral expansion of a stochastic process with an exponential spatial covariance is fully determined.

6.4 Extension to Multiple Dimensions

Suppose that we apply the spectral expansion of equation (6.8) to a 1-dimensional flow model. For simplicity we assume that materials properties are constant, and there is a constant pressure gradient directed along the X-axis (e.g., water flowing through a homogeneous porous medium in a straight pipe slanting downwards). Then the SDE for the motion of a fluid element, subject to stochastic motion superimposed on Darcy's equation, can be written as:

$$dx = -\frac{K}{\varphi}\left(\frac{d\phi}{dx}\right)dt + \sigma^2 \sum_{n=0}^{\infty} \sqrt{\lambda_n} f_n(x) db_n(t,\omega). \tag{6.21}$$

If we have 2-dimensional motion, it is reasonable to assume that the stochastic perturbations for the two spatial directions are independent. It is tempting to conclude that we simply need to add another equation of similar form for the y-coordinate. However, this is too simplistic. It would imply, for example, that since y does not occur in the x equation above, the correlation between x displacements decays only along the x-direction and not along y – something that is clearly implausible if the correlation is supposed to reflect physical interaction between fluid elements inside the volume of a pore.

6.5 Scalar Stochastic Processes in Multiple Dimensions

As mentioned in the introduction, there are two aspects of multi-dimensionality: the fact that the displacement is a vector, and the fact that it is a function of a vector, the position. We attend to the second factor first, by considering a scalar stochastic process defined in two dimensions. In fact, the realted discussion was formulated in general enough terms that it remains applicable if we interpret the variable x as a vector, and the domain D correspondingly as multidimensional. This means, for example, that equation (6.7) is reinterpreted for two dimensions as

$$\int C(x_1, y_1, x_2, y_2) f_k(x_2, y_2) dx_2 dy_2 = \lambda_k f_k(x_1, y_1). \tag{6.22}$$

The straightforward generalization of the exponential covariance function used in section 5.3 with unit variances for simplicity, would be the isotropic expression

$$C(\vec{x}_1,\vec{x}_2)=e^{-\frac{1}{b}|\vec{x}_1-\vec{x}_2|}=e^{-\frac{1}{b}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}. \qquad (6.23)$$

When this is substituted into equation (6.22), the procedure that was applied in section 6.3 leads to the equation

$$\lambda\nabla^2 f(\vec{x})=-\frac{1}{b}\int\frac{e^{-\frac{1}{b}|\vec{x}-\vec{x}_2|}}{|\vec{x}-\vec{x}_2|}f(\vec{x}_2)dx_2dy_2+\left(\frac{1}{b}\right)^2\lambda f(\vec{x}). \qquad (6.24)$$

Unfortunately this equation is, unlike equation (6.14), not a simple differential equation and is not very helpful to solve equation (6.22). However, progress can be made if we retrace our steps and write the 2D version of equation (6.4) in the form

$$\alpha(\vec{x},\omega)=\sum_{n,m=0}^{\infty}\sqrt{\lambda_{nm}}f_n(x)g_m(y)\xi_{nm}(\omega). \qquad (6.25)$$

This expression results from the observation that if $f_n(x)$ and $g_m(x)$ are two orthonormal sets that form a complete basis for the 1-dimensional subspaces associated with the X-domain and Y-domain separately, their outer product forms a complete basis for the 2-dimensional space. As before the $\xi_{nm}(\omega)$ are required to be independent, and hence we find

$$C(\vec{x}_1,\vec{x}_2)=\sum_{n,m}\lambda_{nm}f_n(x_1)f_n(x_2)g_m(x_1)g_m(x_2) \qquad (6.26)$$

which, by using the 1-dimensional orthonormality twice, leads to

$$\int C(\vec{x}_1,\vec{x}_2)f_k(x_2)g_l(y_2)dx_2dy_2=\lambda_{kl}f_k(x_1)g_l(y_1). \qquad (6.27)$$

This equation is no easier to solve than equation (6.22), except if we make the assumption that C is separable into factors that depend on the space directions separately:

$$C(\vec{x}_1,\vec{x}_2)=C_x(x_1,x_2)C_y(y_1,y_2). \qquad (6.28)$$

If this holds, the double integral in equation (6.27) factorizes and it is easy to see that it will be solved by choosing $f_k(x)$ and $g_l(y)$ to be the eigenfunctions of the 1-dimensional integral kernels C_x and C_y respectively, with corresponding eigenvalues λ_{xk} and λ_{yl} found from equations such as

(6.19). Moreover, the eigenvalues of the 2-dimensional equation are then given by a simple product

$$\lambda_{kl} = \lambda_{xk}\lambda_{yl}, \tag{6.29}$$

and as a result, the double summation in equation (6.26) also factorizes and reduces to an expression consistent with equation (6.28) if the eigenfunction expansions of the 1-dimensional kernels according to equation (6.6) is applied.

So by imposing equation (6.28), the 2-dimensional eigenvalue problem is reduced to solving two 1-dimensional problems and a 2-dimensional spectral expansion of the form of equation (6.25) directly follows. We now may consider how realistic this simplifying assumption is, by applying it to the exponential covariance function.

Since equation (6.23) does not factorize, the next best choice for a factorized exponentially decreasing covariance kernel would be

$$C_f(\vec{x}_1, \vec{x}_2) = e^{-\frac{1}{b}|x_1 - x_2|} e^{-\frac{1}{b}|y_1 - y_2|}. \tag{6.30}$$

The difference between these two is highlighted by making contour plots as functions of x_1, for a fixed reference point x_2. All points on a contour share the same degree of correlation with the reference point in the middle of the picture.

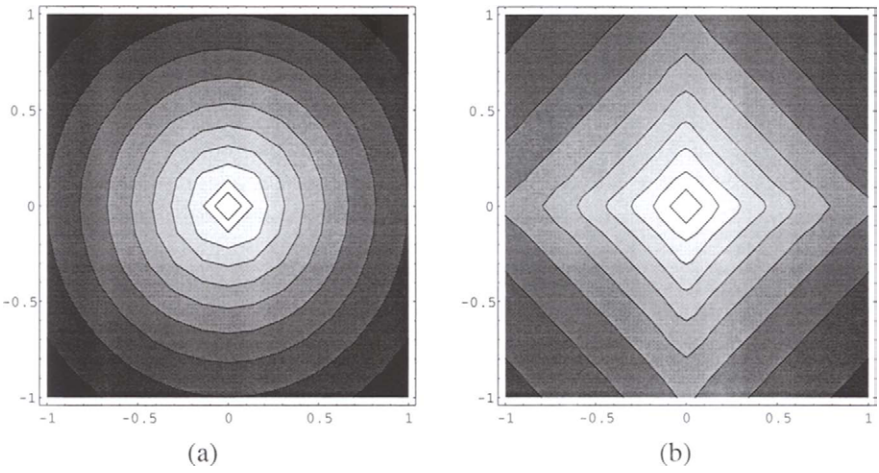

Figure 6.1 Contour maps of (a) 2D exponentially decreasing covariance function C (b) Factorised approximation C_f .

The plots in Figure 6.1 shows that the price paid for the mathematical simplification brought about by requiring C to be factorized, is a loss of symmetry. Instead of the fully isotropic correlation, it becomes only square symmetric, which is hard to justify on physical grounds. On the other hand, the choice of an exponentially decreasing covariance was itself made on the grounds of plausibility and tractability rather than a fundamental justification. Even if anisotropic, the factorized form does retain the qualitative feature of an exponential decrease at a similar rate in all directions. It therefore seems a reasonable compromise to make in a first attempt to model 2-dimensional stochastic flow.

Also note that it is possible to avoid making this approximation for some choices of the functional form of C – for example, a Gaussian choice would be naturally factorizable. In this case, the 1-dimensional problem is more difficult, however, and will not be further pursued here.

The results of this section can be extended to 3-dimensional space in a straightforward manner.

6.6 Vector Stochastic Processes in Multiple Dimensions

A vector stochastic process is obviously constructed by applying an expansion like equation (6.25) to each vector component separately. It is plausible to assume that the correlation length is the same for each vector component, and consequently to use the same sets of 1-dimensional eigenfunctions as the basis for expansion in each component, while the stochastic variations are independent for different components. Writing the vector process as $\alpha(x,\omega) = (\alpha_1,\alpha_2)$ in 2 dimensions, we have

$$\alpha_i(\vec{x},\omega) = \sum_{n,m=0}^{\infty} \sqrt{\lambda_{nm}} f_n(x) g_m(y) \xi_{inm}(\omega), \tag{6.31}$$

and the independence condition is

$$E[\xi_{inm}(\omega)\xi_{jkl}(\omega)] = \delta_{ij}\delta_{nk}\delta_{ml}. \tag{6.32}$$

The covariance is not a scalar function any more, but becomes a matrix defined by

$$C_{ij}(\vec{x}_1,\vec{x}_2) = E[\alpha_i(\vec{x}_1,\omega)\alpha_j(\vec{x}_2,\omega)]. \tag{6.33}$$

By combining equations (6.31), (6.32) and (6.33) it is easily seen that C_{ij} is a diagonal matrix, and in fact all the diagonal elements are equal and each has the value given by equation (6.26). This means that the addition of independent stochastic variations do not introduce any cross-correlation between vector components. The deterministic differential equations for the components of a vector can each be augmented by a similar stochastic term, and only the stochastic variable in each needs to be generated independently. Therefore the case of a vector stochastic process is not mathematically significantly more complex than the scalar case, and we may directly apply equation (6.31) with the eigenfunctions and eigenvalues as found in the previous sections.

6.7 Simulation of Stochastic Flow in 1 and 2 Dimensions

6.7.1 1-D case

At this point in our discussion on the stochastic flow, it would be interesting to obtain an intuitive feel about how the covariance parameter σ^2 and the correlation length b affect the flow of a fluid particle in one dimension. The obvious way of accomplishing this would be to simulate the behavior on computer.

For the 1-dimensional case, we confine our attention to a domain D=[-1,1]. Then for the correlation function given by equation (6.11), the discrete set of k values is given by

$$\begin{aligned} &1-bk\ \tan(k)=0, \\ &bk+\tan(ka)=0. \end{aligned} \tag{6.34}$$

A program can be written to obtain the k values for a given b, then from equation (6.19),we obtain the eigenvalue for a given σ^2:

$$\lambda_n=\frac{2\sigma^2 b}{1+b^2k_n^2}. \tag{6.35}$$

The corresponding eigenfunction can be obtained by substituting a=1 into equation (6.20):

$$f_n(x)=\begin{cases} \dfrac{\cos(k_n\ x)}{\sqrt{1+\dfrac{\sin(2k_n)}{2k_n}}}\ ;\ n\ is\ even \\ \dfrac{\sin(k_n\ x)}{\sqrt{1-\dfrac{\sin(2k_n)}{2k_n}}}\ ;\ n\ is\ odd \end{cases} \tag{6.36}$$

and then we use equation (6.21) to simulate the stochastic flow.

6.7.2 2-D Case

For the simple case of stochastic flow in a rectangular 2-dimensional region where the porous medium properties are constant and there is a constant pressure gradient directed along the X-axis, the 1-dimensional model of equation (6.21) is replaced by the set:

$$\begin{cases} dx = \dfrac{-K}{\varphi}\left(\dfrac{d\phi}{dx}\right)dt + \sigma^2 \sum\limits_{n,m=0}^{\infty} \sqrt{\lambda_{nm}}\ \ f_n(x)\ \ g_m(y)\ \ db_{1nm}(f,\omega), \\ dy = \sigma^2 \sum\limits_{n=0}^{\infty} \sqrt{\lambda_{nm}}\ \ f_n(x)\ \ g_m(y)\ \ db_{2nm}(f,\omega). \end{cases} \tag{6.37}$$

where the numeric index on the Wiener process simply identifies the x or y component to which it belongs. Equation (6.37) can be solved numerically by using, for example, an Euler scheme or a Milstein scheme which are strong Taylor approximations (Kloeden and Platen, 1992).