

Processes with volatility-induced stationarity: an application for interest rates

João Nicolau*

*Instituto Superior de Economia e Gestão/Univers. Técnica de Lisboa,
ISEG, Rua do Quelhas 6, 1200-781 Lisboa, Portugal*

In this paper we propose a refinement of the existing definition of volatility-induced stationarity that allows us to distinguish between processes with drift and diffusion induced stationarity and processes with pure volatility-induced stationarity. We also propose a classification of stationary processes with volatility-induced stationarity according to the volatility that is needed to inject stationarity. Processes with volatility-induced stationarity are potentially applicable to interest rate time-series since, as has been acknowledged, mean-reversion effects occur mainly in periods of high volatility. As such, we provide evidence that the logarithm of the Fed funds rate can be modelled as a local martingale with volatility-induced stationarity.

Key Words and Phrases: stochastic differential equations, diffusion processes, parametric estimation, non-parametric estimation.

1 Introduction

Estimation of short-term interest rate processes has recently received much of attention. Several models have been proposed, ranging from discrete-time processes such as GARCH, stochastic volatility and regime-switching models, to diffusion (with or without stochastic volatility) and jump-diffusion processes (see HONG *et al.*, 2004, for a recent overview). Most studies on short-term interest rate processes have shown at least two main facts. Firstly, the mean-reverting effect is very weak (see, for example, CHAN *et al.*, 1992 or BANDI, 2002). In fact, the stationarity of short-term interest rate processes is quite dubious. The usual unit root tests neither clearly reject nor the hypothesis of stationarity. Since interest rate processes are bounded by a lower (zero) and upper (finite) value a pure unit root hypothesis seems impossible since a unit root process goes to ∞ or $-\infty$ with probability one as time goes to ∞ . Some authors have addressed this question. The issue is how to reconcile an apparent absence of mean-reverting effects with the fact that the interest rate is a

*nicolau@iseg.utl.pt

I would like to thank two anonymous referees for valuable suggestions, and Tom Kundert for helpful comments. This research was supported by the Fundação para a Ciência e a Tecnologia (FCT) and by POCTI.

bounded (and possibly stationary) process. AÏT-SAHALIA (1996) using nonparametric methods suggests a nonlinear drift coefficient. The drift is zero in some “central” region of the state space (so the process behaves like a martingale in this region) and mean-reverting at the edges of the range of the process, which is sufficient to assure stationarity. PRITSKER (1998) and CHAPMAN and PEARSON (2000), however, have pointed out that the findings of nonlinear drifts could be spurious due to the poor finite sample performance of the nonparametric estimators at the boundary region of the interest rate data. Nevertheless, the martingale behaviour of the interest rate process seems clear over most of its range. While AÏT-SAHALIA (1996) suggests that stationarity can be drift-induced, CONLEY *et al.* (1997) (CHLS, henceforth) suggest that stationarity is primarily volatility induced. Their point is that what matters for reversion is a “pull” measure defined as the ratio $a(x)/(2b^2(x))$, where a and b are the drift and diffusion infinitesimal coefficients, respectively. In fact, it has been observed that higher volatility periods are associated with mean reversion effects. Thus, the CHLS hypothesis is that higher volatility injects stationarity in the data.

The second (well known) fact is that the volatility of interest rates is mainly level dependent and highly persistent. The higher (lower) the interest rate is, the higher (lower) the volatility. The volatility persistence can thus be partially attributed to the level persistence of the interest rate.

The hypothesis of CHLS is interesting since volatility-induced stationarity can explain martingale behaviour (fact one), level volatility persistence (fact two), and mean-reversion.

To illustrate these ideas and show how volatility can inject stationarity we present in Figure 1 a simulated path from the SDE

$$dX_t = (1 + X_t^2)dW_t. \quad (1)$$

It is worth mentioning that the Euler scheme

$$Y_i = Y_{i-1} + (1 + Y_{i-1}^2)\sqrt{t_i - t_{i-1}}\varepsilon_i, \quad \varepsilon_i \sim \text{i.i.d.}N(0, 1)$$

cannot be used since Y explodes as $t_i \rightarrow \infty$ (see RICHTER, 2002). For a method to simulate X , see NICOLAU (2005).

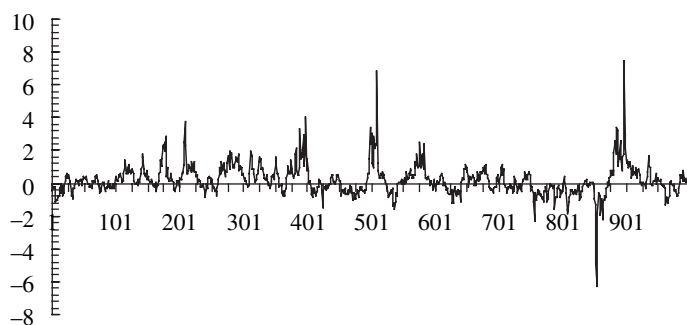


Fig. 1. Simulated path from the SDE $dX_t = (1 + X_t^2)dW_t$.

Since the SDE (1) has zero drift, we could expect a random walk behaviour. Nevertheless, Figure 1 shows that the simulated trajectory of X exhibits reversion effects towards zero, which is assured solely by the structure of the diffusion coefficient. It is the volatility that induces stationarity. In the neighbourhood of zero the volatility is low so the process tends to spend more time in this interval. If there is a shock, the process moves away from zero and the volatility increases (since the diffusion coefficient is $1 + x^2$) which, in turn, increases the probability that X crosses zero again. The process can reach extreme peaks in a very short time but quickly returns to the neighbourhood of zero. It can be proved, in fact, that X is a stationary process with stationary density $\bar{p}(x) = 2/(\pi(1 + x^2)^2)$ and stationary moments $E[X] = \int x\bar{p}(x)dx = 0$ and $\text{Var}[X] = \int x^2\bar{p}(x)dx = 1$ and that the conditional expectation decays exponentially fast to the stationary mean (see the example in appendix 1). Thus, X is a stationary local martingale but not a martingale since $E[X_t|X_0]$ converges to the stationary mean as $t \rightarrow \infty$ and not to X_0 as would be required if X was a martingale (BIBBY and SØRENSEN, 1995, give another example of a stationary local martingale that is not a martingale).

The main objectives of this paper are the following: to discuss volatility-induced stationarity (VIS), refine the existing definition of VIS (section 2) and illustrate that the logarithm of the Fed funds rate can be modelled as a local martingale process with VIS (section 3).

2 Definitions of volatility-induced stationarity

In this section we present a refinement of the existing VIS definition.

For simplicity, we always assume that the initial value of an ergodic process has a stationary density; thus, we do not distinguish between ergodic and stationary processes.

To our knowledge, CHLS were the first to discuss VIS ideas. RICHTER (2002) generalizes the definition of CHLS. Basically, their definition states that the stationary process X (solution of the stochastic differential equation (SDE) $dX_t = a(X_t)dt + b(X_t)dW_t$) has VIS at boundaries $l = -\infty$ and $r = \infty$ if

$$\lim_{x \rightarrow l} s_X(x) < \infty \text{ and } \lim_{x \rightarrow r} s_X(x) < \infty \quad (2)$$

where s_X is the scale density,

$$s_X(x) = \exp \left\{ - \int_{z_0}^x \frac{2a(u)}{b^2(u)} du \right\} \quad (z_0 \text{ is an arbitrary value}).$$

We note that a process with VIS is a stationary process (by definition). In order to guarantee stationarity one has to assume that $S_X(l, x] = \lim_{x_1 \rightarrow l} \int_{x_1}^x s_X(u)du = \infty$, $S_X[x, r) = \lim_{x_2 \rightarrow r} \int_x^{x_2} s_X(u)du = \infty$ for $x \in (l, r)$ and $\int_l^r m_X(x)dx < \infty$ where $m_X(u) = (b^2(u)s_X(u))^{-1}$ (see appendix 1, assumptions A1 and A2). If these conditions hold with $s_X(x)$ finite then, according to previous definition,

X has VIS at the boundaries. The example we saw in the previous section, the solution of $dX_t = (1 + X_t^2)dW_t$, has VIS at boundaries $l = -\infty$ and $r = \infty$ since X is a stationary process and $s_X(x) = 1$ (in appendix 1 we analyse this case). In contrast, some well-known stationary diffusions do not satisfy the VIS condition of CHLS and Richter. For example, the CIR process, introduced by Cox *et al.* (1985), to model instantaneous spot rate, with infinitesimal coefficients, $a(x) = \beta(\tau - x)$ and $b(x) = \sigma\sqrt{x}$ ($\beta > 0$, $\sigma > 0$) does not verify the VIS condition since

$$s_X(x) = \exp\left\{-\int_{z_0}^x 2a(u)/b^2(u)du\right\} = \frac{e^{2\beta(x-1)/\sigma^2}}{x^{2\beta/\sigma^2}} \quad (z_0 = 1)$$

and $s_X(x) \rightarrow \infty$ as $x \rightarrow \infty$ or $x \rightarrow 0$ (in this case $l = 0$).

Our point is that the VIS definition of CHLS and Richter has the disadvantage that the source of stationarity is not clearly identified. To be more precise, consider the following example. According to the CHLS definition, the solution of $dX_t = -X_t dt + \sqrt{1 + X_t^4}dW_t$ has VIS at boundaries $l = -\infty$ and $r = \infty$, since X is a stationary process (the A1 and A2 conditions in appendix 1 are satisfied) and the limit of $s_X(x) = \exp\{\pi/4 - 1/4\arctan(1/x^2)\}$ is finite when $x \rightarrow \pm\infty$ ($\lim_{x \rightarrow \pm\infty} s_X(x) = \exp(\pi/4)$). However, we cannot conclude that stationarity is only ensured by the diffusion coefficient, as the solution of $dY_t = -Y_t dt + \sigma dW_t$ is stationary as well. Neither can we conclude that stationarity is ensured by the drift coefficient, as the solution of $dY_t = \sqrt{1 + Y_t^4}dW_t$ is also stationary. In this case both infinitesimal coefficients independently assure stationarity.

The example above shows that the problem with the CHLS definition is that it does not exclude mean-reversion effects and thus stationarity can also be drift-induced. Another way to confirm this idea is verifying that $\lim_{x \rightarrow \pm\infty} xa(x) < 0$ (which implies mean-reversion effects) is not ruled out by condition (2).

To characterize this type of VIS, i.e. the case where both infinitesimal coefficients independently induce stationarity we state the following. Whenever a stationary process X satisfies condition (2) and $\lim_{x \rightarrow \pm\infty} xa(x) < 0$ we say that X has VIS of type one (VIS1). The condition $\lim_{x \rightarrow \pm\infty} xa(x) < 0$ implies mean-reversion effects and this condition jointly with condition (2) implies that $b^2(x)$ dominates asymptotically $a(x)$ in the sense that $a(x)/b^2(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Otherwise, the integral $-\int_{z_0}^x 2a(u)/b^2(u)du$ is divergent.

We now discuss other forms of VIS and propose a more intuitive version of VIS.

Consider the following SDEs

$$dX_t = a(X_t)dt + b(X_t)dW_t \quad (3)$$

$$dY_t = a(Y_t)dt + \sigma dW_t, \quad (4)$$

where $0 < \sigma < \varepsilon$ (ε is a fixed but controllable small value). We call Y the associated process of X . For example, if $dX_t = \beta(\tau - X_t)dt + \sqrt{1 + X_t^2}dW_t$, the associated process is the solution of $dY_t = \beta(\tau - Y_t)dt + \sigma dW_t$. Thus, both processes have the same drift.

We say that a stationary process X has VIS of type two (VIS2) if the associated process Y does not possess a stationary distribution. In other words, X has VIS2 if X satisfies the A1 and A2 conditions in appendix 1 and Y does not satisfy at least one of these conditions. We now explain the intuition behind this definition and why in the VIS2 definition only the diffusion coefficient induces stationarity.

Although the process Y has the same drift as that of the process X , Y is nonstationary (by definition) whereas X is stationary. That is, the substitution of σ by $b(x)$ transforms a nonstationary process Y (equation (4)) into a stationary process (equation (3)). Thus, the stationarity of X can only be attributed to the role of the diffusion coefficient (volatility) and in this case we have in fact a pure VIS process.

The following is a simple criterion to identify VIS2, in the case $l = -\infty$ and $r = \infty$. We say that a stationary X process with boundaries $l = -\infty$ and $r = \infty$ has VIS2 if

$$\lim_{x \rightarrow \infty} xa(x) \geq 0 \text{ or } \lim_{x \rightarrow -\infty} xa(x) \geq 0. \quad (5)$$

It is easy to show that the solution of the associated process $dY_t = a(Y_t)dt + \sigma dW_t$, under condition (5), does not possess a stationary distribution. Condition (5) implies an absence of mean-reversion of the process Y since when y is 'high' ('low') the drift $a(y)$ is zero or positive (negative), and as a consequence the process moves further and further away from a 'central region'. If the solution of $dX_t = a(X_t)dt + b(X_t)dW_t$ turns out to be stationary, where the drift satisfies (5), this can only be due to the role of the instantaneous volatility $b(X_t)$.

One example of VIS2 was presented in section 1 ($dX_t = (1 + X_t^2)dW_t$). Another example is the following:

$$dX_t = (1 + X_t)dt + (1 + X_t^2)dW_t.$$

The associated process Y , solution of the SDE $dY_t = (1 + Y_t)dt + \sigma dW_t$, is clearly nonstationary (the drift satisfies condition (5)). Nevertheless, it can be proved (using either the conditions A1 and A2 in appendix 1 or Proposition 1 (a), see below) that X is stationary. To appreciate the differences between X and Y we present two simulated paths in Figure 2. The initial condition is $Y_0 = 1$, and in both cases the

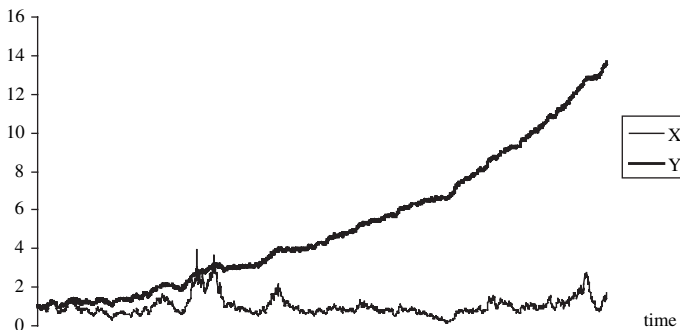


Fig. 2. A path of Y and X where $Y: dY_t = (1 + Y_t)dt + dW_t$, $X: dX_t = (1 + X_t)dt + (1 + X_t^2)dW_t$ (the initial condition is $Y_0 = 1$ and both the processes are driven by the same Wiener path W).

processes are driven by the same Wiener path W . The simulation was carried out following the ideas in NICOLAU (2005). Figure 2 shows that the associated process is entirely dominated by the structure of the drift, which causes the process to reach infinity as $t \rightarrow +\infty$. However, if the constant diffusion coefficient is substituted by a function like $b(x) = 1 + x^2$, the volatility prevents the process from moving away from its stationary mean. The mechanism to induce stationarity is very similar to that described in section 1.

We have so far stressed some differences between the VIS definition of CHLS and Richter and the proposed definition VIS2. However, there are common features in these definitions. In both cases, (a) X is stationary and (b) the condition $\lim_{x \rightarrow \infty} s_X(x) < \infty$ or $\lim_{x \rightarrow -\infty} s_X(x) < \infty$ must be verified. Therefore, if a process satisfies the VIS2 condition it also satisfies the VIS condition of CHLS and Richter.

The behaviour of the Y process corresponding to a VIS2 process can be quite distinct. For example, consider the stochastic differential equations in Table 1, below. All processes satisfy the VIS2 condition. They are stationary and satisfy (5). Nevertheless, the associated processes (all nonstationary) exhibit a quite distinct behaviour. The process $dY_t = Y_t^2 dt + \sigma dW_t$ is explosive (reaches infinity in finite time with probability one) and non-recurrent; the third one ($dY_t = (1 + Y_t)dt + \sigma dW_t$) is still non-recurrent but the expected time to reach ∞ is infinity; the fifth ($dY_t = \sigma dW_t$) is the Wiener process (which is recurrent). In Figure 3 we show the typical behaviour of these three processes: an explosive process (path A), a process that drifts away from a central region eventually reaching infinity, but without blowing out in finite time (path B) and a Wiener process (path C).

It seems obvious that the volatility that is needed to inject stationarity varies according to the nature of the associated process. Typically, it has to be higher when the associated process is explosive (path A in Figure 3) than when the associated process is only null-recurrent (path C in Figure 3). For example, consider the SDE $dX_t = (\alpha_0 + \alpha_1 X_t + \alpha_2 X_t^2)dt + (\sigma + \beta X_t^2)^\gamma dW_t$, $\sigma > 0$, $\beta > 0$. Following the conditions for stationarity presented in appendix 1 we have the following. If $\alpha_1 < 0$ and $\alpha_2 = 0$ the drift induces mean-reversion effects and it is not necessary to impose a restriction on γ in order that X is stationary. If $\alpha_0 = \alpha_1 = \alpha_2 = 0$ the

Table 1. Examples of application of proposition 1.

Stoch. Diff. Equations	(1)	(2)	(3)	(4)	(5)	(6)	(7)	VIS2-
$dX_t = X_t^2 dt + (1 + X_t^4) dW_t$	2	∞	$-\infty$	0	0	-4	-4	a
$dX_t = \frac{X_t}{1+X_t^2} dt + \sqrt{1+X_t^2} dW_t$	-1	1	1	0	0	-1	-1	b
$dX_t = (1 + X_t) dt + (1 + X_t^2) dW_t$	1	∞	∞	0	0	-2	-2	b
$dX_t = \alpha X_t dt + \sqrt{1+X_t^2} dW_t$	1	∞	∞	α	α	$\alpha - 1$	$\alpha - 1$	(*)
$dX_t = (1 + X_t^2) dW_t$	-	0	0	0	0	-2	-2	c
$dX_t = \frac{1}{1+X_t^2} dt + (1 + X_t^2)^{1/3} dW_t$	-2	0	0	0	0	-2/3	-2/3	c

(1) $\alpha : a(x) = O(|x|^2)$; (2) $\lim_{x \rightarrow \infty} x a(x)$; (3) $\lim_{x \rightarrow -\infty} x a(x)$; (4) $\lim_{x \rightarrow \infty} x \frac{a(x)}{b^2(x)}$;

(5) $\lim_{x \rightarrow \infty} x \frac{a(x)}{b^2(x)}$; (6) $\lim_{x \rightarrow \infty} x \left(\frac{a(x)}{b^2(x)} - \frac{b'(x)}{b(x)} \right)$; (7) $\lim_{x \rightarrow -\infty} x \left(\frac{a(x)}{b^2(x)} - \frac{b'(x)}{b(x)} \right)$.

(*) b if $0 < \alpha < \frac{1}{2}$.

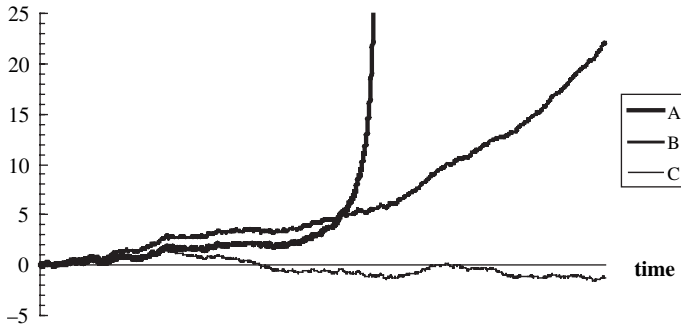


Fig. 3. Three paths from the following processes: A – $dY_t = Y_t^2 dt + dW_t$; B – $dY_t = (1 + Y_t)dt + dW_t$; C – $Y_t = W_t$ (the initial condition is $Y_0 = 0$ and all the processes are driven by the same Wiener path W).

associated process is null-recurrent and one has to impose $\gamma > 1/4$. If $\alpha_0 > 0$, $\alpha_1 > 0$ and $\alpha_2 = 0$ we have to impose $\gamma > 1/2$ or $\gamma = 1/2$ and $\beta > 2$. Finally, if $\alpha_0 > 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$ the associated process is explosive and one has to impose $\gamma > 3/4$ or $\gamma = 3/4$ and $\beta > 2\sqrt{2}$. Thus, roughly speaking, the lower is the mean-reversion effect the greater the volatility needed to inject stationarity.

To distinguish the volatility that is needed to inject stationarity we consider three sub-types of VIS2 according to the behaviour of the associated Y process: from explosive processes (first case) to null-recurrent processes (third case). Consider again SDEs (3) and (4) and the following functionals (see KARLIN and TAYLOR, 1981) related to the associated Y process:

$$S_Y[x, x_0] = \int_{x_0}^x s_Y(u) du$$

$$\Sigma_Y(l) = \lim_{\alpha \downarrow l} \int_{\alpha}^{x_0} S_Y[\alpha, u] m_Y(u) du, \Sigma_Y(r) = \lim_{\beta \uparrow r} \int_{x_0}^{\beta} S_Y[u, \beta] m_Y(u) du$$

where $s_Y(x) = \exp\{-\int_{x_0}^x [2a(u)/\sigma^2] du\}$ (z_0 is an arbitrary value) and $m_Y(u) = 1/\sigma^2 s_Y(u)$. We say that the stationary X process has volatility-induced stationarity of type 2a, 2b and 2c (VIS2-a, VIS2-b and VIS2-c) if it has, respectively:

$$\text{VIS2-a } \Sigma_Y(l) < \infty \text{ or } \Sigma_Y(r) < \infty.$$

$$\text{VIS2-b } S_Y(l, x] < \infty \text{ or } S_Y[x, r) < \infty \text{ and } \Sigma_Y(l) = \Sigma_Y(r) = \infty.$$

$$\text{VIS2-c } S_Y(l, x] = S_Y[x, r) = \infty, \int_l^r m_Y(u) du = \infty.$$

Figure 3 shows possible behaviours of the associated processes under conditions VIS2-a, VIS2-b and VIS2-c, respectively paths A, B and C. In appendix 2 we discuss these conditions in detail and we provide some more examples. Also, see Table 1.

We notice that if $l = -\infty$ then $S_Y(l, x]$ should be interpreted as $\int_l^x s_Y(u) du = \lim_{k \rightarrow -l} \int_k^x s_Y(u) du$ (likewise for $S_Y[x, r)$ if $r = \infty$).

The following proposition establishes sufficient conditions for the three sub-types of VIS2 in the case where the state space is \mathbb{R} .

PROPOSITION 1. *Let \mathbb{R} be the state space of X . Suppose that functions a and b have continuous derivatives and that $b(x) > 0$ for all $x \in (-\infty, \infty)$. If (a) $\lim_{x \rightarrow \pm\infty} x a(x)/b^2(x) < 1/2$ and $\lim_{x \rightarrow \pm\infty} x[a(x)/b^2(x) - b'(x)/b(x)] < -1/2$ then X is ergodic and the stationary density is $\bar{p}(x) = m_X(x)/\int_{-\infty}^{\infty} m_X(x)dx$ where $m_X(x) = 1/(s_X(x)b^2(x))$ is the speed density. Furthermore, (b) if $a(x) = O(|x|^\alpha)$, $\alpha > 1$ and $\lim_{x \rightarrow \infty} a(x) = \infty$, $\lim_{x \rightarrow -\infty} a(x) = -\infty$ then X has VIS of type 2a (VIS2-a); (c) if $a(x) = O(|x|^\alpha)$, $\alpha \leq 1$ and $\lim_{x \rightarrow \pm\infty} xa(x) > 0$ then X has VIS of type 2b (VIS2-b); (d) if $\lim_{x \rightarrow \pm\infty} xa(x) = 0$ then X has VIS of type 2c (VIS2-c);*

PROOF. See appendix 3. □

The application of proposition 1 is straightforward, as Table 1 shows.

3 Modelling the fed funds rate with VIS

Processes with VIS are potentially applicable to interest rate time-series since, as has been acknowledged, reversion effects (towards a central measure of the distribution) occur mainly in periods of high volatility. CHLS were the first (to the best of my knowledge) to propose a VIS specification for interest rates. To model the Fed funds rate CHLS consider the following general specification for the infinitesimal coefficients:

$$a(x) = \sum_{i=-k}^L \alpha_i x^i, \quad b^2(x) = \kappa x^\gamma, \quad \kappa, \gamma > 0. \quad (6)$$

With this specification it is obvious that the drift cannot be zero (i.e., $\alpha_i = 0$, $i = -k, \dots, L$); otherwise the left boundary (zero) would be attracting (i.e. $S_X(0, x] < \infty$) and the model would be incorrect. To impose stationarity via volatility, CHLS consider a number of restrictions on α_i and γ . Some of these restrictions are explicitly defined to avoid the process reaching the left boundary with positive probability. The model of CHLS follows a VIS2-a specification.

In this section we discuss an alternative model to the CHLS for interest rates. The main objective is to illustrate VIS ideas and provide further evidence to support VIS.

3.1 Data and identification

We consider monthly sampling of the Fed funds rate between January 1962 and December 2002 (see Figure 4). The source for the data is the H-15 Federal Reserve Statistical Release. At high frequencies (for example, daily frequency) the Fed funds rate series exhibits strong microstructure effects reflecting the institutional features of the operational framework, including the beginning and the end of the reserve

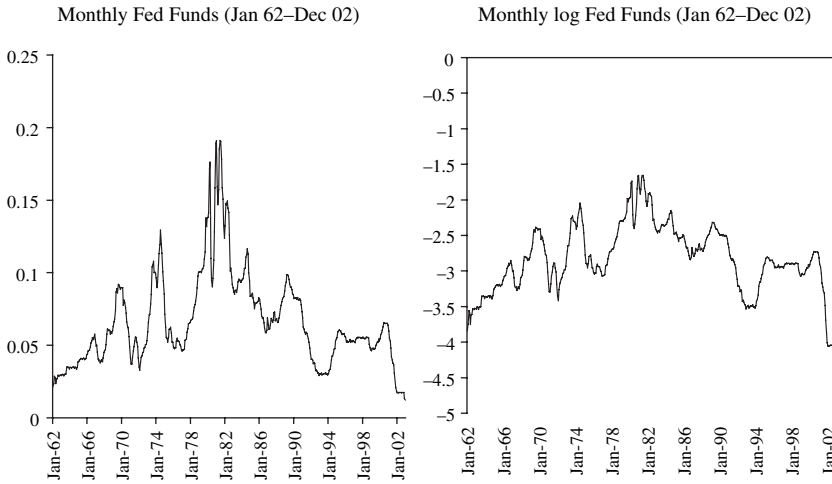


Fig. 4. Fed Funds and log Fed Funds in the Period Jan 1962–Dec 2002.

maintenance period and the open market operations allotment and settlement days. These effects almost disappear at monthly frequency.

We now present evidence that supports the specification

$$dX_t = \exp\{\alpha/2 + \beta/2(X_t - \mu)^2\}dW_t \quad (7)$$

where $X_t = \log r_t$ and r represents the Fed funds rate (original data was divided by 100) (see Figure 4). We denote $\{r_{t_i}\}_{i \geq 1}$ and $\{X_{t_i}\}_{i \geq 1} = \{\log r_{t_i}\}_{i \geq 1}$ the data (observed at equidistant instants t_1, t_2, \dots, t_n).

We observe that the state space of r is $(0, \infty)$ and X is $(-\infty, \infty)$. That is, X can assume any value in \mathbb{R} . This transformation preserves the state space of r , since $r_t = \exp(X_t) > 0$. By Itô's formula, equation (7) implies a VIS2-a specification for interest rates

$$dr_t = r_t \frac{1}{2} e^{\alpha + \beta(\log r_t - \mu)^2} dt + r_t e^{\alpha/2 + \beta/2(\log r_t - \mu)^2} dW_t. \quad (8)$$

A direct application of Proposition 1(a) to equation (7) allows us to conclude that X has non-attracting boundaries and the speed density function m_X is integrable in $(-\infty, \infty)$ (in fact, $\lim_{x \rightarrow \pm\infty} xa(x)/b^2(x) = 0$ and $\lim_{x \rightarrow \pm\infty} x(a(x)/b^2(x) - b'(x)/b(x)) = -\infty$). Thus X is an ergodic process with stationary density

$$\bar{p}(x) = \frac{m_X(x)}{\int m_X(x) dx} = \frac{\sqrt{\beta}}{\sqrt{\pi}} e^{-\beta(x-\mu)^2} \quad (9)$$

i.e. $X = \log r \sim N(\mu, 1/(2\beta))$. By the continuous mapping theorem, $r = \exp(X)$ is an ergodic process. Furthermore, it has a log-normal stationary density. There is some empirical evidence that supports model (7). It is based on three facts.

(1) The empirical marginal distribution of $X_t = \log r_t$ matches the (marginal) distribution that is implicit in model (7). If models (7) and (8) are correct then the

empirical marginal (or unconditional) distribution of $X_t = \log r_t$ should be Gaussian (or approximately Gaussian). Table 2 and Figure 5 show that the hypothesis that the Fed funds rate logarithm is normally distributed cannot be rejected by the data in the period considered. Thus, the empirical marginal distribution of $X_t = \log r_t$ matches the (marginal) distribution that is implicit in model (7). It is worth mentioning that the first difference sequence of r and $\log r$ (respectively, $\{r_{t_i} - r_{t_{i-1}}\}_{i \geq 1}$ and $\{\log r_{t_i} - \log r_{t_{i-1}}\}_{i \geq 1}$) presents a completely different picture: fat tails, strong mean-reversion effects, etc.

(2) The results of Dickey–Fuller tests are compatible with a zero drift function for X , as specified in model (7). The logarithm of the Fed funds rate displays high persistence and the Augmented Dickey–Fuller tests do not reject an integrated process at all conventional levels (see Table 3). This pattern of persistence is similar to the Fed funds rate. As pointed out in section 1, a pure unit root process is impossible so the Dickey–Fuller test results can be misleading. We conjecture that these tests have very low power against certain forms of stationary local martingales. Although the Dickey–Fuller results are unreliable in deciding whether X is stationary

Table 2. Summary Statistics for log interest rates $\{\log r_{t_i}\}_{i \geq 1}$.

Mean	Variance	Kurtosis	Skewness	Bera–Jarque (p value)
−2.83	0.227	3.28	−0.182	0.113

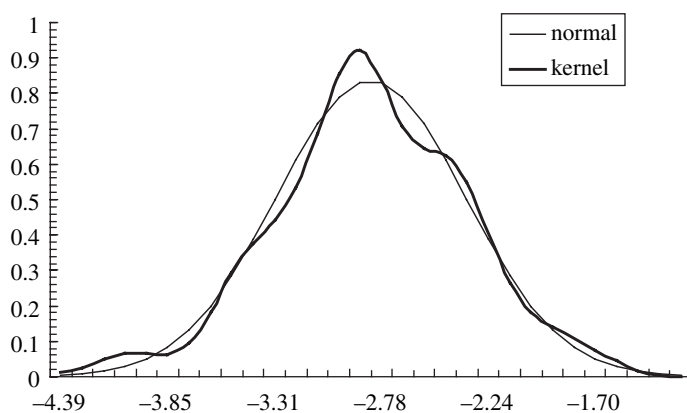


Fig. 5. Kernel estimates of the log interest rates vs. normal density.

Table 3. Augmented Dickey–Fuller for log Fed funds $\{\log r_{t_i}\}_{i \geq 1}$.

	Autoregr.param.	ADF <i>t</i> -statistic	10% CV
random walk without drift	1.000	0.177	−1.63
random walk with drift	0.991	−1.43	−2.59
random walk with drift and trend	0.991	−1.458	−3.16

or not, we consider that they give strong evidence of a zero drift diffusion. In effect, the null hypothesis of the Dickey–Fuller test is compatible with the integral equation $X_{t_i} = X_{t_i} + \int_{t_{i-1}}^{t_i} b(X_u) dW_u$ (zero drift).

(3) Nonparametric estimates of $a(x)$ and $b^2(x)$ do not reject specification (7). The conditional infinitesimal dynamics $X = \log(r)$ can be analysed through a nonparametric approach. This approach helps to identify the infinitesimal coefficients. We use the following nonparametric estimators

$$\hat{a}(x) = \frac{\sum_{i=0}^{n-1} K\left(\frac{x-X_{t_i}}{h}\right) \frac{(X_{t_{i+1}}-X_{t_i})}{\Delta}}{\sum_{i=0}^{n-1} K\left(\frac{x-X_{t_i}}{h}\right)}, \quad \hat{b}^2(x) = \frac{\sum_{i=0}^{n-1} K\left(\frac{x-X_{t_i}}{h}\right) \frac{(X_{t_{i+1}}-X_{t_i})^2}{\Delta}}{\sum_{i=0}^{n-1} K\left(\frac{x-X_{t_i}}{h}\right)}$$

for $a(x)$ and $b^2(x)$ respectively, where $\Delta = (t_i - t_{i-1}) = 1/12$ is the discretization step, K is the Gaussian kernel and $h = (4/3)^{0.2} \sigma n^{-1/5}$ (σ is the standard deviation of the X). The properties of these estimators are discussed in BANDI and PHILLIPS (2003) and NICOLAU (2003).

In Figure 6 we present nonparametric estimates of the infinitesimal coefficients using $\log(r)$ data.

It seems clear that a zero drift exists for the logarithm of interest rates. This has already been suggested by Dickey–Fuller tests. The presence of an exponential quadratic specification for the diffusion is, however, not so clear. But we cannot reject it either (in the sense that we can select α , β and μ such that the specification $b^2(x) = \exp\{\alpha + \beta(x - \mu)^2\}$ approximately matches the nonparametric estimates and is inside the confidence band). Nonparametric estimates for the diffusion also suggest a quadratic type specification for $\log r$. There is perhaps a more convincing

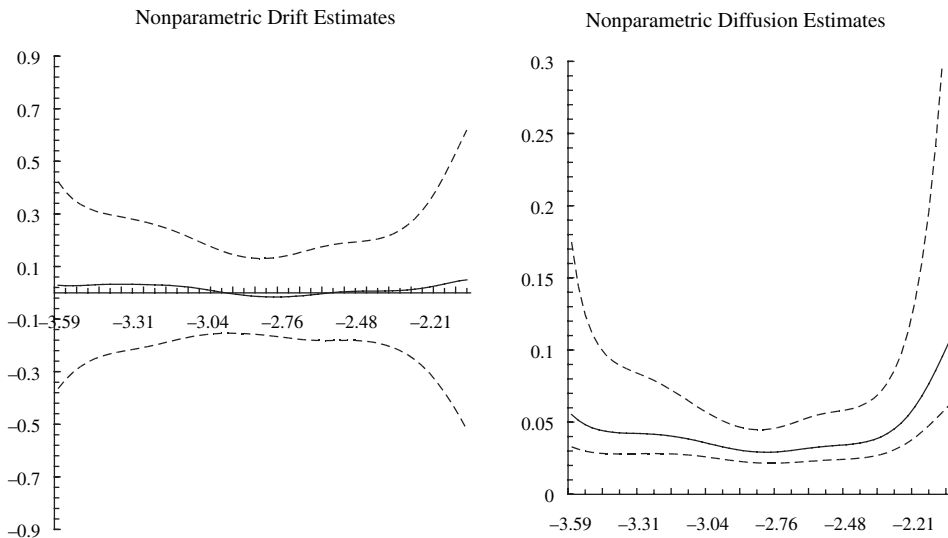


Fig. 6. Nonparametric estimates of drift and diffusion coefficients based on $\log(r)$ data (with 95% confidence band).

argument to support the exponential quadratic specification. A zero drift coefficient of a stationary diffusion process governed by a SDE $dX_t = b(X_t)dW_t$ has a stationary density proportional to $b^{-2}(x)$ (this is due to the fact that $a(x) = 0 \Rightarrow s_X(x) = 1 \Rightarrow m_X(x) = b^{-2}(x)$, and thus if X is a stationary process then $\bar{p}(x)$ is proportional to $m_X(x) = b^{-2}(x)$ – see appendix 1). For example, if we choose a quadratic specification for $b^2(x)$ (i.e. $b^2(x) = \alpha + \beta(x - \mu)^2$, $\alpha, \beta > 0$), the solution of the SDE $dX_t = b(X_t)dW_t$ will have a Cauchy stationary density. If we choose an exponential quadratic specification for $b^2(x)$ (i.e. $b^2(x) = \exp\{\alpha + \beta(x - \mu)^2\}$, $\beta > 0$), the solution of the SDE $dX_t = b(X_t)dW_t$ will have a Gaussian stationary density (see equation (9)). Since, as we pointed out, the data give evidence of a Gaussian (marginal) distribution for $X = \log r$, the most appropriate choice for $b^2(x)$ is the exponential quadratic specification.

We observe that the logarithmic transformation induces the volatility to rise when the interest rate is low and to decrease when the interest rate is high. This explains why non-parametric estimates of the logarithm of Fed funds volatility has a “U” form (smiling volatility).

3.2. Parametric estimation

The transition (or conditional) densities of X required to construct the exact likelihood function are unknown. Several estimation approaches have been proposed under these circumstances (see NICOLAU, 2002, for a brief survey). To estimate the parameters of equation (7) we considered the simulated maximum likelihood estimator suggested in NICOLAU (2002) (with $N = 20$ and $S = 20$). The method proposed by AÏT-SAHALIA (2002) with $J = 1$ (Aït-Sahalia’s notation for the order of expansion of the density approximation) gives similar results (to apply Aït-Sahalia’s method we use a program developed by Aït-Sahalia, which is available on his homepage). The approximation of the density based on $J = 2$ is too complicated to implement (it involves dozens of intricate expressions that are difficult to evaluate).

The quasi-MLE based on the Euler transition density approximation also gives similar results. As we have pointed out, although X is a stationary process, X cannot be simulated by standard discretized methods (e.g. Euler, Milstein or Platen) since these schemes are explosive. Thus, simulation-based methods using discretized solutions such as Pedersen’s method cannot, in principle, be used. In Table 4 we report the estimation results.

We observe that the parametric (and non-parametric) estimates of $a(r) = r^{\frac{1}{2}}e^{\alpha + \beta(\log r - \mu)^2}$ are approximately zero when $r \in (0.02, 0.12)$. Thus, in this interval, a quasi-martingale behaviour for r is allowed by the model. The minimum point of $a(r)$ is $\exp(\frac{1}{2\beta}(2\beta\mu - 1))$. An estimate of this quantity (based on parameter

Table 4. Estimates of (7) (simulated MLE).

	α	β	μ
Estimate	-3.49	1.59	-2.92
St Error	0.11	0.30	0.07

Table 5. Log-likelihood of some parametric models for the monthly federal funds data, 1963–1998.

Models	$\ln L$	no. of parameters
$dr_t = \kappa(\tau - r_t)dt + \sigma dW_t$	1569.9	3
$dr_t = \kappa(\tau - r_t)dt + \sigma\sqrt{r_t}dW_t$	1692.6	3
$dr_t = r_t(\kappa - (\sigma^2 - \kappa\tau)r_t)dt + \sigma r_t^{3/2}dW_t$	1801.9	3
$dr_t = \kappa(\tau - r_t)dt + \sigma r_t^\rho dW_t$	1802.3	4
$dr_t = (\beta_1 r_t^{-1} + \beta_2 + \beta_3 r_t + \beta_4 r_t^2)dt + \sigma r_t^{3/2}dW_t$	1802.7	5
$dr_t = r_t^{\frac{1}{2}}e^{x+\beta(\log r_t - \mu)^2}dt + r_t e^{x/2+\beta/2(\log r_t - \mu)^2}dW_t$	1805.1	3

Source (first five models): Table VI of AïT-SAHALIA (1999). In Aït-Sahalia's Table VI, $\ln L$ means $\ln L/n$.

estimates) is 0.04 (4%) and the minimum value is $a(0.04) = 0.0007$. In the neighbourhood of $r = 4\%$ the drift is approximately zero and only when $\log r_t$ moves significantly away from μ , that is, when the quantity $(\log r - \mu)^2$ is large, does the drift increase. The drift increases especially sharply when r approaches zero since in this case the quantity $(\log r - \mu)^2$ becomes arbitrarily large. This behaviour prevents r from reaching the zero boundary.

Our model (7) compares extremely favourably with other proposed one-factor continuous-time models. In Table 5 we compare the proposed model with other relevant models for interest rates. Only the proposed method was estimated by us. The remaining information was obtained from Table VI of AïT-SAHALIA (1999). For comparison purposes the proposed model was estimated using the same method applied to the other models (we considered the density approximation proposed by AïT-SAHALIA, 2002, with $J = 1$, in the period January 1963 to December 1998). Table 5 indicates that the proposed model outperforms the others in terms of accuracy and parsimony.

4 Conclusion

In this paper we stress the role of volatility in ensuring stationarity. We suggest a refinement of the existing definition of volatility-induced stationarity that allows us to distinguish between processes with drift and diffusion induced stationarity and processes with pure volatility-induced stationarity. We also emphasize that the volatility needed to inject stationarity varies according to the nature of the associated process. Typically, it has to be higher when the associated process is explosive than when the associated process is only null-recurrent. This remark leads to a further classification of volatility-induced stationarity.

As an application we consider the Fed funds rate. We provide evidence that the logarithm of the Fed funds rate can be modelled as a local martingale with volatility-induced stationarity.

We believe that there are other economic applications where volatility can inject stationarity, either as in VIS1 where both infinitesimal coefficients induce stationarity, or as in VIS2 where only the diffusion coefficient induces stationarity.

Appendix 1 – Stationary and Ergodic conditions for univariate diffusions

Let $X = \{X_t, t \geq 0\}$ be a diffusion process, with state space $I = (l, r)$, governed by the stochastic differential equation

$$dX_t = a(X_t)dt + b(X_t)dW_t, \quad X_0 = x$$

where $\{W_t, t \geq 0\}$ is a (standard) Wiener process, a and b are the infinitesimal coefficients and x is either a constant value or a random value \mathcal{F}_0 -measurable independent of W_t . We assume that a and b have continuous derivatives.

Let $s_X(z) = \exp\{-\int_{z_0}^z 2a(u)/b^2(u)du\}$ be the scale density function (z_0 is an arbitrary point inside I) and $m_X(u) = (b^2(u)s_X(u))^{-1}$ the speed density function. Let $S_X(l, x] = \lim_{x_1 \rightarrow l} \int_{x_1}^x s_X(u)du$ and $S_X[x, r) = \lim_{x_2 \rightarrow r} \int_x^{x_2} s_X(u)du$ where, $l < x_1 < x < x_2 < r$. To characterize the X process we present four assumptions. The first one is:

$$S_X(l, x] = S_X[x, r) = \infty \text{ for } x \in I. \quad (\text{A1})$$

According to ARNOLD (1974, page 114), if the infinitesimal coefficient a and b have continuous derivatives with respect to x , then there exists a unique continuous process that is defined up to a random explosion time η in the interval $t_0 < \eta \leq \infty$. The A1 condition assures that $P[\eta = \infty | X_0 = x] = 1$ (IKEDA and WATANABE, 1981, pp. 362–363). Furthermore, the boundaries l and r are neither attracting, nor attainable (see KARLIN and TAYLOR, 1981, chapter 15) and the process is recurrent, i.e. $P[T_y < \infty | X_0 = x] = 1$ for every $x, y \in I$ where $T_y = \inf\{t \geq 0, X_t = y\}$ (IKEDA and WATANABE, 1981, Theorem 3.1, Chapter VI). Roughly speaking, the boundaries l and r are never attained although every finite point can be reached with probability one in finite time. Global Lipschitz and growth conditions, which fail to be satisfied for many interesting models in economics in finance (AÏT-SAHALIA, 1996), are not needed in the presence of the previous assumptions. The A1 condition is not very strong: for example, the standard Brownian motion satisfies the A1 condition (KARLIN and TAYLOR, 1981, page 228). Actually, every process with zero drift $a(x) = 0$ (and $b(x) > 0$) satisfies the A1 condition.

$$\int_l^r m_X(x)dx < \infty. \quad (\text{A2})$$

A1 and A2 conditions assure that X is ergodic and the invariant distribution P^0 has density $\bar{p}(x) = m_X(x) / \int_l^r m_X(u)du$ with respect to the Lebesgue measure. They are also necessary conditions (SKOROKHOD, 1989, theorem 16). The expression $\bar{p}(x)$ is usually denoted as stationary density.

$$X_0 = x \text{ has distribution } P^0. \quad (\text{A3})$$

Assumption A3, together with A1–A2 implies that X is stationary (ARNOLD, 1974). Assumption A3 can, in some cases, be replaced by: X_0 is a random variable with mean μ and variance σ^2 such that $\int x\bar{p}(x)dx = \mu < \infty$ and

$\int (x - \mu)^2 \bar{p}(x) dx = \sigma^2 < \infty$. In this case, X is a covariance-stationary process. Even if the A3 condition does not hold, it is known that when t is sufficiently large, the distribution of the ergodic process X_t is well approximated by the distribution with density $\bar{p}(x)$.

$$\limsup_{x \rightarrow r} \left(\frac{a(x)}{b(x)} - \frac{b'(x)}{2} \right) < 0, \limsup_{x \rightarrow l} \left(\frac{a(x)}{b(x)} - \frac{b'(x)}{2} \right) > 0. \quad (\text{A4})$$

These conditions are discussed in CHEN *et al.* (1998) and are similar to ones proposed by HANSEN and SCHEINKMAN (1995). Under the A4 assumption the process is ρ -mixing (see CHEN *et al.*, 1998). Technically, for a Markov process, the notion of ρ -mixing requires the conditional expectations operator for any interval of time to be a strong contraction for all functions with zero mean and finite variance. As a consequence, the j th autocovariance of $f(X_t)$ tends to zero at exponential rate as $j \rightarrow \infty$, for all functions f such that $\int f(x) \bar{p}(x) dx = 0$ and $\int f^2(x) \bar{p}(x) dx < \infty$ (see HANSEN and SCHEINKMAN, 1995, proposition 8). We notice that even if the drift is zero or converges to zero, the A4 assumption can hold, provided that volatility grows at least linearly.

As an example, consider the SDE discussed in section 1: $dX_t = (1 + X^2) dW_t$. Thus, $a(x) = 0$ and $b(x) = 1 + x^2$ and the state space of X is $(l, r) = (-\infty, \infty)$. We have,

$$\begin{aligned} s_X(z) &= \exp\left\{-\int_0^z 2a(u)/b^2(u) du\right\} = 1, \\ S_X(l, x] &= \lim_{x_1 \rightarrow -\infty} \int_{x_1}^x s_X(u) du = \lim_{x_1 \rightarrow -\infty} \int_{x_1}^x 1 du = \infty, \\ S_X[x, r) &= \lim_{x_2 \rightarrow \infty} \int_x^{x_2} 1 du = \infty \\ m_X(x) &= (b^2(x) s_X(x))^{-1} = \frac{1}{(1+x^2)^2}, \\ \int_l^r m_X(x) dx &= \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx = \frac{1}{2} \pi < \infty. \end{aligned}$$

Therefore, the stationary density is

$$\bar{p}(x) = \frac{m_X(x)}{\int_l^r m_X(x) dx} = \frac{1/(1+x^2)^2}{\frac{1}{2}\pi} = \frac{2}{(\pi(1+x^2)^2)}$$

and the stationary (or unconditional) moments are

$$E[X] = \int_{\mathbb{R}} x \bar{p}(x) dx = 0, \quad \text{Var}[X] = \int_{\mathbb{R}} x^2 \bar{p}(x) dx = 1.$$

In this example the A4 assumption still holds.

Proposition 1(a) establishes sufficient conditions (easy to verify) for A1 and A2. We can use Proposition 1 (a) in the previous example. Since

$$\lim_{x \rightarrow \pm\infty} x \frac{a(x)}{b^2(x)} = 0 < \frac{1}{2}$$

$$\lim_{x \rightarrow \pm\infty} x \left(\frac{a(x)}{b^2(x)} - \frac{b'(x)}{b(x)} \right) = \lim_{x \rightarrow \pm\infty} x \left(-\frac{2x}{1+x^2} \right) = -2 < -\frac{1}{2},$$

proposition 1(a) guarantees that X has a stationary distribution.

Appendix 2 – VIS: discussion and examples

Consider SDEs (3) and (4) and the following functionals (see KARLIN and TAYLOR, 1981) of the associated Y process:

$$S_Y[x, x_0] = \int_{x_0}^x s_Y(u) du$$

$$\Sigma_Y(l) = \lim_{\alpha \downarrow l} \int_{\alpha}^{x_0} S_X[\alpha, u] m_Y(u), \Sigma_Y(r) = \lim_{\beta \uparrow r} \int_{x_0}^{\beta} S_Y[u, \beta] m_Y(u)$$

where $s_Y(x) = \exp\{-\int_{x_0}^x 2a(u)/\sigma^2 du\}$ and $m_Y(u) = 1/\sigma^2 s_Y(u)$. Let $l < \alpha < \beta < r$, $T_{\alpha} = \inf\{t \geq 0: Y_t = \alpha\}$ and $T_{\alpha} \wedge T_{\beta} = \min\{T_{\alpha}, T_{\beta}\}$. We say that the stationary process X has volatility-induced stationary of type $2a$, $2b$ and $2c$ (VIS2-a, VIS2-b and VIS2-c) if it has, respectively:

VIS2-a $\Sigma_Y(l) < \infty \Leftrightarrow \lim_{\alpha \rightarrow -\infty} E[T_{\alpha} \wedge T_{\beta} | X_0 = x] < \infty$ or

$\Sigma_Y(r) < \infty \Leftrightarrow \lim_{\beta \rightarrow \infty} E[T_{\alpha} \wedge T_{\beta} | X_0 = x] < \infty$.

VIS2-b $S_Y(l, x] < \infty \Leftrightarrow \lim_{\alpha \rightarrow -\infty} P_x[T_{\alpha} \leq T_{\beta} | X_0 = x] > 0$ or

$S_Y[x, r) < \infty \Leftrightarrow \lim_{\beta \rightarrow \infty} P_x[T_{\beta} \leq T_{\alpha} | X_0 = x] > 0$ and

$\Sigma_Y(l) = \Sigma_Y(r) = \infty$.

VIS2-c $S_Y(l, x] = S_Y[x, r) = \infty, \int_l^r m_Y(u) du = \infty$.

We briefly analyse these conditions and relate them to VIS.

In VIS2-a at least one of the boundaries is attainable, i.e. the expected time for one of the boundaries to be reached is finite (notice that $\Sigma_Y(l) < \infty$ implies $S_Y(l, x] < \infty$). When $l = -\infty$ and $r = \infty$, this case characterizes explosive processes (the expected time to reach $l = -\infty$ or $r = \infty$ is finite). Furthermore, the process is not recurrent. As an example, consider $dY_t = Y_t^3 dt + \sigma dW_t$. It can be proved that $\Sigma_Y(l) < \infty$ and $\Sigma_Y(r) < \infty$. Any trajectories of this process quickly go to infinity in a finite expected time. Further, the conditions $\Sigma_Y(l) < \infty$ and $\Sigma_Y(r) < \infty$ imply $P[\eta < \infty | Y_0 = y] = 1$ where η is the random explosion time of Y defined in the interval $t_0 < \eta \leq \infty$ (IKEDA and WATANABE, 1981, theorem 3.2). That is, the process blows up in finite time with probability one.

When one or both boundaries are finite one has an absorbing phenomenon (reflecting or sticky barrier is possible if further assumptions regarding the behaviour

at the boundaries is specified; in any case, this would imply a discontinuity of the trajectory).

The solution of $dX_t = X_t^3 dt + (1 + X_t^4) dW_t$ is an example of VIS of type 2a, since it can be proved that $S_X(-\infty, x] = S_X[x, \infty) = \infty$, $\int_{-\infty}^{\infty} m_X(u) du < \infty$. Another example of VIS2-a is $dX_t = (1 + X_t + X_t^2) dt + X_t^4 dW_t$, with state space $(0, \infty)$. This case is analysed in CHLS.

To explain the VIS2-b case, suppose $S_Y[x, r) < \infty$ and $r = \infty$. Then the boundary $r = \infty$ can be reached prior to reaching an arbitrary state β with positive probability from any interior starting point $x < \beta$, although not in a finite expected time, as by hypothesis, $\Sigma_Y(r) = \infty$. This condition can characterize processes that drift away from a central region in the state space, eventually reaching the boundaries but without blowing out in finite time with probability one. Consider for example the SDE $dY_t = Y_t/(1 + Y_t^2) dt + dW_t$, $Y_0 = 0$ with state space $(-\infty, \infty)$. It can be proved that $S_Y(l, x] < \infty$, $S_Y[x, r) < \infty$ and $\Sigma_Y(l) = \Sigma_Y(r) = \infty$. Any realization of Y tends to move further and further away from $Y_0 = 0$ (since $a(x) > 0$ if $x > 0$ and $a(x) < 0$ if $x < 0$). These trajectories are accompanied by a reduction of the drift magnitude. Despite $a(x) \rightarrow 0$ as $|x| \rightarrow \infty$, we have $|Y_t| \rightarrow \infty$ as $t \rightarrow \infty$ with positive probability, although the random explosion time occurs in infinite time with probability one. The expected time to reach $r = \infty$ is infinite, i.e. $\Sigma_Y(r) = \infty$. As in the previous case, these processes are not recurrent. This second case can be thought of as a moderate version of case one: although non-recurrent, the expected time to reach ∞ or $-\infty$ is infinite.

The solution of $dX_t = X_t/(1 + X_t^2) dt + \sqrt{1 + X_t^2} dW_t$ is an example of VIS2-b since it can be shown $S_X(l, x] = \infty$, $S_X[x, r) = \infty$ and $\int_l^r m_X(u) du < \infty$. Another example of VIS2-b is the solution of $dX_t = (1 + X_t) dt + (1 + X_t^2) dW_t$. In fact, while X is an ergodic process, the solution of $dY_t = (1 + Y_t) dt + dW_t$ has boundaries that are attracting, although the expected time to reach them is infinite.

In the third case (VIS2-c) Y is a null-recurrent process. If $\beta > x$, the recurrence (every finite point can be reached with probability one in finite time) follows from $P[T_\beta < \infty | X_0 = x] \geq P[T_\beta < T_l | X_0 = x] = 1$ (we notice that $S_Y(l, x] = \infty$ is equivalent to $P[T_\beta < T_l | X_0 = x] = 1$). By similar arguments, if $\alpha < x$, then $S_Y[x, r) = \infty$ implies $P[T_\alpha < \infty | X_0 = x] = 1$. The boundaries cannot be reached from the interior of the state space (observe that $S_Y[x, r) = \infty \rightarrow \Sigma_Y(r) = \infty$). The case $\int_l^r m_Y(u) du = \infty$ can occur either with $M_Y(l, x] = \infty$ or $M_Y[x, r) = \infty$. The quantity $M_Y(l, x]$ can be interpreted as a measure of the speed of the process near l (see KARLIN and TAYLOR, 1981, page 231). The higher this quantity is, the stickier the boundary. Generally speaking, the condition $\int_l^r m_Y(u) du = \infty$ can be thought of as an absence of a general drifting towards a central set for large excursions of the stochastic paths.

The solution of $dX_t = \sqrt{1 + X_t^2} dW_t$ is an example of VIS2-c, since the X solution admits an ergodic distribution and the associated Y process (the Brownian motion) is a null-recurrent process and nonstationary.

Appendix 3 – Proof of Proposition 1

(a) If the infinitesimal coefficients a and b have continuous derivatives with respect to x , then there is a unique continuous process that is defined up to a random explosion time η in the interval $t_0 < \eta \leq \infty$ (ARNOLD, 1974, page 114). This random explosion time is infinite with probability one if $S_X(l, x_0) = S_X[x_0, r) = \infty$ for $x_0 \in \mathbb{R}$ (IKEDA and WATANABE, 1981, pp. 362–363) where

$$S_X[c, d] = S_X(d) - S_X(c) \text{ and } S_X(x) = \int_{x_0}^x s_X(u) du.$$

Firstly, we show that $\lim_{x \rightarrow \pm\infty} x[a(x)/b^2(x)] < 1/2$ implies $S_X(l, x] = S_X[x, r) = \infty$. Consider

$$S_X[x_0, r) = \lim_{\xi \rightarrow \infty} \int_{x_0}^{\xi} s_X(u) du = \lim_{\xi \rightarrow \infty} \int_{x_0}^{\xi} \exp \left\{ - \int_{z_0}^u 2 \frac{a(s)}{b^2(s)} ds \right\} du$$

If there exists some $x_1 > x_0$ such that

$$\exp \left\{ - \int_{z_0}^x 2 \frac{a(s)}{b^2(s)} ds \right\} > \frac{1}{x} \Leftrightarrow - \int_{z_0}^x 2 \frac{a(s)}{b^2(s)} ds > -\log x \Leftrightarrow \int_{z_0}^x 2 \frac{a(s)}{b^2(s)} ds < \log x$$

(z_0 is an arbitrary value) for all $x > x_1 > 0$ then $S_X[x_0, r) = \infty$ (observe that $\lim_{\xi \rightarrow \infty} \int_{x_0}^{\xi} 1/x dx = \infty$). Equivalently, if

$$\lim_{x \rightarrow \infty} \frac{\int_{z_0}^x 2 \frac{a(s)}{b^2(s)} ds}{\log x} < 1$$

then s_X is not integrable. Using L'Hôpital rule we have

$$\lim_{x \rightarrow \infty} \frac{\int_{z_0}^x 2 \frac{a(s)}{b^2(s)} ds}{\log x} < 1 \Leftrightarrow \lim_{x \rightarrow \infty} \frac{2 \frac{a(x)}{b^2(x)}}{\frac{1}{x}} < 1 \Leftrightarrow \lim_{x \rightarrow \infty} \frac{a(x)}{b^2(x)} x < \frac{1}{2}.$$

Thus, if $[a(x)/b^2(x)]x < 1/2$ as $x \rightarrow \infty$, then $S_X[z_0, r) = \infty$. Using similar arguments (with a change of variable $y = -x$) we conclude that if $[a(x)/b^2(x)]x < 1/2$ as $x \rightarrow -\infty$ then $S_X(l, z_0] = \infty$.

We now show that $\lim_{x \rightarrow \pm\infty} x[a(x)/b^2(x) - b'(x)/b(x)] < -1/2$ implies $M_X[x_0, r) < \infty$ and $M_X(l, x_0] < \infty$, i.e. $\int_{-\infty}^{\infty} m_X(x) dx < \infty$ where $m_X(x) = 1/(s_X(x)b^2(x))$.

Consider

$$M_X[x_0, r) = \lim_{\xi \rightarrow \infty} \int_{x_0}^{\xi} \frac{1}{s_X(u)b^2(u)} du = \lim_{\xi \rightarrow \infty} \int_{x_0}^{\xi} \frac{\exp \left\{ \int_{z_0}^u 2 \frac{a(s)}{b^2(s)} ds \right\}}{b^2(u)} du$$

If there exists some $x_1 > x_0$ and $\alpha > 1$ such that

$$\begin{aligned} \frac{\exp\left\{\int_{z_0}^x 2 \frac{a(s)}{b^2(s)} ds\right\}}{b^2(x)} < \frac{1}{x^\alpha} &\Leftrightarrow \int_{z_0}^x 2 \frac{a(s)}{b^2(s)} ds - \log b^2(x) < -\alpha \log x \\ &\Leftrightarrow \frac{\int_{z_0}^x 2 \frac{a(s)}{b^2(s)} ds - \log b^2(x)}{\alpha \log x} < -1 \end{aligned}$$

for all $x > x_1 > 0$ then $M_X[x_0, r) < \infty$. Equivalently, if

$$\lim_{x \rightarrow \infty} \frac{\int_{z_0}^x 2 \frac{a(s)}{b^2(s)} ds - \log b^2(x)}{\alpha \log x} < -1$$

then $M_X[x_0, r) < \infty$. Using L'Hôpital rule we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\int_{z_0}^x 2 \frac{a(s)}{b^2(s)} ds - \log b^2(x)}{\alpha \log x} < -1 &\Leftrightarrow \lim_{x \rightarrow \infty} \frac{2 \frac{a(x)}{b^2(x)} - 2 \frac{b'(x)}{b(x)}}{\frac{\alpha}{x}} < -1 \\ &\Leftrightarrow \lim_{x \rightarrow \infty} \left(\frac{a(x)}{b^2(x)} - \frac{b'(x)}{b(x)} \right) x < -\frac{\alpha}{2} \end{aligned}$$

Thus $\lim_{x \rightarrow \infty} [a(x)/b^2(x) - b'(x)/b(x)]x < -1/2$ is a sufficient condition for $M_X[x_0, r) < \infty$.

Using similar arguments, one can conclude that

$$\lim_{x \rightarrow -\infty} \left(\frac{a(x)}{b^2(x)} - \frac{b'(x)}{b(x)} \right) x < -\frac{1}{2} \Rightarrow M_X(l, x_0] < \infty.$$

Note that for any constants x_0 and x_1 we have $\int_{-\infty}^{\infty} m_X(x) dx = M_X(l, x_0] + M_X[x_0, x_1] + M_X[x_1, \infty)$. Since a and b are continuous functions and $b(x) > 0$ we always have $M_X[x_0, x_1] < \infty$. Therefore, $M_X[x_0, r) < \infty$ and $M_X(l, x_0] < \infty$ imply $\int_{-\infty}^{\infty} m_X(x) dx < \infty$. Now, it is known that if $S_X(l, x) = S_X[x, r) = \infty$ and $\int_{-\infty}^{\infty} m_X(x) dx < \infty$ then X is ergodic and the invariant distribution P^0 has density $\bar{p}(x) = m_X(x) / \int_l^r m_X(u) du$ with respect to the Lebesgue measure (see SKOROKHOD, 1989, theorem 16).

(b) It is sufficient to show that $\Sigma_Y(\infty) = \Sigma_Y(-\infty) < \infty$ with respect to the process $dY_t = X_t^\alpha dt + \sigma dW_t$ (without loss of generality we fix $\sigma = 1$). We have

$$\Sigma_Y(\infty) = \lim_{\xi \rightarrow \infty} \int_{x_0}^{\xi} S_Y[u, \beta] m_Y(u) = \lim_{\xi \rightarrow \infty} \int_{x_0}^{\xi} f(u) du$$

where

$$f(x) = 2^{-\frac{1}{1+\alpha}} \left(-\frac{1}{1+\alpha} \right)^{\frac{-1}{1+\alpha}} e^{\frac{-2x^{1+\alpha}}{1+\alpha}} \left(\text{gamma} \left[\frac{1}{1+\alpha}, -\frac{2}{1+\alpha} \right] - \text{gamma} \left[\frac{1}{1+\alpha}, -\frac{2x^{1+\alpha}}{1+\alpha} \right] \right)$$

and

$$\text{gamma}(a, x) = \int_x^\infty t^{a-1} e^{-t}.$$

Simple but cumbersome calculations show that if $\alpha \leq 1$ then $f(x) > 1/x$ which implies that $\Sigma_Y(\infty) = \infty$ and that if $\alpha > 1$ there exists a $\delta > 1$ such that $f(x) < 1/x^\delta$ which implies that $\Sigma_Y(\infty) < \infty$. Similar arguments can be used to conclude that $\alpha \leq 1$ implies that $\Sigma_Y(-\infty) = \infty$ and $\alpha > 1$ implies that $\Sigma_Y(-\infty) < \infty$.

(c) Following the arguments in (b) we conclude that $a(x) = O(|x|^\alpha)$, $\alpha \leq 1$ implies that $\Sigma_Y(\infty) = \Sigma_Y(-\infty) < \infty$. It remains to be shown that $S_Y(l, x_0] < \infty$ and $S_Y[x_0, r) < \infty$.

Consider

$$S_Y[x_0, r) = \lim_{\xi \rightarrow \infty} \int_{x_0}^{\xi} s_Y(u) du = \lim_{\xi \rightarrow \infty} \int_{x_0}^{\xi} \exp\left\{-\frac{1}{\sigma^2} \int_{z_0}^u 2a(s) ds\right\} du$$

If there exists some $x_1 > x_0$ and $\alpha > 1$ such that

$$\exp\left\{-\frac{1}{\sigma^2} \int_{z_0}^x 2a(s) ds\right\} < \frac{1}{x^\alpha} \Leftrightarrow \frac{1}{\sigma^2} \int_{z_0}^x 2a(s) ds > \alpha \log x$$

for all $x > x_1$ then s_Y is integrable on (x_0, r) , i.e. $S_Y[x_0, r) < \infty$. Equivalently, if

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{\sigma^2} \int_{z_0}^x 2a(s) ds}{\alpha \log x} > 1$$

then s_Y is integrable on (x_0, r) . Using L'Hôpital rule we have

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{\sigma^2} \int_{z_0}^x 2a(s) ds}{\alpha \log x} > 1 \Leftrightarrow \lim_{x \rightarrow \infty} \frac{\frac{1}{\sigma^2} 2a(x)}{\alpha/x} > 1 \Leftrightarrow \lim_{x \rightarrow \infty} xa(x) > \frac{\alpha \sigma^2}{2}.$$

Since σ is a small fixed controlled parameter, it follows that $\lim_{x \rightarrow \infty} xa(x) > 0$ implies $S_Y[x_0, r) < \infty$. Similar argument, allow us to conclude that $\lim_{x \rightarrow -\infty} xa(x) > 0$ implies $S_Y(l, x_0] < \infty$.

(d) We sketch the proof since it is similar to the above ones. We have

$$\begin{aligned} \lim_{x \rightarrow \pm\infty} xa(x) < \frac{\sigma^2}{2} &\Rightarrow S_Y[z_0, r) = \infty \text{ and } S_Y(l, z_0] = \infty \\ \lim_{x \rightarrow \pm\infty} a(x)x > -\frac{\sigma^2}{2} &\Rightarrow M_Y[x, r) = \infty \text{ and } M_Y(l, x_0] = \infty. \end{aligned}$$

Thus $\lim_{x \rightarrow \pm\infty} xa(x) = 0$ is sufficient to assure the results stated. \square

References

- AÏT-SAHALIA, Y. (1996), Testing Continuous-Time Models of the Spot Interest Rate, *The Review of Financial Studies* **9**, 385–426.
 AÏT-SAHALIA, Y. (1999), Transition Densities for Interest Rate and Other Nonlinear Diffusions, *The Journal of Finance* **LIV**, 1361–1395.

- AÏT-SAHALIA, Y. (2002), Maximum Likelihood Estimation of Discretely Sampled Diffusions: a Closed-Form Approximation Approach, *Econometrica* **70**, 223–262.
- ARNOLD, L. (1974), *Stochastic Differential Equations: Theory And Application*, John Wiley & Sons, New York.
- BANDI, F. (2002), Short-Term Interest Rate Dynamics: A Spatial Approach, *Journal of Financial Economics* **65**, 73–110.
- BANDI, F. and P. PHILLIPS (2003), Fully Nonparametric Estimation of Scalar Diffusion Models, *Econometrica* **71**, 241–283.
- BIBBY, B. and M. SØRENSEN (1995), Martingale Estimation Function for Discretely Observed Diffusion Process, *Bernoulli* **1**, 17–39.
- CHAN, K., G. KAROLYI, F. LONGSTAFF and A. SANDERS (1992), An Empirical Comparison of Alternative Models of the Short-Term Interest Rate, *The Journal of Finance* **XLVII**, 1210–1227.
- CHEN, X., HANSEN and M. CARRASCO (1998), Nonlinearity and Temporal Dependence, Unpublished.
- CHAPMAN, D. and N. PEARSON (2000), Is the Short Rate Drift Actually Nonlinear? *Journal of Finance* **55**, 355–388.
- CONLEY, T., L. HANSEN, E. LUTTMER and J. SCHEINKMAN (1997), Short-term interest rates as subordinated diffusions, *The Review of Financial Studies* **10**, 525–577.
- COX, J., J. INGERSOLL and S. ROSS (1985), A theory of the Term Structure of Interest Rates, *Econometrica* **53**, 385–407.
- HANSEN, L. and J. SCHEINKMAN (1995), Back to the Future: Generating Moment Implications for Continuous-Time Markov Processes, *Econometrica* **63**, 767–804.
- HONG, Y., L. HAITAO and F. ZHAO (2004), Out-of-sample performance of spot interest rate models, *Journal Business and Economic Statistics* **22**, 457–473.
- IKEDA, N. and S. WATANABE (1981), *Stochastic Differential Equations and Diffusion Processes*, North-Holland Mathematical Library, Amsterdam.
- KARLIN, S. and H. TAYLOR (1981), *A Second Course in Stochastic Processes*, Academic Press, New York.
- NICOLAU, J. (2002), A New Technique for Simulating the Likelihood of Stochastic Differential Equations, *The Econometrics Journal* **5**, 91–103.
- NICOLAU, J. (2003), Bias Reduction in Nonparametric Diffusion Coefficient Estimation, *Econometric Theory* **19**, 774–777.
- NICOLAU, J. (2005), A Method for Simulating Non-Linear Stochastic Differential Equations in \mathbb{R}^1 , *Journal of Statistical Computation and Simulation*, Forthcoming.
- PRITSKER, M. (1998), Nonparametric Density Estimation and Tests of Continuous Time Interest Rate Models, *Review of Financial Studies*, **11**, 449–487.
- RICHTER, M. (2002), A Study of Stochastic Differential Equations with Volatility Induced Stationarity, Unpublished.
- SKOROKHOD, A. (1989), *Asymptotic Methods in the Theory of Stochastic Differential Equation*, Translation of Mathematical Monographs 78, American Mathematical Society, Providence, Rhode Island.

Received: January 2004. Revised: March 2005.