Wind project inference

Khaoula Ben Chaabane

Ecole Polytechnique de Tunisie

26/05/2020

Introduction

In these slides, I will explain the inference process with which we estimate the parameters of our model.

The wind power production is modeled as follows, where X_t is the normalized real production :

$$\begin{cases} dX_t = \left(\dot{p}_t - \theta_t \left(X_t - p_t\right)\right) dt + \sqrt{2\alpha\theta_0 X_t \left(1 - X_t\right)} dW_t, \quad t \in [0, T] \\ X_0 = x_0 \in [0, 1] \end{cases}$$

We may introduce the following model for the forcecast error of the normalized wind power production where X_t is the real production, p_t the forecast and $V_t = X_t - p_t$ is the error :

$$\begin{cases} dV_t = -\theta_t V_t dt + \sqrt{2\alpha\theta_0 \left(V_t + p_t\right)\left(1 - V_t - p_t\right)} dW_t, \quad t \in [0, T] \\ V_0 = v_0 \in [-1 + \varepsilon, 1 - \varepsilon] \end{cases} \tag{1}$$

Model

To guarantee a unique solution for the process X_t, θ_t needs to be bounded for $t \in [0, T]$. We have that :

$$\theta_t = \max\left(\theta_0, \frac{\alpha\theta_0 + |2\dot{p}_t|}{2\min(1-p_t, p_t)}\right)$$

This is not true for θ_t if $p_t \to 0^+$ or $p_t \to 1^-$. Therefore we need to ensure that $p_t \in [\varepsilon, 1-\varepsilon]$ for some $0 < \varepsilon < \frac{1}{2}$, $\forall\ t \in [0, T]$.

Model

We define then the corrected forecast :

$$p_t^\varepsilon = \begin{cases} \varepsilon & \text{if} \quad p_t < \varepsilon \\ p_t & \text{if} \quad \varepsilon \le p_t < 1 - \varepsilon \\ 1 - \varepsilon & \text{if} \quad p_t \ge 1 - \varepsilon \end{cases}$$

and the corrected (and bounded) drift coefficient is therefore :

$$\theta_t^\varepsilon = \max\left(\theta_0, \frac{\alpha\theta_0 + 2\,|\dot{p}_t^\varepsilon|}{2\min\left(1 - p_t^\varepsilon, p_t^\varepsilon\right)}\right)$$

Likelihood

We sample each of our M continuous-time Itô process $V = (V_t)_{t\in[0,T]}$ at $N+1$ equidistant discrete points with a given length interval Δ. $V^{M,N+1} = \left\{V_{t_1}^{N+1}, V_{t_2}^{N+1}, \dots, V_{t_{M+1}}^{N+1}\right\}$ denotes this random sample, with $V_{t_j}^{N+1} = \left\{V_{t_j+i\Delta}, i = 0, \dots, N\right\}, \forall j \in \{1, \dots, M\}$.

Let $\rho\left(v|v_{j,i-1};\boldsymbol{\theta}\right)$ be the conditional probability density of $V_{t_j+i\Delta} \equiv V_{j,i}$, given $V_{j,i-1}$ where $\boldsymbol{\theta} = (\theta_0, \alpha)$ are the unknown model parameters.

The Itô process V defined by the SDE (1) is Markovian, then the likelihood function of the sample $V^{M,N+1}$ can be written as follows :

$$\mathcal{L}\left(\boldsymbol{\theta}; V^{M,N+1}\right) = \prod_{j=1}^{M}\left\{\prod_{i=1}^{N}\rho\left(V_{j,i}|V_{j,i-1}; p_{[t_{j,i-1},t_{j,i}]}, \boldsymbol{\theta}\right)\right\}$$

where $t_{j,i} \equiv t_j + i\Delta$ for any $j \in \{1, \dots, M\}$ and $i \in \{0, \dots, N\}$

Likelihood approximation

In order to compute the exact likelihood function, we need a closed-form expression of the transition probability of V which can be found using the Fokker-Planck equation :

$$\frac{\partial f}{\partial t}\rho\left(v,t|v_{j,i-1},t_{j,i-1};\theta\right) = -\frac{\partial}{\partial v}\left(-\theta_t v\rho\left(v,t|v_{j,i-1},t_{j,i-1};\theta\right)\right) \\ +\frac{1}{2}\frac{\partial^2}{\partial v^2}\left(2\theta_0\alpha\left(v+p_t\right)\left(1-v-p_t\right)\rho\left(v,t|v_{j,i-1},t_{j,i-1};\theta\right)\right)$$

However, solving this equation is not always possible and is computationally costly. For this reason, we approximated the likelihood using a proxy distribution for V.
In our case we used a Beta distribution (ξ_1,ξ_2) as for the family of diffusion term in our SDE (1) (Pearson diffusion), it has been proved to be the best approximation. In order to find the parameters (ξ_1,ξ_2) of this proxy distribution we will match its first and second moments with the ones of the exact distribution deduced from the SDE (1).

Moment matching : First moment

For a time s $\in [t_n, t_{n+1}]$ the exact first moment $m_1(s)$ deduced from the SDE (1) is the solution of the following ODE :

$$\begin{cases} \mathrm{d}m_1(s) = [-m_1(s)\theta(s)]\,\mathrm{d}s \\ m_1\left(t_{n-1}\right) = v_{t_{n-1}} \end{cases}$$

We want to compute $m_1(t_n)$:

- If $\theta(t_n) = \theta_(t_{n+1}) = \theta$ then the exact solution is : $m_1(t_n) = m_1(t_{n-1})\exp(-\theta(t_n - t_{n-1})$
- else, we compute a linear approximation of $\theta(s)$ and approximate the ODE using Forward-Euler :

$$m_1\left(s_n\right) = m_1\left(s_{n-1}\right)\left(1 - \theta\left(s_{n-1}\right)\Delta s\right)$$

Moment matching : Second moment

Using Ito's formula, we find, for a time $s \in [t_n, t_{n+1}]$, the exact second moment $m_2(s)$ deduced from the SDE (1) is the solution of the following ODE :

$$\left\{\begin{array}{l} \mathrm{d}m_2(s) = [-2m_2(s)\,(\theta(s) + \alpha\theta_0) + 2\alpha\theta_0 m_1(s)(1 - 2p(s)) \\ +2\alpha\theta_0 p(s)(1 - p(s))\mathrm{d}s \\ m_2\left(t_{n-1}\right) = v_{t_{n-1}}^2 \end{array}\right. .$$

We compute a linear interpolation for the functions $\theta(s)$ and $p(s)$. After, we solve the ODE using Forward-Euler :
$m_2\left(s_n\right) = m_2\left(s_{n-1}\right) + \left[-2m_2\left(s_{n-1}\right)\left(\theta\left(s_{n-1}\right) + \alpha\theta_0\right)\right.$
$+2\ \alpha\theta_0 m_1\left(s_{n-1}\right)\left(1 - 2p\left(s_{n-1}\right)\right) + 2\alpha\theta_0 p\left(s_{n-1}\right)\left(1 - p\left(s_{n-1}\right)\right)\Delta s$

We use the same discretization points for both $m_1(s)$ and $m_2(s)$.

Moment matching

V is approximated by a new proxy random variable : $V = a + (b - a)X$ with support in $[a, b] = [-1, 1]$, where $X \sim \beta(\xi_1, \xi_2)$ and PDF $f_V(v)$. We find the two first moments :

- $\mathbb{E}[V] = a + (b - a)\mathbb{E}[X] = a + (b - a)\frac{\xi_1}{\xi_1+\xi_2} = \mu_V$
- $\mathbb{V}[V] = (b - a)^2\mathbb{V}[X] = \frac{(b-a)^2\xi_1\xi_2}{(\xi_1+\xi_2)^2(\xi_1+\xi_2+1)} = \sigma_V^2$

We want the first two moments of the true random variable and its approximation to be equal $\forall t$.
Therefore, $\mu(t) = m_1(t)$ and $\sigma^2(t) = m_2(t) - m_1^2(t)$.

For each measurement $V_{t_{n-1}}$, we can find the analytical moments at time t_n solving the ODEs from the previous slides. We can then find the parameters ξ_1 and ξ_2 of the proxy.

Evaluation of (ξ_1, ξ_2)

- $\xi_1 = -\frac{(1+\mu)\left(\mu^2+\sigma^2-1\right)}{2\sigma^2}$.
- $\xi_2 = \frac{(\mu-1)\left(\mu^2+\sigma^2-1\right)}{2\sigma^2}$.

all evaluated at time t_n

Log-density of the proxy random variable V

We want to compute the PDF $f_V(v)$ of the random variable : $V = a + (b-a)X$.

For $[a,b] = [-1,1]$, we have that : $f_V(v) = f_X\left(g^{-1}(v)\right)\left|\frac{\mathrm{d}}{\mathrm{d}v}g^{-1}(v)\right|$ where $f_X(x) = \text{Beta}\left(\xi_1, \xi_2\right)$ and $g(x) = a + (b-a)x$.

Then, $f_V(v) = \frac{1}{|(b-a)|}\frac{1}{B(\xi_1,\xi_2)}\left(\frac{v-a}{b-a}\right)^{\xi_1-1}\left(1-\frac{v-a}{b-a}\right)^{\xi_2-1}$, because $g^{-1}(v) = \frac{v-a}{b-a}$.

Therefore : $\log\left(f_V(v)\right) = \log\left(\frac{1}{B(\xi_1,\xi_2)}\right) + (\xi_1-1)\log\left(\frac{v-a}{b-a}\right) + (\xi_2-1)\log\left(\frac{b-v}{b-a}\right)$

Log-likelihood

We introduce the number of paths (days) M, and the number of measurements per path $N+1$(N transitions). We have a total of $M \times N$ samples. The log-likelihood is :

$$\mathfrak{L}\left(\{V\}_{M,N}\right) = \sum_{i=1}^{M} \sum_{j=2}^{N+1} \log\left[\rho_{i,j}\left(V_{i,j} | V_{i,j-1}\right)\right]$$

where $\rho_{i,j}\left(V_{i,j} | V_{i,j-1}\right) = \rho_{i,j}\left(V_{i,j} | V_{i,j-1}; \xi_{1,j}, \xi_{2,j}\right)$.

Initial Estimation of the parameters

In order to evaluate the initial parameters of our model we apply the least square method on the forecast error V_t.
We consider the transition $\Delta V_i = V_{i+1} - V_i$ with $\Delta t = t_{i+1} - t_i$.
$(V_{i+1}|V_i)$ is a random variable which conditional mean can be approximated by the solution of the following system :

$$\begin{cases} \mathrm{d}\mathbb{E}[V] = -\theta_t^{\varepsilon}\mathbb{E}[V]\mathrm{d}t \\ \mathbb{E}\left[V\left(t_i\right)\right] = V_i \end{cases}$$

evaluated in t_{i+1} (i.e., $\mathbb{E}\left[V\left(t_{i+1}\right)\right]$).

Then, the random variable $\left(V_{i+1} - \mathbb{E}\left[V\left(t_{i+1}\right)\right]\right)$ has a mean equal to 0 approximately.
If we assume that $\theta_t^{\varepsilon} = c \in \mathbb{R}^+$ for all $t \in \left[t_i, t_{i+1}\right]$, then $\mathbb{E}\left[V\left(t_{i+1}\right)\right] = V_i e^{-c\Delta t}$.
If we have a total of n transitions, we can write the regression problem for the conditional mean with L^2 loss function as :

$$\begin{aligned} c^* &\approx \arg\min_{c \geq 0} \left[\sum_{i=1}^{n} \left(V_{i+1} - \mathbb{E}\left[V\left(t_{i+1}\right)\right]\right)^2\right] \\ &= \arg\min_{c \geq 0} \left[\sum_{i=1}^{n} \left(V_{i+1} - V_i e^{-c\Delta t}\right)^2\right] \end{aligned} \quad (2)$$

Least Square Minimization : LSM

We take the first order approximation of $e^{-c\Delta t}$ w.r.t. c :

$$e^{-c\Delta t} = 1 - c\Delta t + O\left((c\Delta t)^2\right)$$

and introduce it in equation (1). We get

$$c^* \approx \arg\min_{c\geq 0} \underbrace{\left[\sum_{i=1}^{n} \left(V_{i+1} - V_i(1 - c\Delta t)\right)^2\right]}_{=f(c)}$$

As $f(c)$ is convex in c, solving (5) (finding c^*) is equivalent to solving

$$\frac{\partial f}{\partial c}\left(c^{**}\right) = 0$$

and choosing $c^* = \max\left\{0, c^{**}\right\}$

Least Square Minimization : LSM

$$\frac{\partial f}{\partial c} = \sum_{i=1}^{n} 2\,(-V_i)\,(-\Delta t)\,(V_{i+1} - V_i\,(1 - \theta_0 \Delta t))$$
$$= \sum_{i=1}^{n} 2 V_i \Delta t\,(V_{i+1} - V_i(1 - c\Delta t))$$
$$= \sum_{i=1}^{n} 2 V_{i+1} V_i \Delta t - 2 V_i^2 \Delta t + 2 V_i^2 \Delta t^2 c$$

Then, c^{**} satisfies the following :

$$c^{**} \approx \frac{\sum_{i=1}^{n} V_i\,(V_i - V_{i+1})}{\Delta t \cdot \sum_{i=1}^{n} (V_i)^2}$$

Quadratic variation

We approximate the SDE by its E-M scheme. In particular, we approximate the Itô quadratic variation with the discrete one :

- Itô process quadratic variation : $[V]_t = \int_0^t \sigma_s^2 \mathrm{d}s$
- Discrete process quadratic variation : $[V]_t = \Sigma_{0<s\leq t} (\Delta V_s)^2$

Then, considering Δt the time between the measurements, we approximate :

$$\theta_0^* \alpha^* \approx \frac{\sum_{i=1}^{n} (\Delta V_i)^2}{2\Delta t \sum_{i=1}^{n} (V_i + p_i)(1 - V_i - p_i)}$$

Estimation of $(\theta_0, \alpha, \varepsilon)$

In this section, we will use the approximation made previously to estimate the parameters $(\theta_0, \alpha, \varepsilon)$ of the SDE. Let us define $(\theta_0^*, \alpha^*, \varepsilon^*)$ as their estimators.

If we fix ε, we define the forecast error $\forall i \in 1...n \; V_i = X_i - p_i^{\varepsilon}$.
If we also fix θ_0 and α, we can define the set of indexes :

$\mathrm{I} = \{i \in \{1, \dots, n\} : \text{ the LSM estimation will estimate } \theta_0\}$
$\mathrm{J} = \left\{j \in \{1, \dots, n\} : \text{ the } LSM \text{ estimation will estimate } \frac{\theta_0 \alpha}{\varepsilon}\right\}$
We will proceed then to approximate these sets in order to estimate our parameters.

Estimation of $(\theta_0, \alpha, \varepsilon)$

To use the LSM estimation, we assumed that $\theta_t^\varepsilon = c \in \mathbb{R}^+$, and we defined θ_t^ε :

$$\theta_t^\varepsilon = \max\left(\theta_0, \frac{\alpha\theta_0 + 2\left|\dot{p}_t^\varepsilon\right|}{2\min\left(1 - p_t^\varepsilon, p_t^\varepsilon\right)}\right)$$

From the definition of θ_t^ε : We have that for $\varepsilon << 1$, and $p_t = \varepsilon$ or $p_t = 1 - \varepsilon$, the approximation $\theta_t^\varepsilon \approx \frac{\theta_0\alpha}{\varepsilon}$ holds. Then, for ε small enough, J can be approximated by the following :

$$\mathrm{J} \approx \mathrm{J} = \left\{j \in \{1, \ldots, n\} : p_j^\varepsilon \in \{\varepsilon, 1 - \varepsilon\}\right\}$$

and θ_t^ε, we have that it is more likely that $\theta_t^\varepsilon = \theta_0$ if $p_t^\varepsilon \approx \frac{1}{2}$. Then, we can approximate I by

$$\mathrm{I} \approx \tilde{\mathrm{I}} = \left\{i \in \{1, \ldots, n\} : p_i \in (\gamma, 1 - \gamma)\right\}, \gamma \approx \frac{1}{2}, \gamma < \frac{1}{2}$$

Estimation of α^*

With the previous approximation made of the quadratic variation we can estimate $\theta_0 * \alpha* = 0.094$ therefore, with our given estimation of θ_0* we find that : $\alpha* = 0.08$

Estimation of ε^*

Now that we have an approximated value of $\theta_0\alpha$, if we can estimate $\frac{\theta_0\alpha}{\varepsilon}$, then we can estimate ε. We showed previously that for $\varepsilon << 1$, the LSM estimation using indexes from J is an estimator for $\frac{\theta_0\alpha}{\varepsilon} =: k$
The goal is to find values for ε that satisfy $\varepsilon << 1$. For that we start by randomly choosing a small initial value for ε (that we will call ε_0), and iterating we aim to converge to some local minimum.
We proceed with the following steps :

- We sample ε_0 from $U[0.01, 0.1]$ and load $\varepsilon \leftarrow \varepsilon_0$
- We create $\tilde{J}$ and use the LSM estimation to find k.
  - If $k < \theta_0^*$, then the assumption $\theta_t^\varepsilon = c \in \mathbb{R}^+$ is wrong and we reduce the value of ε, i.e., $\varepsilon \leftarrow \varepsilon * 0.999$.
  - If $k \geq \theta_0^*$, we load $\varepsilon \leftarrow \frac{\theta_0^*\alpha^*}{k}$(we allow a maximum relative change of 1%).

  We repeat this step 100 times.
- We repeat steps 1 and 2, 50 times.

To conclude, the estimations of the SDE parameters that we found are : $(\theta_0^*, \alpha^*, \varepsilon^*) = (1.25, 0.08, 0.018)$.

The code computing this process can be found in the file **Wind_project_intial_guess.ipynb**.

Log-likelihood optimization

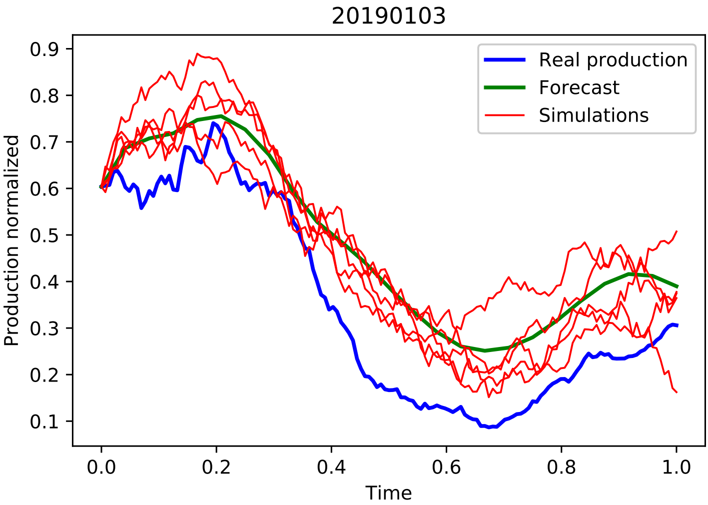
I minimized the negative log-likelihood using the function fmin from the library scipy.optimize in Python to estimate the parameters (θ_0, α). I found the following results for the different datasets provided :

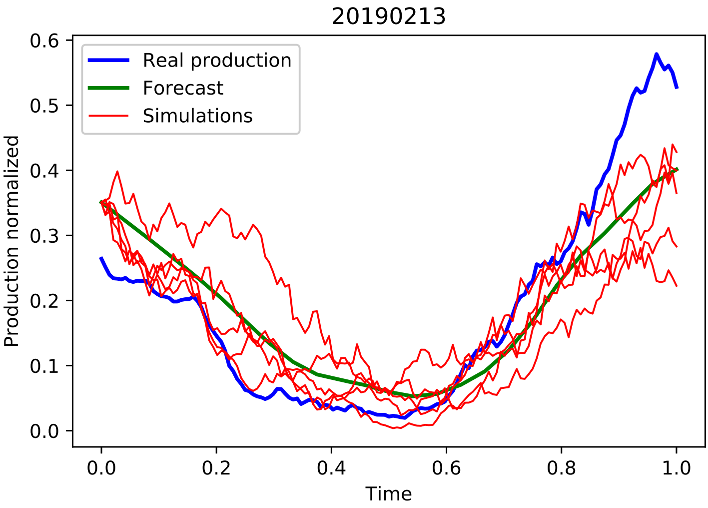
Data providers	θ_0	α	$\theta_0\alpha$
Complete	1.161	0.0718	0.083
UTEP5	1.357	0.0809	0.108
MTLOG	1.175	0.0856	0.100
AWSTEP	1.196	0.0846	0.101

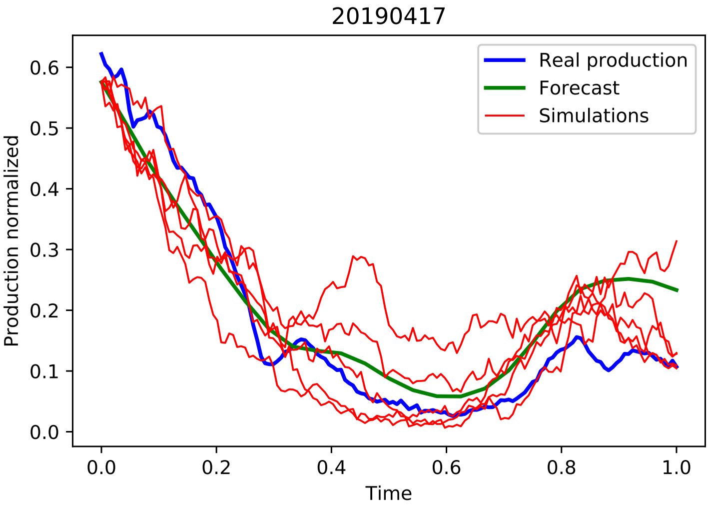
The code of this optimization can be found in the file **Wind_project_optimization.ipynb**

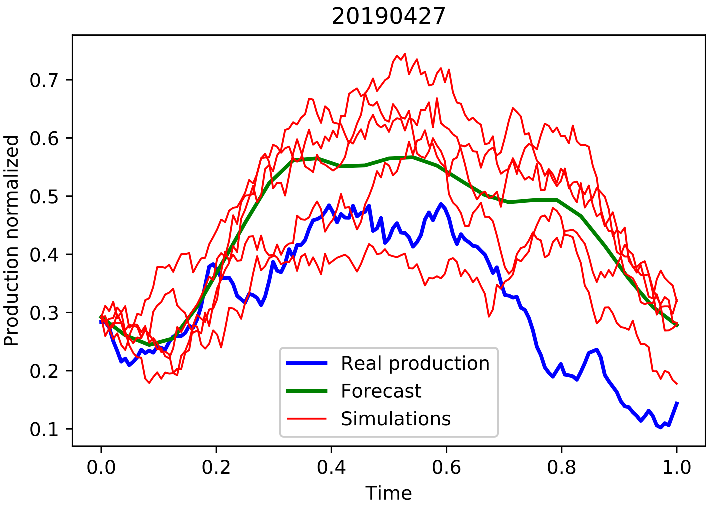
Path simulation

I plotted the path for the real production, the forecast and some simulations of the production using the model.




20190213
Real production
Forecast
Simulations
Production normalized
Time
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.0
0.2
0.4
0.6
0.8
1.0

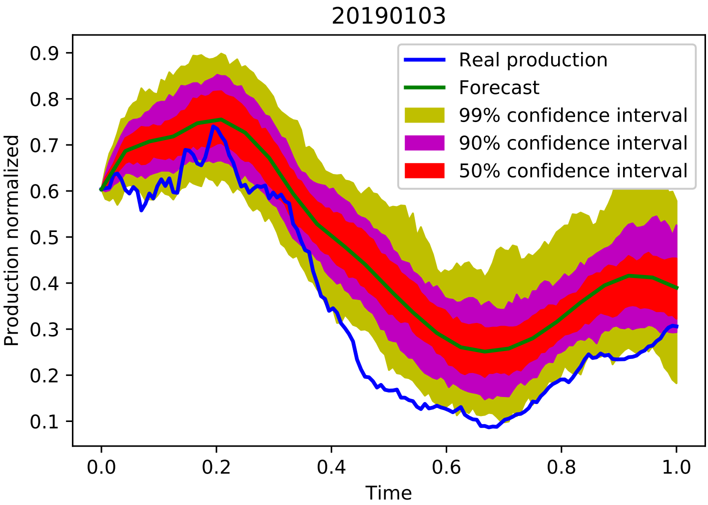

20190417
Real production
Forecast
Simulations
Production normalized
Time
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0

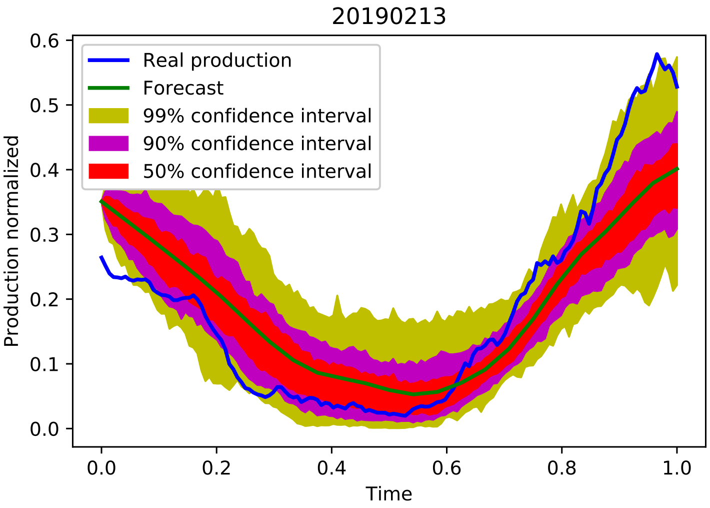


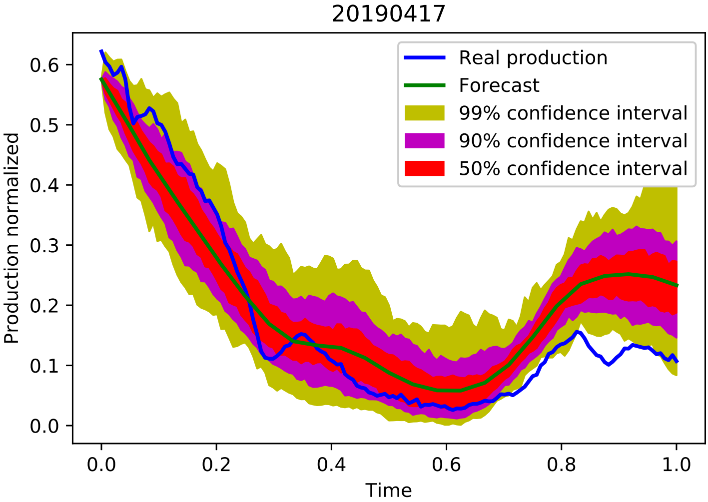
20190427
Production normalized
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.2
0.4
0.6
0.8
1.0
Time
Real production
Forecast
Simulations

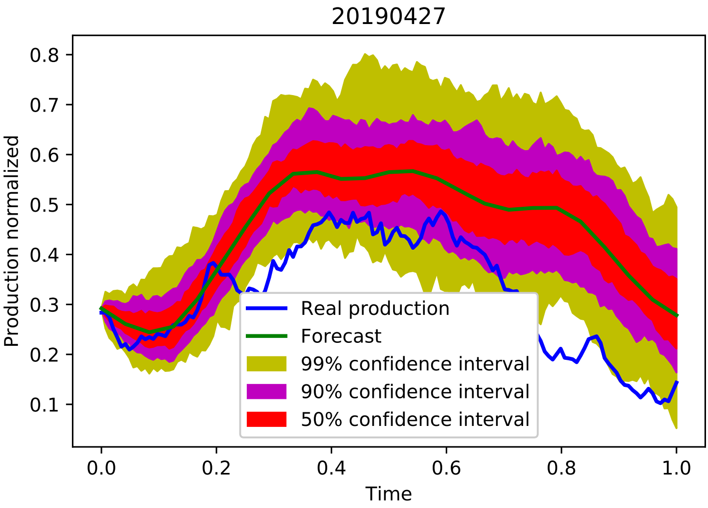
Confidence intervals

I plotted the 99%, 90% and 50% confidence intervals using 100 simulations per day.




20190213
Real production
Forecast
99% confidence interval
90% confidence interval
50% confidence interval
Production normalized
Time
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.0
0.2
0.4
0.6
0.8
1.0


20190417
Real production
Forecast
99% confidence interval
90% confidence interval
50% confidence interval
Production normalized
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.0
0.2
0.4
0.6
0.8
1.0
Time


20190427
Production normalized
Time
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.2
0.4
0.6
0.8
1.0
Real production
Forecast
99% confidence interval
90% confidence interval
50% confidence interval