1 The model

For a time horizon $T > 0$ and a parameter $\alpha > 0$ and $(\theta_t)_{t\in[0,T]}$ a positive deterministic function, let us consider the model given by

$$\begin{aligned} dX_t &= \big(\dot{p}_t - \theta_t(X_t - p_t)\big)dt + \sqrt{2\alpha\theta_t X_t(1-X_t)}dW_t \quad t \in [0,T] \\ X_0 &= x_0 \in [0,1], \end{aligned} \tag{1}$$

where $(p_t)_{t\in[0,T]}$ denotes the prediction function that satisfies $0 \le p_t \le 1$ for all $t \in [0,T]$. This prediction function is assumed to be a smooth function of the time so that

$$\sup_{t\in[0,T]} \big(|p_s| + |\dot{p}_s|\big) < +\infty.$$

The following proofs are based on standard arguments for stochastic processes that can be found e.g. in Alfonsi [1] and Karatzas and Shreve [2] that we adapted to the setting of our model (1).

Theorem 1.1. Assume that

$$\forall t \in [0,T], \;\; 0 \le \dot{p}_t + \theta_t p_t \le \theta_t, \quad \text{and} \quad \sup_{t\in[0,T]} |\theta_t| < +\infty. \tag{A}$$

Then, there is a unique strong solution to (1) s.t. for all $t \in [0,T]$, $X_t \in [0,1]$ a.s.

Proof. Let us first consider the following SDE for $t \in [0,T]$

$$X_t = x_0 + \int_0^t \big(\dot{p}_s - \theta_s(X_s - p_s)\big)ds + \int_0^t \sqrt{2\alpha\theta_s|X_s(1-X_s)|}dW_s, \quad x_0 > 0. \tag{2}$$

According to Proposition 2.13, p. 291 of [2], under assumption (A) there is a unique strong solution X to (2). Moreover, as the diffusion coefficient is of linear growth, we have for all $p > 0$

$$\mathbb{E}[\sup_{t\in[0,T]} |X_t|^p] < \infty. \tag{3}$$

Then, it remains to show that for all $t \in [0,T]$, $X_t \in [0,1]$ a.s. For this aim, we need to use the so-called Yamada function ψ_n that is a $\mathcal{C}^2$ function that satisfies a bench of useful properties:

$$|\psi_n(x)| \underset{n\to+\infty}{\to} |x|, \;\; x{\psi'}_n(x) \underset{n\to+\infty}{\to} |x|, \;\; |\psi_n(x)| \wedge |x{\psi'}_n(x)| \le |x|$$

$${\psi'}_n(x) \le 1, \quad \text{and } {\psi''}_n(x) = g_n(|x|) \ge 0 \quad \text{with} \quad g_n(x)x \le \frac{2}{n} \quad \text{for all } x \in \mathbb{R}.$$

See the proof of Proposition 2.13, p. 291 of [2] for the construction of such fucntion. Applying Itô's formula we get

$$\psi_n(X_t) = \psi_n(x_0) + \int_0^t \psi'_n(X_s)\big(\dot{p}_s + \theta_s p_s - \theta_s X_s\big)ds + \int_0^t \psi'_n(X_s)\sqrt{2\alpha\theta_s |X_s(1-X_s)|}dW_s$$
$$+ \alpha \int_0^t \theta_s g_n(|X_s|)|X_s(1-X_s)|ds.$$

Now, thanks to (A), (3) and to the above properties of ψ_n and g_n, we get

$$\mathbb{E}[\psi_n(X_t)] \leq \psi_n(x_0) + \int_0^t \big(\dot{p}_s + \theta_s p_s - \theta_s \mathbb{E}[\psi'_n(X_s)X_s]\big)ds + \frac{2\alpha}{n}\int_0^t \theta_s \mathbb{E}|1-X_s|ds.$$

Therefore, letting n tends to infinity we use Lebesgue's theorem to get

$$\mathbb{E}[|X_t|] \leq x_0 + \int_0^t \big(\dot{p}_s + \theta_s p_s - \theta_s \mathbb{E}|X_s|\big)ds.$$

Besides, taking the expectation of (2), we get

$$\mathbb{E}X_t = x_0 + \int_0^t \big(\dot{p}_s + \theta_s p_s - \theta_s \mathbb{E}X_s\big)ds$$

and thus we have

$$\mathbb{E}[|X_t| - X_t] \leq \int_0^t \theta_s \mathbb{E}[|X_s| - X_s]ds.$$

Then, Gronwall's lemma gives us $\mathbb{E}[|X_t|] = \mathbb{E}X_t$ and thus for any $t \in [0,T]$ $X_t \geq 0$ a.s. The same arguments work to prove that for any $t \in [0,T]$ $Y_t := 1 - X_t \geq 0$ a.s. since the process $(Y_t)_{t\in[0,T]}$ is solution to

$$dY_t = \big(\theta_t(1-p_t) - \dot{p}_t - \theta_t Y_t\big)dt - \sqrt{2\alpha\theta_t Y_t(1-Y_t)}dW_t,$$

Then similarly, we need to assume that $\dot{p}_t + \theta_t p_t \geq 0$. This completes the proof. □

Theorem 1.2. Assume that assumptions of Theorem 1.1 hold with $x_0 \in]0,1[$. Let $\tau_0 := \inf\{t \in [0,T],\ X_t = 0\}$ and $\tau_1 := \inf\{t \in [0,T],\ X_t = 1\}$ with the convention that $\inf \emptyset = +\infty$. Assume in addition that

$$0 < \alpha < \frac{1}{2} \quad \text{and} \quad \forall t \in [0,T],\ \alpha\theta_t \leq \dot{p}_t + \theta_t p_t \leq (1-\alpha)\theta_t. \tag{B}$$

Then, $\tau_0 = \tau_1 = +\infty$ a.s.

Proof. For $t \in [0, \tau_0[$, we have

$$\frac{dX_t}{X_t} = \frac{\dot{p}_t + \theta_t p_t}{X_t} - \theta_t dt + \sqrt{\frac{2\alpha\theta_t(1-X_t)}{X_t}} dW_t$$

so that

$$X_t = x_0 \exp\Big(\int_0^t \frac{\dot{p}_s + \theta_s(p_s - \alpha)}{X_s} ds - (1-\alpha) \int_0^t \theta_s ds + M_t \Big),$$

where $M_t = \int_0^t \sqrt{\frac{2\alpha\theta_s(1-X_s)}{X_s}} dW_s$ is a continuous martingale. Then as for all $t \in [0,T]$, we have $\dot{p}_t + \theta_t(p_t - \alpha) \geq 0$, we deduce that

$$X_t \geq x_0 \exp\Big(-(1-\alpha) \int_0^t \theta_s ds + M_t \Big).$$

By way of contradiction let us assume that $\{\tau_0 < \infty\}$, then letting $t \to \tau_0$ we deduce that $\lim_{t\to\infty} \mathbf{1}_{\{\tau_0<\infty\}} M_{t\wedge\tau_0} = -1_{\{\tau_0<\infty\}}\infty$ a.s. This leads to a contradiction since we know that continuous martingales likewise the Brownian motion cannot converge almost surely to $+\infty$ or $-\infty$. It follows that $\tau_0 = \infty$ almost surely. Next, recalling that the process $(Y_t)_{t\geq 0}$ given by $Y_t = 1 - X_t$ is solution to

$$dY_t = \big(\theta_t(1-p_t) - \dot{p}_t - \theta_t Y_t\big)dt - \sqrt{2\alpha\theta_t Y_t(1-Y_t)} dW_t$$

we deduce using similar arguments as above $\tau_1 = \infty$ a.s. provided that $\theta_t(1-p_t) - \dot{p}_t - \alpha\theta_t \geq 0$. □

When using Lamperti, the form

2 New model

In what follows we adapt the proof of the above theorem for the new model

$$\begin{aligned} dX_t &= \big(\dot{p}_t - \theta_t(X_t - p_t)\big)dt + \sqrt{2\alpha\theta_0 X_t(1-X_t)} dW_t \quad t \in [0,T] \\ X_0 &= x_0 \in [0,1], \end{aligned} \tag{4}$$

Theorem 2.1. Assume that assumptions of Theorem 1.1 hold with $x_0 \in]0,1[$. Let $\tau_0 := \inf\{t \in [0,T],\ X_t = 0\}$ and $\tau_1 := \inf\{t \in [0,T],\ X_t = 1\}$ with the convention that $\inf \emptyset = +\infty$. Assume in addition that for all $t \in [0,T]$, $p_t \in]0,1[$ and that

$$\theta_t \geq \max\left(\frac{\alpha\theta_0 + \dot{p}_t}{1-p_t}, \frac{\alpha\theta_0 - \dot{p}_t}{p_t} \right). \tag{C}$$

Then, $\tau_0 = \tau_1 = +\infty$ a.s.

Proof. For $t \in [0, \tau_0[$, we have

$$\frac{dX_t}{X_t} = \left(\frac{\dot{p}_t + \theta_t p_t}{X_t} - \theta_t\right) dt + \sqrt{\frac{2\alpha\theta_0(1 - X_t)}{X_t}} dW_t$$

so that

$$X_t = x_0 \exp\Big(\int_0^t \frac{\dot{p}_s + \theta_s p_s - \theta_0\alpha)}{X_s} ds + \alpha\theta_0 t - \int_0^t \theta_s ds + M_t\Big),$$

where $M_t = \int_0^t \sqrt{\frac{2\alpha\theta_s(1-X_s)}{X_s}} dW_s$ is a continuous martingale. Then as for all $t \in [0, T]$, we have $\dot{p}_t + \theta_t p_t - \theta_0\alpha \geq 0$, we deduce that

$$X_t \geq x_0 \exp\Big(\alpha\theta_0 t - \int_0^t \theta_s ds + M_t\Big).$$

By way of contradiction let us assume that $\{\tau_0 < \infty\}$, then letting $t \to \tau_0$ we deduce that $\lim_{t\to\infty} \mathbf{1}_{\{\tau_0<\infty\}} M_{t\wedge\tau_0} = -1_{\{\tau_0<\infty\}}\infty$ a.s. This leads to a contradiction since we know that continuous martingales likewise the Brownian motion cannot converge almost surely to $+\infty$ or $-\infty$. It follows that $\tau_0 = \infty$ almost surely. Next, recalling that the process $(Y_t)_{t\geq 0}$ given by $Y_t = 1 - X_t$ is solution to

$$dY_t = \big(\theta_t(1 - p_t) - \dot{p}_t - \theta_t Y_t\big)dt - \sqrt{2\alpha\theta_0 Y_t(1 - Y_t)}dW_t$$

we deduce using similar arguments as above $\tau_1 = \infty$ a.s. provided that $\theta_t(1-p_t)-\dot{p}_t-\alpha\theta_0 \geq 0$. $\square$

3 Answers to Renzo's questions

So, Theorem 2.1 answers the second question in slide 11. The new condition (C) that guarantees for $(X_t)_{t\geq 0}$ to stay in $]0, 1[$ depends now on α. I would keep condition (C) as we have a proof for it. Moreover, if condition (C) is satisfied, then the orange condition in slide 8 is also satisfied since we have

$$\max\left(\frac{\alpha\theta_0 + \dot{p}_t}{1 - p_t}, \frac{\alpha\theta_0 - \dot{p}_t}{p_t}\right) \geq \max\left(\frac{\frac{\alpha\theta_0}{2} + \dot{p}_t}{1 - p_t}, \frac{\frac{\alpha\theta_0}{2} - \dot{p}_t}{p_t}\right)$$

which ensures that Z don't hit the boundaries too according to Renzo's intuitive arguments.

Concerning, the first question on slide 11: we can't apply Itô's formula, unless we are sure that the process don't hit the boundaries, that's why we have to assume (C) before using Itô's formula. In addition, as the diffusion coefficient of X given by $x \mapsto \sqrt{\frac{2\alpha\theta_0(1-x)}{x}}$

is strictly positive for all x in the state space of X, namely $]0,1[$ under assumption (C), this is enough to ensure that the transformation between Z and X is bijective so that we deduce the existence and uniqueness of Z from those of X. Otherwise the Lamperti transform is not valid if the state space of X is $[0,1]$.

References

[1] Alfonsi, A. (2015) Affine diffusions and related processes: simulation, theory and applications. *Springer, Cham; Bocconi University Press, Milan, 2015.*

[2] Karatzas, I. and Shreve, S. E. (1991), Brownian motion and stochastic calculus *Springer-Verlag, New York.*