3D Body Metric

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Motivation

Let $A, B \subset \mathbb{R}^3$ two sets describing 3D bodies.

We want a mathematical way to describe when both bodies are equal. As an example, let $A = B_{([0,0,0],1)}$ and $B = B_{([1,1,1],1)}$ the unitary balls with centers in $(0,0,0) \in \mathbb{R}^3$ and $(1,1,1) \in \mathbb{R}^3$, respectively.

Can we say $A = B$**?** Since $(0,0,0) \in A$ but $(0,0,0) \notin B$, we have that $A \neq B$.
Can we say $A = B$ **almost everywhere?** We know the answer is NO.

However, both A and B are unitary balls in $\mathbb{R}^3$, so there exists an isometry $I : \mathbb{R}^3 \to \mathbb{R}^3$ such that $I(A) = B$. Then, we can say they are equal under some isometry.

Definitions

Let $A, B \subset \mathbb{R}^3$ two sets describing 3D bodies.

Definition

An equivalent class of A contains all the sets $X \subset \mathbb{R}^3$ for with exists $I : \mathbb{R}^3 \to \mathbb{R}^3$ isometry such that $I(X) = A$ almost everywhere w.r.t. the Lebesgue measure.

Then, we have that $A \sim B$ if and only if $\exists I : \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $A = I(B)$ a.e. We use $\mathscr{A}$ to denote the equivalent class of A.

Let $\mathscr{A}$ and $\mathscr{B}$ two non-necessarily equal equivalent classes as described before, and let $A \in \mathscr{A}$ and $B \in \mathscr{B}$ some elements from that classes. The next inequalities are trivial:

- $A \cup B \supseteq A$ and $A \cup B \supseteq B$. We reach qualities if and only if $A = B$.
- $A \cap B \subseteq A$ and $A \cap B \subseteq B$. We reach qualities if and only if $A = B$.

Definitions

$A \cup B \supseteq A \cap B \implies \mu(A \cup B) \geq \mu(A \cap B) \implies \mu(A \cup B) - \mu(A \cap B) \geq 0$, and we reach the equality if and only if $A = B$ almost everywhere. Then we can define our metric:

Definition

Let $\mathscr{A}$ and $\mathscr{B}$ two equivalent classes over isometries, and let A and B arbitrary elements from that classes, we define the metric $d(\mathscr{A}, \mathscr{B})$ as

$$d(\mathscr{A}, \mathscr{B}) = \min_{A \in \mathscr{A}, B \in \mathscr{B}} \left[\mu(A \cup B) - \mu(A \cap B) \right]. \tag{1}$$

Notice that, $d(\mathscr{A}, \mathscr{B}) = 0$ if and only if, $\mathscr{A} = \mathscr{B}$ (which implies that, for each pair A and B, there exists an isometry $I : \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $I(A) = B$ almost everywhere).