

CHAPTER 3

WHICH ISOMETRIES DO IT?

3.0 Congruent sets

3.0.1 Congruence. We call two sets **congruent** to each other if and only if there exists an isometry that maps one to the other; in simpler terms, if and only if one is a **copy** of the other. For example, this is the case with the quadrilaterals ABCD and A′B′C′D′ in either of figures 1.18 & 1.30. It is correct to say that this definition extends the familiar definition of congruent triangles and, more

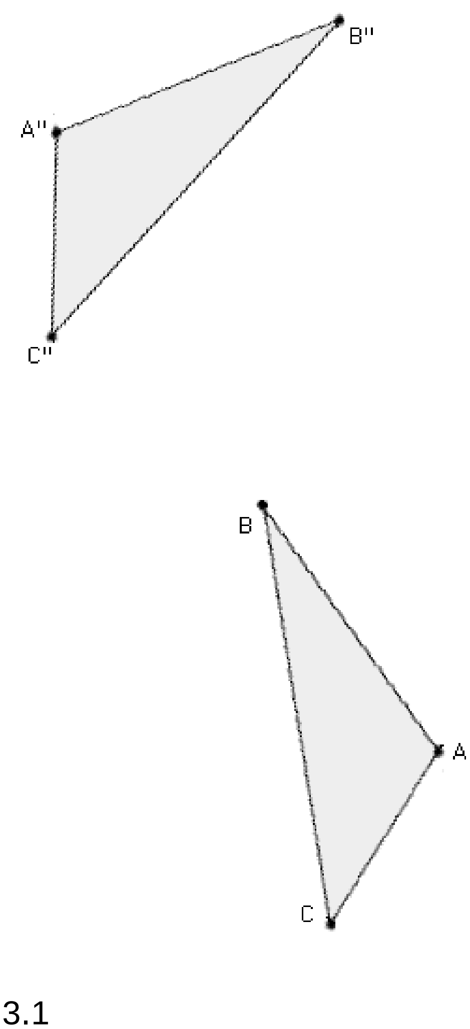

Fig. 3.1

generally, congruent polygons. In more practical terms, two sets are congruent if and only if one of them can be brought to perfectly **'match'** the other point by point. Let us for example have a look at the triangles ABC, A′B′C′, and A″B″C″ of figure 3.1: both triangles A′B′C′ and A″B″C″ are congruent to ABC as |A″B″| = |A′B′| = |AB|, |A″C″| = |A′C′| = |AC|, and |B″C″| = |B′C′| = |BC|. The relation of each triangle to ABC is somewhat **different** though: while A′B′C′ may be **slid** (i.e., glided and turned as needed) until it matches ABC point by point, A″B″C″ may **not** be brought back to ABC by mere sliding. How do we demonstrate the congruence of ABC and A″B″C″ in a hands-on way then? One needs to be clever enough to observe that A″B″C″ may in fact be slid back to ABC after it gets **flipped**: if that is not obvious to you, simply trace A″B″C″ on tracing paper, then flip the tracing paper and slide the flipped A″B″C″ back to ABC -- it works!

Revisiting the pairs of quadrilaterals in figures 1.18 & 1.30, we make similar observations: in figure 1.18 A′B′C′D′ (image of ABCD under reflection) must be flipped in order to be slid back to the original ABCD, while in figure 1.30 A′B′C′D′ (image of ABCD under rotation) can be slid back to ABCD without any flipping. You have probably suspected this one by now: flipping is required in the case of reflection but not in the case of rotation. But let us now take a look at the two triangles of figure 1.14, mirror images of each other:

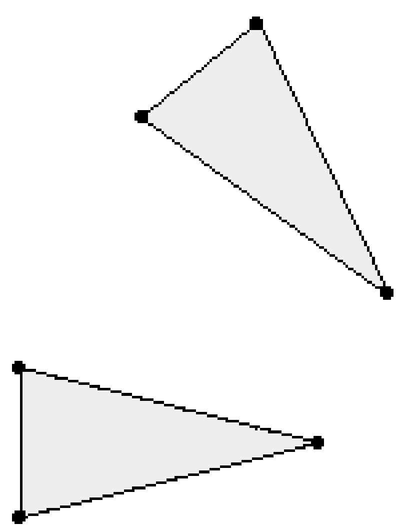

Fig. 3.2

Despite the reflection, you can easily check, using tracing paper if necessary, that the two triangles may easily be slid back to each other (without any flipping, that is). What makes the difference?

3.0.2 Homostrophy and heterostrophy. Before addressing the issues raised by figure 3.2, we need some terminology. We call two congruent sets **homostrophic** ('of same turning') if and only if they can match each other via mere sliding; and we call two congruent sets **heterostrophic** ('of opposite turning') if and only if they can only match each other via a combination of flipping and sliding. For example, ABCD and A′B′C′D′ are homostrophic in figure 1.30, but heterostrophic in figure 1.18. And, in figure 3.1 above, A′B′C′ and A″B″C″ are homostrophic and heterostrophic to ABC, respectively.

3.0.3 Labeling. Let us now return to the 'puzzle' of figure 3.2 and reinstate the vertex labels from figure 1.14 as below:

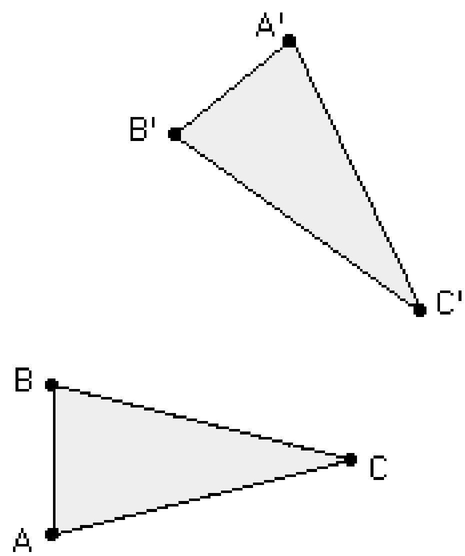

Fig. 3.3

Can you now slide A′B′C′ 'back' to ABC in a way that A′, B′, C′ 'return' to A, B, C, **respectively**? After a shorter or longer effort -- that depends on your personality -- you are bound to give up: it is simply impossible! That is, the **labeling** of the vertices has made the two congruent triangles heterostrophic: A′B′C′ needs to be flipped before it can slide to ABC. And, once again, heterostrophy seems to be associated with **reflection**.

Back in 3.0.1, and figure 3.2, you were able to slide the triangle **now** labeled A′B′C′ to match ABC. What would happen if you repeat that same sliding? The two triangles would still match each other, except that now A′ 'returns' to B and B′ 'returns' to A. This is not quite a perfect match, but it would obviously be one if we **swap** A′ and B′. Indeed such an action leads to the following situation:

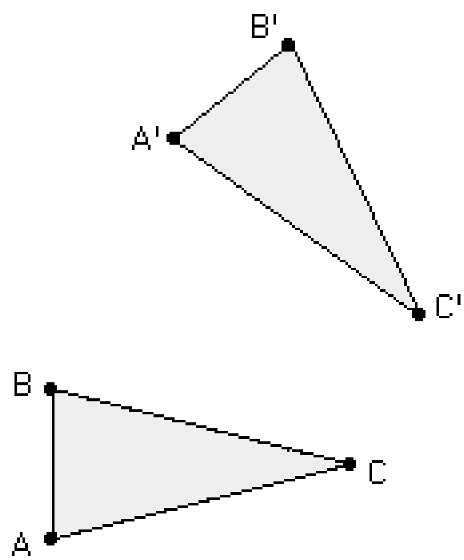

Fig. 3.4

Clearly, it is now possible to simply slide A′B′C′ to ABC: the two triangles are now homostrophic! A rushed conclusion is that **homostrophy and heterostrophy are concepts 'defined' by labeling**; this is a rule with its fair share of **exceptions**, as we will see in 3.2.6 and 3.5.4. And, in view of our entire discussion so far, an obvious question would be: is there a **rotation** that maps ABC to A′B′C′ in figure 3.4? We **knew** ahead of time, thanks to figure 1.14, of a reflection that mapped ABC to A′B′C′ in figure 3.3; it is not unreasonable now to **suspect** that there is a rotation that maps ABC to A′B′C′ in figure 3.4: but **how** do we determine such a rotation, how do we come up with a center and an angle that would work?

3.0.4 The 'reverse' problem. Let us now consider a situation similar to the one discussed in 3.0.3, departing from rotation and figure 1.24 this time; we leave the familiar triangle ABC untouched but we **swap** A′ and B′ as shown in figure 3.5 below:

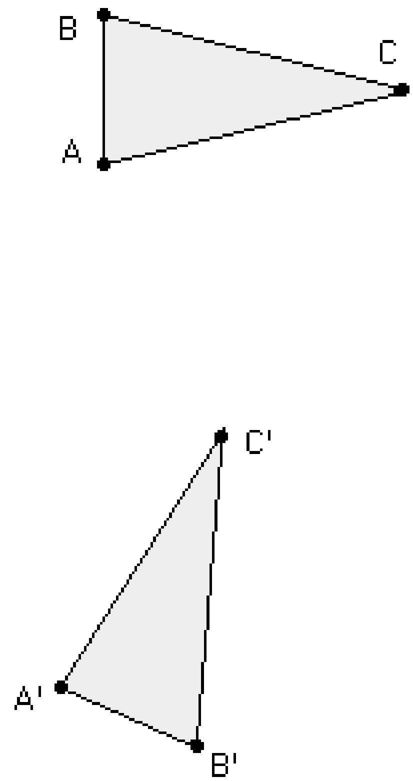

Fig. 3.5

It is clear that the homostrophy (created by rotation) in figure 1.24 has now been eliminated. Does that mean that there exists a reflection that maps ABC to A′B′C′? The answer is “no”: in every reflection the segments PP′ that join every point P of the original figure to its image point P′ are perpendicular to the reflection axis, hence they must all be **parallel** to each other; and that is clearly **not** the case in figure 3.5! For exactly the same reason there is no translation mapping ABC to A′B′C′. Nor is a rotation plausible, as we do suspect, without proof so far, that rotation is **always** associated with homostrophy. There only remains one possibility: **glide reflection**!

That glide reflection can be associated with heterostrophy is suggested by the effect of the two opposite glide reflections on ABC in figure 1.34: both A′B′C′ and A″B″C″ are easily seen to be

heterostrophic to ABC! So yes, there is hope, if not certainty, that there exists a glide reflection that maps ABC to A′B′C′ in figure 3.5; but **how** do we determine such a glide reflection, how do we come up with an axis and a vector that would work?

This last question sounds very similar to the one posed at the end of 3.0.3, doesn't it? The two questions are indeed the two faces of a broader question that **reverses** the tasks you learned in chapter 1: back then you were given a set and an isometry and you had to determine the image; here you are given the 'original' set and an 'image' set congruent to it, and you are asked to determine **all** the isometries that send the original to the image. That there may be **more than one isometries** 'between' two congruent sets should be clear in view of the examples discussed in this section, and has in fact been explicitly demonstrated in figure 2.22. Chapter 3 is devoted to this 'reverse' question.

3.1 Points

3.1.1 <u>Infinite flexibility.</u> Points do not take much room at all, hence they ought to be rather easy to deal with! In our context, given any two points A and A′, we can at once find not one but two isometries that map A to A′. These are a **translation** defined by the **vector AA′** and a **reflection** whose axis is the **perpendicular bisector of AA′**:

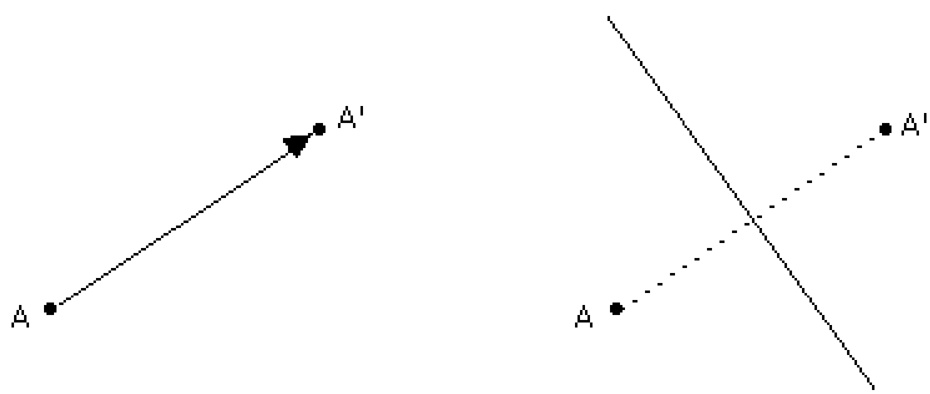

Fig. 3.6

We are much ‘luckier’ than that though! There exist in fact **infinitely many rotations** and **infinitely many glide reflections** that map A to A′. Getting them all turns out to be mostly a matter of remembering the ways of rotation and glide reflection from chapter 1: we will also need to learn to play the game backwards, and employ a bit of high school geometry as well.

3.1.2 Rotations. Let us revisit figure 1.21, where we defined rotation, and make a fundamental observation: since $|KP| = |KP'|$, K must lie on the **perpendicular bisector** of PP′! Indeed if M is the **midpoint** of PP′ then the two triangles MKP and MKP′ have three pairs of equal sides, hence they are congruent; but then $\angle KMP' = \angle KMP = 180^0/2 = 90^0$, hence KM is **perpendicular** to PP′:

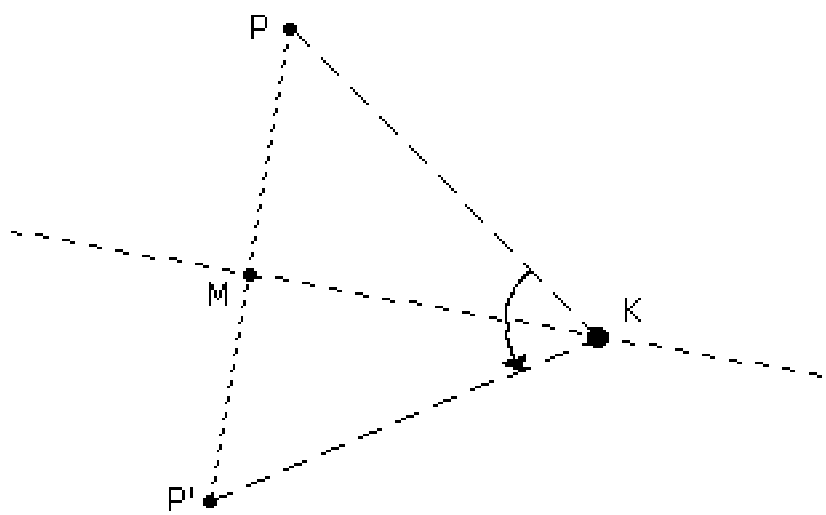

Fig. 3.7

Returning to A and A′ of 3.1.1, we may now obtain infinitely many rotations that map A to A′; simply apply the previous argument **backwards**, pick an **arbitrary** point K on the **perpendicular bisector** of AA′ to be the rotation center, and then observe that the rotation angle is none other than the **oriented angle ∠AKA′, opening from A toward A′ by way of K**:

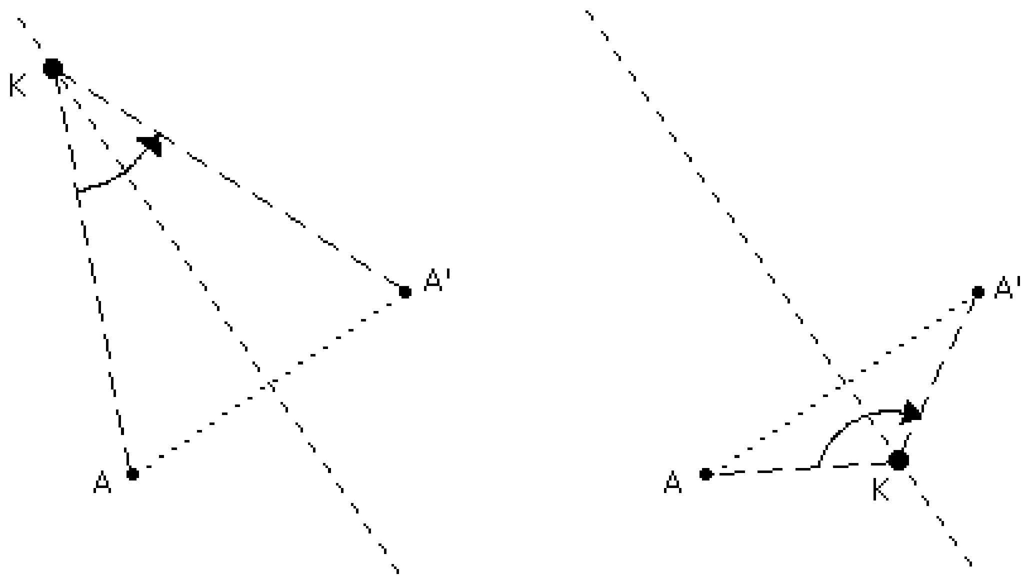

Fig. 3.8

Notice that the rotation angle could be either clockwise or counterclockwise, depending on the relative position of A, K, and A′: this information should **always** be part of your answer! Notice also that the rotation angle is 180^0 when K is the **midpoint** of AA′, and approaches 0^0 as K moves far away from (and on either side of) AA′ (with the rotation itself ‘approaching’ -- near AA′ at least -- the **translation** of figure 3.6).

3.1.3 <u>Glide reflections.</u> It’s time now to revisit figure 1.31, where we defined glide reflection, and make a crucial observation: the glide reflection axis L does intersect PP′ at its **midpoint**!

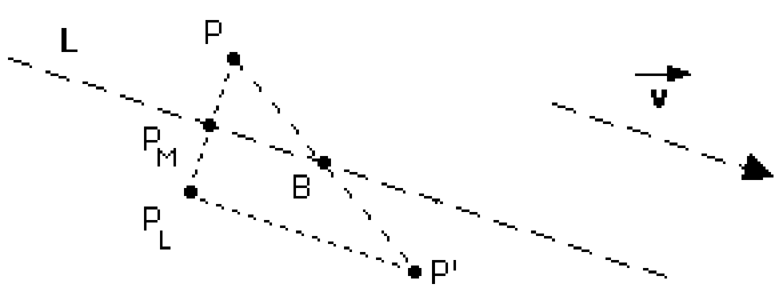

Fig. 3.9

While this is made ‘obvious’ by figure 3.9, it is not difficult to offer a rigorous proof. Indeed, since P_LP' is **parallel** to P_MB (by the very definition of glide reflection), $\frac{|PB|}{|PP'|} = \frac{|PP_M|}{|PP_L|} = \frac{1}{2}$.

How do we take advantage of this crucial observation and, in the context of 3.1.1 in particular, how could we use it to obtain glide reflections that map A to A′? All we have to do is to play the game **backwards**! Simply draw an **arbitrary** line L through the **midpoint** M of AA′ and then find the image A_L of A under reflection about L; it is easy then to check that the line L and the vector A_LA' (**pointing from the ‘intermediate’ mirror image toward the actual glide reflection image**) are the axis and vector of a glide reflection that maps A to A′:

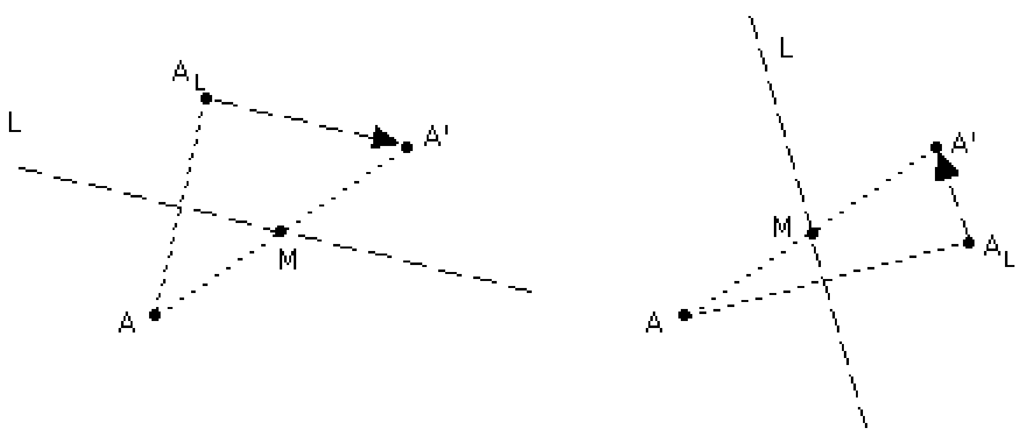

Fig. 3.10

Figure 3.10 offers two out of infinitely many possibilities for a glide reflection that maps A to A′. Notice that the glide reflection vector can be of every possible direction, but its length cannot exceed |AA′|; in the **special case** where L is the perpendicular bisector of AA′, the length of the glide reflection vector is equal to **zero** and the glide reflection is ‘reduced’ to the **reflection** of figure 3.6.

3.2 Segments

3.2.1 Two possibilities. Consider two straight line segments of **equal length**, one of them already labeled as AB:

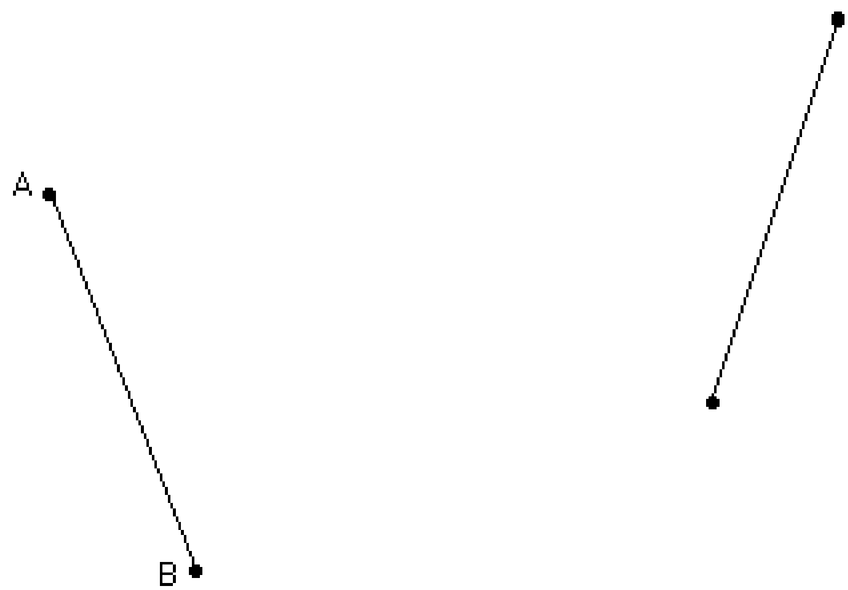

Fig. 3.11

You are probably certain that there exist isometries that map AB to the segment on the right, but you probably cannot guess how many and you are not sure how to find them, right? Well, one departing point is to realize that there exist **only two possibilities** for A and B: either A gets mapped to the 'top endpoint' and B gets mapped to the 'bottom endpoint' of the segment on the right, or vice versa. We will begin with the first possibility.

3.2.2 Two perpendicular bisectors, one center. Now that we have for the time being decided where A and B are mapped by the isometry we are trying to determine ('first possibility' in 3.2.1), we may recall (3.1.2) that there exist infinitely many **rotations** that map A to A′ and infinitely many rotations that map B to B′. The obvious question is: could some of those rotations perform **both** tasks, mapping A to A′ **and** B to B′? This question is answered if we also recall **how** all those rotations were determined! That is, let us recall (3.1.2) that the set of **centers** of all the rotations that map A to A′ is the **perpendicular bisector** of AA′, and likewise the set of centers of all the rotations that map B to B′ is the perpendicular bisector of BB′. Isn't it reasonable then to guess that the **intersection of the two perpendicular bisectors**, lying on **both** of them, will be the **unique rotation center** that achieves both goals? This guess is correct, as shown in figure 3.12, where we also determine the **rotation angle**:

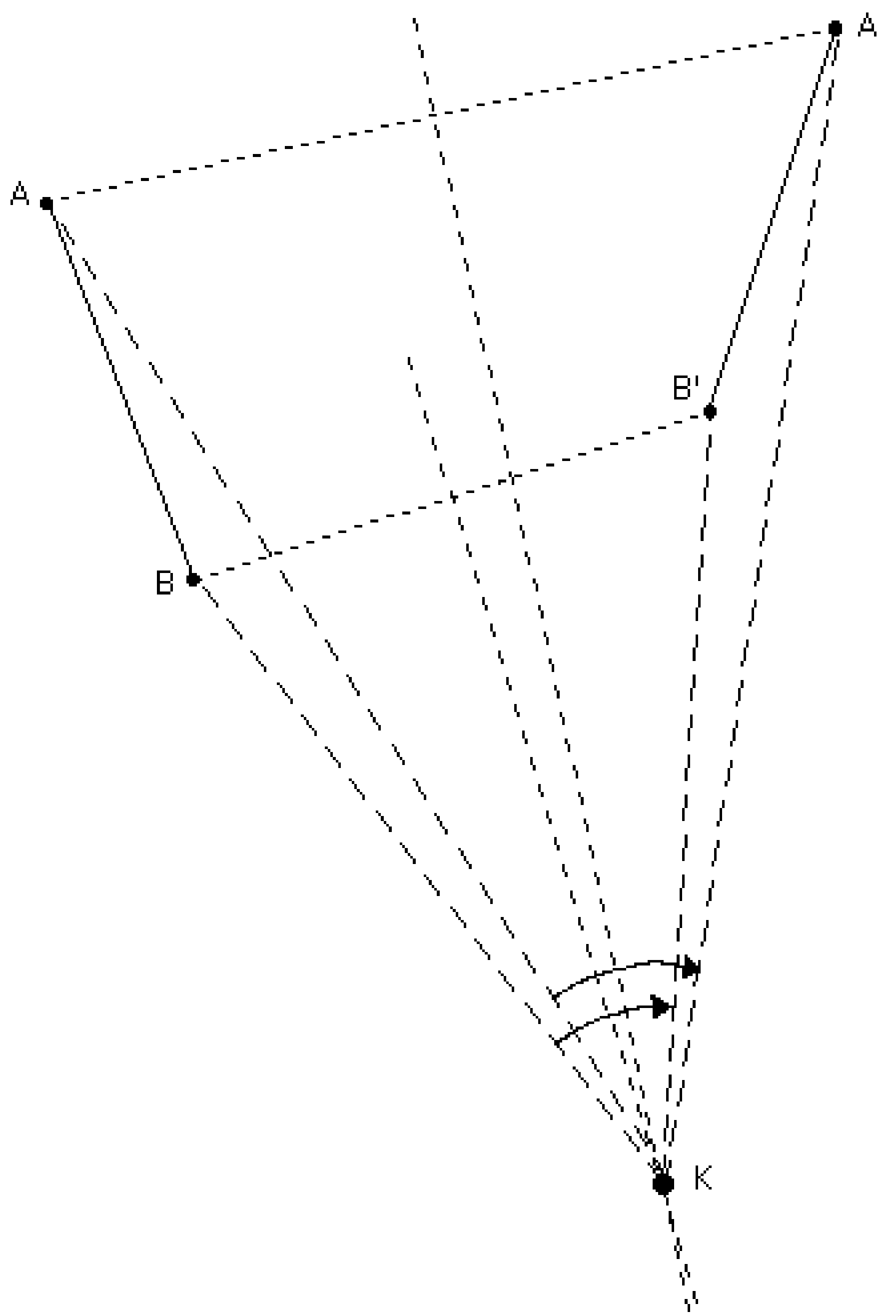

Fig. 3.12

Although it is next to impossible to achieve perfect precision, we see that approximately the same clockwise angle (and center) does indeed work for both A and B; in fact the rotation works for **all** points on AB -- we elaborate on this in 3.3.1.

3.2.3 Two midpoints, one axis. You can almost guess the game now: always sticking with that ‘first possibility’ of 3.2.1 (or just the placement of A′ and B′ in figure 3.12 if you wish), we would like to determine a **glide reflection** that maps **both** A to A′ and B to B′. We may at this point recall (3.1.3) that a glide reflection maps A to A′ (and B to B′) if and only if it passes through the **midpoint** of AA′ (and the midpoint of BB′). Arguing as in 3.2.2, we conclude that there exists a **unique glide reflection** mapping both A to A′ and B to B′, the axis of which is no other than the **line connecting the two midpoints**. The whole affair is presented in figure 3.13, where we also determine the **glide reflection vector**:

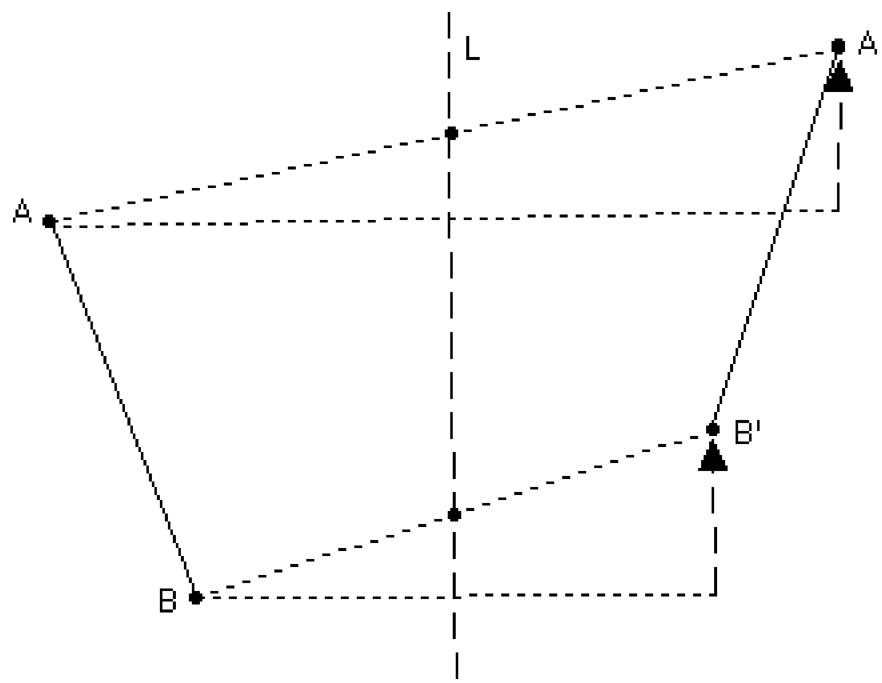

Fig. 3.13

Again we see that approximately the same S-N vector works for both A and B. The glide reflection **must** in fact work for **all** points on AB, as we are going to see in 3.3.1.

3.2.4 The 'second possibility'. We now take care of the second possible labeling of the segment on the right in figure 3.11 (3.2.1) and obtain two more isometries between the two segments as shown in figures 3.14 (rotation) and 3.15 (glide reflection):

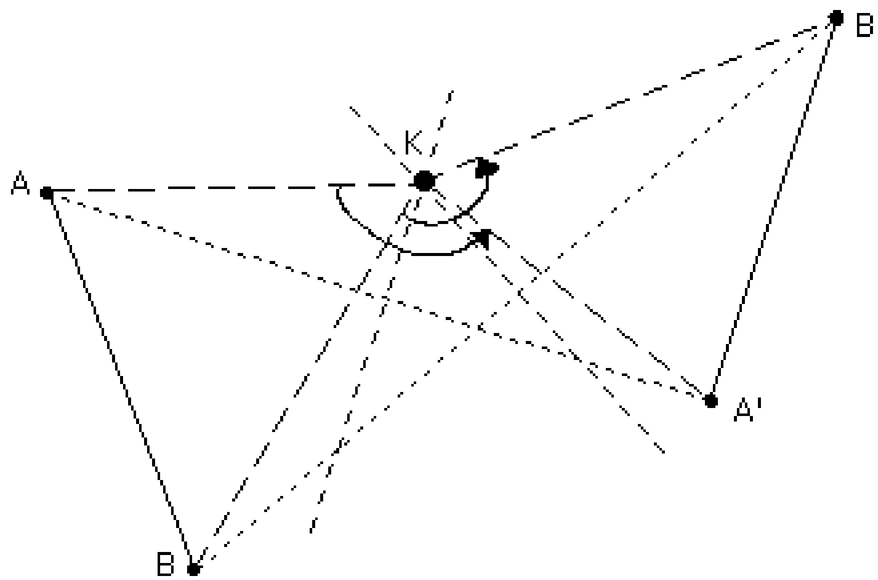

Fig. 3.14

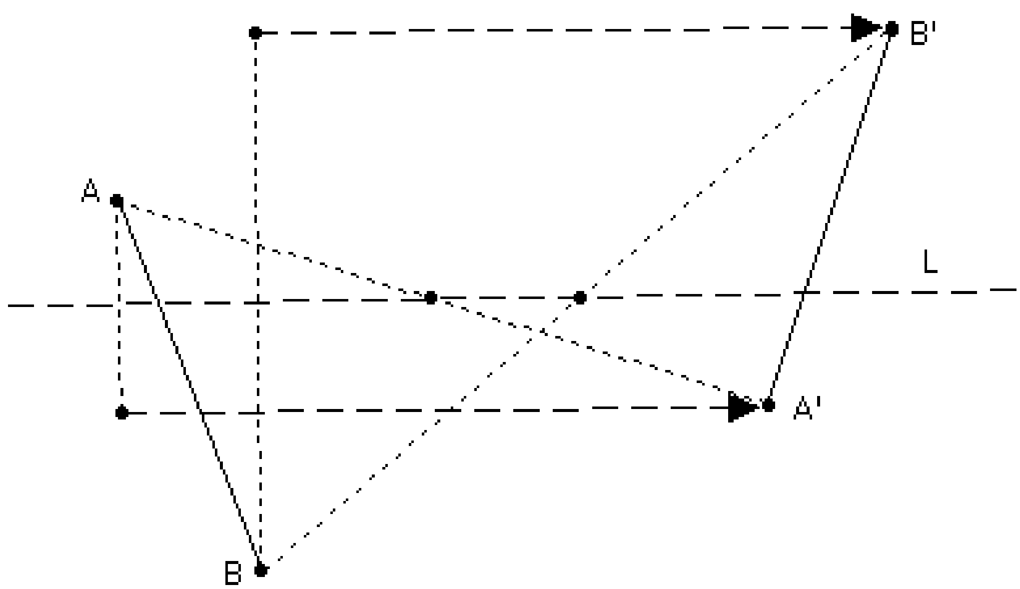

Fig. 3.15

3.2.5 <u>Four isometries!</u> Putting everything together, we see that there exist **two rotations and two glide reflections** mapping every two congruent straight line segments to each other. **One** of the rotations may be **'deformed'** into a **translation** (in case AB and A′B′, hence the perpendicular bisectors of AA′ and BB′ as well, are **parallel** to each other, 'meeting at infinity' -- see also concluding remark in 3.1.2); likewise, **one** of the glide reflections may be **'reduced'** to a **reflection** (in case the line connecting the midpoints of AA′, BB′ is **perpendicular** to one -- hence, by a geometrical argument, both -- of them). Both situations occur for example in the case of any two adjacent hexagons in a beehive!

3.2.6 <u>Homostrophic segments.</u> We conclude by pointing out that, **under either labeling**, the segment on the right (in figures 3.11 through 3.15) is homostrophic to AB. This is further indicated by the two rotations determined in figures 3.12 & 3.14, of course. But notice here that, quite uniquely as we will see later on, the segment A′B′, homostrophic to AB, is **at the same time** the image of AB under the two glide reflections determined in figures 3.13 & 3.15!

3.3 Triangles

3.3.1 <u>Two points almost determine it all.</u> Here is a simple question you could have already asked in section 3.2: how do we really know that each of the four isometries mapping A and B to the endpoints of the 'image segment' on the right do actually map (every point P on) the **segment** AB to (a point P′ on) the segment on the right? Good question! Luckily, circles come to the rescue of segments in figure 3.16 below.

Indeed, P′ (the image of P under whatever isometry maps A to A′ and B to B′) must lie on **both** the circle **$C_{A'}$ = (A′; |AP|)** of **center A′** and **radius |AP|** and the circle **$C_{B'}$ = (B′; |BP|)** of **center B′** and **radius |BP|**: the distances of P from both A and B must be preserved. But these two circles can have only one **'tangential'** point in common, **lying on A′B′**, due to |A′B′| = |AB| = |AP| + |BP| (figure 3.16).

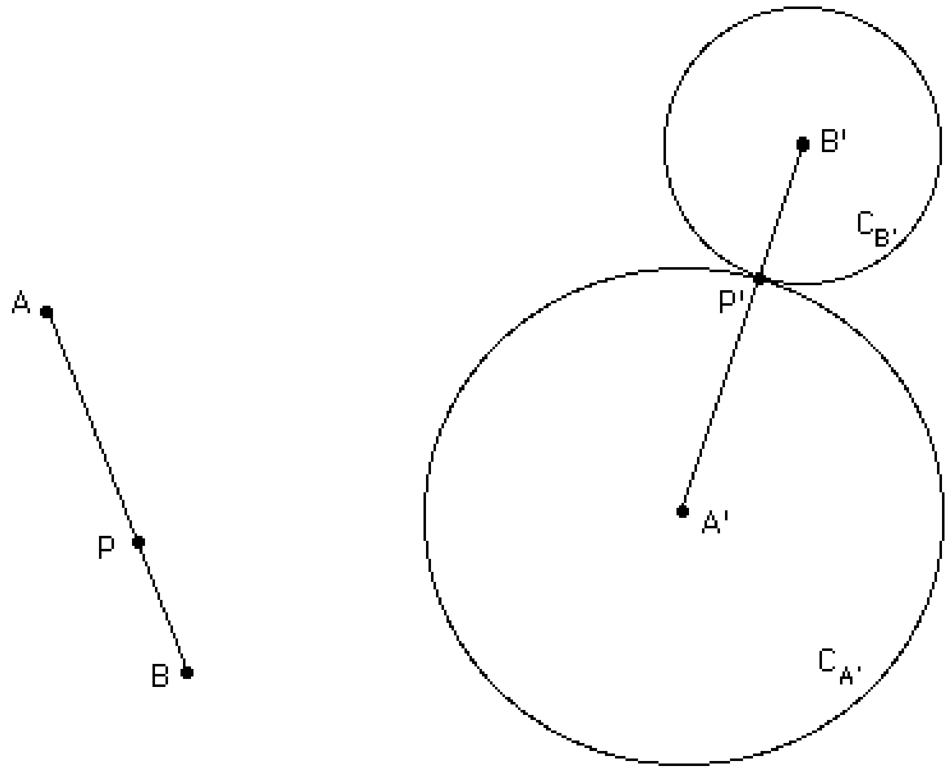

Fig. 3.16

There are of course **precisely two** possibilities for the exact location of P′ on AB's image on the right, depending on the two possibilities for A′ in 3.2.1; but in both cases every point P on AB is indeed mapped to a point P′ on A′B′ (with |PA| = |P′A′| and |PB| = |P′B′|, of course), therefore the **entire segment** AB is mapped to A′B′.

Now that you have seen that the images of the endpoints A, B completely determine the image of every point P on the segment AB (and in fact of every point on the **entire line** of AB, thanks to a similar argument involving ‘exterior points’ and ‘interior tangency’), you may wonder: what if P lies **outside** that line? Once again the circles $C_{A'}$ and $C_{B'}$ can be of great help, except that this time, with $|A'B'| < |AP| + |BP|$ instead of $|A'B'| = |AP| + |BP|$, they do **intersect** each other instead of being tangent to each other; hence there are **two possibilities** for P′, indicated by P'_1 and P'_2 in figure 3.17:

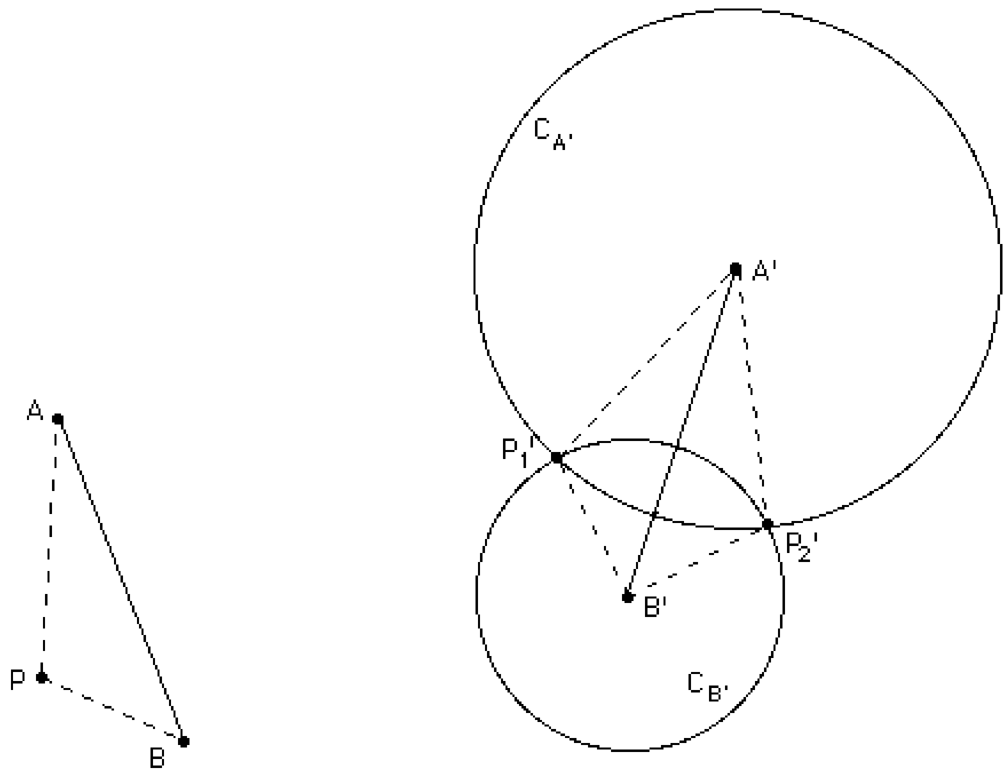

Fig. 3.17

3.3.2 <u>Congruent triangles.</u> It doesn't take long to observe that, in figure 3.17, $A'B'P'_1$ is (congruent and) **homostrophic** to ABP while $A'B'P'_2$ is (congruent and) **heterostrophic** to ABP. Reversing this observation, we notice that whenever a triangle $A'B'P'$ is congruent to a triangle ABP there exists **precisely one isometry** mapping ABP to $A'B'P'$: a rotation (or translation) in case $A'B'P'$ is homostrophic to ABP and a glide reflection (or reflection) in case $A'B'P'$ is heterostrophic to ABP. Indeed, with **A′ and B′ determined** on AB's image **by P''s position** (and $|AP| \neq |BP|$, the case $|AP| = |BP|$ being deferred to section 3.4), there are precisely two isometries mapping AB to $A'B'$, one rotation and one glide reflection (section 3.2): P' may then be **only one** of the **two intersection points** of the two circles shown in figure 3.17 (and corresponding to the **two isometries** mapping AB to $A'B'$).

We illustrate this in figure 3.18: labeling the 'original' triangle as DEF, we easily determine the images D', E', F' (homostrophic copy of DEF) and D'', E'', F'' (heterostrophic copy of DEF); it is then clear that **only one** of the two intersection points (F′) of the circles (D′; |DF|) and (E′; |EF|) corresponds to a homostrophic copy of DEF, and likewise **only one** of the two intersection points (F″) of the circles (D″; |DF|) and (E″; |EF|) corresponds to a heterostrophic copy of DEF.

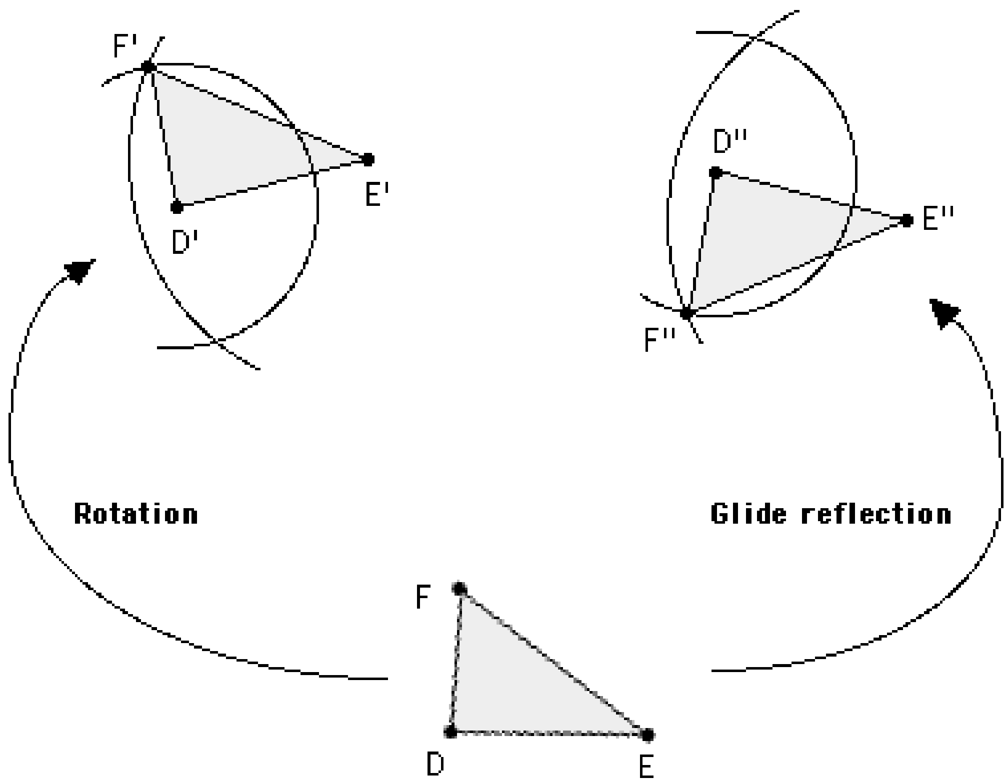

Fig. 3.18

3.3.3 <u>Circular orientation revisited.</u> Implicit in the discussion above is the assumption that translation and rotation are associated with homostrophy (**'same turning'**), while reflection and glide reflection are associated with heterostrophy (**'opposite turning'**). We can offer a quick justification for this assumption (and naming) as follows.

Returning to figure 3.17, let us replace the circles $C_{A'}$ and $C_{B'}$ by the three **congruent** circles C_0, C_1, and C_2, **circumscribed** to the triangles ABP, $A'B'P'_1$, and $A'B'P'_2$, respectively:

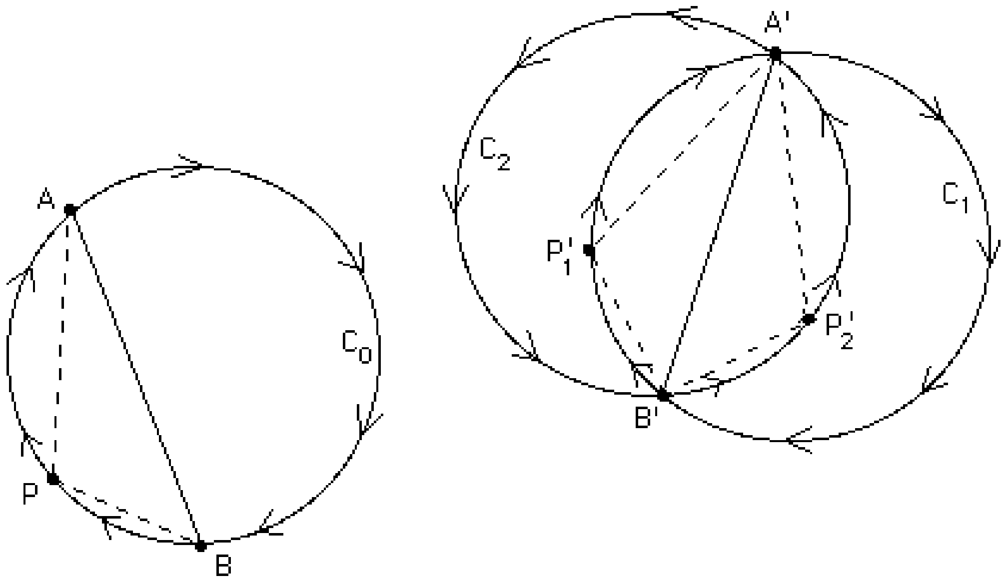

Fig. 3.19

Clearly the points A, B, P, traced **alphabetically**, are **clockwise** placed on C_0; their images are **clockwise** placed on C_1 (A′, B′, P'_1) and **counterclockwise** placed on C_2 (A′, B′, P'_2). As this **'circular order'** among A, B, P on C_0 has been **preserved** among their images on C_1 but **reversed** on C_2, it is easy to see that C_1 **can** slide back to C_0 returning images to originals, while C_2 **cannot** (without flipping, that is). So, homostrophy is associated with preservation of circular order, while heterostrophy is associated with reversal of circular order. But we have already seen in 1.5.4 -- and could certainly verify from scratch by extending section 3.1 from points to circles! -- that preservation of circular order is associated with translations and rotations, while reversal of circular order is associated with reflections and glide reflections.

3.3.4 Triangles determine everything! We just saw that, in the case of two congruent triangles, homostrophy is indeed associated with translation or rotation, and heterostrophy with reflection or glide reflection. This holds true for every pair of congruent sets on the plane, and relies on a broader fact, demonstrated in figure 3.20 below: every isometry on the plane is uniquely determined by its effect on **any three non-collinear points**!

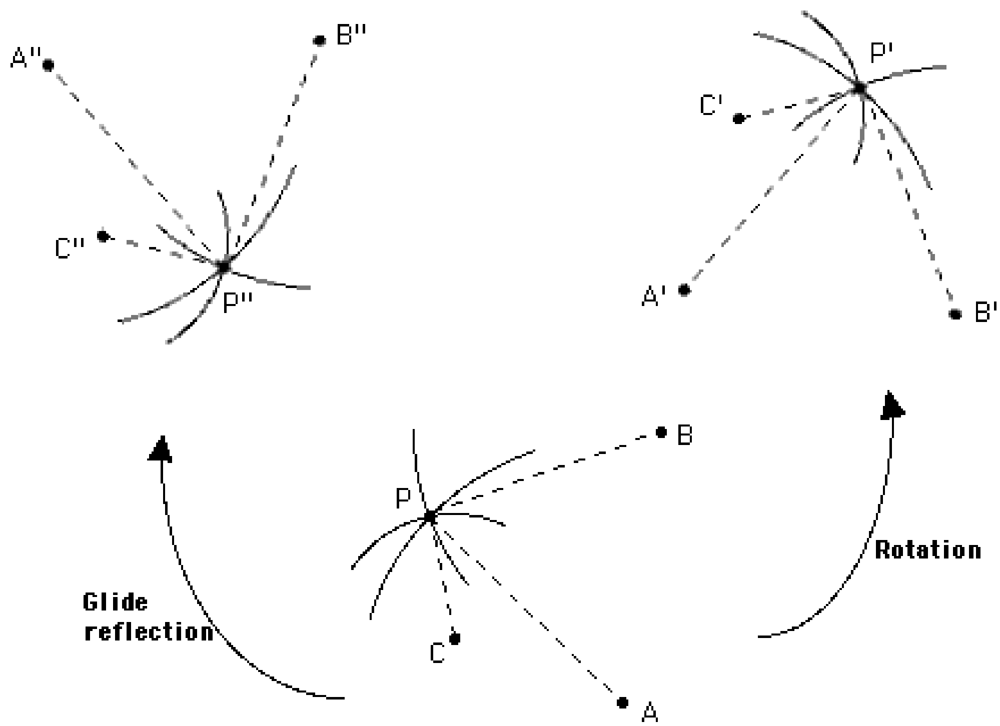

Fig. 3.20

Indeed, any ‘fourth point’ P lies at the intersection of **three circles** of centers A, B, C and radii |AP|, |BP|, |CP|, respectively. So the image of P is forced by the radius-preserving isometry to lie at the intersection of the **three image circles** (of centers A′, B′, C′ (rotation) or A″, B″, C″ (glide reflection) and radii |AP|, |BP|, |CP|); but every three circles with **non-collinear centers** can have **at most one** point in common, hence the image of P -- P′ under the rotation, P″ under the glide reflection -- is **uniquely** determined!

3.3.5 From theory to practice. Returning to section 3.0 and figure 3.1, we demonstrate in figure 3.21 how to find the **rotation** that maps ABC to A′B′C′ (homostrophic pair) and the **glide reflection** that maps ABC to A″B″C″ (heterostrophic pair).

As you can see, determining the isometries in question **reduces**, in view of 3.3.2, to picking the right type of isometry (rotation or glide reflection) that maps AB to A′B′; the rotation center or glide reflection axis is subsequently located as in section 3.2. It is always **wise** to use a **third point** (like C in figure 3.21) and its image to determine the rotation angle or glide reflection vector, as shown in figure 3.21 -- and even **wiser** to check that the same angle or vector **indeed** works for a **fourth** point, as well as for A and B!

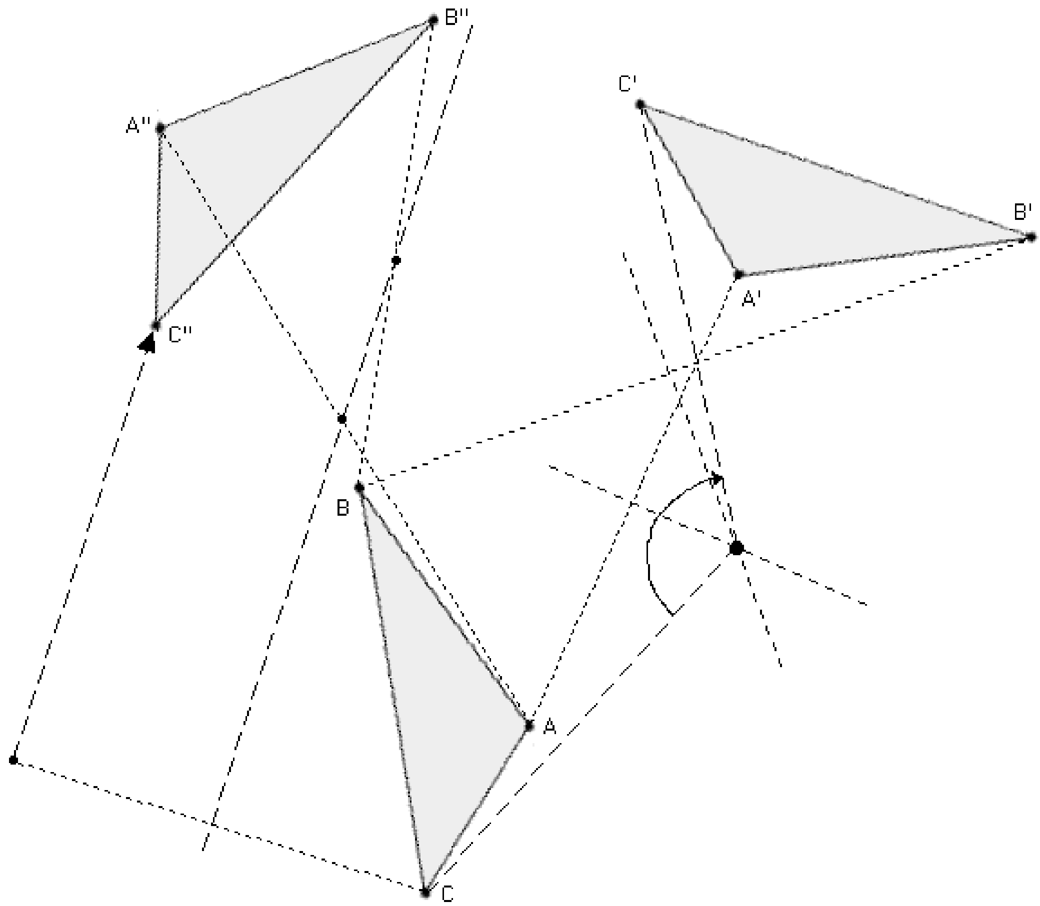

Fig. 3.21

3.4 Isosceles triangles

3.4.1 <u>The 'second possibility' revived.</u> As we pointed out in 3.3.2, there is generally only one isometry mapping a 'randomly chosen' triangle ABC to a congruent triangle A′B′C′: this is because C′ **both** allows only one possibility for the **positions** of A′ and B′ on the image of AB and determines the **kind of isometry** that maps AB to A′B′. While homostrophy/heterostrophy considerations never allow us to avoid the second limitation, it is possible to escape from the first one in case ABC happens to be isosceles (with **|AC| = |BC|**); there exist then again, as in 3.2.1, **two** possibilities for the images of A and B, associated with homostrophy (A'_1, B'_1) and heterostrophy (A'_2, B'_2). And there exist therefore one **rotation** mapping ABC to $A'_1B'_1C'$ (figure 3.22) **and** one **glide reflection** mapping ABC to $A'_2B'_2C'$ (figure 3.23), determined as in 3.3.5; but, of course, $A'_1B'_1C'$ and $A'_2B'_2C'$ are **one and the same** triangle, congruent to ABC!

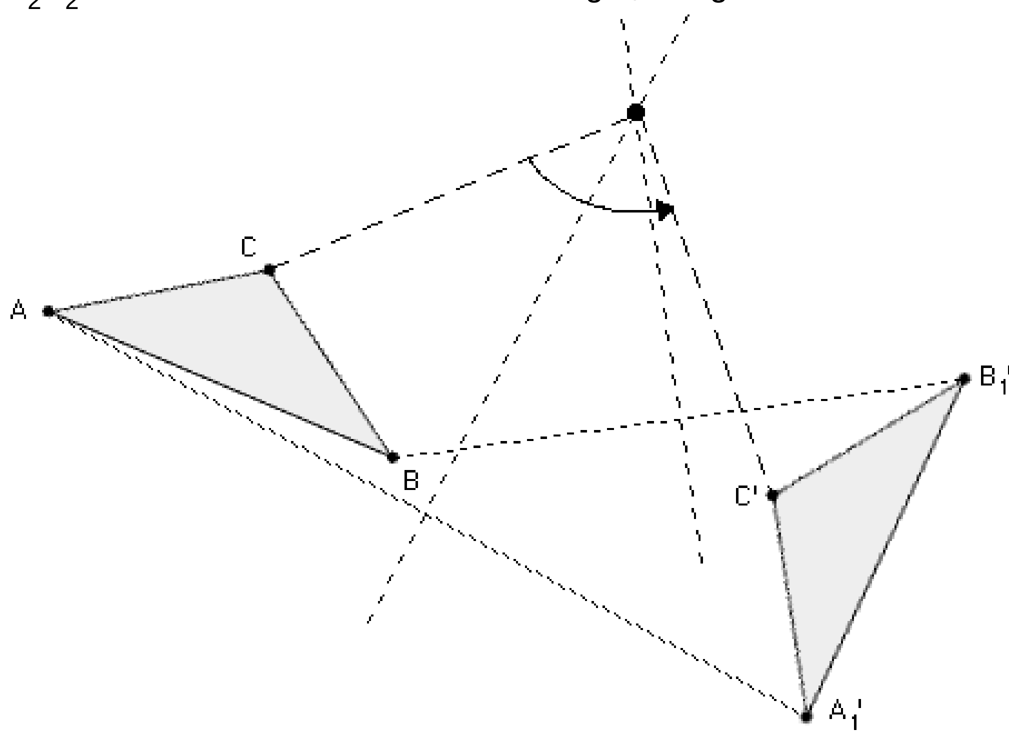

Fig. 3.22

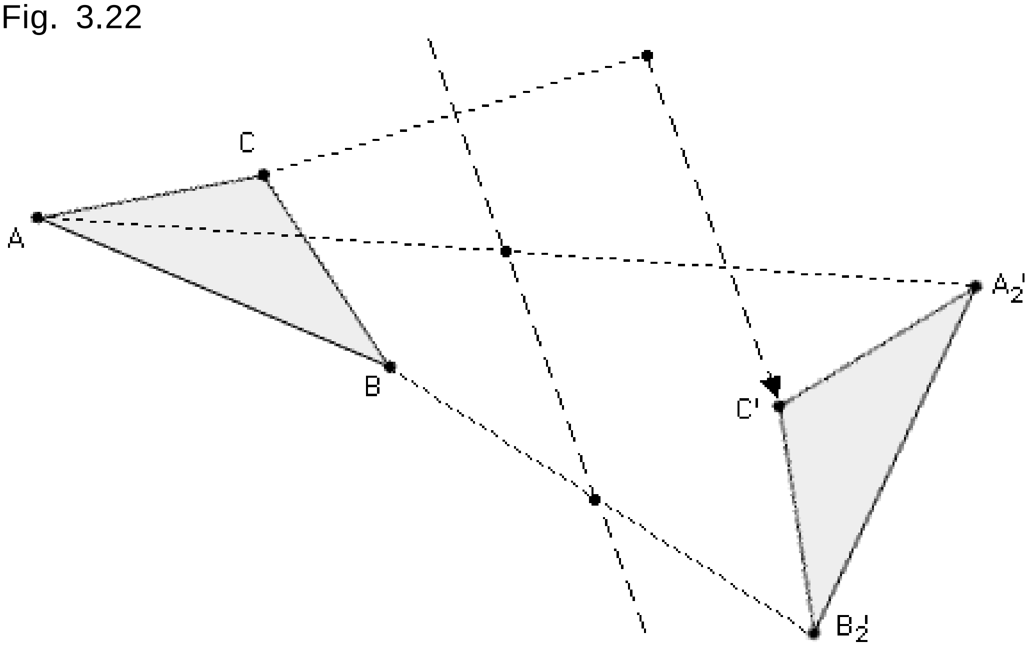

Fig. 3.23

3.4.2 <u>Old examples revisited.</u> We can at long last confirm and justify what we suspected in 3.0.3 and 3.0.4: there exists a rotation that achieves what reflection did in figure 1.14, and there also exists a glide reflection that rivals the rotation in figure 1.23. We demonstrate our findings in figures 3.24 & 3.25:

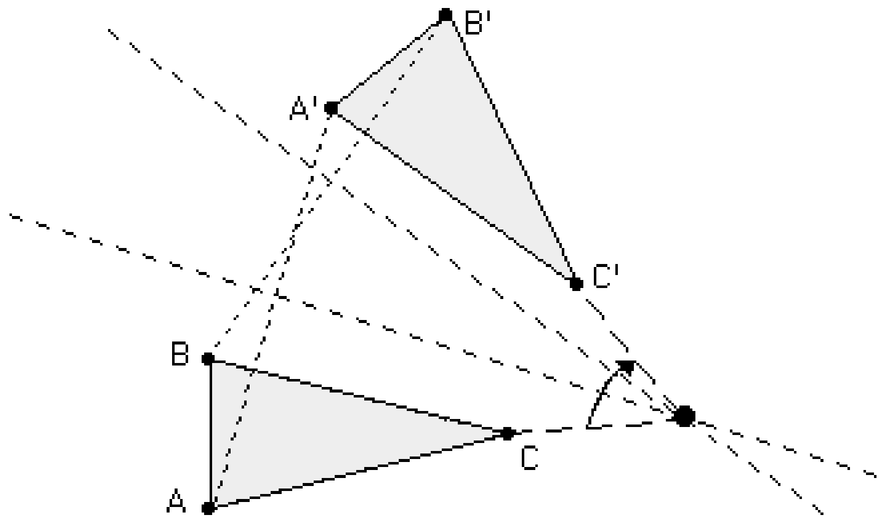

Fig. 3.24

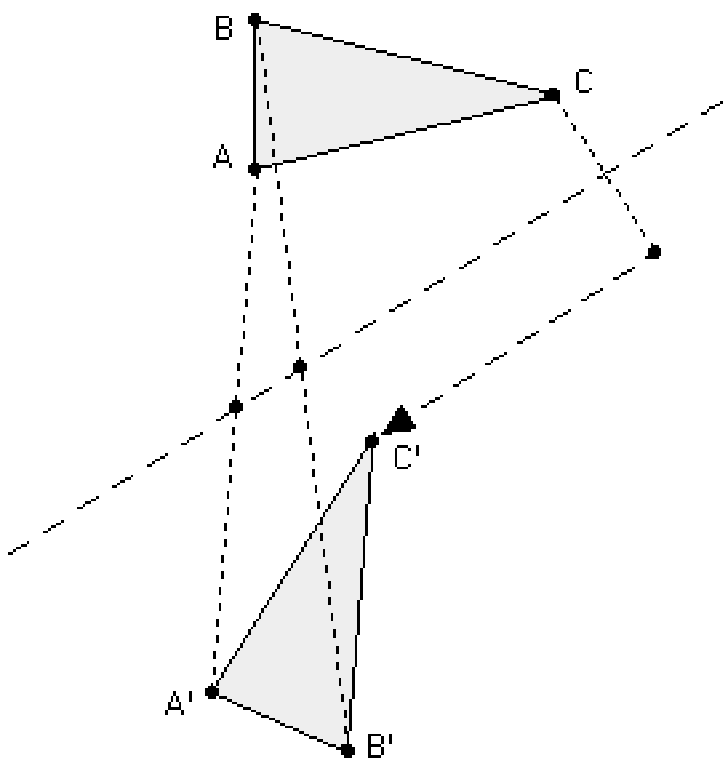

Fig. 3.25

3.5 Parallelograms, 'windmills', and C_n sets

3.5.1 Two triangles to go to! Consider the congruent, **heterostrophic parallelograms** ABCD, EFGH of figure 3.26:

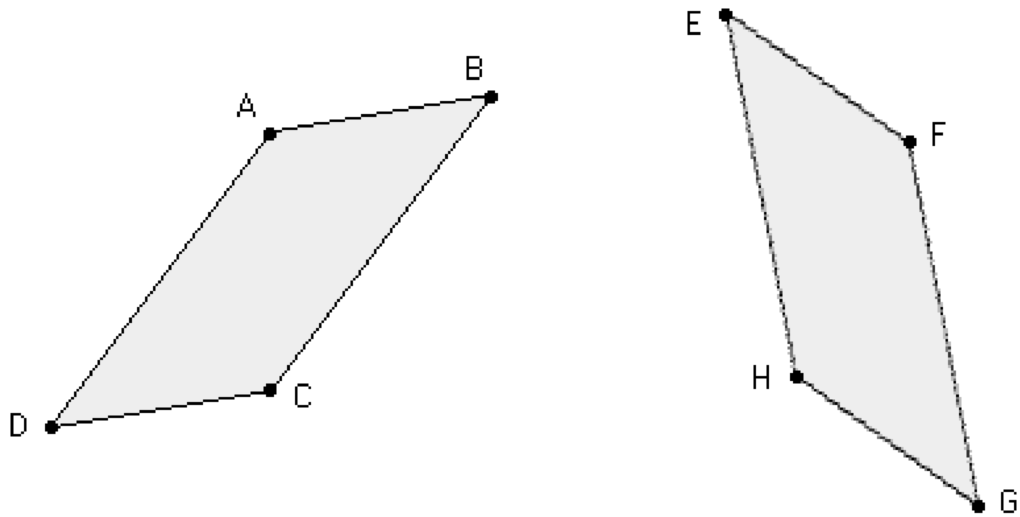

Fig. 3.26

The **congruent** triangles ABC and HGF are **heterostrophic**, so there certainly exists a glide reflection mapping ABC to HGF, hence ABCD to EFGH as well (3.3.4), obtained in figure 3.27:

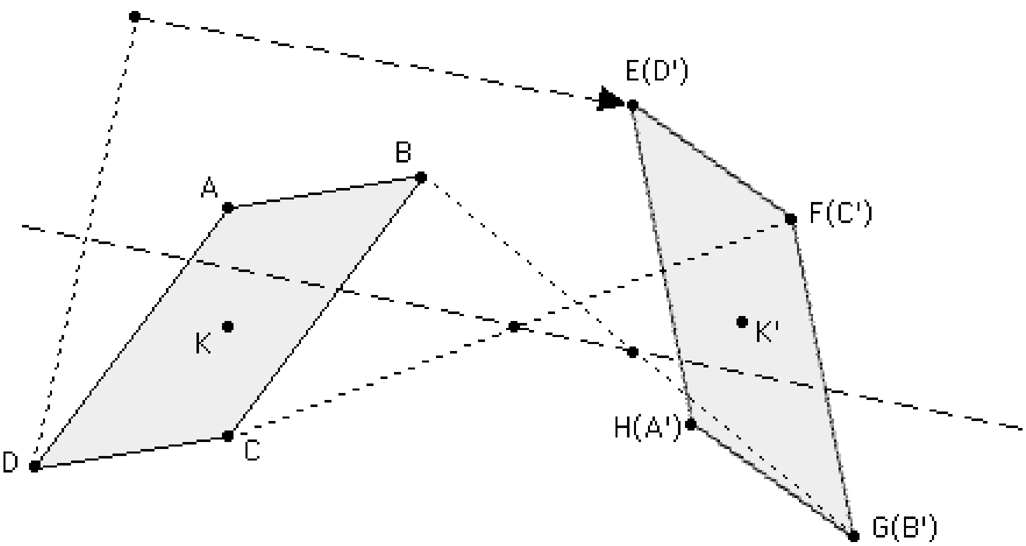

Fig. 3.27

On the other hand ... don't you think that ABC could have gone to FEH instead? Indeed ABC and FEH **also** happen to be congruent and heterostrophic, so there must exist a glide reflection mapping ABC to FEH, hence ABCD to EFGH as well (3.3.4); and such a glide reflection is obtained in figure 3.28:

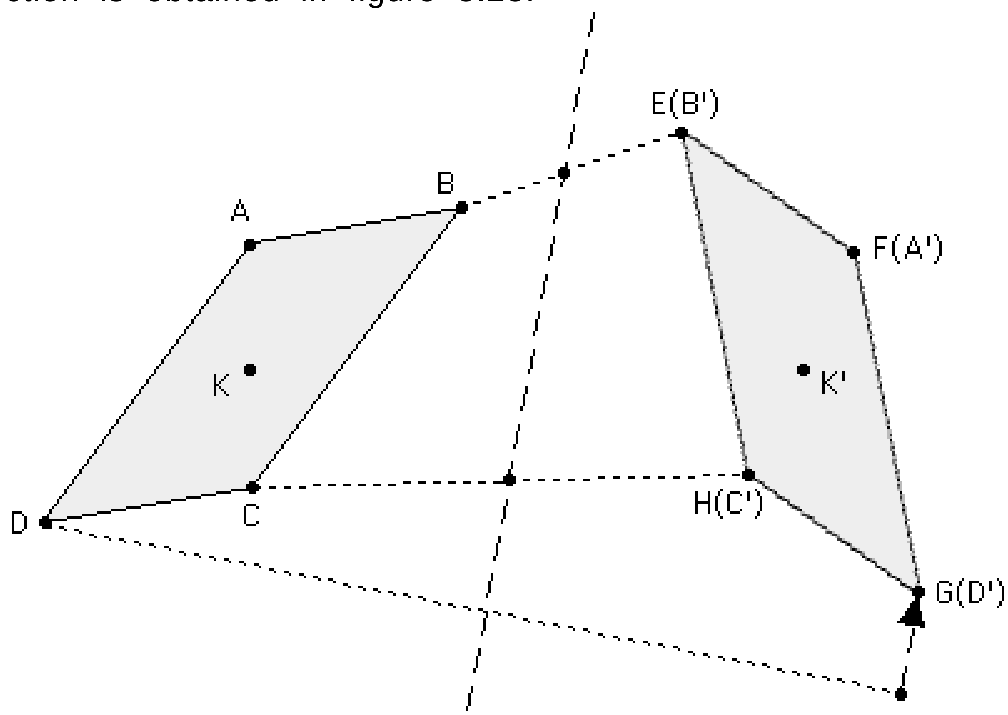

Fig. 3.28

So, there exist **two glide reflections** mapping ABCD to EFGH! What has happened? Clearly, the extra flexibility we have here is due to the existence of two **congruent** triangles **within** EFGH, HGF and FEH. Digging a bit deeper into this, what has made these two 'components' of the parallelogram EFGH congruent to each other? Could there be an 'obvious' isometry mapping one to the other? The answer is "yes": there exists an isometry mapping HGF to FEH (or vice versa) and that is ... no other than the **half turn** about the **parallelogram's center**, K′! To put it in more familiar terms (1.3.9), the parallelogram EFGH has **rotational symmetry**: indeed a **twofold** (180^0) rotation about K′ maps the parallelogram to itself, **swapping** HGF and FEH; and this twofold rotation is in fact **'combined'** (section 7.8) with the glide reflection of figure 3.27 to produce the glide reflection of figure 3.28!

3.5.2 How about more triangles? It is not that difficult to come up with situations involving more than two glide reflections between two congruent sets. Indeed, and in view of the discussion in 3.5.1, all we need is two copies of a set with 'richer' rotational symmetry than that of the parallelogram, a set with more than two 'triangles' rotating around a center. How about the following pair of **heterostrophic** windmill-like sets:

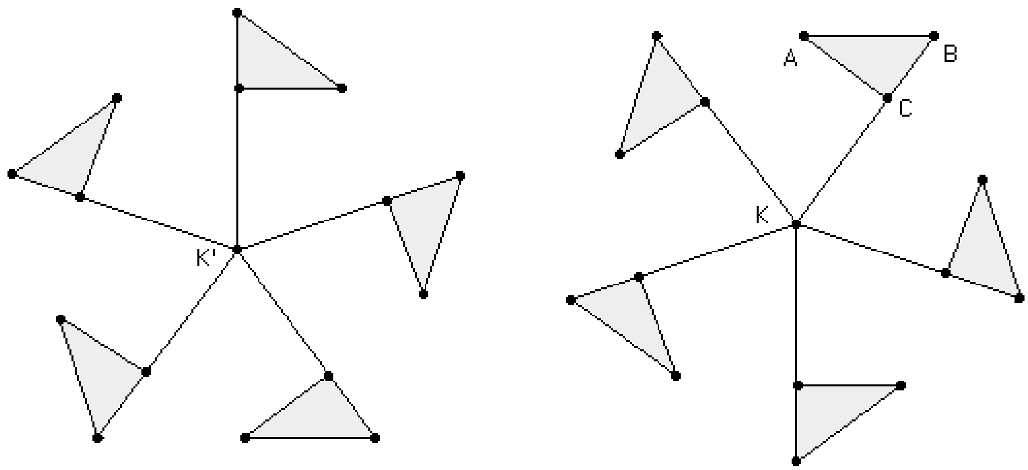

Fig. 3.29

You should have no trouble realizing that, with **five** 'options' for 'blade' ABC, there exist indeed **five glide reflections** mapping the 'windmill' on the right to the 'windmill' on the left. We leave it to you to determine these glide reflections; and if you go through this task with the great precision that is typical of you by now, you are going to see **all** five glide reflection axes **passing through the same point**, that is the **midpoint of KK′**: that should not surprise you if you care to notice that all five glide reflections **must** map K to K′! (You may of course decide to **'cheat'** by choosing K, K′ as one of your two pairs of points needed to determine each one of the five glide reflections!)

3.5.3 How about rotations? Returning to the parallelograms of figure 3.26, let us 'rectify' EFGH a bit, so that ABCD and EFGH are now **homostrophic**:

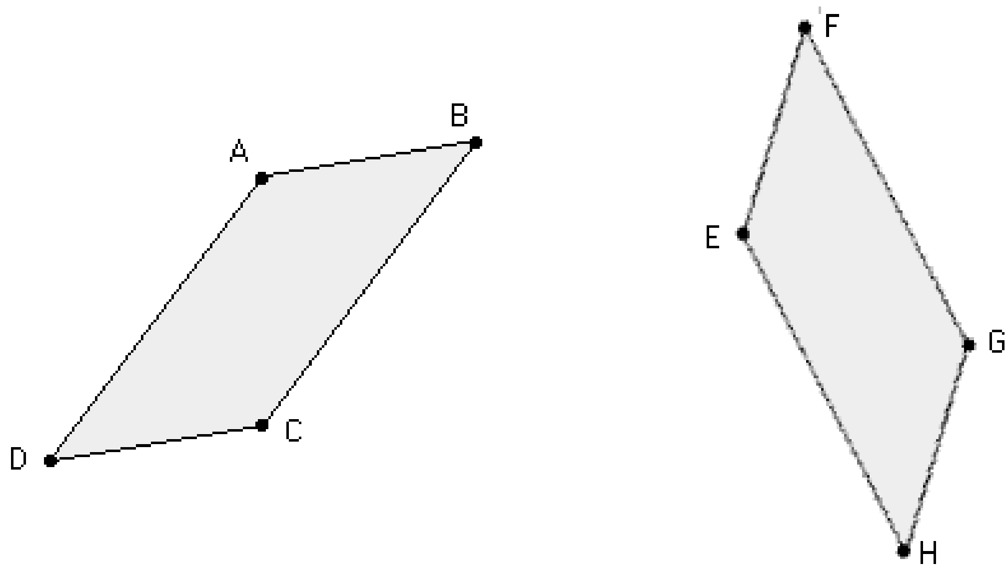

Fig. 3.30

Repeating the thought process of 3.5.1 we see that ABC is homostrophic to both GHE and EFG, hence there exist **two rotations** mapping ABCD to EFGH: one of them, by a clockwise 116^0, maps ABC to GHE (figure 3.31), the other one, by a counterclockwise 64^0, maps ABC to EFG (figure 3.32). Notice that $116^0 + 64^0 = 180^0$, in the same way, say, that the two glide reflection axes in figures 3.27 & 3.28 are perpendicular to each other: please check and, perhaps, **think** about such 'phenomena'!

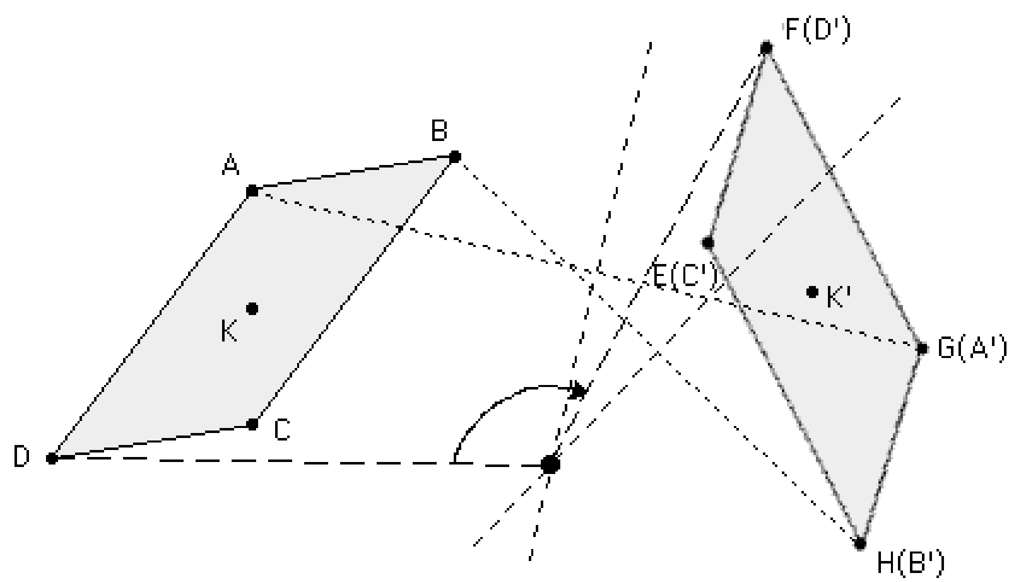

Fig. 3.31

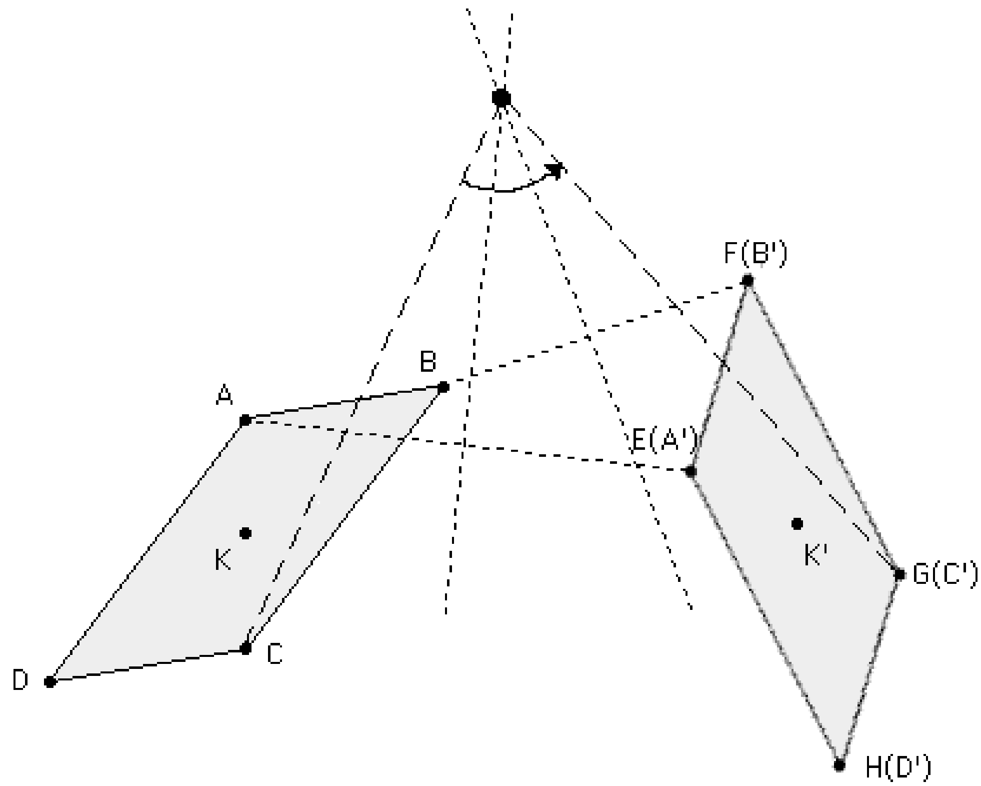

Fig. 3.32

Just as we turned the two glide reflections between the two heterostrophic parallelograms (3.5.1) into two rotations between homostrophic parallelograms, we can now turn the five glide reflections between the heterostrophic ‘windmills’ of figure 3.29 into **five** rotations between homostrophic ‘windmills’:

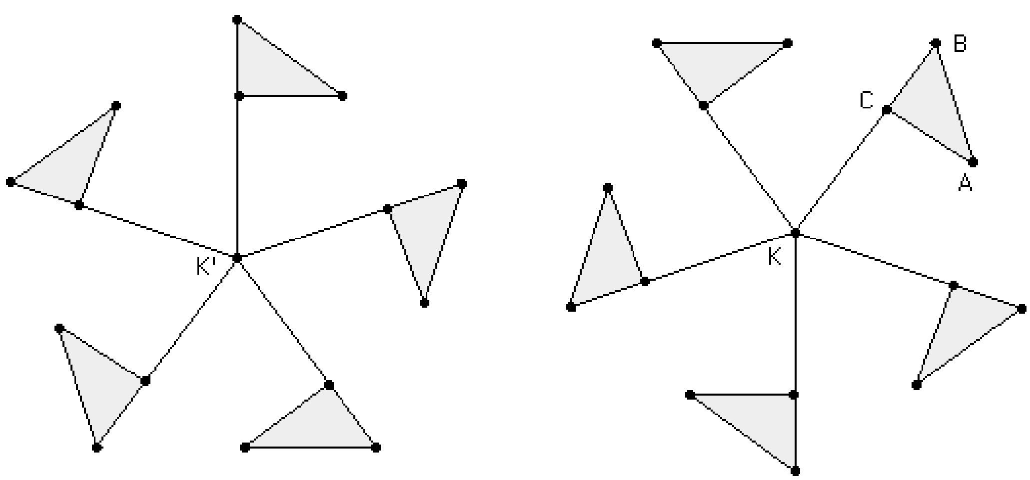

Fig. 3.33

Again, it is left to you to check that there exist **five rotations** mapping the 'windmill' on the right to the 'windmill' on the left, determined by the 'blade' to which ABC is mapped. As in 3.5.2, **all** five rotations **must** map K to K′, hence **all** five rotation centers must lie on the same line, the **perpendicular bisector of KK′**!

3.5.4 C_n sets and the role of orientation. Let us revisit figures 3.27, 3.28, 3.31, and 3.32, where we labeled EFGH as D′C′B′A′, B′A′D′C′, C′D′A′B′, and A′B′C′D′, respectively -- which, by the way, are the only possible labelings induced by isometries mapping ABCD to EFGH. Still, what made the difference was not labeling but orientation: **regardless of labeling**, we obtained a **glide reflection** between the **heterostrophic** parallelograms in **both** figure 3.27 and figure 3.28, and, likewise, a **rotation** between the **homostrophic** parallelograms in **both** figure 3.31 and figure 3.32.

Similarly, revisiting figures 3.29 & 3.33, we see that the existence of five glide reflections **or** five rotations between the two 'windmills' is associated **solely** with heterostrophy and homostrophy, respectively: labeling plays no role whatsoever.

Such observations always hold true between every two congruent **C_n sets**, that is, sets with **n-fold rotational symmetry without mirror symmetry**: there exist **either n rotations** mapping one to the other (in case they are **homostrophic**) **or n glide reflections** mapping one to the other (in case they are **heterostrophic**).

In addition to 'n-blade windmills', examples of C_n sets, known as **chiral sets** -- "hand(s)-like", from Greek "chir" = "hand" -- in Molecular Chemistry or Particle Physics, include: non-isosceles triangles (C_1), parallelograms (C_2), the triskelion (three human legs joining each other at 120^0 angles) from, among several other places, Isle of Man (C_3), the heterostrophic and culturally unrelated Hindu and Nazi swastikas (C_4), the Star of David (C_6), etc. An excellent collection of C_n sets and likewise of D_n sets (studied in section 3.6 right below), from various regions of the world and historical periods, is available in Peter S. Stevens' book (pages 16-93) already cited in section 2.9.

3.6 Rhombuses, ‘daisies’, and D_n sets

3.6.1 Two triangles and two ways. Let us consider the special case |AB| = |BC| = |CD| = |DA| (hence |FE| = |EH| = |HG| = |GF| as well) in **either** of figures 3.26 or 3.30; that is, let us consider the special case where each of the two congruent parallelograms is a **rhombus**:

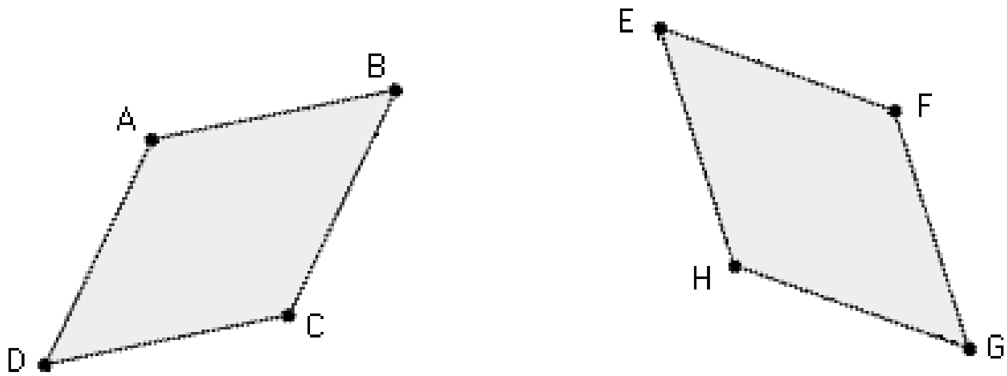

Fig. 3.34

How many isometries map ABCD to EFGH? Arguing in the spirit of section 3.5, we notice that every isometry mapping ABCD to EFGH has to map ABC to **either** FGH **or** HEF. But we also notice that ABC and either of FGH or HEF are **isosceles** triangles, and we do as well recall (section 3.4) that there always exist **two** isometries mapping an isosceles triangle to a congruent to it isosceles triangle. We conclude that there exist **two** $\times$ **two = four** isometries mapping ABCD to EFGH: **two rotations** (one mapping ABC to FGH (figure 3.35), another mapping ABC to HEF (figure 3.36)) and **two glide reflections** (one mapping ABC to HGF (figure 3.37), another mapping ABC to FEH (figure 3.38)); D’s image is simply determined by those of A, B, C (3.3.4).

More **rigorously**, and with composition of isometries (chapter 7) in mind, we could see how, for example, the rotation in figure 3.35 is **‘combined’** with the rhombus’ internal reflection swapping E, G to produce the glide reflection of figure 3.38, etc.

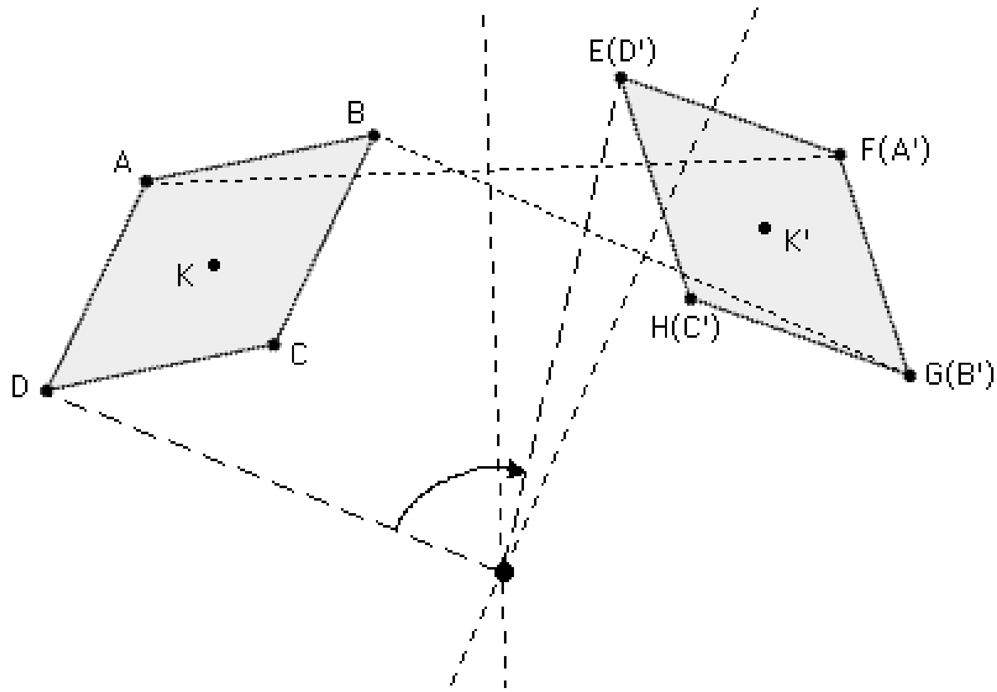

Fig. 3.35

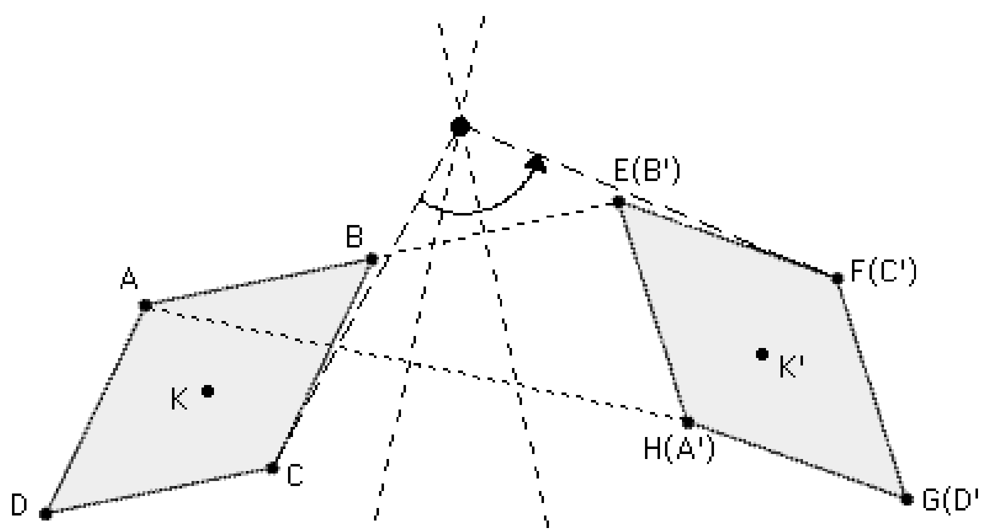

Fig. 3.36

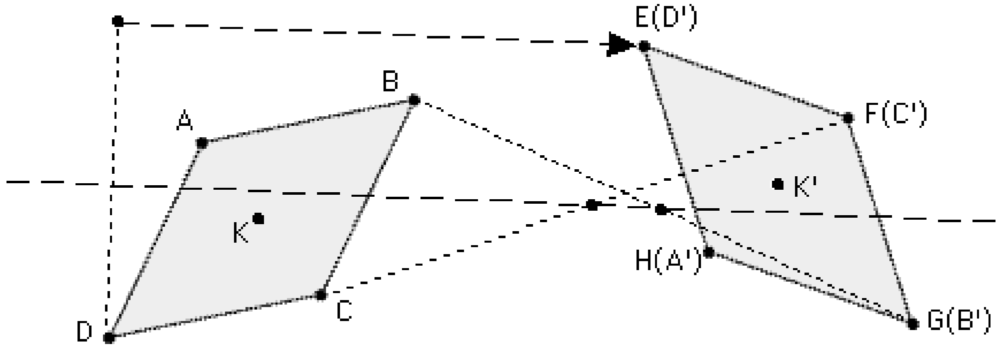

Fig. 3.37

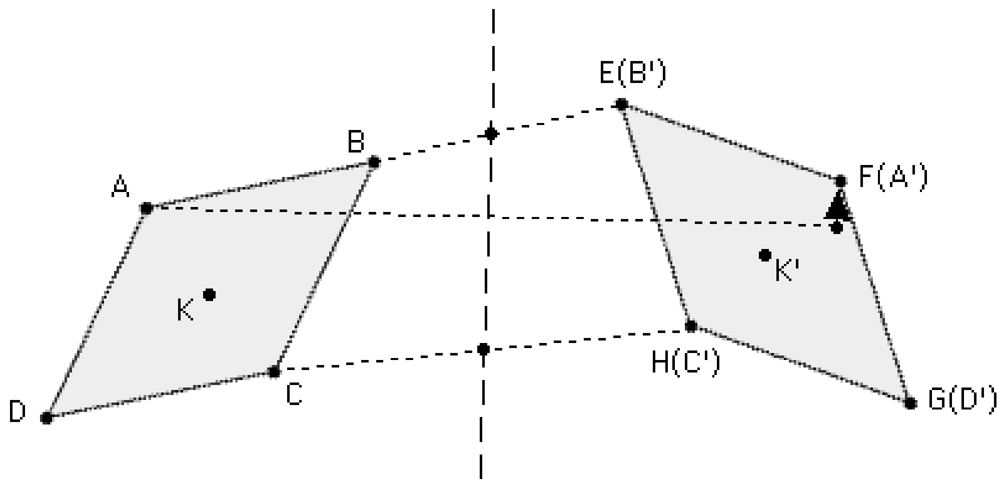

Fig. 3.38

Notice that **homostrophy or heterostrophy induced by labeling** has once again become important, just as in section 3.4: we obtained rotations in the cases of homostrophic copies (with EFGH labeled in effect as either **D′A′B′C′** (figure 3.35) or **B′C′D′A′** (figure 3.36), both of them **homostrophic to ABCD**) and glide reflections in the cases of heterostrophic copies (with EFGH labeled in effect as either **D′C′B′A′** (figure 3.37) or **B′A′D′C′** (figure 3.38), both of them **heterostrophic to ABCD**).

3.6.2 Everything in double! What happens if we apply the same 'symmetrization' process applied to the parallelograms of figures

3.26 & 3.30 to those **‘5-blade windmills’** of figures 3.29 & 3.33? We need to **replace** non-isosceles triangles by isosceles ones, arriving at a pair of congruent **‘5-petal daisies’**:

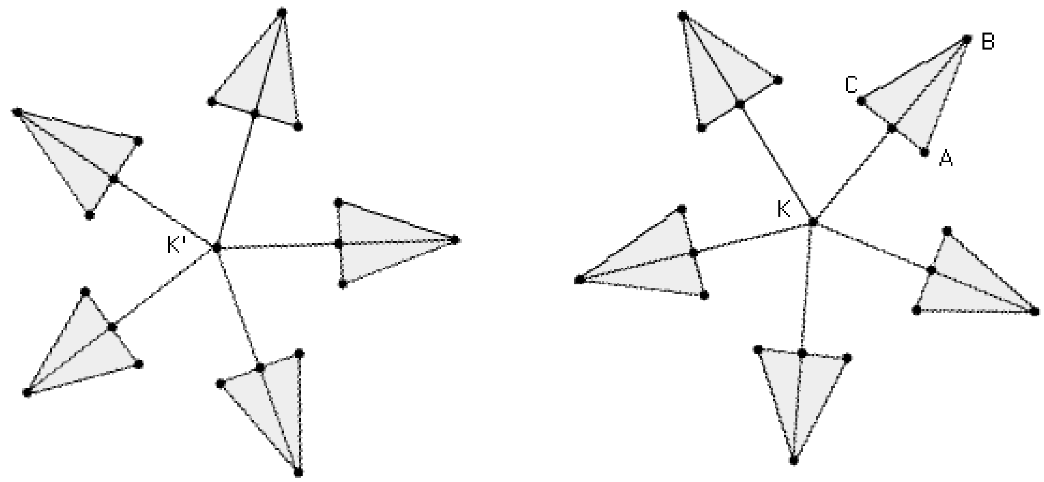

Fig. 3.39

How many isometries map the ‘daisy’ on the right to the ‘daisy’ on the left? Well, you almost know the game by now: there are **five** ‘petals’ to which ‘petal’ ABC can be mapped, and in each case this can be done by **both** a rotation and a glide reflection mapping the ‘daisy’ on the right to the ‘daisy’ on the left; putting everything together, we see that there exist **five** × **two = ten** isometries between the two congruent ‘daisies’, **five rotations and five glide reflections**! We leave it to you to provide the right labeling for each one of these ten isometries: you should then be able to check that all five glide reflection axes pass through the midpoint of KK′ (3.5.2) and that all five rotation centers lie, despite falling off this page on occasion, on the perpendicular bisector of KK′ (3.5.3).

3.6.3 D_n sets and the role of labeling. A closer look at 3.6.1 and 3.6.2 explains the abundance of isometries between the rhombuses and the ‘5-petal daisies’: in addition to **rotational symmetry** (by 180^0 and 72^0, respectively), they are both blessed by at least one **isosceles triangle** the reflection axis of which acts as a reflection axis for the entire set -- that is, they also have **mirror symmetry**!

Summarizing and generalizing our findings in this section, we may say that between every two congruent **D_n sets**, that is, sets with **both mirror symmetry and n-fold rotational symmetry**, there exist **n rotations** (allowed by ‘homostrophic labeling’) **and n glide reflections** (allowed by ‘heterostrophic labeling’).

In addition to ‘n-petal daisies’, examples of D_n sets, known to scientists as **achiral sets**, include: isosceles triangles (D_1), rhombuses and straight line segments (D_2), equilateral triangles (D_3), squares and the Red Cross symbol (D_4), the ‘pentagram’ (D_5), snowflakes (D_6), regular n-gons (D_n), circles and points (D_∞), etc.

3.6.4 <u>‘Practical’ issues.</u> You must have observed by now that, once we know what type of isometry (rotation or glide reflection) between two congruent sets we are looking for, and any and all issues of homostrophy/heterostrophy and labeling have been decided, the actual determination of the isometry simply reduces to constructing one between two congruent **segments** and choosing the relevant **endpoints**. Such an observation is of course a natural consequence of our discussion in the entire chapter; see in particular 3.3.5.

When looking for a rotation, it is advisable to choose two pairs of points such that the perpendicular bisectors of the corresponding segments will **not** be **‘nearly parallel’** to each other. The idea here is that tiny, almost inevitable, errors in the location of the two midpoints and/or the direction of the perpendicular bisectors are propagated in case the two lines run nearly parallel to each other, hence the rotation center could be greatly misplaced. For example, choosing to work with B, B′ and C, C′ would have been a bad idea in figure 3.36 but not in figure 3.35.

Likewise, in the case of a glide reflection, it is **not** advisable to work with two segments the midpoints of which are **‘too close’** to each other: again, tiny, almost inevitable, errors in the location of the two midpoints are likely to lead to a considerably misplaced glide reflection axis; this has in fact happened to some extent with B, B′ and C, C′ in figure 3.37 (why?), but probably not in figure 3.38.

Prior to choosing your pair of points, you should decide whether you need a rotation or a glide reflection. If both are possible, then homostrophy/heterostrophy issues become important. You must carefully choose your **labeling** so that it will be both **possible** (avoid a disaster situation where, for example, $|A'B'| \neq |AB|$!) and **appropriate** (homostrophic or heterostrophic as needed) for the isometry you are looking for. Figures 3.35-3.38 should provide sufficient illustration in this direction. Another useful tip for labeling the image set, suggested by **Erin MacGivney** (Spring 1998), is this: trace the original set, including original labels (A, B, C, ...) on tracing paper, then slide it in every possible way until it matches (and labels!) the image set, **with or without flipping** the tracing paper (and inducing heterostrophic or homostrophic labeling, respectively).

Various 'labeling' errors can often be caught with the use of a **'third point'** (in determining the rotation angle or glide reflection vector) already advocated in 3.3.5: **watch out** in particular for **unequal** angle legs or for an axis and vector **not parallel** to each other!

Of course, you should first of all answer the following question about the given pair of congruent sets: **are they C_n sets or D_n sets, and what is n?** If they are **C_n sets** then you must decide whether they are homostrophic or heterostrophic, allowing for **either n rotations or n glide reflections**, respectively. If they are **D_n sets** you should keep in mind that, with fully developed labeling skills, you ought to be able to get all **n rotations and n glide reflections** between the two sets; and keep in mind that **one** rotation could be 'reduced' to a translation (in case the two sets are side-by-side **'parallel'** to each other) and **one** glide reflection might 'merely' be a reflection (3.2.5).

3.7* Cyclic (C_n) and dihedral (D_n) groups

3.7.1 <u>'Turning the windmills'.</u> Let us revisit that '5-blade windmill' C_5 set of figure 3.29, redrawn and relabeled in figure 3.40 below; more specifically, the five 'blades' are now labeled as T_0, T_1, T_2, T_3, and T_4.

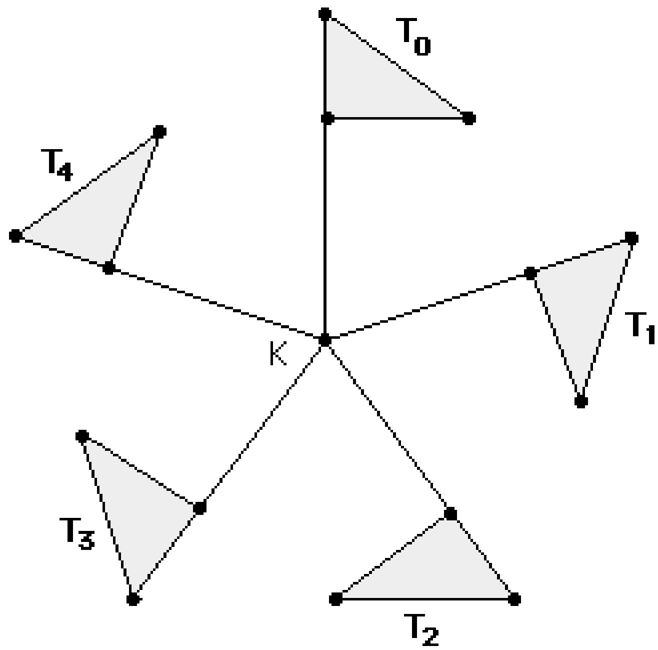

Fig. 3.40

As we noticed in 3.5.2, a clockwise $360^0/5 = 72^0$ rotation **r** about K maps the 'windmill' to itself by moving the 'blades' T_s around according to the formula $\mathbf{r}(T_s) = T_{(s+1)mod5}$, where, for every integer t, **tmod5** is the **remainder** of the division of t by 5. For example, $\mathbf{r}(T_2) = T_3$, $\mathbf{r}(T_4) = T_0$, etc. What happens when we apply **r** **twice in a row**? Clearly, $\mathbf{r}(\mathbf{r}(T_2)) = \mathbf{r}(T_3) = T_4$, $\mathbf{r}(\mathbf{r}(T_4)) = \mathbf{r}(T_0) = T_1$, and so on; we write $\mathbf{r}^2(T_2) = T_4$, $\mathbf{r}^2(T_4) = T_1$, and so on, and we notice that $\mathbf{r}^2$ is a clockwise rotation by $2 \times 72^0 = 144^0$, 'rigorously' defined via $\mathbf{r}^2(T_s) = T_{(s+2)mod5}$. And likewise we can go on and define $\mathbf{r}^3$ and $\mathbf{r}^4$ as clockwise rotations by $3 \times 72^0 = 216^0$ and $4 \times 72^0 = 288^0$, respectively, subject to the rule, for l = 3 and l = 4, respectively, $\mathbf{r}^l(T_s) = T_{(s+l)mod5}$. For example, for l = 3 and s = 4, (s+l)mod5 = 7mod5 = 2, therefore $\mathbf{r}^3(T_4) = T_2$, a result that you may certainly confirm **geometrically** (by rotating T_4 about K by a clockwise 216^0).

All this extends naturally to the 'n-wing windmill' (of 'blades' T_0, T_1, ... , T_{n-1}), where the clockwise $360^0/n$ rotation **r** satisfies $\mathbf{r^l}(T_s) = T_{(s+l)modn}$ for all integers s, l between 0 and n–1. We may in fact extend this formula for all $l \geq n$; for **l = n**, in particular, we notice that (s+n)modn = (s+0)modn, hence $\mathbf{r^n}(T_s) = T_s$ for all s: that is, $\mathbf{r^n} = \mathbf{r^0} = \boldsymbol{I}$ is a 'dead' rotation (***Identity map***) that leaves all the 'blades' unchanged. Moreover, we can easily compute the **product** of the rotations $\mathbf{r^k}$ and $\mathbf{r^l}$ (a clockwise rotation of $l \times 360^0/n$ followed by a clockwise rotation by $k \times 360^0/n$) via $\mathbf{r^k * r^l}(T_s) = \mathbf{r^k}(\mathbf{r^l}(T_s)) = \mathbf{r^k}(T_{(s+l)modn}) = T_{(s+l+k)modn} = \mathbf{r^{k+l}}(T_s)$; that is, $\mathbf{r^k * r^l = r^{(k+l)modn}}$, due to (s+l+k)modn = (s+(l+k)modn)modn: please check!

You may also verify this 'rotation multiplication' by **adding the angles** via $l \times 360^0/n + k \times 360^0/n = (l+k) \times 360^0/n$ and noticing that a $(l+k) \times 360^0/n$ rotation is the same as a $((l+k)modn) \times 360^0/n$ rotation. Moreover, you may certainly confirm this multiplication rule geometrically by returning to our '5-blade windmill' and checking that, for example, the $\mathbf{r^3}$ rotation of 216^0 followed by the $\mathbf{r^4}$ rotation of 288^0 does indeed produce the $\mathbf{r^2}$ rotation of 144^0, precisely as (4+3)mod5 = 2 would predict!

The relations $\mathbf{r^n} = \mathbf{r^0} = \boldsymbol{I}$ and $\mathbf{r^k * r^l = r^{(k+l)modn}}$ derived above define the **cyclic group of order n**, denoted by $\mathbf{C_n}$: an algebraic structure whose elements are $\boldsymbol{I}$, **r**, ..., $\mathbf{r^{n-1}}$ and whose importance in Mathematics is inversely proportional to its simplicity! It is in fact a **commutative** group: the order of 'multiplication' does not matter ($\mathbf{r^k * r^l = r^l * r^k = r^{(k+l)modn}}$). Its identity element is the 'dead map' (rotation) $\mathbf{r^n} = \boldsymbol{I}$ we already discussed, and the **inverse** of $\mathbf{r^l}$ is simply $\mathbf{r^{n-l}}$ (with $\mathbf{r^l * r^{n-l} = r^{n-l} * r^l = r^0} = \boldsymbol{I}$).

3.7.2 'Bisecting the daisies'. Let us now apply the notation of 3.7.1 to the 'leaves' of that '5-petal daisy' from figure 3.39, drawing its reflection axes ('bisectors') at the same time; we end up with the axis $\mathbf{m_s}$ bisecting 'petal' T_s for s = 1, 2, 3, 4, and with axis $\mathbf{m_5}$ bisecting 'petal' T_0:

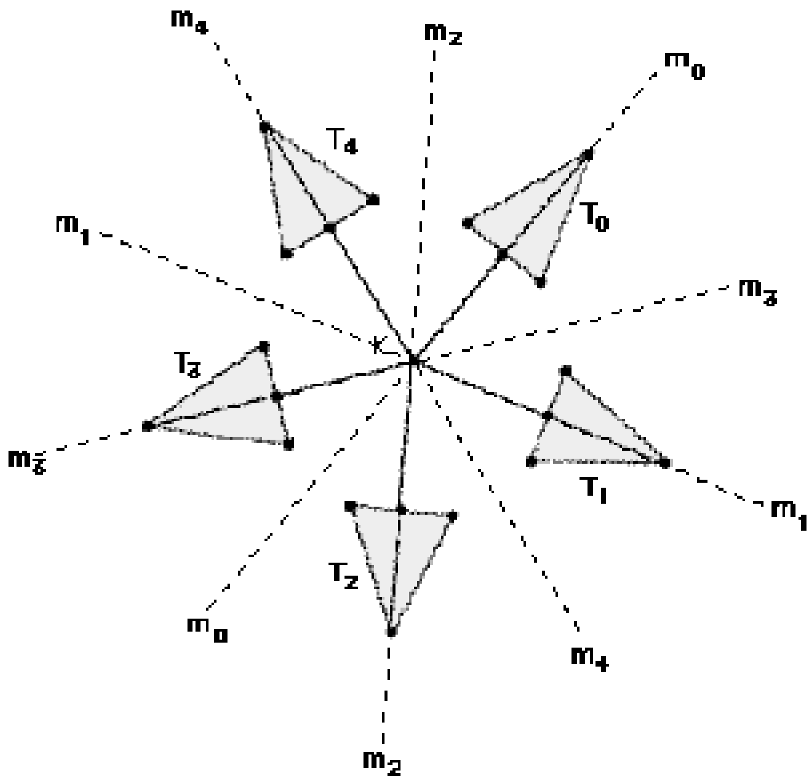

Fig. 3.41

In addition to the reflections, our ‘5-petal daisy’ has five rotations, including the trivial one, all ‘inherited’ from 3.7.1. In total, there are ten isometries mapping the daisy to itself: $\boldsymbol{I}$, $\mathbf{r}$, $\mathbf{r}^{2}$, $\mathbf{r}^{3}$, $\mathbf{r}^{4}$, $\mathbf{m}_{1}$, $\mathbf{m}_{2}$, $\mathbf{m}_{3}$, $\mathbf{m}_{4}$, and $\mathbf{m}_{5}$. Could these isometries possibly form a group? The answer would be “yes” if we could show that the product (successive application) of every two of them is still one of the ten isometries listed above, and that each one of the ten isometries has an inverse. Starting from the latter, recall (1.4.4) that every reflection is the inverse of itself, hence the inverse of $\mathbf{m}_{s}$ is $\mathbf{m}_{s}$ with $\mathbf{m}_{s}*\mathbf{m}_{s} = \boldsymbol{I}$ for s = 1, ... , 5; moreover, the inverse of $\mathbf{r}^{l}$ for l = 1, ... , 4 is $\mathbf{r}^{5-l}$ (3.7.1), and $\boldsymbol{I}$ is of course the inverse of itself. Observe next that the product of every two rotations is indeed a rotation, with $\mathbf{r}^{k}*\mathbf{r}^{l} = \mathbf{r}^{l}*\mathbf{r}^{k} = \mathbf{r}^{(k+l)\mathrm{mod}5}$ (3.7.1), and same holds for the product of every two reflections (as we will see in section 7.2 and as you could probably verify even now). Keeping the latter in mind and observing also that $\mathbf{m}_{t}(T_{s}) = T_{(2t-s)\mathrm{mod}5}$, let us now compute $\mathbf{m}_{t}*\mathbf{m}_{s}(T_{0}) = \mathbf{m}_{t}(\mathbf{m}_{s}(T_{0})) = \mathbf{m}_{t}(T_{(2s-0)\mathrm{mod}5}) = \mathbf{m}_{t}(T_{2s\mathrm{mod}5}) = T_{(2t-2s)\mathrm{mod}5} = \mathbf{r}^{(2t-2s)\mathrm{mod}5}(T_{0})$ -- the last step follows from $\mathbf{r}^{k}(T_{0}) = T_{k}$ -- and conclude that $\mathbf{m_t * m_s = r^{(2t-2s) mod 5}}$: if two **rotations** have the same effect on any one of the 'petals' (in this case T_0) then they must be one and the same! 'Multiplying' both sides of the derived identity by $\mathbf{m_t}$ (from the left) and by $\mathbf{m_s}$ (from the right), we obtain, with some details omitted, the identities $\mathbf{r^l * m_s = m_k}$ (where $\mathbf{2k = (2s+l) mod 5}$) and $\mathbf{m_t * r^l = m_k}$ (where $\mathbf{2k = (2t-l) mod 5}$), respectively. (You may not be able to derive the missing details right now, but you can certainly verify them **geometrically**: for l = 3 and s = 2, for example, k = 1 satisfies 2k = (2s+l)mod5, hence the product $\mathbf{r^3 * m_2}$ (reflection bisecting T_2 followed by clockwise 216^0 rotation) ought to be $\mathbf{m_1}$ (reflection bisecting T_1), etc.)

Replacing 5 by **any odd n**, we can extend the results and formulas of the preceding paragraph to arbitrarily large groups of 2n elements (and yet very similar structure). For **even n** some slight modifications, as well as a '6-petal daisy', are in order:

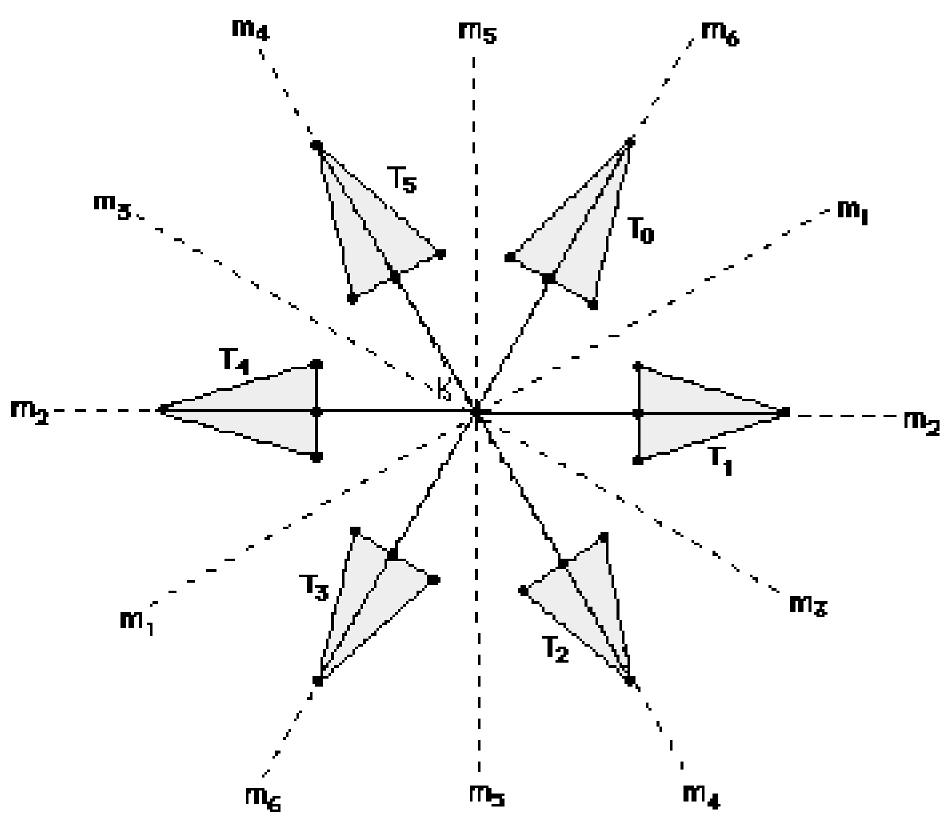

Fig. 3.42

Observe that our new ‘daisy’ is now bisected in two different ways, with some axes (**m_2**, **m_4**, **m_6**) cutting through ‘petals’ and other axes (**m_1**, **m_3**, **m_5**) passing right between ‘petals’. The effect of the reflections on the ‘petals’ is now somewhat different, with $\mathbf{m_t}(T_s) = T_{(2t-s)\text{mod}5}$ replaced by $\mathbf{m_t}(T_s) = T_{(t-s)\text{mod}6}$; that leads in turn to $\mathbf{m_t * m_s = r^{(t-s)mod6}}$, as opposed to $\mathbf{m_t * m_s = r^{(2t-2s)mod5}}$. At the same time, $\mathbf{r^k * r^l = r^l * r^k = r^{(k+l)modn}}$ (3.7.1) remains intact, while the other two kinds of products are in fact **simplified**: proceeding as in the preceding paragraph, we now establish $\mathbf{r^l * m_s = m_{(s+l)mod6}}$ and $\mathbf{m_t * r^l = m_{(t-l)mod6}}$. (Once again you should be able to verify these formulas **geometrically**; for example, $\mathbf{m_3 * r^4}$ (clockwise $4\times 360^0/6 = 240^0$ rotation followed by ‘in-between’ bisection $\mathbf{m_3}$) ought to be equal to $\mathbf{m_{(3-4)mod6} = m_{(-1)mod6} = m_5}$, another ‘in-between’ bisection.) Notice that all formulas obtained in this paragraph for n = 6 can be easily modified for **arbitrary even n**.

To summarize, we have just proven, by going through two distinct cases (odd n and even n), that, for every n, the set of 2n isometries {***I***, **r**, ..., $\mathbf{r^{n-1}}$, $\mathbf{m_1}$, ... , $\mathbf{m_n}$} forms indeed a **group** under **composition of isometries**, subject to the rules and formulas established in this section. This **non-commutative** group, which contains **C_n** as a **subgroup**, is well known in the literature as **dihedral group of order 2n,** denoted by **D_n**. It may be shown -- see for example chapter 8 in George E. Martin’s ***Transformation Geometry: An Introduction to Symmetry*** (Springer, 1982) -- that every **finite group** of isometries in the plane **must** be **C_n** or **D_n** for some n: this result is attributed to none other than Leonardo da Vinci and is known as **Leonardo’s Theorem**!

[How about **infinite** such groups? Well, those are actually studied in chapters 2 and 4, but our approach tends to be informal and geometrical (even in chapter 8) rather than group-theoretic -- a group-theoretic approach is available in Martin’s book above, as well as in several **Abstract Algebra** texts, such as M. A. Armstrong’s ***Groups and Symmetry*** (Springer, 1997), for example.]