Journal of Sound and Vibration (1988) **121**(3), 383-411

MATHEMATICAL MODELS USED IN GEAR DYNAMICS—A REVIEW

H. NEVZAT ÖZGÜVEN† AND D. R. HOUSER

Gear Dynamics and Gear Noise Research Laboratory, Department of Mechanical Engineering, The Ohio State University, Columbus, Ohio 43210, *U.S.A.*

(*Received* 14 *January* 1987)

With increased demand for high speed machinery, the mathematical modelling of the dynamic analysis of gears has gained importance. Numerous mathematical models have been developed for different purposes in the past three decades. In this paper the mathematical models used in gear dynamics are discussed and a general classification of these models is made. First, the basic characteristics of each class of dynamic models along with the objectives and different parameters considered in modeling are discussed. Then, the early history of the research made on gear dynamics is summarized and a comprehensive survey of the studies involved in mathematical modelling of gears for dynamic analysis is made. Generally, a chronological order is followed in each class studied. The goal is not just to refer to several papers published in this field, but also to give brief information about the models and, sometimes, about the approximations and assumptions made. A considerable number of publications were reviewed and 188 of them are included in the survey.

1. INTRODUCTION

There is a vast amount of literature on gear dynamics and dynamic modelling of gear systems. The objectives in dynamic modelling of a gear system vary from noise control to stability analysis. The ultimate goals in dynamic modelling of gears may be summarized as the study of the following: stresses (bending stresses, contact stresses); pitting and scoring; transmission efficiency; radiated noise; loads on the other machine elements of the system (especially on bearings); stability regions; natural frequencies of the system; vibratory motion of the system; whirling of rotors; reliability; life.

The concern with loads on gears goes back at least to the eighteenth century. However, the first systematic efforts to analyze gear dynamics occurred in the 1920s and early 1930s. In these studies the concern was the determination of dynamic loads on gear teeth through both analytical and experimental methods. In the 1950s, the first simple mass-spring models were introduced, the major aim still being the estimation of dynamic tooth loads. More involved models representing the dynamic behavior of gears in mesh appeared in the mid 1950s with the main objectives including many of the other considerations. The complicated models suggested in the 1970s and 1980s include effects such as three-dimensional stiffness of gear teeth, non-linearity of system elements, and damping and excitation effects of friction between teeth. In recent works, torsional, lateral, axial and even plate mode vibrations of geared systems are considered, and steady state and transient responses of the system to several gear errors are determined.

The models proposed by several investigators show considerable variations, not only in the effects included, but also in the basic assumptions made. The interesting point is

†On leave from The Middle East Technical University, Ankara, Turkey; presently, in the Mechanical Engineering Department at the Middle East Technical University, Ankara, Turkey.

0022-460X/88/060383+29 $03.00/0

that considerably different models have been claimed to be in good agreement with experimental observations. This is simply due to two reasons: (1) the systems modelled show different dynamic properties—for instance, while a gear mounted on a very short shaft might be assumed to be rigidly mounted in the transverse direction, the same assumption cannot be made for a gear on a long, slender shaft; (2) the purpose of mathematical modelling differs—while only the lower vibrational modes of a gear system might be sufficient in a model constructed to study only the dynamic tooth stresses, this model might not be sufficient in the study of a noise problem in the same system. Another reason for having good agreement of experimental results with basically different models is that investigators have usually constructed their experimental rigs such that the basic assumptions of their models could be satisfied.

Although it is quite difficult to group the mathematical models developed in gear dynamics, the following classification seems appropriate.

(1) Simple Dynamic Factor Models. This group includes most of the early studies in which a dynamic factor that can be used in gear root stress formulae is determined. These studies include empirical and semi-empirical approaches as well as recent dynamic models constructed just for the determination of a dynamic factor.

(2) Models with Tooth Compliance. There is a very large number of studies which include only the tooth stiffness as the potential energy storing element in the system. That is, the flexibility (torsional and/or transverse) of shafts, bearings, etc., are all neglected. In such studies the system is usually modelled as a single degree of freedom spring-mass system. There is an overlap between the first group and this group since such simple models are sometimes developed for the sole purpose of determining the dynamic factor.

(3) Models for Gear Dynamics. Such models include the flexibility of the other elements as well as the tooth compliance. Of particular interest have been the torsional flexibility of shafts and the lateral flexibility of the bearings and shafts along the line of action.

(4) Models for Geared Rotor Dynamics. In some studies, the transverse vibrations of a gear-carrying shaft are considered in two mutually perpendicular directions, thus allowing the shaft to whirl. In such models, the torsional vibration of the system is usually considered.

(5) Models for Torsional Vibrations. The models in the third and fourth groups consider the flexibility of gear teeth by including a constant or time varying mesh stiffness in the model. However, there is also a group of studies in which the flexibility of gear teeth is neglected and a torsional model of a geared system is constructed by using torsionally flexible shafts connected with rigid gears. The studies in this group may be viewed as pure torsional vibration problems, rather than gear dynamic problems.

Although the discussion of previous studies will be made according to the above classification, it should be remembered that sometimes it may be very difficult to label a certain study and some models might be considered in more than one group.

In the solution of the system equations, numerical techniques have usually been employed. Although most of the models for which numerical techniques are used are lumped parameter models, some investigators have introduced continuous system or finite element models. While closed form solutions are given for some simple mathematical models, analog computer solutions have sometimes been preferred for non-linear and more complicated models, particularly in the earlier studies.

In some studies the main objective has been to find the system natural frequencies and mode shapes and, therefore, only free vibration analyses are made. However, usually the dynamic response of the system is analyzed for a defined excitation. In most of the studies, the response of the system to forcing due to gear errors and to parametric excitation due to tooth stiffness variation during the tooth contact cycle is determined. The models constructed to study the excitations due to gear errors and/or tooth stiffness variation provide either a transient vibration analysis or a harmonic vibration analysis by first determining the Fourier series coefficients of the excitation. Some studies also include the non-linear effects caused by loss of tooth contact or by the friction between meshing teeth. The excitation then is taken as an impact load and a transient vibration analysis is made.

2. HISTORY AND SIMPLE DYNAMIC FACTOR MODELS

The actual tooth load of gears in mesh consists of two main components: a static component corresponding to the transmitted power (which is almost equal to the total load at low speeds of rotation) and a dynamic component which provides a fluctuating increment due to dynamic action. The earliest studies were investigations of empirical dynamic factors which included this dynamic increment and which could be used to place a penalty on the load-carrying capacity of gears when their speeds increase. Therefore the history of the dynamic modeling of gears starts with the studies determining dynamic factors.

Although the history of loads on gears dates back almost two centuries, the dynamic factor (which was then called speed factor) was originally suggested in 1868 by Walker [1]. It is defined as

$$DF = SL/DL, \tag{1}$$

where DF is the dynamic factor, SL is the static load, and DL is the dynamic load. The concept of speed factor was originally introduced on the basis of strength considerations. In the earliest studies, the dynamic factors were determined empirically by comparing the gear size and strength calculations with records of tooth failures at different speeds. Carl G. Barth first expressed the dynamic factor, based on Walker's original factors, as

$$DF = 600/(600 + V), \tag{2}$$

where V is the pitch line velocity in fpm (feet per minute). More reliable tests made between 1910 and 1915 showed that this formula was too conservative for gears running at speeds which were then considered to be high (about 2000 fpm) [2]. It was concluded after these tests that in addition to pitch line velocity the major factors in establishing the dynamic gear forces were tooth errors and the effect of gear and pinion inertias.

In 1924, Franklin and Smith [3] confirmed the formula given by Barth for gears with a certain pitch error and showed that the constants in the formula should be changed for gears with different pitch errors. In 1927, Ross [4] found that even the lowest dynamic factor (which gives the highest stress) was too conservative for velocities over 4000 fpm, and recommended the modified form

$$DF = 78/(78 + \sqrt{V}). \tag{3}$$

Both the Barth equation and several modified forms of it are still used in some fields of design and are given in design books. The formulae which are widely used in gear rating standards of the American Gear Manufacturing Association are modified versions of these equations. However, these modifications were made after the 1950s.

The results of several works conducted by the ASME Research Committee on Strength of Gear Teeth were published in 1931 by Buckingham [5]. The report stated that for speeds over 5000 fpm the load-carrying capacity of gears changes very little. It was shown that the tooth load variation depends largely upon the effective masses, the effective errors and the speed of the gears [6]. After the development of the dynamic load equation given in this report, which is more popularly known as Buckingham's Equation, little was done until 1950. A detailed discussion of these pre-1950 studies was given by Fisher [1] and Buckingham [6].

In 1950 a new era in gear dynamics was initiated which incorporated the use of vibratory models in the dynamic analysis of gears. Such mathematical models made it possible to study other dynamic properties of geared systems in addition to the dynamic loads. However, the earlier vibratory models were very simple and therefore could provide little additional dynamic information beyond the dynamic load for the gear system. Therefore, some of these earlier vibratory models were only used to determine the dynamic factors.

In the first spring-mass model, which was introduced by Tuplin [7-9], an equivalent constant mesh stiffness was considered and gear errors were modelled by the insertion and withdrawal of wedges with various shapes at the base of the spring. Thus, the dynamic loads due to transient excitation were approximated for various forms of errors. As the problem was considered as a transient excitation problem disregarding the periodicity of the excitation, this simple spring-mass model can be used to estimate dynamic factors only at conditions well below resonance. The dynamic model employed by Tuplin is shown in Figure 1.

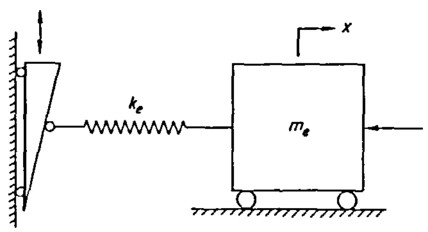

Figure 1. Dynamic model employed by Tuplin. k_e = equivalent constant tooth mesh stiffness; m_e = equivalent mass; w = transmitted load.

The effect of various forms of assumed tooth error was also discussed by Reswick [10] who used a simple dynamic model consisting of two masses constrained to move in a horizontal direction and excited by a parabolic or constant acceleration cam which was suddenly moved downward at the pitch line velocity of engagement. The tooth stiffness was assumed to be constant, but the fact that at certain times the load was carried by two or more pairs of teeth while at other times only a single pair of teeth may support the load was taken into consideration. Transient excitation was considered in the model of Reswick; again it was acceptable only for predicting dynamic loads below resonance. The results showed a general agreement with the predictions of Buckingham's equations for lightly loaded gears, but differed somewhat in the case of heavily loaded gears. Reswick concluded that dynamic loads might be ignored in many heavily loaded gears, while dynamic loads provided an important basis for the design of lightly loaded gears.

The work of Strauch [11] published in 1953 seems to be the first study in which periodic excitation was considered. He considered the step changes in mesh stiffness due to changing from single pair to double pair tooth contact. He analyzed the forced vibrations which might build up as a result of the continuous error between two unmodified involute gears. In 1957 Zeman [12] considered the effects of periodic profile errors, assuming a constant mesh stiffness. He analyzed the transient effects of four different forms of error, and the steady state effect of one form of repeated error.

After the studies made in the 1930s to improve Barth's equation and in the 1950s to determine the dynamic factor for systems operating below the resonance, Seireg and

Houser [13], in 1970, used the experimental test results they had reported previously [14] and used a geometrical analysis of the tooth meshing action to develop a semi-empirical formula for dynamic tooth load. The formula developed for a generalized dynamic factor for spur and helical gears, which was meant to be used below system resonances, takes into account gear geometries, manufacturing errors, and operating loads and speeds. The equation was compared with the formula widely used in gear rating standards of the American Gear Manufacturing Association (Wellauer [15]) which is given below and with the equation of Ross [4] (equation (3)):

$$DF=\sqrt{78/(78+\sqrt{V})} \tag{4}$$

In 1971, Tucker [16, 17] used the modified Tuplin equation [18] with the spur and helical gear tooth stiffnesses approximated from cantilever beam theory by Seireg and Houser [13] and the equivalent mass at the pitch line defined by Buckingham [6]. Tucker aimed to replace several dynamic factors that were then in use: three different factors presented by AGMA [19, 20], one used for aerospace gearing [21], another one used for industrial use [22], and another for marine applications [23]. He concluded that Tuplin's equation, derived from the assumption that the external gear tooth acts as a cantilevered spring, could be applied to either lightly or heavily loaded gears and could be used as a design aid by the gear engineer.

Rettig [24] suggested a vibratory model for determining the dynamic forces on gear teeth, and presented a different simplified formula for the dynamic factor in each of the following speed regions: subcritical, main resonance and supercritical. His model will be discussed later.

While several complicated mathematical models were developed in the 1970s and 1980s, the search for a simple dynamic factor formula which could easily be used to determine gear dynamic tooth load has continued. The dynamic factor equations in current AGMA standards (AGMA 218.01, December 1982) are functions of gear pitch line velocity and the quality of the gears. These equations do not account for gear inertias, loading effects, specific tooth error patterns and other system-dependent characteristics. In a recently proposed ISO method (ISO TC/60), however, three different approaches are suggested for the calculation of the dynamic factor. The most complex of them requires a comprehensive dynamic analysis so that the vibration resonance can be considered, while the simplified method predicts the dynamic factor for gears running only in the subcritical zone. A recent method for calculating dynamic loads by using fairly simple equations was developed by Wang [25] in 1985. The method is based on the laws of mechanics for rigid bodies and theoretically requires the dynamic transmission error as the input. As a first order approximation, Wang suggested that the static transmission error could be assumed to be proportional to the dynamic transmission error. Wang's model consisted of rigid disks representing equivalent inertias of gears shifted to a common shaft. Then Newton's second law of motion was applied to each rigid disk in the equivalent system by using the gear tooth force induced torques and a given output torque. From these equations together with the equations obtained from the definition of angular transmission error and from the force equilibrium condition, the unknown gear tooth forces were determined by simply solving a set of algebraic equations for each time increment.

3. MODELS WITH TOOTH COMPLIANCE

The basic characteristic of the models in this group is that the only compliance considered is due to the gear tooth and that all other elements are assumed to be perfectly rigid. The resulting models are either translational (Figure 2) or torsional (Figure 3). The

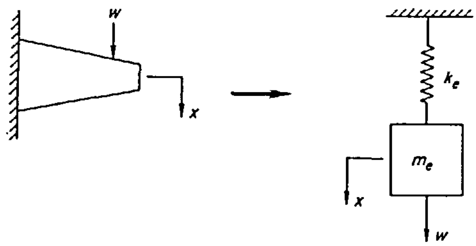

Figure 2. A general dynamic model for a gear tooth. k_e = equivalent stiffness of gear tooth; m_e = equivalent mass of gear tooth; w = transmitted load.

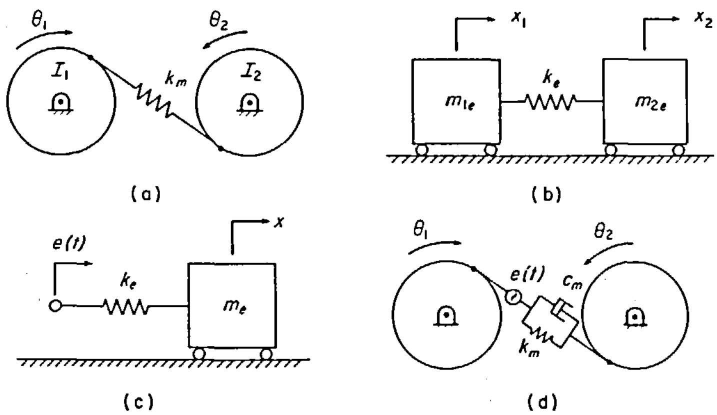

Figure 3. Some torsional models with tooth compliance. I_i = mass moment of inertia of gear i; m_{ie} = equivalent mass of inertia I_i; m_e = equivalent mass of all inertias; k_e = equivalent tooth mesh stiffness; k_m = tooth mesh stiffness; c_m = tooth mesh damping; $e(t)$ = displacement input representing gear errors.

distinction between translational and torsional models is not made according to the appearance of the model, but according to whether the translational motion of the tooth or the rotational motion of gear is modelled. As can be seen from Figure 3(b) and (c) some torsional models are presented in the form of their translational equivalents, while, in general, in such models the system is idealized as a pair of inertias coupled by a spring which permits relative motion. With torsional models one can study the torsional vibrations of gears in mesh, whereas with translational models the tooth of a gear is considered as a cantilever beam and one can study the forced vibrations of the teeth. In either of these models transmission error excitation is simulated by a displacement excitation at the mesh.

The first model which might be considered in this group is Tuplin's model discussed in the previous section [7-9]. In 1956, Nakada and Utagawa [26] considered varying elasticities of the mating teeth in their vibratory model. In their model the torsional vibrations of two mating gears were simulated by introducing an equivalent translational vibratory system. The time variation of stiffness was approximated as a rectangular wave, and closed form solutions of piecewise linear equations were obtained for different damping cases for accurately manufactured gear tooth profiles. Another mass and equivalent spring model was introduced in 1957 by Zeman [12]. He neglected the variation of stiffness and analyzed the transient effects of periodic profile errors. Harris's work [27], published in 1958, was an important contribution in which the importance of transmission error in gear trains was discussed and photo-elastic gear models were used. In his single degree of freedom model, Harris considered three internal sources of vibration: manufacturing errors, variation in the tooth stiffness and non-linearity in tooth stiffness due to the loss of contact. He treated the excitation as periodic and employed a graphical phase-plane technique for the solution. Harris seems to have been the first to point out the importance of transmission error by showing that the behavior of spur gears at low speeds can be summarized in a set of static transmission error curves. Harris also appears to have been the first to predict the dynamic instability due to parametric excitation of the gear mesh.

Johnson [28] argued in 1958 that the model of Tuplin in which the excitation due to profile errors is described as a number of isolated transients could apply only to a slow-speed gear system. He showed that in heavily loaded precision gears the elastic deformation was considerably larger than possible inaccuracies of manufacture and that the departure from constant velocity ratio under these conditions is, therefore, a continuously varying periodic function. Johnson discussed the characteristics of the frequency spectrum of this function. He assumed constant stiffness and took the measured transmission error as a forcing function. Similar work has been reported by Kohler [29] and Wood [30].

Utagawa [31] considered gradually changing stiffness and predicted individual tooth load cycles by adopting a piecewise solution. Predicted values showed good correlation with experimental work. In the early 1960s, Utagawa and Harada [32, 33] tested gears at higher velocities and compared measured dynamic loads with their calculated results. Their undamped single degree of freedom model consisted of an effective mass representing the inertias of pinion and gear, and a time varying tooth stiffness with which they investigated ground gears having pressure angle errors [32] and pitch errors [33].

Tobe [34] presented what appears to be the first model in which the dynamics of a tooth were considered separately from the dynamics of the gear wheel, with the resulting equations then being coupled. He modelled a gear tooth as a cantilever beam and formulated tooth deflection as a function of dynamic load, and combined these equations with the equations of motion for gear wheels which are under the action of external torques and the unknown dynamic tooth load. Thus he combined translational and torsional models, and calculated the dynamic loads on spur gear teeth. An approximate method was used for the solution of the resulting integral equation. The errors introduced into the results due to the approximate solution made it impossible to study the effect of the transmitted load on the dynamic load. In 1967, Tobe and Kato [35] analyzed the same problem with a simpler model, but used a numerical integration technique for the solution of the equations.

In 1963, Gregory, *et al.* [36, 37] extended the theoretical analysis of Harris [27] and made comparisons with experimental observations. The torsional vibratory model of Grebory *et al.* included sinusoidal-type stiffness variation as an approximation. They treated the excitation as periodic, and solved the equations of motion analytically for zero damping and on an analog computer for non-zero damping. The experimental data [36] and the computational results [37] generally confirmed Harris's contention that non-linear effects are insignificant when damping is more than about 0·07 of critical. It was claimed that when damping is heavy the simple theory of damped linear motion can be used. Between 1965 and 1970, Rettig [38], Bosch [39, 40] and Aida *et al.* [41–43] presented the examples of other studies in this area. Each author modelled the vibration characteristics of gears by considering the excitation terms due to tooth profile errors and pitch errors, and by including the variation of teeth mesh stiffness. In the model of Aida *et al.* time varying mesh stiffness and periodic tooth errors were considered, and the model was used for determining stability regions and steady state gear vibrations. A comparison with experimental measurements was also made. In 1967, Opitz [44] used the equations derived by Bosch [39, 40] to investigate the dynamic tooth forces in spur and helical gears. In the single degree of freedom model used, viscous damping, time varying mesh stiffness, backlash and gear error were considered. The non-linear equation of motion of the model was solved by using an analog computer. The theoretical results were confirmed by measurements and were also compared with the results of other analytical methods that were then available.

In 1967, Nakamura [45] investigated the separation of tooth meshing with a single degree of freedom model. He accounted for single and double tooth pair contact with a square wave tooth mesh stiffness variation and used a sinusoidal representation of tooth errors. He adopted a numerical piecewise solution, and concluded that the largest dynamic load occurs immediately after the separation which happens at the specific speed defined by the amount of transmission error and tangential load.

Bollinger and Harker [46] investigated the dynamic instability that may arise due to varying mesh stiffness. They used a simple single degree of freedom model with an equivalent mass representing the inertias of the gear and pinion. Mesh stiffness variation was assumed to be harmonic, which resulted in a form of the damped, forced Mathieu equation. The solution of the resulting equation of motion was obtained by using an analog computer, and it was shown that the dynamic load may be reduced by increasing the damping between the gear teeth or by reducing the amount of stiffness variation.

In 1967, Tordion and Gerardin [47] used an equivalent single degree of freedom dynamic model to determine transmission error from experimental measurements of angular vibrations. They first constructed a torsional multi-degree of freedom model for a general rotational system with a gear mesh. Then, only the equations of the gears were considered for obtaining an equivalent single degree of freedom model with a constant mesh stiffness and a displacement excitation representing the transmission error. An analog computer solution was used to obtain the transmission error from the measured angular accelerations. In this paper, transmission error was proposed to be used as a new concept for determining the gear quality, rather than using individual errors.

In 1972 Wallace and Seireg [48] used a finite element model to study the stress, deformation and fracture in gear teeth when subjected to dynamic loading. Impulsive loads applied at different points on the tooth surface and moving loads normal to the tooth profile were studied. Rather than lumping the inertia of the gear and treating the teeth in contact as massless springs connecting the two wheel bodies, in their dynamic model they treated the gear as a continuum and included the mass of the tooth investigated. It was shown for the case considered that a normal mode analysis of the cantilever beam subjected to the Hertzian impact was inadequate while the results found by the finite element model were in good agreement with the measured strains. In 1973 Tobe and Takatsu [49] studied gear tooth impact by coupling a torsional model for the relative rotary motion of a gear pair with a rectilinear model for the flexural vibration of the gear teeth. The mesh stiffness in this model was taken to be a constant which depends upon the point of impact. This work was based on an earlier work of Tobe and Kato [35] where the solution was obtained by numerical integration. However, in this work, an analog computer was used to solve the problem and an approximate solution was obtained by considering only the fundamental mode of vibration of a beam which represented a gear tooth.

Ichimaru and Hirano [50] presented a vibratory model composed of an effective mass of the gear blanks and the stiffness of meshing teeth. Manufacturing errors under a given operating condition were taken as the main excitation and the interaction between the tooth deformation and the dynamic load was theoretically investigated. Although the model resembled the ones presented previously, the solution technique employed made it possible to consider the mesh stiffness as a function of the position of the mesh point along the line of action and, at the same time, linearized the equation.

Cornell and Westervelt [51] extended and improved the dynamic models of Richardson [52] and Howland [53] to cover contact ratios up to four. The effects of system damping, the non-linearity of the tooth pair stiffness during mesh, the tooth errors, and the tooth profile modifications were included in the analysis. The non-linear equation of the single degree of freedom model was obtained as a piecewise continuous, closed-form solution. The results of their analysis showed that the tooth profile modification, system inertia and damping, and system critical speed can significantly affect the dynamic load.

In 1978, Kubo [54] used a torsional vibratory model to predict tooth fillet stress and to study the vibration of helical gears with manufacturing and alignment error. Periodic change of total tooth stiffness was included in the model. Also, an analytical method for the calculation of the contact pattern on tooth flank was presented. A good agreement was obtained between calculated and measured values of both tooth fillet stress and the vibratory behavior of helical and spur gears.

Remmers [55] presented a damped vibratory model in which the transmission error of a spur gear was expressed as a Fourier series. He used viscous damping and constant tooth pair stiffness, and considered the effects of spacing errors, load, design contact ratio and profile modifications.

Benton and Seireg [56] presented a simple single degree of freedom model with variable mesh stiffness and studied the steady state response, resonances and instabilities of a pinion-gear system subjected to harmonic excitation. They used the phase-plane method to integrate numerically the equation of motion. The numerical results were compared with experimental data and were used to discuss the influence of harmonic excitation on the system response.

In a single degree of freedom model Ishida and Matsuda [57, 58] placed emphasis on the sliding friction between mating teeth and they studied the effect of friction force variation on noise and vibration [57] when the meshing stiffness was assumed to be constant. The phase-plane method was used for the solution. They also studied the effect of surface roughness on gear noise [58] by first determining the radial vibrations of the gear. Then the axial displacement of the gear was determined from the calculated radial displacement by considering the flexibility of the gear-carrying shaft. (In this respect, their mathematical model does not fit in this group of models.) The noise radiation was calculated from the axial vibrations of the gear.

In 1981, Wang and Cheng [59, 60] used a torsional vibratory model; however, their primary concern was developing a numerical solution to predict the minimum film thickness, the bulk surface temperature, and the total contact temperature in spur gear teeth contacts. The computer code developed also predicted the dynamic tooth load by assuming that the dynamic load was not influenced by the lubricant film thickness or by the surface temperature. This assumption made an independent dynamic load analysis possible. Although their dynamic model was a single degree of freedom lumped model similar to previous ones, the variable tooth stiffness of the model were obtained by a finite element method. In a later publication [61], they discussed some typical results to illustrate the effects of gear geometry, velocity, load, lubricant viscosity, and surface convective heat transfer coefficient on the performance of spur gears. In this study they determined the effects of load sharing between a pair of teeth, variable mesh stiffness, and tooth profile error on the variations of dynamic tooth load by using the mathematical model they had developed.

It is interesting that the most recent models in this group still do not show much difference from the pioneer models. The complications usually arise from the inclusion of various effects such as damping and friction which were usually neglected in most of the early models. Rebbechi and Crisp [62, 63] considered the material damping of the gear-wheel shafts, while the compliance of the shafts was neglected. The three degree of freedom model is reduced to a two degree of freedom model for the study of the torsional vibrations of a gear pair, and an uncoupled equation which gives tooth deflection. The other effects included in the model were material damping inherent to the tooth, perturbations of input and output torques, arbitrary tooth profile error, time variation of that error due to deformation, and perturbations of the base circle due to profile errors. The effects of kinetic sliding friction at the contact point and the sliding velocity on the dynamics of continuous meshing were also studied [63].

In one of several studies on gear dynamics, Mark modelled a gear fatigue test apparatus [64] by assuming rigid shafts, rigid gear bodies, and rigid bearing supports. As only the gear teeth were modelled as elastic members, his model can be considered in the group of "models with tooth compliance". However, he included the inertia of the shafts, and the damping between the slave gear and its shafting into the model. Thus, the system which was composed of four gears and two shafts was assumed to have three degrees of freedom. He used a Fourier series representation of the excitation which was discussed in detail in his earlier publications [65, 66], and thus the computations were carried out, for the most part, in the frequency domain by using the fast Fourier transform computational algorithm.

In 1984, Spotts [67] used the famous spring-wedge analogy of Tuplin to estimate simply the dynamic load for use in gear design problems. The dynamic load was calculated by considering constant stiffness in a single degree of freedom model and by assuming that it can be expressed as the multiplication of some powers of velocity, stiffness and mass. The equation for dynamic load was then obtained by using the condition that the expression was to be dimensionally homogeneous.

Another translational model was suggested by Lin, Huston and Coy [68] in 1984. They investigated the effect of load speed on straight and involute tooth forms by using finite element tooth models, and showed that for stubby tooth forms there is considerable difference between results obtained with finite element models and results obtained with Timoshenko beam models. Also it was shown that the tooth form itself induces gear vibrations which becomes increasingly significant at higher speeds. In the same years, Ostiguy and Constaninescu [69] also made a finite element analysis of a gear tooth to evaluate the natural frequencies and mode shapes, and to study the transient response during the meshing period. Modal analysis was used in determining the transient response of the system under time varying moving loads. Other significant translational models for tooth dynamics which followed the work of Wallace and Seireg [48] that was discussed earlier have been suggested by Wilcox and Coleman [70], Chabert *et al.* [71], Ramamurti and Gupta [72], and Nagaya and Uematsu [73]. In most of these studies finite element models were used. Umezawa *et al.* [74] used a single degree of freedom vibratory model including periodic variation of tooth meshing stiffness and constant damping. In their simple model for torsional vibrations they used the errors of gears measured with a newly developed automatic gear accuracy measuring instrument, and accurately predicted the dynamic behavior of the tested spur gear pair. The numerical solutions in this study were obtained by using the Runge-Kutta-Gill method. Umezawa and Sato [75, 76] used a model developed to study the influence of pressure angle error, normal pitch error and sinusoidal tooth profile error on the vibration of the profile corrected spur gear. In a very recent paper Umezawa, Suzuki and Sato [77] modelled a helical gear pair with narrow facewidth. In their single degree of freedom model developed for the study of torsional vibrations they considered variable tooth mesh stiffness, damping and tooth errors. They proposed an approximate equation for the tooth mesh stiffness which is based on the theoretical deflections calculated by using the finite difference method. The meshing resonance frequencies were also calculated by using the average stiffness in a meshing period, and it was observed that the calculated resonance frequencies were in good agreement with experimental values. Sato *et al* [78] also used a single degree of freedom model in which they assumed that the torsional shaft stiffness is small compared to the torsional effect of the tooth mesh stiffness to study the torsional vibrations of a gear pair subjected to random excitation. Periodic variation of tooth mesh stiffness, damping and sinusoidal transmission error were considered in the model, and the forced response resulting from a randomly changing external torque was calculated by using an approximate technique. Although the accuracy of the solution was not found to be sufficient, the method was suggested to be useful in certain applications such as random fatigue problems.

One of the recent publications in this group belongs to Masuda *et al.* [79]. The main objective of this study was to predict the gear noise by adding a dynamic term to Kato's semi-empirical equation. The dynamic model they developed is not much different from the other torsional models in which variable mesh stiffness, damping, and profile error of the meshing tooth are considered. The analysis was also expanded to helical gears.

Recently, Lewicki [80] used the classical single degree of freedom dynamic model for a gear pair in modifying a NASA computer program prepared for predicting gear life. In this work he combined the models of previous investigators for tooth mesh stiffness, dynamic load calculations and gear life. Due to the Hertzian compression considered in the computation of tooth mesh stiffness, the tooth mesh stiffness is not independent of dynamic load, which required an iteration cycle for the computation of dynamic load. By using the model, dynamic loads and gear mesh life predictions were performed over a range of speeds, numbers of teeth, gear sizes, diametral pitches, pressure angles and gear ratios.

In another recent publication, Yang and Lin [81] modified the torsional model of Yang and Sun [82] by adding the torque due to the friction force between the mating teeth and by considering the bending deflection and axial compression of a gear tooth in deriving the mesh stiffness. In their model they also included Hertzian damping and backlash, and the Runge–Kutta method was used for solution.

4. MODELS FOR GEAR DYNAMICS

Although the mathematical models in which the stiffness and mass contribution of the shafts carrying gears in mesh were ignored showed good agreement with the experimental measurements, it was realized in the late 1960s and early 1970s that dynamic models in which the shaft and bearing flexibilities were considered were necessary for more general models. Unless the stiffnesses of these elements are relatively high or low compared to the effective mesh stiffness, the vibration coupling of different elements cannot be neglected. The good correlation that was obtained between the experimental results and the predictions provided by many of the single degree of freedom models of the previous section can be explained by the fact that the experimental rigs used in such studies satisfied the basic assumptions made in the mathematical modelling. However, in practical applications, these assumptions may not always be satisfied. One then needs more general models in which the flexibility and mass of the other elements are considered as well.

The models that can be considered in this group are either torsional models in which only the torsional stiffness of the gear-carrying shafts is included, or torsional and translational models in which both the torsional and transverse flexibility of the gear-carrying shafts are considered. In some models the lateral vibrations of gear blanks in two mutually perpendicular directions are considered. However, considering two coupled lateral vibrations of a gear shaft system makes the problem a rotor dynamics problem. Such models will be discussed in a subsequent section. Still, some of the models for studying the lateral vibrations in two directions will be mentioned in this group because of the difficulty in drawing absolute divisions between mathematical model types. Typical models used for torsional, and torsional and lateral vibrations of gears are shown in Figure 4.

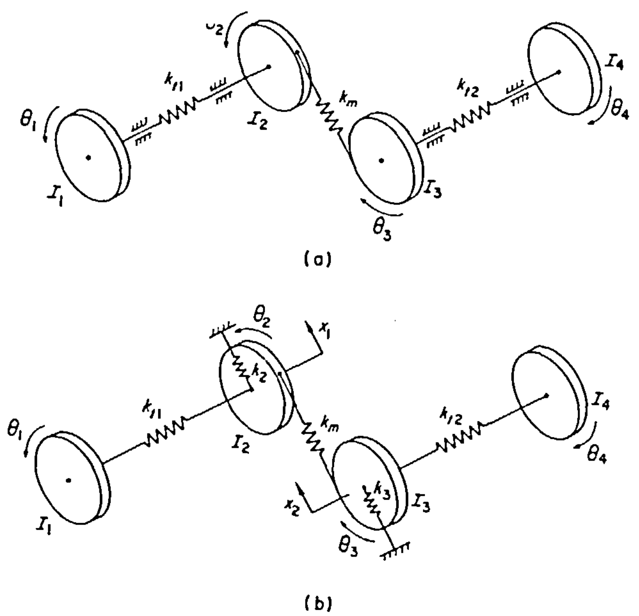

Figure 4. Typical models for gear dynamics: (a) torsional model; (b) torsional and translational model. I_1, I_4 = mass moment of inertias of prime mover and load; I_2, I_3 = mass moment of inertias of gears; k_{ti} = torsional stiffness of shaft i; k_m = tooth mesh stiffness; k_2, k_3 = stiffnesses representing lateral flexibility due to shafts and bearings.

In an earlier work Johnson [83] used a receptance coupling technique to calculate the natural frequencies from the receptance equation obtained by first separately finding the receptances at the meshing point of each of a pair of geared shafts. In the model, the varying mesh stiffness was replaced by a constant stiffness equal to the mean value of the varying stiffness and thus, a linear system was obtained. His work was one of the first attempts at using a mesh stiffness in coupling the vibration of gear shafts. In 1963, Tordion [84] presented a torsional model in which the torsional vibrations of two gear shafts were coupled by a constant mesh stiffness. In his model all non-linear effects including backlash were neglected and the general receptance technique was used to obtain the system response when there is a periodic transmission error (which was then called "error in action").

Seager [85] modelled a test gear-shaft system with three degrees of freedom by assuming laterally flexible bearings, rigid shafts and flexible gear teeth. Thus, the rocking motion of the gear was considered in addition to the torsional and transverse motion of the gear.

An important contribution in this area came from Kohler, Pratt and Thomson [86] in 1970. Concluding from their experimental results that dynamic loads and noise result primarily from the steady state vibration of the gear system when forced by transmission error, they developed a six degree of freedom dynamic model with four torsional degrees of freedom and one lateral degree of freedom in the direction of the tooth force on each shaft. They assumed the tooth mesh stiffness to be constant in their model. The spectrum analysis of the static transmission error for the single-stage reduction gear unit used was also given. In 1971, Remmers [87] suggested a similar model and made a harmonic vibration analysis. He prepared a program to calculate vibratory bearing forces, dynamic tooth load, and oscillatory motion of the gears as a function of the frequency of tooth meshing errors. Theoretical predictions were verified by the experimental results. Kasuba [88] used one and two degree of freedom models based on his previous work [89] to determine dynamic load factors for gears which were heavily loaded. He used a torsional vibratory model which considered the torsional stiffness of the shaft. He also argued that the rigidity of the connection shafts is much lower than the rigidity of the gear teeth in meshing, and then decoupled the meshing system. The tooth error in mesh was represented by a pure sine function having the frequency of tooth meshing. In his model meshing stiffness was time varying.

Wang and Morse [90] constructed a torsional model including shaft and gear web stiffnesses as well as a constant mesh stiffness. The torsional response of a general gear train system to an external torque was obtained by the transfer matrix method. The torsional natural frequencies and mode shapes determined from a free vibration analysis correlated well with experimental results at low frequencies. Later, Wang [91] extended this work to the linear and non-linear transient analysis of complex torsional gear train systems. In this later model he considered the variation of tooth stiffness, and included gear tooth backlash, linear and non-linear damping elements and multi-shock loadings. Three different numerical methods that can be used in the solution of non-linear systems that cannot be approximated piecewise linearly were also briefly discussed in his work.

Fukuma *et al.* [92] published a series of reports on gear noise and vibration in which both experimental and analytical studies were reported. In these studies three-dimensional vibrations of the gears were studied by including the flexibility of the shafts and bearings. The mass of the shafts was lumped and a multi-degree of freedom model was obtained. The gear mesh was modeled as a translational and a torsional spring with time varying stiffness. The Runge–Kutta–Merson process was employed to solve the system equations.

In 1975 Salzer and Smith [93], and in 1977 Salzer, Smith and Welbourn [94] discussed the real time modelling of gearboxes and offered analog computer solutions, claiming that digital computers sometimes suffer frequency or memory limitations. They proposed a six degree of freedom model for a car gearbox which included time dependent gear tooth stiffness, non-linear bearing stiffness and loss of tooth contact. An audible output was obtained by driving an amplifier and loudspeaker. Transmission error excitation was generated by using a rectified sine wave representing periodic tooth profile errors. Spacing errors were also included in the model. Salzer [95] developed a more detailed model for a commercial gearbox in his dissertation published in 1977. Still, a minimum complexity mathematical model was developed. The dynamic behavior of the gears, shafts and bearings was represented by three torsional and five lateral freedoms. Lateral freedoms were granted only at bearing locations by assuming that the intermediate shafting was rigid. However, the flexibility of both the lay shaft and the output shaft was considered by adjusting the parameters employed for bearing stiffnesses. A constant mesh stiffness, which was allowed to drop to zero on tooth separation, was assumed for tooth mesh. It was shown that relatively small adjacent pitch errors were of greater significance than usually imagined, and significant reductions in dynamic loads were predicted by reducing bearing stiffnesses. In the same year Astridge and Salzer [96] analyzed a spiral bevel gearbox, as a first step towards the analysis of the more complex rotor gearboxes of helicopters. A 78 degree of freedom lumped-mass model was used for the dynamic analysis in which the natural frequencies, mode shapes, and forced responses to displacement excitation represented by transmission errors were determined. The model included linear representations of bearing and housing stiffnesses.

In 1975, Rettig [24] modeled a single gear stage with six degrees of freedom, four lateral and two torsional, with all lateral freedoms being in the same direction. He considered a variable tooth mesh stiffness and presented simplified formulae for the calculation of dynamic factors in three different regions: subcritical, main resonance and supercritical regions. A comparison of the theoretical values with the experimental measurements was also given.

Tobe, Sato and Takatsu [97] in 1976 presented a statistical method of finding the relation between transmission errors and dynamic loads by using a torsional model with a periodic tooth mesh stiffness and torsional stiffnesses for gear carrying shafts. Monte Carlo simulation was employed to find maximum dynamic loads by means of an analog computer. In 1977, Tobe and Sato [98] made a similar, but more detailed analysis in which they assumed that the transmission error of a gear pair is composed of a certain random component due to the irregularities of clearances and elastic deformations of wear of bearings and gear teeth, and a harmonic component caused by the eccentricity or pressure angle error. The effect of the random components of the error on dynamic load was investigated by using a torsional model. In addition to time varying mesh stiffness, backlash was also included in the model. Analog computer solutions were obtained.

In 1977, Drosjack and Houser [99] modelled three gears in mesh to simulate pitch line pitting. The torsional dynamic model developed included an equivalent circuit of an electric generator as well as variable tooth mesh stiffnesses. Both time and frequency domain responses were obtained by using Runge–Kutta integration techniques. A good correlation between theoretical predictions and experimental observations suggested that such a modelling process might be successfully utilized in diagnostic procedures for geared systems.

The four degree of freedom torsional model for a lightly loaded geared system by Azar and Crossley [100] was directed towards studying tooth impact in spur gears. They considered sinusoidally changing tooth mesh stiffness, and included the effects of backlash, torsional stiffness of shafts, and tooth form error. The digital simulation results compared well with the experimental values found for the unloaded case, and it was concluded that the model could be used to predict the torsional vibrations of lightly loaded spur gear systems.

Tordion and Gauvin [101] studied the dynamic stability of a two-stage gear system by using a torsional dynamic model. The parametric vibrations due to variable meshing stiffness were studied and the influence of the phase angle between meshing stiffnesses was presented. An intermediate shaft carrying a gear at each of its ends was assumed to be rigid and the system was modeled with three degrees of freedom. Their result showed that the phase angle between the two meshing stiffnesses acting on the shaft strongly influences the range of frequencies over which instability occurs. In 1980, Benton and Seireg [102] used a multi-degree of freedom torsional model to study the influence of several factors on the stability and resonances of geared systems. These factors were system inertia, variation in tooth mesh stiffness, contact ratio and damping in the mesh. Assuming that the tooth mesh stiffness in most geared systems is considerably higher than the torsional stiffness of the shafting connecting gears, the gear pair was uncoupled from the rest of the system, and the effect of the system on the gear pair was included as external loads which were calculated from the analysis of the system assuming rigid teeth.

The object of the study of Kiyono, Aida and Fujii [103] was to obtain a simple model of helical gear pairs which would show the differences between dynamic behaviors of helical and spur gears. In the model constructed they considered the torsional, lateral, longitudinal and rotational freedoms. The natural frequencies of the system were calculated by neglecting the dynamic-coupling terms and by assuming constant tooth stiffness. The influence of gear carrying shafts was included by using equivalent frequency dependent stiffnesses and masses of the shafts. Later, in 1981, Kiyono, Fujii and Suzuki [104] developed a two degree of freedom model to study the transverse vibrations of bevel gears. This appears to be the first study in modelling bevel gears mathematically so that the difference between the vibrations of bevel gears and spur and helical gears can be investigated. Some fundamental characteristics of the vibrations of bevel gears in free vibrations, such as the effects of contact ratio, flexibility of shafts and damping ratios on the stability of vibrations, were studied. It was found that the fundamental difference between the dynamics of bevel gears and spur and helical gears was caused by the change in the mesh direction which was considered in the model with a rotating spring-damper element.

Kishor [105] constructed a four degree of freedom torsional model of the gear train which consisted of two gears, two disks and two shafts. The non-linear vibrations due to gear errors were studied for the constant tooth mesh stiffness model. An approximate solution method was employed to solve the system equations. Toda and Tordion [106] proposed a four degree of freedom torsional model for a gear system similar to the one analyzed by Kishor. However, they included the non-linearity of the tooth mesh stiffness, damping and tooth separation, and studied the effects of the transmission error excitation on the dynamic response of the system. The results were obtained with a hybrid computer and were given in the form of the tooth separation charts.

In the 1980s more and more complicated models have been developed in order to include several other effects and to obtain more accurate predictions, while some simple models were still developed for the purpose of simplifying dynamic load prediction for gear standards. In 1980, Smith [107] used a four degree of freedom model for a gear pair in which he considered two rotations and two transverse motion along the pressure line. However, he assumed uncoupled vibrations and calculated the force between teeth from a given transmission error. The dynamic force predicted by a relatively simple method at particular frequencies was found to be sufficiently accurate for the study of gear impact noise in diesel drives.

Furya *et al.* [108] suggested a torsional model in which a lumped parameter system was used for gears and a distributed parameter system for transmission shafts. Thus the natural frequencies of the gear system were examined over higher orders. The model, in which constant tooth mesh stiffness is assumed and the excitation caused by tooth profile errors and pitch errors is considered, was used to investigate the dependency of the natural frequencies of a gear system on the dynamic tooth load. Analytical and experimental observations led to the conclusion that under a regulated operating condition the dynamic increment in the tooth load is dominated by a specific natural frequency of the system, which makes it possible to model a gear pair as a single degree of freedom vibratory system. However, it was concluded that this is not the case under rough operating conditions.

The model of Kubo and Kiyono [109] for a helical gear pair included torsional and translational degrees of freedom. Shaft stiffness, as well as variable tooth mesh stiffness were considered. The model was used to estimate the dynamic exciting force due to both profile and lead errors and due to periodic change of tooth stiffness with progress of meshing. Several tooth error forms were investigated and it was concluded that the convex tooth form error is the most harmless among the different kinds studied.

In an interesting study of Lees and Pandey [110], in 1980, the mathematical model of a gear system and bearing vibrations measured at the bearing were used to estimate the gear errors and resulting tooth forces. Estimated profile errors were found to be in good agreement with measured values. A finite element model of a gearbox was used to establish a direct link between vibrations and gear forces. Additional components were used in this finite element model to represent a gear mesh. It was also shown how tooth pitch errors give rise to harmonic components in the spectrum at frequencies which are independent of shaft speed.

Sakai *et al.* [111] developed a torsional model for the rattling noise analysis of an automotive gearbox. The non-linear characteristics of backlashes of the gear teeth and also backlashes of the clutch hub splines were considered in this five degree of freedom non-linear model. Analog computer solutions were obtained and the effects of several parameters of the gear train on the noise level were studied.

The torsional model of Hlebanja and Duhovnik [112] was used to determine the dynamic tooth forces due to pitch errors. A special emphasis was placed on systems with large inertias (e.g., systems with flywheels) and high contact ratio gear pairs. In addition to variable mesh stiffness, torsional stiffness of shafts and bearings were also included in their four degree of freedom model. In this study it was concluded that the effect of pitch error on the power transmitted is more pronounced at small loads. Winter and Kojima [113] considered also the translational vibrations of gears in the pressure line direction, and used a four degree of freedom model for a gear pair. Tooth backlash was included in the model which was combined with a dynamic model of a practical system to study the gear tooth loads when tooth separation occurs.

In an extensive work of Kasuba and Evans [114] the gear mesh stiffness in engagement was calculated as a function of transmitted load, gear profile errors, gear tooth deflections and gear hub deformation, as well as the position of contact. A unique feature of this model is that off-line-of-action contact was computed. Also they introduced the distinction between "fixed-variable gear mesh stiffness" which is calculated by making several simplifications, and "variable-variable gear mesh stiffness" which they calculated. Their torsional vibratory model was not much different from previous torsional models except for the variable-variable mesh stiffness. With the model, in which the mesh damping was included as well, the response to variable-variable mesh stiffness and the profile error-induced interruptions of the stiffness function were calculated. Their computer program calculates the mesh stiffness, the static and dynamic loads, the variations in transmission ratios, sliding velocities and the maximum contact pressures acting on the gear teeth as they move through the contact zone. Kasuba [115, 116] used the same model to study the tooth mesh stiffness and dynamic load characteristics for several cases of normal contact ratio and high contact ratio gearing. Later, Pintz and Kasuba [117] extended this method to internal spur gears with high contact ratios. While the Runge-Kutta integration method was used to integrate the differential equations of motion, an iterative procedure was applied to solve the statically indeterminate problem of multi-tooth pair contacts, load sharing, and operational contact ratios as influenced by both the gear mesh and the radial deflections of components. It was concluded in this study that internal spur gear drives have lower dynamic load factors than the equivalent external spur gear drives.

Troeder *et al.* [118] constructed a model considering torsional, lateral and axial vibrations of a helical gear pair-shaft-bearing system. Fourier expansion of tooth mesh stiffness in the form of a square wave was used in the model. Tooth profile errors, as well as pitch errors were considered in the model developed for a parametric study. The effect of torque change was studied and the numerical results were compared with the results of an approximate study [119].

In 1983, Bahgat, Osman and Sankar [120] analyzed the dynamic loads on spur gears by using an approach similar to the one employed by Tobe [34]. They first formulated the vibration of a gear tooth by considering a moving load on a cantilever beam shaped like an involute profile. Then the equations of motion for the torsional vibrations of a gear pair were expressed in terms of an unknown dynamic load, and the resulting equations were solved simultaneously. The solution was obtained by using a harmonic series expansion satisfied at three discrete positions during a very small period of time within the contact period. A numerical example was solved to illustrate the procedure.

In 1984, Lees [121] suggested a simple torsional model consisting of four inertias and two torsional springs which represent a machine with a pair of gears. Although the tooth mesh stiffness was assumed to be infinite, this model is included in this group rather than in the last group, simply because the gear profile errors were considered in the model. The model was used to predict dynamic loads in gear teeth and it was shown that, although the formulation was nonlinear, a linearized version was adequate in many instances.

An eight degree of freedom model of Küçükay [122] for single stage spur and helical gears included the axial vibrations of rigid disks which represented gear blanks, as well as torsional, transverse and tipping motions. Periodic tooth mesh stiffness, tooth errors and external torques were considered, as were load dependent contact ratio and non-linearities due to the separation of the teeth. However, stability analysis was made by using a linearized model. Steady state solutions for the determination of dynamic tooth displacements and loads were found by using perturbation methods and the linearized model. The behavior of the non-linear model was also investigated. It was concluded that the approximate solution obtained for the linearized model was very appropriate for the determination of dynamic load.

In 1985, Kumar, Sankar and Osman [123] used a torsional model for a single stage spur gear system in order to determine the dynamic tooth load and to study the stability of the system. A new state-space approach was developed for the solution. This straightforward method was found to be less time-consuming for obtaining a time domain solution of the mathematical model of the gear system. The model was used to study the effects of changes in contact position, operating speed, backlash, damping and stiffness upon the dynamic load.

In 1985 Iida, Tamura and Yamada [124] studied the excitation effect of friction between gear teeth by considering only the vibration in the tooth sliding direction and ignoring the vibration in the other directions in order to simplify the mathematical model. In their single degree of freedom model they considered only the flexibility of the gear carrying shaft. Their harmonic analysis revealed that the peak value of the vibrational amplitude response curve caused by friction is almost independent of lubricant viscosity and transmitted power.

Ohnuma *et al.* [125] used a non-linear four degree of freedom torsional model of a diesel engine drive shaft system in order to study idling rattle of manual transmission. The dynamics of the flywheel and clutch, backlash between the driven shaft gears and drive shaft pinion and various idler gears as well as clutch hub spline backlash were considered in the mathematical model. The Runge-Kutta-Gill technique was employed for numerical integration. Thus, rattle noise was estimated for a defined engine torque output and the calculated values were compared with measured ones. In this model, equivalent values for tooth mesh stiffness and damping were used.

In two recent papers of Nielsen, Pearce and Rouverol [126, 127] gear noise induced by transmission error was investigated. They analyzed the torsional vibrations of a gear with a single degree of freedom mathematical model obtained by assuming that torsional modes can be uncoupled from other vibration modes. Their aim was to see which design parameter would be most effective for noise control.

The mathematical models developed by Sato and Matsuhisa [128] in 1981, by Oda, Koide and Miyachika [129] in 1985 and by Oda *et al.* [130] in 1986 are quite different from previous discrete models. It was noted by the authors that to advance the study on the dynamic behavior of thin-rimmed gears it would be necessary to investigate the flexural vibrations of the gear body. In these models, the gear body was taken as a circular plate and its flexural vibrations were studied by using Mindlin's method. The effects of the gear teeth were considered in the proper boundary conditions. Natural frequencies and frequency responses were calculated and compared with measured values. In the later two works by Oda *et al.* [129, 130], the circumferential, radial and axial accelerations and stresses were also measured under several running conditions, and the results were used to investigate the effects of web arrangement in spur and helical gears on vibration and dynamic loads.

Recently, Lin and Huston [131] used a torsional model to develop a computer program for the design of spur gear systems. Variable tooth mesh stiffness was calculated by taking a tooth as a cantilever beam and by considering also the flexibility of the fillet and foundation and the local compliance due to contact forces. Damping due to lubrication of gears and shafts were expressed with constant damping coefficients, and the friction between gear teeth was included in the model with a frictional torque. The model was developed for low contact ratio gear pairs and the transverse flexibilities of the shafts and bearings were not considered. A linearized-iterative procedure was used for the numerical solution. The model was used to study the effects of several parameters such as friction, damping, tooth geometry, stiffnesses, etc., upon the system behavior.

5. MODELS FOR GEARED ROTOR DYNAMICS

Pioneer models of this group are those for studying whirling of gear-carrying shafts, rather than the dynamics of the gear itself. Although investigators have studied whirling of disk-carrying shafts for many years, it was not until the 1960s that the influence of the constraint imposed by the gear on the whirling of geared shafts was considered in rotor dynamics problems. Seireg [132], in 1966, investigated the whirling of geared shafts experimentally, but he did not develop any model for the analytical study of the problem, although he gave an empirical procedure for predicting the main resonance frequency. The receptance model of Johnson [83] discussed in the previous section, however, might be considered to be the first attempt to include the constraints imposed by gears in rotor dynamics. The extensive model of Fukuma *et al.* [92] which is also discussed in the previous section, could also be included in this group, since it is a three-dimensional model and several possible motions of the gear and shaft are considered. However, their model was not developed for rotor dynamics studies, but for gear dynamic problems.

In 1975, Mitchell and Mellen [133] presented experimental data indicating the torsional-lateral coupling in a geared high speed rotor system. They pointed out that mathematical models based on uncoupled lateral-torsional effects fail to provide the necessary information for a proper design of high-performance machinery.

In 1977, Lund [134] developed a rather simple model to study critical speeds, stability and forced response of a geared rotor. His analysis was based on the development of a set of influence coefficients at each gear mesh by using the Holzer method for torsional vibrations and the Myklestad-Prohl method for lateral vibrations. The results were coupled through impedance matching at the gear meshes. He assumed constant tooth mesh stiffness, and calculated the forced response of the system caused by mesh errors or by mass unbalance. He treated the excitation terms in his analysis such that they were assumed to be at the same frequency. A linear model was used to determine the natural frequencies of the system as well as the dynamic loads.

Hamad and Seireg [135, 136] investigated the whirling of pinion-gear systems supported on hydrodynamic bearings. First they considered the shaft of the gear to be rigid and ignored the effect of the transmitted load [129]. The model was extended by assuming that the gear rotor was also supported on isoviscous fluid bearings [136]. In this later work, they also considered the transmitted gear load and its effect on whirl amplitude and stability of balanced and unbalanced gears. However, the model developed did not take account of the torsional vibrations. The solution was obtained by using a digital phase-plane method.

Daws and Mitchell [137, 138] analyzed gear coupled rotors by developing a three-dimensional model in which variable mesh stiffness was considered as a time varying three dimensional stiffness tensor. The "force coupling" caused by the interaction of gear deflection and the time varying stiffness was considered in their model which predicted the forced response of the system to excitations due to unbalanced rotors and mesh errors. The transfer matrix method extended to branched gear systems was used for the solution. Daws and Mitchell, however, did not consider the "dynamic coupling" terms in their model. Later, Mitchell and David [139] showed that the magnitude of the dynamic coupling terms is potentially as large as the magnitude of the linear terms that are included in most rotor analyses. David [140] investigated the dynamic coupling in non-linear geared rotor systems. He improved the model of Daws, in particular by including the second order coupling terms. It was found that the inclusion of dynamic coupling effects changed the predicted response amplitudes of a trial system by four to eight orders of magnitude at some frequencies. It was also shown that the dynamic coupling is capable of producing system responses of the same magnitude as the unbalanced response. With the same model, David and Mitchell [140, 141] also studied the effects of linear dynamic coupling terms by solving a trial problem and concluded that these terms produced significant changes in the predicted response at all of the frequencies associated with tooth passing. In all of these studies, the transfer matrix method was employed by developing the method for nonlinear systems whenever it was necessary. Blanding [142] also used the transfer matrix method in his dissertation published in 1985. The main emphasis in this work was placed on the derivation of the time varying stiffness tensor representing the involute spur gear mesh. The effects considered in the stiffness derivation were bending, shear, compression and local contact deformation.

In the model developed by Buckens [143] in 1980 several simplifying assumptions are made but the elasticity and damping of the bearings as well as those of the shafts, and the damping due to friction at the contact between the gear teeth are retained. In his model it was assumed that the contact between the gears is never interrupted.

Iida *et al.* [144–147] have published a series of papers between 1980 and 1986 on the coupled torsional-transverse vibrations of geared rotors. Transverse vibrations in tooth sliding and power transmitting directions were considered and it was shown that transverse vibrations couple with torsional vibrations even though gyroscopic effects are neglected. Ignoring the compliance of the gear tooth and other non-linear effects resulted in a linear model which was used to determine the natural frequencies and mode shapes. In their early work [144], a two shaft - two gear system was analyzed by assuming that one of the shafts was rigid, and the response to gear eccentricity and mass unbalance was determined. In later papers, a two shaft - four gear system was modeled by considering the torsional flexibilities of all shafts, but the transverse flexibility of only one shaft. In their recent paper [147], however, all shafts were assumed rigid in the transverse direction but the countershaft was assumed to be softly supported. The theoretically determined natural frequencies were compared with experimental values. The change in the natural frequencies with the angle between the power transmitting directions of two gear pairs placed on a shaft was studied by using their models.

In 1981, Hagiwara, Ida and Kikuchi [148] used a simple model to study the vibration of geared shafts due to unbalanced and run-out errors. The lateral flexibilities of shafts were considered using discrete stiffness values. Journal bearings were represented by damping and stiffness matrices of order two which were calculated from Reynolds equation as a function of constant tooth force, rotating speed, clearance and oil viscosity. A constant mesh stiffness was assumed and the backlash and tooth separation were not considered in the analysis. It was both analytically and experimentally observed that unbalance forces and gear errors can excite both torsional and lateral modes, and large displacements can be observed in torsional modes.

Iwatsubo, Arii and Kawai [149] studied the rotor dynamics problem of geared shafts by including a constant mesh stiffness and the forcing due to unbalanced mass but by neglecting the tooth profile error and backlash. The transfer matrix method was employed in the solution and free and forced vibration analyses were made. In a subsequent paper [150] the authors solved a similar problem by including the effects of periodic variation of tooth mesh stiffness and a tooth profile correction. In this study a stability analysis was also made by assuming a rectangular mesh stiffness variation.

Neriya, Bhat and Sankar [151, 152] found the finite element formulation very useful in the dynamic analysis of geared trained rotors, since the coupling action in the gear pairs could be easily incorporated into the mass and stiffness matrices. They modeled a single gear as a two mass - two spring - two damper system, one of the set representing a tooth and the other the gear itself. In their earlier work [153] the shafts were assumed to be massless and an equivalent discrete model including lateral and torsional stiffness of shafts was used. In the later studies [151, 152], the shafts in the system were modeled by finite elements, and the coupling action between torsion and flexure was introduced in the model at the pair locations. A constant mesh stiffness was assumed and the natural frequencies of the resulting linear system were obtained. The response of the system to mass unbalance and to geometric eccentricity in the gear, and the resulting dynamic tooth load were calculated by using undamped modes of the system and equivalent modal damping values. Several numerical results were presented and discussed.

6. OTHER MATHEMATICAL MODELS

Another extreme in the dynamic modelling of gears is to neglect the flexibility of gear teeth and to consider the problem as a torsional vibration problem. A model for such an analysis consists of torsional springs representing the torsional flexibility of gear-carrying shafts, and rigid disks representing the interia of gears and shafts. Although such models were generally used to determine the natural frequencies of multi-gear-shaft systems [154–156], some investigators have used the rigid gear tooth assumption even in determining dynamic loads or the effect of gear errors upon the dynamic behavior of the system. For instance, in 1968 Rieger [157, 158] modelled a drive train for torsional vibrations by assuming rigid gears, and studied the effect of various types of gear errors. In this model, even the inertia of each gear was neglected. Mahalingam and Bishop [159] used modal analysis in the solution of a torsional model of a pair of gears. The response of the system to a displacement excitation representing periodic or transient static transmission error was calculated. Radzimovsky and Mirarefi [160] modelled a gear testing machine for torsional vibrations by assuming rigid gear teeth. They studied the effects of several factors on the efficiency of gear drives and the coefficient of friction. Although some further assumptions such as no geometric errors in gears and equal load sharing between gear teeth in mesh were made, a close correlation was obtained between experimental and theoretical results. Ikeda and Muto [161] studied the vibrations of a gear pair due to transmission errors and tooth frictions by again using the rigid gear tooth assumption. Their single degree of freedom model included gear inertias, torsional flexibility of the shaft, and damping. The calculated gear vibrations compared well with the experimental values in the frequency range tested. Also in the models of Wang [162, 163] rigid gear teeth were assumed. He developed two models [162]: a two mass model without an elastic element and a three mass - one spring system. In both models time varying backlash, impact and displacement excitations were considered, and dynamic loads due to backlash impact were calculated by using a piecewise linear iteration technique. The theoretical predictions were experimentally verified [162, 163] and it was shown that severe tooth loads may occur in lightly loaded gears due to impact. In another paper, Wang [164] derived Hertzian impact formulas for a crossed helical gear pair as an example for Hertzian impact loads arising in rotational systems with backlash. These studies and his models, which cannot easily be grouped according to the classification made here lead to his interesting model which was discussed previously [25]. Another example for a mathematical model with rigid teeth assumption to calculate dynamic gear tooth load is the model of Osman, Bahgat and Sankar [165, 166]. They studied the effect of bearing clearances on the dynamic response and dynamic tooth loads of spur gears and reached the conclusion that bearing clearances have considerable effects on the dynamics of gears, especially at high speed. Their analysis basically relies on the geometric computation of some angles and the use of rigid body dynamics.

There are also various studies mainly aimed at modelling complex systems containing several gears as well as other mechanical or electrical elements, such as rolling mill drives [167-169], machine tool gear drives [170], aircraft transmissions [96, 171-179] and other gear units [111, 125, 155, 180-184] some of which have already been discussed in the preceding sections. While lumped-parameter models with several simplifying assumptions were employed in most of these analyses, models with finite elements were also employed in some analyses [174, 175, 179]. Inclusion of other elements, especially the elements with non-linear properties, usually complicated the models. Among such studies the extensive program of gearbox modelling carried out by Badgley, Laskin, *et al.* [171-173, 176] and by Sciarra, Howells *et al.* [174, 175, 179] are especially important. The models developed in these studies are specifically for a helicopter rotor-drive, although the methods employed are general and can be applied to different gearboxes. Several papers published during the first stage of the research for this program have been summarized by Badgley and Hartman [173]. More recently Sciarra *et al.* [174] described the results of this extensive research. Dynamic tooth forces were calculated by using a torsional model. The calculated tooth forces were then used in a finite element model of the structural part of the gearbox in order to determine its forced response.

There have also been numerous studies on the dynamic modelling of planetary gear trains which are not included in this review.

Before closing, it is worth mentioning that several different aspects of the researches in gear dynamics were discussed by giving further references in several papers, in particular in references [68, 74, 102, 106, 114, 123, 185-188]. Finally, papers written in languages other than English have also been discussed in several references [74, 86, 88, 114, 122, 182].

REFERENCES

1. A. FISHER 1961 *Machinery* 98, 545-552. Factors in calculating the load-carrying capacity of helical gears.

2. G. H. MARX and L. E. CUTTER 1915 *Transactions of the American Society of Mechanical Engineers* 37, Strength of gear teeth.
3. L. J. FRANKLIN and C. H. SMITH 1924 *Transactions of The American Society of Mechanical Engineers* 46, Effect of inaccuracies on strength of gears.
4. A. A. ROSS 1927 *American Gear Manufacturers Association Paper.* High speed gears.
5. E. BUCKINGHAM 1931 *Dynamic Loads on Gear Teeth.* New York: American Society of Mechanical Engineers Special Publication.
6. E. BUCKINGHAM 1949 *Analytical Mechanics of Gears.* New York: McGraw-Hill.
7. W. A. TUPLIN 1950 *Proceedings of the Institution of Mechanical Engineers* 16, 162-167. Gear tooth stresses at high speed.
8. W. A. TUPLIN 1953 *Machine Design* 25, 203-211. Dynamic loads on gear teeth.
9. W. A. TUPLIN 1958 *Proceedings of the International Conference on Gearing, Institution of Mechanical Engineers*, 24-30. Dynamic load on gear teeth.
10. J. B. RESWICK 1955 *Transactions of the American Society of Mechanical Engineers* 77, 635-644. Dynamic loads on spur and helical gear teeth.
11. H. STRAUCH 1953 *Zeitschrift des Vereines Deutscher Ingiere* 95, 159-163. Zahnradschwingungen (Gear vibrations).
12. J. ZEMAN 1957 *Zeitschrift des Vereines Deutscher Ingiere* 99, 244. Dynamische Zusatzkrafter in Zahnbradgetrieben.
13. A. SEIREG and D. R. HOUSER 1970 *Journal of Engineering for Industry. Transactions of the American Society of Mechanical Engineers* 192, 504-515. Evaluation of dynamic factors for spur and helical gears.
14. D. R. HOUSER and A. SEIREG 1970 *Journal of Engineering for Industry, Transactions of the American Society of Mechanical Engineers* 192, 495-503. Investigation of dynamic factors in spur and helical gears.
15. E. J. WELLAUER 1959 *American Society of Mechanical Engineers Paper* 59-*A*-121. An analysis of factors used for strength rating helical gears.
16. A. I. TUCKER 1971 *American Society of Mechanical Engineers Paper* 71-*DE*-1. Dynamic loads on gear teeth, design applications.
17. A. I. TUCKER 1971. *Mechanical Engineering*, 29-33. Gear design: dynamic loads.
18. W. A. TUPLIN 1961 *Gear Load Capacity.* New York: John Wiley.
19. *American Gear Manufacturers Association Paper* 215.01. Information sheet for surface durability (pitting) of spur, helical, herringbone, and bevel gear teeth.
20. *American Gear Manufacturers Association Paper* 225.01. Information sheet for strength of spur, helical, herringbone, and bevel gear teeth.
21. *American Gear Manufacturers Association Paper* 411.02. Design procedure for aircraft engine and power takeoff spur and helical gears.
22. D. W. DUDLEY 1969 *American Gear Manufacturers Association Paper* 159.02. Proposed application factors for industrial and high speed gearing.
23. F. A. THOMA 1969 *Society of Naval Architects and Marine Engineers Paper.* An up-do-date look at marine gear tooth loading.
24. H RETTIG 1975 *Proceedings of the International Federation of the Theory of Machines and Mechanisms 4th World Congress, Newcastle.* Vibrations in gear drives; test results and calculation method for dynamic tooth forces.
25. C. C. WANG 1985 *Journal of Mechanisms, Transmissions, and Automation in Design, Transactions of the American Society of Mechanical Engineers* 107, 301-311. On analytical evaluation of gear dynamic factors based on rigid body dynamics.
26. T. NAKADA and M. UTAGAWA 1956 *Proceedings of the Sixth Japanese National Congress on Applied Mechanics*, 493-497. The dynamic loads on gears caused by the varying elasticity of the mating teeth of spur gears.
27. S. L. HARRIS 1958 *Proceedings of the Institution of Mechanical Engineers* 172, 87-112. Dynamic loads on the teeth of spur gears.
28. D. C. JOHNSON 1958 *Proceedings of the International Conference on Gearing, Institution of Mechanical Engineers*, 18-23. Excitation of resonant vibrations by gear tooth meshing effects.
29. H. KOHLER 1959. *Ph.D. Thesis, University of Sheffield.* The mechanism and measurement of dynamic loading in spur gears.
30. B. WOOD 1960 *Ph.D. Thesis, University of Leeds.* Sources of vibration excitation in spur gears.
31. M. UTAGAWA 1958 *Bulletin of the Japanese Society of Mechanical Engineers* 1, 397-403. Dynamic load on spur gear teeth.

32. M. UTAGAWA and T. HARADA 1961 *Bulletin of the Japanese Society of Mechanical Engineers* **4**, 706–713. Dynamic loads on spur gear teeth at high speed (influence of the pressure angle errors and comparison between the reduction gear and the speed-up gears).
33. M. UTAGAWA and T. HARADA 1962 *Bulletin of the Japanese Society of Mechanical Engineers* **5**, 374–381. Dynamic loads on spur gear teeth having pitch errors at high speed.
34. T. TOBE 1961 *Bulletin of the Japanese Society of Mechanical Engineers* **4**, 417–422. Dynamic loads on spur gear teeth.
35. T. TOBE and M. KATO 1967 *Proceedings of the Japanese Society of Mechanical Engineers Semi-International Gearing Symposium*, 189–198. Dynamic loads on the teeth of spur gears.
36. R. W. GREGORY, S. L. HARRIS and R. G. MUNRO 1963–1964 *Proceedings of the Institution of Mechanical Engineers* **178**, 207–226. Dynamic behavior of spur gears.
37. R. W. GREGORY, S. L. HARRIS and R. G. MUNRO 1963–1964 *Proceedings of the Institution of Mechanical Engineers* **178**, 166–173. Torsional motion of a pair of spur gears.
38. H. RETTIG 1965 *Konstruktion* **1**, 41–53. Zahnkrafte und Schwingungen in Stirnradgetrieben.
39. M. BOSCH 1965 *Industrie-Anzeiger* **87**, 2469–2478. Das dynamische Verhalten von Stirnradgetrieben unter besonderer Berucksichtigung der Verzaqhnungsgenauigkeit.
40. M. BOSCH 1966 *Industrie-Anzeiger* **88**, 267–274. Das dynamische Verhalten von Stirnradgetrieben unter besonderer Berucksichtigung der Verzahnungsgenauigkeit (II).
41. T. AIDA, 1968 *Transactions of the Japanese Society of Mechanical Engineers* **34**, 2226–2264. Fundamental research on gear noise and vibration.
42. T. AIDA, 1969 *Transactions of the Japanese Society of Mechanical Engineers* **35**, 2113–2119. Fundamental research on gear noise and vibration, II.
43. T. AIDA, S. SATO and H. FUKUMA 1967 *Proceedings of the Japanese Society of Mechanical Engineers Semi-International Gearing Symposium*, 151–160. Properties of gear noise and its generating mechanism.
44. H. OPITZ 1967 *Proceedings of the Japanese Society of Mechanical Engineers Semi-International Gearing Symposium*, 199–209. Dynamic behaviour of spur and helical gears.
45. K. NAKAMURA 1967 *Bulletin of the Japanese Society of Mechanical Engineers* **10**, 846–854. Tooth separations and abnormal noise on power-transmission gears.
46. J. G. BOLLINGER and R. J. HARKER 1967 *The Journal of the Industrial Mathematics Society* **17**, 39–55. Instability potential of high speed gearing.
47. G. V. TORDION and H. GERARDIN 1967 *Proceedings of the Japanese Society of Mechanical Engineers Semi-International Gearing Symposium*, 279–284. Dynamic measurement of the transmission error in gears.
48. D. B. WALLACE and A. SEIREG 1973 *Journal of Engineering for Industry, Transactions of the American Society of Mechanical Engineers* **95**, 1108–1115. Computer simulation of dynamic stress, deformation and fracture of gear teeth.
49. T. TOBE and N. TAKATSU 1973 *Bulletin of the Japanese Society of Mechanical Engineers* **16**, 1031–1037. Dynamic loads on spur gear teeth caused by teeth impact.
50. K. ICHIMARU and F. HIRANO 1974. *Journal of Engineering for Industry, Transactions of the American Society of Mechanical Engineers* **96**, 373–381. Dynamic behavior of heavy-loaded spur gears.
51. R. W. CORNELL and W. W. WESTERVELT 1981 *Journal of Mechanical Design, Transactions of the American Society of Mechanical Engineers* **100**, 69–76. Dynamic tooth loads and stressing for high contact ratio spur gears.
52. H. RICHARDSON 1958 *Massachusetts Institute of Technology Dynamics and Controls Laboratory Research Memorandum* 7454. Static and dynamic load, stress, and deflection cycles in spur-gear systems.
53. J. HOWLAND 1958 *M.S. Thesis, Massachusetts Institute of Technology*. An investigation of dynamic loads in spur-gear teeth.
54. A. KUBO 1978 *Journal of Mechanical Design, Transactions of the American Society of Mechanical Engineers* **100**, 77–84. Stress condition, vibrational exciting force, and contact pattern of helical gears with manufacturing and alignment errors.
55. E. P. REMMERS 1978 *Journal of Mechanical Design, Transactions of the American Society of Mechanical Engineers* **10**, 715–722. Gear mesh excitation spectra for arbitrary tooth spacing errors, load, and design contact ratio.
56. M. BENTON and A. SEIREG 1978 *Journal of Mechanical Design, Transactions of the American Society of Mechanical Engineers* **100**, 26–32. Simulation of resonances and instability conditions in pinion-gear systems.

57. K. ISHIDA and T. MATSUDA 1980 *American Society of Mechanical Engineers Paper* 80-C2/DET-69. Study on pitch circle impulse noise of gear.
58. K. ISHIDA and T. MATSUDA 1980 *American Society of Mechanical Engineers Paper* 80-C2/DET-70. Effect of tooth surface roughness on gear noise and gear noise transmitting path.
59. K. L. WANG and H. S. CHENG 1981 *Journal of Mechanical Design, Transactions of the American Society of Mechanical Engineers* **103**, 177–187. A numerical solution to the dynamic load, film thickness and surface temperature of spur gears: Part 1 Analysis.
60. K. L. WANG and H. S. CHENG 1981 *Journal of Mechanical Design, Transactions of the American Society of Mechanical Engineers* **103**, 188–194. A numerical-solution to the dynamic load, film thickness, and surface temperatures in spur gears, Part 2, results.
61. K. L. WANG and H. S. CHENG 1983 *NASA Lewis Research Center Advanced Power Transmission and Technology* 503–518. A computer code for performance of spur gears.
62. B. REBBECHI and J. D. C. CRISP 1981 *Proceedings of the International Symposium on Gearing and Power Transmissions, Japanese Society of Mechanical Engineers* **2**, 61–66. A new formulation of spur-gear vibration.
63. B. REBBECHI and J. D. C. CRISP 1983 *Proceedings of the International Federation of the Theory of Machines and Mechanisms Sixth World Congress, New Delhi*, 802–806. The kinetics of the contact point and oscillatory mechanisms in resilient spur gears.
64. W. D. MARK 1982 *NASA CR* 3626. Transfer function method for gear system dynamics applied to conventional and minimum excitation gearing designs.
65. W. D. MARK 1978 *Journal of the Acoustical Society of America* **63**, 1409–1430. Analysis of the vibratory excitation of gear systems: basic theory.
66. W. D. MARK 1979 *Journal of the Acoustical Society of America* **66**, 1758–1787. Analysis of the vibratory excitations of gear systems. II: Tooth error representations, approximations, and applications.
67. M. F. SPOTTS 1984 *Machine Design* **56**, 84–88. Estimating dynamic loads on gear teeth.
68. H. LIN, R. L. HUSTON and J. J. COY 1984 *American Society of Mechanical Engineers Paper* 84-DET-226. Dynamic analysis of straight and involute tooth forms.
69. G. L. OSTIGUY and I. CONSTANTINESCU 1984 *Proceedings of the Second International Modal Analysis Conference, Union College* **1**, 286–291. Dynamic response of gear teeth.
70. L. WILCOX and W. COLEMAN 1973 *Journal of Engineering for Industry, Transactions of the American Society of Mechanical Engineers* **95**, 1139–1148. Application of finite elements to the analysis of gear tooth stresses.
71. G. CHABERT, T. T. DANG and R. MATHIS 1978 *Journal of Engineering for Industry, Transactions of the American Society of Mechanical Engineers* **96**, 85–93. An evaluation of stresses and deflection of spur gear teeth under strain.
72. V. RAMAMURTI and L. S. GUPTA, 1979 *American Society of Mechanical Engineers Paper* 79-DET-38. Dynamic stress analysis of a spur gear tooth.
73. K. NAGAYA and S. UEMATSU 1981 *Journal of Mechanical Design, Transactions of the American Society of Mechanical Engineers* **103**, 357–363. Effects of moving speeds of dynamic loads on the deflections of gear teeth.
74. K. UMEZAWA, T. SATO and J. ISHIKAWA 1984 *Bulletin of the Japanese Society of Mechanical Engineers* **27**, 102–109. Simulation on rotational vibration of spur gears.
75. K. UMEZAWA, S. SATO and K. KOHNO 1984 *Bulletin of the Japanese Society of Mechanical Engineers* **27**, 569–575. Influence of gear errors on rotational vibration of power transmission spur gears, 1. Pressure angle error and normal pitch error.
76. K. UMEZAWA and T. SATO 1985 *Bulletin of Japanese Society of Mechanical Engineers* **28** 2143–2148. Influence of gear error on rotational vibration of power transmission spur gear (2nd report—waved form error).
77. K. UMEZAWA, T. SUZUKI and T. SATO 1986 *Bulletin of the Japanese Society of Mechanical Engineers* **29**, 1605–1611. Vibration of power transmission helical gears (approximate equations of tooth stiffness).
78. K. SATO, S. YAMAMOTO, O. KAMADA and N. TAKATSU 1986 *Bulletin of the Japanese Society of Mechanical Engineers* **29**, 1586–1589. Approximate solution of a gear system subjected to random excitation.
79. T. MASUDA, T. ABE and K. HATTORI 1986 *Journal of Vibration Acoustics, Stress and Reliability in Design, Transactions of the American Society of Mechanical Engineers Paper* 85-*DET*-2. Prediction method of gear noise considering the influence of the tooth flank finishing method.
80. D. LEWICKI 1986 *NASA Technical Paper* 2610. Predicted effect of dynamic load on pitting fatigue life for low-contact ratio spur gears.

81. D. C. H. YANG and J. Y. LIN 1986, *Journal of Mechanisms, Transmissions, and Automation in Design, Transactions of the American Society of Mechanical Engineers Paper* 86-*DET*-64. Hertzian damping, tooth friction and bending elasticity in gear impact dynamics.
82. D. C. H. YANG and T. SUN 1985 *Journal of Mechanisms, Transmissions, and Automation in Design, Transactions of the American Society of Mechanical Engineers Paper* 85-*DET*-2. A rotary model for spur gear dynamics.
83. D. C. JOHNSON 1962 *Journal of Mechanical Engineering Science* 4, 241-250. Modes and frequencies of shafts coupled by straight spur gears.
84. G. V. TORDION 1963 *American Gear Manufacturers Association Paper* 209.04. The mechanical impedance approach to the dynamics (torsional vibrations) of geared systems.
85. D. SEAGER 1969 *American Society of Mechanical Engineers Paper* 69-*VIBR*-16. Dynamic behavior of helical gears.
86. H. K. KOHLER, A. PRATT and A. M. THOMSON 1970 *Proceedings of the Institution of Mechanical Engineers* **184**, 111-121. Dynamics and noise of parallel axis gearing.
87. E. P. REMMERS 1971 *American Society of Mechanical Engineers Paper* 71-*DE*-23. The dynamics of gear pair systems.
88. R. KASUBA 1971 *American Society of Mechanical Engineers Paper* 71-*DE*-26. Dynamic loads on spur gear teeth by analog computation.
89. R. KASUBA 1961 *Ph.D. Thesis, University of Illinois.* An analytical and experimental study of dynamic loads on spur gear teeth.
90. S. M. WANG and I. E. MORSE 1972 *Journal of Engineering for Industry, Transactions of the American Society of Mechanical Engineers* **94**, 583-594. Torsional response of a gear train system.
91. S. M. WANG 1974 *Journal of Engineering for Industry, Transactions of the American Society of Mechanical Engineers* **96**, 51-59. Analysis of non-linear transient motion of a geared torsional.
92. H. FUKUMA, T. FURUKAWA and T. AIDA 1973 *Bulletin of the Japanese Society of Mechanical Engineers* **16**, 1094-1107. Fundamental research on gear noise and vibration (6th report, generation mechanism of radial and axial vibration of spur gears).
93. M. W. SALZER and J. D. SMITH 1975 *Proceedings of the International Federation of the Theory of Machines and Mechanisms Fourth World Congress, Newcastle* **1**, 175-177. Real time simulation of gearboxes.
94. M. W. SALZER, J. D. SMITH and D. B. WELBOURN 1977 *Proceedings of the World Congress on Gearing, Paris* **1**, 297-308. Simulation of noise from gears when varying design and manufacturing parameters.
95. M. W. SALZER 1977 *Ph.D. Thesis, Cambridge University.* The dynamics of a layshaft gearbox.
96. A. ASTRIDGE and M. SALZER 1977 *Proceedings of the Third European Rotorshaft and Power Lift Aircraft*, 1-10. Gearbox dynamics—modelling of a spiral bevel gearbox.
97. T. TOBE, K. SATO and N. TAKATSU 1976 *Bulletin of the Japanese Society of Mechanical Engineers* **19**, 808-813. Statistical analysis of dynamic loads on spur gear teeth (effect on shaft stiffness).
98. T. TOBE and K. SATO 1977 Bulletin of the Japanese Society of Mechanical Engineers **20**, 882-889. Statistical analysis of dynamic loads on spur gear teeth.
99. M. J. DROSJACK and D. R. HOUSER 1977 *American Society of Mechanical Engineers Paper* 77-*DET*-123. An experimental and theoretical study of the effects of simulated pitch line pitting on the vibration of a geared system.
100. R. C. AZAR and F. R. E. CROSSLEY 1977 *Journal of Engineering for Industry, Transactions of the American Society of Mechanical Engineers* **99**, 792-798. Digital simulation of impact phenomenon in spur gear systems.
101. G. V. TORDION and R. GAUVIN 1977 *Journal of Engineering for Industry, Transactions of the American Society of Mechanical Engineers* **99**, 785-791. Dynamic stability of a two-stage gear train under the influence of variable meshing stiffnesses.
102. M. BENTON and A. SEIREG 1981 *Journal of Mechanical Design, Transactions of the American Society of Mechanical Engineers* **103**, 372-378. Factors influencing instability and resonances in geared systems.
103. S. KIYONO, T. AIDA and Y. FUJII 1978 *Bulletin of the Japanese Society of Mechanical Engineers* **21**, 915-922. Vibration of helical gears 1—theoretical analysis.
104. S. KIYONO, Y. FUJII and Y. SUZUKI 1981 *Bulletin of the Japanese Society of Mechanical Engineers* **24**, 441-446. Analysis of vibration of bevel gears.

105. B. KISHER 1979 *Proceedings of the International Federation of the Theory of Machines and Mechanisms Fifth World Congress, Montreal*, 1122-1125. Effect of gear errors on non-linear vibrations of a gear train system.
106. A. TODA and G. V. TORDION 1979 *Proceedings of the International Federation of the Theory of Machines and Mechanisms Fifth World Congress, Montreal*, 1130-1133. Dynamic behavior of a mechanical system with gear transmission error.
107. J. D. SMITH 1980 *Proceedings of the Institute of Acoustics Conference, Loughborough* 151-160. Gear impact noise in diesel drives.
108. Y. FURUYA, K. SETO, K. YAMADA and M. YAMANOUCHI 1980 *Bulletin of the Japanese Society of Mechanical Engineers* **23**, 300-308. Influence of torsional vibrations of a gear train system on tooth load.
109. A. KUBO and S. KIYONO 1980 *Bulletin of the Japanese Society of Mechanical Engineers* **23** 1536-1543. Vibration excitation of cylindrical involute gears due to tooth form error.
110. A. W. LEES and P. C. PANDEY 1980 *Proceedings of the Conference on Rotor Dynamics, Institution of Mechanical Engineers*, 103-108. Vibration spectra from gear drives.
111. T. SAKAI, Y. DOI, K. YAMAMOTO, T. OGASAWARA and M. NARITA 1981 *Society of Automotive Engineers Paper* 810773. Theoretical and experimental analysis of rattling noise of automotive gearbox.
112. J. HLEBANJA and J. DUHOVNIK 1981 *Proceedings, International Symposium of Gearing and Power Transmissions, Tokyo*, 43-48. Power transmission between teeth at small loads.
113. H. WINTER and M. KOJIMA 1981 *Proceedings, International Symposium of Gearing and Power Transmissions, Tokyo*, 67-72. A study on the dynamics of geared system—estimation of overload on gears in system
114. R. KASUBA and J. W. EVANS 1981 *Journal of Mechanical Design, Transactions of the American Society of Mechanical Engineers* **103**, 398-409. An extended model for determining dynamic loads in spur gearing.
115. R. KASUBA 1981 *NASA CP* 2210 403-419. A method for static and dynamic load analysis of standard and modified spur gears.
116. R. KASUBA 1981 *Proceedings, International Symposium of Gearing and Power Transmissions, Tokyo*, 49-54. Dynamic loads in normal and high contact ratio spur gearing.
117. A. PINTZ and R. KASUBA 1985 *Journal of Mechanisms, Transmissions, and Automation in Design, Transactions of the American Society of Mechanical Engineers* **107**, 424-429. Dynamic load factors in internal spur gear drives.
118. C. TROEDER, H. PEEKEN, A. LASCHET and K. TOOTEN 1983 *Proceedings of the International Federation of the Theory of Machines and Mechanisms Sixth World Congress, New Delhi*, **2**, 936-943. Causes and effect of torque change in hammering of toothed gears.
119. W. PINNEKAMP 1982 *Konstruktion* **34**. Dynamische Analyse eines Antriebes mit Zahnspiel.
120. B. M. BAHGAT, M. O. M. OSMAN and T. S. SANKAR 1983 *Journal of Mechanisms, Transmissions, and Automation in Design, Transactions of the American Society of Mechanical Engineers* **105**, 302-309. On the spur-gear dynamic tooth-load under consideration of system elasticity and tooth involute profile.
121. A. W. LEES 1984 *Proceedings of the Third International Conference on Vibrations in Rotating Machinery, Institution of Mechanical Engineers*, 73-79. Dynamic loads in gear teeth.
122. F. KÜÇÜKAY 1984 *Proceedings of the Third International Conference on Vibrations in Rotating Machinery, Institution of Mechanical Engineers*, 81-90. Dynamic behaviour of high speed gears.
123. A. S. KUMAR, T. S. SANKAR and M. O. M. OSMAN 1985 *Journal of Mechanisms, Transmissions, and Automation in Design, Transactions of the American Society of Mechanical Engineers* **107**, 54-60. On dynamic tooth load and stability of a spur-gear system using the state-space approach.
124. H. LIDA, A. TAMURA and Y. YAMADA 1985 *Bulletin of the Japanese Society of Mechanical Engineers* **28**, 1512-1519. Vibrational characteristics of friction between gear teeth.
125. S. OHNUMA, S. YAHATA, M. INAGAWA and T. FUJIMOTO 1985 *Society of Automotive Engineers Paper* 850979. Research on idling rattle of manual transmission.
126. M. A. NIELSEN, W. J. PEARCE and W. S. ROUVERAL 1985 *American Gear Manufacturers Association Paper* 85*FTM*12. Minimizing gear noise excitation.
127. W. J. PEARCE, M. A. NIELSON and W. S. ROUVERAL 1986 *Proceedings of the Second World Congress on Gearing*, Paris, 587-598. Reducing gear noise excitation.
128. S. SATO and H. MATSUHISA 1981 *Proceedings, International Symposium of Gearing and Power Transmissions, Tokyo*, 19-24. Noise and vibration of a thin helical gear.

129. S. ODA, T. KOIDE and K. MIYACHIKA 1985 *Bulletin of the Japanese Society of Mechanical Engineers* **28**, 2434–2441. Dynamic behavior of thin-rimmed helical gears with various web arrangements, I.
130. S. ODA, K. MIYACHIKA, T. KOIDE and T. FUJII 1986 *Bulletin of the Japanese Society of Mechanical Engineers* **29**, 241–248. Dynamic behavior of thin-rimmed spur gears with various web arrangements.
131. H. H. LIN and R. HUSTON 1986 *NASA CR* 179473. Dynamic loading on parallel shaft gears.
132. A. SEIREG 1966 *American Society of Mechanical Engineers Paper* 66-*WA*/*MD*-6. Whirling of shafts in geared systems.
133. L. D. MITCHELL and D. M. MELLEN 1975 *American Society of Mechanical Engineers Paper* 75-DET-75. Torsional-lateral coupling in a geared high-speed rotor system.
134. J. W. LUND 1978 *Journal of Mechanical Design, Transactions of the American Society of Mechanical Engineers* **100**, 535–538. Critical speeds, stability and response of a geared train of rotors.
135. B. M. HAMAD and A. SEIREG 1979 *Proceedings of the International Federation on the Theory of Machines and Mechanisms, Fifth World Congress, Montreal*, 1109–1112. Whirl of geared systems supported on hydrodynamic bearings.
136. B. M. HAMAD and A. SEIREG 1980 *Journal of Engineering for Power, Transactions of the American Society of Mechanical Engineers* **102**, 508–510. Simulation of whirl interaction in pinion-gear systems supported on oil film bearings.
137. J. W. DAWS 1979 *Ph.D. Dissertation, Virginia Polytechnic Institute and State University*. An analytical investigation of three dimensional vibration in gear-coupled rotor systems.
138. L. D. MITCHELL and J. W. DAWS 1983 *Society of Automotive Engineers Transactions* **9**, 3366–3379. A basic approach to gearbox noise prediction.
139. L. D. MITCHELL and J. W. DAVID 1985 *Journal of Vibration, Stress, and Reliability in Design, Transactions of the American Society of Mechanical Engineers* **107**, 112–116. Proposed solution methodology for the dynamically coupled nonlinear geared rotor mechanics equation.
140. J. W. DAVID 1984 *Ph.D. Thesis, Virginia Polytechnic Institute and State University*. Analytical investigation of dynamic coupling in nonlinear geared rotor systems.
141. J. W. DAVID and L. D. MITCHELL 1985 *Journal of Vibration, Acoustics, Stress, and Reliability in Design, Transactions of the American Society of Mechanical Engineers Paper* 85-*DET*-11. Dynamic coupling in geared rotor systems.
142. J. M. BLANDING 1985 *Ph.D. Thesis, Virginia Polytechnic Institute and State University*. An analytical study and computer analysis of three-dimensional steady state vibration of multishaft geared-rotor systems.
143. F. BUCKENS 1980 *American Society of Mechanical Engineers Paper* 80-*C*2/*DET*-6. On the damped coupled torsional and flexural vibrations of gear-controlled parallel shafts.
144. H. IIDA, A. TAMURA, K. KIKUCH and H. AGATA 1980 *Bulletin of the Japanese Society of Mechanical Engineers* **23**, 2111–2117. Coupled torsional-flexural vibration of a shaft in a geared system of rotors (1st report).
145. H. IIDA and A. TAMURA 1984 *Proceedings of the Conference on Vibrations in Rotating Machinery, Institution of Mechanical Engineers* 67–72. Coupled torsional-flexural vibration of a shaft in a geared system.
146. H. IIDA, A. TAMURA and M. OONISHI 1985 *Bulletin of the Japanese Society of Mechanical Engineers* **28**, 2694–2698. Coupled dynamic characteristics of a counter shaft in a gear train system.
147. H. IIDA, A. TAMURA and H. YAMAMOTO 1986 *Bulletin of the Japanese Society of Mechanical Engineers* **29**, 1811–1816. Dynamic characteristics of a gear train system with softly supported shafts.
148. N. HAGIWARA, M. IDA and K. KIKUCHI 1981 *Proceedings, International Symposium of Gearing and Power Transmissions, Tokyo*, 85–90. Forced vibration of a pinion-gear system supported on journal bearings.
149. T. IWATSUBO, S. ARII and R. KAWAI 1984 *Bulletin of the Japanese Society of Mechanical Engineers* **27**, 271–277. Coupled lateral-torsional vibration of rotor system trained by gears (1. Analysis by transfer-matrix method).
150. T. IWATSUBO, S. ARII and P. KAWAI 1984 *Proceedings of the Third International Conference on Vibrations in Rotating Machinery, Institution of Mechanical Engineers*, 59–66. The coupled lateral torsional vibration of a geared rotor system.
151. S. V. NERIYA, R. B. BHAT and T. S. SANKAR 1985 *American Society of Mechanical Engineers Paper* 85-*DET*-124. Vibration of a geared train of rotors with torsional-flexural coupling.

152. S. V. NERIYA, R. B. BHAT and T. S. SANKAR 1985. *The Shock and Vibration Bulletin* **55**, 13-25. Coupled torsional-flexural vibration of a geared shaft system using finite element analysis.
153. S. V. NERIYA, R. B. BHAT and T. S. SANKAR 1984 *The Shock and Vibration Bulletin* **54**, 67-75. Effect of coupled torsional-flexural vibration of a geared shaft system on the dynamic tooth load.
154. S. MAHALINGHAM 1968 *Aeronautical Journal, Royal Aeronautical Society* **72**, 522-526. Coupled vibration of geared systems.
155. T. A. H. PIXTON and R. ALI 1978 *Proceedings of the Institution of Mechanical Engineers, August/September*, 64-66. Analysis of torsional vibration of an automotive gearbox.
156. S. SANKAR 1979 *Journal of Mechanical Design, Transactions of the American Society of Mechanical Engineers* **101**, 546-553. On the torsional vibration of branched systems using extended transfer matrix method.
157. N. F. RIEGER 1968 *American Gear Manufacturers Association Paper* P109.20. Gear-excited torsional vibrations of machine drive systems.
158. N. F. RIEGER 1969 *Machine Design* **4**, 115-119. Drive train vibrations.
159. S. MAHALINGAM and R. E. D. BISHOP 1974 *Journal of Sound and Vibration* **36**, 179-189. Dynamic loading of gear teeth.
160. E. RADZIMOVSKY and A. MIRAREFI 1975 *Journal of Engineering for Industry, Transactions of the American Society of Mechanical Engineers* **97**, 1274-1281. Dynamic behavior of gear systems and variation of coefficient of friction and efficiency during the engagement cycle.
161. H. IKEDA and E. MUTO 1981 *Proceedings, International Symposium of Gearing and Power Transmissions, Tokyo*, 31-36. The simulations of the gear vibrations based on the actual transmission errors and the tooth frictions.
162. C. C. WANG 1978 *Journal of Mechanical Design, Transactions of the American Society of Mechanical Engineers* **100**, 363-373. Rotational vibration with backlash: Part 1.
163. C. C. WANG 1981 *Journal of Mechanical Design, Transactions of the American Society of Mechanical Engineers* **103**, 387-397. Rotational vibration with backlash: Part 2.
164. C. C. WANG 1980 *American Society of Mechanical Engineers Paper* 80-*C2/DET*-55. On derivation of Hertzian impact formula in the geared rotational systems.
165. M. O. M. OSMAN, B. M. BAHGAT and T. S. SANKAR 1981 *Proceedings, International Symposium of Gearing and Power Transmissions, Tokyo*, 79-84. The effect of bearing clearances on the dynamic response in gearing.
166. B. M. BAHGAT, M. O. M. OSMAN and T. S. SANKAR 1982 *Journal of Mechanical Design, Transactions of The American Society of Mechanical Engineers* **104**, 724-730. On the dynamic gear tooth loading as affected by bearing clearances in high speed machinery.
167. N. F. RIEGER and A. J. SMALLEY 1970 *American Gear Manufacturers Association Paper* 229.14. Dynamic loads on gear teeth from transient shock.
168. J. SLAVIK 1983 *Proceedings of the International Federation on the Theory of Machines and Mechanisms Sixth World Congress, New Delhi*, 1327-1330. Dynamics of torsional driving systems of heavy mills.
169. A. K. TIEU and P. J. GOODISON 1985 *Journal of Mechanisms, Transmissions, and Automation in Design, Transactions of the American Society of Mechanical Engineers Paper* 84-*DET*-14. The effects of dynamic loads on the gearing of a rolling mill drive.
170. E. I. RIVIN 1980 *American Society of Mechanical Engineers Paper* 80-*DET*-104. Compilation and compression of mathematical model for a machine transmission.
171. I. LASKIN, F. K. ORCUTT and E. E. SHIPLEY 1968 *U.S. Army Aviation Laboratories, Ft. Eustis, Virginia, Technical Report* 68-41 (*AD*675457). Analysis of noise generated by UH-1 helicopter transmission.
172. R. H. BADGLEY and I. LASKIN 1970 *US Army Air Mobility Research and Development Laboratories, Ft. Eustis, Virginia, Technical Report* 70-12 (*AD* 869822). Program for helicopter gearbox noise prediction and reduction.
173. R. H. BADGLEY and R. M. HARTMAN 1974 *Journal of Engineering for Industry, Transactions of the American Society of Mechanical Engineers* **96**, 567-577. Gearbox noise reduction: prediction and measurement of mesh-frequency vibrations within an operating helicopter rotor-drive gearbox.
174. J. J. SCIARRA, R. W. HOWELLS, J. W. LENSKI JR., R. J. DRAGO and E. G. SCHAFFER 1978 *U.S. Army Research and Technology Laboratories, Ft. Eustis, Virginia, Technical Report* 78-2A. Helicopter transmission vibration and noise reduction program.

175. R. W. HOWELLS 1978 *Power Transmission Design* 20, 46-49. Transmission design with finite element analysis: Parts 1, 2, 3 and 4.
176. R. M. HARTMAN and R. H. BADGLEY 1974 *U.S. Army Air Mobility Research and Technology Laboratory, Ft. Eustis, Virginia, Technical Report* 74-58 (*AD* 784132). Model 301 HLH/ATC transmission noise reduction program.
177. M. A. BOWES, N. GIANSANTE, R. B. BOSSLER and A. BERMAN 1977 *U.S. Army Air Mobility Research and Technology Laboratory, Ft. Eustis, Virginia, Technical Report* 77-14. Helicopter transmission vibration and noise reduction program.
178. M. A. BOWES 1977 *American Helicopter Society Preprint* 77.33-76. Development and evaluation of a method for predicting the vibration and noise characteristics of helicopter transmissions.
179. R. H. DRAGO, J. W. LENSKI and A. ROYAL 1979. *Proceedings of the International Federation of the Theory of Machines and Mechanisms Fifth World Congress, Montreal*, 910-913. An analytical approach to the source of noise and vibration in highly loaded mechanical power-transmission systems.
180. J. THEISSEN and G. DIEKHANS 1983 *American Gear Manufacturers Association Paper* 429.07. Dynamic gear forces in gear units with alternating output torques.
181. Y. INOUE and A. SEIREG 1984 *Journal of Mechanisms, Transmissions, and Automation in Design, Transactions of the American Society of Mechanical Engineers, Paper* 84-*DET*-209. Simulation of dual drive branched gear systems during selective engagement of the prime movers.
182. J. BARAN and K. MARCHELEK 1973 *Nonlinear Vibration Problems* 1, 547-559. The dynamic investigation of the multi-degree gear transmission model with variable stiffness coefficient.
183. F. KÜÇÜKAY 1986 *Zeitschrift für Angewandte Mathematik und Mechanit* 66, 57-59. Rasselschwingungen in PKW-Getrieben.
184. A. LASCHET and C. TROEDER 1984 *Computers in Mechanical Engineering* 3, 32-43. Torsional and flexural vibrations in drive systems: A computer simulation.
185. E. ERRICHELLO 1979 *Journal of Mechanical Design, Transactions of the American Society of Mechanical Engineers* 101, 369-371. State-of-the-art review: gear dynamics.
186. D. B. WELBOURN 1979 *Proceedings of the Conference on Noise and Vibrations of Engines and Transmissions, Institution of Mechanical Engineers*, 9-14. Fundamental knowledge of gear noise—a survey.
187. D. R. HOUSER 1985 *OEM Conference Proceedings, Philadelphia.* Gear noise sources and their prediction using mathematical models.
188. D. R. HOUSER 1986 *Gear Noise Short Course Notes.* Ohio State University: Columbus, Ohio.