# Experiment in robotic self-repair: Section IV extension

Renzo Caballero (renzo.caballerorosas@kaust.edu.sa)
Eric Feron (eric.feron@kaust.edu.sa)

KAUST, KSA

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Please, instead of citing this document or the Git repository [1], cite our main article [].

## 1 Introduction

In this document, we explain in detail the mathematical modeling behind our article *Experiment in robotic self-repair, Section IV* []. The simulation contains three main blocks, as can be seen in Figure 1.

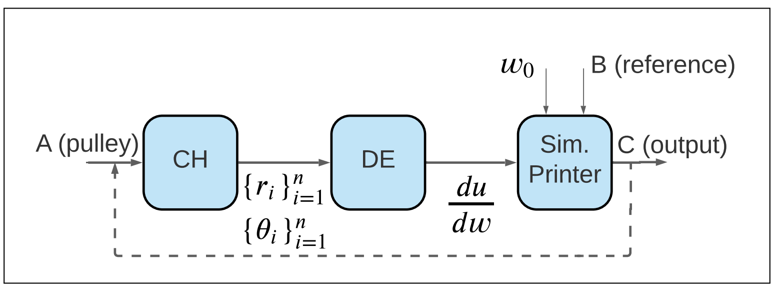


Figure 1: Block diagram of the simulation system. The three blocks are: left) convex hull finder, middle) $\frac{du}{dw}$ calculator, and right) printer simulator.

In the following sections, we explain in detail the mathematics behind each block and the data structure utilized during the simulations.

## 2 Notation

1. $C \subset \mathbb{R}^2$ is a convex subset in $\mathbb{R}^2$ where $(0,0)$ is an interior point of $C$.
2. $\{P_1, P_2, \ldots, P_n\}$ the set of vertices for a convex hull $C \subset \mathbb{R}^2$ with $n \in \mathbb{N}$ vertices.
3. $\mathcal{I}_n = \{1, 2, \ldots, n\}$ is the set of indexes from 1 to $n$.
4. $\{\triangle_1, \triangle_2, \ldots, \triangle_n\}$ the set of triangles in $\mathbb{R}^2$ such that the triangle $\triangle_i$ is conformed by the points $\{0, P_i, P_{i+1}\}$ for $i \in \mathcal{I}_{n-1}$ and $\triangle_n$ by $\{0, P_n, P_1\}$. Also, we have that:
   (a) for all non-equal $i, j \in \mathcal{I}_n$, $\triangle_i \cap \triangle_j$ has zero measure, and
   (b) $\bigcup_{i=1}^n \triangle_n = C$.
5. $D \in \mathcal{M}^{\ell\times\ell}$ is the discrete representation of a convex set $C \subset \mathbb{R}^2$, where $\mathcal{M}^{\ell\times\ell}$ is the space of $\ell \times \ell$ square matrices with binary 0-1 entries.

## 3 Convex hull (block CH)

As we explain in [], there are three main timing pulleys involved in the simulation:

1. The pulley controlling the $x$-axis movement for the nozzle. We call it $A$.
2. The pulley which we desire to print (equivalent to the .STL file in a real 3D printer). We call it $B$.
3. The printed pulley. We call it $C$.

Since the only irregularity is presented in the $x$-axis, it is enough to utilize two-dimensional bodies to represent the pulleys, in which case we utilize matrices. We utilize $2000 \times 2000$ binary $(0, 1)$-matrices to represent the pulleys' geometry, where 1 indicates the physical space that the pulley is occupying.

In Figure 2 we observe an example of a matrix representing a triangular pulley (left) and its convex hull (right). In the example, the matrix's dimension is $40 \times 40$ to help the visualization. The original input has .bmp format, it must be in black and withe, and the file must be square. Observe that the convex cover is conformed by a discrete number of non-intercepting triangles, all of which share the vertex $(0, 0)$. This property is essential to find an expression for $\frac{\mathrm{d}u}{\mathrm{d}w}$ under the proposed model.

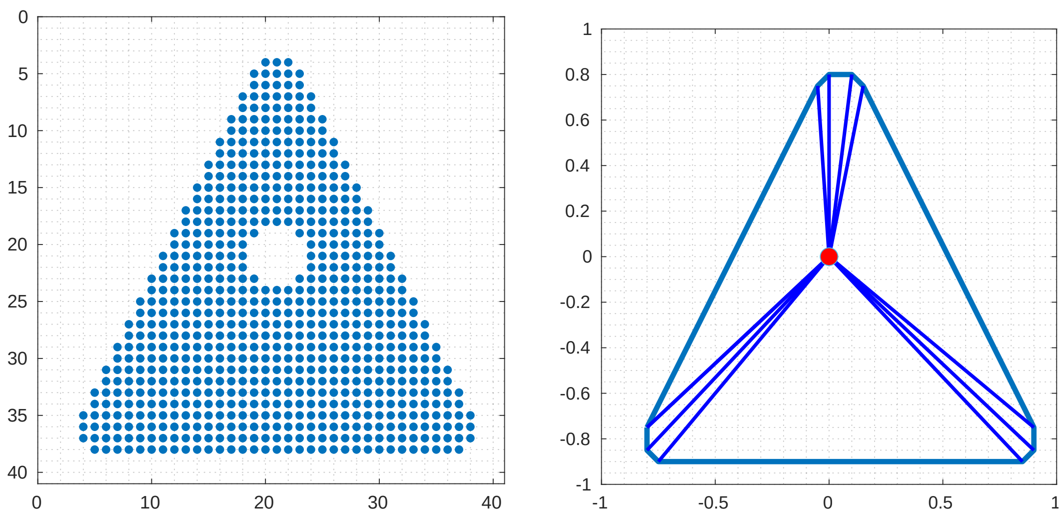


Figure 2: Example of input pulley $A$ (left), and its convex hull (right).

## 4 Algorithm to find $\frac{\mathrm{d}u}{\mathrm{d}w}$ (block DE)

In the following construction, we will only consider the clockwise rotation. We may relax this assumption in the future, but furder investigation is required to ensure accurate mathematical modeling.

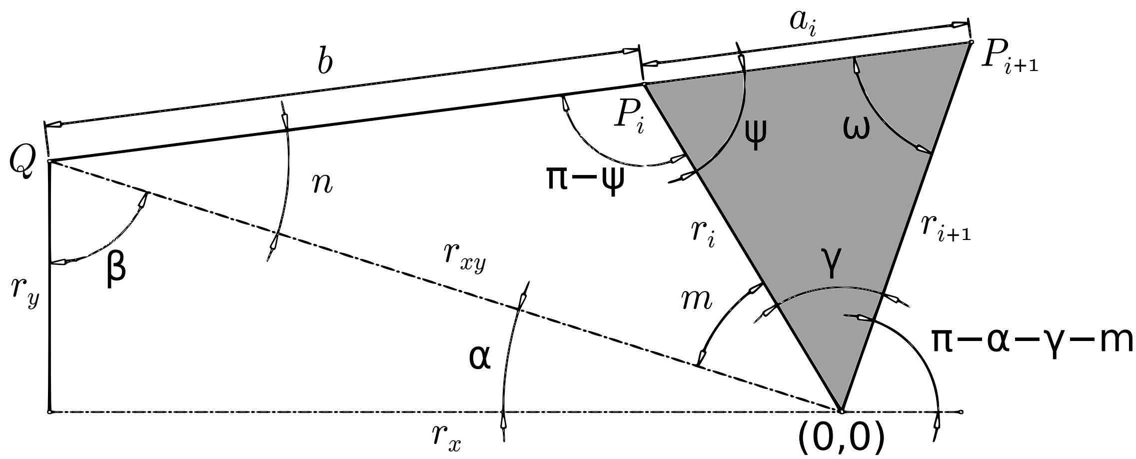


Figure 3: Alignment position for an arbitrary triangle from the convex hull.

Let $\{P_1, P_2, \ldots, P_n\}$ be the vertices for the convex set $C$ in $\mathbb{R}^2$ with $n \in \mathbb{N}$ vertices. Let $\{\triangle_1, \triangle_2, \ldots, \triangle_n\}$ the set of triangles which defines the convex hull $C$. Since the set $C$ is convex, at each time one and only one vertex is pulling from the timing belt. Also, only one vertex in controlling the belt until the next vertex takes its place []. As a consequence, we are interested in finding the set of transition angles $\{w_1, w_2, \ldots, w_n\}$, with the corresponding initial and final values for $m$: $\{m_1^-, m_2^-, \ldots, m_n^-\}$ and $\{m_1^+, m_2^+, \ldots, m_n^+\}$, i.e., $m \in [m_i^-, m_i^+)$ while the vertex $P_i$ is controlling the belt's movement.

In Figure 3, we can observe a picture of the instant moment of alignment for the triangle $\triangle_i$ with vertices $P_i$ and $P_{i+1}$, with the point $Q$ which is the contact belt-bearing. At the aliment position, the triangle $\triangle_{i+1}$ stops dominating the belt's movement, and the triangle $\triangle_i$ starts controlling it. In the same way, the point $P_{i+1}$, which was controlling the belt, is substituted by the point $P_i$. The value $r_y$ is the bearing's radius, and $r_x$ is the distance between bearing and pulley rotation points.

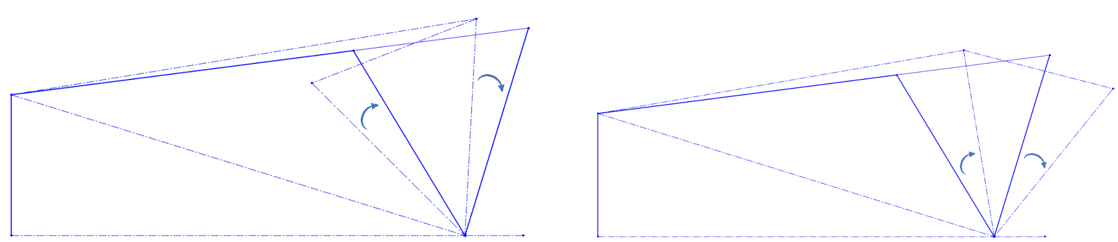

Figure 4: Pictures before (left) and after (right) the transition time.

In the Appendix we can find the trigonometric properties used to find the value of all variables during the transition time. From position in Figure 3, vertex $P_i$ is dominating the timing belt, and we want to know how a small rotation of the pulley affects the belt's position.

Since $\frac{\mathrm{d}m}{\mathrm{d}w} = 1$ and $\frac{\mathrm{d}u}{\mathrm{d}b} = 1$, we have that $\frac{\mathrm{d}u}{\mathrm{d}w} = \frac{\mathrm{d}u}{\mathrm{d}b}\frac{\mathrm{d}b}{\mathrm{d}m}\frac{\mathrm{d}m}{\mathrm{d}w} = \frac{\mathrm{d}u}{\mathrm{d}w}$. Then, for the triangle $\triangle_i$, we derive

$$b(m) = \sqrt{r_i^2 + r_{xy}^2 - 2r_i r_{xy}\cos(m)} \tag{1}$$

to find

$$\frac{\mathrm{d}b}{\mathrm{d}m}(m) = \frac{1}{2}\frac{2r_i r_{xy}\sin(m)}{\sqrt{r_i^2 + r_{xy}^2 - 2r_i r_{xy}\cos(m)}}, \tag{2}$$

where $m \in \left[m_i^-, m_i^+\right)$. Recall that $\sum_{i=1}^n \mu\left(\left[m_i^-, m_i^+\right)\right) = 2\pi$ where $\mu(\cdot)$ is the Lebesgue measure, so we can construct an analytic expression of $\frac{\mathrm{d}u}{\mathrm{d}w}(w)$ for the complete rotation by connecting the value of (2) for each segment $\left[m_i^-, m_i^+\right)$ with $i \in \mathcal{I}_n$.We can observe an example in Figure 5, where the pulley $A$ has a triangular shape, which defines a convex set with essentially three main triangles.

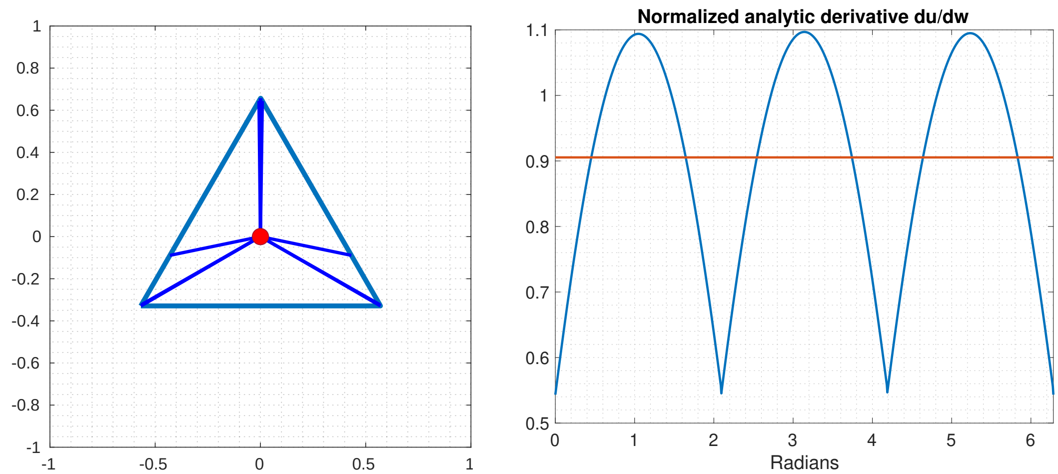


Figure 5: Convex set (left) and plot of $\frac{\mathrm{d}u}{\mathrm{d}w}$ for a triangular shape pulley. The plot is normalized with respect to the ideal value of $\frac{\mathrm{d}u}{\mathrm{d}w}$ from the ideal pulley, and also shows mean value.

# 5 Printing simulation (block Sim. Printer)

The printing simulation utilizes the function $\frac{\mathrm{d}u}{\mathrm{d}w}(w)$, which was constructed from pulley $A \in \mathcal{M}^{\ell\times\ell}$, and the discretized reference $B \in \mathcal{M}^{\ell\times\ell}$ to simulate an output $C \in \mathcal{M}^{\ell\times\ell}$. Since the step motor has 200 steps, the effect of using an non-uniform pulley is averaged over each step, i.e., given an initial position $w_0$ and the displacement in the timing pulley $\Delta w$ corresponding to a single step, the belt's change of position after the step is

$$\Delta u = \int_{w_0}^{w_0+\Delta w} \frac{\mathrm{d}u}{\mathrm{d}w}(s)\,\mathrm{d}s. \tag{3}$$

# 6 Example

Assume we use an oval shaped pulley as timing pulley $A$. We construct the convex hull with its respective triangular discretization, and the instant displacement $\frac{\mathrm{d}u}{\mathrm{d}w}(w)$ for all $w \in [0, 2\pi)$. In Figure 6, we can see both, convex hull (left) and derivative (right).

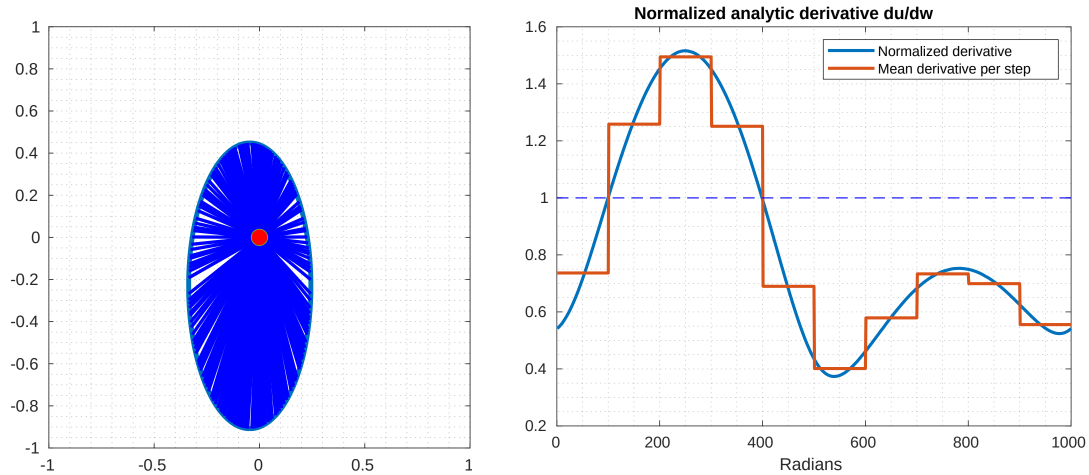


Figure 6: Convex set (left) and plot of $\frac{du}{dw}$ (right) for a oval shape pulley. The plot is normalized with respect to the ideal pulley derivative, it also shows mean values per step.

Once we have the instant displacement $\frac{du}{dw}$, we can simulate the printing process described in Section 5. In this example, we will assume the step motor has only ten steps, in order to create more intuitive graphics. Recall Equation (3), where we average the derivative over the step size. We observe this average in Figure 6-(right). In Figure 7, we observe the reference pulley $B$ on the left, and the printed pulley $C$ on the right. Notice how each step either expands or contracts the reference's design. Compare the output pulley from Figure 7-(right) with the instant derivative from figure 6-(right).

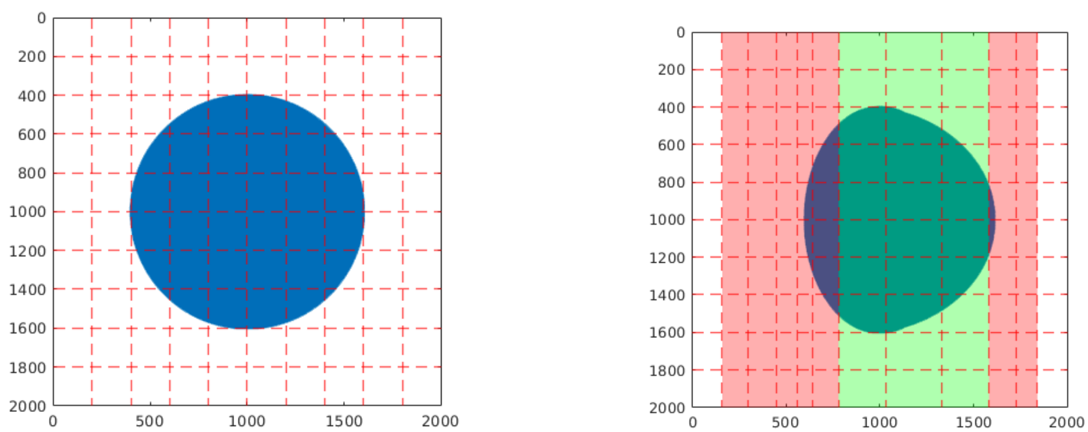

Figure 7: Reference pulley $B$ on the left, and printed pulley $C$ on the right. Both corresponding to example from Section 6.

# 7 Appendix

$r_x, r_y, r_i, r_{i+1}, a_i$ are well defined, and they are considered data provided by the block CH (see Section 3). We assume aliment between $Q, P_i$ and $P_{i+1}$. The following deduction reflects what is in the code:

1. Calculate fixed parameters: $r_{xy} = \sqrt{r_y^2 + r_x^2}$, $\alpha = \arctan\left(\frac{r_y}{r_x}\right)$, and $\beta = \pi - \alpha - \pi/2$.
2. Calculate triangle $\triangle_i$ angles: $\gamma_i = \arccos\left(\frac{r_i^2 + r_{i+1}^2 - a_i^2}{2 r_i r_{i+1}}\right)$, $\psi_i = \arccos\left(\frac{r_i^2 + a_i^2 - r_{i+1}^2}{2 r_i a_i}\right)$, and $\omega_i = \pi - \gamma_i - \psi_i$.

3. Find value of $b$: $n = \arcsin\left(\frac{r_i}{r_{xy}} \sin(\pi - \psi_i)\right)$, $m = \psi_i - n$, and $b = r_i \frac{\sin(m)}{\sin(n)}$.

4. Find transition angles: $m_i^+ = m_i^- + \Delta m_i$, where $\Delta m_i = m_i^- + \gamma_i - m_{i+1}^-$ for $i \in \mathcal{I}_{n-1}$, and $\Delta m_n = m_n^- + \gamma_n - m_1^-$.

# References


[1] Experiment in robotic self-repair: Git repository, `https://github.com/RenzoCab/Simulation-Gear-Belt`.