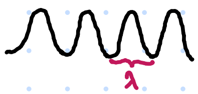
Lecture 3: The Wave Function

* Aside: Polarizing filters a physical example of our boxes in lecture 1 and interference effects (lecture 2)
  - see 3Blue1Brown YouTube video on this
  - Another example of our boxes: the Stern-Gerlach experiment
  - The polarization example is a canaconical one in QM
* Comment on lecture 2:
  - we know a given EM wave will have a wavelength λ [m] and angular frequency ω [rad/s]



  - The photoelectric effect tells us that in addition to having a specific λ and ω, a given EM wave carries an energy E proportional to ω:

    $E \sim \hbar\omega = h\nu$ where h = Planck's constant and $\hbar = \frac{h}{2\pi}$ and $\omega = 2\pi\nu$ and ν = frequency [Hz]
    - in practice, $\hbar$ = reduced Planck constant
    - Conclusion: the energy associated w/ an EM wave is linearly proportional to its frequency ν by a factor of h
  - The photoelectric effect also says that the momentum p associated w/ an EM wave is inversely proportional to its λ by a factor of h:

    $p = \hbar k = \frac{h}{\lambda}$ where $k = \frac{2\pi}{\lambda}$ = wavenumber [rad/m]

  $\Rightarrow$ Basic relations for light:

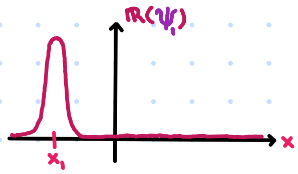
  $\boxed{E = \hbar\omega}$ and $\boxed{p = \hbar k}$ deBroglie relations *Important for next few lectures

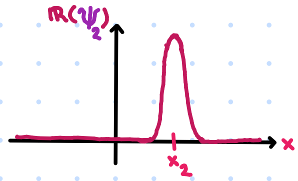
  $\Rightarrow$ the claim of the photoelectric effect: for a given EM wave (which we assume has a frequency/wavelength), both its associated energy and momentum are quantized

  $\Rightarrow$ the wave-like properties are coupled to the corpuscule-like properties via energy-frequency and momentum-wavelength relations

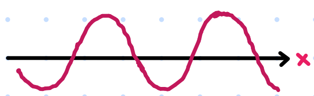
* Shortly after Einstien proposed special relativity, a young French guy named deBroglie proposed this wave-particle relation holds true for all objects
  - Note: he didn't have any experimental evidence at all (example of physical intuition turning out to be right
  - Moreover, he said any object moving w/ a momentum p has a wavelength λ associated w/ it s.t. $p = \frac{h}{\lambda}$ AND every object that has energy E has a frequency ν associated w/ it s.t. $E = h\nu$
  - The Davisson-Germer experiment (lecture 2) was experimental confirmation of this prediction
    - Specifically, shooting out discrete electrons w/ definite E at a crystal resulted in wave-like behavior

* Postulate 1: $\psi(x)$ completely defines the state of a quantum object where ψ is a complex function and x = position
  - Example: Consider the following wavefunctions:





★ Importantly, these plots only depict the real part of our wavefunction



★ Note: any arbitrarily complicated function can be depicted in $\mathbb{R}(\psi)$ vs. x plot
$\Rightarrow$ need a way to specify functions that aren't "stupid"

* Postulate 2: The wavefunction from postulate 1 can be interpreted as follows

$$\mathbb{P}(x) = |\psi(x)|^2 = \text{the probability of measuring a particle's position at position } x$$

- Note: Here, $|\vec{y}|$ denotes the norm of a vector $\vec{y}$
  ~ For a scalar $\in \mathbb{C}$, this specifies to the absolute value
- Probability density for finding our object of interest somewhere btwn x and $x+dx$:

$$\mathbb{P}(x, x+dx) = \mathbb{P}(x)dx = |\psi(x)|^2 dx$$

- For this to make sense, we have to make sure $\psi(x)$ is properly normalized
  - We know total probabilities sum to 1
    $\Rightarrow$ Lets assume the probability that our object is located somewhere is 1
      $\Rightarrow$ i.e., the integral of $\mathbb{P}(x)$ over all possible x-values must be equal to 1:

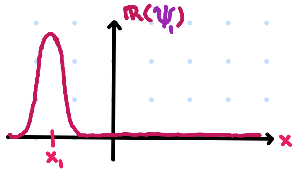
$$\int_D \mathbb{P}(x)\,dx = \int_D |\psi(x)|^2 dx = 1 \quad \text{where } D \text{ denotes the domain of } x$$

★ Note: $\mathbb{P}(x, x+dx)$ denotes the probability density along the interval $[x, x+dx]$ and $\mathbb{P}(x)$ denotes the probability for a given x-value

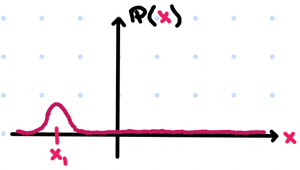
* Aside: the dimensions of the wavefunction are $[1/\sqrt{L}]$ where L denotes unit length
  $\Rightarrow$ this causes our integral expressions to evaluate to a unitless quantity that corresponds w/ probability
  - An essential thing for succeeding in this field of study is dimensional analysis
    ~ allows us to sanity check calculations
    ~ sometimes, dimensional analysis will take away the need to do calculations in the first place
      $\Rightarrow$ can show that there is only one possible way to build something using a particular dimension (e.g., length)
  - Always ask "what are the dimensions of all the objects in my system?

*Q: Using our def'n of a wavefunction in postulate 2, how can we physically interpret knowing a wavefunction?

- Lets compare the probability distribution to the following real part of a wave function:

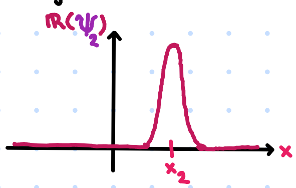


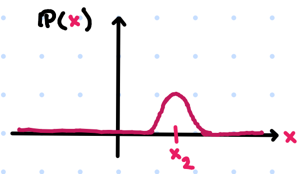
=>



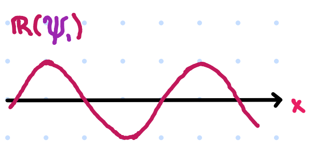
- For this wavefunction, we have high confidence that our object of interest is located at position x_1
  => $x \sim x_1$ and Δx is small

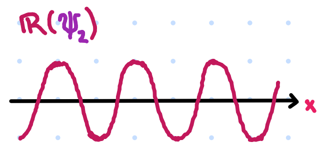
- Simillarly, the wavefunction depicted below says... $x \sim x_2$ and Δx is small:





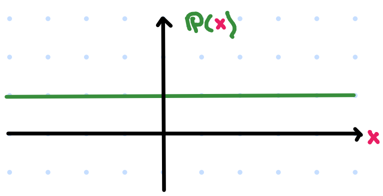
- Example 2: Draw out the $\mathbb{P}(x)$ vs. x graph for the following wavefunctions:
  - Let $\psi_1(x) = e^{ik_1 x}$ and $\psi_2(x) = e^{ik_2 x}$





- Recall: the absolute value of a complex number squared is 1, i.e., $|e^{i\alpha}|^2 = 1$ where $\alpha \in \mathbb{R}$
  => For $\beta \in \mathbb{C}$, we can say $|\beta|^2 = \beta^* \beta$ where β^* is the complex conjugate associated w/ β

- The corresponding $\mathbb{P}(x)$ for both of these wave functions ends up being $1 \;\forall\; x$:



here, we get a uniform probability distribution
=> $x \sim ??$, Δx is large

* Importantly, we can't write out a probability distribution as defined in postulate 2 if our wavefunction is a multivalued/parametric (i.e., "stupid") functions
  => our wavefunction must be single-valued

- Note: the wavefunctions ψ_1 and ψ_2 are sinusoidal wavefunctions
  - Since they give a uniformily distributed $\mathbb{P}(x)$, they give us no info. about the position of an object of interest
  - However, the deBroglie relations tell us that any object/particle has a wavelength $\lambda = \frac{2\pi}{k}$ that is related to its momentum and a frequency $\nu = \frac{\omega}{2\pi}$ that is related to its energy E: $p = \hbar k$ and $E = \hbar\omega$

=> although ψ_1 and ψ_2 don't give any info. about position, they are both periodic functions w/ definite wavelengths $\lambda_1 = \frac{2\pi}{k_1}$ and $\lambda_2 = 2\pi/k_2$

=> if we measure the momentum of a particle described by ψ_1 or ψ_2, we will have high confidence due to its well-defined wavelength

=> $p_i \sim \hbar k_i$ and Δp_i is small

*Note: our diagrams tell us $\lambda_1 > \lambda_2$ => $k_1 < k_2$ => $p_1 < p_2$

* Aside:
  - we still need an official definition for Δx and Δp
  - Any given photon can exist in a superposition of different frequencies
    - ⇒ this extends to all particles/objects

* One of the implications of the deBroglie relations is any particle has some $E=\hbar\omega$ and $p=\hbar k$ associated w/ it
  - ⇒ the wavefunction that satisfies this is the plane wave of the form...

$$\psi(\vec{x},t)=e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

  - However, until otherwise specified, we are going to focus on example that occur in 1D space
    - ⇒ our equation above simplifies to...

$$\boxed{\psi(x,t)=e^{i(kx-\omega t)}}$$

plane wave satisfying the deBroglie relations

  - ⇒ only issue is not all wave functions are plane waves

* Postulate 3: Given two possible configurations/states of a quantum system corresponding to two distinct wavefunctions, $\psi_1(x)$ and $\psi_2(x)$, the system can also be in a superposition of $\psi_1(x)$ and $\psi_2(x)$:

$$\boxed{\psi(x)=\alpha\psi_1(x)+\beta\psi_2(x)}$$

where $\alpha,\beta\in\mathbb{C}$ and are subject to the normalization condition

  - Most important postulate ***
    - ⇒ encompasses all of QM; "the beating soul" of QM
  - In other words, for any two possible configurations of the system, there is also an allowed configuration of the system corresponding to being in an arbitrary superposition of them
    - e.g., if an e^- can be hard and it can also be soft, it can also be in an arbitrary superposition of being hard and soft
      - ~ "hard" corresponds to some particular wavefunction and "soft" corresponds to another particular wavefunction
      - ~ the "hard/soft" superposition corresponds to a different wavefunction which is a linear combination of them

* Aside: Alternative way to denote a probability distribution:

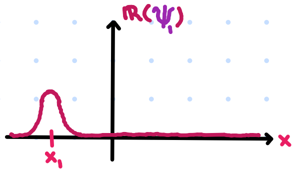
$$\mathbb{P}(x)=\frac{|\psi(x)|^2}{\int_0 |\psi|^2 dx}$$

⇒ if our wavefunction is properly normalized, the denominator evaluates to 1

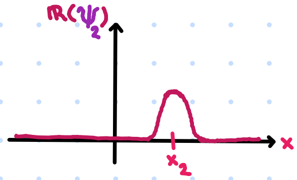
⇒ if our wavefunction isn't properly normalized, our $\mathbb{P}(x)$ distribution is automatically normalized by the denominator

* Sometimes its easier to normalize our wavefunction first and then use the $\mathbb{P}(x)=|\psi(x)|^2$ identity and other times its easier to use the above identity (sometimes normalizing first is too much work)

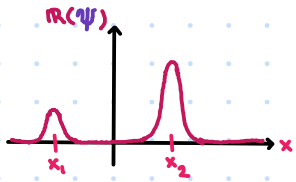
* Example: Lets evaluate the superposition of the following two wavefunctions:



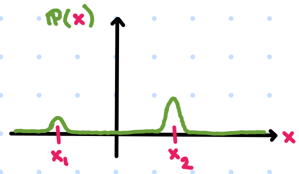
and



- Superposition of ψ_1 and ψ_2: $\psi(x) = \alpha\psi_1(x) + \beta\psi_2(x)$
  - Depending on our α, β one of our possible superposition diagrams can have the following form:



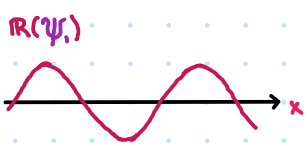
$\Rightarrow$



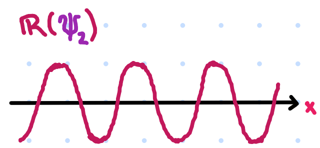
$\Rightarrow$ if we measured x for many samples, our avg. value of x should end up somewhere btwn x_1 and x_2

$\Rightarrow$ our uncertainty in x is of order $x_1 - x_2$: $\Delta x \sim x_1 - x_2$

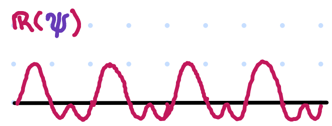
- Now lets consider these two wavefunctions:



+



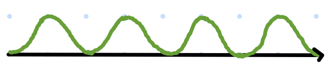
=



- Here, our ψ is comprised of two definite wavelengths, λ_1 and λ_2

  $\Rightarrow \psi \sim e^{ik_1x} + e^{ik_2x}$

- Probability distribution associated w/ ψ:



$$\mathbb{P}(x) = |\alpha\psi_1(x) + \beta\psi_2(x)|^2 = \underbrace{|\alpha|^2|\psi_1|^2}_{=|\alpha\psi_1|^2 = \mathbb{P}_1(x)} + \underbrace{\alpha^*\psi_1^*\beta\psi_2 + \alpha\psi_1\beta^*\psi_2^*}_{\text{Interference Terms}} + \underbrace{|\beta|^2|\psi_2|}_{=|\beta\psi_2|^2 = \mathbb{P}_2(x)}$$

$\Rightarrow \mathbb{P}(x) = \mathbb{P}_1(x) + \mathbb{P}_2(x) +$ Interference Terms

- Conclusion: The superposition principle together w/ the interpretation of the probability distribution as the norm squared of the wavefunction gives us a correction wrt the way we classically add probabilities $\Rightarrow$ includes interference terms
  - Note: $\mathbb{P}_1(x)$ and $\mathbb{P}_2(x)$ are the prod's of squared complex numbers $\Rightarrow \mathbb{P}_1, \mathbb{P}_2 \geq 0$ (i.e., nonegative real #'s)
  - Moreover, our interference terms are complex conjugates of each other

    $\Rightarrow$ the sum of our interference terms will be real, but not necessarily positive
  - Recall: Bell's inequality tells us that the probability that a system is simultaneously in two independent states (w/ probability $\mathbb{P}_1$ of being in our first state of interest and probability $\mathbb{P}_2$ of being in our second state of interest) is $\mathbb{P} = \mathbb{P}_1 + \mathbb{P}_2$

    ^ However, in QM, we add our wavefunctions and our associated probability is the norm squared of our summed wave function

    ★ wave functions add, but NOT our probabilities $\Rightarrow$ this underlies all the interference effects we've seen and is at the heart of the rest of QM

- Importantly, the last probability distribution we drew gives us some info. about the state of our system (Recall: page 3 shows a uniform distribution for $\mathbb{P}_1$ and $\mathbb{P}_2$
  - we still don't have enough info. to tell us the definite location (i.e., state) of our particle, but we at least have some info

    $\Rightarrow x \sim$ some info and $\Delta x \sim$ huge

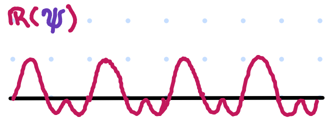
* Recall: $|e^{ia} + e^{ib}|^2 = |e^{ia}(1+e^{i(b-a)})|^2$

- Also, the norm squared of a product of different terms is equal to the product of the norm squared of each term:

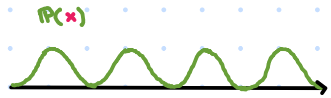
$\Rightarrow$ $|e^{ia}(1+e^{i(b-a)})|^2 = \underbrace{|e^{ia}|^2}_{=1} |1+e^{i(b-a)}|^2 = |1+e^{i(b-a)}|^2$

$\Rightarrow$ the $\mathbb{R}(\psi)$ plot associated w/ $\mathbb{P} = |1+e^{i(b-a)}|^2$ is a cosine function whose output are all nonnegative real numbers

e.g., example from last page:

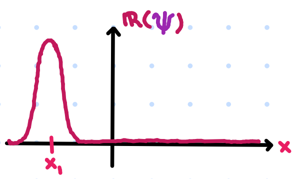


$\Rightarrow$



* Tool this class uses: Fourier analysis w/ Mathematica

- Used to demonstrate an increase in localization as we increase the number of wave functions we are superimposing
  - $\Rightarrow$ i.e., our Δx decreases (caused by an interference btwn momenta described by individual wave functions
- However, since we are superimposing wave functions w/ different momenta (i.e., a superposition of different momenta)
  - $\Rightarrow$ we lose info. about the object's momentum $\Rightarrow$ Δp increases
- Note: we can think of a wave function whose real components are purely sinusoidal as being a superposition of many different positions
- This gives rise to the uncertainty relation
  - Q: why does the following wave function have a large uncertainty in momentum?



- A: The ψ associated w/ this graph doesn't have a definite λ

$\Rightarrow$