Undergraduate Seminar Presentation

Introduction to Continuous Differential Equations

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Dynamical Systems

"Dynamics is primarily the study of the time-evolutionary process and the corresponding system of equations is known as dynamical system."

Continuous Dynamical Processes

A continuous dynamical process are represented by differential equations. For example:

$$\frac{dx}{dt} = \dot{x} = f(x, t),$$

where x is the position, t is the time, and f is generally non-linear.

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Continuous dynamical systems can be **autonomous** or **non-autonomous**.

Linear and Non-Linear Dynamical Systems

Definition (Linear System)

Function which is describing the system behavior must satisfy the following properties

1. Additivity: $f(x+y) = f(x) + f(y)$
2. Homogenity: $f(\alpha x) = \alpha f(x)$

Definition (Non-Linear System)

The function describing the system behavior is non-linear. It does not satisfy the previously specified properties.

Autonomous and Non-autonomous

Definition

If a dynamical system is explicitly time **independent**, then it is called *autonomous*.

$$\dot{x} = 12 \cdot x(t)$$

Definition

If a dynamical system is explicitly time **dependent**, then it is called *non-autonomous*.

$$\dot{x} = t \cdot x(t)$$

Lotka-Volterra

A famous continous dynamical system is the population model of two competing species formulated by *Alfred J. Lotka* in 1910 and *Vito Volterra* in 1926.

$$\begin{cases} \dot{x} = \alpha x - \beta xy = x(\alpha - \beta y) \\ \dot{y} = -\gamma y + \delta xy = -y(\gamma - \delta x) \end{cases}, \tag{1}$$

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This system is **Autonomous**.

Linear Oscillator with External Time-Dependant Force

Observe a dynamical system

$$\ddot{x} + \alpha\dot{x} + \beta x = f \cdot \cos(\omega t),$$

where f is the frequency of the driving force, and ω is the amplitude.

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It is obvious that this system is time dependent; therefore, it is **non-autonomous**.

Flows

Definition (Flow)

The time evolutionary process can be described as a flow of a vector field. It is frequently used for discussing the dynamics as a whole rather than the evolution of a system at a particular point.

A given solution $\underset{\sim}{x}(t)$ of a system $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$, which satisfies the condition $\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0$ gives both the past $(t < t_0)$ and the future $(t > t_0)$ evolutions of the system.

Flows Contd.

Mathematically, the flow of a dynamical system is defined by

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Where $\phi_t(\underset{\sim}{x}) = \phi(t, \underset{\sim}{x})$ is a smooth vector function of $x \in U \subseteq \mathbb{R}^n$ and $t \in I \subseteq \mathbb{R}$ that satisfies

$$\frac{d}{dt}\phi_t(\underset{\sim}{x}) = \underset{\sim}{f}(\phi_t(\underset{\sim}{x}))$$

for all t such that the solution through $\underset{\sim}{x}$ exists and $\phi(0, \underset{\sim}{x}) = x$.

Flows Contd.

Flows must also satisfy the properties

1. $\phi_0 = I_d$.
2. $\phi_{t+s} = \phi_t \circ \phi_s$.

Flows Contd.

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Some flows may also satisfy the property,

$$\phi(t+s, \underset{\sim}{x}) = \phi(t, \phi(s, \underset{\sim}{x})) = \phi(s, \phi(t, \underset{\sim}{x})) = \phi(s+t, \underset{\sim}{x}).$$

Flows in $\mathbb{R}$

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1. Imagine a fluid which is flowing along the real line with local velocity $f(x)$.
2. The fluid is called the **phase fluid**.
3. The real line is called the **phase line**.

Flows in $\mathbb{R}$ Contd.

For solution of the system $\dot{x} = f(x)$ starting from an arbitrary initial position x_0, we place an imaginary particle, called a **phase point**, at x_0 and watch how it moves along with the flow in phase line along the changing time t.

Flows in $\mathbb{R}$ Contd.

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As time goes on, the phase point (x, t) in the one dimensional system $\dot{x} = f(x)$ with $x(0) = x_0$ moves along the x-axis according to some function $\phi(t, x_0)$. The function $\phi(t, x_0)$ is called the **trajectory** for a given initial state x_0, and the set $\{\phi(t, x_0) | t \in I \subseteq \mathbb{R}\}$ is the orbit of $x_0 \in \mathbb{R}$.

Flows in $\mathbb{R}^2$

Consider a two-dimensional system represented by

$$\dot{x} = f(x, y) \text{ and } \dot{y} = g(x, y) \text{ where } (x, y) \in \mathbb{R}^2.$$

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Definition

The succession of states given parametrically by $x = x(t)$ and $y = y(t)$ trace out the curve through some initial point $P(x(t_0), y(t_0))$ is called a **phase path**.

Flows in $\mathbb{R}^n$

We can define an autonomous system representing n ordinary differential equations as

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \dots, x_n) \\ \dot{x}_2 = f_2(x_1, x_2, \dots, x_n) \\ . \\ . \\ . \\ \dot{x}_n = f_n(x_1, x_2, \dots, x_n) \end{cases} \tag{2}$$

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For ease of comprehension we rewrite the system as $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$, where $\underset{\sim}{x} = (x_1, x_2, \dots, x_n)$ and $\underset{\sim}{f} = (f_1, f_2, \dots, f_n)$.

Flows in $\mathbb{R}^n$ Contd.

The solution of the system with the initial condition $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ can be thought as a continous curve in the phase space $\mathbb{R}^n$ parameterized by time $t \in I \subseteq \mathbb{R}$.

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Thus, the set of all states of the evolutionary process is represented by an n-valued vector field in $\mathbb{R}^n$.

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The solutions of the system with different initial conditions describe a family of phase curves in the phase space, called the **phase portrait** of the system.

Evolution

Let's consider a system $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(x)$, $\underset{\sim}{x} \in \mathbb{R}^n$ with initial condition $\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0$.

Let $E \subset \mathbb{R}^n$ be an open set and $\underset{\sim}{f} \in C^1(E)$.

For $\underset{\sim}{x}_0 \in E$, let $\phi(t, \underset{\sim}{x}_0)$ be a solution of the above system on the maximum interval of existence $I(\underset{\sim}{x}_0) \subset \mathbb{R}$.

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Definition

The mapping $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\phi_t(\underset{\sim}{x}_0) = \phi(t, \underset{\sim}{x}_0)$ is known as the **evolution operator** of the system.

Evolution Contd.

The mappings ϕ_t for both linear and nonlinear systems satisfy the following properties:

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(iii) $\phi_t(\phi_{-t}(\underset{\sim}{x})) = \phi_{-t}(\phi_t(\underset{\sim}{x})) = \underset{\sim}{x},\ \forall t \in \mathbb{R}$

Evolution Contd.

A dynamical system may be viewed as a group of nonlinear / linear operators evolving as $\{\phi_t(\underset{\sim}{x}), t \in \mathbb{R}, \underset{\sim}{x} \in \mathbb{R}^n\}$ under composition.

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In some cases a flow may also satisfy the commutative property $\phi_t \phi_s = \phi_s \phi_t$.

Fixed Points

Definition

A point is a fixed point of the flow generated by an autonomous system $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$, $\underset{\sim}{x} \in \mathbb{R}^n$, if and only if $\phi(t, \underset{\sim}{x}) = \underset{\sim}{x}$ for all $t \in \mathbb{R}$.

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Informally put, at a fixed point, x does not change as time increases, i.e. $\frac{dx}{dt} = 0$

Fixed Points Contd.

A fixed point is also known as a **critical point** or a **eqilibrium point** or a **stationary point**. It is also often called a **stagnation point** with respect to the flow ϕ_t in $\mathbb{R}^n$.

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Example 2: $\dot{x} = x$ has only one fixed point.

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Example 3: $\dot{x} = x^2 - 1$ has two fixed points.

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Example 4: $\dot{x} = \sin(x)$ has infinitely many fixed points.

Classification of One Dimensional Dynamical Systems

By taking the derivative of f at a fixed point, we can classify the system behavior as one of the following:

Definition (Stable Fixed Point)

A fixed point $\underset{\sim}{x}$ is stable if for all starting values x_0 near $\underset{\sim}{x}$, the system $f(t, x)$ converges to $\underset{\sim}{x}$ as $t \to \infty$.

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A fixed point $\underset{\sim}{x}$ is unstable if for all starting values x_0 near $\underset{\sim}{x}$, the system $f(t, x)$ diverges far away from $\underset{\sim}{x}$ as $t \to \infty$.

A simple rule to identify the type of fixed point is stable or not is to look at the sign of the derivative of the flow at the fixed point.
If $f'(\underset{\sim}{x}) < 0$ then the fixed point is a stable fixed point.
If $f'(\underset{\sim}{x}) > 0$ then the fixed point is an unstable fixed point.

Fixed Point Analysis Example 1

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$$f(x) = \dot{x} = x(1 - x)$$

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Setting the right-hand side of the differential equation to zero, we get:

$$f(x) = \dot{x} = x(1 - x) = 0 \Longrightarrow x = 0 \text{ or } 1$$

Fixed Point Analysis Example 1 Contd.

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Using the notion of the flow, we may now discuss the stability of the system around the fixed points. We denote the direction of the flow with arrows. A flow to the right would be $\rightarrow$, when $\dot{x} > 0$. A flow to the left would be $\leftarrow$, when $\dot{x} < 0$. We mark a stable fixed point by a solid circle and an unstable fixed point by a hollow circle.

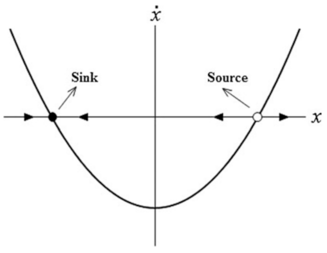

Figure 1: Graphical representation of $f(x) = x(1-x)$

Fixed Point Analysis Example 2: Bacteria Growth Model

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Thus, $\frac{dx}{dt} = bx - px^2$.

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$$\underset{\sim}{x}(b - p\underset{\sim}{x}) = 0$$

Thus we can see that there are two fixed points:

$$\underset{\sim}{x}_0 = 0 \text{ is } \textbf{stable},$$

$$\underset{\sim}{x}_1 = \frac{b}{p} \text{ is } \textbf{unstable}.$$

Bacteria Growth Model Solution Graph

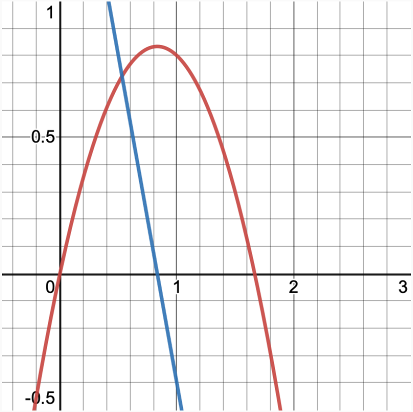

Figure 2: Bacteria Growth Model ($b = 2, p = 1.2$)

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We can see that when $x > 0$, $\dot{x} > 0$ and when $x < 0$, $\dot{x} < 0$.
Therefore, the fixed point $x = 0$ is unstable.

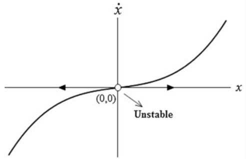

Figure 3: Graphical representation of $f(x) = x + x^3$

Fixed Point Analysis Example 4

Consider the following system:

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Fixed Point Analysis Example 4

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We can see that when $x < -1$, $\dot{x} > 0$. When $-1 < x < 0$, $\dot{x} < 0$. When $0 < x < 1$, $\dot{x} > 0$. When $x > 1$, $\dot{x} < 0$.

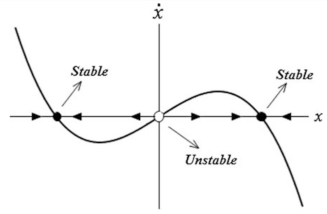

Figure 4: Graphical representation of $f(x) = x - x^3$

Summary

In this presentation, we have

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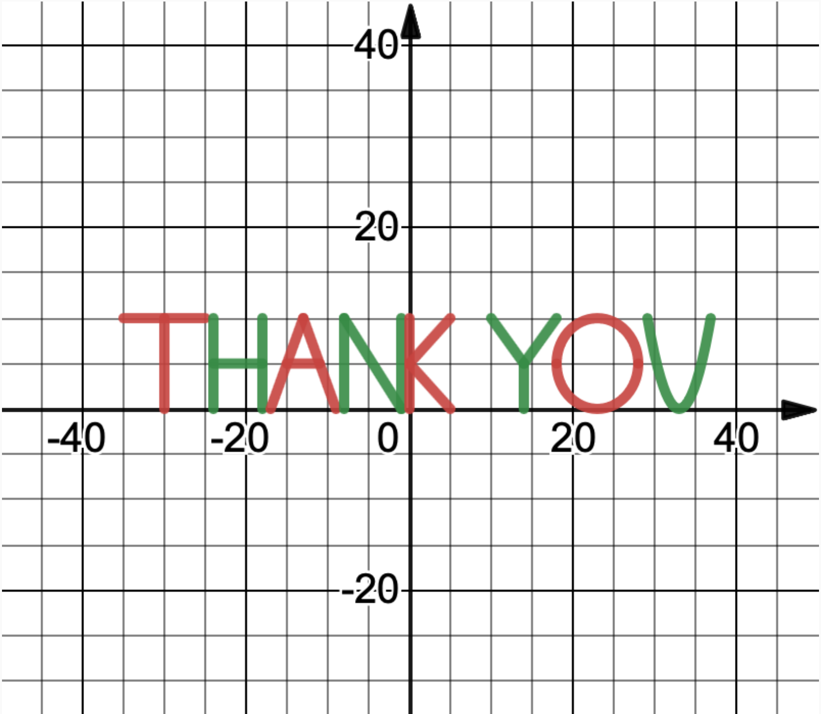
- covered several illustrative examples for above listed concepts.

Work Cited

G.C. Layek, An Introduction to Dynamical Systems and Chaos, Springer, 2015.

R. Devaney and L. Devaney, An Intrduction To Chaotic Dynamical Systems, Second Edition, Avalon Publishing, 1989.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THANK YOU