Undergraduate Seminar Presentation

Number of Primes

Karen Arzumanyan

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The number of primes until x, the function $\pi(x)$

Definition

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$$\pi(x) = \sum_{p \leq x} 1 = \#\{p \leq x | p \text{ is prime.}\}$$

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This was proven by *J. Hadamard* and *C. de la Vallé Poussin* in 1896.
The proof is too complicated and is beyond the scope of this presentation.

Chebyshev's Theorem

Theorem (Chebyshev)

There exist constants $0 < c_1 < 1 < c_2$ *such that for all* $x \geq 1$ *we have that*

$$c_1 \frac{x}{\log x} \leq \pi(x) \leq c_2 \frac{x}{\log x}$$

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We will prove the theorem for weaker bounds.

Sum of logs

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Therefore,

$$\frac{\theta(x)}{x} \leq \frac{\pi(x)}{\left(\frac{x}{\log x}\right)}$$

Small ϵ

Suppose we have $0 < \epsilon < 1$, then

$$\theta(x) = \sum_{p \le x} \log p = \sum_{p \le x^{1-\epsilon}} \log p + \sum_{x^{1-\epsilon} < p \le x} \log p$$

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If we divide both sides by x, we will get

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Remark: Recall that we also had

$$\frac{\theta(x)}{x} \leq \frac{\pi(x)}{\left(\frac{x}{\log x}\right)}$$

Since the term $\frac{\log x}{x^{\epsilon}}$ vanishes as $x \to \infty$, we can go back and forth between asymptotic formulas of $\pi(x)$ and $\theta(x)$.

Lemma for $\theta(x)$

Lemma

For all $x \geq 1$ we have,

$$\theta(x) < (4 \log 2) x$$

Lemma for $\theta(x)$

Proof.

Let's consider the binomial coefficient

$$\binom{2n}{n} = \frac{(2n)(2n-1)\dots(n+1)}{n!} < 2^{2n}$$

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Therefore,

$$\prod_{n<p\leq 2n} p \leq \binom{2n}{n} < 2^{2n}$$

Proof for Lemma for $\theta(x)$

Proof.

Hence, taking the log of both sides

$$\sum_{n<p\leq 2n} \log p < \log 2^{2n}$$

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$$\theta(2n) - \theta(n) < (2\log 2)n$$

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In particular, taking $n = 2^{k-1}$, we get that

$$\theta(2^k) - \theta(2^{k-1}) < (2\log 2)2^{k-1}$$

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$$= \Big(\theta(2) - \theta(1)\Big) + \Big(\theta(4) - \theta(2)\Big) + \ldots + \Big(\theta(2^m) - \theta(2^{m-1})\Big)$$

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$$= \theta(2^m) - \theta(1) = \theta(2^m)$$

Thus, together with the inequality from before

$$\theta(2^m) \leq (2 \log 2) \sum_{k=1}^{m} 2^{k-1} < (2 \log 2) 2^m$$

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Proof.

For an arbitrary $x \geq 1$, choose an integer m such that $2^{m-1} \leq x < 2^m$.

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For an arbitrary $x \geq 1$, choose an integer m such that $2^{m-1} \leq x < 2^m$. Then,

$$\theta(x) \leq \theta(2^m) < (2\log 2)2^m \leq (4\log 2)x$$

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Proof.

For an arbitrary $x \geq 1$, choose an integer m such that $2^{m-1} \leq x < 2^m$. Then,

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that is

$$\theta(x) \leq (4\log 2)x$$

Upper bound of Chebyshev

Proof.

We already know that,

$$(4 \log 2)x \geq \theta(x) \geq \sum_{\sqrt{x} < p \leq x} \log p$$

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We already know that,

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$$\geq \log(\sqrt{x})(\pi(x) - \pi(\sqrt{x})) \geq \frac{1}{2}\log(x)(\pi(x) - \pi(\sqrt{x}))$$

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$$\geq \frac{1}{2}\log(x)\pi(x) - \frac{1}{2}\log(x)\sqrt{x}$$

where the last inequality follows from $\pi(\sqrt{x}) \leq \sqrt{x}$.

Equivalently,

$$\frac{1}{2}\log(x)\pi(x) \leq (4\log 2)x + \frac{1}{2}\log(x)\sqrt{x}$$

Upper bound of Chebyshev

Proof.

Dividing both sides of the last inequality by $\frac{1}{2}\log(x)$, we get

$$\pi(x) \leq (8\log 2)\frac{x}{\log x} + \sqrt{x} \leq (8\log 2 + 2)\frac{x}{\log x}$$

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$$\pi(x) \leq (8\log 2)\frac{x}{\log x} + \sqrt{x} \leq (8\log 2 + 2)\frac{x}{\log x}$$

where we used the simple inequality $\sqrt{x} < \frac{2x}{\log x}$ for all $x \geq 2$.

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where we used the simple inequality $\sqrt{x} < \frac{2x}{\log x}$ for all $x \geq 2$. Therefore

$$c_2 = 8\log 2 + 2 \approx 4.41$$

Proof of the Lower Bound

For the lower bound, let us consider the following binomial coefficient inequality

$$\binom{2n}{n} = \prod_{1 \le k \le n} \frac{k+n}{k} \ge 2^n$$

Lower bound of Chebyshev

Proof of the Lower Bound

For the lower bound, let us consider the following binomial coefficient inequality

$$\binom{2n}{n} = \prod_{1 \leq k \leq n} \frac{k+n}{k} \geq 2^n$$

We now write

$$\binom{2n}{n} = \prod_{p < 2n} p^{\alpha_p} \geq 2^n$$

for $\alpha_p > 0$

Lemma

The largest power of p dividing $n!$ is given by

$$\nu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor$$

Proof.

As $\binom{2n}{n} = \frac{2n!}{(n!)^2}$, then the power of p in the decomposition of $\binom{2n}{n}$ can be calculated by subtracting twice the power of p in the decomposition of $n!$ from the power of p in the decomposition of $(2n)!$. Applying the previous lemma,

$$\alpha_p = \sum_{j=1}^{t_p} \left(\left\lfloor \frac{2n}{p^j} \right\rfloor - 2 \left\lfloor \frac{n}{p^j} \right\rfloor \right) = \sum_{j=1}^{t_p} \left\lfloor \frac{2n}{p_j} \right\rfloor - 2 \sum_{j=1}^{t_p} \left\lfloor \frac{n}{p_j} \right\rfloor$$

where t_p is the largest integer such that $p^{t_p} \leq 2n$.

Lower bound of Chebyshev

Proof.

Taking the log of both sides gives $t_p = \left\lfloor \frac{\log 2n}{\log p} \right\rfloor$, implying

$$n \log 2 \leq \sum_{p<2n} t_p \log p = \sum_{p<2n} \left\lfloor \frac{\log 2n}{\log p} \right\rfloor \log p$$

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Now, we need to separate the right hand side (RHS) into two separate sums and deal with them individually.

Lower bound of Chebyshev

Proof.

We have,

$$RHS = \sum_{p<\sqrt{2n}} \left\lfloor \frac{\log 2n}{\log p} \right\rfloor \log p + \sum_{\sqrt{2n} \le p \le 2n} \log p \le \sqrt{2n} \log 2n + \theta(2n)$$

Lower bound of Chebyshev

Proof.

We have,

$$RHS = \sum_{p<\sqrt{2n}} \left\lfloor \frac{\log 2n}{\log p} \right\rfloor \log p + \sum_{\sqrt{2n} \leq p \leq 2n} \log p \leq \sqrt{2n} \log 2n + \theta(2n)$$

We have proven that for sufficiently large $n \geq 1$,

$$\theta(2n) \geq n \log 2 - \sqrt{2n} \log 2n \geq Cn$$

Lower bound of Chebyshev

Proof.

For $x = 2n + 1$, we also have that

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It is now clear that

$$\theta(x) = \sum_{p \leq x} \log p \leq \pi(x) \log x$$

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For $x = 2n + 1$, we also have that

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It is now clear that

$$\theta(x) = \sum_{p \leq x} \log p \leq \pi(x) \log x$$

Therefore,

$$\frac{C}{4} \frac{x}{\log x} \leq \frac{\theta(x)}{\log x} \leq \pi(x)$$

Work Cited

Mallahi-Karai, K., Number Theory Complete Lecture Notes.