Undergraduate Seminar Presentation

Expander Graphs

Karen Arzumanyan

November 1, 2021

Graphs

What is a graph?

Graphs

What is a graph?

Graph

A graph G is determined by a set of vertices V and a set of edges E that connect the elements of V together.
An edge $e \in E$ that connects the vertices $v_1, v_2 \in V$ is denoted by (v_1, v_2).

Graphs

Example:

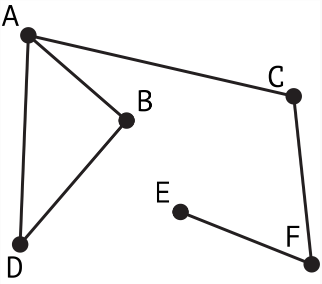

Figure 1: A graph consisting of 6 vertices and 6 edges.

Example:

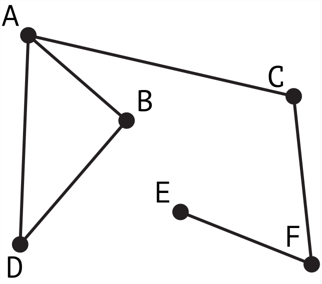

Figure 1: A graph consisting of 6 vertices and 6 edges.

$V = \{A, B, C, D, E, F\}$

Graphs

Example:

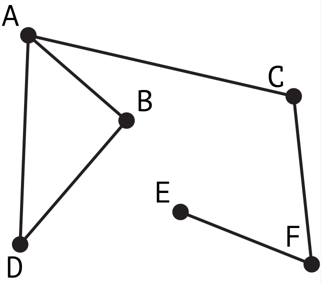

Figure 1: A graph consisting of 6 vertices and 6 edges.

$V = \{A, B, C, D, E, F\}$
$E = \{(A, B), (A, C), (A, D), (C, F), (D, B), (E, F)\}$

Expander Graphs

Informal Definition: Expander Graph

A graph G = (V, E) is called an *expander graph* if every $S \subset V$ has its vertices connected to many verticies in $\bar{S} \subset V$.
Where $\bar{S} = V - S$.

d-Regular Graphs

Definition: d-Regular Graphs

A graph G is called *d-Regular* if and only if every $v \in V$ is connected to d other vertices in V.

d-Regular Graphs

Definition: d-Regular Graphs

A graph G is called *d-Regular* if and only if every $v \in V$ is connected to d other vertices in V.

Example:

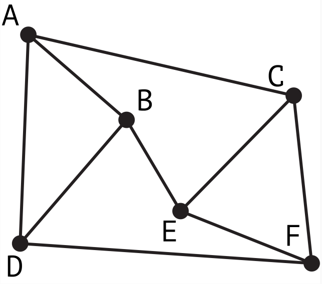

Figure 2: A d-Regular graph where d is 3.

Edge Boundary and Edge Expansion

Edge Boundary

For a subset S of V we define the *edge boundary* of $S \subset V$ to be the set of edges connecting S to its complement, $\bar{S}$. We denote the *edge boundary* of S as ∂S.

$$\partial S = \{\ (v, w) \mid v \in S \text{ and } w \notin S\ \}$$

Edge Boundary and Edge Expansion

Edge Boundary

For a subset S of V we define the *edge boundary* of $S \subset V$ to be the set of edges connecting S to its complement, $\bar{S}$. We denote the *edge boundary* of S as ∂S.

$$\partial S = \{\ (v, w) \mid v \in S \text{ and } w \notin S\ \}$$

Edge Expansion Parameter (also known as the Cheeger Constant)

We define a function

$$h(G) \equiv \min_{S : |S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|}.$$

$h(G)$ is the smallest possible ratio between the size of the *edge boundary* of $S \subset V$ and the size of S with $|S| \leq \frac{n}{2}$.

Edge Boundary and Edge Expansion

Example 1: Suppose G is a graph consisting of n vertices, such that each vertex is connected to every other vertex (also known as a complete graph). Then for any two vertices $v \in S$ and $w \in \bar{S}$ there exists an edge $e \in E$ such that $e = (v, w)$.

Edge Boundary and Edge Expansion

Example 1: Suppose G is a graph consisting of n vertices, such that each vertex is connected to every other vertex (also known as a complete graph). Then for any two vertices $v \in S$ and $w \in \bar{S}$ there exists an edge $e \in E$ such that $e = (v, w)$.

Thus, we can compute that $|\partial S| = |S| \times |\bar{S}| = |S|(n - |S|)$.

Edge Boundary and Edge Expansion

Example 1: Suppose G is a graph consisting of n vertices, such that each vertex is connected to every other vertex (also known as a complete graph). Then for any two vertices $v \in S$ and $w \in \bar{S}$ there exists an edge $e \in E$ such that $e = (v, w)$.

Thus, we can compute that $|\partial S| = |S| \times |\bar{S}| = |S|(n - |S|)$.

Substituting this to the formula of $h(G)$ we get

$$h(G) = \min_{S:|S| \leq \frac{n}{2}} n - |S| = \left\lceil \frac{n}{2} \right\rceil.$$

Edge Boundary and Edge Expansion

Example 2: Suppose that G is an $n \times n$ lattice in 2 dimensions, with periodic boundary conditions (so that G is 4-regular). Then if we consider a large connected subset $S \subset V$, it ought to be plausible that the *edge boundary* set ∂S will contain roughly one edge for each vertex on the perimeter of the region S. We expect there to be roughly $\sqrt{|S|}$ such vertices.

Edge Boundary and Edge Expansion

Example 2: Suppose that G is an $n \times n$ lattice in 2 dimensions, with periodic boundary conditions (so that G is 4-regular). Then if we consider a large connected subset $S \subset V$, it ought to be plausible that the *edge boundary* set ∂S will contain roughly one edge for each vertex on the perimeter of the region S. We expect there to be roughly $\sqrt{|S|}$ such vertices. Thus

$$\frac{|\partial S|}{|S|} \approx \frac{\sqrt{|S|}}{|S|} = \frac{1}{\sqrt{|S|}}$$

Edge Boundary and Edge Expansion

Example 2: Suppose that G is an $n \times n$ lattice in 2 dimensions, with periodic boundary conditions (so that G is 4-regular). Then if we consider a large connected subset $S \subset V$, it ought to be plausible that the *edge boundary* set ∂S will contain roughly one edge for each vertex on the perimeter of the region S. We expect there to be roughly $\sqrt{|S|}$ such vertices. Thus

$$\frac{|\partial S|}{|S|} \approx \frac{\sqrt{|S|}}{|S|} = \frac{1}{\sqrt{|S|}}$$

We know that S may contain up to $O(n^2)$ elements.

Edge Boundary and Edge Expansion

Example 2: Suppose that G is an $n \times n$ lattice in 2 dimensions, with periodic boundary conditions (so that G is 4-regular). Then if we consider a large connected subset $S \subset V$, it ought to be plausible that the *edge boundary* set ∂S will contain roughly one edge for each vertex on the perimeter of the region S. We expect there to be roughly $\sqrt{|S|}$ such vertices. Thus

$$\frac{|\partial S|}{|S|} \approx \frac{\sqrt{|S|}}{|S|} = \frac{1}{\sqrt{|S|}}$$

We know that S may contain up to $O(n^2)$ elements. Therefore

$$h(G) = O\left(\frac{1}{n}\right)$$

Edge Boundary and Edge Expansion

Example 2: Suppose that G is an $n \times n$ lattice in 2 dimensions, with periodic boundary conditions (so that G is 4-regular). Then if we consider a large connected subset $S \subset V$, it ought to be plausible that the *edge boundary* set ∂S will contain roughly one edge for each vertex on the perimeter of the region S. We expect there to be roughly $\sqrt{|S|}$ such vertices. Thus

$$\frac{|\partial S|}{|S|} \approx \frac{\sqrt{|S|}}{|S|} = \frac{1}{\sqrt{|S|}}$$

We know that S may contain up to $O(n^2)$ elements. Therefore

$$h(G) = O\left(\frac{1}{n}\right)$$

NOTE: In this example, $\lim_{n \to \infty} h(G) = 0$

Edge Boundary and Edge Expansion

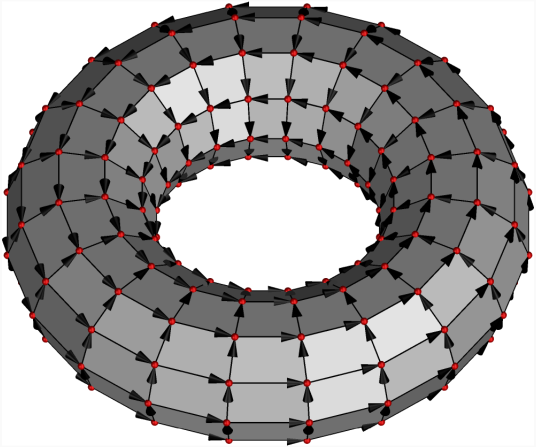

Figure 3: A lattice graph with a periodic boundary condition in 3D.

Edge Boundary and Edge Expansion

Example 3: Consider a d-regular graph G. Let all vertices in V be connected to d other *random* vertices. Let $S \subset V$ consisting of at most $\frac{n}{2}$ vertices.

Edge Boundary and Edge Expansion

Example 3: Consider a d-regular graph G. Let all vertices in V be connected to d other *random* vertices. Let $S \subset V$ consisting of at most $\frac{n}{2}$ vertices. Then, a vertex in S will be connected to roughly $d \times \frac{|\bar{S}|}{n}$ vertices in $\bar{S}$. We expect $|\partial S| \approx d\frac{|S| \times |\bar{S}|}{n}$,

Edge Boundary and Edge Expansion

Example 3: Consider a d-regular graph G. Let all vertices in V be connected to d other *random* vertices. Let $S \subset V$ consisting of at most $\frac{n}{2}$ vertices. Then, a vertex in S will be connected to roughly $d \times \frac{|\bar{S}|}{n}$ vertices in $\bar{S}$. We expect $|\partial S| \approx d\frac{|S| \times |\bar{S}|}{n}$, thus

$$\frac{|\partial S|}{|S|} \approx \frac{1}{|S|} \times d\frac{|S| \times |\bar{S}|}{n} = d\frac{|\bar{S}|}{n}$$

Example 3: Consider a d-regular graph G. Let all vertices in V be connected to d other *random* vertices. Let $S \subset V$ consisting of at most $\frac{n}{2}$ vertices. Then, a vertex in S will be connected to roughly $d \times \frac{|\bar{S}|}{n}$ vertices in $\bar{S}$. We expect $|\partial S| \approx d\frac{|S|\times|\bar{S}|}{n}$, thus

$$\frac{|\partial S|}{|S|} \approx \frac{1}{|S|} \times d\frac{|S| \times |\bar{S}|}{n} = d\frac{|\bar{S}|}{n}$$

But since $|\bar{S}|$ has its minimum at approximately $\frac{n}{2}$,

Example 3: Consider a d-regular graph G. Let all vertices in V be connected to d other *random* vertices. Let $S \subset V$ consisting of at most $\frac{n}{2}$ vertices. Then, a vertex in S will be connected to roughly $d \times \frac{|\bar{S}|}{n}$ vertices in $\bar{S}$. We expect $|\partial S| \approx d\frac{|S| \times |\bar{S}|}{n}$, thus

$$\frac{|\partial S|}{|S|} \approx \frac{1}{|S|} \times d\frac{|S| \times |\bar{S}|}{n} = d\frac{|\bar{S}|}{n}$$

But since $|\bar{S}|$ has its minimum at approximately $\frac{n}{2}$, we plug that value into our $h(G)$ approximation and get

$$h(G) \approx d\frac{|\bar{S}|}{n} = d\frac{n}{2} \times \frac{1}{n} = \frac{d}{2}$$

Edge Boundary and Edge Expansion

Example 3: Consider a d-regular graph G. Let all vertices in V be connected to d other *random* vertices. Let $S \subset V$ consisting of at most $\frac{n}{2}$ vertices. Then, a vertex in S will be connected to roughly $d \times \frac{|\bar{S}|}{n}$ vertices in $\bar{S}$. We expect $|\partial S| \approx d\frac{|S| \times |\bar{S}|}{n}$, thus

$$\frac{|\partial S|}{|S|} \approx \frac{1}{|S|} \times d\frac{|S| \times |\bar{S}|}{n} = d\frac{|\bar{S}|}{n}$$

But since $|\bar{S}|$ has its minimum at approximately $\frac{n}{2}$, we plug that value into our $h(G)$ approximation and get

$$h(G) \approx d\frac{|\bar{S}|}{n} = d\frac{n}{2} \times \frac{1}{n} = \frac{d}{2}$$

$$h(G) \approx \frac{d}{2}$$

Expander Graphs

Definition: Expander Graph

If we have a family $G_j = (V_j, E_j)$ of d-regular graphs, indexed by j, and such that $|V_j| = n_j$ for some increasing sequence n_j. Then we say that the family $\{G_j\}$ is a *family of expander graphs* if the edge expansion parameter is bounded strictly away from 0, i.e, there is some (small) constant c such that $h(G_j) \geq c > 0$ for all G_j in the family.

Expander Graphs

Example: Consider a family of graphs indexed by a prime number p. The set of vertices for the graph G_p is just the set of points in Z_p, the field of integers modulo p.

Expander Graphs

Example: Consider a family of graphs indexed by a prime number p. The set of vertices for the graph G_p is just the set of points in Z_p, the field of integers modulo p. We construct a 3-regular graph by connecting each vertex $x \neq 0$ to $x - 1$, $x + 1$, and x^{-1}.

Expander Graphs

Example: Consider a family of graphs indexed by a prime number p. The set of vertices for the graph G_p is just the set of points in Z_p, the field of integers modulo p. We construct a 3-regular graph by connecting each vertex $x \neq 0$ to $x - 1$, $x + 1$, and x^{-1}. The vertex $x = 0$ is connected to $p - 1$, 0, and 1.

Expander Graphs

Example: Consider a family of graphs indexed by a prime number p. The set of vertices for the graph G_p is just the set of points in Z_p, the field of integers modulo p. We construct a 3-regular graph by connecting each vertex $x \neq 0$ to $x - 1$, $x + 1$, and x^{-1}. The vertex $x = 0$ is connected to $p - 1$, 0, and 1. This is proven to be a family of expander graphs by Lubotsky, Philips, and Sarnak in 1988.

Adjacency Matrix

The rows and columns of the Adjacency Matrix are labelled by the vertices of V.
More precisely, $A(G)$ is the adjacency matrix of the graph $G = (V, E)$.
Then for vertices $v, w \in V$ the entry $A(G)_{vw} = 1$ if (v, w) is an edge, i.e $(v, w) \in E$; likewise, $A(G)_{vw} = 0$ if (v, w) is not an edge, i.e. $(v, w) \notin E$.

Eigenvalues

Assuming that the graph G is a finite graph with n vertices, then A(G) is an $n \times n$ symmetric matrix.

Eigenvalues

Assuming that the graph G is a finite graph with n vertices, then A(G) is an $n \times n$ symmetric matrix. Therefore, $A(G)$ has n real eigenvalues, counting multiplicities.

Eigenvalues

Assuming that the graph G is a finite graph with n vertices, then A(G) is an $n \times n$ symmetric matrix. Therefore, $A(G)$ has n real eigenvalues, counting multiplicities.

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$

Eigenvalues

Assuming that the graph G is a finite graph with n vertices, then A(G) is an $n \times n$ symmetric matrix. Therefore, $A(G)$ has n real eigenvalues, counting multiplicities.

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$

Furthermore, if G is d-regular, then the largest eigenvalue of $A(G)$ is d.

Eigenvalues

Assuming that the graph G is a finite graph with n vertices, then A(G) is an $n \times n$ symmetric matrix. Therefore, $A(G)$ has n real eigenvalues, counting multiplicities.

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$

Furthermore, if G is d-regular, then the largest eigenvalue of $A(G)$ is d.

Definition: Spectrum of a Graph

The spectrum of the graph G is the set of all *eigenvalues* of $A(G)$.

Connectivity of a d-regular Graph

Proposition

A d-regular graph G is connected if and only if $\lambda_1(G) > \lambda_2(G)$.

Connectivity of a d-regular Graph

Proposition

A d-regular graph G is connected if and only if $\lambda_1(G) > \lambda_2(G)$.

Proof

We prove by contrapositive, namely, that a d-regular *disconnected* graph has $\lambda_1(G) = \lambda_2(G)$.

Connectivity of a d-regular Graph

Proposition

A d-regular graph G is connected if and only if $\lambda_1(G) > \lambda_2(G)$.

Proof

We prove by contrapositive, namely, that a d-regular *disconnected* graph has $\lambda_1(G) = \lambda_2(G)$. We break up the graph G into disconnected components G_1 and G_2. We can observe that $A(G) = A(G_1) \bigoplus A(G_2)$.

Connectivity of a d-regular Graph

Proposition

A d-regular graph G is connected if and only if $\lambda_1(G) > \lambda_2(G)$.

Proof

We prove by contrapositive, namely, that a d-regular *disconnected* graph has $\lambda_1(G) = \lambda_2(G)$. We break up the graph G into disconnected components G_1 and G_2. We can observe that $A(G) = A(G_1) \bigoplus A(G_2)$. Since both G_1 and G_2 are d-regular, their largest eigenvalues are d, which means that d appears (at least) twice in spectrum of $A(G)$.

Gap and Expansion

Definition: Gap of the graph G

We define the *gap for the graph G* to be the difference $\Delta(G) \equiv \lambda_1(G) - \lambda_2(G)$ between the largest and the second-largest eigenvalues.

Gap and Expansion

Definition: Gap of the graph G

We define the *gap for the graph G* to be the difference $\Delta(G) \equiv \lambda_1(G) - \lambda_2(G)$ between the largest and the second-largest eigenvalues.

Theorem

The expansion parameter $h(G)$ for a d-regular graph G is related to the gap $\Delta(G)$ by:

$$\frac{\Delta(G)}{2} \leq h(G) \leq \sqrt{2d\Delta(G)}$$

Gap and Expansion

Definition: Gap of the graph G

We define the *gap for the graph G* to be the difference $\Delta(G) \equiv \lambda_1(G) - \lambda_2(G)$ between the largest and the second-largest eigenvalues.

Theorem

The expansion parameter $h(G)$ for a d-regular graph G is related to the gap $\Delta(G)$ by:

$$\frac{\Delta(G)}{2} \leq h(G) \leq \sqrt{2d\Delta(G)}$$

Remark

If the gap for a family of d-regular graphs is bounded below by a positive constant, then the expansion parameter must also be bounded below by a positive constant, thus showing that the family is expander.

Gap and Expansion

Proof Sketch of the Theorem

We already know that $\lambda_1(G) = d$ with eigenvector $\vec{1} = (1, 1, \ldots, 1)$.

Proof Sketch of the Theorem

We already know that $\lambda_1(G) = d$ with eigenvector $\vec{1} = (1, 1, \ldots, 1)$.
So we will now try to understand the behavior of $\lambda_2(G)$.

Proof Sketch of the Theorem

We already know that $\lambda_1(G) = d$ with eigenvector $\vec{1} = (1, 1, \ldots, 1)$. So we will now try to understand the behavior of $\lambda_2(G)$. A way to gain control over $\lambda_2(G)$ is to observe that it is just the maximum of the expression

$$v^T A(G) v / v^T v$$

where we maximize over all vectors v orthogonal to the eigenvector $\vec{1}$.

Gap and Expnansion

Proof Sketch of the Theorem Continued

Using the fact that if v is orthogonal to $\vec{1}$ is only possible if the sum of v's entries is equal to 0.

Gap and Expnansion

Proof Sketch of the Theorem Continued

Using the fact that if v is orthogonal to $\vec{1}$ is only possible if the sum of v's entries is equal to 0. Thus we have,

$$\lambda_2(G) = \max_{v:tr(v)=0} \frac{v^T A(G) v}{v^T v}$$

where $tr(v)$ is the sum of entries of the vector v.

Gap and Expnansion

Proof Sketch of the Theorem Continued

Using the fact that if v is orthogonal to $\vec{1}$ is only possible if the sum of v's entries is equal to 0. Thus we have,

$$\lambda_2(G) = \max_{v:tr(v)=0} \frac{v^T A(G) v}{v^T v}$$

where $tr(v)$ is the sum of entries of the vector v.
We can provide a lower bound on $\lambda_2(G)$ by simply guessing a good choice of v satisfying $tr(v) = 0$, in addition to using the fact

$$\lambda_2(G) \geq \frac{v^T A(G) v}{v^T v}.$$

Proof Sketch of the Theorem Continued

To make a good guess, we need a convenient way to look at $v^T A(G) v$, where $tr(v) = 0$.

Gap and Expnansion

Proof Sketch of the Theorem Continued

To make a good guess, we need a convenient way to look at $v^T A(G)v$, where $tr(v) = 0$. Let's think of v as the difference of two disjoint probability distributions, p and q, which means that $v = p - q$, where p and q are non-negative vectors each summing to 1 and having disjoint support.

Gap and Expnansion

Proof Sketch of the Theorem Continued

To make a good guess, we need a convenient way to look at $v^T A(G)v$, where $tr(v) = 0$. Let's think of v as the difference of two disjoint probability distributions, p and q, which means that $v = p - q$, where p and q are non-negative vectors each summing to 1 and having disjoint support. Now let's define a vector $\vec{1}_S$ to be the vector whose entries are 1 on S, and 0 elsewhere.

Proof Sketch of the Theorem Continued

Now we can observe that

$$\vec{1}_S^T A(G)\vec{1}_T = |E(S, T)|,$$

where $E(S, T)$ is the number of edges between the vertex sets S and T.

Proof Sketch of the Theorem Continued

Now we can observe that

$$\vec{1}_S^T A(G)\vec{1}_T = |E(S,T)|,$$

where $E(S,T)$ is the number of edges between the vertex sets S and T. This means that we should choose p and q in terms of vectors like $\vec{1}_S$, since it will enable us to relate expressions such as $v^T A(G)v$ to the sizes of various edge sets, which are how we will relate to the expansion parameter.

Gap and Expnansion

Proof Sketch of the Theorem Continued

Now we can observe that

$$\vec{1}_S^T A(G) \vec{1}_T = |E(S, T)|,$$

where $E(S, T)$ is the number of edges between the vertex sets S and T. This means that we should choose p and q in terms of vectors like $\vec{1}_S$, since it will enable us to relate expressions such as $v^T A(G) v$ to the sizes of various edge sets, which are how we will relate to the expansion parameter. Suppose we chose

$$v = \frac{\vec{1}_S}{|S|} - \frac{\vec{1}_{\bar{S}}}{|\bar{S}|}$$

Proof Sketch of the Theorem Continued

Now we can observe that

$$\vec{1}_S^T A(G) \vec{1}_T = |E(S,T)|,$$

where $E(S,T)$ is the number of edges between the vertex sets S and T. This means that we should choose p and q in terms of vectors like $\vec{1}_S$, since it will enable us to relate expressions such as $v^T A(G) v$ to the sizes of various edge sets, which are how we will relate to the expansion parameter. Suppose we chose

$$v = \frac{\vec{1}_S}{|S|} - \frac{\vec{1}_{\bar{S}}}{|\bar{S}|}$$

This satisfies the condition tr(v)=0, and leaves us with

$$v^T v = \frac{1}{|S|} + \frac{1}{|\bar{S}|}$$

Gap and Expnansion

Proof Sketch of the Theorem Continued

This in fact shows us that

$$v^T A(G) v = \frac{1}{|S|^2} E(S,S) + \frac{1}{|\bar{S}|^2} E(\bar{S},\bar{S}) - \frac{2}{|S||\bar{S}|} E(S,\bar{S})$$

Proof Sketch of the Theorem Continued

This in fact shows us that

$$v^T A(G) v = \frac{1}{|S|^2} E(S,S) + \frac{1}{|\bar{S}|^2} E(\bar{S},\bar{S}) - \frac{2}{|S||\bar{S}|} E(S,\bar{S})$$

By the definition of an expander graph, we have control over $E(S,\bar{S})$, so we can rewrite $E(S,S)$ and $E(\bar{S},\bar{S})$ in terms of $E(S,\bar{S})$, using the d-regularity of the graph.

Gap and Expnansion

Proof Sketch of the Theorem Continued

This in fact shows us that

$$v^T A(G) v = \frac{1}{|S|^2} E(S,S) + \frac{1}{|\bar{S}|^2} E(\bar{S},\bar{S}) - \frac{2}{|S||\bar{S}|} E(S,\bar{S})$$

By the definition of an expander graph, we have control over $E(S,\bar{S})$, so we can rewrite $E(S,S)$ and $E(\bar{S},\bar{S})$ in terms of $E(S,\bar{S})$, using the d-regularity of the graph. We get,

$$E(S,S) + E(S,\bar{S}) = d|S| \text{ and } E(\bar{S},\bar{S}) + E(S,\bar{S}) = d|\bar{S}|$$

Gap and Expnansion

Proof Sketch of the Theorem Continued

Now if we substitute our findings into $v^T A(G)v$, we get

$$v^T A(G)v = d\Big(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\Big) - \Big(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\Big)^2 E\Big(S, \bar{S}\Big)$$

Proof Sketch of the Theorem Continued

Now if we substitute our findings into $v^T A(G)v$, we get

$$v^T A(G)v = d\Big(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\Big) - \Big(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\Big)^2 E\Big(S, \bar{S}\Big)$$

And then comparing with the earlier expression for the denominator $v^T v$, we obtain

$$\lambda_2(G) \geq d - \Big(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\Big) E(S, \bar{S})$$

Gap and Expnansion

Proof Sketch of the Theorem Continued

Now if we substitute our findings into $v^T A(G)v$, we get

$$v^T A(G)v = d\Big(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\Big) - \Big(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\Big)^2 E\Big(S, \bar{S}\Big)$$

And then comparing with the earlier expression for the denominator $v^T v$, we obtain

$$\lambda_2(G) \geq d - \Big(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\Big) E(S, \bar{S})$$

Now we choose S such that $E(S, \bar{S}) = h(G)|S|$ and $|S| \leq \frac{n}{2}$.

Gap and Expnansion

Proof Sketch of the Theorem Continued

Now if we substitute our findings into $v^T A(G)v$, we get

$$v^T A(G)v = d\Big(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\Big) - \Big(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\Big)^2 E\Big(S, \bar{S}\Big)$$

And then comparing with the earlier expression for the denominator $v^T v$, we obtain

$$\lambda_2(G) \geq d - \Big(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\Big) E(S, \bar{S})$$

Now we choose S such that $E(S, \bar{S}) = h(G)|S|$ and $|S| \leq \frac{n}{2}$. After substituting and performing a little bit of algebra, we get

$$\lambda_2(G) \geq d - 2h(G),$$

Gap and Expnansion

Proof Sketch of the Theorem Continued

Now if we substitute our findings into $v^T A(G)v$, we get

$$v^T A(G)v = d\left(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\right) - \left(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\right)^2 E\left(S, \bar{S}\right)$$

And then comparing with the earlier expression for the denominator $v^T v$, we obtain

$$\lambda_2(G) \geq d - \left(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\right) E(S, \bar{S})$$

Now we choose S such that $E(S, \bar{S}) = h(G)|S|$ and $|S| \leq \frac{n}{2}$. After substituting and performing a little bit of algebra, we get

$$\lambda_2(G) \geq d - 2h(G),$$

and thus

$$\frac{\Delta(G)}{2} \leq h(G).$$

Gap and Expnansion

We only proved the lower bound.

Gap and Expnansion

We only proved the lower bound.
The proof of the upper bound is too complicated and time consuming, so it can be found in the notes of Linial and Wigderson.

What comes next?

Random Walks on Expanders

You choose some vertex in an expander graph G, and then repeadetly move to one of its d neighbors, while choosing uniformly at random, as well as, independent of previous choices.

For more details see [1]

Most applications of expanders utilize the random walks.

Applications of Expander Graphs

- **Reduce the need for randomness:** That is, expanders can be used to reduce the number of random bits needed to make a probabilistic algorithm work with some desired probability.
- **Find good error-correcting codes:** Expanders can be used to construct error correcting codes for protecting information against noise. Expanders can be used to find error correcting codes which are efficiently encodable and decodable. Finding codes with such properties was one of the milestone in information theory.

Work Cited

Nielsen, M. A., Introduction to expander graphs.

G. Davidoff, P. Sarnak, and A. Valette. *Elementary number theory, group theory, and Ramanujan graphs*, volume 55 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2003.