1. Introduction

Definition 1. Let $X \subset \mathbb{R}^d$ be a convex set. We say that $f : X \to \mathbb{R}$ is log-convex if

$$\log f(tx + (1-t)y) \leq t \log f(x) + (1-t) \log f(y),$$

$\iiint_V \mu(u,v,w)\, du\, dv\, dw forall$x,y $\in X$ and $0 \leq t \leq 1$.

Theorem 1. *Let $I \subseteq \mathbb{R}$ be an interval. Suppose that $T : I \to \mathbb{R}^{n\times n}$ is entrywise log-convex. Then, $r(T) : I \to \mathbb{R}$ is log-convex.*

2. Main result

The purpose of this note is to extend Theorem 1 to higher dimensions. In particular, we will prove the following corollary.

Corollary 1 (Extension to $\mathbb{R}^d$)**.** *Suppose that $X \subseteq \mathbb{R}^d$ is a convex set. Suppose that $T : X \to \mathbb{R}^{n\times n}$ is entrywise log-convex. Then, $r(T) : X \to \mathbb{R}$ is log-convex.*

Proof of Corollary 1. Suppose not, that is, suppose that $r(T)$ is not log-convex. Then, there exists $x_0, y_0 \in X$ and $t_0 \in [0,1]$ such that

$$\log r(T)(t_0 x_0 + (1-t_0)y_0) > t_0 \log r(T)(x_0) + (1-t_0) \log r(T)(y_0). \tag{1}$$

Define the function $g : [0,1] \to \mathbb{R}$ by

$$g(u) = r(T)(ux_0 + (1-u)y_0).$$

We claim that the function g is log-convex. Indeed, define $h : [0,1] \to \mathbb{R}^{n\times n}$ by $h(u) = T(ux_0 + (1-u)y_0)$. By assumption, T is entrywise log-convex, so it follows from Lemma 1 that the function h is entrywise log-convex. Thus, by Theorem 1, the function g is log-convex. Thus,

$$\log g(t_0) \leq t_0 \log g(1) + (1-t_1) \log g(0),$$

which by the definition of g is equivalent to

$$\log r(T)(t_0 x_0 + (1-t_0)y_0) \leq t_0 \log r(T)(x_0) + (1-t_0) \log r(T)(y_0),$$

which contradicts (1). This completes the proof. $\square$

Lemma 1. *Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is log-convex. Fix any $x, y \in \mathbb{R}^n$ and define $g : [0,1] \to \mathbb{R}$ by*

$$g(u) = f(ux + (1-u)y).$$

Then, g is log-convex

Proof of Lemma 1. We want to show that

$$\log g(tu + (1-t)v) \leq t \log g(u) + (1-t) \log g(v),$$

for all $v, u \in \mathbb{R}$ and $t \in [0,1]$. Observe that

$$\begin{aligned}
\log g(tu + (1-t)v) &= \log f\left(\big(tu + (1-t)v\big)x + \big(1 - (tu + (1-t)v)\big)y\right), \\
&= \log f\left(\big(ux + (1-u)y\big)t + \big(vx + (1-v)y\big)(1-t)\right), \\
&\leq (t \log f\left(ux + (1-u)y\right) + (1-t) \log f\left(vx + (1-v)y\right), \\
&= (t \log g(u) + (1-t) \log g(v),
\end{aligned}$$

where the inequality follows from the fact that f is log-convex.

$\square$