# Sperm competition games when males invest in paternal care

## List of Variables and Parameters

c = Energetic expenditure to obtain each mating event

n = Number of mating events for the mutant male with allocation strategy (s, p)

s = Allocation strategy adopted by the mutant male in terms of the energetic expenditure of the one ejaculate transferred after each mating event

p = Allocation strategy adopted by the mutant male in terms of the energetic expenditure of the parental activities towards only one clutch produced during the breeding season

nHat = Number of mating events for the wild-type males with allocation strategy (sHat, pHat)

sHat = Allocation strategy adopted by the wild-type males in terms of the energetic expenditure of the one ejaculate transferred after each mating event

pHat = Allocation strategy adopted by the wild-type males in terms of the energetic expenditure of the parental activities towards only one clutch produced during the breeding season

alpha = Shape parameter of the exponential function relating male allocation to parental care (p or pHat) and the probability of offspring survival

b = Baseline offspring survival without any paternal care, represented by the combined effects of biotic and abiotic conditions that affects offspring development and performance

F = Average number of mating events for females

# Equations

```
ClearAll["Global`*"];
(*
Relative gain in mating success of a mutant male with allocation strategy
 (s, p) in a population of wildtype males with strategy (sHat,pHat)
*)
n[s_, sHat_, p_, pHat_] = (nHat * (c + sHat) + pHat - p)/(c + s)

(*
Offspring survival exponential function
*)

offspringSurvival[alpha_, p_, b_] = 1 - E^(-(b + alpha*p))

(*
Fitness function of a mutant male with allocation strategy
 (s, p) in a population of wildtype males with strategy (sHat,pHat)
*)
fitness[s_, sHat_, p_, pHat_, alpha_, b_, F_] =
 s/(s + (F - 1) * sHat) * offspringSurvival[alpha, p, b] +
  (n[s, sHat, p, pHat] - 1) * s/(s + (F - 1) * sHat) * offspringSurvival[alpha, pHat, b]

Simplify[fitness[s, sHat, p, pHat, alpha, b, F]]
```

$$\frac{-p + pHat + nHat\,(c + sHat)}{c + s}$$

$$1 - e^{-b-alpha\,p}$$

$$\frac{\left(1 - e^{-b-alpha\,p}\right)\,s}{s + (-1 + F)\,sHat} + \frac{\left(1 - e^{-b-alpha\,pHat}\right)\,s\,\left(-1 + \frac{-p+pHat+nHat\,(c+sHat)}{c+s}\right)}{s + (-1 + F)\,sHat}$$

$$\frac{s\left(1 - e^{-b-alpha\,p} + \left(1 - e^{-b-alpha\,pHat}\right)\left(-1 + \frac{-p+pHat+nHat\,(c+sHat)}{c+s}\right)\right)}{s + (-1 + F)\,sHat}$$

# Finding the Evolutionary Stable Allocation Strategies (ESAS)

## Step I: Partial Derivative in *s*

```
Clear[c, n, nHat, s, sHat, p, pHat, alpha, b, F, diffFitness];

(*
We take the partial derivative of the fitness function
 with respect to male allocation to ejaculate production (s)
*)
diffFitness = D[fitness[s, sHat, p, pHat, alpha, b, F], s] ;
Simplify[diffFitness]
```

$$\frac{1}{\left(s+(-1+F)\,sHat\right)^{2}}\Bigg(\left(-1+e^{-b-alpha\,p}\right)s+\left(1-e^{-b-alpha\,p}\right)\left(s+(-1+F)\,sHat\right)-\frac{1}{(c+s)^{2}}\left(1-e^{-b-alpha\,pHat}\right)s\left(s+(-1+F)\,sHat\right)\left(-p+pHat+nHat\,(c+sHat)\right)-\left(1-e^{-b-alpha\,pHat}\right)s\left(-1+\frac{-p+pHat+nHat\,(c+sHat)}{c+s}\right)+\left(1-e^{-b-alpha\,pHat}\right)\left(s+(-1+F)\,sHat\right)\left(-1+\frac{-p+pHat+nHat\,(c+sHat)}{c+s}\right)\Bigg)$$

```
(*
We solve the partial derivative of the
  fitness function with respect to s at sHat=s and pHat=p
*)
sHat = s;
pHat = p;
Solve[diffFitness == 0, s]
```

$$\{\{s\to c\,(-1+F)\}\}$$

## Step 2: Partial Derivative in *p*, when s=s*

```
Clear[c, n, nHat, s, sHat, p, pHat, alpha, b, F, diffFitness2];

(*
We take the partial derivative of the fitness
 function with respect to male allocation to care (p)
*)
diffFitness2 = D[fitness[s, sHat, p, pHat, alpha, b, F], p]
Simplify[diffFitness2]
```

$$\frac{\text{alpha}\, e^{-b-\text{alpha}\, p}\, s}{s+(-1+F)\, \text{sHat}} - \frac{\left(1-e^{-b-\text{alpha}\, \text{pHat}}\right) s}{(c+s)\,(s+(-1+F)\, \text{sHat})}$$

$$\left(e^{-b-\text{alpha}\,(p+\text{pHat})}\, s\, \left(e^{\text{alpha}\, p} - e^{b+\text{alpha}\,(p+\text{pHat})} + \text{alpha}\, e^{\text{alpha}\, \text{pHat}}\,(c+s)\right)\right) / \left((c+s)\,(s+(-1+F)\, \text{sHat})\right)$$

```
(*
We solve the partial derivative of the fitness function with respect to p at the
  solution for s found in step 1 [i.e., sHat = s = c*(-1+F)] and pHat = p
*)
sHat = s = c (-1 + F);
pHat = p;
Solve[diffFitness2 == 0, p, Reals]
```

$$\left\{\left\{p \to \text{ConditionalExpression}\left[\frac{-b+\text{Log}[1+\text{alpha}\, c\, F]}{\text{alpha}},\right.\right.\right.$$
$$\left(F>0 \,\&\&\, \text{alpha}>0 \,\&\&\, \frac{1}{\text{alpha}\, c}+F<0 \,\&\&\, c<0\right) \,||\, (F>0 \,\&\&\, \text{alpha}>0 \,\&\&\, c>0) \,||$$
$$\left(F>0 \,\&\&\, \text{alpha}<0 \,\&\&\, \frac{1}{\text{alpha}\, c}+F<0 \,\&\&\, c>0\right) \,||\, (F>0 \,\&\&\, \text{alpha}<0 \,\&\&\, c<0) \,||$$
$$\left(F<0 \,\&\&\, \text{alpha}>0 \,\&\&\, \frac{1}{\text{alpha}\, c}+F>0 \,\&\&\, c>0\right) \,||\, (F<0 \,\&\&\, \text{alpha}>0 \,\&\&\, c<0) \,||$$
$$\left.\left.\left.\left(F<0 \,\&\&\, \text{alpha}<0 \,\&\&\, \frac{1}{\text{alpha}\, c}+F>0 \,\&\&\, c<0\right) \,||\, (F<0 \,\&\&\, \text{alpha}<0 \,\&\&\, c>0)\right]\right\}\right\}$$

```
(*
We set biological constraints to parameters in order to help solving the
 partial derivative of the fitness function with respect to p. Biologically,
the average number of mating events for females must be positive (i.e., F>0),
the energetic expenditure to obtain each mating event must also
 be positive (i.e. c>0), and the shape parameter of the offspring
 survival exponential must also be positive (i.e., alpha>0)
*)
Solve[diffFitness2 == 0 && F > 0 && alpha > 0 && c > 0, p, Reals]
```

$$\left\{\left\{p \to \text{ConditionalExpression}\left[\frac{-b+\text{Log}[1+\text{alpha}\, c\, F]}{\text{alpha}},\ \text{alpha}>0 \,\&\&\, c>0 \,\&\&\, F>0\right]\right\}\right\}$$

```
(*
It is worth noting that the same solutions are found whether we
 follow the order presented here or solve the partial derivative
 of the fitness function with respect to p first and, then,
solve the partial derivative of the fitness function with respect to s
  at the solution for p [i.e., pHat = p = (-b+Log[1+alpha c F])/alpha] and sHat = s
*)
```

## Step 3: Checking for Consistency (in c)

```
(*
Using the solutions for s* and p*,
we rearrange the equation for the number of mating events
 for males from [c*n + s*n + p = 1], leading to [n=(1-p)/(c+s)]
*)
s = c (-1 + F);
p = (-b + Log[1 + c F alpha])/alpha;
Simplify[(1 - p) / (c + s)]
```

$$\frac{\text{alpha} + \text{b} - \text{Log}[1 + \text{alpha c F}]}{\text{alpha c F}}$$

```
(*
We constrain the parametric space to ensure self-
 consistency in the energetic expenditure to obtain each mating event,
setting n = nHat = n* = F (Fisher condition when sex ratio 1:1)
*)
Solve[(alpha + b - Log[1 + alpha c F])/(alpha c F) - F == 0, c]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\left\{\left\{\text{c} \rightarrow \frac{-\text{F} + \text{ProductLog}\left[e^{\text{alpha}+\text{b}+\text{F}}\ \text{F}\right]}{\text{alpha F}^2}\right\}\right\}$$

# Step 4: Checking for Fitness Maxima

## Obtaining the Hessian Matrix for the Fitness Function

```
Clear[c, n, nHat, s, sHat, p, pHat, alpha, b, F];
(*
For the equilibrium for the variables allocation to ejaculate production (s) and
 allocation to care (p) (found in steps 1 and 2) to represent fitness maxima,
the Hessian matrix of the fitness function
 (i.e.,the square matrix of the second-order partial derivatives
   of the fitness function), must be negative semidefinite.
  Therefore, we first calculate the Hessian matrix of the fitness function
*)
HessianH[f_, x_List?VectorQ] := D[f, {x, 2}];
hess = HessianH[f = fitness[s, sHat, p, pHat, alpha, b, F], x = {s, p}]
```

$$\left\{\left\{\frac{2\left(1-e^{-b-\text{alpha}\,p}\right)s}{\left(s+(-1+F)\,\text{sHat}\right)^3}-\frac{2\left(1-e^{-b-\text{alpha}\,p}\right)}{\left(s+(-1+F)\,\text{sHat}\right)^2}+\frac{2\left(1-e^{-b-\text{alpha}\,\text{pHat}}\right)s\left(-p+\text{pHat}+\text{nHat}\,(c+\text{sHat})\right)}{(c+s)^2\left(s+(-1+F)\,\text{sHat}\right)^2}+\frac{2\left(1-e^{-b-\text{alpha}\,\text{pHat}}\right)s\left(-p+\text{pHat}+\text{nHat}\,(c+\text{sHat})\right)}{(c+s)^3\left(s+(-1+F)\,\text{sHat}\right)}-\frac{2\left(1-e^{-b-\text{alpha}\,\text{pHat}}\right)\left(-p+\text{pHat}+\text{nHat}\,(c+\text{sHat})\right)}{(c+s)^2\left(s+(-1+F)\,\text{sHat}\right)}+\frac{2\left(1-e^{-b-\text{alpha}\,\text{pHat}}\right)s\left(-1+\frac{-p+\text{pHat}+\text{nHat}\,(c+\text{sHat})}{c+s}\right)}{\left(s+(-1+F)\,\text{sHat}\right)^3}-\frac{2\left(1-e^{-b-\text{alpha}\,\text{pHat}}\right)\left(-1+\frac{-p+\text{pHat}+\text{nHat}\,(c+\text{sHat})}{c+s}\right)}{\left(s+(-1+F)\,\text{sHat}\right)^2},\right.$$

$$-\frac{\text{alpha}\,e^{-b-\text{alpha}\,p}\,s}{\left(s+(-1+F)\,\text{sHat}\right)^2}+\frac{\left(1-e^{-b-\text{alpha}\,\text{pHat}}\right)s}{(c+s)\left(s+(-1+F)\,\text{sHat}\right)^2}+\frac{\text{alpha}\,e^{-b-\text{alpha}\,p}}{s+(-1+F)\,\text{sHat}}+\frac{\left(1-e^{-b-\text{alpha}\,\text{pHat}}\right)s}{(c+s)^2\left(s+(-1+F)\,\text{sHat}\right)}-\frac{1-e^{-b-\text{alpha}\,\text{pHat}}}{(c+s)\left(s+(-1+F)\,\text{sHat}\right)}\Bigg\},$$

$$\left\{-\frac{\text{alpha}\,e^{-b-\text{alpha}\,p}\,s}{\left(s+(-1+F)\,\text{sHat}\right)^2}+\frac{\left(1-e^{-b-\text{alpha}\,\text{pHat}}\right)s}{(c+s)\left(s+(-1+F)\,\text{sHat}\right)^2}+\frac{\text{alpha}\,e^{-b-\text{alpha}\,p}}{s+(-1+F)\,\text{sHat}}+\frac{\left(1-e^{-b-\text{alpha}\,\text{pHat}}\right)s}{(c+s)^2\left(s+(-1+F)\,\text{sHat}\right)}-\frac{1-e^{-b-\text{alpha}\,\text{pHat}}}{(c+s)\left(s+(-1+F)\,\text{sHat}\right)},-\frac{\text{alpha}^2\,e^{-b-\text{alpha}\,p}\,s}{s+(-1+F)\,\text{sHat}}\right\}\right\}$$

```
(*
Then, we ensure self-
 consistency for the solutions s* p* (found in steps 1 and 2),
using the solution for the the energetic expenditure
 to obtain each mating event (c) (found in step 3)
*)
```

$$c = \frac{-F + \text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]}{\text{alpha}\,F^2};$$

$$n[s\_,\,sHat\_,\,p\_,\,pHat\_] = \frac{nHat * (c + sHat) + pHat - p}{c + s};$$

$$s = c\,(-1 + F);$$

$$p = \frac{-b + \text{Log}[1 + c\,F\,\text{alpha}]}{\text{alpha}};$$

```
Simplify[s]
Simplify[p]
```

$$\frac{(-1+F)\,\left(-F + \text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\right)}{\text{alpha}\,F^2}$$

$$\frac{-b + \text{Log}\left[\frac{\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]}{F}\right]}{\text{alpha}}$$

```
(*
Next, we set the conditions for the equilibrium s = sHat = s*, and p = pHat = p*
*)
sHat = s;
pHat = p;
nHat = F;
FullSimplify[hess][[1]][[1]]
FullSimplify[hess][[1]][[2]]
FullSimplify[hess][[2]][[1]]
FullSimplify[hess][[2]][[2]]
```

$$\left(2\,\text{alpha}^2\,F^2\right) \Big/ \left((-1+F)\,\left(F - \text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\right)\,\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\right)$$

$$\frac{\text{alpha}^2\,F}{\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\,\left(-F + \text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\right)}$$

$$\frac{\text{alpha}^2\,F}{\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\,\left(-F + \text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\right)}$$

$$-\frac{\text{alpha}^2}{\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]}$$

```
(*
We present the denominator of the first element of the
 Hessian matrix of the fitness function in a different way,
so it would be clear the similarities among all the elements
 Here, we show that they are erquivalent
*)
test1 = Expand[(-1 + F) (F - ProductLog[e^(F+fe+shape) F]) ProductLog[e^(F+fe+shape) F]]
test2 = Expand[ProductLog[e^(F+fe+shape) F] * (-F + ProductLog[e^(F+fe+shape) F]) * (1 - F) ]
test1 - test2
```

$$-F\,\text{ProductLog}\left[e^{F+fe+shape}\,F\right] + F^2\,\text{ProductLog}\left[e^{F+fe+shape}\,F\right] + \text{ProductLog}\left[e^{F+fe+shape}\,F\right]^2 - F\,\text{ProductLog}\left[e^{F+fe+shape}\,F\right]^2$$

$$-F\,\text{ProductLog}\left[e^{F+fe+shape}\,F\right] + F^2\,\text{ProductLog}\left[e^{F+fe+shape}\,F\right] + \text{ProductLog}\left[e^{F+fe+shape}\,F\right]^2 - F\,\text{ProductLog}\left[e^{F+fe+shape}\,F\right]^2$$

0

```
(*
Since the Hessian matrix of the fitness function at equilibrium is symmetric,
it will only be negative semidefinite
 if its eigenvalues are nonpositive. Therefore,
we need to calculate the eigenvalues of the Hessian matrix
*)
FullSimplify[Eigenvalues[hess]]
```

$$\Big\{\Big(\text{alpha}^2\,\Big(F + F^2 + (-1+F)\,\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right] - \sqrt{\left(F^2\,(5 + F\,(-14 + 13\,F)) + (-1+F)\,\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\,\left(2\,(1 - 3\,F)\,F + (-1+F)\,\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\right)\right)}\Big)\Big) \Big/ \left(2\,(-1+F)\,\left(F - \text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\right)\,\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\right),$$
$$\Big(\text{alpha}^2\,\Big(F + F^2 + (-1+F)\,\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right] + \sqrt{\left(F^2\,(5 + F\,(-14 + 13\,F)) + (-1+F)\,\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\,\left(2\,(1 - 3\,F)\,F + (-1+F)\,\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\right)\right)}\Big)\Big) \Big/ \left(2\,(-1+F)\,\left(F - \text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\right)\,\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\right)\Big\}$$

## Finding Solutions for Parameters to Satisfy the NonPositive Condition for the First Eigenvalue

```
(*
For the first eigenvalue of the Hessian matrix of the fitness function,
we try to find the analytical solution for each parameter
 (i.e., the shape parameter of the offspring survival function alpha,
  the baseline offspring survival without any paternal care
   b and the average number of mating events for females F)
 that satisfies the condition that the eigenvalue is non positive
*)
eig1 = (alpha^2 (F + F^2 + (-1 + F) ProductLog[e^(alpha+b+F) F] - √(F^2 (5 + F (-14 + 13 F)) + (-1 + F)
            ProductLog[e^(alpha+b+F) F] (2 (1 - 3 F) F + (-1 + F) ProductLog[e^(alpha+b+F) F])))) /
    (2 (-1 + F) (F - ProductLog[e^(alpha+b+F) F]) ProductLog[e^(alpha+b+F) F]);
Solve[eig1 ≤ 0, alpha, Reals]
Solve[eig1 ≤ 0, b, Reals]
Solve[eig1 ≤ 0, F, Reals]
```

••• Solve: This system cannot be solved with the methods available to Solve.

```
Solve[(alpha^2 (F + F^2 + (-1 + F) ProductLog[e^(alpha+b+F) F] - √(F^2 (5 + F (-14 + 13 F)) + (-1 + F)
            ProductLog[e^(alpha+b+F) F] (2 (1 - 3 F) F + (-1 + F) ProductLog[e^(alpha+b+F) F])))) /
    (2 (-1 + F) (F - ProductLog[e^(alpha+b+F) F]) ProductLog[e^(alpha+b+F) F]) ≤
  0, alpha, Reals]
```

••• Solve: When parameter values satisfy the condition

$(alpha > 0 \&\& F > 1) || (alpha > 0 \&\& F > 1) || \left(alpha > 0 \&\& \frac{1}{3} < F < 1\right) || \left(alpha > 0 \&\& \frac{1}{3} < F < 1\right) || \left(alpha > 0 \&\& 0 < F \leq \frac{1}{3}\right) || \left(alpha > 0 \&\& -\frac{1}{3} < F < 0\right) || \left(alpha > 0 \&\& -1 < F \leq -\frac{1}{3}\right) || (alpha < 0 \&\& F > 1) || (alpha < 0 \&\& F > 1) || \left(alpha < 0 \&\& \frac{1}{3} < F < 1\right) || \left(alpha < 0 \&\& \frac{1}{3} < F < 1\right) || \left(alpha < 0 \&\& 0 < F \leq \frac{1}{3}\right) || \left(alpha < 0 \&\& -\frac{1}{3} < F < 0\right) || \left(alpha < 0 \&\& -1 < F \leq -\frac{1}{3}\right)$,

the solution set contains a full–dimensional component; use Reduce for complete solution information.

```
{}
```

••• Solve: This system cannot be solved with the methods available to Solve.

```
Solve[(alpha^2 (F + F^2 + (-1 + F) ProductLog[e^(alpha+b+F) F] - √(F^2 (5 + F (-14 + 13 F)) + (-1 + F)
            ProductLog[e^(alpha+b+F) F] (2 (1 - 3 F) F + (-1 + F) ProductLog[e^(alpha+b+F) F])))) /
    (2 (-1 + F) (F - ProductLog[e^(alpha+b+F) F]) ProductLog[e^(alpha+b+F) F]) ≤ 0, F, Reals]
```

```
(*
In a first general attempt,
we could not solve for the parameters alpha and F that satisfies the
 condition that the eigenvalue is non positive. On the other hand,
any value of the parameter b results in non positive eigenvalues,
as long as the parameters alpha and F respect some constraints. In particular,
the constraint [alpha > 0 and F > 1] is the only one that is biologically valid,
since the offspring survival function should approaches 1
 as allocation to care (p) increases in a diminishing returns
 format (for the constraint for the parameter alpha) and females
 must mate multiply (for the constraint for the parameter F)
*)

(*
In fact, if females do not mate (i.e., F = 0) or if they are
 monandric (i.e., F = 1), the first eigenvalue is indeterminate and,
therefore, does not sartisfy the condition to represent a fitness maximum
*)

F = 0;
eig1
```

Power: Infinite expression $\frac{1}{0}$ encountered.

Power: Infinite expression $\frac{1}{0}$ encountered.

Infinity: Indeterminate expression 0 alpha$^2$ ComplexInfinity ComplexInfinity encountered.

```
Indeterminate

F = 1;
eig1
```

Power: Infinite expression $\frac{1}{0}$ encountered.

Infinity: Indeterminate expression $\frac{0\ \text{alpha}^2\ \text{ComplexInfinity}}{\left(1-\text{ProductLog}\left[e^{1+\text{alpha}+\text{b}}\right]\right)\ \text{ProductLog}\left[e^{1+\text{alpha}+\text{b}}\right]}$ encountered.

```
Indeterminate
```

```
(*
On the other hand, if females mate multiply,
the first eigenvalue can be calculated and whether it
 satisfies the conditions that represent fitness maximum or
 not depends on the particular values for the parameter alpha
*)
F = 2;
eig1
```

$$\left(\text{alpha}^2\left(6+\text{ProductLog}\left[2\,e^{2+\text{alpha}+b}\right]-\sqrt{\left(116+\left(-20+\text{ProductLog}\left[2\,e^{2+\text{alpha}+b}\right]\right)\text{ProductLog}\left[2\,e^{2+\text{alpha}+b}\right]\right)}\right)\right)\Big/\left(2\left(2-\text{ProductLog}\left[2\,e^{2+\text{alpha}+b}\right]\right)\text{ProductLog}\left[2\,e^{2+\text{alpha}+b}\right]\right)$$

```
F = 3;
eig1
```

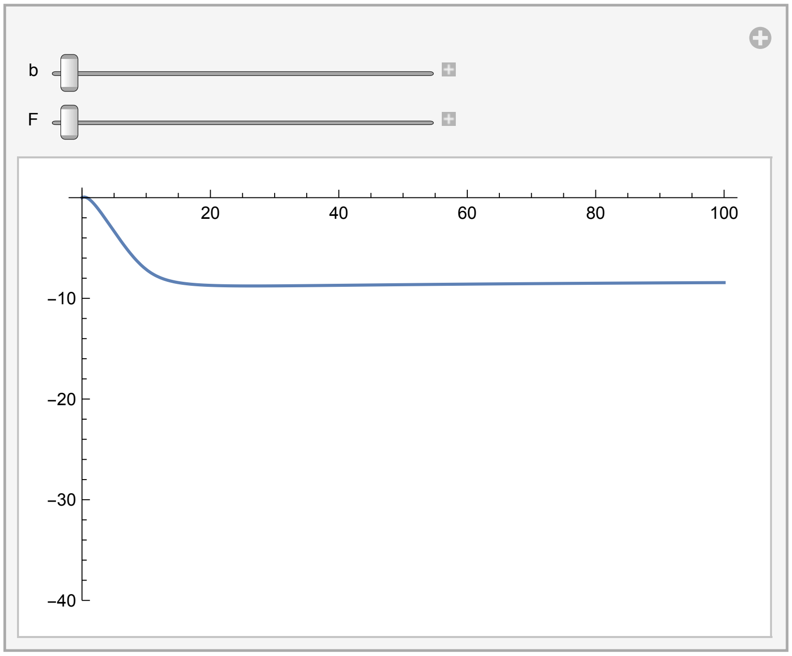
$$\left(\text{alpha}^2\left(12+2\,\text{ProductLog}\left[3\,e^{3+\text{alpha}+b}\right]-\sqrt{\left(720+2\,\text{ProductLog}\left[3\,e^{3+\text{alpha}+b}\right]\left(-48+2\,\text{ProductLog}\left[3\,e^{3+\text{alpha}+b}\right]\right)\right)}\right)\right)\Big/\left(4\left(3-\text{ProductLog}\left[3\,e^{3+\text{alpha}+b}\right]\right)\text{ProductLog}\left[3\,e^{3+\text{alpha}+b}\right]\right)$$

```
F = 30;
eig1
```

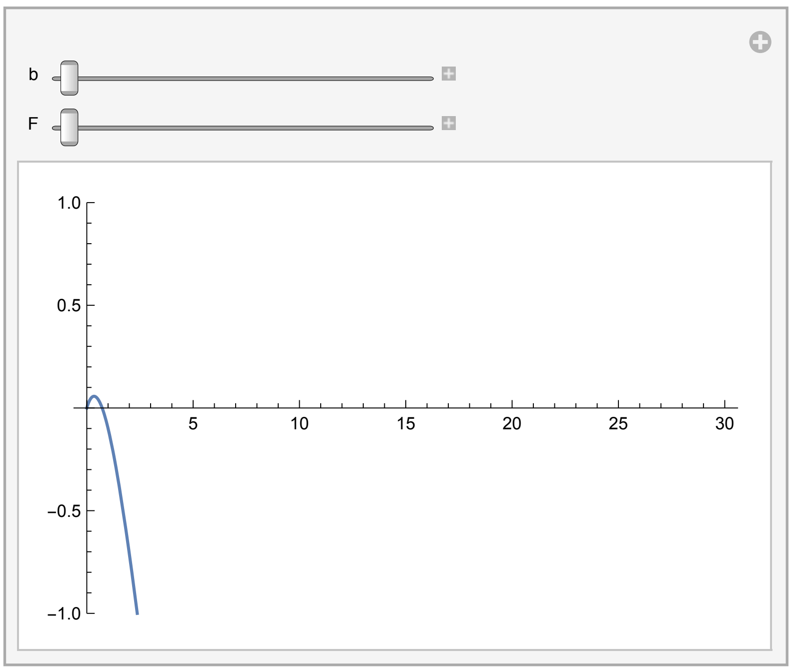
$$\left(\text{alpha}^2\left(930+29\,\text{ProductLog}\left[30\,e^{30+\text{alpha}+b}\right]-\sqrt{\left(10\,156\,500+29\,\text{ProductLog}\left[30\,e^{30+\text{alpha}+b}\right]\left(-5340+29\,\text{ProductLog}\left[30\,e^{30+\text{alpha}+b}\right]\right)\right)}\right)\right)\Big/\left(58\left(30-\text{ProductLog}\left[30\,e^{30+\text{alpha}+b}\right]\right)\text{ProductLog}\left[30\,e^{30+\text{alpha}+b}\right]\right)$$

In[285]:=
```
(*
Alternatively, we can use a graphical tool to visualize
 the behavior of the first eingenvalue in the parametric space,
considering the bounds to parameter F (i.e., F ≥ 2),
parameter alpha (i.e., alpha > 0) and parameter b (i.e., (0 ≤ b ≤ 1)
*)
Clear[F];
eigen1[alpha_, F_, b_] :=
  (alpha^2 (F + F^2 + (-1 + F) ProductLog[e^(alpha+b+F) F] - √(F^2 (5 + F (-14 + 13 F)) + (-1 + F)
           ProductLog[e^(alpha+b+F) F] (2 (1 - 3 F) F + (-1 + F) ProductLog[e^(alpha+b+F) F])))) /
   (2 (-1 + F) (F - ProductLog[e^(alpha+b+F) F]) ProductLog[e^(alpha+b+F) F]);
Manipulate[Plot[N[eigen1[alpha, F, b]], {alpha, 0.0001, 100}, PlotRange → {-40, 1}],
 {b, 0, 1}, {F, 2, 30}]
Manipulate[Plot[N[eigen1[alpha, F, b]], {alpha, 0.0001, 30}, PlotRange → {-1, 1}],
 {b, 0, 1}, {F, 2, 50}]
```

Out[286]=




Out[287]=



```
(*
We can see that, as the average number of mating events for females F increases,
the range of values for the shape parameter of the offspring survival function
 alpha that results in a positive eigenvalue also increases. For example,
considering the baseline offspring survival without any male care b = 0,
when F = 2,alpha must be greater than approximately
 0.73 to satisfy the condition for fitness maxima, when F = 5,
alpha must be greater than approximately 2.4, when F = 10,
alpha must be greater than approximately 4.9, when F = 20,
alpha must be greater than approximately 9.9, and so on.
  Additionally, we can also see that, as the the parameter b increases,
the range of values for alpha that results in a positive
 eigenvalue also decreases. For example, considering F = 10,
when b = 0, , alpha must be greater than approximately 4.9,
when b = 0.5, alpha must be greater than approximately 4.4,
when b = 1, alpha must be greater than approximately 3.9.
*)
```

```
Clear[F, b, alpha, eig1];
eig1 = (alpha^2 (F + F^2 + (-1 + F) ProductLog[E^(alpha+b+F) F] - √(F^2 (5 + F (-14 + 13 F)) + (-1 + F)
        ProductLog[E^(alpha+b+F) F] (2 (1 - 3 F) F + (-1 + F) ProductLog[E^(alpha+b+F) F])))) /
  (2 (-1 + F) (F - ProductLog[E^(alpha+b+F) F]) ProductLog[E^(alpha+b+F) F]);
Solve[eig1 == 0, alpha]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\left\{\{\text{alpha} \to 0\},\ \left\{\text{alpha} \to -b - F + \text{Log}\left[\frac{e^{\frac{1}{2}(-1+3F)}\,(-1+3F)}{2F}\right]\right\}\right\}$$

```
b = 0;
F = 2;
```

$$\text{alpha} = -b - F + \text{Log}\left[\frac{e^{\frac{1}{2}(-1+3F)}(-1+3F)}{2F}\right]$$

```
eig1
```

$$\text{alpha} = -b - F + \text{Log}\left[\frac{e^{\frac{1}{2}(-1+3F)}(-1+3F)}{2F}\right] + 0.000001$$

```
eig1
```

$$\text{alpha} = -b - F + \text{Log}\left[\frac{e^{\frac{1}{2}(-1+3F)}(-1+3F)}{2F}\right] - 0.000001$$

```
eig1
```

$$-2 + \text{Log}\left[\frac{5\, e^{5/2}}{4}\right]$$

0

0.723145

$-2.81244 \times 10^{-7}$

0.723143

$2.81243 \times 10^{-7}$

```
(*
Here, we show that, for the first eigenvalue,
the condition for the parameter alpha to result in non positive values is
```

$$\text{alpha} \geqslant -b - F + \text{Log}\left[\frac{e^{\frac{1}{2}(-1+3F)}(-1+3F)}{2F}\right]$$

```
*)
```

## Finding Solutions for Parameters to Satisfy the NonPositive Condition for the Second Eigenvalue

```
(*
We follow the same protocol for the second
 eigenvalue of the Hessian matrix of the fitness function,
in order to find the analytical solution for each parameter that
 satisfies the condition that the eigenvalue is non positive
*)
Clear[F, b, alpha, eig1];
eig2 =
  (alpha^2 (F + F^2 + (-1 + F) ProductLog[e^(alpha+b+F) F] + √(F^2 (5 + F (-14 + 13 F)) + (-1 + F)
          ProductLog[e^(alpha+b+F) F] (2 (1 - 3 F) F + (-1 + F) ProductLog[e^(alpha+b+F) F])))) /
   (2 (-1 + F) (F - ProductLog[e^(alpha+b+F) F]) ProductLog[e^(alpha+b+F) F]);
Solve[eig2 ≤ 0, alpha, Reals]
Solve[eig2 ≤ 0, b, Reals]
Solve[eig2 ≤ 0, F, Reals]
```

Solve: This system cannot be solved with the methods available to Solve.

```
Solve[(alpha^2 (F + F^2 + (-1 + F) ProductLog[e^(alpha+b+F) F] + √(F^2 (5 + F (-14 + 13 F)) + (-1 + F)
          ProductLog[e^(alpha+b+F) F] (2 (1 - 3 F) F + (-1 + F) ProductLog[e^(alpha+b+F) F])))) /
   (2 (-1 + F) (F - ProductLog[e^(alpha+b+F) F]) ProductLog[e^(alpha+b+F) F]) ≤
  0, alpha, Reals]
```

Solve: When parameter values satisfy the condition (alpha > 0 && F > 1) || (F < -1 && alpha > 0) || (0 < F < 1 && alpha > 0) || (-1 < F < 0 && alpha > 0) || (alpha < 0 && F > 1) || (alpha < 0 && F < -1) || (0 < F < 1 && alpha < 0) || (-1 < F < 0 && alpha < 0), the solution set contains a full-dimensional component; use Reduce for complete solution information.

```
{}
```

Solve: This system cannot be solved with the methods available to Solve.

```
Solve[(alpha^2 (F + F^2 + (-1 + F) ProductLog[e^(alpha+b+F) F] + √(F^2 (5 + F (-14 + 13 F)) + (-1 + F)
          ProductLog[e^(alpha+b+F) F] (2 (1 - 3 F) F + (-1 + F) ProductLog[e^(alpha+b+F) F])))) /
   (2 (-1 + F) (F - ProductLog[e^(alpha+b+F) F]) ProductLog[e^(alpha+b+F) F]) ≤ 0, F, Reals]

(*
Once again, we could not solve for the parameters alpha and F,
while any value of the parameter b results in non positive eigenvalues,
as long as the parameters alpha and F respect some constraints,
with the most biologically valid constraint being [alpha > 0 and F > 1]
*)
```

```
F = 0;
eig2
```

Power: Infinite expression $\frac{1}{0}$ encountered.

Power: Infinite expression $\frac{1}{0}$ encountered.

Infinity: Indeterminate expression 0 alpha$^2$ ComplexInfinity ComplexInfinity encountered.

```
Indeterminate
```

```
F = 1;
eig2
```

Power: Infinite expression $\frac{1}{0}$ encountered.

```
ComplexInfinity
```

```
(*
On the other hand, if females mate multiply,
the first eigenvalue can be calculated and whether it
 satisfies the conditions that represent fitness maximum or
 not depends on the particular values for the parameter alpha
*)
F = 2;
eig2
```

$$\left(\text{alpha}^2 \left(6+\text{ProductLog}\left[2\, e^{2+\text{alpha}+b}\right]+\sqrt{\left(116+\left(-20+\text{ProductLog}\left[2\, e^{2+\text{alpha}+b}\right]\right)\text{ProductLog}\left[2\, e^{2+\text{alpha}+b}\right]\right)}\right)\right)\Big/\left(2\left(2-\text{ProductLog}\left[2\, e^{2+\text{alpha}+b}\right]\right)\text{ProductLog}\left[2\, e^{2+\text{alpha}+b}\right]\right)$$

```
F = 3;
eig2
```

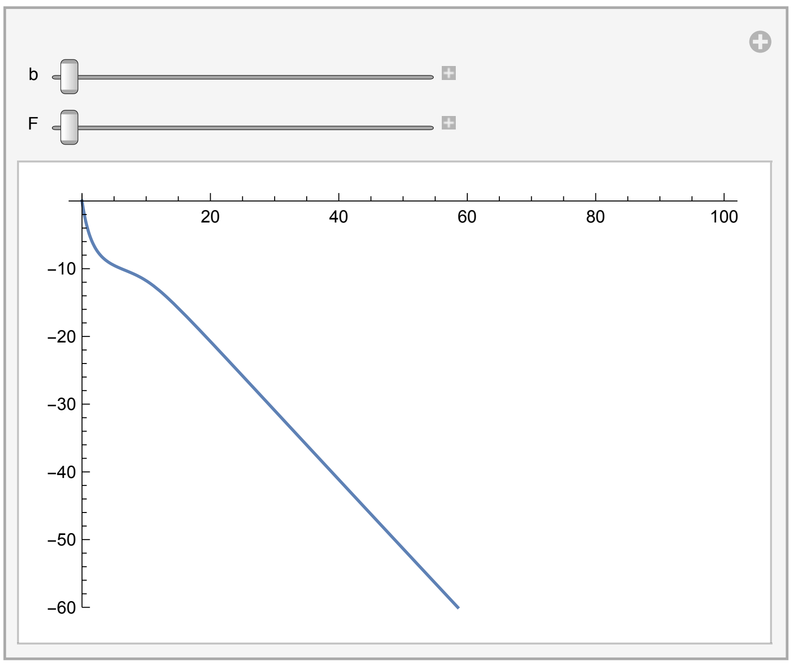
$$\left(\text{alpha}^2 \left(12+2\,\text{ProductLog}\left[3\, e^{3+\text{alpha}+b}\right]+\sqrt{\left(720+2\,\text{ProductLog}\left[3\, e^{3+\text{alpha}+b}\right]\left(-48+2\,\text{ProductLog}\left[3\, e^{3+\text{alpha}+b}\right]\right)\right)}\right)\right)\Big/\left(4\left(3-\text{ProductLog}\left[3\, e^{3+\text{alpha}+b}\right]\right)\text{ProductLog}\left[3\, e^{3+\text{alpha}+b}\right]\right)$$

```
F = 30;
eig2
```

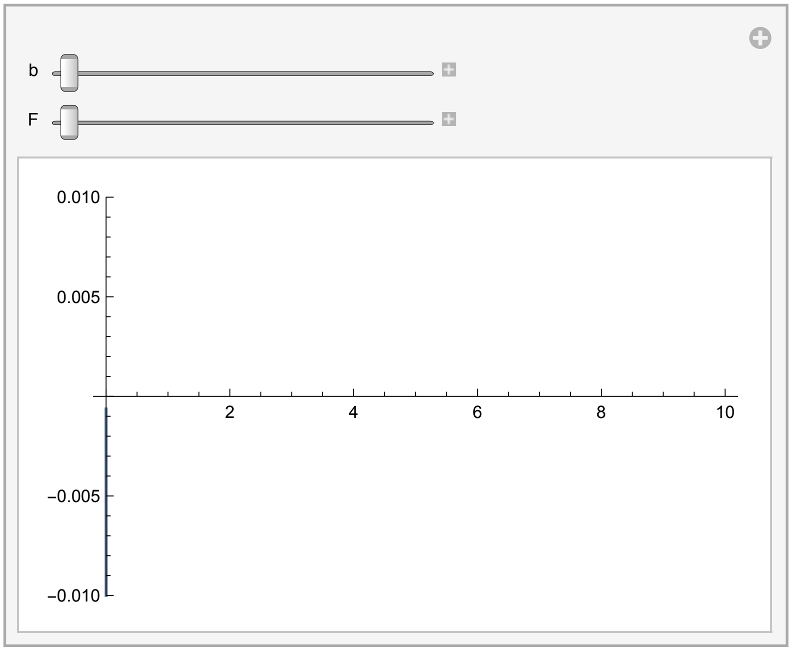
$$\left(\text{alpha}^2 \left(930+29\,\text{ProductLog}\left[30\, e^{30+\text{alpha}+b}\right]+\sqrt{\left(10\,156\,500+29\,\text{ProductLog}\left[30\, e^{30+\text{alpha}+b}\right]\left(-5340+29\,\text{ProductLog}\left[30\, e^{30+\text{alpha}+b}\right]\right)\right)}\right)\right)\Big/\left(58\left(30-\text{ProductLog}\left[30\, e^{30+\text{alpha}+b}\right]\right)\text{ProductLog}\left[30\, e^{30+\text{alpha}+b}\right]\right)$$

In[288]:=
```
(*
We also use a graphical tool to visualize the
 behavior of the second eingenvalue in the parametric space,
considering the bounds to parameter F (i.e., F ≥ 2),
parameter alpha (i.e., alpha > 0) and parameter b (i.e., (0 ≤ b ≤ 1)
*)
Clear[F];
eigen2[alpha_, F_, b_] :=
 (alpha^2 (F + F^2 + (-1 + F) ProductLog[e^(alpha+b+F) F] + √(F^2 (5 + F (-14 + 13 F)) + (-1 + F)
         ProductLog[e^(alpha+b+F) F] (2 (1 - 3 F) F + (-1 + F) ProductLog[e^(alpha+b+F) F])))) /
   (2 (-1 + F) (F - ProductLog[e^(alpha+b+F) F]) ProductLog[e^(alpha+b+F) F]);
Manipulate[Plot[N[eigen2[alpha, F, b]], {alpha, 0.0001, 100}, PlotRange → {-60, 1}],
 {b, 0, 1}, {F, 2, 30}]
Manipulate[Plot[N[eigen2[alpha, F, b]], {alpha, 0.0001, 10},
  PlotRange → {-0.01, 0.01}], {b, 0, 1}, {F, 2, 50}]
```

Out[289]=



Out[290]=




```
(*
Different from the pattern for the first eigenvalue, we can see that,
within the parametric space delimited by [F ⩾ 2, alpha > 0, and 0 ⩽ b ⩽ 1],
every value for the parameter alpha results in a positive value for
 the second eigenvalue of the Hessian matrix of the fitness function
*)

Clear[F, b, alpha, eig2];
eig2 =
  (alpha^2 (F + F^2 + (-1 + F) ProductLog[e^(alpha+b+F) F] + √(F^2 (5 + F (-14 + 13 F)) + (-1 + F)
           ProductLog[e^(alpha+b+F) F] (2 (1 - 3 F) F + (-1 + F) ProductLog[e^(alpha+b+F) F])))) /
   (2 (-1 + F) (F - ProductLog[e^(alpha+b+F) F]) ProductLog[e^(alpha+b+F) F]);
Solve[eig2 == 0, alpha]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\left\{\{\text{alpha}\to 0\},\ \left\{\text{alpha}\to -b-F+\text{Log}\left[\frac{e^{\frac{1}{2}(-1+3F)}\,(-1+3F)}{2F}\right]\right\}\right\}$$

```
b = 0;
F = 2;
```

$$\text{alpha} = N\left[-b - F + \text{Log}\left[\frac{e^{\frac{1}{2}(-1+3F)}(-1+3F)}{2F}\right]\right]$$

```
N[eig2]
```

$$\text{alpha} = N\left[-b - F + \text{Log}\left[\frac{e^{\frac{1}{2}(-1+3F)}(-1+3F)}{2F}\right] + 2\right]$$

```
N[eig2]
0.723144
-3.55597
2.72314
-7.837
```

```
(*
Nevertheless, the condition for the parameter alpha to result in non positive
  values for the second eigenvalue is the same as for the first eigenvalue
  alpha ⩾ -b -F + Log[e^(1/2 (-1+3 F)) (-1+3 F) / (2 F)]
*)
```

# Calculating Values

## Energetic Expenditure to Obtain Each Mating Event (c) to Ensure Self-Consistency

### Case I: Offspring survival depends entirely on paternal care (i.e., b = 0)

```
(*
We explore the parametric space delimited by 10 ⩽ alpha ⩽ 60 and 2 ⩽ F ⩽ 10,
keeping the baseline offspring survival without care constant at zero
*)
```

```
Clear[cc, F, alpha, b, shape, cStar, h, female, baseline1, shape1, m];
```

$$\text{cc[F\_, alpha\_, b\_] = N}\left[\frac{-\text{F} + \text{ProductLog}\left[e^{\text{alpha+b+F}}\ \text{F}\right]}{\text{alpha F}^2}\right];$$

```
shape = Range[10, 60, 0.2];
female = Range[2, 10, 0.05];
cStar = {};

For[h = 1, h < 252, h++, {

  baseline1 = PadLeft[{}, Length[female], 0];
  shape1 = PadLeft[{}, Length[female], shape[[h]]];

  m = MapThread[cc, {F = female, alpha = shape1, b = baseline1}];

  AppendTo[cStar, m];
 }]

cStar
```

```
{{0.20889, 0.199256, 0.190281, 0.181904, 0.174075, 0.166745,
  0.159874, 0.153423, 0.147358, 0.14165, … 141 …, 0.0102179,
  0.0101144, 0.0100125, 0.00991211, 0.00981322, 0.00971582,
  0.00961986, 0.00952532, 0.00943217, 0.00934039}, … 250 …}
```

large output | show less | show more | show all | set size limit...

## Case 2: Offspring survival depends partially on paternal care (i.e., b > 0)

```
(*
First, we explore the parametric space delimited by 10 ⩽ alpha ⩽ 60 and 2 ⩽ F ⩽
 10, keeping the baseline offspring survival without care constant at 0.25
*)
```

```
Clear[cc,  F, alpha, b, shape, cStar, h, female, baseline1, shape1, m];
```

$$cc[F\_,\ alpha\_,\ b\_] = N\left[\frac{-F + \mathrm{ProductLog}\left[e^{alpha+b+F}\ F\right]}{alpha\ F^2}\right];$$

```
shape = Range[10, 60, 0.2];
female = Range[2, 10, 0.05];
cStar = {};

For[h = 1, h < 252, h++, {

  baseline1 = PadLeft[{}, Length[female], 0.25];
  shape1 = PadLeft[{}, Length[female], shape[[h]]];

  m = MapThread[cc, {F = female, alpha = shape1, b = baseline1}];

  AppendTo[cStar, m];
 }]

cStar
```

```
{{0.214596, 0.20469, 0.195461, 0.186849, 0.1788,
  0.171265, 0.164202, 0.15757, 0.151337, 0.145469, … 141 …,
  0.0104783, 0.0103722, 0.0102676, 0.0101646, 0.0100632, 0.00996327,
  0.00986484, 0.00976786, 0.00967232, 0.00957817}, … 250 …}
```

large output | show less | show more | show all | set size limit...

```
(*
Then, we explore the parametric space delimited by 2 ⩽ F ⩽ 10 and 0 ⩽ b ⩽ 0.8,
keeping the shape parameter of the offspring
 survival exponential function constant at 20
*)
```

```
Clear[cc, F, alpha, b, shape, baseline, cStar, h, female, baseline1, shape1, m];
```

$$cc[F\_,\ alpha\_,\ b\_] = N\left[\frac{-F + ProductLog\left[e^{alpha+b+F}\ F\right]}{alpha\ F^2}\right];$$

```
baseline = Range[0, 0.8, 0.005];
female = Range[2, 10, 0.05];
cStar = {};

For[h = 1, h < 162, h++, {

  shape1 = PadLeft[{}, Length[female], 20];
  baseline1 = PadLeft[{}, Length[female], baseline[[h]]];

  m = MapThread[cc, {F = female, alpha = shape1, b = baseline1}];

  AppendTo[cStar, m];
 }]

cStar
```

```
{{0.221399, 0.210982, 0.201287, 0.192251, 0.183813,
  0.175922, 0.168532, 0.161601, 0.155091, 0.14897, ··· 141 ···,
  0.010366, 0.0102601, 0.0101559, 0.0100532, 0.00995207, 0.00985247,
  0.00975437, 0.00965773, 0.00956252, 0.00946872}, ··· 160 ···}
```

large output | show less | show more | show all | set size limit...

# Mating, Fertilization and Parental Effort at ESA

## Calculating the General Expressions for Each Effort

```
(*
First, we ensure self-consistency for the solutions s* p*, using the
 solution for the the energetic expenditure to obtain each mating event (c)
*)

ClearAll["Global`*"];
```

$$c = \frac{-F + \text{ProductLog}\left[e^{\text{alpha}+b+F}\ F\right]}{\text{alpha}\ F^2};$$

$$n[s\_,\ sHat\_,\ p\_,\ pHat\_] = \frac{nHat * (c + sHat) + pHat - p}{c + s};$$

$$s = c\ (-1 + F);$$

$$p = \frac{-b + \text{Log}[1 + c\ F\ \text{alpha}]}{\text{alpha}};$$

```
Simplify[s]
Simplify[p]
```

$$\frac{(-1 + F)\ \left(-F + \text{ProductLog}\left[e^{\text{alpha}+b+F}\ F\right]\right)}{\text{alpha}\ F^2}$$

$$\frac{-b + \text{Log}\left[\frac{\text{ProductLog}\left[e^{\text{alpha}+b+F}\ F\right]}{F}\right]}{\text{alpha}}$$

```
sHat = s;
pHat = p;
nHat = F;

(*
Then, we calculate mating effort ( M* ) as the relative
 allocation to obtaining matings over the breeding season
*)
mStar = c * n[s, sHat, p, pHat]
```

$$\frac{-F + \text{ProductLog}\left[e^{\text{alpha}+b+F}\ F\right]}{\text{alpha}\ F}$$

```
(*
Then, we calculate fertilization effort ( E* ) as the relative
 allocation to ejaculate production over the breeding season
*)
eStar = s * n[s, sHat, p, pHat]
```

$$\frac{(-1 + F)\ \left(-F + \text{ProductLog}\left[e^{\text{alpha}+b+F}\ F\right]\right)}{\text{alpha}\ F}$$

```
eStar0 = s * n[s, sHat, p, pHat]
```

$$\frac{(-1+F)\left(-F+\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]\right)}{\text{alpha}\,F}$$

```
(*
Finally, we calculate paternal effort ( P* ) as the
 relative allocation to care of the one clutch produced
*)
pStar = Simplify[p]
```

$$\frac{-b+\text{Log}\left[\frac{\text{ProductLog}\left[e^{\text{alpha}+b+F}\,F\right]}{F}\right]}{\text{alpha}}$$

## Case 1: Offspring survival depends entirely on paternal care (i.e., b = 0)

```
(*
We explore the parametric space delimited by 10 ≤ alpha ≤ 60 and 2 ≤ F ≤ 15,
keeping the baseline offspring survival without care constant at zero
*)
```

```
(*
First, we calculate the mating effort ( M* )
*)
Clear[mmm,  F, alpha, b, shape, female, mStar, h, shape1, baseline1, m];

mmm[F_, alpha_, b_] = N[(-F + ProductLog[e^(alpha+b+F) F])/(alpha F)];

shape = Range[10, 60, 0.2];
female = Range[2, 15, 0.1];
mStar = {};

For[h = 1, h < 252, h++, {

  shape1 = PadLeft[{}, Length[female], shape[[h]]];
  baseline1 = PadLeft[{}, Length[female], 0];

  m = MapThread[mmm, {F = female, alpha = shape1,  b = baseline1}];

  AppendTo[mStar, m];
 }]

mStar
```

```
{{0.417781, 0.39959, 0.382965, 0.36771, 0.353659, 0.340675,
  0.328637, 0.317445, 0.307012, 0.297261, 0.288128, 0.279554, 0.271489,
  0.263889, ··· 103 ···, 0.0691558, 0.0686753, 0.0682015, 0.0677342,
  0.0672733, 0.0668187, 0.0663703, 0.0659279, 0.0654914, 0.0650607,
  0.0646356, 0.0642161, 0.0638021, 0.0633934}, ··· 249 ···, {··· 1 ···}}
```

large output | show less | show more | show all | set size limit...

```
(*
Then, we calculate the fertilization effort ( E* )
*)
Clear[eee,  F, alpha, b, shape, female, eStar, h, shape1, baseline1, e];

eee[F_, alpha_, b_] = N[((-1 + F) (-F + ProductLog[ⅇ^(alpha+b+F) F]))/(alpha F)];

shape = Range[10, 60, 0.2];
female = Range[2, 15, 0.1];
eStar = {};

For[h = 1, h < 252, h++, {

  shape1 = PadLeft[{}, Length[female], shape[[h]]];
  baseline1 = PadLeft[{}, Length[female], 0];

  e = MapThread[eee, {F = female, alpha = shape1,  b = baseline1}];

  AppendTo[eStar, e];
 }]

eStar
```

```
{{0.417781, 0.439549, 0.459558, 0.478023, 0.495123, 0.511012, 0.525819, 0.539657,
  0.552621, 0.564796, 0.576256, 0.587063, 0.597276, 0.606944, 0.616111, … 102 …,
  0.878279, 0.879044, 0.879799, 0.880545, 0.881281, 0.882007, 0.882725, 0.883434,
  0.884134, 0.884825, 0.885508, 0.886183, 0.886849, 0.887508}, … 250 …}
```

large output | show less | show more | show all | set size limit...

```
(*
Then, we calculate the paternal effort ( P* )
*)
Clear[ppp,  F, alpha, b, shape, female, mStar, h, shape1, baseline1, pp];

ppp[F_, alpha_, b_] = N[(-b + Log[ProductLog[E^(alpha+b+F) F]/F])/alpha];

shape = Range[10, 60, 0.2];
female = Range[2, 15, 0.1];
pStar = {};

For[h = 1, h < 252, h++, {

  shape1 = PadLeft[{}, Length[female], shape[[h]]];
  baseline1 = PadLeft[{}, Length[female], 0];

  pp = MapThread[ppp, {F = female, alpha = shape1, b = baseline1}];

  AppendTo[pStar, pp];
 }]

pStar
```

```
{{0.164438, 0.160862, 0.157477, 0.154268, 0.151218, 0.148314, 0.145544, 0.142898,
  0.140367, 0.137942, 0.135616, 0.133383, 0.131235, 0.129168, 0.127176,
  0.125255, 0.123402, 0.121611, 0.119879, 0.118204, 0.116582, 0.11501,
  0.113486, 0.112008, 0.110573, 0.109179, 0.107824, 0.106507, 0.105226,
  0.103979, 0.102764, 0.101581, 0.100428, 0.0993039, 0.0982073, 0.0971373,
  0.0960928, 0.0950728, 0.0940764, 0.0931027, 0.092151, 0.0912203, 0.0903101,
  0.0894195, 0.0885478, 0.0876945, 0.086859, 0.0860405, 0.0852387, 0.0844528,
  0.0836825, 0.0829273, 0.0821866, 0.0814601, 0.0807472, 0.0800477, 0.0793611,
  0.078687, 0.0780251, … 14 …, 0.0693497, 0.0688432, 0.0683445, 0.0678534,
  0.0673697, 0.0668931, 0.0664236, 0.0659611, 0.0655052, 0.0650559, 0.064613,
  0.0641765, 0.063746, 0.0633216, 0.0629031, 0.0624903, 0.0620831, 0.0616815,
  0.0612853, 0.0608943, 0.0605085, 0.0601278, 0.0597521, 0.0593812, 0.0590151,
  0.0586537, 0.0582968, 0.0579444, 0.0575965, 0.0572528, 0.0569134, 0.0565782,
  0.056247, 0.0559199, 0.0555966, 0.0552772, 0.0549616, 0.0546497, 0.0543415,
  0.0540368, 0.0537357, 0.053438, 0.0531437, 0.0528527, 0.052565, 0.0522806,
  0.0519992, 0.051721, 0.0514459, 0.0511738, 0.0509046, 0.0506383, 0.0503749,
  0.0501143, 0.0498564, 0.0496013, 0.0493489, 0.0490991}, … 249 …, {… 1 …}}
```

large output | show less | show more | show all | set size limit...

```
(*
Finally, we calculate the parental effort
 ( P* ) for particular combinations of F and alpha
*)
Clear[ppp, F, alpha, b, shape, female, mStar, h, shape1, baseline1, pp];

ppp[F_, alpha_, b_] = N[(-b + Log[ProductLog[e^(alpha+b+F) F]/F])/alpha];

shape = {10, 20, 40, 60, 80};
female = Range[2, 15, 1];
pESPECIAL = {};

For[h = 1, h < 6, h++, {

  shape1 = PadLeft[{}, Length[female], shape[[h]]];
  baseline1 = PadLeft[{}, Length[female], 0];

  pp = MapThread[ppp, {F = female, alpha = shape1, b = baseline1}];

  AppendTo[pESPECIAL, pp];
 }]

pESPECIAL
{{0.164438, 0.135616, 0.116582, 0.102764, 0.092151, 0.0836825, 0.0767364,
  0.0709175, 0.0659611, 0.0616815, 0.0579444, 0.0546497, 0.051721, 0.0490991},
 {0.114404, 0.0974183, 0.0858753, 0.077281, 0.0705279, 0.065028, 0.0604319,
  0.0565157, 0.0531276, 0.0501598, 0.0475335, 0.0451894, 0.0430817, 0.0411744},
 {0.0742788, 0.0650053, 0.0585793, 0.0537077, 0.049815, 0.0465944, 0.0438631,
  0.0415032, 0.0394345, 0.0375999, 0.0359572, 0.0344745, 0.0331271, 0.0318953},
 {0.0562994, 0.04993, 0.0454835, 0.0420887, 0.0393577, 0.0370835, 0.0351428,
  0.033456, 0.0319688, 0.0306425, 0.0294485, 0.0283653, 0.027376, 0.0264673},
 {0.0458476, 0.0409989, 0.0376007, 0.0349965, 0.0328939, 0.0311369, 0.0296323,
  0.0283203, 0.0271598, 0.0261217, 0.0251842, 0.0243312, 0.0235499, 0.0228302}}
```

## Case 2: Offspring survival depends partially on paternal care (i.e., b > 0)

```
(*
First, we explore the parametric space delimited by 10 ≤ alpha ≤ 60 and 2 ≤ F ≤
 15, keeping the baseline offspring survival without care constant at 0.25
*)
```

```
(*
We calculate the mating effort ( M* )
*)
Clear[mmm,  F, alpha, b, shape, female, mStar, h, shape1, baseline1, m];

mmm[F_, alpha_, b_] = N[(-F + ProductLog[e^(alpha+b+F) F])/(alpha F)];

shape = Range[10, 60, 0.2];
female = Range[2, 15, 0.1];
mStar = {};

For[h = 1, h < 252, h++, {

  shape1 = PadLeft[{}, Length[female], shape[[h]]];
  baseline1 = PadLeft[{}, Length[female], 0.25];

  m = MapThread[mmm, {F = female, alpha = shape1, b = baseline1}];

  AppendTo[mStar, m];
 }]

mStar
```

```
{{0.429191, 0.410469, 0.39336, 0.377664, 0.363208, 0.34985,
  0.337468, 0.325957, 0.315227, 0.3052, 0.295808, 0.286992, 0.278701,
  0.270887, … 103 …, 0.0709055, 0.0704127, 0.0699266, 0.0694473,
  0.0689746, 0.0685083, 0.0680484, 0.0675946, 0.0671469, 0.0667051,
  0.0662691, 0.0658389, 0.0654142, 0.064995}, … 249 …, {… 1 …}}
```

large output | show less | show more | show all | set size limit...

```
(*
Then, we calculate the fertilization effort ( E* )
*)
Clear[eee,  F, alpha, b, shape, female, eStar, h, shape1, baseline1, e];

eee[F_, alpha_, b_] = N[((-1 + F) (-F + ProductLog[e^(alpha+b+F) F]))/(alpha F)];

shape = Range[10, 60, 0.2];
female = Range[2, 15, 0.1];
eStar = {};

For[h = 1, h < 252, h++, {

  shape1 = PadLeft[{}, Length[female], shape[[h]]];
  baseline1 = PadLeft[{}, Length[female], 0.25];

  e = MapThread[eee, {F = female, alpha = shape1,  b = baseline1}];

  AppendTo[eStar, e];
 }]

eStar
```

```
{{0.429191, 0.451515, 0.472033, 0.490963, 0.508491, 0.524775, 0.539949, 0.554126,
  0.567408, 0.579879, 0.591616, 0.602684, 0.613142, 0.62304, … 103 …, 0.9005,
  0.901282, 0.902054, 0.902815, 0.903568, 0.90431, 0.905043, 0.905768, 0.906483,
  0.907189, 0.907887, 0.908576, 0.909258, 0.90993}, … 249 …, {… 1 …}}
```

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```
(*
Then, we calculate the paternal effort ( P* )
*)
Clear[ppp,  F, alpha, b, shape, female, mStar, h, shape1, baseline1, pp];

ppp[F_, alpha_, b_] = N[(-b + Log[ProductLog[e^(alpha+b+F) F]/F])/alpha];

shape = Range[10, 60, 0.2];
female = Range[2, 15, 0.1];
pStar = {};

For[h = 1, h < 252, h++, {

  shape1 = PadLeft[{}, Length[female], shape[[h]]];
  baseline1 = PadLeft[{}, Length[female], 0.25];

  pp = MapThread[ppp, {F = female, alpha = shape1, b = baseline1}];

  AppendTo[pStar, pp];
 }]

pStar
```

```
{{0.141618, 0.138016, 0.134607, 0.131374, 0.128301, 0.125374,
  0.122583, 0.119917, 0.117365, 0.114921, 0.112576, 0.110323,
  0.108158, 0.106073, ··· 104 ···, 0.0283053, 0.0280197, 0.0277372,
  0.0274578, 0.0271815, 0.0269082, 0.0266378, 0.0263703, 0.0261056,
  0.0258438, 0.0255846, 0.0253283, 0.0250745}, ··· 249 ···, {··· 1 ···}}
```

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```
(*
Finally, we calculate the parental effort
 ( P* ) for particular combinations of F and alpha
*)
Clear[ppp,  F, alpha, b, shape, female, mStar, h, shape1, baseline1, pp];

ppp[F_, alpha_, b_] = N[(-b + Log[ProductLog[E^(alpha+b+F) F]/F])/alpha];

shape = {10, 20, 40, 60, 80};
female = Range[2, 15, 1];
pESPECIAL = {};

For[h = 1, h < 6, h++, {

   shape1 = PadLeft[{}, Length[female], shape[[h]]];
   baseline1 = PadLeft[{}, Length[female], 0.25];

   pp = MapThread[ppp, {F = female, alpha = shape1,  b = baseline1}];

   AppendTo[pESPECIAL, pp];
  }]

pESPECIAL
{{0.141618, 0.112576, 0.0933739, 0.0794212, 0.0686946, 0.0601292, 0.0530987,
   0.0472056, 0.042183, 0.0378442, 0.0340536, 0.0307103, 0.0277372, 0.0250745},
  {0.102504, 0.0854821, 0.0739095, 0.0652896, 0.0585141, 0.052994, 0.0483797,
   0.0444469, 0.0410434, 0.0380615, 0.0354221, 0.0330657, 0.0309466, 0.0290286},
  {0.0681845, 0.0589058, 0.0524754, 0.0475998, 0.0437034, 0.0404795, 0.037745,
   0.0353822, 0.0333107, 0.0314734, 0.0298281, 0.028343, 0.0269932, 0.0257593},
  {0.0522025, 0.0458315, 0.0413836, 0.0379876, 0.0352553, 0.0329801, 0.0310383,
   0.0293505, 0.0278623, 0.0265351, 0.0253403, 0.0242562, 0.0232661, 0.0223565},
  {0.042762, 0.0379125, 0.0345137, 0.031909, 0.0298059, 0.0280483, 0.0265434,
   0.0252309, 0.02407, 0.0230314, 0.0220935, 0.0212401, 0.0204584, 0.0197383}}

(*
Then, we explore the parametric space delimited by 2 ⩽ F ⩽ 15 and 0 ⩽ b ⩽ 0.8,
keeping the shape parameter of the offspring
 survival exponential function constant at 20
*)
```

```
(*
We calculate the mating effort ( M* )
*)
Clear[mmm, F, alpha, b, baseline, female, mStar, h, shape1, baseline1, m];

mmm[F_, alpha_, b_] = N[(-F + ProductLog[e^(alpha+b+F) F])/(alpha F)];

baseline = Range[0, 0.8, 0.005];
female = Range[2, 15, 0.1];
mStar = {};

For[h = 1, h < 162, h++, {

  baseline1 = PadLeft[{}, Length[female], baseline[[h]]];
  shape1 = PadLeft[{}, Length[female], 20];

  m = MapThread[mmm, {F = female, alpha = shape1, b = baseline1}];

  AppendTo[mStar, m];
 }]

mStar
```

```
{{0.442798, 0.422704, 0.404388, 0.387623, 0.372219, 0.358014,
  0.344874, 0.332682, 0.321338, 0.310756, 0.300861, 0.291587, 0.282879,
  0.274684, … 103 …, 0.0698035, 0.0693126, 0.0688286, 0.0683513,
  0.0678807, 0.0674165, 0.0669587, 0.0665072, 0.0660617, 0.0656222,
  0.0651886, 0.0647607, 0.0643385, 0.0639217}, … 159 …, {… 1 …}}
```

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```
(*
Then, we calculate the fertilization effort ( E* )
*)
Clear[eee,  F, alpha, b, baseline, female, eStar, h, shape1, baseline1, e];

eee[F_, alpha_, b_] = N[((-1 + F) (-F + ProductLog[e^(alpha+b+F) F]))/(alpha F)];

baseline = Range[0, 0.8, 0.005];
female = Range[2, 15, 0.1];
eStar = {};

For[h = 1, h < 162, h++, {

  baseline1 = PadLeft[{}, Length[female], baseline[[h]]];
  shape1 = PadLeft[{}, Length[female], 20];

  e = MapThread[eee, {F = female, alpha = shape1, b = baseline1}];

  AppendTo[eStar, e];
 }]

eStar
```

```
{{0.442798, 0.464974, 0.485266, 0.50391, 0.521106, 0.537022, 0.551799,
  0.565559, 0.578408, 0.590436, 0.601721, 0.612334, 0.622333, 0.631774,
  0.640703, 0.649162, 0.657189, 0.664816, 0.672075, 0.678993, 0.685593,
  0.691899, 0.69793, 0.703704, 0.709238, 0.714547, 0.719646, 0.724547,
  0.729262, 0.733801, ⋯ 71 ⋯, 0.873976, 0.874845, 0.875701, 0.876545,
  0.877377, 0.878197, 0.879005, 0.879802, 0.880588, 0.881364, 0.882128,
  0.882882, 0.883626, 0.88436, 0.885085, 0.8858, 0.886505, 0.887201,
  0.887888, 0.888567, 0.889237, 0.889898, 0.890551, 0.891196, 0.891833,
  0.892462, 0.893084, 0.893698, 0.894305, 0.894904}, ⋯ 159 ⋯, {⋯ 1 ⋯}}
```

large output | show less | show more | show all | set size limit...

```
(*
Then, we calculate the paternal effort ( P* )
*)
Clear[ppp,  F, alpha, b, baseline, female, mStar, h, shape1, baseline1, pp];

ppp[F_, alpha_, b_] = N[(-b + Log[ProductLog[E^(alpha+b+F) F]/F])/alpha];

baseline = Range[0, 0.8, 0.005];
female = Range[2, 15, 0.1];
pStar = {};

For[h = 1, h < 162, h++, {

  baseline1 = PadLeft[{}, Length[female], baseline[[h]]];
  shape1 = PadLeft[{}, Length[female], 20];

  pp = MapThread[ppp, {F = female, alpha = shape1,  b = baseline1}];

  AppendTo[pStar, pp];
 }]

pStar
```

```
{{0.114404, 0.112322, 0.110346, 0.108467, 0.106675, 0.104964, 0.103327, 0.101759,
  0.100254, 0.0988089, 0.0974183, 0.096079, 0.0947878, 0.0935415, 0.0923376,
  0.0911734, 0.0900467, 0.0889554, 0.0878975, 0.0868714, 0.0858753,
  0.0849078, 0.0839674, 0.0830528, 0.0821628, 0.0812963, 0.0804522,
  0.0796295, 0.0788273, … 74 …, 0.046803, 0.0465649, 0.0463293, 0.0460964,
  0.045866, 0.045638, 0.0454125, 0.0451894, 0.0449687, 0.0447503, 0.0445341,
  0.0443202, 0.0441085, 0.0438989, 0.0436915, 0.0434862, 0.0432829,
  0.0430817, 0.0428825, 0.0426852, 0.0424899, 0.0422965, 0.042105, 0.0419153,
  0.0417274, 0.0415413, 0.041357, 0.0411744}, … 159 …, {… 1 …}}
```

large output | show less | show more | show all | set size limit...

```
(*
Finally, we calculate the parental effort
 ( P* ) for particular combinations of F and alpha
*)
Clear[ppp,  F, alpha, b, shape, female, mStar, h, shape1, baseline1, pp];

ppp[F_, alpha_, b_] = N[(-b + Log[ProductLog[E^(alpha+b+F) F]/F])/alpha];

baseline = {0.1, 0.2, 0.4, 0.6};
female = Range[2, 15, 1];
pESPECIAL = {};

For[h = 1, h < 5, h++, {

   baseline1 = PadLeft[{}, Length[female], baseline[[h]]];
   shape1 = PadLeft[{}, Length[female], 20];

   pp = MapThread[ppp, {F = female, alpha = shape1, b = baseline1}];

   AppendTo[pESPECIAL, pp];
  }]

pESPECIAL
{{0.109645, 0.0926445, 0.0810897, 0.072485, 0.0657229, 0.0602149, 0.0556115,
   0.0516887, 0.0482943, 0.0453208, 0.0426893, 0.0403403, 0.038228, 0.0363164},
 {0.104884, 0.0878698, 0.0763031, 0.0676883, 0.0609172, 0.0554012, 0.0507905,
   0.046861, 0.0434605, 0.0404814, 0.0378446, 0.0354907, 0.0333738, 0.0314579},
 {0.0953607, 0.0783175, 0.0667274, 0.0580925, 0.0513036, 0.0457717, 0.0411466,
   0.0372039, 0.0337914, 0.030801, 0.0281538, 0.0257902, 0.0236643, 0.0217399},
 {0.0858327, 0.0687614, 0.0571482, 0.0484935, 0.0416872, 0.0361396, 0.0315003,
   0.0275446, 0.0241201, 0.0211187, 0.0184612, 0.016088, 0.0139531, 0.0120203}}
```