Augmenting search trees

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- It's supposed to be easy-to-understand and maybe rough
- But presentation is not completed yet so neither of these goals is achieved

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And also there's direct and straightforward applications of such topics
Often you don't have a choice, but to use scientific approach

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```
def rsq(l, r):
  return cumsum[r] - (cumsum[l-1] if l != 0 else 0)
```

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A bit more complicated?
Actually much-much larger class of similar problems is solvable with generic approach! (stay tuned)

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Let `cache(l, r)` be precomputed minimal value in subarray `a[l..r]` where l and r are powers of 2
Begining with $tl = 1, tr = n$ we can solve task with simple recursive logic

```
def rmq(l, r, tl, tr):
  if not intersects(l, r, tl, tr) or tl>tr: return INFINITY;
  if contains(l, r, tl, tr): return cache(l,r)
  m = (tl+tr)//2
  infimum = INFINITY
  if intersects(l, r, tl, m):
    infimum = min(infimum, rmq(l, r, tl, m))
  if intersects(l, r, m+1, tr):
     infimum = min(infimum, rmq(l, r, m+1, tr))
  return infimum
```

RMQ

Such cache can be easily represented with binary tree, where firt node assigned to segment $1..n$, it's children to segments $1..m$ and $m+1..n$ and so on.

RSQ using array-based tree

Let's represent binary tree with array indices arithmetics:

- Each node is assigned a number v, which is index in array
- Root's index is 0
- Left child of v has index $2v + 1$
- Right child of v has index $2v + 2$
- Parent of v has index floor$((v - 1)/2)$

RSQ using array-based tree (API)

```
template<typename T>
class RSQ {
  private:
    int n;
    int buffsize;
    T*  tree;
  public:
    RSQ(int n);
    ~RSQ();
    void put(int k, T v);
    T get(int k);
    T sum(int l, int r);
  private:
    // i
    put(int i, T v, int ti, int tl, int tr);
    T sum(int l, int r, int ti, int tl, int tr);
```

Constructor

```
RSQ(int n) {
  this->n    = n;
  this->buffsize=4*n;
  this->tree = new T[this->buffsize];
  for (int i = this->buffsize - 1; i >= 0; --i) this->tree[i] = 0;
}
~RSQ() {
  delete[] this->tree;
}
```

Retrieving

```
T get(int l, int r, int ti, int tl, int tr) {
  if (ti >= this->buffsize) return 0;
  if (tl > tr) return 0;
  if (l <= tl && tr <= r) return this->tree[ti];
  int m = (tl+tr)/2;
  T s = 0;
  if (intersects(l, r, tl, m))   s += get(l, r, left(ti),  tl,  m);
  if (intersects(l, r, m+1, tr)) s += get(l, r, right(ti), m+1, tr);
  return s;
}
```

Updating

```
void set(int i, T v, int ti, int l, int r) {
  if (ti >= this->buffsize) return;
  if (l>r)  return;
  if (l==r) {
    this->tree[ti] = v;
  }
  else {
    int m = (l+r)/2;
    if (i <= m) set(i, v, left(ti),  l,   m);
    else        set(i, v, right(ti), m+1, r);
    this->tree[ti] = this->tree[left(ti)] + this->tree[right(ti)];
  }
}
```

Lazy propagation

Online problem

Given n — maximal number of elements, and q — number of queries, handle q queries of following types:

- `put(k, v)`: set k'th item to be equal to v
- `get(l, r)`: find sum/minimum/whatever in subarray

Arbitrary keys

Now let's make our task a bit trickier and say, that we want, for example, say that keys are points in time or whatever with total order defined on it, instead of indices of array
And like before we want perform

- `put(k, v)`
- `get(l, r)`

reasonably fast

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What to do?
Binary Search Tree, it's straightforward
Can we handle range queries anyhow but bruteforcing?
YES!

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And what the heck is magma?
Magma is just a set with some binary operation on it, w/o any restrictions

Monoid

Monoid is a set with an associative binary operation on it and an identity element regarding this operation

Augmentation

The idea is following:
If you're considering node n, which represents range `[n.lk, n.rk]`, and you know the multiple `n.l.mul` of values with keys in range `n.l.lk, n.l.rk` and multiple of `n.r.mul` of values with keys in range `n.r.lk, n.r.lk`, then simply `n.mul = n.l.mul * n.v * n.r.mul`

Augmentation

More generally, if range $l..r$ is requested, and we're in node n, then we either

- return 0 if $[n.lk, n.rk] \cap [l, r] = \emptyset$
- return `n.mul` if $[n.lk, n.rk] \subset [l, r]$
- recursively go into childrens and combine answers and value in current node

Augmentation theorem

Let K be totally-ordered set and $(M, \circ, 1)$ be monoid and let $T \subset M$ note values in tree.
Then following operations can be performed in time logarithmic in input size just by storing additional data in the nodes of tree and maintaining this data during rotations:

- `put(k, v)` Associate key k with value v
- `mul(l, r)` Calculate $v_{k_1} \circ v_{k_2} \circ \cdots \circ v_{k_m}$, where $v_{k_j} \in M \cap T$ and $l, r, k_j \in K$ and $l \leq k_j \leq r$ and $k_i \leq k_j$ for all $i \leq j = \overline{1, m}$

Augmenting search trees

Let's say we implemented red-black BST with following API:

```
class Node:
    def __init__(self, k, v): pass
class RBBST:
    def __init__(self):
      self.Node = Node # to be overriden
      pass
    def restore(self, h): pass # to be overriden
    def balance(self, h): pass
    def put(self, k, v):  pass
    def get(self, k):     pass
    def rotate_left(self, h): pass
    def flip_colors(self, h): pass
    def rotation_fix(self, x, h):  pass
```

Augmenting search trees

Here `balance` performs required rotations calls `restore` if necessary and returns new root of subtree It is called during insertion in following manner:

Augmenting search trees

```
    def __put__(self, h, k, v):
        """__put__(h, k, v)
h:  root of subtree
k:  key
v:  value
recursive method to insert new kv-pair into tree
and maintain balance"""
        if h is None: return self.Node(k, v)
        if k < h.k:      h.l = self.__put__(h.l, k, v)
        elif h.k < k:    h.r = self.__put__(h.r, k, v)
        else:            h.v = v
        self.restore(h)
        h = self.balance(h)
        return h
```

We will augment tree by overriding `Node` class and `restore` method

SegmentTree (API)

```
class SegmentTree(rbbst.RBBST):
    """SegmentTree(mul, id)
mul:    multiplication operation (callable, binary)
id:     identity element
*   put(k, v) where $k\\in K, v\\in M$
        associates key 'k' with value 'v'
*   get(k)  where $k\\in K$
        retrieve value associated with key 'k' if any
*   mul(l, r) where $l,r\\in K$ and $l\\leq r$"""
```

SegmentTree (API)

```
def add(x, y): return x+y
t = SegmentTree(add, 0)
t.put('a', 1) # associate value 1 with key 'a'
t.put('b', 3)
t.put('c', 22)
t.mul('a', 'c') # -> 26
t.mul('a', 'a') # -> 1
```

SegmentTree (API)

```
def add(x, y): return x+y
t = SegmentTree(add, '')
t.put(1, 'some ')
t.put(10**32, 'strings') # we can use some large 'indices'* * *
t.put(-10**9, 'concat ')
t.mul(-10**64, 10**64)   # yields 'concat some strings'
```

SegmentTree Node

```
class Node:
    def __init__(self, k, v):
        self.k, self.v = k, v # key, value
        self.c         = RED  # color
        self.l, self.r = None, None # segment [l,r] represented by
    def __str__(self):
        return '(%s, %s, %s)'%(k, v, 'RED' if self.c else 'BLACK')
    def __repr__(self): return self.__str__()
```

SegmentTree maintaining aux data

```
    def restore(self, h):
        """restore(h)
overrided `restore` will update cumulative
after insertions and balancing"""
        assert not h is None
        h.lk, h.rk = h.k, h.k
        m = h.v
        if h.l:
            m = self.mulbin(h.l.mul, m)
            h.lk = h.l.lk
        if h.r:
            m = self.mulbin(m, h.r.mul)
            h.rk = h.r.rk
        h.mul = m
```

SegmentTree query

```
    def __mul__(self, h, l, r):
        """__mul__(h, l, r)
calculates cumulative in intersection of (h.lk, h.rk) and (l, r)"""
        s = self.id
        if l <= h.lk <= h.rk <= r: return h.mul
        if h.l and intersects(h.l.lk, h.l.rk, l, r):
            s = self.mulbin(self.__mul__(h.l, l, r), s)
        if l <= h.k <= r:
            s = self.mulbin(s, h.v)
        if h.r and intersects(h.r.lk, h.r.rk, l, r):
            s = self.mulbin(s, self.__mul__(h.r, l, r))
        return s
```

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