

MANUAL SOLUTION FOR THE TRANSPORT PROBLEM

(% i1) ratprint: false\$

1 Preparations

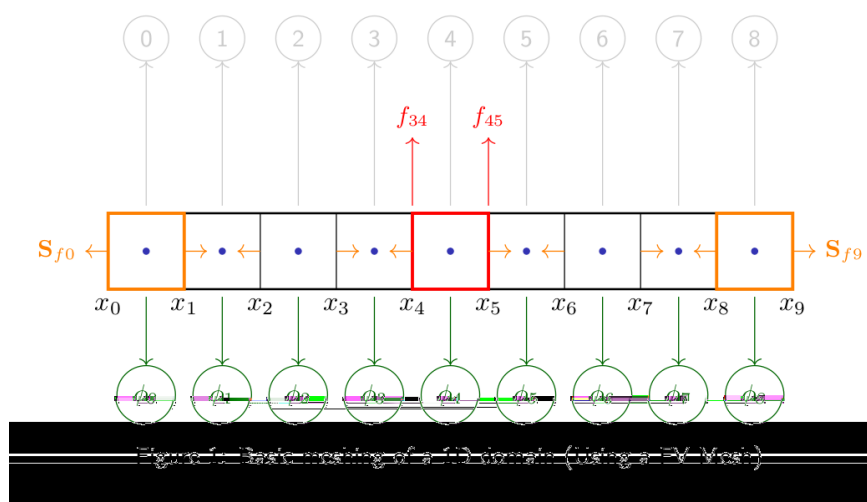


Figure 1: The Mesh we'll work on

How the gradient of ϕ at the face should be calculated (between cells i and j)?

(% i2) $\text{grad}(\phi, i, j, dx) := (\phi[i] - \phi[j]) / dx;$

(% o2) $\text{grad}(\phi, i, j, dx) := \frac{\phi_i - \phi_j}{dx}$

How to interpolate ϕ values from cell centers (between cells i and j) to face centers?

(% i3) $\text{face_inter}(\phi, i, j) := (\phi[i] + \phi[j]) / 2;$

(% o3) $\text{face_inter}(\phi, i, j) := \frac{\phi_i + \phi_j}{2}$

Gauss-Seidel iterations

```
(% i4) gauss_seidel(A,b,itors):=block(
  n:rank(A),
  x:float(zeromatrix(n,1)),
  y:float(zeromatrix(n,1)),
  dif:1,
  for m:1 thru iters do(
    for i:1 thru n do(
      y[i,1]:(b[i,1]-sum(A[i,j]*y[j,1],j,1,i-1)-sum(A[i,j]*x[j,1],j,i+1,n))/A[i,i]
    ),
    dif:sort (makelist(abs(x[i,1]-y[i,1]),i,1,n), 'ordergreatp),
    print ("Iteration ",m, ": ", transpose(y), ",dif=", dif[1]),
    x:copymatrix(y)
  ), return(x))$
```

Convective and diffusive setup

```
(% i7) U:0.03$ K:0.01$ dx:0.1$
```

2 Algebraic Equations for cells

For all internal cells (i going from 1 to 8):

```
(% i9) X:[\phi[i-1], \phi[i], \phi[i+1]]$ icell: U*face_inter(X,3,2)-K*grad(X, 3,2,dx) -
  U*face_inter(X,2,1) +K*grad(X, 2,1,0.1),ratsimp;
```

$$(icell) \quad -\frac{17\phi_{i+1} - 40\phi_i + 23\phi_{i-1}}{200}$$

```
(%      icell, expand, float;
i10)
```

$$(\% o10) \quad -0.085\phi_{i+1} + 0.2\phi_i - 0.115\phi_{i-1}$$

```
(%      X:[\phi[0], \phi[1]]$ xb:1$ cell0: U*face_inter(X,1,2)-K*grad(X, 2,1,0.1) - U*xb
i13) - K*(xb-X[1])/(dx/2), expand, float;
```

$$(cell0) \quad -0.085\phi_1 + 0.315\phi_0 - 0.23$$

```
(%      X:[\phi[8], \phi[9]]$ xb:0$ cell9: -U*face_inter(X,1,2)+K*grad(X, 2,1,0.1) +
i16) U*xb - K*(xb-X[2])/(dx/2), expand, float;
```

$$(cell9) \quad 0.285\phi_9 - 0.115\phi_8$$

3 Matrix Resolution

Collect coefficients from cells equations:

```
(%      A:matrix( [0.315 , -0.085, 0, 0, 0, 0, 0, 0, 0], [-0.115, 0.2, -0.085, 0, 0, 0, 0, 0, 0],
i17)    [0, -0.115, 0.2, -0.085, 0, 0, 0, 0, 0], [0, 0, -0.115, 0.2, -0.085, 0, 0, 0, 0], [0,
0, 0, -0.115, 0.2, -0.085, 0, 0, 0], [0, 0, 0, 0, -0.115, 0.2, -0.085, 0, 0], [0, 0, 0, 0,
0, -0.115, 0.2, -0.085, 0], [0, 0, 0, 0, 0, 0, -0.115, 0.2, -0.085], [0, 0, 0, 0, 0, 0, 0,
-0.115, 0.285] );
```

(A)

$$\begin{pmatrix} 0.315 & -0.085 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.115 & 0.2 & -0.085 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.115 & 0.2 & -0.085 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.115 & 0.2 & -0.085 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.115 & 0.2 & -0.085 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.115 & 0.2 & -0.085 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.115 & 0.2 & -0.085 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.115 & 0.2 & -0.085 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.115 & 0.285 \end{pmatrix}$$

Source term present only in first cell

```
(%      B:matrix([0.23],[0],[0],[0],[0],[0],[0],[0],[0]);
i18)
```

(B)

$$\begin{pmatrix} 0.23 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solve the system with Gauss-Seidel Method (run for 40 iterations and watch max solution change in an iteration)

```
(%      iterative_sol:gauss_seidel(A,B,40);
i19)
```

... Goes on about iteration results ...

$$\begin{array}{l}
 \text{(iterative_sol)} \\
 \left(\begin{array}{c}
 0.989130803064315 \\
 0.9598697431668418 \\
 0.9205309580963118 \\
 0.8676405533439283 \\
 0.7964673452312362 \\
 0.7005668030132802 \\
 0.5711678657073875 \\
 0.3963489778943668 \\
 0.1599302893257971
 \end{array} \right)
 \end{array}$$

4 Solution from Theory

Forget some variable so we can continue using the same names

```
(%      kill(U, K, x, y)$ assume(U> 0);
i21)
```

```
(% o21)      [U 0]
```

Compose and solve the ODE:

```
(%      ode: U*'diff(y, x) - K*'diff(y,x,2) = 0;
i22)
```

$$\text{(ode)} \quad U \left(\frac{d}{dx} y \right) - K \left(\frac{d^2}{dx^2} y \right) = 0$$

```
(%      gsol: ode2(ode, y, x);
i23)
```

$$\text{(gsol)} \quad y = \%k1 \%e^{\frac{Ux}{K}} + \%k2$$

Figure out k1 and k2 coeffs based on boundary values (involves solving a system of two equations)

```
(%      f0: rhs(ev(gsol, x=0))$ f9: rhs(ev(gsol, x=0.9))$
i25)
```

```
(%      sol:solve([f0=1, f9=0], [%k1, %k2]);
i26)
```

$$\text{(sol)} \quad \left[\left[\%k1 = -\frac{1}{\%e^{\frac{0.9U}{K}} - 1}, \%k2 = \frac{\%e^{\frac{0.9U}{K}}}{\%e^{\frac{0.9U}{K}} - 1} \right] \right]$$

```
(%      coeffs:ev(sol, U=0.03, K=0.01);
i27)
```

```
(coeffs)      [[%k1 = -0.07204750205711648, %k2 = 1.072047502057116]]
```

```
(%      theory_sol:makelist( ev(rhs(gsol), x=i/10.0+0.05, U=0.03, K=0.01,
i28)      %k1=rhs(coeffs[1][1]), %k2=rhs(coeffs[1][2])), i, 0, 8);
```

```
(theory_sol)
```

```
[0.9883402470641222, 0.9590545266468127, 0.9195229390052993, 0.8661608772499396, 0.7941296281990526, 0.69
```

```
(%      iterative_sol-theory_sol;
i29)
```

```
(% o29)      
$$\begin{pmatrix} 7.90556000192843610^{-4} \\ 8.15216520029138410^{-4} \\ 0.001008019091012557 \\ 0.001479676093988691 \\ 0.00233771703218355 \\ 0.003669190766267927 \\ 0.005519746571607254 \\ 0.007869143024669012 \\ 0.01060262339579992 \end{pmatrix}$$

```