Manual Solution for the transport Problem

(% i1) ratprint: false$

1 Preparations

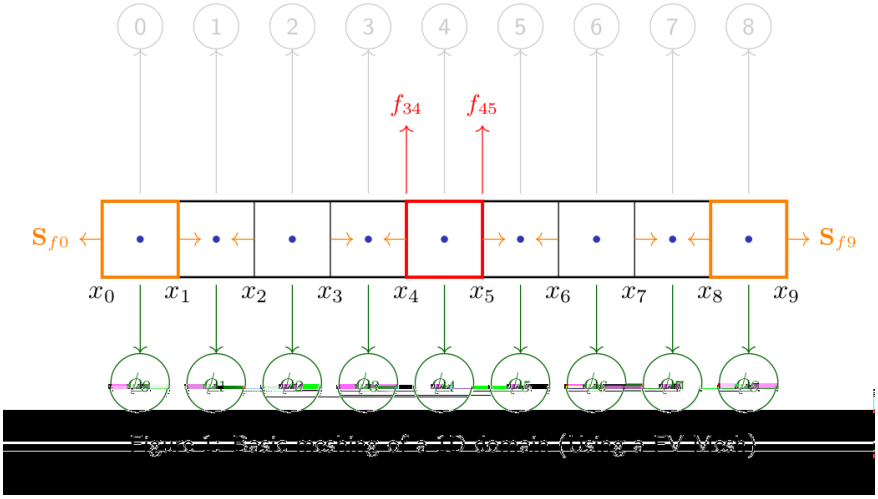

Figure 1: The Mesh we'll work on

How the gradient of ϕ at the face should be calculated (between cells i and j)?

(% i2) grad(\phi,i,j,dx) := (\phi[i]-\phi[j])/dx;

(% o2) $$\text{grad}\,(\phi, i, j, dx) := \frac{\phi_i - \phi_j}{dx}$$

How to interpolate ϕ values from cell centers (between cells i and j) to face centers?

(% i3) face_inter(\phi,i,j):= (\phi[i]+\phi[j])/2;

(% o3) $$\text{face\_inter}\,(\phi, i, j) := \frac{\phi_i + \phi_j}{2}$$

Gauss-Seidel iterations

```
(% i4) gauss_seidel(A,b,iters):=block(
          n:rank(A),
          x: oat(zeromatrix(n,1)),
          y: oat(zeromatrix(n,1)),
          dif:1,
          for m:1 thru iters do(
            for i:1 thru n do(
              y[i,1]:(b[i,1]-sum(A[i,j]*y[j,1],j,1,i-1)-sum(A[i,j]*x[j,1],j,i+1,n))/A[i,i]
            ),
            dif:sort (makelist(abs(x[i,1]-y[i,1]),i,1,n), 'ordergreatp),
            print ("Iteration ",m, ": ", transpose(y), ",dif=", dif[1]),
            x:copymatrix(y)
        ), return(x))$
```

Convective and di usive setup

```
(% i7) U:0.03$ K:0.01$ dx:0.1$
```

2 Algebraic Equations for cells

For all internal cells (i going from 1 to 8):

```
(% i9) X:[\phi[i-1], \phi[i], \phi[i+1]]$ icell: U*face_inter(X,3,2)-K*grad(X, 3,2,dx) -
       U*face_inter(X,2,1) +K*grad(X, 2,1,0.1),ratsimp;
```

(icell)
$$-\frac{17\phi_{i+1} - 40\phi_i + 23\phi_{i\ \ 1}}{200}$$

```
(% i10) icell, expand, oat;
```

(% o10)
$$-0.085\phi_{i+1} + 0.2\phi_i - 0.115\phi_{i\ \ 1}$$

```
(% i13) X:[\phi[0], \phi[1]]$ xb:1$ cell0: U*face_inter(X,1,2)-K*grad(X, 2,1,0.1) - U*xb
        - K*(xb-X[1])/(dx/2), expand, oat;
```

(cell0)
$$-0.085\phi_1 + 0.315\phi_0 - 0.23$$

```
(% i16) X:[\phi[8], \phi[9]]$ xb:0$ cell9: -U*face_inter(X,1,2)+K*grad(X, 2,1,0.1) +
        U*xb - K*(xb-X[2])/(dx/2), expand, oat;
```

(cell9)
$$0.285\phi_9 - 0.115\phi_8$$

3 Matrix Resolution

Collect coe cients from cells equations:

(%i17)
```
A:matrix( [0.315 , -0.085, 0, 0, 0, 0, 0, 0, 0], [-0.115, 0.2, -0.085, 0, 0, 0, 0, 0,
0], [0, -0.115, 0.2, -0.085, 0, 0, 0, 0, 0], [0, 0, -0.115, 0.2, -0.085, 0, 0, 0, 0], [0,
0, 0, -0.115, 0.2, -0.085, 0, 0, 0], [0, 0, 0, 0, -0.115, 0.2, -0.085, 0, 0], [0, 0, 0, 0,
0, -0.115, 0.2, -0.085, 0], [0, 0, 0, 0, 0, 0, -0.115, 0.2, -0.085], [0, 0, 0, 0, 0, 0, 0,
-0.115, 0.285] );
```

(A)

$$\begin{pmatrix} 0.315 & -0.085 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.115 & 0.2 & -0.085 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.115 & 0.2 & -0.085 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.115 & 0.2 & -0.085 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.115 & 0.2 & -0.085 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.115 & 0.2 & -0.085 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.115 & 0.2 & -0.085 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.115 & 0.2 & -0.085 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.115 & 0.285 \end{pmatrix}$$

Source term present only in rst cell

(%i18)
```
B:matrix([0.23],[0],[0],[0],[0],[0],[0],[0],[0]);
```

(B)

$$\begin{pmatrix} 0.23 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solve the system with Gauss-Seidel Method (run for 40 iterations and watch max solution change in an iteration)

(%i19)
```
iterative_sol:gauss_seidel(A,B,40);
```

... Goes on about iteration results ...

(iterative_sol)
$$\begin{pmatrix} 0.989130803064315 \\ 0.9598697431668418 \\ 0.9205309580963118 \\ 0.8676405533439283 \\ 0.7964673452312362 \\ 0.7005668030132802 \\ 0.5711678657073875 \\ 0.3963489778943668 \\ 0.1599302893257971 \end{pmatrix}$$

4 Solution from Theory

Forget some variable so we can continue using the same names

(%i21)
```
kill(U, K, x, y)$ assume(U> 0);
```

(% o21) $[U > 0]$

Compose and solve the ODE:

(%i22)
```
ode: U*'di (y, x) - K*'di (y,x,2) = 0;
```

(ode)
$$U\left(\frac{d}{dx}y\right) - K\left(\frac{d^2}{dx^2}y\right) = 0$$

(%i23)
```
gsol: ode2(ode, y, x);
```

(gsol)
$$y = \%k1\ \%e^{\frac{Ux}{K}} + \%k2$$

Figure out k1 and k2 coe s based on boundary values (involves solving a system of two equations)

(%i25)
```
f0: rhs(ev(gsol, x=0))$ f9: rhs(ev(gsol, x=0.9))$
```

(%i26)
```
sol:solve([f0=1, f9=0], [%k1, %k2]);
```

(sol)
$$[[\%k1 = -\frac{1}{\%e^{\frac{0.9U}{K}} - 1}, \%k2 = \frac{\%e^{\frac{0.9U}{K}}}{\%e^{\frac{0.9U}{K}} - 1}]]$$

(%i27) coe s:ev(sol, U=0.03, K=0.01);

(coe s) [[%k1 = −0.07204750205711648, %k2 = 1.072047502057116]]

(%i28) theory_sol:makelist(ev(rhs(gsol), x=i/10.0+0.05, U=0.03, K=0.01, %k1=rhs(coe s[1][1]), %k2=rhs(coe s[1][2])), i, 0, 8);
(theory_ sol)
[0.9883402470641222, 0.9590545266468127, 0.9195229390052993, 0.8661608772499396, 0.7941296281990526, 0.69

(%i29) iterative_sol-theory_sol;

(% o29)
$$\begin{pmatrix} 7.90556000192843610^{\ 4} \\ 8.15216520029138410^{\ 4} \\ 0.001008019091012557 \\ 0.001479676093988691 \\ 0.00233771703218355 \\ 0.003669190766267927 \\ 0.00519746571607254 \\ 0.00786914302466901 2 \\ 0.01060262339579992 \end{pmatrix}$$