TD2

Part I

Q1

A: « A person is old »
B: « A person suffers from Alzheimer's disease»

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$\frac{P(A|B)}{p(B|A)} = \frac{P(A)}{p(B)} \gg 1$ because $P(A) > P(B)$

Then $P(A|B) > P(B|A)$

Part II

Q2

Our subject is to report with a cursor the location from which we perceived the flash to emanate. So, if we performed many times trails, we can get the likelihood function like:

$$P(X|\theta) = \prod_{i=1}^{n} P(x_i|\theta)$$

Where $\theta = s$. According to the theory of Bayesian, we should find the $P(s|X)$ which is equal to:

$$\hat{\theta}_{MAP} = \underset{x}{\operatorname{argmax}}\, P(X|\theta)P(\theta)$$

From the slide, we know that the prior function and likelihood function follow the normal distribution. Our aim is to find a prior that would give rise to a posterior with two local maxima. Besides, we know that $P(s|X)$ also follows a normal distribution. We assume that this condition is unchanged. If our prior function follows the Mixture Gaussian Distribution (GMM), that's:

$$P(s) = \pi_1 G(\mu_1, \sigma_{s1}) + \pi_2 G(\mu_2, \sigma_{s2})$$

Where the π_1 and π_2 are the probability of the corresponding Gaussian distribution. So, if we assume that $\mu_1 = 1/3$ and $\mu_2 = 2/3$, we will get $P(s|X)$ that satisfied:

$$P(s|x) \propto P(X|s)P(s) = \pi_1 K_1 G\left(\mu_{p1}, \sigma_{p1}\right) + \pi_2 K_2 G\left(\mu_{p2}, \sigma_{p2}\right)$$

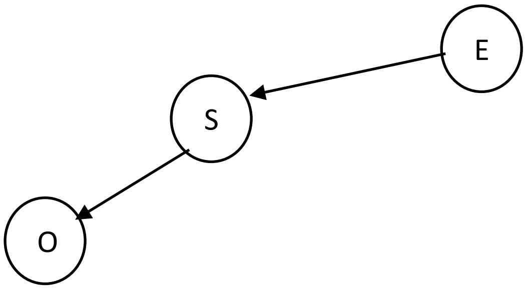
$$K_1 > 0, K_2 > 0$$

In summary, the prior function that we found is:

$$P(s) = \pi_1 G(1/3\,, \sigma_{s1}) + \pi_2 G(2/3\,, \sigma_{s2})$$

Q3

E: Easter
O: October
S: Stimulus: all the animals



Q4

From the class, we know that $P(s|x_A, x_V) \propto P(x_A|s)P(x_V|s)$. So firstly, we calculate $P(x_A|s)P(x_V|s)$.

$$P(x_A|s)P(x_V|s) = \frac{1}{\sqrt{2\pi{\sigma_A}^2}} exp\left(-\frac{(s-\mu_A)^2}{2{\sigma_A}^2}\right) \cdot \frac{1}{\sqrt{2\pi{\sigma_V}^2}} exp\left(-\frac{(s-\mu_V)^2}{2{\sigma_V}^2}\right)$$
$$= \frac{1}{2\pi\sqrt{{\sigma_A}^2{\sigma_V}^2}} exp\left(-\frac{(s-\mu_A)^2}{2{\sigma_A}^2} - \frac{(s-\mu_V)^2}{2{\sigma_V}^2}\right)$$

We can see clearly that $P(s|x_A, x_V)$ will also be a Gaussian distribution. As a distribution, it needs to be integrated to 1 and change the constant multiple of the existing part. Therefore, we only focus on the index part to obtain $\mu_{combined}$ and $\sigma_{combined}$.

$$\frac{(s-\mu_A)^2}{2{\sigma_A}^2} + \frac{(s-\mu_V)^2}{2{\sigma_V}^2} = \frac{({\sigma_A}^2 + {\sigma_V}^2)s^2 - (2\mu_A{\sigma_V}^2 + 2\mu_V{\sigma_A}^2)s + ({\mu_A}^2{\sigma_V}^2 + {\mu_V}^2{\sigma_A}^2)}{2{\sigma_A}^2{\sigma_V}^2}$$
$$= \frac{s^2 - \frac{2\mu_A{\sigma_V}^2 + 2\mu_V{\sigma_A}^2}{{\sigma_A}^2 + {\sigma_V}^2}s + \frac{{\mu_A}^2{\sigma_V}^2 + {\mu_V}^2{\sigma_A}^2}{{\sigma_A}^2 + {\sigma_V}^2}}{\frac{2{\sigma_A}^2{\sigma_V}^2}{{\sigma_A}^2 + {\sigma_V}^2}} = \frac{(s-\mu_{combined})^2 + C}{2{\sigma_{combined}}^2}$$

So, we can get:

$$\mu_{combined} = \frac{\mu_A{\sigma_V}^2 + \mu_V{\sigma_A}^2}{{\sigma_A}^2 + {\sigma_V}^2} = \frac{\frac{\mu_A}{{\sigma_A}^2} + \frac{\mu_V}{{\sigma_V}^2}}{\frac{1}{{\sigma_A}^2} + \frac{1}{{\sigma_V}^2}}$$
$$\sigma_{combined} = \frac{2{\sigma_A}^2{\sigma_V}^2}{{\sigma_A}^2 + {\sigma_V}^2} = \frac{1}{\frac{1}{{\sigma_A}^2} + \frac{1}{{\sigma_V}^2}}$$

Q5

a)

Measurement x_A and x_V are centered in the stimulus s. The expectation of the estimator is:

$$E(\hat{s}) = E(\omega x_A + (1-\omega)x_V) = \omega E(x_A) + (1-\omega)E(x_V) = \omega s + (1-\omega)s = s$$

So, it's an unbiased estimator.

b)

Measurement x_A and x_V are independent, so the variance of the estimator is:

$$Var(\hat{s}) = Var(\omega x_A + (1-\omega)x_V) = \omega^2 Var(x_A) + (1-\omega)^2 Var(x_V)$$
$$= \omega^2 {\sigma_A}^2 + (1-\omega)^2 {\sigma_V}^2$$

This is a quadratic function, its minimal is obtained at

$$\omega = \frac{{\sigma_V}^2}{{\sigma_A}^2 + {\sigma_V}^2}$$

This value makes sense because it's the proportion of one variance and the sum of two variances so that the ratio of the coefficients is the inverse of the variances:

$$\frac{\omega}{1-\omega} = \frac{{\sigma_V}^2}{{\sigma_A}^2}$$

c)

In that case, the conclusion in a) breaks down. The expectation of the estimator is:

$$E(\hat{s}) = E(\omega_A x_A + \omega_V x_V) = \omega_A E(x_A) + \omega_V E(x_V) = \omega_A s + \omega_V s = (\omega_A + \omega_V)s$$

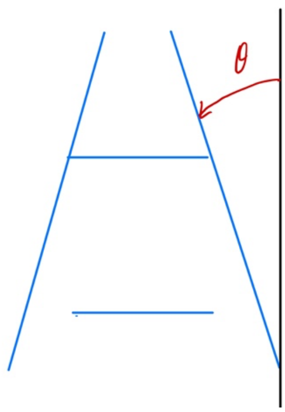
It's no longer unbiased unless $\omega_A + \omega_V = 1$.

The conclusion in b) also breaks down. The variance of the estimator is:

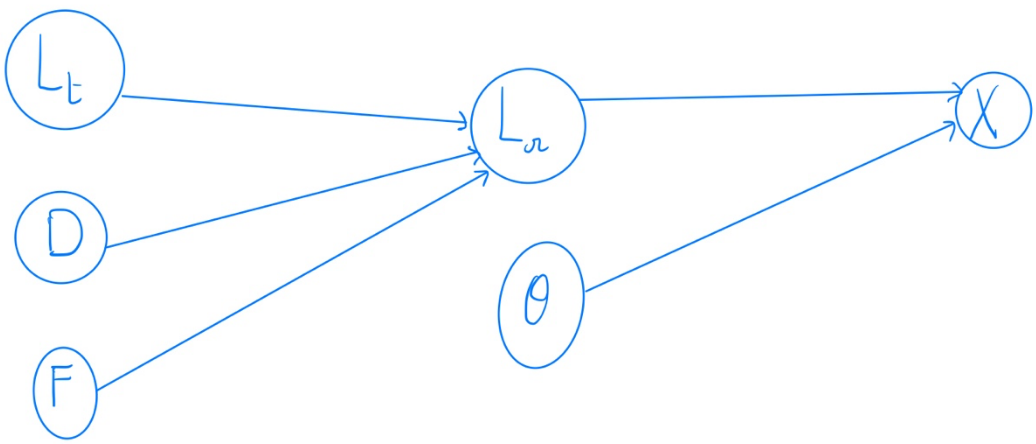
$$Var(\hat{s}) = Var(\omega_A x_A + \omega_V x_V) = {\omega_A}^2 Var(x_A) + {\omega_V}^2 Var(x_V) = {\omega_A}^2 {\sigma_A}^2 + {\omega_V}^2 {\sigma_V}^2$$

We can't clarify its minimum value unless we know the relationship between ω_A and ω_V.

Q6



L_t : *true length of the upper horizontal line*
L_r : *length on the retina*
Y : *Vertical position*
D : *Distance of the line from the observer*
F : *true field of view*
θ : *Context*
X : *Perceived length of the upper line*



Part III

$$\boldsymbol{D_N} = \boldsymbol{log}\left(\frac{p(x = left|d_1 \dots d_n)}{P(x = Right|d_1 \dots d_n)}\right)$$

$$= \boldsymbol{log}\left(\frac{p(x = left|d_1 \dots d_n)p(d_n|x = left)}{P(x = Right|d_1 \dots d_n)p(d_n|x = Right)}\right)$$

$$\boldsymbol{=D_{N-1}} + \boldsymbol{log}\left(\frac{p(d_n|x=left)}{P(d_n|x=Right)}\right)$$