Foundations of Analytical Mechanics: Generalised Coordinates to D'Alembert's Principle

Shaik Mytheensha
Physics Undergraduate Society

August 02, 2025

Abstract

In the previous session, we examined the concept of degrees of freedom in thermodynamics and derived the equipartition theorem. In this article, we discuss the foundational ideas of analytical mechanics: generalised coordinates, constraints, virtual displacement, virtual work, and D'Alembert's principle.

To build intuition, we begin by recalling standard coordinate systems before introducing the concept of generalised coordinates.

1 Coordinate Systems

A coordinate system in physics is a mathematical tool that is used to specify the position of a point in space by assigning it a unique set of numerical parameters (coordinates) relative to a chosen origin and basis. Common coordinate systems used in physics include:

1. **Cartesian coordinates**: For a Cartesian coordinate system, x, y, z are the parameters that define a vector $\boldsymbol{r}$ which is given by:

$$r = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$$

2. **Polar (planar) coordinates**: In two dimensions, the parameters r, θ define a vector $\boldsymbol{r}$. It is expressed in terms of Cartesian coordinates as:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta \end{aligned}$$

3. **Cylindrical coordinates**: An extension of polar coordinates to three dimensions, where the parameters r, θ, z define the vector $\boldsymbol{r}$ and in terms of Cartesian coordinate system it is represented as:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z. \end{aligned}$$

4. **Spherical coordinates**: A vector $\boldsymbol{r}$ is represented by the parameters r, θ, ϕ, where θ is the polar angle and ϕ is the azimuthal angle. In Cartesian form:

$$\begin{aligned} x &= r\sin\theta\,\cos\phi, \\ y &= r\sin\theta\,\sin\phi, \\ z &= r\cos\theta. \end{aligned}$$

The set of parameters used to define different coordinate systems is related to **degrees of freedom**.

1.1 Degrees of Freedom

Degrees of freedom of a mechanical system is the number of **independent parameters** required to uniquely specify the configuration of a system. The word "independent" is important to understand and will be discussed in the section of constraints.

For example, to uniquely specify a particle in a two-dimensional plane, we require only two parameters, say, (x, y). Similarly, for a particle in a three-dimensional space, we require three parameters.

For a system of N particles in a three-dimensional space, we require $3N$ independent parameters.

In general, for a system of N particles in an n dimensional space, the number of degrees of freedom is:

$$f = nN$$

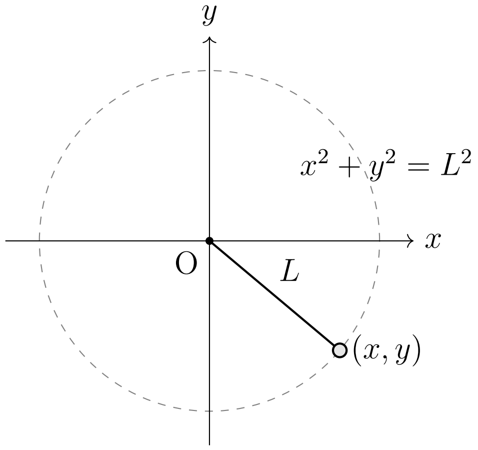
1.2 Constraints

Constraints are mathematical conditions that restrict the motion of a system by imposing relationships between coordinates (and possibly time). A constraint is classified as **holonomic**, if it can be expressed in an algebraic equation that relates the coordinates of a particle and, possibly, time. The general form of a holonomic constraint is:

$$f(r_1, r_2, \ldots, t) = 0$$

Example: A Simple Pendulum

Consider a point mass located at (x, y), attached to a pivot at the origin by a rigid, massless rod of length L. The particle is constrained to move on a circle. This is a holonomic constraint expressed by the equation $x^2 + y^2 - L^2 = 0$



In this case, (x, y) are no longer independent, specifying x will determine y from the constraint. Similarly, specifying y will determine x. Hence, although two coordinates are used, only one independent parameter is needed to describe the configuration of the system. Therefore, the actual degree of freedom for this system is not two, but **one**.

Holonomic constraints reduce the number of independent coordinates needed to describe the system's configuration, or in other words, holonomic constraints, in principle, reduce degrees of freedom.

In general, for a system of N particles in an n dimensional space subject to k holonomic constraints, the number of degrees of freedom is:

$$f = nN - k \tag{1}$$

For any system without constraints, all coordinates are independent, and the number of coordinates equals the number of degrees of freedom.

2 Generalised Coordinates

During our discussion, the definition of generalised coordinates was interpreted differently by different members. Therefore, instead of stating the definition directly, we will begin with common ground and then build the definition from there.

Our goal is to uniquely specify the configuration of a mechanical system. To achieve this, we need to know the **degrees of freedom** of the system. Let us take a system having n degrees of freedom, then we require n independent parameters to completely specify the system, let us denote these parameters as:

$$(q_1, q_2, \ldots, q_n).$$

The important thing to note here is that we need all the n parameters, q_1 to q_n, to completely specify the system. This is the definition of degrees of freedom. If we specify the same system with more than n parameters, then the parameters will not be independent. So n is the *minimum* number of independent parameters required to completely specify the system. For example, a free particle in n dimensional space has n DOF, specified directly by its n Cartesian coordinates.

Now, if we subject the system to k holonomic constraints, then the degrees of freedom reduce to $n - k$. Similar to how the degrees of freedom decrease for a simple pendulum. Now we require only $n-k$ independent parameters to completely specify the new system, which is given by

$$(q_1, q_2, \ldots, q_{n-k})$$

All the parameters from q_{n-k+1} to q_n will not be independent, they are dependent on the independent parameters by the constraint equation. For example, in a simple pendulum, the Cartesian coordinates (x, y) are related by $x^2 + y^2 = L^2$ and therefore are not independent.

In both cases, we completely specify a system using the set of *minimum independent parameters*. This set is called the **generalised coordinates**. There might be multiple generalised coordinates for a particular system.

Important Notes

1. A *generalised coordinate* is a single choice of parameters that specifies the system. It is **not** a large set of all possible such choices, which can be called the *set of all generalised coordinates*, but each element of this larger set is itself a set of parameters. It is the element of this large set that we call generalised coordinates and not the larger set.

2. The number of elements in a generalised coordinates will always be equal to degrees of freedom. This follows directly from the definition of both generalised coordinates and degrees of freedom. Degrees of freedom of a mechanical system is the minimum number of parameters required to completely specify the system, if we add all these parameters to a set, this set is called the generalised coordinates.

3. Generalised coordinates are chosen so that no constraint equations are needed among them. So, for a system initially described by n original parameters (e.g., Cartesian coordinates) with k holonomic constraints, we change the variables so that only $n - k$ independent coordinates remain.

Condensed Definition

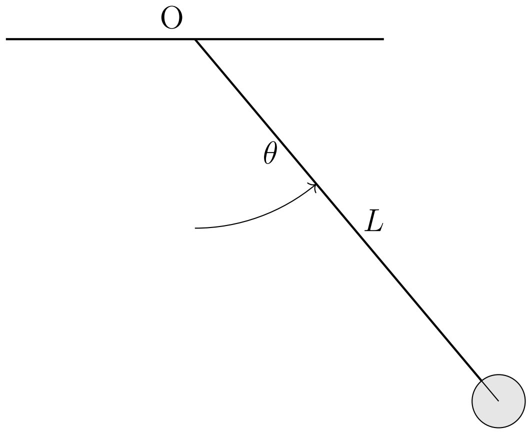
Generalised coordinates are any set of f independent parameters,

$$(q_1, q_2, \ldots, q_f)$$

that uniquely specify the configuration of a system, where f is its degrees of freedom. They are chosen so that no constraint equations are required among them. Different sets of generalised coordinates may exist for the same system.

Back to Simple Pendulum:

For a simple pendulum, we have seen that we require only one parameter to completely specify the system i.e. swinging of the bob. It is redundant to use a Cartesian coordinate with two parameters x and y and one constraint $x^2 + y^2 = L^2$ to solve such a system. Therefore, we consider θ as the generalised coordinate for this system which eliminates the need for constraint equations.



3 Virtual Displacement and Virtual Work

To understand D'Alembert's principle, we should understand the **principle of virtual work** and **virtual displacement**.

3.1 Virtual Displacement

A **virtual displacement**, denoted δr_i, is a hypothetical infinitesimal change in the position of the particle i that is consistent with all the constraints of the system. Crucially, this displacement is imagined to occur at a single instant in time, which means $\delta t = 0$. It represents a possible movement, not an actual movement over time.

Definition includes:

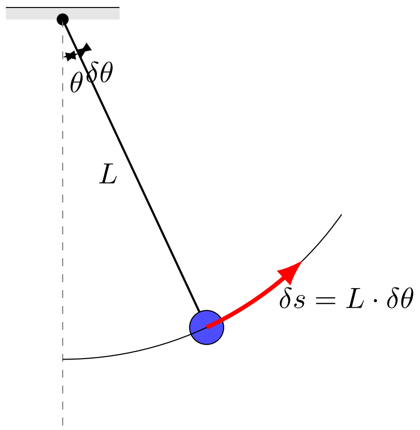
1. Hypothetical motion: It is not a displacement over time, but an imaginary instantaneous movement.
2. Infinitesimally small: This allows us to use calculus to analyze the system.
3. Consistent with constraints: The motion will not violate any of the constraints of the system's motion.

Back to Simple Pendulum:

For a simple pendulum, the constraint is that the distance from the pivot to the bob is fixed. This means that the bob can only move along the arc of the circle. A virtual displacement in such a case would be a small nudge along the circular arc. This virtual displacement is represented as δs.

For an infinitesimally small change in angle $\delta\theta$, the arc length of the virtual displacement is:

$$\delta s = L.\delta\theta$$



3.2 Virtual Work

The **virtual work** is the work done by the actual forces acting on the system as it undergoes this virtual displacement. It is a mathematical construct, like virtual displacement, not a physical quantity of work.

$$\delta W = F.\delta r$$

We divide the total forces into applied forces ($F_i^{(a)}$) and constraint forces ($F_i^{(c)}$). The total virtual work becomes:

$$\sum_i F_i^{(a)}.\delta r_i + \sum_i F_i^{(c)}.\delta r_i$$

For ideal constraints, the net virtual work done by constraint forces is zero:

$$\sum_i F_i^{(c)}.\delta r_i = 0$$

This is because the virtual displacement is consistent with the constraint equations, and for ideal constraints, the constraint forces either act perpendicular to the allowed motion or their contributions cancel out.

The principle of virtual work is concerned with the total virtual work done by all the applied forces.

Principle of Virtual Work

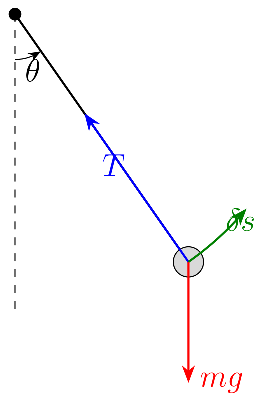
The principle of virtual work states that for a system to be in static equilibrium, the total virtual work done by all applied forces on the system must be zero for any and all possible virtual displacements. Mathematically:

$$\sum_i \delta W = \sum_i (F_i^{(a)}.\delta r_i) = 0$$

Back to Simple Pendulum:

There are two forces that act on the pendulum bob.

1. Gravity(mg): Acts vertically downward.
2. Tension (T): Acts along the string, towards the pivot.



Here, tension is the constraint force and gravity is the applied force.

1. Work done by tension δW_T: The tension force T is directed along the string. The virtual displacement δs is along the arc, perpendicular to the string. Since force and displacement are perpendicular ($\theta = 90°$), the work done by tension is zero.

$$\delta W_T = 0$$

2. Work done by gravity δW_g: Taking the component of force along the virtual displacement gives work done as:

$$\delta W_g = -mgL\sin(\theta).\delta\theta$$

From the principle of virtual work, we get

$$-mgL\sin(\theta).\delta\theta = 0$$

which gives $\sin(\theta) = 0$ as m, g, L are constants and $\delta\theta \neq 0$, implying $\theta = 0°$ or $\theta = 180°$ which are the static equilibrium positions of a simple pendulum.

4 D'Alembert's principle

The principle of virtual work applies to systems in static equilibrium, where the net force on the particle is zero. D'Alembert's principle is an extension of it to dynamic systems.

From Newton's second law of motion for the i-th particle,

$$F_i - \dot{p}_i = 0$$

D'Alembert proposed to treat the term $-\dot{p}_i$ as an inertial force. By adding inertial force to the actual forces acting on the particle, we can create a system where the effective total force is always zero. This allows the dynamic system to be in dynamic equilibrium at every instant in time.

Let the total force on the particle i be $F_i = F_i^{(a)} + F_i^{(c)}$, where $F^{(a)}$ denotes the applied forces, and $F^{(c)}$ denotes the constraint forces. Newton's law is $F_i^{(a)} + F_i^{(c)} = \dot{p}_i$. Rearranging gives:

$$(F_i^{(a)} - \dot{p}_i) + F_i^{(c)} = 0$$

Now, we take the dot product with virtual displacement δr_i and sum over all particles:

$$\sum_i (F_i^{(a)} - \dot{p}_i).\delta r_i + \sum_i F_i^{(c)}.\delta r_i = 0$$

As in the static case, we assume that the forces of constraint do no work, so the second term vanishes. This leaves us with the final form of D'Alembert's principle:

$$\sum_i (F_i^{(a)} - \dot{p}_i).\delta r_i = 0$$

D'Alembert's principle is the statement that the total virtual work done by the sum of the applied forces and the inertial forces is zero for any virtual displacement consistent with the constraints of the system.

Conclusion

In this article, we explored the foundational principles of analytical mechanics, beginning with coordinate systems and degrees of freedom. We then introduced the concept of constraints and generalised coordinates, which allow us to describe mechanical systems in the most efficient way. Building on these, we examined virtual displacement, virtual work, and finally D'Alembert's principle, which extends the principle of virtual work to dynamic systems. These ideas lay the groundwork for Lagrangian and Hamiltonian mechanics, which will be developed in future discussions.

References

1. H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics*, 3rd Edition, Addison-Wesley, 2001.