Unit 4: RELATIVITY

Q4.1. What do you mean by an inertial and non-inertial frame of reference?

A Reference frame: -

Three space coordinates x, y, z and time t define a reference frame. All physical phenomenon's need a reference frame with respect to which they can be studied.

Inertial frame: -

If a reference frame is either at rest or moving with a uniform velocity then it is called an inertial frame of reference. Example: A person standing on a railway station and a person standing in a train that is moving with constant velocity. Both persons are in inertial frames.

Non inertial frame: -

An accelerating frame of reference is called non inertial frame of reference. Example: If the train in the previous example does not move with constant velocity, then the train would have been an example of non-inertial frame of reference.

Q4.2. State the fundamental postulates of Special theory of relativity

1. The Principle of Relativity
   All physical laws are the same in all inertial frames that are moving relative to each other with constant velocity.
2. Principal of independence of velocity of light
   The speed of light in free space has the same value 'c' in all inertial reference frames.

Q4.3. Discuss the Galilean transformations for space and time.

Different inertial frames may have different values of physical quantities, but laws of physics do not depend on the choice of the frame. Galilean transformations are used to relate the physical quantities in one inertial frame to another inertial frame.

Let S and S' be Representations of two inertial frames with coordinate systems (x, y, z) and (x' y' z') respectively. Let S' be moving with uniform velocity v with respect to S reference frame. Let an event take place at point P. The coordinates of point P with respect to individual frame is S (x, y, z, t) and S' (x', y', z', t') for simplicity let us assume that the x, y and z axes of coordinate systems are parallel to each other. We start counting from the time when origin of S i.e. O and origin of S' i.e., O' coincide as shown in *Figure 4.3.1*

After lapse of time 't', 'S' moving with velocity 'v' would have covered a distance 'd = v t'. Let coordinates system move along x axis with respect to each other.

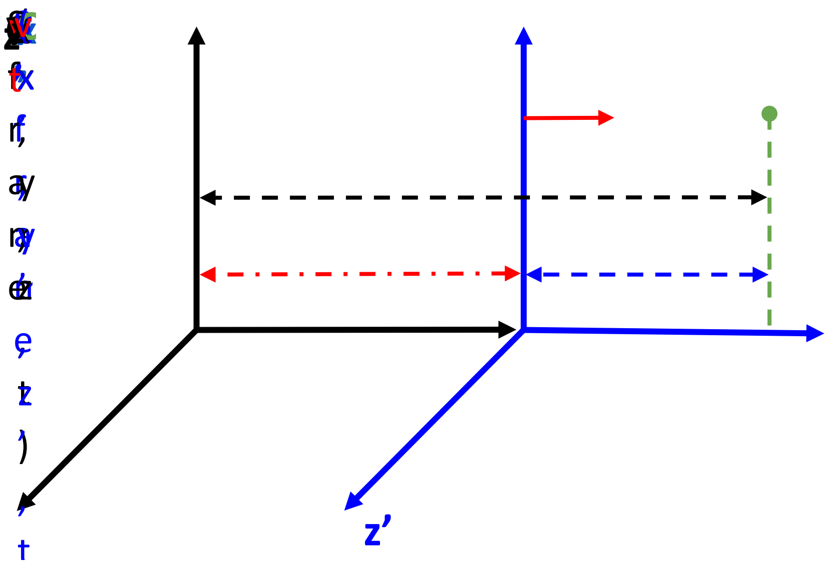

Figure 4.3.1: Galilean Transformation

Hence, the relation between coordinates of P in one frame is related to those in the other frame by the equations called the Galilean transformation equations.

The transformations for space and time are given as:

$$x' = x - vt \quad \_\_\_\_\_\_ (1)$$

As the object does not undergo any relative motion in the y and z direction:

$y' = y$ ______ (2)

$z' = z$ ______ (3)

Also,

$t' = t$ ______ (4)

The Galilean Transformations for velocity are as follows:

Assuming the object is moving with a velocity u, let u_x, u_y, and u_z be the velocities in the x, y and z directions respectively, with respect to frame S and u_x', u_y' and u_z' with respect to frame S'.

Consider equation 1, upon differentiating with respect to time, we get the following result:

$(dx'/dt) = (dx/dt) - v_x$ ______ (5) or,

$u_x' = u_x - v_x$ ______ (6)

Similarly, we get,

$u_y' = u_y - v_y$ ______ (7)

$u_z' = u_z - v_z$ ______ (8)

However, Galilean Transformations do not satisfy the postulates of the Special Theory of Relativity. Galilean Transformations give very different results while converting some physical quantities in different frames which violates the first postulate. Also, according to Galilean Transformations, the speed of light would be different in different inertial frames, which violates the second postulate.

Q4.4. Derive the Lorentz transformation equations for space and time

The Galilean transformations relate space time coordinates in one inertial frame to those in the other frame. But these transformations are not valid for the case where velocity of motion of one frame with respect to other frame approaches 'c' i.e., velocity of light.

The transformations that apply to all speeds and are valid up to 'c' i.e., velocity of light is known as Lorentz transformation.

Consider two inertial frames S and S' where standards of measuring distance and time are the same. Let S be stationary and S' be moving with a constant velocity 'v' with respect to frame S along x' and x axes of frames S' and S respectively. Let us assume x and x' are the same line while y and y' also z and z' are parallel. Let at initial time t=t'=0 and this is when Origin O and O' of frames S and S' coincide as shown in *Figure 4.4.1*.

As space and time are regarded as homogeneous. The relation between their coordinate and time in different frames is linear

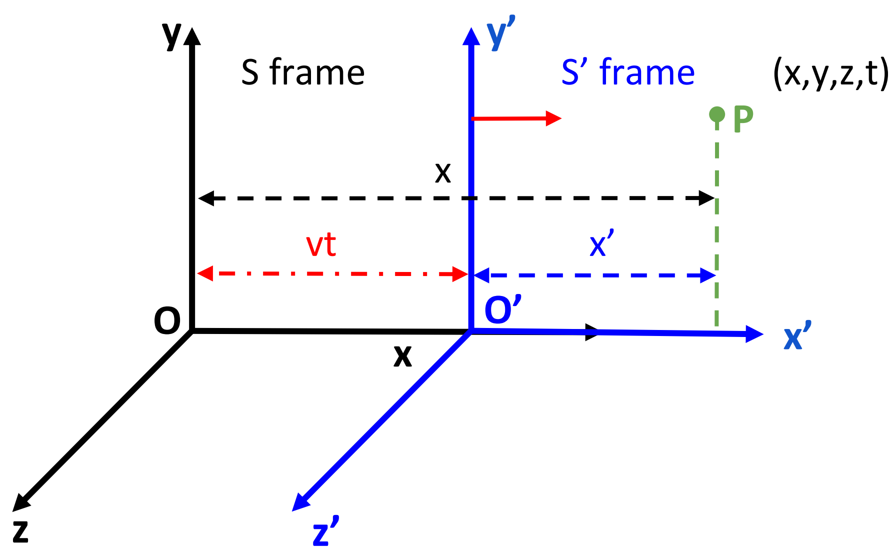

Figure 4.4.1: Lorentz transformation

x'= ax + bt...(1)

t'= fx + gt...(2)

Equation (1) can be written as:

x'= a (x+ $\frac{b}{a}$ t) ....(3)

After time t, origin O' of frame S' is at position x=vt with respect to frame S

And x' for origin O' corresponds to x'=0, putting these values in Equation (3)

$0=a\,(vt+\frac{b}{a}t)$ this gives,

$v= -\frac{b}{a}$ ....(4)

Substituting (4) in Equation (3) we get,

x'=a(x-vt) ....(5)

Similarly, we can write, x= a (x'+ vt) ....(6)

Since S' moves along x axis only, the y and z coordinates do not change, hence:

y= y' and z'=z ....(7)

We know that for light velocity 'c',

x' =c t', whereas x = ct putting these two in Equation (5) we have:

c t'=a (c t - v t)

$\therefore$ c t' = a c t $(1-\frac{v}{c})$

t' = a t $(1-\frac{v}{c})$ ....(8)

Hence for t we can write, t = a t' $(1+\frac{v}{c})$ ....(9)

Using Equation (9) in Equation (8);

t' = a a t' $(1+\frac{v}{c})$ $(1-\frac{v}{c})$ this becomes

$$a^2=\frac{1}{1-\frac{v^2}{c^2}}$$

More often the constant 'a' is denoted by $'\gamma'$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \text{ ....(10)}$$

Using Equation (10) in Equation (8) and Equation (5),

$$x' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} \text{ ....(11)}$$

$$t' = \frac{t\left(1-\frac{v}{c}\right)}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{t-\frac{vtc}{c\,c}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$ (using ct=x) ....(12)

Thus, the Lorentz transformation of coordinate from system S' to frame S are

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

y'= y; z'= z

$$t' = \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$ . . ..(13)

Similarly, the Inverse Lorentz transformation equations can be written as:

$x = \frac{x'+vt}{\sqrt{1-\frac{v^2}{c^2}}}$; y= y'; z= z'; $t = \frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$. . ...(14)

Q4.5. Explain length contraction and time dilation

Relativistic effects appear to conflict common sense. This is due to the fact that the velocities that we have observed in our everyday experience are very small as compared to velocity of light 'c'. Special theory of relativity predicts observers will measure different time and length in different inertial frames only when the frames are moving with a velocity comparable to 'c' with respect to each other. Length contraction and time dilation are two relativistic effects explained below:

1. Length Contraction:
   Consider two inertial frames S and S', where S' is moving with a constant velocity 'v' with respect to S. Let x_1' and x_2' be two ends of a rod at rest as seen by an observer in S' frame. Let L be the length of the rod measured by the observer in the S frame as shown in *Figure 4.5.1*.

$L = x_2 - x_1$

$L' = x_2' - x_1'$

Both measurements are done simultaneously: $t_2 = t_1 = t$

Using Lorentz transformation equations, we get:

$L' = x_2' - x_1' = \gamma\, [\,(x_2 - vt) - (x_1 - vt)]$

$L' = \gamma\,(x_2 - x_1)$

$L' = \gamma\, L$

The Rod is at rest in S': L' =Lo true length of the rod and the rod appears to be in motion to an observer in S frame so L= apparent length

$$L = \frac{L_o}{\gamma} = L_o\sqrt{1 - \frac{v^2}{c^2}}$$

So, length contraction is explained by the equation $L = L_o\sqrt{1 - \frac{v^2}{c^2}}$

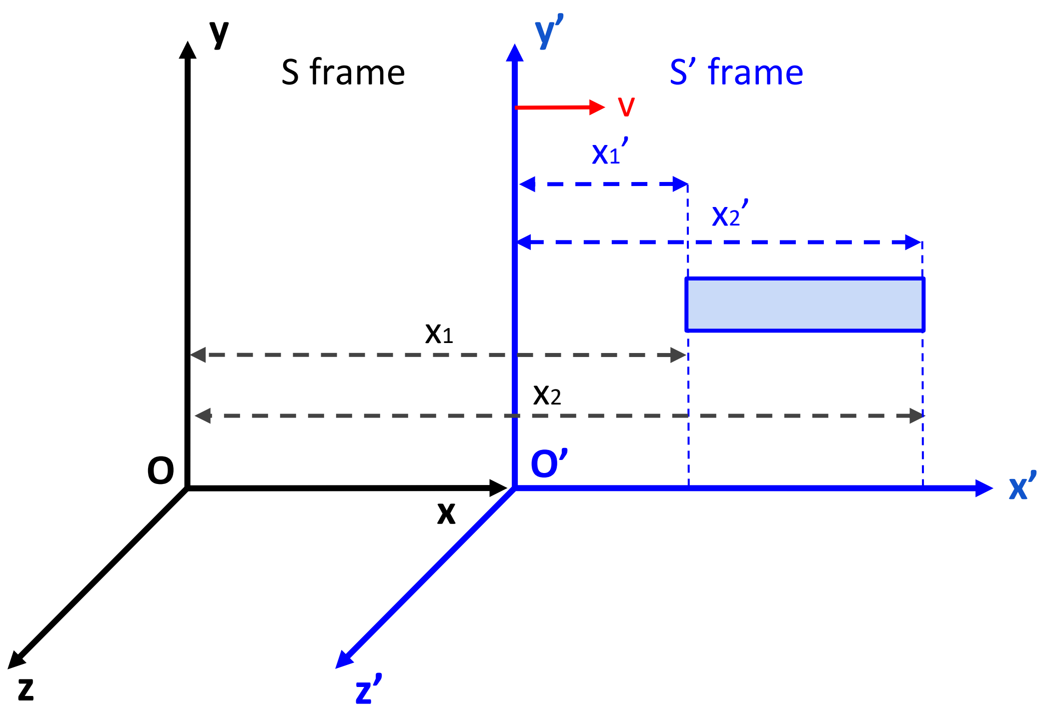

Figure 4.5.1: Length contraction

2. Time dilation:

Let a clock be at rest in S frame at point x. Suppose it produces two ticks at times t_1' and t_2' in frame S'. The time interval between these two ticks may be given by,

$$\Delta t' = t_2' - t_1'$$

Using Lorentz transformation equations,

$$\Delta t' = t_2' - t_1' = \gamma \, [\, (t_2 - \frac{v}{c^2} x) - (t_1 - \frac{v}{c^2} x)\}$$

$$\Delta t' = \gamma \, (t_2 - t_1) = \gamma \, \Delta t$$

Let $\Delta t = t_o$ as the clock was at rest in the S frame

$\Delta t' = t$ as the apparent time interval measured in the S' frame

$$\therefore t = \gamma t_o$$

So, the time dilation is explained by equation $t = \frac{t_o}{\sqrt{1-\frac{v^2}{c^2}}}$

Q4.6. Derive Einstein's Mass energy relation.

Relativistic mass:

Mass of a body is supposed to be independent of its velocity. Due to momentum conservation, we require that momentum of an isolated system be conserved.

Relativistic to an isolated system to conserve momentum it is observed that mass must depend on velocity and the relation that govern this dependence is,

$$m = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} \text{..........................} (1)$$

Where,

m= moving mass; m_o= rest mass; v= velocity of motion; c = velocity of light

Relativistic momentum:

$$P = mv = \frac{m_o \times v}{\sqrt{1-\frac{v^2}{c^2}}} \text{........................} (2)$$

This is the resultant relativistic momentum of the particle after substituting the relativistic mass that is dependent on velocity of the particle ‘v’.

Kinetic energy:

Newton's second law states that force is equal to the rate of change of momentum.

$$F=\frac{d\,(mv)}{dt}=m\frac{d\,(v)}{dt}+v\frac{d\,(m)}{dt} \text{ ....(3)}$$

Kinetic energy of a moving body is force into displacement,

$$dE_k = F.\,dx=\left(m\frac{d\,(v)}{dt}+v\frac{d\,(m)}{dt}\right).dx$$

$$dE_k = mv\,dv+ v^2\,dm \text{ .. ..(4). using } \frac{dx}{dt}=v$$

We know that, $m=\frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}$ and hence

$$dm = m_o\left(\frac{-1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{\frac{-3}{2}}\left(\frac{-2v}{c^2}dv\right)$$

$$dm=\frac{m_0\,vdv}{c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

m_o can be replaced by $m\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}$......using (1)

$$\therefore dm = \frac{mvdv}{c^2\left(1-\frac{v^2}{c^2}\right)}=\frac{mvdv}{(c^2-v^2)} \text{ ……….…......................(5)}$$

Rearranging (5) we can write,

$$c^2\,dm - v^2dm = m\,vdv \text{ ....(6)}$$

Comparing Equations (6) and (4) we get

$$dE_k= c^2dm$$

$$E_k=\int_0^{E_k} dE_k = c^2\int_{m_0}^{m} dm$$

$$E_k = c^2\,(m-m_o)$$

Thus, relativistic kinetic energy of a body is equal to the gain in mass multiplied by square of speed of light.

:. $m_o c^2$ is the energy of the body at rest.

Total energy of a body: $E=E_k+$ rest energy

$E= c^2 (m-m_o)+m_o c^2$

This is the Energy mass relation of Einstein i.e., $\mathbf{E = m c^2}$