QUANTUM PHYSICS

Q1. Explain de Broglie's Hypothesis of matter waves and deduce the expression for λ.

(M.U. May 2008; Dec 2010, 2011; June 2019) [3 Marks]

De Broglie introduced the concept of matter waves. He suggested that the wave particle duality of radiation should be extended to micro-particles as well.

From the theory of radiations, it is known that the energy of a photon, a zero mass particle representation of light radiation is given by

$$E = h\upsilon = h\frac{c}{\lambda} \text{-----(1)}$$

where,

c = velocity of light

υ = frequency

λ = wavelength

On the other hand, according to Einstein's mass energy relation, the energy of a particle with non-zero mass particle mass 'm' is given by

$$E = mc^2 \text{-----(2)}$$

Comparing (1) and (2),

$$h\frac{c}{\lambda} = mc^2$$

this can be rearranged for 'λ' as;

$$\lambda = \frac{h}{mc} = \frac{h}{p} \text{-----(3)}$$

where, p is the momentum of a particle of mass 'm' moving with velocity 'c'.

The kinetic energy 'E' can be written in terms of its momentum 'p' as follows;

$$E = \frac{1}{2}mc^2 = \frac{(mc)^2}{2m} = \frac{p^2}{2m} \text{-----}(4)$$

Rearranging the above equation for ‘p’ we have

$$p = \sqrt{2mE} \text{-----}(5)$$

Potential is defined as work done (Energy ‘E’ consumed) per unit charge ‘q’;

Hence

$$\mathrm{E} = \mathrm{qV} \text{ ----- } (6)$$

Thus ‘λ’ can be written in terms of momentum, energy and potential using equations (3), (5) and (6) as follows:

$$\boldsymbol{\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2mqV}}}$$

This is the expression for De-Broglie’s wavelength in various forms.

Q2. Why is the wave nature of matter not visible in our daily life?

(M.U. Dec 2009) [3 Marks]

According to De-Broglie’s hypothesis every matter particle of mass 'm' and velocity 'v' is associated with a wave of wavelength

$$\lambda = \frac{h}{mv}$$

where $h = 6.63 \times 10^{-34} J.sec$ i.e. of the order of 10^{-34}

The matter waves are not visible for

1. A particle at rest for which $v = 0$ and hence $\lambda = \infty$
2. A macroscopic particle for which $m >> 0$ moving with moderate velocity and hence $\lambda \cong 0$.

Eg. Mass= 100 kg ; v=100 m/s then λ is of the order of 10^{-30} i.e. very small

The matter waves are visible for

1. A very tiny particle moving with faster velocity.

Eg. Mass= 10^{-31} kg ; v=10^4m/s then λ is of the order of $10^{-7}m$ i.e. visible.

In daily life we generally encounter bodies that are at massive, at rest or moving with moderate velocity. This is the reason why wave nature of matter is not visible in our day to day life.

Q3. If an electron is accelerated at potential V, find the wavelength of matter wave associated with it. Give its importance.

(M.U. Dec 2008) [3 Marks]

Let an electron with charge 'e' be accelerated through a potential 'V' then the kinetic energy 'E' of that electron in terms of the potential is ;

$$E = \frac{1}{2}mv^2 = eV \text{ -----(1)}$$

The wavelength associated with such an electron wave can be written as;

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2meV}} \text{ -----(2)}$$

Substituting values for Plank's constant $(h = 6.63 \times 10^{-34} J.sec)$, mass of an electron (m=$9.1 \times 10^{-31}kg$) and charge on an electron (e=$1.6 \times 10^{-19}C$),

$$\lambda = \frac{12.25}{V} \text{ -----(3)}$$

This is the theoretical wavelength prediction for electron.

The importance of this wavelength is that the theoretical value almost matches the experimental value confirming the de Broglie's hypothesis.

Eg. When an electron is accelerated through a potential of 54 V the value of wavelength using (3) is 1.67 $\mathring{A}$. This is very close its experimental value.

Q4. What are properties of matter waves?

(M.U. May 2015; Dec 2018; Dec 2019) [3-5 Marks]

Properties of matter waves are the following:

1. Matter waves are neither mechanical nor electromagnetic waves, they are hypothetical waves.
2. Matter waves with different de Broglie wavelengths travel with different velocities whereas electromagnetic waves of all wavelengths travel with the same velocity i.e. the speed of light ‘c’.
3. The de Broglie wavelength depends on the momentum; kinetic energy; accelerating potential for the matter particle as shown in the following formulae: $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m}} = \frac{h}{\sqrt{2mqV}}$
4. Matter waves travel faster than light.

Q5. Explain wave packet, phase velocity and group velocity of matter waves.

(M.U. May 2011; Dec 2016, 2018) [5 Marks]

Phase Velocity: The rate at which the phase of the wave changes at a point in the medium is called the **phase velocity** ‘v_p’ of the wave in that medium.

$$v_p = \frac{\omega}{k}$$

Where, ω is the angular velocity of the wave and the k is the wave vector. *Figure 5a* shown representation of phase velocity.

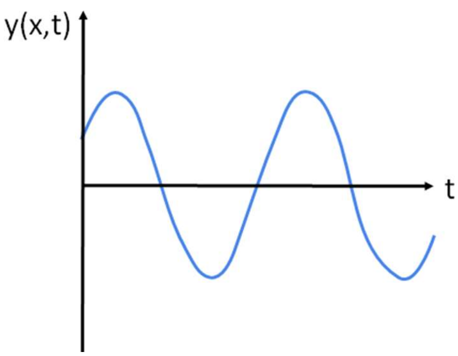

Figure 5a: Phase Velocity

Group Velocity: The rate at which the energy propagates through the medium while a group of waves within a frequency band $d\omega$ is travelling through it is called the **group velocity.**

$$v_g = \frac{d\omega}{dk}$$

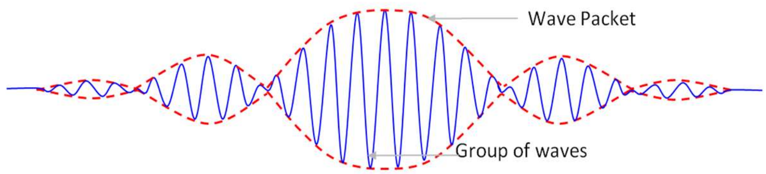

Figure 5b: Group Velocity

A particle confined to an extremely small volume cannot be represented by a single wave. Therefore, it is represented by a combination of multiple waves of slightly different frequencies known as **wave group the resultant is called** a **wave packet.**

Briefly, **phase velocity** is velocity of a monochromatic wave whereas, the **group velocity** refers to the velocity with which a **group** of waves within a frequency band $d\omega$ would travel.

In the *Figure 5b* **red wave packet** travels with **group velocity** whereas the carrier that is the **blue wave** travels with the velocity called as **phase velocity.**

Q6. What is the significance of the wave function Ψ of matter waves?

(M.U. May 2011) [3 Marks]

De Broglie's hypothesis shows that particles exhibit wave properties. The wave function $\Psi(x, t)$ represents a de Broglie wave function. This function describes the behavior of a matter wave (wave corresponding to a particle) as a function of position (x) and time (t). The wave function $\Psi(x, t)$ has no direct physical significance as it is not an observable quantity. However, the wave function provides information about the probability of finding a particular particle at a given position and at a given time.

If the particle is present in the space available, then the probability of finding it in the entire available space should be one. This is called normalizing condition and is given as:

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \; dx = 1$$

The wave function $\Psi(x,t)$ also helps in finding the average value of position, momentum and other physical properties 'P' of a matter particle as follows:

$$< P > = Pavg = \int_{-\infty}^{\infty} P \; |\Psi(x,t)|^2 \; dx$$

Q7. Derive one dimensional time dependent Schrödinger for matter waves.

(M.U. May 2009, 2015, 2015, 2017; Dec 2012, 2013, 2014, 2015, 2017, 2018)

[5/8 Marks]

For a wave propagating a medium the displacement of its particles at any instant 't', at any point 'x' is given by a plane wave which is represented as:

$y(x,t) = Ae^{i(kx-\omega t)}$ -----(1)

Where, ω is the angular velocity of the wave and the k is the wave vector given as : $\omega = 2\pi v$ & $k = \frac{2\pi}{\lambda}$ -----(2)

Here, v is the angular frequency and λ is the wavelength.

In analogy with the description above for the plane wave, the wave function which describes the behavior of the matter particle at any instant 't', at any point 'x' in space can be written as

$$\Psi(x,t) = Ae^{i(kx\,-\,\omega t)} \text{ -----(3)}$$

Where again, ω is the angular velocity of the wave and the k is the wave vector.

For a wave its energy $E = h\nu$; this implies $\nu = \frac{E}{h}$ and hence using (3) we get;

$$\omega = 2\pi\nu = 2\pi\frac{E}{h} \text{-----(4)}$$

For a matter wave, De Broglie's hypothesis gives $\lambda = \frac{h}{p}$ and hence ;

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{h}\cdot p \text{ -----(5)}$$

Let us denote $\hbar = \frac{h}{2\pi}$ a new constant, putting this in (4) & (5) we get:

$$\omega = \frac{E}{\hbar} \quad \& \quad k = \frac{p}{\hbar} \text{ -----(7)}$$

Substituting (7) into (3) for a matter wave function we have;

$$\Psi(x,t) = Ae^{\frac{i}{\hbar}(px\,-\,Et)} \text{ -----(8)}$$

We know that the total energy of the particle 'E' is given by

$$\text{E} = \text{Kinetic energy} + \text{Potential energy}$$

$$E = \frac{1}{2}m\nu^2 + V$$

$$E = \frac{(m\nu)^2}{2m} + V$$

Therefore using $p = m\nu$ we get,

$$E = \frac{p^2}{2m} + V \text{-----(9)}$$

In quantum mechanics all measurable quantities like energy momentum potential and position are operators; hence they do not have any significance until they are operated on a state or a wave function. Equation (9) is an operator equation operating it on the wave function Ψ(x, t) gives;

$$E\,\Psi(x,t\,) = \frac{p^2}{2m}\Psi(x,t) + V\,\Psi(x,t) \text{ -----(10)}$$

We have set out to find the differential equation of the wave function representing the matter wave $\Psi(x,t)$. Differentiating $\Psi(x,t) = Ae^{\frac{i}{\hbar}(px\,-\,Et)}$ with respect to 't' and 'x' we get;

$$\frac{\partial\Psi(x,t)}{\partial t} = -i\frac{E}{\hbar}Ae^{\frac{i}{\hbar}(px\,-\,Et)} = -i\,\frac{E}{\hbar}\,\Psi(x,t) \text{ -----(11)}$$

$$\frac{\partial\Psi(x,t)}{\partial x} = i\frac{p}{\hbar}Ae^{\frac{i}{\hbar}(px\,-\,Et)} = i\,\frac{p}{\hbar}\,\Psi(x,t) \text{ -----(12)}$$

In order to obtain a square of momentum ‘p’ we differentiate with ‘x’ twice

$$\frac{\partial^2\Psi(x,t)}{\partial x^2} = \left(i\frac{p}{\hbar}\right)^2 Ae^{\frac{i}{\hbar}(px\,-\,Et)} = -\left(\frac{p}{\hbar}\right)^2\Psi(x,t)\text{-----(13)}$$

Obtain $E\,\Psi$ and $p^2\,\Psi$ in terms of the differentials from (11) and (13)

$$p^2\,\Psi(x,t) = -\hbar^2\frac{\partial^2\Psi(x,t)}{\partial x^2} \text{ -----(14)}$$

and,

$$E\,\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t} \quad \text{ -----(15)}$$

Putting (14) and (15) in (10) we get

$$\boldsymbol{i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V\,\Psi(x,t)}$$

This is called the one-dimensional time dependent Schrödinger equation, the first and second term on the right-hand side represent the kinetic and potential energies respectively of the particle and the left-hand side represents the total energy.

Q8. Derive Schrödinger's Time Independent Schrödinger Equation.

(M.U. May 2016; Dec 2019) [5 Marks]

For stationary fields, the time dependent part of the wave function for matter waves is eliminated by using the method of separation of variables. The resultant differential equation thus obtained is called the Schrodinger's time independent equation.

Consider wave function $\Psi(x,t)$, doing variable separation we can write it as a product of two independent functions $\Psi(x)$ and $\phi(t)$ as follows:

$$\Psi(x,t) = \Psi(x)\phi(t) \text{ -----(1)}$$

If the potential energy of the particle varies only with the position of the particle and is independent of time, then the potential is only function of position:

$$V = V(x) \text{ -----(2)}$$

Schrödinger's time dependent equation can be written as:

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} + V\,\Psi(x,t) \text{----(3)}$$

Substituting (1) and (2) in (3) we have

$$i\hbar\Psi(x)\frac{d\phi(t)}{dt} = -\frac{\hbar^2}{2m}\phi(t)\frac{d^2\Psi(x)}{dx^2} + V(x)\Psi(x)\phi(t) \text{ -----(4)}$$

Dividing (4) by $\Psi(x)\phi(t)$, we get

$$i\hbar\frac{1}{\phi(t)}\frac{d\phi(t)}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\Psi(x)}\frac{d^2\Psi(x)}{dx^2} + V(x) \text{ -----(5)}$$

The left-hand side of the above equation is a function of 'x' only and the right hand side is a function of only 't' and thus both the sides can be equated to a constant. With all possibilities the constant should be total energy 'E'. Hence

$$i\hbar \frac{1}{\phi(t)} \frac{d\phi(t)}{dt} = E = -\frac{\hbar^2}{2m} \frac{1}{\Psi(x)} \frac{d^2\Psi(x)}{dx^2} + V(x)$$

Taking the position dependent side with 'E' and multiplying both sides with $\Psi(x)$ to obtain time independent equation; we get

$$E \cdot \Psi(x) = -\frac{\hbar^2}{2m} \cdot \frac{d^2\Psi(x)}{dx^2} + V(x)\Psi(x)$$

This can be rearranged into proper order as follows:

$$\boldsymbol{\frac{d^2\Psi(x)}{dx^2} - \frac{2m}{\hbar^2}(E - V)\Psi(x) = 0}$$

This is called the **one-dimensional time independent wave equation.**

Q9. Write Schrödinger's time dependent and time independent wave equations and state the physical significance of these equations.

(M.U. Dec 2016) [7 Marks]

A wave function $\Psi(x, t)$ describes the behavior of a De Broglie's matter wave. Schrodinger's equations are differential equations of that wave function.

The one-dimensional time dependent Schrödinger's wave equation is given as

$$-\frac{\hbar}{2m} \cdot \frac{\partial^2\Psi(x,t)}{\partial x^2} + V\Psi(x,t) = i\hbar \frac{\partial\Psi(x,t)}{\partial t}$$

The Schrodinger's time dependent equation describes the dynamical behavior of the particle. By solving Schrodinger's time dependent equation, we get a wave function Ψ(x,t) which can tell us the probability of finding the particle in some region in space and how it varies as a function of time.

The one-dimensional time independent Schrödinger's wave equation is:

$$-\frac{\hbar^2}{2m}\cdot\frac{d^2(x)}{dx^2}+V(x)\Psi(x)=E\cdot\Psi(x)$$

By solving Schrodinger's time independent equation, we get a wave function Ψ(x) which can tell us the probability of finding the particle in some region in space at a particular time. This probability for one-dimensional space of length 'L' can be found by:

$$\int_0^L |\Psi(x)|^2 \ dx = 1$$

Q10. Derive the expression for energy eigenvalues for free particles in one dimensional potential well.

(M.U. May 2015, 2018; Dec 2015, 2017, 2019) [5 Marks]

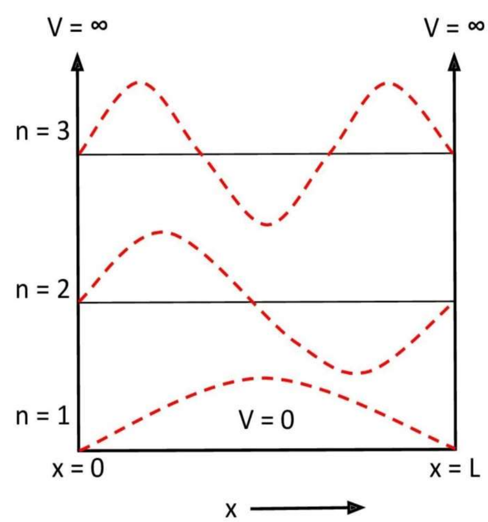

Figure 10: Particle in a Box

Consider a particle confined to an infinite potential well. The potential energy is zero inside the well and infinite at the boundaries as shown in *Figure 10*. A particle inside this well can propagate freely only along x-axis and gets reflected from the boundaries at x=0 and x=L but it cannot leave well, such state is called bound state. We have to derive energy eigenvalues for such a free particle confined to a one-dimensional potential well.

This particle remains inside the well for all the time so its position is time independent hence its wavefunction will be the solution of time independent Schrodinger's equation where we have substituted V=0:

$$\frac{d^2\Psi(x)}{dx^2} - \frac{2m}{\hbar^2}\Psi(x) = 0\text{----}(1)$$

Let us assume $\frac{2mE}{\hbar^2} = k^2$ ------(2)

Then the differential equation looks very similar to a simple harmonic motion

$$\frac{d^2\Psi(x)}{dx^2} - k^2\Psi(x) = 0\text{-----}(3)$$

The most general solution for plane waves corresponding to D.E. in (3) is

$$\Psi(x) = Ae^{i\,k\,x} + Be^{-i\,k\,x}\text{-----}(4)$$

In order to make this general solution suitable to our case we apply boundary conditions specific to our problem:

- **First boundary condition: Ψ(x) = 0 at x = 0**
- **Second boundary condition: Ψ(x) = 0 at x = L**

The above conditions come from the fact that the particle cannot move out of the well and hence its wave function vanishes at the boundaries.

First boundary condition: Ψ(x) = 0 at x = 0

Application of this boundary condition to (4) $\Psi(x) = Ae^{i\,k\,x} + Be^{-i\,k\,x}$ gives

$$0 = Ae^{i\,k\,0} + Be^{-i\,k\,0}$$

This reduces to

$$A + B = 0$$

Therefore, B= -A-----(5)

Second boundary condition: Ψ(x) = 0 at x = L

Application of this boundary condition to (4) $\Psi(x) = Ae^{ikx} + Be^{-ikx}$ gives

$$0 = Ae^{ikL} + Be^{-ikL}$$

This reduces to,

$$Ae^{ikL} + Be^{-ikL} = 0 \text{ -----(6)}$$

Using (5) B=-A here we get

$$A(e^{ikL} - e^{-ikL}) = 0 \text{ -----(7)}$$

Using, $\sin kL = \frac{e^{ikL} - e^{-ikL}}{2i}$ we obtain in equation (7) we obtain

$$2i\, A \sin kL = 0$$

Which makes us conclude that either A=0 or $\sin kL = 0$

'A' cannot be zero because if A=0 then $\Psi(x) = 0$ everywhere which is not true.

Hence, we conclude $\sin kL = 0$

$\sin kL = 0$ implies that $kL = n\pi$ ------(8)

with n = 0, 1, 2, ... ------

Using (2) $\frac{2m}{\hbar^2} = k^2$ to substitute the value of 'k' in equation (8) we get

$$\sqrt{\frac{2mE}{\hbar^2}}L = n\pi$$

Squaring both sides, and after rearranging the terms we have expression for 'E'

$$E = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

This is the expression for energy eigen values of a free particle in an infinite potential well. It proves that in quantum theory, the lowest available energy for the particle called the energy quantum is $E = \frac{\pi^2\hbar^2}{2mL^2}$ and the higher energies are just multiples of it.

Q11. Explain Heisenberg's uncertainty principle with an example and give its physical significance.

(M.U. Dec 2008, 2009, 2013; May 2013) [5 Marks]

In wave mechanics a matter particle in motion is associated with a group of waves in the form of a wave packet.

When the wave packet is small, it is very **easy to locate** (error in measurement is small) the particle, but it is **difficult to determine** (error in measurement is large) the wavelength (λ) associated with it and hence its momentum ($p = h/\lambda$) as shown in *Figure 11a*. In this case, the particle characteristic of the matter wave is dominant over the wave characteristic.

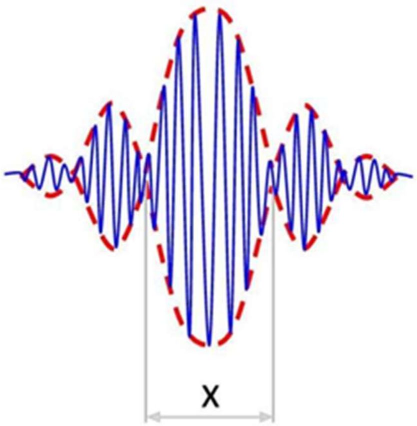

Figure 11a: Δx is small Δp is big

On the other hand, if the wave packet is large the wavelength and hence the momentum of the particle can be **easily determined** but it is **difficult to locate** the particle which can be anywhere within the wave packet. This is shown in *Figure 11b*. In this case, the wave characteristic of the matter wave is dominant over the particle characteristic.

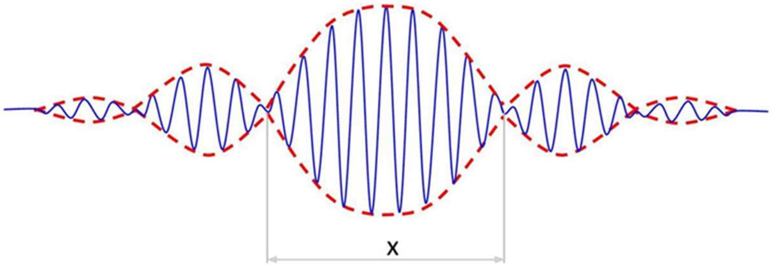

Figure 11b: Δx is large Δp is small

All this is concluded in the form of Heisenberg's Uncertainty Principle that the position and momentum of a particle cannot be determined precisely and simultaneously. Let the inaccuracies in determination of position and momentum be represented as 'Δx' and 'Δp' respectively. Then Heisenberg's Uncertainity principle relates them as follows:

$$\Delta x \cdot \Delta p \geq \hbar$$

where, $\hbar = \frac{h}{2\pi}$, h being the Planck's constant.

The Heisenberg's uncertainty principle puts a limit to the accuracy with which measurements can be made in quantum mechanics. Measurable quantities come in pairs called conjugate pairs such that, the product of their dimensions are energy x time i.e. in the units of Planck's constant. For example, energy and time, position and momentum and angle an angular momentum. Each quantity is measured with some error but according to uncertainty principle the product of errors in measurement of conjugate quantities must be smaller than Planck's constant irrespective of the sophistication of the measuring instrument.

Q12. Using Heisenberg's principle show that electrons cannot exist within the nucleus.

(M.U. May 2009, 2011, 2014, 2015, 2016; Dec 2012, 2014, 2017, 2018, 2019)

[5 Marks]

Initially assume that an electron is a part of the nucleus.

The size of a nucleus is about 1 fermi = 10^{-15} m.

If an electron is confined within a nucleus, the uncertainty in its position must not be greater than the dimension of the nucleus, i.e., 10^{-15}m. Hence,

$$\Delta x_{max} = 10^{-15} m \text{ -----(1)}$$

From the limiting condition of Heisenberg's uncertainty principle, it can be written as,

$$\Delta x_{min} \cdot \Delta p_{min} = h \text{ -----(2)}$$

Substituting values,

$$\Delta p_{min} = \frac{\hbar}{\Delta x_{min}} = \frac{6.63\times10^{-34}}{2\times3.14\times10^{-15}}$$

$$= 1.055\times10^{-19} \text{ kg-m/sec. -----(3)}$$

Now,

$$\Delta p_{min} = m\Delta v_{min} \text{ -----(4)}$$

Hence,

$$\Delta v_{min} = \frac{\Delta p_{min}}{m} = \frac{1.055\times10^{-19}}{9.1\times10^{-31}}$$

$$= 1.159\times10^{11} \text{ m/s} > \text{c -----(5)}$$

As no particle can have a velocity greater than light, it is evident that an electron cannot exist in the nucleus.

As

$$\Delta v_{min} < v, \quad v > 1.159 \times \frac{10^{11} m}{s} > c$$

Therefore, the electron inside the nucleus behaves as a relativistic particle.

The relativistic energy of the electron is

$$E = \sqrt{m_0^2 c^4 + p^2 c^2} \text{ -----(6)}$$

Since, the actual momentum of the electron p >> Δp_{min}, $p^2c^2 \gg m_0^2c^2$ (the rest mass energy of the electron the value of which is 0.511 MeV). Hence,

$$\text{E} = \text{pc -----(7)}$$

Assuming p = Δp_{min} , the least energy that an electron should possess within a nucleus is given by

$E_{min} = \Delta p_{min} \cdot c$

$= 1.055 \times 10^{-19} \times 3 \times 10^{8}$

$= 3.165 \times 10^{11}$ J

Therefore,

$$E_{min} = \frac{3.165 \times 10^{-11}}{1.6 \times 10^{-19}} = 197\ MeV$$

In reality, the only source of generation of electrons within a nucleus is the process of β - decay. The maximum kinetic energy possessed by the electrons during β - decay is about 100 KeV. This shows that an electron cannot exist within a nucleus.

Q13. Write a brief note on Quantum Computing.

The area of computing which is mainly focused on using the principles of quantum theory to develop various computing-based technologies is **Quantum Computing.**

Normal computers use the binary system where each bit of information is coded as '0' or '1'. However, in Quantum Computing, information is stored as an "atom" or a "photon" (a quantum system). A qubit (or a quantum bit) is the fundamental unit of information.

Quantum Computing has endless applications in the real world. The main benefits include much less computation time and less bulky computers.