ELECTRODYNAMICS

A) Problems on Cartesian, Cylindrical and Spherical Co-ordinate System

Formulae Used:

1. Cylindrical- Cartesian

$$\boldsymbol{r = \sqrt{x^2 + y^2}}$$

$$\emptyset = \boldsymbol{tan^{-1}(\frac{y}{x})}$$

2. Spherical- Cartesian

$$\mathbf{r = \sqrt{x^2 + y^2 + z^2}}$$

$$\boldsymbol{\theta} = \mathbf{tan^{-1}}\left(\frac{\sqrt{\mathbf{x^2+y^2}}}{\mathbf{z}}\right)$$

$$\emptyset = \mathbf{tan^{-1}(\frac{y}{x})}$$

Where,

a) $x \rightarrow the\ x\ co\ ordinate$
b) $y \rightarrow the\ y\ co\ ordinate$
c) $z \rightarrow the\ z\ co\ ordinate$
d) $\emptyset \rightarrow angle\ between\ the\ vector\ and\ z\ axis$
e) $\theta \rightarrow angle\ between\ the\ vector\ and\ x\ axis$

Solved Problems

1. Convert the point P(1,3,5) from Cartesian to Cylindrical and Spherical polar coordinates

Solution:

Data: x=1,y=3,z=5 in Cartesian coordinates

In cylindrical polar coordinates, (r, $\emptyset, z$)

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 3^2} = \sqrt{10} = 3.162$$

$$\emptyset = \tan^{-1}\frac{y}{x} = \tan^{-1} 3 = 71.57^{\circ}$$

Z=5

In Spherical polar coordinates (r,$\theta, \emptyset$)

r=$\sqrt{x^2+y^2+z^2} = \sqrt{1^2+3^2+5^2} = \sqrt{35}$ =5.916

$$\emptyset = \tan^{-1}\frac{y}{x} = \tan^{-1} 3 = 71.519^{\circ}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\frac{\sqrt{1^2+3^2}}{5} = 32.3^{\circ}$$

Answer: Coordinates of p in cylindrical polar coordinates are (3.162, 71.57°, 5) and in spherical polar coordinates are (5.916, 71.57°, 32.3°)

2. Given vector $\vec{A}(x,y,z) = y\widehat{\iota_x} + (x+z)\widehat{\iota_y}$ in Cartesian coordinate system at point P(-2,6,3). Convert the vector $\vec{A}$ into cylindrical and spherical coordinates.

Solution:

Data: $\vec{A} = y\widehat{\iota_x} + (x+z)\widehat{\iota_y}$, x=-2, y=6, z=3

Formula: Cylindrical coordinates

$$r = \sqrt{x^2+y^2}, \emptyset = \tan^{-1}\frac{y}{x}, z = z$$

Spherical coordinates

$$r = \sqrt{x^2+y^2+z^2}, \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \emptyset = \tan^{-1}\frac{y}{x}$$

Calculations:

Cylindrical coordinates:

r=$\sqrt{(-2)^2+6^2 = 6.32}$

$$\emptyset = \tan^{-1}\frac{6}{-2} = 108.43^{\circ}$$

Z=3

Spherical coordinates:

$$r = \sqrt{(-2)^2 + 6^2 + 3^2} = 7$$

$$\theta = \tan^{-1}\frac{\sqrt{(-2)^2+6^2}}{3} = 64.62^{\circ}$$

$$\emptyset = \tan^{-1}\frac{6}{-2} = 108.43^{\circ}$$

Answer: $\vec{A} = \vec{A}(6.32, 108.43^{\circ}, 3)$ is cylindrical coordinates

$\vec{A} = \vec{A}(7, 64.62^{\circ}, 108.43^{\circ})$ is spherical coordinates.

B) Problems on Divergence and Curl

Formulae Used:

1) Divergence

$$(\vec{\nabla}.\vec{A}) = \frac{\partial}{\partial x}\widehat{\iota_x} + \frac{\partial}{\partial y}\widehat{\iota_y} + \frac{\partial}{\partial z}\widehat{\iota_z}$$

2) Curl

$$(\vec{\nabla} \times \vec{A}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{A_x} & \mathbf{A_y} & \mathbf{A_z} \end{vmatrix}$$

Solved Problems

1) Find the divergence and curl of the fielf $F = 30\widehat{\iota_x} + 2xy\widehat{\iota_y} + 5xz^2\widehat{\iota_z}$ in Cartesian coordinates.

   Solution:

   Divergence, $\vec{\nabla}.\vec{F} = \left(\frac{\partial}{\partial x}\widehat{\mathbf{i_x}} + \frac{\partial}{\partial y}\widehat{\mathbf{i_y}} + \frac{\partial}{\partial z}\widehat{\mathbf{i_z}}\right).(30\widehat{\iota_x} + 2xy\widehat{\iota_y} + 5xz^2\widehat{\iota_z})$

   $$= \frac{\partial}{\partial x}(\mathbf{30}) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(2xz^2) = 2x + 10xz$$

   $$\vec{\nabla}.\vec{F} = 2x(1 + 5z)$$

$$\vec{\nabla} \times \vec{F} = -5z^2\widehat{\mathbf{i_y}} + 2y\widehat{\mathbf{i_z}}$$

2) In cylindrical coordinates, find the divergence and curl of the field, $F = \frac{150}{r^2}\widehat{\iota_r} + 10\widehat{\iota_\emptyset}$

Solution:

Here, $F_r = \frac{150}{r^2}, F_\emptyset = 10, F_z = 0$

Divergence of $\vec{F} = \vec{\nabla}.\vec{F} = \frac{1}{r}.\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\emptyset}{\partial \emptyset} + \frac{\partial F_r}{\partial z}$

$$\vec{\nabla}.\vec{F} = \frac{1}{r}.\frac{\partial}{\partial r}\left(r.\frac{150}{r^2}\right) + \frac{1}{r}\frac{\partial}{\partial \emptyset}(10) + \frac{\partial}{\partial z}(0)$$

$$\vec{\nabla}.\vec{F} = -\frac{150}{r^3}$$

Curl of $\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \frac{\widehat{\iota_r}}{r} & \widehat{\iota_\emptyset} & \frac{\widehat{\iota_z}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \emptyset} & \frac{\partial}{\partial z} \\ \mathrm{F_x} & \mathrm{F_y} & \mathrm{F_z} \end{vmatrix} = 0$

3) Given a general field $\vec{B} = 10\sin\theta\ \widehat{\iota_\theta}$ in spherical coordinates. Find curl B at $(2, \frac{\pi}{2}, 0)$.

Solution:

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \frac{\widehat{\iota_r}}{r^2\sin\theta} & \frac{\widehat{\iota_\theta}}{r\sin\theta} & \frac{\widehat{\iota_\emptyset}}{r} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{B_r} & \mathrm{rB_\theta} & r\sin\theta\, B_\emptyset \end{vmatrix}$$

Here $\vec{B} = 10\sin\theta$, hence $\mathrm{B_r} = 0, B_\emptyset = 0$ and $\mathrm{B_\theta} = 10\sin\theta$

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \frac{\widehat{\iota_r}}{r^2\sin\theta} & \frac{\widehat{\iota_\theta}}{r\sin\theta} & \frac{\widehat{\iota_\emptyset}}{r} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \mathrm{rB_\theta} & 0 \end{vmatrix}$$

$$\vec{\nabla} \times \vec{B} = \frac{10}{r}\sin\theta\ \widehat{\iota_\emptyset}$$

Therefore, at $(2, \frac{\pi}{2}, 0)$, $\vec{\nabla} \times \vec{B} = 5\widehat{\iota_\emptyset}$