Algebra, Logic and Qubits: Quantum $\check{\alpha}\beta\alpha\xi$.

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Abstract

The canonical anticommutation relations (CAR) for fermion systems can be represented by finite-dimensional matrix algebra, but it is impossible for canonical commutation relations (CCR) for bosons. After description of more simple case with representation of CAR and (bounded) quantum computational networks via Clifford algebras in the paper are discussed CCR. For representation of the algebra it is not enough to use quantum networks with fixed number of qubits and it is more convenient to consider Turing machine with essential operation of appending new cells for description of infinite tape in finite terms — it has straightforward generalization for quantum case, but for CCR it is necessary to work with symmetrized version of the quantum Turing machine. The system is called here *quantum abacus* due to understanding analogy with the ancient counting devices ($\check{\alpha}\beta\alpha\xi$).

1 Introduction

In his first article about quantum computers [1] David Deutsch wrote in relation with Feynman's work [2]:

> Feynman (1982) went one step closer to a true quantum computer with his 'universal quantum simulator'. This consists of a lattice of spin systems with nearest-neighbour interactions that are freely specifiable. Although it can surely simulate any system with a finite-dimensional state space (I do not understand why Feynman doubts that it can simulate fermion systems), it is not a computing machine in the sense

of this article. ‘Programming’ the simulator consists of endowing it by *fiat* with the desired dynamical laws, and then placing it in a desired initial state. But the mechanism that allows one to select arbitrary dynamical laws is not modelled. The dynamics of a true ‘computer’ in my sense must be given once and for all, and programming it must consist entirely of preparing it in a suitable *state* (or mixed case).

The cited article (together with [3]) forms some standard for many works about quantum information science and here is necessary to say few words for explanation of some difference in methods and purposes of present paper.

First, it maybe even more interesting to understand simulation of boson systems, because due to infinite-dimensional state space (see Sec. 2) it is impossible to apply directly approach mentioned by David Deutsch. Is it possible to extend the ideas from systems with finite number of states to *countable* number (discrete spectrum)? Quantization of harmonic oscillator can be considered as simple example — quantum model of natural numbers, ‘quantum abacus ($\breve{\alpha}\beta\alpha\xi$)’.

Second, the cited methods of description of quantum computations are similar with attempts to learn classical information science only by ‘wiring schemes’ of processors with huge number of gates. But usually it is efficient to consider general methods of mathematical logic, algebra *etc.* together or even before binary numbers and logical gates.

The algebraic methods currently are common for quantum theory and if we are interesting in boson and fermion systems, we should consider canonical commutation and anticommutation relations.

2 (Anti)commutation relations

The using *annihilation* and *creation* operators a, a^* for definition of quantum gates was introduced by Feynman [2, 4]. The operators meet anticommutation relation $\{a, a^*\} \equiv a\, a^* + a^*\, a = 1$ for one qubit. For system with n qubits we may use CAR for n fermions, *i.e.* introduce operators a_i, a_i^* for each qubit with properties:

$$\{a_i, a_j\} = \{a_i^*, a_j^*\} = 0, \quad \{a_i, a_j^*\} = \delta_{ij} \tag{1}$$

The algebra generated by $2n$ operators Eq. (1) is isomorphic with algebra of all complex $2^n \times 2^n$ matrices and so can be used for representation of any quantum gate [5].

In basis $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ the operators can be expressed as:

$$\begin{aligned} a &= \frac{\sigma_x + i\sigma_y}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ a^* &= \frac{\sigma_x - i\sigma_y}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} a_i &= \underbrace{1 \otimes \cdots \otimes 1}_{n-i-1} \otimes a \otimes \underbrace{\sigma_z \otimes \cdots \otimes \sigma_z}_{i} \\ a_i^* &= \underbrace{1 \otimes \cdots \otimes 1}_{n-i-1} \otimes a^* \otimes \underbrace{\sigma_z \otimes \cdots \otimes \sigma_z}_{i} \end{aligned}$$

The representation of CAR also can be expressed in more invariant and clear way with using of Clifford algebras. Let us recall it briefly [5].

We can start with complex Clifford algebra $\mathbb{C}\mathbf{l}(n,\mathbb{C})$ with n generators e_i with relations $e_i e_j + e_j e_i = 2\delta_{ij}$. For even $n = 2k$ there are two useful properties: first, there are k pairs $a_l = (e_{2l} + ie_{2l+1})/2$, $a_l^* = (e_{2l} - ie_{2l+1})/2$ those satisfy Eq. (1), *i.e.* CAR. Second, there is recursive construction of $\mathbb{C}\mathbf{l}(n+2,\mathbb{C}) = \mathbb{C}\mathbf{l}(n,\mathbb{C}) \otimes \mathbb{C}\mathbf{l}(2,\mathbb{C})$: if $e_i^{(n)}$ $(i \geqslant 0)$ are n generators of $\mathbb{C}\mathbf{l}(n,\mathbb{C})$ and $e_0^{(2)}$, $e_1^{(2)}$ are generators of $\mathbb{C}\mathbf{l}(2,\mathbb{C})$ with $e_{01}^{(2)} \equiv ie_0^{(2)} e_1^{(2)}$, then $e_i^{(n+2)} = e_i^{(n)} \otimes e_{01}^{(2)}$, $e_n^{(n+2)} = \mathbf{1}^{(n)} \otimes e_0^{(2)}$, $e_{n+1}^{(n+2)} = \mathbf{1}^{(n)} \otimes e_1^{(2)}$ are $n+2$ generators of $\mathbb{C}\mathbf{l}(n+2,\mathbb{C})$. Because $\mathbb{C}\mathbf{l}(2,\mathbb{C})$ is isomorphic with algebra of Pauli matrices, the constructions correspond to equations above if $e_0^{(2)} \mapsto \sigma_x$, $e_1^{(2)} \mapsto \sigma_y$, $e_{01}^{(2)} \mapsto \sigma_z$.

Let us now consider similar possibility for Bose particles with *annihilation* and *creation* operators c, c^* with commutation relation $[c, c^*] \equiv c\,c^* - c^*\,c = 1$.

$$[c_i, c_j] = [c_i^*, c_j^*] = 0, \quad [c_i, c_j^*] = \delta_{ij} \tag{2}$$

The relation $[A, B] = 1$ cannot be true for finite-dimensional matrices A, B because $\operatorname{trace}[A, B] = \operatorname{trace}(AB) - \operatorname{trace}(BA) = 0 \neq \operatorname{trace}(1)$. On the other hand infinite-dimensional algebra with the property can be simple found. For example for algebra of functions on line it may be linear operators[1]:

$$\begin{gathered} \mathbf{D}\colon \psi(x) \mapsto \psi'(x), \quad \mathbf{X}\colon \psi(x) \mapsto x\,\psi(x); \\ [\mathbf{D}, \mathbf{X}]\,\psi(x) \equiv \mathbf{D}\Big(\mathbf{X}\big(\psi(x)\big)\Big) - \mathbf{X}\Big(\mathbf{D}\big(\psi(x)\big)\Big) = \\ = \big(x\psi(x)\big)' - x\psi'(x) = \big(\psi(x) + x\psi(x)'\big) - x\psi'(x) = \mathbf{1}\,\psi(x) \end{gathered} \tag{3}$$

[1] *Cf.* with well known representation of ip and q operators in quantum mechanics.

It is possible to define commutation relations on algebra of square-integrable functions with scalar product:

$$\langle \psi \mid \varphi \rangle = \int\limits_{-\infty}^{+\infty} \overline{\psi(x)}\, \varphi(x)\, \mathrm{d}x$$

Then $\mathbf{X}^* = \mathbf{X}$, $\mathbf{D}^* = -\mathbf{D}$ and so $c = (\mathbf{X} + \mathbf{D})/\sqrt{2}$ and $c^* = (\mathbf{X} - \mathbf{D})/\sqrt{2}$ satisfy commutation relation for simplest case of one variable $[c, c^*] = 1$. The operators is also related with quantization of harmonic oscillator [6] — there is recurrence relation for eigenfunctions $\psi_n(x)$ of the 1D oscillator: $\psi_n(x) = c^* \psi_{n-1}(x)/\sqrt{n}$ and so c^* really corresponds to 'rudimentary' creation operator *i.e.* excitation of the system on next level.

For more general case of Eq. (2) it is possible to define n pairs of operators c_i, c_i^* on algebra of functions with n variables $\psi(x_1, \ldots, x_2)$:

$$c_i = \frac{1}{\sqrt{2}}\Big(x_i + \frac{\partial}{\partial x_i}\Big), \quad c_i^* = \frac{1}{\sqrt{2}}\Big(x_i - \frac{\partial}{\partial x_i}\Big) \tag{4}$$

Now let us consider more formal representation of CCR used in secondary quantization for case of Bose statistics [6]. Here is used *occupation numbers* representation $|\,\Psi\rangle = |n_1, n_2, \ldots\rangle$ with n_i is number of particles in i-th state. Then operators c_i and c_i^* are formally defined as:

$$\begin{array}{rcl} c_i\,|n_1, n_2, \ldots, n_i, \ldots\rangle & = & \sqrt{n_i}\,|n_1, n_2, \ldots, n_i - 1, \ldots\rangle \\ c_i^*\,|n_1, n_2, \ldots, n_i, \ldots\rangle & = & \sqrt{n_i + 1}\,|n_1, n_2, \ldots, n_i + 1, \ldots\rangle \end{array} \tag{5}$$

where definition of conjugated operator meets the standard condition *i.e.* $\langle n_i | c_i^* | n_i - 1\rangle = \langle n_i - 1 | c_i | n_i \rangle^*$.

In computational terminology it can be considered as quantum version of ancient[2] counting device with 'unary'[3] number system (*vs.* binary or decimal) — number n is represented as '$\underbrace{/\!/\!/ \cdots /\!/}_{n}$'. Let us call it $\breve{\alpha}\beta\alpha\xi$ (abacus).

[2]Very ancient, because in *Abriss der geschichte der mathematik* (*A Brief Review of the History of Mathematics*) by von Dirk J. Struik (Berlin 1963) is mentioned that *The Rhind Mathematical Papyrus* written about 3650 years ago already used more or less directly both decimal and binary number systems.

[3]I 'borrowed' the term *unary* from Seth Lloyd (exchange about [†]), to avoid ambiguity of word 'unitary' in given context.

3 Quantum infinite Turing machine

In classical theory of recursion the Turing machine is supplied with infinite tape. Usually it is not considered as practical linitation due to argument: let us start with finite tape and we always may add new sections (cells) if head of Turing machine going to reach the end of the tape.

In quantum computation instead of section of the tape with two states we have two-dimensional Hilbert space $\mathcal{H}_2$, instead of finite tape with n sections here is tensor power of $\mathcal{H}$:

$$\mathsf{T}_n(\mathcal{H}) \equiv \mathcal{H}^{\otimes n} = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n}$$

Then an analogue of discussed operation may be construction of space[4] [8]:

$$\mathsf{T}_*(\mathcal{H}) \equiv \bigoplus_{k=1}^{\infty} \mathsf{T}_k(\mathcal{H}) = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \cdots \tag{6}$$

sometime it is convenient to extend summation for $k = 0$ with $\mathsf{T}_0(\mathcal{H}) \equiv \mathbb{C}$.

Here subspace $\mathsf{T}_k(\mathcal{H})$ in direct sum corresponds to Turing machine with k-qubits tape. We considered only tape of quantum Turing machine, because it is enough for future description of CCR. The bounded quantum Turing machine [7] also uses other elements and more than one tape and in addition to the property it should be mentioned that $\mathsf{T}_*(\mathcal{H})$ maybe more appropriate for description of semi-infinite Turing tape with necessity of extension of the summation in Eq. (6) for negative k (*cf.* also K-theory) or using two half-tapes for each infinite tape.

Because it is quantum system we could have superposition of two states with different number of qubits[5], and here is important that states with different number of qubits belong to *orthogonal* subspaces.

It is possible to introduce Hermitian scalar product by summation of each component, *i.e.* for $\Psi = \sum_k \psi_k$, $\Phi = \sum_k \varphi_k$ where $\Psi, \Phi \in \mathsf{T}_*(\mathcal{H})$ and $\psi_k, \varphi_k \in \mathsf{T}_k(\mathcal{H})$:

$$\langle \Psi \mid \Phi \rangle = \sum_{k=0}^{\infty} \langle \psi_k \mid \varphi_k \rangle \tag{7}$$

[4]It is *tensor algebra* of $\mathcal{H}$.

[5]In physics may exist superselection laws those prohibit some superpositions.

4 Symmetric qubits

Let us now consider symmetric spaces $\mathsf{S}_k(\mathcal{H})$ [8, 9]. For two-dimensional $\mathcal{H}$ with basis e_0, e_1 the space is produced by $k+1$ elements $e_{00\ldots00}$, $e_{00\ldots01}$, …, $e_{01\ldots11}$, $e_{11\ldots11}$. Let us use notation $e_{\{i,k-i\}}$ for the elements: $e_{\{k,0\}}$, $e_{\{k-1,1\}}$, …, $e_{\{0,k\}}$. It could be enough to use only one index $e_{\{i\}}$, $i=0,\ldots,k$ if we would work with space $\mathsf{S}_k(\mathcal{H})$, but it is not enough for $\mathsf{S}_*(\mathcal{H})$ defined as:

$$\mathsf{S}_*(\mathcal{H}) \equiv \bigoplus_{k=1}^{\infty} \mathsf{S}_k(\mathcal{H}) = \mathcal{H} \oplus (\mathcal{H} \odot \mathcal{H}) \oplus (\mathcal{H} \odot \mathcal{H} \odot \mathcal{H}) \oplus \cdots \tag{8}$$

where '$\odot$' is used for symmetric product.

Due to isomorphism of symmetric space S_k with space of homogeneous k-polynomials [8] elements of $\mathsf{S}_k(\mathcal{H})$ can be represented as homogeneous polynomials with two variables ξ, η: $p_k(\xi,\eta) = a_k\xi^k + a_{k-1}\xi^{k-1}\eta + \cdots + a_0\eta^k$ and then element $\mathsf{S}_*(\mathcal{H})$ corresponds to arbitrary (non-homogeneous) polynomial with two variables $p(\xi,\eta)$; $e_{\{i,j\}} \mapsto \xi^i\eta^j$.

It should be mentioned also, that $\mathsf{S}_k(\mathcal{H})$ can be considered as space of polynomials with one variable and with degree less or equal than k *i.e.* $p_k(\zeta) = a_k\zeta^k + a_{k-1}\zeta^{k-1} + \cdots + a_0$ and there is special case with one or more higher coefficients are zeros $a_i = 0$, $k \geqslant i > l$. If $\zeta_1,\ldots,\zeta_l$, $l \leqslant k$ are roots of the polynomial, then factorization of the $p_k(\zeta) = a_l\Pi_{i=1}^{l}(\zeta-\zeta_i)$ corresponds to factorization of $p_k(\xi,\eta)$ on k terms:

$$a_k\xi^k + a_{k-1}\xi^{k-1}\eta + \cdots + a_0\eta^k = (\alpha_1\xi - \beta_1\eta) \times \cdots \times (\alpha_k\xi - \beta_k\eta)$$

where $\alpha_i\zeta_i = \beta_i$ and maybe $\alpha_i = 0$ if $l < k$ and $i > l$ *i.e.* formally $\zeta_i = \infty$, but here is more rigorously to use projective spaces[6] — spaces of rays in terminology more usual for quantum mechanics. Projective coordinate ζ corresponds to ray $(\xi,\eta) \sim (\lambda\xi,\lambda\eta)$ in $\mathcal{H}$ or point on Riemann (Bloch) sphere and roots ζ_i correspond to pairs $(\alpha_i,\beta_i) \sim (\lambda\alpha_i,\lambda\beta_i)$.

Due to the property element of $\mathsf{S}_k(\mathcal{H})$ up to multiplier is defined by k points (α_i,β_i) from $\mathcal{H}$: $\Pi_{i=1}^{k}(\alpha_i\xi - \beta_i\eta) = \lambda p_k(\xi,\eta)$ and because multiplication of each pair $(\alpha_i,\beta_i) \mapsto (\lambda_i\alpha_i,\lambda_i\beta_i)$ changes only common multiplier λ for same p_k, it is correct map from k rays in $\mathcal{H}$ to ray in $\mathsf{S}_k(\mathcal{H})$.

So, symmetric qubits are never entangled, each element of $\mathsf{S}_k(\mathcal{H})$ can be represented as symmetrical product of k qubits, elements of $\mathcal{H}$. Because

[6] Projective spaces make possible to get rid of special cases like $\zeta_i = \infty$ if $a_k \cdots = 0$.

$\mathsf{S}_k(\mathcal{H})$ is $k+1$-dimensional linear space and can be used as space of states for particle with spin $k/2$, the factorization described before explains why state of the particle always can be described as k points on Riemann sphere[7].

Note: the model of symmetric qubits may looks like some violation of Pauli's spin-statistics principle. Really, one qubit can be considered as fermion, especially in the context of the paper with anticommutation relation and Pauli's matrices. It should be said for justification, that it is very convenient mathematical model of spin-$\frac{k}{2}$ system originated by Weyl, Majorana, Penrose and the symmetric spin-$\frac{1}{2}$ subsystems can be considered as formal math 'ghosts' (or 'colored' like quarks). Yet another reason — the qubit is an abstract two-states system and it is not quite correct to talk about spins and statistics, it maybe electron with spin half or photon with spin one, or some model described by Schrödinger equation with potential well, *i.e.* by one-component, scalar wave function. And next, the coordinate dependence is not considered in usual models of qubit and so Pauli's exclusion principle sometime can be formally avoided by suggestion about different locations for each qubit. *Cf.* also misc. 2 at end of Sec. 5.

Now let us define scalar product on S_k and S_*. It is convenient together with basis $e_{\{i,j\}}$, $i+j=k$ to consider:

$$\tilde{e}_{\{i,j\}} \equiv \frac{e_{\{i,j\}}}{\sqrt{i!j!}} = \sqrt{\frac{C_k^i}{k!}}\, e_{\{i,j\}} \tag{9}$$

The basis is convenient by following reasons: First, in 'more physical' definition [9] the $\mathsf{S}_k(\mathcal{H})$ is *subspace* ($\mathsf{S}_k \subset \mathsf{T}_k$) of symmetrical tensors with operation of *symmetrization* by summation of $k!$ transpositions $\sigma(T)$ of indexes for given $T \in \mathsf{T}_k$: $\mathsf{S}(T) = \frac{1}{k!}\sum_\sigma \sigma(T)$, and if to consider $e_{\{i,k-i\}}$ as element $e_{00\ldots11}$ of T_k, then $|\mathsf{S}(e_{\{i,k-i\}})| = 1/\sqrt{C_k^i}$ (it is sum of all C_k^i possible transpositions with coefficient $1/C_k^i$) and needs for normalizing multiplier proportional to $\sqrt{C_k^i}$ (in [8] is used second definition of symmetric space as *quotient space* $\mathsf{S} = \mathsf{T}/\mathfrak{S}$, where $\mathfrak{S}$ is equivalence relation: $T \sim \sigma(T)$).

Second, $\tilde{e}_{\{i,j\}}$ form representation of $SU(2)$ group in $SU(k+1)$ [11] in such a way, that if we use other basis $U\colon (e_0, e_1) \mapsto (e'_0, e'_1)$, $U \in SU(2)$ then $\tilde{e}'_{\{i,j\}}$ are also connected with $\tilde{e}_{\{i,j\}}$ by some *unitary* transformation from $SU(k+1)$.

Let us use for S_k basis $\tilde{e}_{\{i,j\}}$ with standard $\langle \cdot \mid \cdot \rangle$ (or $\langle \cdot \mid \cdot \rangle / k!$, see Eq. (10) below) together with $e_{\{i,j\}}$ considered as transformation to other basis with Hermitian scalar product defined by diagonal matrix $h_{ii} = C_k^i$ (or

[7]There is popular introduction and two references in [10, §6; Objects with large spin].

$h_{ii} = C_k^i/k! = \frac{1}{i!(k-i)!}$) — the basis is convenient for representation of S_k as space of polynomials.

The scalar product on $\mathsf{S}_*(\mathcal{H})$ may be defined as in Eq. (7), but it is more convenient to use also:

$$\langle \Psi^{\mathsf{S}} \mid \Phi^{\mathsf{S}} \rangle_{\exp} = \sum_{k=0}^{\infty} \frac{1}{k!} \langle \psi_{_k}^{\mathsf{S}} \mid \varphi_{_k}^{\mathsf{S}} \rangle \tag{10}$$

5 Quantum $\breve{\alpha}\beta\alpha\xi$

It is possible to use $\mathsf{S}_*(\mathcal{H})$ as an example of *quantum* $\breve{\alpha}\beta\alpha\xi$ (let us denote it as $|\breve{\alpha}\beta\alpha\xi\rangle$ or $|\breve{\alpha}\rangle$) introduced in Sec. 2. The approach makes possible to link it with $\mathsf{T}_*(\mathcal{H})$ and quantum Turing machine, $\mathsf{S}(|\text{TM tape}\rangle) \to |\breve{\alpha}\beta\alpha\xi\rangle$

The elements $e_{\{i,j\}}$ or $\tilde{e}_{\{i,j\}}$ of basis $\mathsf{S}_*(\mathcal{H})$ can be used as basis $|i,j\rangle$ of $|\breve{\alpha}\rangle$ with two different kinds of states $n_0 = i$, $n_1 = j$ that can be considered also as composite system of two $|\breve{\alpha}\rangle$ with infinite series of states for each one: $\mathsf{S}_*(\mathcal{H}) \cong |\breve{\alpha}\rangle_0 \otimes |\breve{\alpha}\rangle_1$.

It is possible to introduce operators c_i, c_i^*; $i = 0, 1$ by Eq. (5). Let us $\tilde{e}_{\{n_0,n_1\}} \equiv |n_0, n_1\rangle$. Then:

$$\begin{array}{l} c_0\,|n_0,n_1\rangle = \sqrt{n_0}\,|n_0-1,n_1\rangle,\ \ c_0^*\,|n_0,n_1\rangle = \sqrt{n_0+1}\,|n_0+1,n_1\rangle \\ c_1\,|n_0,n_1\rangle = \sqrt{n_1}\,|n_0,n_1-1\rangle,\ \ c_1^*\,|n_0,n_1\rangle = \sqrt{n_1+1}\,|n_0,n_1+1\rangle \end{array} \tag{11}$$

Here numbers of zeros and units n_0, n_1 are used instead of $n_1, n_2, \ldots$

As understanding example of such system it is possible to use two-dimensional oscillator:

$$i\hbar\dot{\psi}(x,y,t) = \left(\frac{m\omega^2}{2}(x^2+y^2) - \frac{\hbar^2}{2m}\Delta_{x,y}\right)\psi(x,y,t)$$

If $\phi_k(x)$, $k \geqslant 0$ is stationary solution of one-dimensional oscillator for energy $E = (k+\frac{1}{2})\hbar\omega$, then for 2D oscillator function $\phi_k(x)\phi_j(y)$ is solution for $E = (k+j+1)\hbar\omega$ and so for any natural $n \geqslant 0$ there is $n+1$ dimensional space of solutions for given energy $E = (n+1)\hbar\omega$:

$$\phi_n(x,y) = \sum_{k=0}^{n} \alpha_k\,\phi_k(x)\,\phi_{n-k}(y)$$

and nonstationary solution has form:

$$\phi(x,y,t)=\sum_{n=0}^{\infty}A_n\phi_n(x,y)e^{i\,(n+1)\hbar\omega\,t}$$

The example shows, how tensor product of two infinite-dimensional spaces $\mathcal{H}_\infty\otimes\mathcal{H}_\infty$ is decomposed on direct sum of linear spaces with dimensions $1,2,3,\ldots$ for the simple case in good agreement with formal mathematical constructions discussed above.

It is useful to consider Eq. (11) in basis $e_{\{n_0,n_1\}}$:

$$\begin{aligned}
c_0\,e_{\{n_0,n_1\}} &= c_0|n_0,n_1\rangle\sqrt{n_0!n_1!}=\sqrt{n_0}\,|n_0-1,n_1\rangle\sqrt{n_0!}\sqrt{n_1!}\\
&= n_0\,|n_0-1,n_1\rangle\sqrt{n_0{-}1\,!}\sqrt{n_1!}=n_0\,e_{\{n_0-1,n_1\}}\\
c_0^*\,e_{\{n_0,n_1\}} &= c_0^*|n_0,n_1\rangle\sqrt{n_0!n_1!}=\sqrt{n_0+1}\,|n_0+1,n_1\rangle\sqrt{n_0!}\sqrt{n_1!}\\
&= |n_0+1,n_1\rangle\sqrt{n_0{+}1\,!}\sqrt{n_1!}=e_{\{n_0+1,n_1\}}
\end{aligned}\tag{12}$$

and similarly with c_1, c_1^* and n_1.

The equations Eq. (12) demonstrate relation between CCR in secondary quantization Eq. (5) with CCR in differential algebra Eq. (3). Really, let us consider polynomials with two variables χ_0,χ_1, then $e_{\{i,j\}}\mapsto\chi_0^i\chi_1^j$, $c_i\,p(\chi_0,\chi_1)=\frac{\partial}{\partial\chi_i}p(\chi_0,\chi_1)$, $c_i^*\,p(\chi_0,\chi_1)=\chi_i\cdot p(\chi_0,\chi_1)$.

Miscellany

1. Let us return to initial notation ($\xi=\chi_0$, $\eta=\chi_1$) for polynomial basis $e_{\{i,j\}}\leftrightarrow\xi^i\eta^j$ defined earlier in Sec. 4. The linear operators c_0 and c_1 are isomorphic with two partial derivatives ∂_η and ∂_ξ and have some interesting property, 'linear merging' ('anti-cloning').

It was described in Sec. 4 that state of symmetric qubits up to multiplier, *i.e.* ray in $\mathsf{S}_n(\mathcal{H})$, can be described by n points on Riemann sphere. The operators ∂_ξ and ∂_η can be considered as maps $\mathsf{S}_n(\mathcal{H})\to\mathsf{S}_{n-1}(\mathcal{H})$ and also between spaces of rays — from sphere with n marked points $(\zeta_1,\ldots,\zeta_n)$ and without one pole (the pole maps to zero) to sphere without same pole and with $n-1$ marked points $(\zeta'_1,\ldots,\zeta'_{n-1})$. For $n=2$ they map $\mathsf{S}_2(\mathcal{H})\to\mathcal{H}$ and $(\zeta_1,\zeta_2)\mapsto(\zeta'_1)$. If two points on Riemann sphere coincide $\zeta_1=\zeta_2$, then due to standard property of differential we have $\zeta'_1=\zeta_1=\zeta_2$ and so we have maps $(\zeta_1,\zeta_1)\mapsto(\zeta_1)$ described by *linear* operators ∂_ξ or ∂_η.

Cloning is suggested may not to be linear, but for symmetric qubits 'opposite' operation may be defined as linear one almost everywhere except of one point and it may be arbitrary point of Riemann sphere because we can use operator $\vec{\partial}_v \equiv v_1\partial_\xi + v_2\partial_\eta$.

2. Let us consider even subspace of $\mathsf{S}_*(\mathcal{H})$ defined as: $\mathsf{S}^2_*(\mathcal{H}) \equiv \bigoplus_{k=1}^\infty \mathsf{S}_{2k}(\mathcal{H})$ and $\mathsf{T}^2_*(\mathcal{H}) \equiv \bigoplus_{k=1}^\infty \mathsf{T}_{2k}(\mathcal{H})$. Such spaces may be more appropriate for taking into account the Pauli's exclusion principle, but here is discussed only simplest illustrative example.

Let us introduce operators $\tilde{\mathbf{D}} \equiv c_0\, c_1$ and $\tilde{\mathbf{X}} \equiv \xi\, \eta$ *i.e.* :

$$\tilde{\mathbf{D}}\,|n_0,n_1\rangle = \sqrt{n_0 n_1}\,|n_0-1,n_1-1\rangle, \quad \tilde{\mathbf{X}}\,|n_0,n_1\rangle = |n_0+1,n_1+1\rangle$$

The operators act on $\mathsf{S}^2_*(\mathcal{H})$ and there is isomorphism of subspace with basis $|n,n\rangle$, *i.e.* $n_0 = n_1$ and space of functions with operators Eq. (3), $|n,n\rangle \mapsto x^n$.

The operators are described here not only because of trivial identity $\sqrt{n_0 n_1} = n$ for $n_0 = n_1 = n$, but as an illustrative introduction to more difficult 4D case where momentum operator p_i and differentials $\frac{\partial}{\partial x_i}$ have *two* indexes as spinoral objects. But it is already away from theme of the paper[8].

6 Conclusions and discussion

In the paper is discussed algebraic approach to quantum computational models with unlimited number of discrete states. Similarly with classical recursion theory instead of 'actual infinity' here is considered quantum analogue of Turing machine as sequence of systems with increasing number of states.

Here is also used simplest mathematical model with linear spaces, isomorphic to space of polynomials. It maybe defined also by canonical commutation relations, CCR. It is still far from more rigorous models like relativistic quantum theories of interacting fields, but it is some step in the direction.

Usual quantum networks correspond to physical approach with S-matrix. It is analogue of ideas [1] cited in introduction. There are initial state, scattering process described by 'quantum black box' and final state. In such picture we have only two 'points' (*in* $\mapsto$ *out*) instead of 4D spacetime.

[8] It may be suggested, that in [2] the words about simulation of bosons by spin lattice are also related with same area of research *i.e.* 'Feynman checkerboard model', 'Penrose spin network', 'Finkelstein-Selesnick quantum net' *etc.* [‡].

Because CCR are also related with algebra of smooth functions on space-time (see Eq. (3)) the models discussed in the paper are useful possibility to take into account some properties of temporo-spatial systems in quantum networks approach. It is only necessary to accept infinite or dynamically changing number of qubits.

Does the mathematical models like $\mathsf{T}_*(\mathcal{H}_2)$ with infinite sequence of linear spaces with increasing dimensions devote an attention instead of 'actual' infinite-dimensional space $\mathcal{H}_\infty$? The paper is an attempt to make positive answer. For example operators of creation and annihilation can be simply expressed via transition between spaces $\mathsf{S}_k(\mathcal{H})$ of such sequence.

A nontrivial property of such sequences is orthogonality of any states in different terms. For example a state of tape of quantum Turing machine is orthogonal with state that differs only on extra one empty section, but in classical case they are considered as the same. The quantum $\breve{\alpha}\beta\alpha\xi$, $\mathsf{S}_*(\mathcal{H})$ has same property, but it is more clear from physical model of secondary quantization, because $|n_0, n_1\rangle$ and $|n_0 + 1, n_1\rangle$ are obviously orthogonal. But $\mathsf{S}_*(\mathcal{H})$ is simply symmetrization $\mathsf{S}(\mathsf{T}_*(\mathcal{H}))$ and so model of quantum Turing machine as infinite sequence of orthogonal linear spaces is not much more unusual than 2D harmonic oscillator discussed as a model of $\mathsf{S}_*(\mathcal{H})$ in Sec. 5.

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