INERTIAL PARAMETERS AND SUPERFLUID-TO-NORMAL PHASE TRANSITION IN SUPERDEFORMED BANDS

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The quasiclassically exact solution for the second inertial parameter $\mathcal{B}$ is found in a self-consistent way. It is shown that superdeformation and nonuniform pairing arising from the rotation induced pair density significantly reduce this inertial parameter. The new signature for the transition from pairing to normal phase is suggested in terms of the variation $\mathcal{B}/\mathcal{A}$ versus spin. Experimental data indicate the existence of such transition in the three SD mass regions.

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One of the amazing features of superdeformed (SD) rotational bands is the extreme regularity of their rotational spectra: a SD nucleus is the best quantum rotor known in nature. In spite of the fact that numerous theoretical calculations successfully reproduce the measured intraband γ-ray energies (see e.g. [1–4]) the underlying microscopic mechanism of this phenomenon is still not well understood. To explain the SD rotational spectrum regularity we use in the present paper the parameterization of its energy by the three term formula

$$E(I) = E_0 + \mathcal{A}I(I+1) + \mathcal{B}I^2(I+1)^2, \tag{1}$$

which is valid for an axially symmetric deformed nucleus with $K = 0$. The inertial parameters $\mathcal{A} = \hbar^2/2\Im^{(1)}$ ($\Im^{(1)}$ is the kinematic moment of inertia) and $\mathcal{B}$ are the objects of our investigation. They are determined by the transition energies $E_\gamma(I) = E(I+2) - E(I)$ as follows

$$\mathcal{A}(I)=\frac{1}{4(2I+5)}\left[\frac{I^2+7I+13}{2I+3}E_\gamma(I)-\frac{I^2+3I+3}{2I+7}E_\gamma(I+2)\right],$$

$$\mathcal{B}(I) = \frac{1}{8(2I+5)}\left[\frac{E_\gamma(I+2)}{2I+7} - \frac{E_\gamma(I)}{2I+3}\right]. \tag{2}$$

The coefficient $\mathcal{B}(I)$ characterizes the nonadiabatic properties of a band and is very sensitive to its internal structure. In particular, it realizes the relationship of kinematic and dynamic ($\Im^{(2)}$) moments of inertia. Using the well known expressions for these values [5] and the formulas (2) we get

$$\mathcal{B} = \frac{\hbar^2}{2(2I+3)(2I+7)}\left[\frac{1}{\Im^{(2)}} - \frac{2I}{(2I+5)\Im^{(1)}}\right]. \tag{3}$$

The ratio $\mathcal{B}/\mathcal{A}$ determines the convergence radius [6] of the two parameter formula (1), which is of order 100 for the bands in the 80 and 150 mass regions and 40 in the 130 and 190 ones. Faster convergence is obtained with Harris formula

$$E(\omega) = E_0 + \frac{1}{2}\alpha\omega^2 + \frac{3}{4}\beta\omega^4, \tag{4}$$

which is based on the fourth-order cranking expansion

$$\alpha = \frac{1}{\omega}Sp(\ell_x\rho^{(1)}), \quad \beta = \frac{1}{\omega^3}Sp(\ell_x\rho^{(3)}), \tag{5}$$

where $\rho^{(n)}$ is the nth correction to the nucleus density matrix, ℓ_x is the single-particle angular momentum projection on the rotational axis x perpendicular to the symmetry axis z, and ω is the rotational frequency. It follows from Eqs. (1) and (4) that

$$\alpha = \hbar^2/2\mathcal{A}, \quad \beta = -\hbar^4\mathcal{B}/4\mathcal{A}^4. \tag{6}$$

For simplicity we will deal with the parameter β.

The problem of the microscopic calculation of the parameter $\mathcal{B}$ for normal deformed (ND) nuclei has attracted considerable attention. Its value is formed mainly by four effects: vibration-rotation interaction, centrifugal stretching, perturbation of the quasiparticle motion, and attenuation of pair correlation by the Coriolis force (Mottelson-Valatin effect). As it has already been shown in the first attempts at obtaining $\mathcal{B}$ [7, 8], the latter two effects are dominant for well deformed nuclei. Unfortunately the results of these and many other works cannot be used for the superdeformation. The formulas of Ref. [8] have been obtained in the limit of the monopole pairing interaction (the uniform pairing), which is not adequate at SD shapes as shown in Ref. [9]. In the more sophisticated work [7] the gauge invariant pairing interaction allows to study the effect of nonuniform pairing. However this approach is also limited because it neglects the coupling between major shells. Thus, despite a number of publications on the subject the correct cranking selfconsistent solution for the $\mathcal{B}$ coefficient has not been found.

We used the quasiclassical method of Ref. [7] to derive the following expression for the β parameter in the superfluid phase

$$\beta_s = -\frac{\hbar^4}{4\Delta^2}\sum \ell_{12}^x\ell_{23}^x\ell_{34}^x\ell_{41}^x F(x_{12}, x_{23}, x_{34}, x_{41})\delta(\varepsilon_1 - \varepsilon_F), \tag{7}$$

where the summation indices i=1,2,3,4 refer to the single-particle states i of the nonrotating mean field with the energy ε_i. The δ-function means that the summation over the states 1 is restricted by a small interval at the Fermi energy ε_F [10]. The dimensionless values $x_{ii'} = (\varepsilon_i - \varepsilon_{i'})/2\Delta$, where Δ is the state independent pairing gap at $\omega = 0$, correspond to energy differences between states permitted by the selection rules for the matrix element of ℓ_x. The function F depending on these values may be written as follows

$$F = \sum_{k=0}^{3}\hat{P}_k G_{12} + \sum_{k=0}^{1}\hat{P}_k H_{13} - 8D_2^2 x_{12}x_{23}x_{34}x_{41}h(x_{13}), \tag{8}$$

where the permutation operators in the space of four indices i, $i\ mod 4 = i$,

$$\hat{P}_k x_{i,i'} = x_{i+k,i'+k}, \tag{9}$$

are used to simlify the formulas. The expressions for G_{12} and H_{13} involve the well known functions [10]

$$h(x) = (1+x^2)g(x), \quad g(x) = \frac{argshx}{x\sqrt{1+x^2}}, \tag{10}$$

and have the form:

$$G_{12} = \frac{g(x_{12})}{x_{23}x_{41}x_{13}x_{24}} \Big\{ (1 - D_1 x_{12}^2) \Big[-1 - x_{12}^2 - x_{23}x_{41} + D_1[x_{23}^2(1 - x_{12}x_{24}) + x_{41}^2(1 + x_{12}x_{13})]$$
$$+ D_1^2 x_{23}x_{34}x_{41}(x_{23} + x_{41}) D_1^3 x_{12}x_{23}^2 x_{34}x_{41}^2 \Big] + D_1(x_{34} - D_1 x_{12}x_{23}x_{41})(x_{34}$$
$$+ x_{12}x_{13}x_{24} - D_1 x_{12}x_{23}x_{41}) \Big\},$$
$$H_{13} = \frac{h(x_{13})}{x_{12}x_{23}x_{34}x_{41}} \Big[1 - D_1(x_{12}^2 + x_{23}^2 + x_{34}^2 + x_{41}^2)^2 + D_1^2(x_{12}x_{41} + x_{23}x_{34}) \Big], \tag{11}$$

Eq. (7) multiplied by ω^3 is the third order cranking correction to the total angular momentum of the neutron or proton system. Its derivation will be described in a forthcoming paper. We want now to emphasize that Eq. (7) represents the first theoretically correct expression for the high order effect of the Coriolis-pairing interaction at fixed deformation. The result is obtained by taking into account the effect of rotation on the Cooper pairs in the gauge invariant form. This effect is described by the first and the second corrections to the pairing energy

$$\Delta^{(1)}(\mathbf{r}) = -\frac{i\hbar^2\omega}{2\Delta} D_1 \dot{\ell}_x, \quad \Delta^{(2)}(\mathbf{r}) = \frac{\hbar^4\omega^2}{4\Delta^3} D_2 \dot{\ell}_x^2, \tag{12}$$

where D_1 and D_2 are the amplitudes of the nonuniform pairing fields, which are found in a self-consistent way. Let us note that $\Delta^{(n)}$ is the function of the space coordinates because $\hbar\dot{\ell}_x = y\partial U/\partial z - z\partial U/\partial y$, where U is the potential of a mean field. It is seen that for the oscillator potential $\dot{\ell}_x \sim yz$ (the Y_{21} pairing). The coordinate dependent pairing field is crucial for conservation of a nucleon current. The theory incorporating the nonuniform pairing allows also to consider the different limiting cases for the inertial parameters, which make possible the study of an interplay between rotation, pairing correlations, and mean field deformation in a SD band.

In order to consider this problem quantitatively we will use the axially deformed oscillator potential with the frequencies $\omega_x = \omega_y$ and ω_z on the corresponding axes. In this model the matrix element ℓ_{12}^x is non-zero for the two types of transitions:
(i) transitions inside a single oscillator shell (close transitions), for which $x_{12} = \pm\nu_1$;
(ii) transitions over a shell (distant transitions) with $x_{12} = \pm\nu_2$. The quantities ν_1 and ν_2 are the well known parameters involved in the moment of inertia [10]:

$$\nu_{1,2} = \frac{\hbar(\omega_x \mp \omega_z)}{2\Delta} = \frac{k \mp 1}{2\xi k^{2/3}}, \quad \xi = \frac{\Delta}{\hbar\omega_\circ}, \tag{13}$$

where $\hbar\omega_\circ = 41A^{-1/3}$ MeV. Here and later we use the axis or frequency ratio $k = c/a = \omega_x/\omega_z$ and the volume conservation condition. Both of the values ν_1 and ν_2 are large for superdeformation. For the fixed state 1, there are 36 different combinations of these base transitions in the sum of Eq. (7). The summation over the states 1 is performed in the Thomas-Fermi approximation. The final expression for the parameter β_s in the oscillator potential is

$$\beta_s = \frac{(k+1)^4}{1875\hbar^2 k^{4/3}} AM^3R^6\Phi(\xi,k), \tag{14}$$

where $R = 1.2A^{1/3}$ fm is the radius of the sphere, which volume is equal to that of the spheroid with the half-axes $a < c$, M is the nucleon mass, and A is the number of nucleons. The function Φ along with its limiting cases is shown in Fig. 1. It is seen that nonuniform pairing reduces substantially the parameter β_s in agreement with the estimation of Ref. [9]. On the other hand, the contribution of distant transitions is minor for small ξ. Nevertheless the later are necessary to obtain the hydrodynamic limit (see below). Since $\Phi \sim 1$ for a reasonable pairing gap, $\Delta \sim$0.5 MeV, the order of the value β_s is $\hbar^4(A/\varepsilon_F)^3$. This, along with the estimation $\mathcal{A} \sim \varepsilon_F A^{-5/3}$, gives $\mathcal{B}/\mathcal{A} \sim A^{-2}$, which overestimates the minimal value of this ratio in all the SD mass regions. Thus a small Δ and nonuniform pairing does not solve the problem of the SD band regularity.

Let us consider first the limiting case $\Delta = 0$. The right part of Eq. (7) vanishes for noncorrelated nucleons. This result is the artifact of the quasiclassical approximation used in Eq. (7). The correct expression for the β parameter in the normal phase, obtained with the limiting values of the Bogolubov amplitudes ($u_i = 0$, $v_i = 1$ for $n_i = 1$ and $u_i = 1$, $v_i = 0$ for $n_i = 0$, where n_i is the nucleon occupation numbers), has the form

$$\beta_n = -\hbar^4 \sum \ell^x_{12}\ell^x_{23}\ell^x_{34}\ell^x_{41} \sum_{k=0}^{3} \hat{P}_k \Big\{\frac{n_1}{\varepsilon_{12}\varepsilon_{13}\varepsilon_{14}}\Big\}. \tag{15}$$

The odd function of the differences $\varepsilon_{ii'} = \varepsilon_i - \varepsilon_{i'}$ leads to the cancelation of the main terms in sum (15) that decreases substantially the value of β_n, $\beta_n \sim \hbar^4 A^{7/3}/\varepsilon_F^3$. Note that the centrifugal stretching effect has the same order $\beta_{str} \sim \beta_n$. Its contribution is small compared to that of β_s, but it should not be overlooked for an unpaired system. For the oscillator potential, we have the following expression

$$\beta_n + \beta_{str} = \frac{k^4 - 10k^2 + 1}{6\omega_\circ^2 k^{4/3}} AMR^2. \tag{16}$$

It is seen that $\beta_n + \beta_{str} < 0$ for the prolate nuclei with $c/a < 3.15$, whereas β_s is always positive. Thus, with an increase of the spin I, the ratio $\mathcal{B}/\mathcal{A}$ has to change sign and to approach its limiting value $\mathcal{B}_n/\mathcal{A}_n \sim A^{-8/3}$, $\sim 10^{-6}$ for the SD bands in the 130 and 150 mass region, where rapid rotation destroys pairing correlations. The limiting ratio for a nucleus consisting of Z protons and N neutrons is expressed by

$$\frac{\mathcal{B}_n}{\mathcal{A}_n} = -2.56\,\frac{(k^4 - 10k^2 + 1)k^{2/3}}{(k^2+1)^3 A^{8/3}} \left[\Big(\frac{2Z}{A}\Big)^{1/3} + \Big(\frac{2N}{A}\Big)^{1/3}\right]. \tag{17}$$

One can therefore conclude that there are two distinct regions in the variation of $\mathcal{B}/\mathcal{A}$ versus I. The lower part of a SD band is characterized by a gradual decrease of the pairing gap Δ. According to Eq. (14) the ratio $\mathcal{B}/\mathcal{A}$ should exhibit a sharp increase. Then it changes sign and approaches the plateau (17) at the top of a band because the deformation c/a depends weakly on spin in the normal phase. Such behavior of the $\mathcal{B}/\mathcal{A}$ ratio is the signature of the pairing phase transition.

We have analyzed all the SD bands of Ref. [5] with known or suggested spins of levels. Figure 2 shows the variation of the $\mathcal{B}/\mathcal{A}$ ratio with I for bands with different internal structure and different rotational frequencies. Apart from the bands ^{192}Hg(1) and ^{194}Hg(3), where frequencies are so low that $\mathcal{B}/\mathcal{A}$ rises continuously in the superfluid phase, and ^{84}Zr(1), for which pairing is quenched completely and $\mathcal{B}/\mathcal{A}$ is close to the limiting value (17), all other bands display the behavior described above. It is important to note that such behavior is observed for the ND yrast band of ^{84}Zr, where $\mathcal{B}/\mathcal{A}$ reaches the same limiting value (17) as in the SD band ^{84}Zr(1). There are other ND yrast bands of ^{168}Yb and ^{168}Hf with the phase transition which experimental evidence has been discussed previously in terms of the canonical variables [11] and the spectrum of single-particle states [11, 12]. These bands exhibit the same features. Thus the plots of Fig. 2 demonstrate the universality of the superfluid-to-normal phase transition for SD and ND bands. The manifestation of this universality was previously observed in the anomalous small value $\mathcal{B}/\mathcal{A} \simeq 7\times10^{-6}$ for the ND yrast band of ^{168}Hf [13] and in the weak dependence of $\Im^{(1)}$ in the upper part of the SD band ^{152}Dy(1) [14]. The new feature observed in this article is the small decrease of $\mathcal{B}/\mathcal{A}$ at the top of the ^{152}Dy(1), ^{132}Ce(1), ^{84}Zr(1) and other SD bands. It may be explained by the decrease of the first multiplier in r.h.s. of Eq. (17) due to increase of nuclear deformation at highest spins.

All of the theoretical formulas obtained refer to collective rotation only. The influence of single-particle degrees of freedom on the $\mathcal{B}/\mathcal{A}$ ratio is the subject of a separate investigation. Preliminary estimations show that the contribution of an odd nucleon in $\mathcal{B}_s$ may be positive and irregular. Visible evidences of the nonrotational degree of freedom are shown in Fig. 3. A point of particular interest is the staggering band ^{149}Gd(1). It is seen that the staggering pattern is reproduced very well and the error bars are smaller than the staggering amplitude. We would like to emphasize that the physical values adequate for describing the staggering phenomenon are the $\mathcal{B}$ parameter or the $\mathcal{B}/\mathcal{A}$ ratio. They do not require any smooth reference, which introduces an ambiguousness in the amplitude and the phase of oscillations.

The next limit we want to consider is that of a large pairing gap Δ. In this case, the nonuniform pairing (12) is essential and the leading terms in the function Φ are those proportional to the powers of D_1 and D_2^2. They result in the limiting expression $\Phi \sim (\hbar\omega_\circ/\Delta)^2$. Thus for the very strong pairing ($\Delta \gg \hbar\omega_\circ$), when the size of the Cooper pair $R\hbar\omega_\circ/\Delta$ becomes much less than the nuclear radius, the value $\mathcal{B}_s$ vanishes in agreement with the hydrodynamic equations of the ideal liquid [6]. In the limit of an extremely large deformation, $c/a \to \infty$, a needle shaped nucleus with pairing correlations rotates as a rigid body, $\Im = \Im_{rig}$, $\mathcal{B}_s = 0$. For the finite but large deformation the deviations from these values are proportional to $(a/c)^{4/3}$. This means that all nucleons with the exclusion of a small sphere in the center of a nucleus are completely involved in rotational motion. Finally, for small deformations we have $\mathcal{B}_s \sim (c/a-1)^{-6}$ that is comparable to the vibration-rotation interaction [7].

Unlike the limiting value (17), it is impossible to compare with experiment the ratio $\mathcal{B}_s/\mathcal{A}_s$ because the proton (Δ_π) and neutron (Δ_ν) pairing gaps are unknown for the SD bands. In such a case we try to solve an inverse problem. The equations for the two inertial parameters

$$\alpha A/\Im_{rig} = Z\varphi(\xi_\pi, k) + N\varphi(\xi_\nu, k), \tag{18}$$

$$\beta A/\beta_\circ = Z\Phi(\xi_\pi, k) + N\Phi(\xi_\nu, k), \tag{19}$$

allow in principle to find ξ_π and ξ_ν ($\xi_l = \Delta_l/\hbar\omega_{\circ l}$, $\omega_{\circ l} = \omega_\circ(2A_l/A)^{2/3}$, $l = \pi, \nu$). Here the function φ is taken from Ref. [10],

$$\varphi(\nu_1, \nu_2) = 1 - \frac{(\nu_1^2-\nu_2^2)^2 g(\nu_1)g(\nu_2)}{(\nu_1^2+\nu_2^2)[\nu_1^2 g(\nu_1)+\nu_2^2 g(\nu_2)]}, \tag{20}$$

and the rigid body moment of inertia and the value $\beta_\circ$ have the form:

$$\Im_{rig} = 7.29A^{5/3}(k^2+1)k^{-2/3}10^{-3}\hbar^2 MeV^{-1},$$

$$\beta_\circ = 2.59A^4(k+1)^4k^{-4/3}10^{-8}\hbar^4 MeV^{-3}. \tag{21}$$

Finally the axis ratio $k = c/a$ is determined by the quadrupole moment

$$Q_0 = 6.05A^{2/3}(k^2-1)k^{-2/3}10^{-3}eb. \tag{22}$$

Unfortunately the system (18)–(19) does not have a solution. Table I gives the solutions of Eqs. (18) and (19) under simplifying assumption that the neutron and proton pairing gaps are equal. Different bands reveal the same tendency. The obtained values Δ_1 and Δ_2 show that the oscillator potential overestimates the moment of inertia and, to a greater extent, the β parameter. It is essential to use a more realistic potential for extracting pairing gaps from the inertial parameters.

Table I. Pairing gap energies Δ_1 and Δ_2 (in MeV) found as the independent solutions of Eqs. (18) and (19) correspondingly in the case of $\xi_\pi = \xi_\nu$. The values c/a, $\Im_{rig}$, and $\beta_\circ$ were calculated by using Eqs. (21) and (22). The experimental values of $\alpha = \Im$ and β were taken from Refs. [6] (^{172}Hf(yr)) and [15] (^{192,4}Hg(1,2)). Those of ^{236}U(fi) were determined from Eqs. (2), (6).

Band	c/a	$\Im/\Im_{rig}$	Δ_1	β/β_0	Δ_2
^{172}Hf(yr)	1.32	0.373	1.44	0.309	0.02
^{192}Hg(1)	1.61	0.762	0.95	0.151	0.05
^{194}Hg(1)	1.60	0.793	0.83	0.112	0.04
^{236}U(fi)	1.84	0.816	1.01	0.141	0.08

In summary, the exact solution for the inertial parameter $\mathcal{B}$ in the superfluid phase allows to show that neither superdeformation nor nonuniform pairing arising from rotation induced pair density is responsible for the regularity of the SD rotational spectra. The extreme regularity of the SD bands in the 80, 130 and 150 mass regions is explained by the transition from the superfluid to normal phase. The new signature of this transition reveals itself in the characteristic dependence of the ratio $\mathcal{B}/\mathcal{A}$ with the spin I. Application of this criterion to experimental data indicates the existence of the phase transition in the SD bands of the three SD mass regions. A new extraction method of the proton and neutron pairing gap from the γ-ray energies of interband transitions is discussed.

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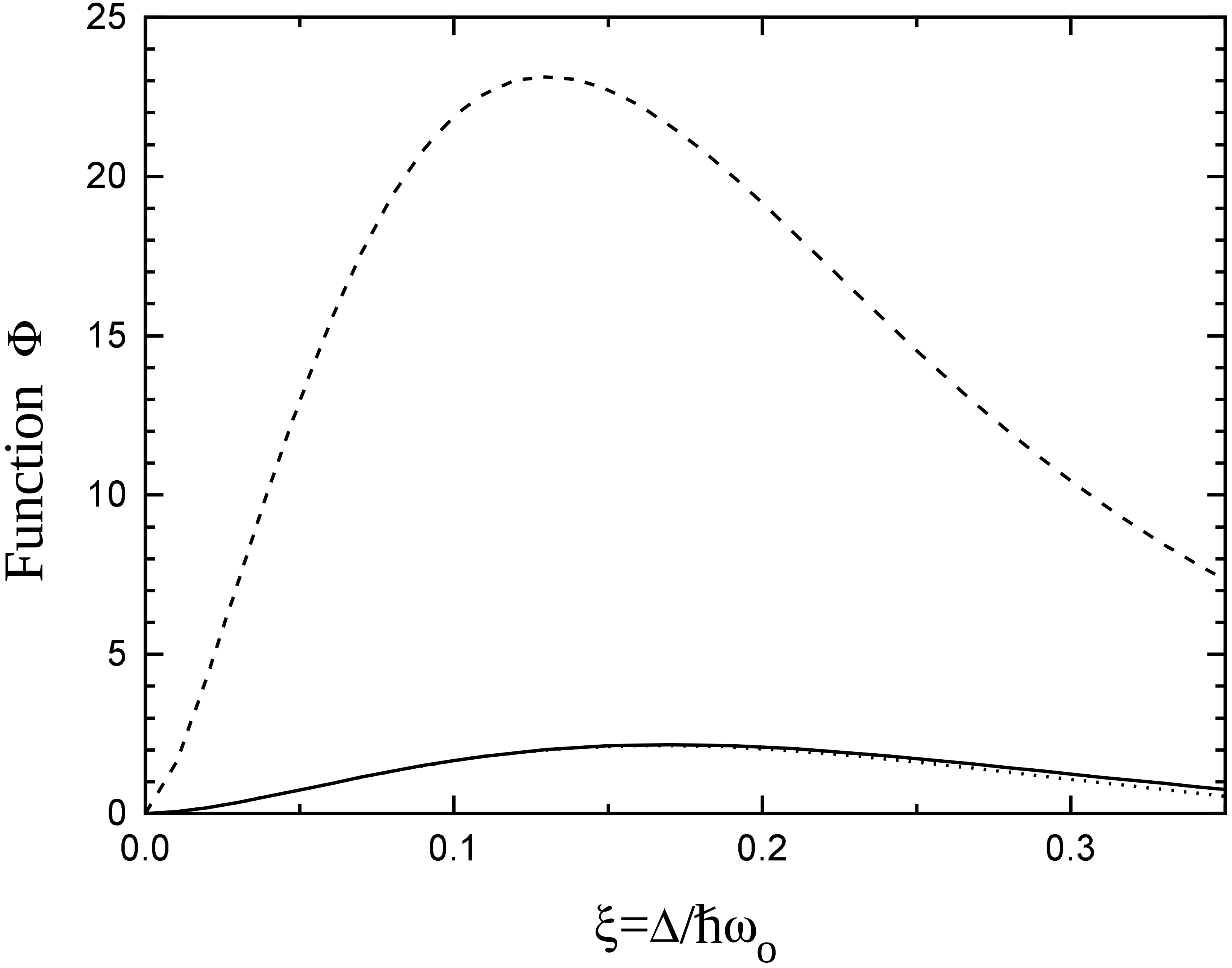
FIGURE CAPTIONS

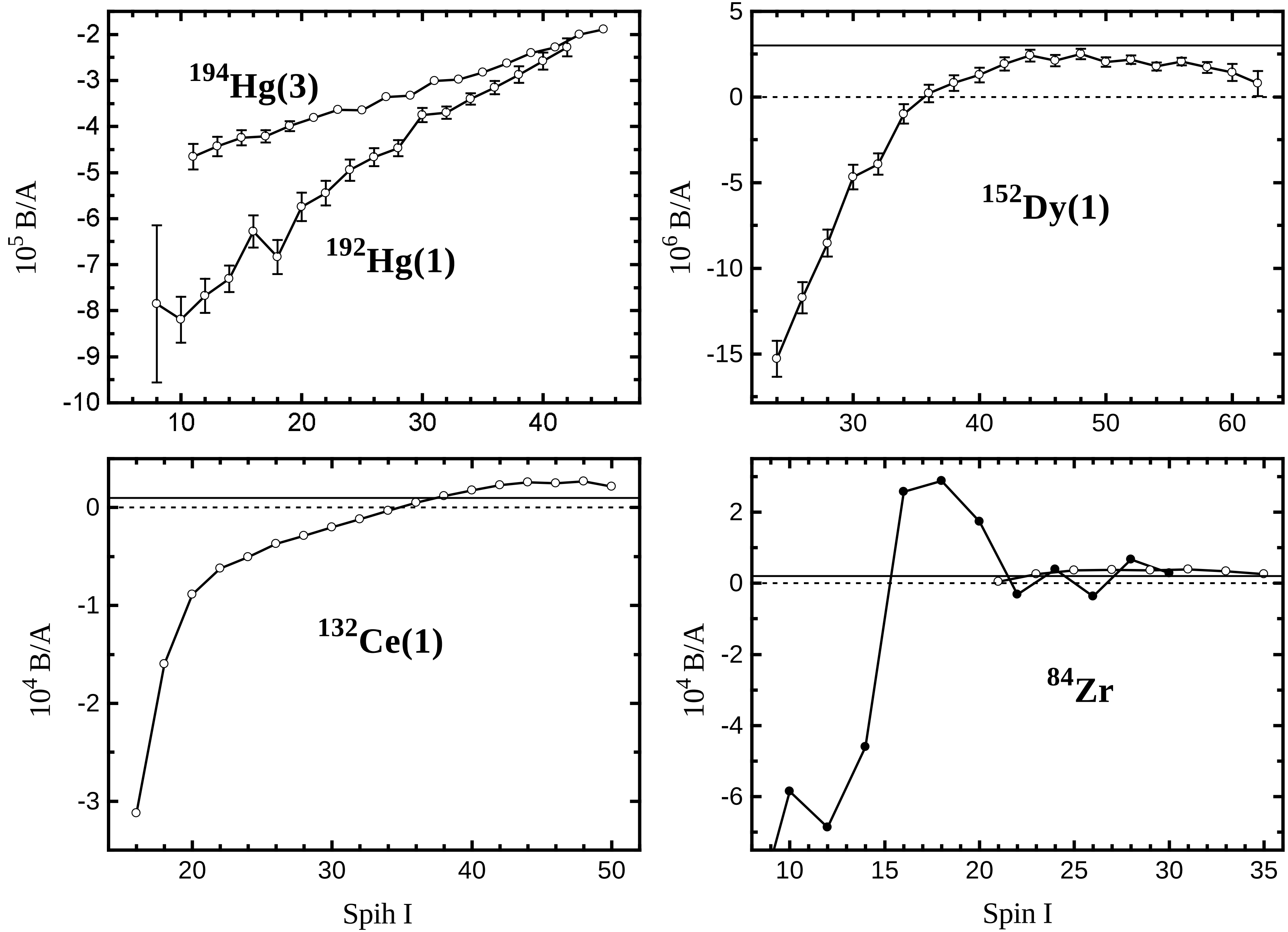
Fig. 1. Plot of the function Φ from Eq. (14) against the dimensionless value ξ for the axis ratio $c/a = 2$. The solid, dotted, and dashed lines correspond respectively to the exact value, the limit of close transitions, and the uniform pairing. The scale on the abscissa should be multiplied by a factor of approximately 7.7 for nuclei in the $A \sim 150$ mass region to obtain a gap energy in MeV.

Fig. 2. $\mathcal{B}/\mathcal{A}$ ratio versus spin for the SD and ND bands with mainly collective behavior. The formulas (2) are used to obtain this ratio from the experimental data for the ND (full circles) and SD (open circles) bands. The solid straight line is the limiting value $\mathcal{B}_n/\mathcal{A}_n$ with the deformation c/a found from the quadrupole moment (22). Error bars (if they are greater than symbols) include γ-ray energy uncertainties only. Uncertainties in a spin assignment are not important for all the SD bands except ^{152}Dy(1), as the variation of spins in $2\hbar$ would merely shift the curves along axis of abscissas.

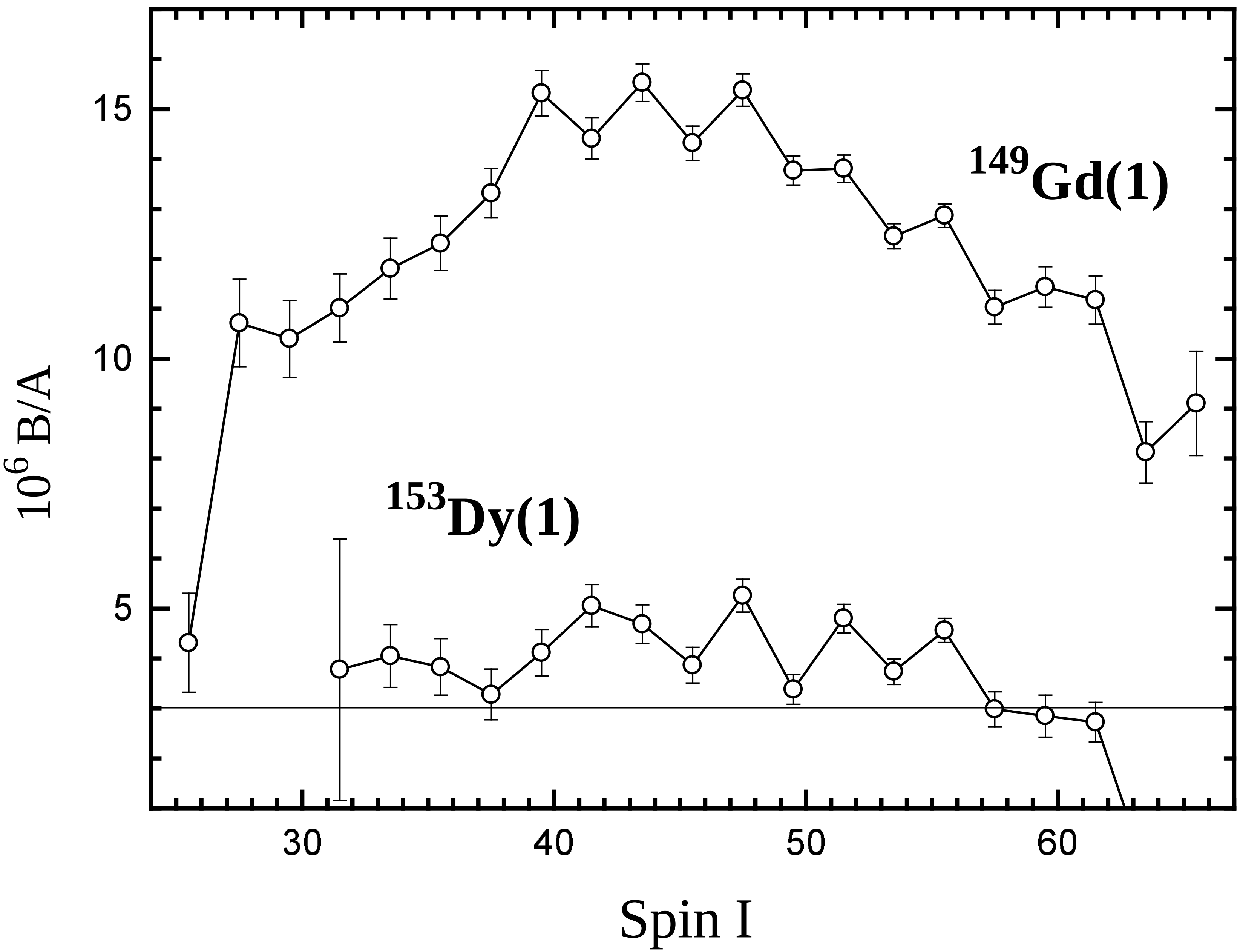
Fig. 3. Same as Fig. 2 for the bands with high-N configurations having the non-zero alignment i. Straight line is the value of $\mathcal{B}_n/\mathcal{A}_n$ for ^{153}Dy(1).



25
20
15
10
5
0
0.0
0.1
0.2
0.3
Function Φ
ξ=Δ/ħω$_o$



^{194}Hg(3)
^{192}Hg(1)
10^5 B/A
^{152}Dy(1)
10^6 B/A
^{132}Ce(1)
10^4 B/A
Spih I
^{84}Zr
10^4 B/A
Spin I



^{149}Gd(1)
^{153}Dy(1)
10^6 B/A
Spin I
15
10
5
30
40
50
60