Difference equations for the higher rank XXZ model with a boundary

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Abstract

The higher rank analogue of the XXZ model with a boundary is considered on the basis of the vertex operator approach. We derive difference equations of the quantum Knizhnik-Zamolodchikov type for $2N$-point correlations of the model. We present infinite product formulae of two point functions with free boundary condition by solving those difference equations with $N = 1$.

1 Introduction

Representation theory of the affine quantum group plays an important role in the description of solvable lattice models and massive integrable quantum field theories in two dimentions [1, 2, 3]. For models with the affine quantum group symmetry the difference analogue of the Knizhnik-Zamolodchikov equations (quantum Knizhnik-Zamolodchikov equations) are satisfied by both correlation functions and form factors [1, 4, 5].

Integrable models with boundary reflection have been also studied in lattice models and massive quantum theories. The boundary interaction is specfied by the reflection matrix K for lattice models [6], and by the boundary S-matrix for massive quantum theories [7]. It is shown in [8] that the space of states of the boundary XXZ model can be described in terms of vertex operators associated with the bulk XXZ model [3]. The explicit bosonic formulae of the boundary vacuum of the boundary XXZ model were obtained by using the bosonization of the vertex operators [8]. This approach is also relevant for other various models [9, 10, 11, 12, 13].

It is shown in [14] that correlation functions and form factors in semi-infinite XXZ/XYZ spin chains with integrable boundary conditions satisfy the boundary analogue of the quantum Knizhnik-Zamolodchikov equation. In this paper we establish the similar results for the $U_q(\widehat{sl_n})$ -analogue of XXZ spin chain with a boundary magnetic field h:

$$\mathcal{H}_B = \sum_{k=1}^{\infty} \left\{ q \sum_{\substack{a,b=0 \\ a>b}}^{n-1} e_{aa}^{(k+1)} e_{bb}^{(k)} + q^{-1} \sum_{\substack{a,b=0 \\ a<b}}^{n-1} e_{aa}^{(k+1)} e_{bb}^{(k)} - \sum_{\substack{a,b=0 \\ a\neq b}}^{n-1} e_{ab}^{(k+1)} e_{ba}^{(k)} \right\} \tag{1.1}$$
$$+\frac{1-q^2}{2q} \left\{ \sum_{a=0}^{L-1} e_{aa}^{(1)} - \sum_{a=M}^{n-1} e_{aa}^{(1)} \right\} + h \sum_{a=L}^{M-1} e_{aa}^{(1)},$$

where $-1 < q < 0$ and $0 \leqq L \leqq M \leqq n-1$. On the basis of the boundary vacuum states constructed in [11] we derive the boundary analogue of the quantum Knizhnik-Zamolodchikov equations for the correlation functions in the higher rank XXZ model with a boundary. We also obtain the two point functions by solving the simplest difference equations for free boundary condition.

The rest of this paper is organized as follows. In section 2 we review the vertex operator approach for the higher rank XXZ model with a boundary. In section 3 we derive the boundary quantum Knizhnik-Zamolodchikov equations for the $2N$-point correlation functions. In section 4 we obtain the two point functions by solving the difference equation with $N = 1$ for free boundary condition. In Appendix A we summarize the results of the bosonization of the vertex operators in $U_q(\widehat{sl_n})$ [15]. In Appendix B we summarize the bosonic formulae of the boundary vacuum states [11].

2 Formulation

The higher rank XXZ model with boundary reflection was formulated in [11] in terms of the vertex operators of the quantum affine group $U_q(\widehat{sl_n})$. For readers' convenience let us briefly review the results in [11].

Throughout this paper we fix $n \in \mathbb{N}_{\geqq 2}$, and also fix q such that $-1 < q < 0$. The model is labeled by the three parameters i, L, M such that $0 \leqq L \leqq M \leqq n-1$ and $i \in \{L, M\}$. In this paper we consider the following three cases:

$$\begin{aligned} &(C1) \quad 0 \leqq L = M = i \leqq n-1, \\ &(C2) \quad 0 \leqq L = i < M \leqq n-1, \\ &(C3) \quad 0 \leqq L < M = i \leqq n-1. \end{aligned}$$

In what follows we denote the q-integer $(q^k - q^{-k})/(q - q^{-1})$ by $[k]$, and we use the following simbols:

$$b(z) = \frac{q - q^{-1}z}{1-z}, \quad c(z) = \frac{q - q^{-1}}{1-z}. \tag{2.1}$$

The nonzero entries of the R-matrix $R^{(i)VV}(z)$ are given by

$$R^{(i)VV}(z)_{j_1,j_2}^{k_1,k_2} = r^{(i)VV}(z) \times \begin{cases} 1, & j_1 = j_2 = k_1 = k_2 \\ b(q^2 z), & j_1 = k_1 \neq j_2 = k_2, \\ -qc(q^2 z), & j_1 = k_2 < j_2 = k_1, \\ -qzc(q^2 z), & j_1 = k_2 > j_2 = k_1. \end{cases} \tag{2.2}$$

Here the scalar functions are

$$r^{(i)VV}(z) = z^{-\delta_{i,0}} \frac{(q^2 z^{-1}; q^{2n})_\infty (q^{2n} z; q^{2n})_\infty}{(q^2 z; q^{2n})_\infty (q^{2n} z^{-1}; q^{2n})_\infty}, \tag{2.3}$$

where

$$(z;p_1,\cdots,p_m)_\infty=\prod_{k_1,\cdots,k_m=0}^{\infty}(1-zp_1^{k_1}\cdots p_m^{k_m}).$$

The boundary K-matrix $K^{(i)}(z)$ is a diagonal matrix, whose diagonal elements are given by

$$K^{(i)}(z)_j^j=\frac{\varphi^{(i)}(z)}{\varphi^{(i)}(1/z)}\times\begin{cases} z^2, & 0\leqq j\leqq L-1,\\ \dfrac{1-rz}{1-r/z}, & L\leqq j\leqq M-1,\\ 1, & M\leqq j\leqq n-1,\end{cases} \tag{2.4}$$

where we have set

$$\varphi^{(i)}(z)=z^{\delta_{i,0}-1}\frac{(q^{2n+2}z^2;q^{4n})_\infty}{(q^{4n}z^2;q^{4n})_\infty}\times\begin{cases} 1, & \text{for } (C1),\\ \dfrac{(rq^{2n}z;q^{2n})_\infty}{(rq^{2n-2M+2L}z;q^{2n})_\infty}, & \text{for } (C2),\\ \dfrac{(r^{-1}z;q^{2n})_\infty}{(r^{-1}q^{2M-2L}z;q^{2n})_\infty}, & \text{for } (C3).\end{cases} \tag{2.5}$$

They satisfy the boundary Yang-Baxter equation:

$$K_2^{(i)}(z_2)R_{21}^{(i)}(z_1z_2)K_1^{(i)}(z_1)R_{12}^{(i)}(z_1/z_2)=R_{21}^{(i)}(z_1/z_2)K_1^{(i)}(z_1)R_{12}^{(i)}(z_1z_2)K_2^{(i)}(z_2). \tag{2.6}$$

Let $V=\mathbb{C}v_0\oplus\cdots\oplus\mathbb{C}v_{n-1}$ be the basic representation of $U_q(sl_n)$, and let V_z be the evaluation representation of $U_q(\widehat{sl_n})$ in the homogeneous picture. Let $V(\Lambda_i)$ be the irreducible highest weight module with the level 1 highest weight Λ_i $(i=0,\cdots,n-1)$. The type-I vertex operator $\Phi^{(i,i+1)}(z)$ is an intertwining operator of $U_q(\widehat{sl_n})$ defined by

$$\Phi^{(i,i+1)}(z):V(\Lambda_{i+1})\to V(\Lambda_i)\hat{\otimes}V_z, \tag{2.7}$$

where the superscripts $i,i+1$ should be interpreted as elements in $\mathbb{Z}_n$. Let us define the component of the vertex operators $\Phi_j^{(i,i+1)}(z)$ as follows.

$$\Phi^{(i,i+1)}(z)|u\rangle=\sum_{j=0}^{n-1}\Phi_j^{(i,i+1)}(z)|u\rangle\otimes v_j,\ \text{for } |u\rangle\in V(\Lambda_{i+1}). \tag{2.8}$$

The dual type-I vertex operator $\Phi^{*(i+1,i)}(z)$ is an intertwining operator of $U_q(\widehat{sl_n})$ defined by

$$\Phi^{*(i+1,i)}(z):V(\Lambda_i)\otimes V_z\to\hat{V}(\Lambda_{i+1}). \tag{2.9}$$

Let us define the components of the dual vertex operators $\Phi_j^{*(i+1,i)}(z)$ as follows.

$$\Phi^{*(i+1,i)}(z)(|u\rangle\otimes v_j)=\Phi_j^{*(i+1,i)}(z)|u\rangle,\quad \text{for } |u\rangle\in V(\Lambda_i). \tag{2.10}$$

Let us summarize here the properties of the vertex operators:

Commutation relations The vertex operators satisfy the following commutation relation:

$$\Phi_{j_2}^{(i-2,i-1)}(z_2)\Phi_{j_1}^{(i-1,i)}(z_1)=\sum_{\substack{j_1',j_2'=0\\ j_1'+j_2'=j_1+j_2}}^{n-1}R^{(i)VV}(z_1/z_2)_{j_1,j_2}^{j_1',j_2'}\Phi_{j_1'}^{(i-2,i-1)}(z_1)\Phi_{j_2'}^{(i-1,i)}(z_2). \tag{2.11}$$

Concerning other commutation relations (3.34), (3.36) and (3.42), see section 3.

Normalizations We adopt the following normalizations:

$$\Phi^{(i,i+1)}(z)|\Lambda_{i+1}\rangle = |\Lambda_i\rangle \otimes v_i + \cdots, \quad \Phi^{*(i+1,i)}(z)|\Lambda_i\rangle \otimes v_i = |\Lambda_{i+1}\rangle + \cdots, \tag{2.12}$$

where $|\Lambda_i\rangle$ is the highest weight vector of $V(\Lambda_i)$.

Invertibility They satisfy the following inversion relation:

$$g_n \Phi_j^{(i-1,i)}(z)\Phi_j^{*(i,i-1)}(z) = \mathrm{id}, \tag{2.13}$$

where

$$g_n = \frac{(q^2;q^{2n})_\infty}{(q^{2n};q^{2n})_\infty}.$$

We define the normarized transfer matrix by

$$T_B^{(i)}(z) = g_n \sum_{j=0}^{n-1} \Phi_j^{*(i,i-1)}(z^{-1})K^{(i)}(z)_j^j \Phi_j^{(i-1,i)}(z), \tag{2.14}$$

Let the space $\mathcal{H}^{(i)}$ be the span of vectors $|p\rangle = \otimes_{k=1}^{\infty} v_{p(k)}$, where $p : \mathbb{N} \to \mathbb{Z}/n\mathbb{Z}$ satisfies the asymptotic condition

$$p(k) = k + i \in \mathbb{Z}/n\mathbb{Z}, \quad \text{for } k \gg 1. \tag{2.15}$$

As usual, the transfer matrix (2.14) and the Hamiltonian (1.1) are related by

$$\left.\frac{d}{dz}T_B^{(i)}(z)\right|_{z=1} = \frac{2q}{1-q^2}\mathcal{H}_B + const, \quad \text{for} \quad h = \frac{r+1}{r-1} \times \frac{1-q^2}{2q}. \tag{2.16}$$

Note that the left hand side act on the space $V(\Lambda_i)$ while the right hand side acts on the space $\mathcal{H}^{(i)}$. Thus we can make the following identification:

$$V(\Lambda_i) \simeq \mathcal{H}^{(i)}. \tag{2.17}$$

The boundary ground state and the dual boundary ground state are characterized by

$$T_B^{(i)}(z)|i\rangle_B = |i\rangle_B, \ (i = 0, \cdots, n-1), \tag{2.18}$$

and

$${}_B\langle i|T_B^{(i)}(z) = {}_B\langle i|, \quad (i = 0, \cdots, n-1). \tag{2.19}$$

Using the inversion relation, the eigenvalue problems (2.18) and (2.19) are reduced to

$$K^{(i)}(z)_j^j \Phi_j^{(i-1,i)}(z)|i\rangle_B = \Phi_j^{(i-1,i)}(z^{-1})|i\rangle_B, \tag{2.20}$$

and

$$K^{(i)}(z)_j^j \ {}_B\langle i|\Phi_j^{*(i,i-1)}(z^{-1}) = {}_B\langle i|\Phi_j^{*(i,i-1)}(z). \tag{2.21}$$

The bosonizations of vertex operators are given in [15]. The bosonic formulae of the boundary vacuum are given in [11]. For readers' convenience we summarize the bosonizations of vertex operators in Appendix A and the bosonic formula of the boundary vacuum in Apendix B.

3 Boundary quantum Knizhnik-Zamolodchikov equations

The purpose of this section is to derive the q-difference equations for the correlation function of the higher rank XXZ spin chain with a boundary magnetic field. For $U_q(\widehat{sl_2})$ case [14], the said difference equations are based on the duality relation of vertex operators

$$\Phi^*_\epsilon(\zeta)=\Phi_{-\epsilon}(-q^{-1}\zeta),$$

in addition to (2.11), (2.20) and (2.21). For $n>2$ case, however, the dual vertex operator $\Phi^*_j(z)$ is written in terms of $(n-1)$-st determinant of $\Phi_j(z)$'s. Thus it is not convenient to use the duality relation for the present case.

For $n>2$, we use the explicit formulae of the boundary states to derive the boundary quantum Knizhnik-Zamolodchikov equations. In this section we establish the following simple relations:

$$\begin{aligned}\Phi_j^{*(i+1,i)}(q^nz)|i\rangle_B &= K^{*(i)}(z)_j^j\Phi_j^{*(i+1,i)}(q^n/z)|i\rangle_B,\ (j=0,\cdots,n-1),\\ {}_B\langle i|\Phi_j^{(i,i+1)}(1/(q^nz)) &= K^{*(i)}(z)_j^j\ {}_B\langle i|\Phi_j^{(i,i+1)}(z/q^n),\ (j=0,\cdots,n-1),\end{aligned}\tag{3.1}$$

where the functions $K^{(i)*}(z)_j^j$ are given by (3.3), (3.11) and (3.18). The relations (3.1) in addition to the commutation relations (2.11), (3.34), (3.36) and (3.42) imply the q-difference equations of the present model.

3.1 Boundary state

In this subsection we use the symbols $P^*(z),Q^*(z),R^-(z),S^-(z)$, which are bosons defined in Appendix A. See Appendix A as for the definitions.

Let us first consider consider "$(C1)\ 0\leqq L=M=i\leqq n-1''$-case. Let us show the following relation:

$$\Phi_j^{*(i+1,i)}(q^nz)|i\rangle_B=K^{*(i)}(z)_j^j\Phi_j^{*(i+1,i)}(q^n/z)|i\rangle_B,\ (j=0,\cdots,n-1),\tag{3.2}$$

where

$$K^{*(i)}(z)_j^j=\frac{\varphi^{*(i)}(z)}{\varphi^{*(i)}(1/z)}\times\left\{\begin{array}{ll}z^2, & (0\leqq j\leqq L-1=i-1),\\ 1, & (i=L\leqq j\leqq n-1),\end{array}\right.\quad \text{for}\ \ (C1),\tag{3.3}$$

and

$$\varphi^{*(i)}(z)=z^{\delta_{i,0}}\frac{(q^{4n}z^2;q^{4n})_\infty}{(q^{2n+2}z^2;q^{4n})_\infty}.\tag{3.4}$$

Multiply the both sides of (3.2) by $k_j^{(i)}(z)\varphi^{*(i)}(1/z)$, where

$$k_j^{(i)}(z)=\left\{\begin{array}{ll}z^{-1}, & 0\leqq j\leqq i-1,\\ 1, & i\leqq j\leqq n-1.\end{array}\right.\tag{3.5}$$

Then the RHS of (3.2) is obtained from the LHS by changing $z\to 1/z$.

Bosonization formulae of $P^*(z),Q^*(z)$ and $|i\rangle_B$ imply the identity

$$e^{Q^*(q^nz)}|i\rangle_B=\frac{(q^{2n+2}z^{-2};q^{4n})_\infty}{(q^{4n}z^{-2};q^{4n})_\infty}e^{P^*(q^n/z)}|i\rangle_B.\tag{3.6}$$

By using this identity we have

$$k_0^i(z)\varphi^{*(i)}(1/z)\Phi_0^{*(i+1,i)}(q^nz)|i\rangle_B = c_0^* e^{P^*(q^nz)+P^*(q^n/z)}e^{\bar{\Lambda}_1}|i\rangle_B, \tag{3.7}$$

where c_0^* is some constant. The relation (3.2) with $j=0$ follows form the fact that RHS of (3.7) is symmetric under $z \to 1/z$.

Invoking the bosonization of the dual vertex operators, we also have for $j>0$ as follows:

$$\begin{aligned} & k_j^{(i)}(z)\varphi^{*(i)}(1/z)\Phi_j^{*(i+1,i)}(q^nz)|i\rangle_B \\ = \; & c_j^* \oint \frac{dw_1}{w_1}\cdots\oint\frac{dw_j}{w_j} k_j^{(i)}(z)\mathrm{Int}(z,w_1,w_2,\cdots,w_j)e^{P^*(q^nz)+P^*(q^n/z)} \\ \times \; & e^{R_1^-(q^{n+1}w_1)+R_1^-(q^{n+1}/w_1)+\cdots+R_j^-(q^{n+1}w_j)+R_j^-(q^{n+1}/w_j)}e^{\bar{\Lambda}_1}|i\rangle_B, \end{aligned} \tag{3.8}$$

where c_j^*'s are some constants. Here we set the integrand:

$$\mathrm{Int}(w_0,w_1,\cdots,w_j) = \frac{w_j\prod_{k=1}^j\left\{(1-w_k^{-2})w_k^{-\delta_{k,i}}(1-qw_{k-1}w_k)\right\}}{\prod_{k=1}^j D(w_{k-1},w_k)}, \tag{3.9}$$

where

$$D(w_1,w_2) = (1-qw_1w_2)(1-qw_1/w_2)(1-qw_2/w_1)(1-q/(w_1w_2)).$$

Thus the relation (3.2) with $j>0$ follows from the identities

$$\sum_{\epsilon_1=\pm,\cdots,\epsilon_j=\pm}\left\{k_j^{(i)}(z^{-1})\mathrm{Int}(z,w_1^{\epsilon_1},\cdots,w_j^{\epsilon_j}) - k_j^{(i)}(z)\mathrm{Int}(z^{-1},w_1^{\epsilon_1},\cdots,w_j^{\epsilon_j})\right\} = 0. \tag{3.10}$$

Let us consider "$(C2)\ 0 \leqq L = i < M \leqq n-1$"-case. From the same arguments as for $(C1)$, we have

$$K^{*(i)}(z)_j^j = \frac{\varphi^{*(i)}(z)}{\varphi^{*(i)}(1/z)} \times \begin{cases} z^2, & (0 \leqq j \leqq L-1), \\ \dfrac{1-q^{n-2M+2L}rz}{1-q^{n-2M+2L}rz^{-1}}, & (L \leqq j \leqq M-1), \\ 1, & (M \leqq j \leqq n-1), \end{cases} \quad \text{for} \quad (C2), \tag{3.11}$$

where we have set

$$\varphi^{*(i)}(z) = z^{\delta_{i,0}}\frac{(q^{4n}z^2;q^{4n})_\infty(rq^nz;q^{2n})_\infty}{(q^{2n+2}z^2;q^{4n})_\infty(rq^{n+2L-2M}z;q^{2n})_\infty}. \tag{3.12}$$

In this case the following relations are useful:

$$e^{Q^*(q^nz)}|0\rangle_B = \frac{(rq^{3n-2M}z^{-1};q^{2n})_\infty(q^{2n+2}z^{-2};q^{4n})_\infty}{(rq^nz^{-1};q^{2n})_\infty(q^{4n}z^{-2};q^{4n})_\infty}e^{P^*(q^n/z)}|0\rangle_B, \tag{3.13}$$

$$e^{Q^*(q^nz)}|i\rangle_B = \frac{(rq^{n+2L-2M}z^{-1};q^{2n})_\infty(q^{2n+2}z^{-2};q^{4n})_\infty}{(rq^nz^{-1};q^{2n})_\infty(q^{4n}z^{-2};q^{4n})_\infty}e^{P^*(q^n/z)}|i\rangle_B, \ (i \geqq 1), \tag{3.14}$$

and

$$e^{S_j^-(w)}|i\rangle_B = g_j^{(i)}(w)e^{R_j^-(q^{2(n+1)}/w)}|i\rangle_B, \tag{3.15}$$

where

$$g_j^{(0)}(q^{n+1}w) = \begin{cases} (1-1/w^2)(1-q^{-n+2M-L}/(rw)), & j=L, \\ (1-1/w^2)(1-q^{n-M}r/w), & j=M, \\ (1-1/w^2), & j \neq L,M, \end{cases} \tag{3.16}$$

and

$$g_j^{(i)}(q^{n+1}w)=\begin{cases}\dfrac{(1-1/w^2)}{(1-rq^{n-2M+L}/w)}, & j=L,\\ (1-1/w^2)(1-q^{n-M}r/w), & j=M,\\ (1-1/w^2), & j\neq L,M,\end{cases}\quad (i\geqq 1). \tag{3.17}$$

Let us consider "$(C3)$ $0\leqq L<M=i\leqq n-1$"-case. Repeating the same procedure as in $(C1)$, we have

$$K^{*(i)}(z)_j^j=\frac{\varphi^{*(i)}(z)}{\varphi^{*(i)}(1/z)}\times\begin{cases} z^2, & (0\leqq j\leqq L-1),\\ \dfrac{1-q^{n+2M-2L}r^{-1}z}{1-q^{n+2M-2L}r^{-1}z^{-1}}, & (L\leqq j\leqq M-1),\\ 1, & (M\leqq j\leqq n-1),\end{cases}\quad \text{for}\ \ (C3), \tag{3.18}$$

where we set

$$\varphi^{*(i)}(z)=\frac{(q^{4n}z^2;q^{4n})_\infty(r^{-1}q^nz;q^{2n})_\infty}{(q^{2n+2}z^2;q^{4n})_\infty(r^{-1}q^{n+2M-2L}z;q^{2n})_\infty}. \tag{3.19}$$

In this case the following relations are useful:

$$e^{Q^*(q^nz)}|i\rangle_B=\frac{(r^{-1}q^{2M-n}z^{-1};q^{2n})_\infty(q^{2n+2}z^{-2};q^{4n})_\infty}{(r^{-1}q^nz^{-1};q^{2n})_\infty(q^{4n}z^{-2};q^{4n})_\infty}e^{P^*(q^n/z)}|i\rangle_B,\ (L=0), \tag{3.20}$$

$$e^{Q^*(q^nz)}|i\rangle_B=\frac{(r^{-1}q^{2M-2L+n}z^{-1};q^{2n})_\infty(q^{2n+2}z^{-2};q^{4n})_\infty}{(r^{-1}q^nz^{-1};q^{2n})_\infty(q^{4n}z^{-2};q^{4n})_\infty}e^{P^*(q^n/z)}|i\rangle_B,\ (L\geqq 1), \tag{3.21}$$

and

$$e^{S_j^-(w)}|i\rangle_B=g_j^{(i)}(w)e^{R_j^-(q^{2(n+1)}/w)}|i\rangle_B, \tag{3.22}$$

where

$$g_j^{(i)}(q^{n+1}w)=\begin{cases}(1-1/w^2)(1-q^{-n+2M-L}/(rw)), & j=L,\\ \dfrac{(1-1/w^2)}{(1-q^{M-n}/(rw))}, & j=M,\\ (1-1/w^2), & j\neq L,M,\end{cases} \tag{3.23}$$

3.2 Dual boundary state

From the same arguments as for the boundary state case, we can show the following relation:

$$_B\langle i|\Phi_j^{(i,i+1)}(1/(q^nz))=K^{*(i)}(z)_j^j\ {}_B\langle i|\Phi_j^{(i,i+1)}(z/q^n). \tag{3.24}$$

For each case the following relations are useful:

$(C1)$-case: $0\leqq L=M=i\leqq n-1$

$$\begin{aligned} {}_B\langle i|e^{P(z/q^n)} &= \frac{(q^{2n+2}z^2;q^{4n})_\infty}{(q^{4n}z^2;q^{4n})_\infty}\ {}_B\langle i|e^{Q(1/(q^nz))},\\ {}_B\langle i|e^{S_j^-(w)} &= g_j^{*(i)}(w)\ {}_B\langle i|e^{R_j^-(q^2/w)},\end{aligned} \tag{3.25}$$

where

$$g_j^{*(i)}(qw)=(1-w^2). \tag{3.26}$$

$(C2)$-case: $0 \leqq L = i < M \leqq n-1$

$$\begin{aligned} {}_B\langle i|e^{P(z/q^n)} &= \frac{(q^{2n+2}z^2;q^{4n})_\infty (rq^{n+2L-2M}z;q^{2n})_\infty}{(q^{4n}z^2;q^{4n})_\infty (rq^n z;q^{2n})_\infty}\ {}_B\langle i|e^{Q(1/(q^n z))}, \\ {}_B\langle i|e^{S_j^-(w)} &= g_j^{*(i)}(w)\ {}_B\langle i|e^{R_j^-(q^2/w)}, \end{aligned} \tag{3.27}$$

where

$$g_j^{*(0)}(qw) = \begin{cases} \dfrac{(1-w^2)}{(1-q^{-L}w/r)}, & j = L, \\ \dfrac{(1-w^2)}{(1-q^{2L-M}rw)}, & j = M, \\ (1-w^2), & j \neq L, M, \end{cases} \tag{3.28}$$

and

$$g_j^{*(i)}(qw) = \begin{cases} (1-w^2)(1-q^L rw), & j = L, \\ \dfrac{(1-w^2)}{(1-q^{2L-M}rw)}, & j = M, \qquad (i \geqq 1). \\ (1-w^2), & j \neq L, M, \end{cases} \tag{3.29}$$

$(C3)$-case: $0 \leqq L < M = i \leqq n-1$

$$\begin{aligned} {}_B\langle i|e^{P(z/q^n)} &= \frac{(q^{2n+2}z^2;q^{4n})_\infty (q^{n+2M-2L}r^{-1}z;q^{2n})_\infty}{(q^{4n}z^2;q^{4n})_\infty (q^n r^{-1} z;q^{2n})_\infty}\ {}_B\langle i|e^{Q(1/(q^n z))} \\ {}_B\langle i|e^{S_j^-(w)} &= g_j^{*(i)}(w)\ {}_B\langle i|e^{R_j^-(q^2/w)}, \end{aligned} \tag{3.30}$$

where

$$g_j^{*(i)}(qw) = \begin{cases} \dfrac{(1-w^2)}{(1-q^{-L}w/r)}, & j = L, \\ (1-w^2)(1-q^{M-2L}w/r), & j = M, \\ (1-w^2), & j \neq L, M, \end{cases} \tag{3.31}$$

3.3 Correlation functions and difference equations

Let us consider the $2N$-point correlation function:

$$\begin{aligned} & G^{(i)}(z_1,\cdots,z_N|z_{N+1},\cdots,z_{2N}) \\ = & \sum_{j_1=0}^{n-1}\cdots\sum_{j_N=0}^{n-1}\sum_{j_{N+1}=0}^{n-1}\cdots\sum_{j_{2N}=0}^{n-1} v_{j_1}^*\otimes\cdots\otimes v_{j_N}^*\otimes v_{j_{N+1}}\otimes\cdots\otimes v_{j_{2N}} \\ \times & G^{(i)}(z_1,\cdots,z_N|z_{N+1},\cdots,z_{2N})_{j_{N+1}\cdots j_{2N}}^{j_1\cdots j_N}, \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} & G^{(i)}(z_1,\cdots,z_N|z_{N+1},\cdots,z_{2N})_{j_{N+1}\cdots j_{2N}}^{j_1\cdots j_N} \\ = & {}_B\langle i|\Phi_{j_1}^{*(i,i-1)}(z_1)\cdots\Phi_{j_N}^{*(i-N+1,i-N)}(z_N)\Phi_{j_{N+1}}^{(i-N,i-N+1)}(z_{N+1})\cdots\Phi_{j_{2N}}^{(i-1,i)}(z_{2N})|i\rangle_B. \end{aligned} \tag{3.33}$$

In order to derive q-difference equations, we use the commutation relations of vertex operators and the action formulae of vertex operators to the boundary state. In what follows we assume that $K^{*(i)}(z)$ is a diagonal matrix whose diagonal elements are given by (3.3), (3.11) and (3.18).

The commutation relations between vertex operators of different types are given as follows [16]:

$$\Phi_j^{(i,i+1)}(z_2)\Phi_j^{*(i+1,i)}(z_1) = \sum_{k=0}^{n-1} R^{(i)V^*V}(z_1/z_2)_{j,j}^{k,k}\Phi_k^{*(i,i-1)}(z_1)\Phi_k^{(i-1,i)}(z_2), \tag{3.34}$$

$$\Phi_k^{(i,i+1)}(z_2)\Phi_j^{*(i+1,i)}(z_1) = r^{(i)V^*V}(z_1/z_2)\Phi_j^{*(i,i-1)}(z_1)\Phi_k^{(i-1,i)}(z_2), \ (j \neq k), \tag{3.35}$$

and

$$\Phi_j^{*(i,i-1)}(z_2)\Phi_j^{(i-1,i)}(z_1) = \sum_{k=0}^{n-1} R^{(i)VV^*}(z_1/z_2)_{j,j}^{k,k}\Phi_k^{(i,i+1)}(z_1)\Phi_k^{*(i+1,i)}(z_2), \tag{3.36}$$

$$\Phi_k^{*(i,i-1)}(z_2)\Phi_j^{(i-1,i)}(z_1) = r^{(i)VV^*}(z_1/z_2)\Phi_j^{(i,i+1)}(z_1)\Phi_k^{*(i+1,i)}(z_2), \quad (j \neq k). \tag{3.37}$$

Here the nonzero components are

$$R^{(i)V^*V}(z)_{j,j}^{k,k} = r^{(i)V^*V}(z) \times \begin{cases} b(z), & j = k, \\ c(z), & j > k, \\ zc(z), & j < k, \end{cases} \tag{3.38}$$

and

$$R^{(i)VV^*}(z)_{j,j}^{k,k} = r^{(i)VV^*}(z) \times \begin{cases} b(q^{2n}z), & j = k, \\ q^{2n}zc(q^{2n}z)q^{2(k-j)}, & j > k, \\ c(q^{2n}z)q^{2(k-j)}, & j < k, \end{cases} \tag{3.39}$$

where

$$r^{(i)V^*V}(z) = -qz^{-\delta_{i,0}}\frac{(z;q^{2n})_\infty(q^{2n+2}z^{-1};q^{2n})_\infty}{(q^2z;q^{2n})_\infty(q^{2n}z^{-1};q^{2n})_\infty}, \tag{3.40}$$

$$r^{(i)VV^*}(z) = -q^{-1}z^{-\delta_{i,0}}\frac{(q^{2n}z;q^{2n})_\infty(q^2z^{-1};q^{2n})_\infty}{(q^{2n+2}z;q^{2n})_\infty(z^{-1};q^{2n})_\infty}. \tag{3.41}$$

The commutation relations between the dual vertex operators are given as

$$\Phi_{j_2}^{*(i+2,i+1)}(z_2)\Phi_{j_1}^{*(i+1,i)}(z_1) = \sum_{\substack{k_1,k_2=0 \\ k_1+k_2=j_1+j_2}}^{n-1} R^{(i)V^*V^*}(z_1/z_2)_{j_1,j_2}^{k_1,k_2}\Phi_{k_1}^{*(i+2,i+1)}(z_1)\Phi_{k_2}^{*(i+1,i)}(z_2). \tag{3.42}$$

Here the nonzero components are

$$R^{(i)V^*V^*}(z_1/z_2)_{j_1,j_2}^{k_1,k_2} = r^{(i)V^*V^*}(z) \times \begin{cases} 1, & j_1 = j_2 = k_1 = k_2, \\ b(q^2z), & j_1 = k_1 \neq j_2 = k_2, \\ -qzc(q^2), & j_1 = k_2 < j_2 = k_1, \\ -qc(q^2), & j_1 = k_2 > j_2 = k_1, \end{cases} \tag{3.43}$$

where

$$r^{(i)V^*V^*}(z) = r^{(i)VV}(z). \tag{3.44}$$

Now we are in a position to derive boundary quantum Knizhnik-Zamolodchikov equations, which is a version of Cherednik's equation [17]. From the commutation relations (2.11), (3.34), (3.36), (3.42) and the boundary state ideitities (3.1) we obtain the following q-difference equations:

$$\begin{aligned}
& G^{(i)}(z_1\cdots q^{-2n}z_j\cdots z_N|z_{N+1}\cdots z_{2N}) \\
= \ & R^{V^*V^*}_{jj-1}(z_j/(q^{2n}z_{j-1}))\cdots R^{V^*V^*}_{j1}(z_j/(q^{2n}z_1))K^{(i)}_j(z_j/q^{2n}) \\
\times \ & R^{V^*V^*}_{1j}(z_1z_j/q^{2n})\cdots R^{V^*V^*}_{j-1j}(z_{j-1}z_j/q^{2n})R^{V^*V^*}_{j+1j}(z_{j+1}z_j/q^{2n})\cdots R^{V^*V^*}_{Nj}(z_Nz_j/q^{2n}) \\
\times \ & R^{VV^*}_{N+1j}(z_{N+1}z_j/q^{2n})\cdots R^{VV^*}_{2Nj}(z_{2N}z_j/q^{2n})K^{*(i)}_j(q^n/z_j)R^{V^*V}_{j2N}(z_j/z_{2N})\cdots R^{V^*V}_{jN}(z_j/z_N) \\
\times \ & R^{V^*V^*}_{jN+1}(z_j/z_{N+1})\cdots R^{V^*V^*}_{jj+1}(z_j/z_{j+1})G^{(i)}(z_1\cdots z_N|z_{N+1}\cdots z_{2N}),
\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}
& G^{(i)}(z_1\cdots z_N|z_{N+1}\cdots q^{-2n}z_j\cdots z_{2N}) \\
= \ & R^{VV}_{jj-1}(z_j/(q^{2n}z_{j-1}))\cdots R^{VV}_{jN+1}(z_j/(q^{2n}z_{N+1}))R^{VV^*}_{jN}(z_j/(q^{2n}z_N))\cdots R^{VV^*}_{j1}(z_j/(q^{2n}z_1)) \\
\times \ & K^{*(i)}_j(q^n/z_j)R^{V^*V}_{1j}(z_1z_j)\cdots R^{V^*V}_{Nj}(z_Nz_j)R^{VV}_{N+1j}(z_{N+1}z_j)\cdots R^{VV}_{j-1j}(z_{j-1}z_j) \\
\times \ & R^{VV}_{j+1j}(z_{j+1}z_j)\cdots R^{VV}_{2Nj}(z_{2N}z_j)K^{(i)}_j(z_j) \\
\times \ & R^{VV}_{j2N}(z_j/z_{2N})\cdots R^{VV}_{jj+1}(z_j/z_{j+1})G^{(i)}(z_1\cdots z_N|z_{N+1}\cdots z_{2N}).
\end{aligned} \tag{3.46}$$

Here the coefficient matrces are given by (2.2),(2.4),(3.3), (3.11),(3.18),(3.38), (3.39) and (3.43).

For $N=1$, the equations (3.45) and (3.46) are as follows:

$$G^{(i)}(q^{-2n}z_1|z_2) = K^{(i)}_1(z_1/q^{2n})R^{VV^*}_{21}(z_2z_1/q^{2n})K^{*(i)}_1(q^n/z_1)R^{V^*V}_{12}(z_1/z_2)G^{(i)}(z_1|z_2), \tag{3.47}$$

$$G^{(i)}(z_1|q^{-2n}z_2) = R^{VV^*}_{21}(z_2/(q^{2n}z_1))K^{*(i)}_2(q^n/z_2)R^{V^*V}_{12}(z_1z_2)K^{(i)}_2(z_2)G^{(i)}(z_1|z_2). \tag{3.48}$$

4 Two point functions

The purpose of this section is to perform explicit calculations of two point functions for free boundary condition. In what follows we consider the case $i=L=M=0$ and $N=1$. In this case the boundary K-matrices $K^{(0)}(z)$ and $K^{*(0)}(z)$ become scalar matrices, i.e.

$$K^{(0)}(z)=\frac{\varphi^{(0)}(z)}{\varphi^{(0)}(z^{-1})}\times \mathrm{id},\quad K^{*(0)}(z)=\frac{\varphi^{*(0)}(z)}{\varphi^{*(0)}(z^{-1})}\times \mathrm{id}.$$

The boundary quantum Knizhnik-Zamolodchikov equations thus reduces to:

$$G^{(0)}(q^{-2n}z_1|z_2) = \frac{\varphi^{(0)}(z_1/q^{2n})}{\varphi^{(0)}(q^{2n}/z_1)}\frac{\varphi^{*(0)}(q^n/z_1)}{\varphi^{*(0)}(z_1/q^n)}R^{VV^*}_{21}(z_2z_1/q^{2n})R^{V^*V}_{12}(z_1/z_2)G^{(0)}(z_1|z_2), \tag{4.1}$$

$$G^{(0)}(z_1|q^{-2n}z_2) = \frac{\varphi^{(0)}(z_2)}{\varphi^{(0)}(1/z_2)}\frac{\varphi^{*(0)}(q^n/z_2)}{\varphi^{*(0)}(z_2/q^n)}R^{VV^*}_{21}(z_2/(q^{2n}z_1))R^{V^*V}_{12}(z_1z_2)G^{(0)}(z_1|z_2). \tag{4.2}$$

Let us now introduce the scalar function $r(z_1|z_2)$ by

$$r(z_1|z_2)=A(z_1)A(q^nz_2)B(z_1z_2)B(z_1/z_2), \tag{4.3}$$

where

$$
\begin{aligned}
A(z) &= \frac{(q^{2n+2}z^2;q^{2n},q^{4n})_\infty (q^{4n+2}/z^2;q^{2n},q^{4n})_\infty}{(q^{4n}z^2;q^{2n},q^{4n})_\infty (q^{6n}/z^2;q^{2n},q^{4n})_\infty}, && (4.4)\\
B(z) &= \frac{(q^{2n}z;q^{2n},q^{2n})_\infty (q^{2n}/z;q^{2n},q^{2n})_\infty}{(q^{2n+2}z;q^{2n},q^{2n})_\infty (q^{2n+2}/z;q^{2n},q^{2n})_\infty}. && (4.5)
\end{aligned}
$$

Note that the function $r(z_1|z_2)$ satisfies

$$
r(q^{-2n}z_1|z_2) = q^{-2n}z_1^2 r^{VV^*}(z_1z_2/q^{2n}) r^{V^*V}(z_1/z_2) \frac{\varphi^{(0)}(z_1/q^{2n})}{\varphi^{(0)}(q^{2n}/z_1)} \frac{\varphi^{*(0)}(q^n/z_1)}{\varphi^{*(0)}(z_1/q^n)} \times r(z_1|z_2), \tag{4.6}
$$

$$
r(z_1|q^{-2n}z_2) = q^{-2n}z_2^2 r^{VV^*}(z_2/(q^{2n}z_1)) r^{V^*V}(z_1z_2) \frac{\varphi^{(0)}(z_2)}{\varphi^{(0)}(1/z_2)} \frac{\varphi^{*(0)}(q^n/z_2)}{\varphi^{*(0)}(z_2/q^n)} \times r(z_1|z_2). \tag{4.7}
$$

Let $\bar{G}(z_1|z_2)_j$ be the auxiliary function defined by

$$
\bar{G}(z_1|z_2)_j = r(z_1|z_2)^{-1} G^{(0)}(z_1|z_2)_j^j. \tag{4.8}
$$

Then we have

$$
\begin{aligned}
\sum_{j=0}^{n-1} \bar{G}(q^{-2n}z_1|z_2)_j &= \frac{1-q^{2n}/(z_1z_2)}{1-z_1z_2} \frac{1-q^{2n}z_2/z_1}{1-z_1/z_2} \sum_{j=0}^{n-1} \bar{G}(z_1|z_2)_j, && (4.9)\\
\sum_{j=0}^{n-1} \bar{G}(z_1|q^{-2n}z_2)_j &= \frac{1-q^{2n}/(z_1z_2)}{1-z_1z_2} \frac{1-q^{2n}z_1/z_2}{1-z_2/z_1} \sum_{j=0}^{n-1} \bar{G}(z_1|z_2)_j. && (4.10)
\end{aligned}
$$

From these we obtain

$$
\begin{aligned}
&\sum_{j=0}^{n-1} {}_B\langle 0|\Phi_j^{*(0,1)}(z_1)\Phi_j^{(1,0)}(z_2)|0\rangle_B = c_0\ r(z_1|z_2) \times && (4.11)\\
&\times \quad \left\{(q^{2n}z_1/z_2;q^{2n})_\infty (q^{2n}z_2/z_1;q^{2n})_\infty (q^{2n}z_1z_2;q^{2n})_\infty (q^{2n}/(z_1z_2);q^{2n})_\infty\right\}^{-1},
\end{aligned}
$$

where c_0 is a constant independent of spectral parameters z_1, z_2. By specializing the spectral parameters $z_1 = z_2$, we have

$$
c_0 = {g_n}^{-1} \times\ {}_B\langle 0|0\rangle_B \times \left\{ \frac{(q^{2n+2};q^{2n},q^{2n})_\infty}{(q^{4n};q^{2n},q^{2n})_\infty} \right\}^2, \tag{4.12}
$$

where the norm ${}_B\langle 0|0\rangle_B$ is given as follows [11]

$$
{}_B\langle 0|0\rangle_B = \frac{1}{\sqrt{(q^{4n};q^{4n})_\infty}} \prod_{j=1}^{n-1} \left\{ \frac{\sqrt{(q^{4n+2-2j};q^{4n})_\infty (q^{4n-2-2j};q^{4n})_\infty}}{(q^{4n-2j};q^{4n})_\infty} \right\}^{j(n-j)}.
$$

Let ω satisfy $\omega^n = 1$ and $\omega \neq 1$. Then we have

$$
\begin{aligned}
\sum_{j=0}^{n-1} (q^2\omega)^j \bar{G}(q^{-2n}z_1|z_2)_j &= q^{2n}z_1^{-2} \sum_{j=0}^{n-1} (q^2\omega)^j \bar{G}(z_1|z_2)_j, && (4.13)\\
\sum_{j=0}^{n-1} (q^2\omega)^j \bar{G}(z_1|q^{-2n}z_2)_j &= q^{2n}z_2^{-2} \sum_{j=0}^{n-1} (q^2\omega)^j \bar{G}(z_1|z_2)_j. && (4.14)
\end{aligned}
$$

From these we obtain

$$\begin{aligned}&\sum_{j=0}^{n-1}(q^2\omega^k)^j\ {}_B\langle 0|\Phi_j^{*(0,1)}(z_1)\Phi_j^{(1,0)}(z_2)|0\rangle_B\\ =\ & c_k\ r(z_1|z_2)\times\{(-q^{2n}z_1^2;q^{4n})_\infty(-q^{2n}/z_1^2;q^{4n})_\infty(-q^{2n}z_2^2;q^{4n})_\infty(-q^{2n}/z_2^2;q^{4n})_\infty\}^{-1}. \quad (4.15)\end{aligned}$$

Here c_k are constants independent of spectral parameters z_1, z_2.

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A Bosonization of vertex operators in $U_q(\widehat{sl_n})$

For readers' convenience, we summarize the results of bosonizations of the vertex operators [15].

Let $\mathbb{C}[\bar{P}]$ be the $\mathbb{C}$-algebra generated by the symbols $\{e^{\alpha_2},\cdots,e^{\alpha_{n-1}},e^{\bar{\Lambda}_{n-1}}\}$ which satisfy the following defining relations:

$$e^{\alpha_i}e^{\alpha_j}=(-1)^{(\alpha_i|\alpha_j)}e^{\alpha_j}e^{\alpha_i},\quad (2\leqq i,j\leqq n-1),$$
$$e^{\alpha_i}e^{\bar{\Lambda}_{n-1}}=(-1)^{\delta_{i,n-1}}e^{\bar{\Lambda}_{n-1}}e^{\alpha_i},\quad (2\leqq i\leqq n-1).$$

For $\alpha=m_2\alpha_2+\cdots+m_{n-1}\alpha_{n-1}+m_n\bar{\Lambda}_{n-1}$, we denote $e^{m_2\alpha_2}\cdots e^{m_{n-1}\alpha_{n-1}}e^{m_n\bar{\Lambda}_{n-1}}$ by e^α. Let $((\alpha_s|\alpha_t))_{1\leqq s,t\leqq n-1}$ stand for the A-type Catran matrix whose matrix element $(\alpha_s|\alpha_t)$ is an integer. Let $\mathbb{C}[\bar{Q}]$ be the $\mathbb{C}$-subalgebra of $\mathbb{C}[\bar{P}]$ generated by the symbols $\{e^{\alpha_1},\cdots,e^{\alpha_{n-1}}\}$ which satisfy the following defining relations:

$$e^{\alpha_i}e^{\alpha_j}=(-1)^{(\alpha_i|\alpha_j)}e^{\alpha_j}e^{\alpha_i},\quad (1\leqq i,j\leqq n-1).$$

Note that

$$\alpha_1=-\sum_{r=2}^{n-1}r\alpha_r+n\bar{\Lambda}_{n-1},\quad \bar{\Lambda}_i=-\sum_{r=i+1}^{n-1}(r-i)\alpha_r+(n-i)\bar{\Lambda}_{n-1}.$$

Let us consider the $\mathbb{C}$-algebra generated by the bosons $a_s(k)$ $(s\in\{1,\cdots,n-1\},k\in\mathbb{Z})$ which satisfy the following defining relations:

$$[a_s(k),a_t(l)]=\delta_{k+l,0}\frac{[(\alpha_s|\alpha_t)k][k]}{k}.$$

The highset weight module $V(\Lambda_i)$ is realized as

$$V(\Lambda_i)=\mathbb{C}[a_s(-k),\ (s\in\{1,\cdots,n-1\},k\in\mathbb{Z}\geqq 0)]\otimes\mathbb{C}[\bar{Q}]e^{\bar{\Lambda}_i}.$$

We consider $\mathbb{C}[\bar{Q}]e^{\bar{\Lambda}_i}$ as a subspace of $\mathbb{C}[\bar{P}]$. Here the actions of the operators $a_s(k),\partial_\alpha,e^\alpha$ on $V(\Lambda_i)$ are defined as follows:

$$a_s(k)f\otimes e^\beta=\begin{cases}a_s(k)f\otimes e^\beta, & (k<0),\\ [a_s(k),f]\otimes e^\beta, & (k>0),\end{cases}$$

$$\begin{aligned}\partial_\alpha f\otimes e^\beta &= (\alpha|\beta) f\otimes e^\beta.\\ e^\alpha f\otimes e^\beta &= f\otimes e^\alpha e^\beta.\end{aligned}$$

The inner product is explicitly given as follows:

$$(\alpha_i|\bar{\Lambda}_j)=\delta_{i,j},\quad (\bar{\Lambda}_i|\bar{\Lambda}_j)=\frac{i(n-j)}{n},\quad (1\leqq i\leqq j\leqq n-1).$$

$$\begin{aligned}\Phi_{n-1}^{(i,i+1)}(z) &= e^{P(z)}e^{Q(z)}e^{\bar{\Lambda}_{n-1}}(q^{n+1}z)^{\partial_{\bar{\Lambda}_{n-1}}+\frac{n-i-1}{n}}(-1)^{(\partial_{\bar{\Lambda}_1}-\frac{n-i-1}{n})(n-1)+\frac{1}{2}(n-i)(n-i-1)},\\ \Phi_0^{*(i+1,i)}(z) &= e^{P^*(z)}e^{Q^*(z)}e^{\bar{\Lambda}_1}((-1)^{n-1}qz)^{\partial_{\bar{\Lambda}_1}+\frac{i}{n}}q^i(-1)^{in+\frac{1}{2}i(i+1)},\end{aligned}$$

$$\begin{aligned}\Phi_j^{(i-1i)}(z) &= c_j\oint\cdots\oint_{C_j}\frac{dw_{j+1}}{2\pi i w_{j+1}}\cdots\frac{dw_{n-1}}{2\pi i w_{n-1}}\frac{w_{j+1}}{z}\frac{1}{(1-qw_{n-1}/z)(1-qz/w_{n-1})}\\ &\times \frac{1}{(1-qw_{n-1}/w_{n-2})(1-qw_{n-2}/w_{n-1})\cdots(1-qw_{j+2}/w_{j+1})(1-qw_{j+1}/w_{j+2})}\\ &\times :\Phi_{n-1}^{(i-1i)}(z)X_{n-1}^-(q^{n+1}w_{n-1})\cdots X_{j+1}^-(q^{n+1}w_{j+1}):,\end{aligned}\tag{A.1}$$

and

$$\begin{aligned}\Phi_j^{*(ii+1)}(z) &= c_j^*\oint\cdots\oint_{C_j^*}\frac{dw_1}{2\pi i w_1}\cdots\frac{dw_j}{2\pi i w_j}\frac{w_j}{z}\frac{1}{(1-qz/w_1)(1-qw_1/z)}\\ &\times \frac{1}{(1-qw_1/w_2)(1-qw_2/w_1)\cdots(1-qw_{j-1}/w_j)(1-qw_j/w_{j-1})}\\ &\times :\Phi_0^{*(ii+1)}(z)X_1^-(qw_1)\cdots X_j^-(qw_j):,\end{aligned}\tag{A.2}$$

where c_j, c_j^* are appropriate constants. The contours C_j, C_j^* encircle $w_l=0$ anti-clockwise in such a way that

$$\begin{aligned}C_j: &\quad |q|<|w_{n-1}/z|<|q^{-1}|,\quad |q|<|w_l/w_{l+1}|<|q^{-1}|,\ (l=j+1,\cdots,n-2),\\ C_j^*: &\quad |q|<|w_1/z|<|q^{-1}|,\quad |q|<|w_{l+1}/w_l|<|q^{-1}|,\ (l=1,\cdots,j-1).\end{aligned}$$

Here we have used

$$X_j^-(w)=e^{R_j^-(w)}e^{S_j^-(w)}e^{-\alpha_j}w^{-\partial_{\alpha_j}},$$

$$\begin{aligned}P(z)=\sum_{k=1}^\infty a_{n-1}^*(-k)q^{\frac{2n+3}{2}k}z^k, &\qquad Q(z)=\sum_{k=1}^\infty a_{n-1}^*(k)q^{-\frac{2n+1}{2}k}z^{-k},\\ P^*(z)=\sum_{k=1}^\infty a_1^*(-k)q^{\frac{3}{2}k}z^k, &\qquad Q^*(z)=\sum_{k=1}^\infty a_1^*(k)q^{-\frac{1}{2}k}z^{-k},\\ R_j^-(w)=-\sum_{k=1}^\infty \frac{a_j(-k)}{[k]}q^{\frac{k}{2}}w^k, &\qquad S_j^-(w)=\sum_{k=1}^\infty\frac{a_j(k)}{[k]}q^{\frac{k}{2}}w^{-k},\\ a_{n-1}^*(k)=\sum_{l=1}^{n-1}\frac{-[lk]}{[k][nk]}a_l(k), &\qquad a_1^*(k)=\sum_{l=1}^{n-1}\frac{-[(n-l)k]}{[k][nk]}a_l(k).\end{aligned}$$

$$[a_j(k),a_{n-1}^*(-k)]=\delta_{j,n-1}\frac{[k]}{k},\quad [a_j(k),a_1^*(-k)]=\delta_{j,1}\frac{[k]}{k}.$$

B Bosonization of the boundary vacuum states

For readers' convenience we summarize the bosonic formulae of the boundary vacuum [11]. Let us set the symmetric matrix as

$$\hat{I}_{s,t}(k) = \begin{cases} 0, & (st = 0), \\ \frac{[sk][(n-t)k]}{[k]^2[nk]}, & (1 \leqq s \leqq t \leqq n-1), \\ \frac{[tk][(n-s)k]}{[k]^2[nk]}, & (1 \leqq t \leqq s \leqq n-1). \end{cases} \tag{B.1}$$

Let us consider the $\mathbb{C}$-algebra generated by the bosons $a_s(k)$ ($s \in \{1, \cdots, n-1\}, k \in \mathbb{Z}$) which satisfy the following defining relations:

$$[a_s(k), a_t(l)] = \delta_{k+l,0} \frac{[(\alpha_s|\alpha_t)k][k]}{k},$$

where I$(\alpha_s|\alpha_t)$ is an element of A-type Cartan matrix.

The boundary state has the form

$$|i\rangle_B = e^{F_i}|i\rangle, \quad F_i = \sum_{s,t=1}^{n-1} \sum_{k=1}^{\infty} \alpha_{s,t}(k) a_s(-k) a_t(-k) + \sum_{s=1}^{n-1} \sum_{k=1}^{\infty} \beta_s^{(i)}(k) a_s(-k).$$

Here the coefficients of the quadratic part are given by

$$\alpha_{s,t}(k) = \frac{-kq^{2(n+1)k}}{2[k]} \times \hat{I}_{s,t}(k), \tag{B.2}$$

and those of the linear part are given by

$$\beta_j^{(i)}(k) = (q^{(n+3/2)k} - q^{(n+1/2)k})\theta_k \sum_{s=1}^{n-1} \hat{I}_{j,s}(k) \tag{B.3}$$

$$+ \begin{cases} 0, & (C1), \\ \hat{I}_{j,L}(k) q^{(2n-2M+L+1/2)k} r^k - \hat{I}_{j,M}(k) q^{(2n-M+1/2)k} r^k, & (C2), \\ -\hat{I}_{j,L}(k) q^{(2M-L+1/2)k} r^{-k} + \hat{I}_{j,M}(k) q^{(M+1/2)k} r^{-k}, & (C3), \end{cases} \tag{B.4}$$

where

$$\theta_k = \begin{cases} 0, & k \text{ is odd}, \\ 1, & k \text{ is even}. \end{cases}$$

The dual boundary state has the form

$$_B\langle i| = \langle i|e^{G_i}, \quad G_i = \sum_{s,t=1}^{n-1} \sum_{k=1}^{\infty} \gamma_{s,t}(k) a_s(k) a_t(k) + \sum_{s=1}^{n-1} \sum_{k=1}^{\infty} \delta_s^{(i)}(k) a_s(k). \tag{B.5}$$

Here the coefficients of the quadratic part are given by

$$\gamma_{s,t}(k) = \frac{-kq^{-2k}}{2[k]} \times \hat{I}_{s,t}(k), \tag{B.6}$$

and those of the linear part are given by

$$\delta_j^{(i)}(k) = -(q^{-k/2} - q^{-3k/2})\theta_k \sum_{s=1}^{n-1} \hat{I}_{j,s}(k) \tag{B.7}$$

$$+ \begin{cases} 0, & (C1), \\ q^{(L-3/2)k} r^k \hat{I}_{j,L}(k) - q^{(2L-M-3/2)k} r^k \hat{I}_{j,M}(k), & (C2), \\ -q^{(-L-3/2)k} r^{-k} \hat{I}_{j,L}(k) + q^{(M-2L-3/2)k} r^{-k} \hat{I}_{j,M}(k), & (C3). \end{cases} \tag{B.8}$$

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