Quantum network architecture of tight-binding models with substitution sequences

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Abstract

We study a two-spin quantum Turing architecture, in which discrete local rotations $\{\alpha_m\}$ of the Turing head spin alternate with quantum controlled NOT-operations. Substitution sequences are known to underlie aperiodic structures. We show that parameter inputs $\{\alpha_m\}$ described by such sequences can lead here to a quantum dynamics, intermediate between the regular and the chaotic variant. Exponential parameter sensitivity characterizing chaotic quantum Turing machines turns out to be an adequate criterion for induced quantum chaos in a quantum network.

1 Introduction

Models described by one-dimensional Schrödinger equations with quasi-periodic potentials display interesting spectra: this kind of potential is intermediate between periodic ones, leading to energy bands and extended states, and truly random potentials, which cause localisation [1]. A super-lattice e.g. may be made of two species of doped semiconductors producing a one-dimensional chain of quantum wells. A qualitative model to describe the corresponding wave-functions is given by the tight-binding model, $\hat{H}\psi(n) = \psi(n+1) + \psi(n-1) + \lambda V(n)\psi(n)$, $\psi \in l^2(\mathbf{Z})$, where $V(n)$ represents the effect of quantum well n , and λ is a positive parameter playing the role of a coupling constant [2]. An interesting description results when this super-lattice is constructed by means of a deterministic rule. The simplest rule obtains when the two species alternate in a periodic way. But in general the rule will be aperiodic. One widely studied example is the Fibonacci sequence, which is quasi-periodic [2]: given two 'letters' a and b, one substitutes $a \to \xi(a) = ab$ and $b \to \xi(b) = a$. Iterating this rule on a one thus generates the sequence $abaababaabaababa\cdots$, in which the frequency of a's is given by the golden mean $(5^{1/2} - 1)/2$. Other examples of such substitution rules are the Thue-Morse sequence (non quasi-periodic), which is obtained through the substitution $a \to \xi(a) = ab$, $b \to \xi(b) = ba$ giving $abbabaabbaababba\cdots$, and the period-doubling sequence (non quasi-periodic), $a \to \xi(a) = ab$, $b \to \xi(b) = aa$ [2]. Accordingly, a piece-wise constant potential $\{V_n; n \in \mathbf{Z}\}$ based on such such a rule, e.g. $V_1 = a, V_2 = b, V_3 = a, \cdots$, is called a 'substitution potential'. The potential of the Fibonacci sequence has one-dimensional quasi-crystalline properties.

In recent years problems of quantum computing (QC) and information processing have received increasing attention. To solve certain classes of problems in a potentially very powerful way, one tries to utilize in QC the quantum-mechanical superposition principle and the (non-classical) entanglement. In the models of QC based on quantum Turing machines (QTM) [3, 4], the computation is characterized by sequences of unitary transformations (i.e. by the corresponding Hamiltonians $\hat{H}$ acting during finite time interval steps). Benioff has studied the tight-binding model in a generalized QTM [5], where QC is associated with different potentials at different steps as an 'environmental effect'. These kinds of influences may introduce *deterministic disorder* which would degrade performance by causing reflections at various steps and decay of the transmitted component [6].

Here we investigate an iterative map on qubits which can be interpreted as a QTM architecture [7]: local transformations of the Turing head controlled by a sequence of rotation angles $\{\alpha_m\}$, $m = 1, 2, \cdots$ (parameter inputs) alternate with a quantum-controlled NOT-operation (QCNOT) with a second spin on the Turing tape. Those angles at steps $n = 2m - 1$ are reminiscent of the potentials V_m introduced before. In the present paper we will investigate the Fibonacci rule and the Thue-Morse case mainly with respect to the local dynamics of the Turing head, which will be shown to reflect the degree of 'randomness' of the substitution sequences. The various types of aperiodic structures have been characterized up to now by the nature of their Fourier spectra only [2].

2 Quantum Turing machine driven by substitution sequences

The quantum network [8] to be considered here is composed of two pseudo-spins $|p\rangle^{(\mu)}$; $p = -1, 1$; $\mu = S, 1$ (Turing-head S, Turing-tape spin 1, see figure 1) so that its network state $|\psi\rangle$ lives in the four-dimensional Hilbert space spanned by the product wave-functions $|j^{(S)}k^{(1)}\rangle = |jk\rangle$. Correspondingly, any (unitary) network operator can be expanded as a sum of product operators. The latter may be based on the $SU(2)$-generators, the Pauli matrices $\hat{\sigma}_j^{(\mu)}$, $j = 1, 2, 3$, together with the unit operator $\hat{1}^{(\mu)}$.

The initial state $|\psi_0\rangle$ will be taken to be a product of the Turing-head and tape wave-functions. For the discretized dynamical description of the QTM we identify the unitary operators $\hat{U}_n$, $n = 1, 2, 3, \cdots$ (step number) with the local unitary transformation on the Turing head S, $\hat{U}_{\alpha_m}^{(S)}$, and the QCNOT on $(S, 1)$, $\hat{U}^{(S,1)}$, respectively, as follows:

$$\hat{U}_{2m-1} \;=\; \exp\left(-i\hat{\sigma}_1^{(S)}\alpha_m/2\right) \tag{1}$$

$$\hat{U}_{2m} \;=\; \hat{U}^{(S,1)} \;=\; \hat{P}_{-1,-1}^{(S)}\,\hat{\sigma}_1^{(1)} + \hat{P}_{1,1}^{(S)}\,\hat{1}^{(1)} \;=\; \left(\hat{U}^{(S,1)}\right)^{\dagger} , \tag{2}$$

where $P_{j,j}^{(S)} = |j\rangle^{(S)(S)}\langle j|$ is a (local) projection operator, and the Turing head is externally driven by substitution sequences $\{\alpha_m\}$ specified by α_1, α_2 . Here we restrict ourselves to the quasi-periodic Fibonacci - (qf) and Thue-Morse sequence (tm), respectively:

$$\alpha_1^{\rm qf} = \alpha_1\,,\; \alpha_2^{\rm qf} = \alpha_2\,,\; \alpha_3^{\rm qf} = \alpha_1\,,\; \alpha_4^{\rm qf} = \alpha_1\,,\; \alpha_5^{\rm qf} = \alpha_2\,,\; \cdots \tag{3}$$

$$\alpha_1^{\rm tm} = \alpha_1\,,\; \alpha_2^{\rm tm} = \alpha_2\,,\; \alpha_3^{\rm tm} = \alpha_2\,,\; \alpha_4^{\rm tm} = \alpha_1\,,\; \alpha_5^{\rm tm} = \alpha_2\,,\; \cdots\,. \tag{4}$$

First, we consider the reduced state-space dynamics of the head S and tape-spin 1, respectively,

$$\sigma_j^{(S)}(n) \;=\; {\rm Tr}\left(\hat{\rho}_n^{(S)}\,\hat{\sigma}_j^{(S)}\right) \;=\; \langle\psi_n|\hat{\sigma}_j^{(S)}\otimes\hat{1}^{(1)}|\psi_n\rangle\,,$$

$$\sigma_k^{(1)}(n) \;=\; \mathrm{Tr}\left(\hat{\rho}_n^{(1)}\,\hat{\sigma}_k^{(1)}\right) \;=\; \langle\psi_n|\hat{1}^{(S)}\otimes\hat{\sigma}_k^{(1)}|\psi_n\rangle\,, \tag{5}$$

where $|\psi_n\rangle$ is the total network state at step n, and $\sigma_j^{(\mu)}(n)$ are the respective Bloch-vectors. Due to the entanglement between the head and tape, both will, in general, appear to be in a 'mixed-state', which means that the length of the Bloch-vectors in (5) is less than 1. However, for specific initial states $|\psi_0\rangle$ the state of head and tape will remain pure: As $|\pm\rangle^{(1)} = \frac{1}{\sqrt{2}}\left(|-1\rangle^{(1)} \pm |1\rangle^{(1)}\right)$ are the eigenstates of $\hat{\sigma}_1^{(1)}$ with $\hat{\sigma}_1^{(1)}|\pm\rangle^{(1)} = \pm|\pm\rangle^{(1)}$, the QCNOT-operation $\hat{U}^{(S,1)}$ of equation (2) cannot create any entanglement, irrespective of the head state $|\varphi\rangle^{(S)}$, i.e.

$$\begin{aligned}\hat{U}^{(S,1)}\,|\varphi\rangle^{(S)}\otimes\,|+\rangle^{(1)} \;&=\; |\varphi\rangle^{(S)}\otimes\,|+\rangle^{(1)}\\ \hat{U}^{(S,1)}\,|\varphi\rangle^{(S)}\otimes\,|-\rangle^{(1)} \;&=\; \hat{\sigma}_3^{(S)}|\varphi\rangle^{(S)}\otimes\,|-\rangle^{(1)}\,.\end{aligned} \tag{6}$$

As a consequence, the state $|\psi_n\rangle$ remains a product state for any step n and initial product state $|\psi_0^{\pm}\rangle = |\varphi_0\rangle^{(S)}\otimes|\pm\rangle^{(1)}$ with $|\varphi_0\rangle^{(S)} = \exp\left(-i\hat{\sigma}_1^{(S)}\varphi_0/2\right)|-1\rangle^{(S)}$, so that the Turing head performs a pure-state trajectory ('primitive', see [7]) on the Bloch-circle $\left(\sigma_1^{(S)}(n)=0\right)$

$$|\psi_n^{\pm}\rangle = |\varphi_n^{\pm}\rangle^{(S)}\otimes|\pm\rangle^{(1)}\,,\quad \left(\sigma_2^{(S)}(n|\pm)\right)^2 + \left(\sigma_3^{(S)}(n|\pm)\right)^2 = 1\,.$$

Here $\sigma_j^{(S)}(n|\pm)$ denotes the Bloch-vector of the Turing head S conditioned by the initial state $|\psi_0^{\pm}\rangle$. From the Fibonacci sequence (3) and the property (6) it is found for $|\varphi_n^{+}\rangle^{(S)}\otimes|+\rangle^{(1)}, n = 2m$, and $\varphi_0^{\pm} = \alpha_0 = 0$ that

$$\sigma_2^{(S)}(2m|+) = \sin\mathcal{C}_{2m}(+)\,,\quad \sigma_3^{(S)}(2m|+) = -\cos\mathcal{C}_{2m}(+)\,, \tag{7}$$

and $\sigma_k^{(S)}(2m-1|+) = \sigma_k^{(S)}(2m|+)$, where the cumulative rotation angle is

$$\mathcal{C}_{2m}(+) \;=\; \sum_{j=1}^{m}\alpha_j^{\mathrm{qf}} \;=\; \alpha_1\,m \;+\; (\alpha_2-\alpha_1)\,m'\,,$$

with $m'\,(\leq m)$ being the total number of angles α_2 up to step $2m$. For the cumulative rotation angle $\mathcal{C}_n(-)$ up to step n we utilize the following recursion relations

$$\mathcal{C}_{2m}(-) = -\mathcal{C}_{2m-1}(-)\,,\quad \mathcal{C}_{2m-1}(-) = \alpha_m^{\mathrm{qf}} + \mathcal{C}_{2m-2}(-)\,.$$

Then it is easy to verify that for $|\varphi_n^{-}\rangle^{(S)}\otimes\,|-\rangle^{(1)}$ and $\varphi_0^{\pm} = \alpha_0 = 0$

$$|\mathcal{C}_n(-)| \;\leq\; 2\,\max\left(|\alpha_1|\,,\,|\alpha_2|\right) \;=:\; M\,, \tag{8}$$

yielding $\sigma_2^{(S)}(n|-) = \sin\mathcal{C}_n(-)$, $\sigma_3^{(S)}(n|-) = -\cos\mathcal{C}_n(-)$.

From any initial state, $|\psi_0\rangle = a^{(+)}|\varphi_0^{+}\rangle^{(S)}\otimes|+\rangle^{(1)} + a^{(-)}|\varphi_0^{-}\rangle^{(S)}\otimes|-\rangle^{(1)}$, we then obtain at step n

$$|\psi_n\rangle = a^{(+)}|\varphi_n^{+}\rangle^{(S)}\otimes\,|+\rangle^{(1)} \;+\; a^{(-)}|\varphi_n^{-}\rangle^{(S)}\otimes\,|-\rangle^{(1)}$$

and, observing the orthogonality of the $|\pm\rangle^{(1)}$,

$$\sigma_k^{(S)}(n) = |a^{(+)}|^2\,\sigma_k^{(S)}(n|+) \;+\; |a^{(-)}|^2\,\sigma_k^{(S)}(n|-)\,. \tag{9}$$

This trajectory of the Turing-head S represents a non-orthogonal pure-state decomposition. By using (7), (8), (9) $\left(\text{with } a^{(+)} = a^{(-)} = 1/\sqrt{2}\right)$ we finally get for $|\psi_0\rangle = |-1\rangle^{(S)} \otimes |-1\rangle^{(1)}$

$$\left(\sigma_2^{(S)}(n),\, \sigma_3^{(S)}(n)\right) \;=\; \cos\mathcal{B}_n \cdot (\sin\mathcal{A}_n,\, -\cos\mathcal{A}_n)\;, \tag{10}$$

where $(\mathcal{C}_n(+) - M)/2 \;\le\; \mathcal{A}_n \;=\; (\mathcal{C}_{2m}(+) + \mathcal{C}_{2m}(-))/2\,,\;\; \mathcal{B}_n \;=\; (\mathcal{C}_{2m}(+) - \mathcal{C}_{2m}(-))/2 \;\le\; (\mathcal{C}_{2m}(+) + M)/2\,,\; n = 2m$ or $2m-1$. Thus the expression (10) indicates that for the local dynamics of the Turing head in the 'non-classical' regime the cumulative control loss due to any small perturbation δ of the given $\alpha_1^{\text{qf}}, \alpha_1^{\text{qf}}$ grows at most linearly with n, so that all periodic orbits on the plane $\left\{0, \sigma_2^{(S)}, \sigma_3^{(S)}\right\}$ are stable (see figure 2*a* - *c*), as in the case of the 'regular' control $\alpha_m = \alpha_1$ [7]. This may be contrasted with the chaotic Fibonacci-rule (cf), $\alpha_{m+1}^{\text{cf}} = \alpha_m^{\text{cf}} + \alpha_{m-1}^{\text{cf}}$ (Lyapunov exponent: $\ln(1+\sqrt{5})/2 > 0$), which can be interpreted as a temporal random (chaotic) analogue to one-dimensional chaotic potentials [9]: each step α_m is controlled by the cumulative information of the two previous steps. For a small perturbation of the initial phase angle α_0 the cumulative angles $\mathcal{A}_m, \mathcal{B}_m$, respectively, grow exponentially with m, and so do the deviation terms $\Delta\mathcal{C}_{2m}^{\text{cf}}(\pm) = \mathcal{C}_{2m}^{\text{cf}\,\prime}(\pm) - \mathcal{C}_{2m}^{\text{cfpo}}(\pm)$ from the periodic orbits (po). Thus the total cumulative control loss induced by the perturbation can show chaotic quantum behaviour on the Turing head [9].

For the Thue-Morse control (4) we easily find that for $|\psi_0\rangle = |-1\rangle^{(S)} \otimes |-1\rangle^{(1)}$ and $n = 8m$

$$\mathcal{C}_n(+) \;=\; 2\,(\alpha_1 + \alpha_2)\,m\,, \quad \mathcal{C}_n(-) \;=\; 0\,,$$

respectively, which is very similar to the result of the 'regular' machine with $\alpha_m^{\text{reg}} = (\alpha_1 + \alpha_2)/2$, implying $\mathcal{C}_{8m}^{\text{reg}}(+) = 2(\alpha_1 + \alpha_2)$ and $\mathcal{C}_{8m}^{\text{reg}}(-) = 0$. In all cases considered we thus find characteristic local invariants with respect to the Turing head (figure 2*d*).

3 Parameter sensitivity

The distance between density operators, $\hat\rho$ and $\hat\rho'$, defined by the so-called Bures metric [10]

$$D^2_{\rho\rho'} := \mathrm{Tr}\left\{(\hat\rho - \hat\rho')^2\right\}\,.$$

lies, independent of the dimension of the Liouville space, between 0 and 2. For pure states we can rewrite

$$\begin{aligned} D^2 \;&=\; 2\,(1 - |\langle\psi|\psi'\rangle|^2) \;=\; 2\,(1 - O') \\ O' &:= |\langle\psi_0(\delta)|\hat U^\dagger(\delta)\,\hat U(0)|\psi_0(0)\rangle|^2\,, \end{aligned}$$

where the perturbed unitary evolution, $\hat U(\delta)$, connects the initial state, $|\psi_0(\delta)\rangle$, and $|\psi'\rangle$. This metric can be applied likewise to the total-network-state space or any subspace. In any case it is a convenient additional means to characterize various QTMs: for the regular case, $\alpha_m^{\mathrm{reg}} = \alpha_1$ (Lyapunov exponent = 0), and any initial perturbation δ for $\hat{\rho}'$ the distance remains almost constant [9]; for the chaotic Fibonacci rule (cf), on the other hand, $\left(\alpha_m(\hat{\rho}) = \alpha_m^{\mathrm{cf}}\, ,\ \alpha_m(\hat{\rho}') = \alpha_m(\hat{\rho}) + \delta_m^{\mathrm{cf}}\right.$, where δ_m^{cf} is the cumulative perturbation of the angle $\left.\alpha_m^{\mathrm{cf}} \text{ at step } n = 2m-1\right)$ we obtain an initial exponential sensitivity [9]. In the case of the present substitution sequences we observe for a small perturbation of the given α_1, α_2 no initial exponential sensitivity in the evolution of D^2, which confirms that any periodic orbit is stable (figure 3a). Finally we display the evolution of D^2 for the total network state $|\psi_n\rangle$, which also shows no exponential sensitivity (figure 3b, cf. figure 3c in [9]). The respective distances for tape-spin 1 are similar to those shown. The corresponding behaviour under the Thue-Morse control is qualitatively the same. This parameter-sensitivity [11] has been proposed as a measure to distinguish quantum chaos from regular quantum dynamics. From the results of the present analysis and those in [9] satisfying this criterion we conclude that, indeed, only classical chaotic input makes the quantum dynamics in QTM architectures chaotic, too (figure 1).

4 Summary

In conclusion, we have studied the quantum dynamics of a small QTM driven by substitution sequences based on a decoherence-free Hamiltonian. As quantum features we utilized the superposition principle and the physics of entanglement. Quantum dynamics manifests itself in the superposition and entanglement of a pair of 'classical' (i.e. disentangled) state-sequences. The generalized QTM under this kind of control connects two fields of much recent interest, quantum computation and motion in one-dimensional structures with 'deterministically aperiodic' potential distributions. No chaotic quantum dynamics results in this case, as shown by the lack of exponential parameter-sensitivity. Local invariants leading to one-dimensional point manifolds (patterns) exist for $\alpha_1 = \alpha_2$ only. For $\alpha_1 \neq \alpha_2$ a continuous destruction of these patterns sets in (figure 2b; hardly visible yet in figure 2a). This reminds us of the disappearing of KAM tori in the classical phase space resulting from a small perturbation (see e.g. [12]). Patterns in reduced Bloch-planes $\left\{0, \sigma_2^{(\mu)}, \sigma_3^{(\mu)}\right\}$ (a quantum version of a Poincaré-cut) should thus be similarly useful to characterize quantum dynamics in a broad class of quantum networks. Due to the entanglement, we can see regular, chaotic, and intermediate quantum dynamics, respectively. Furthermore, the parameter sensitivity gives a sensitive criterion for testing induced quantum chaos in a pure quantum regime. This might be contrasted with the usual quantum chaology, which is concerned essentially only with semiclassical spectrum analysis of classically chaotic systems (e.g. level spacing, spectral rigidity) [13]. It is expected that a QTM architecture with a larger number of pseudo-spins on the Turing tape would still exhibit the same type of dynamical behaviour under the corresponding driving conditions.

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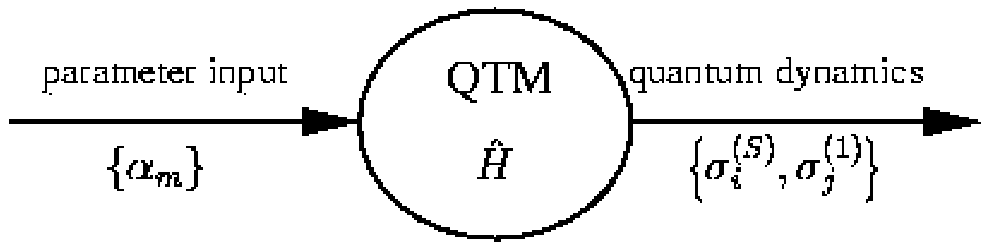
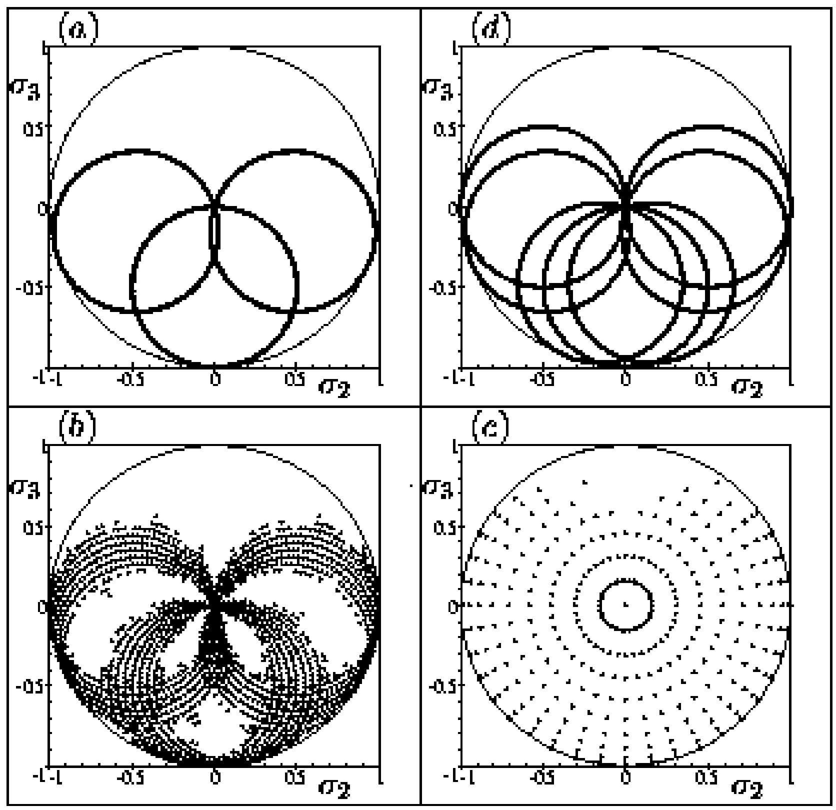
Figure 1: Input-output-scheme of our quantum Turing machine (QTM).

Figure 2: Turing-head patterns $\{0, \sigma_2(n), \sigma_3(n)\}$ for initial state $|\psi_0\rangle = |-1\rangle^{(S)} \otimes |-1\rangle^{(1)}$ under the control of substitution sequences: (a) quasi-periodic Fibonacci (qf) with $\alpha_1 = \frac{2}{5}\pi$, $\alpha_2 = \alpha_1 + 0.0005\pi$, (b) as in $a)$ but for $\alpha_2 = \alpha_1 + 0.03\pi$, (c) as in $a)$ but for $\alpha_2 = \alpha_1 + 0.05\pi$; (d) Thue-Morse (tm) control with $\alpha_1 = \frac{2}{5}\pi$, $\alpha_2 = \alpha_1 + 0.1001\pi$. For each simulation the total step number is $n = 10000$.

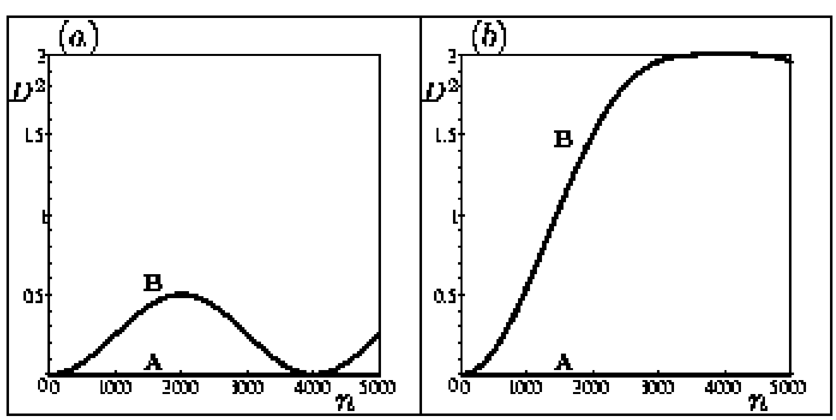
Figure 3: Evolution of the (squared) distance $D^2_{\rho\rho'}$ between the perturbed, $\hat{\rho}'$, and the reference QTM state, $\hat{\rho}$, under the quasi-periodic Fibonacci (qf) control: (a) for Turing head; (b) for total network state $|\psi_n\rangle$. For $\hat{\rho}$ we take $|\psi_0\rangle = |-1\rangle^{(S)} \otimes |-1\rangle^{(1)}$ and $\alpha_1 = \frac{2}{5}\pi$, $\alpha_2 = \alpha_1 + 0.03\pi$. Line **A**: $|\psi_0'\rangle = \exp\left(-i\hat{\sigma}_1^{(S)}\delta/2\right)|\psi_0\rangle$ for $\hat{\rho}'$, $\delta = 0.001$. Line **B**: $|\psi_0'\rangle = |\psi_0\rangle$, but $\alpha_1' = \alpha_1 + 0.001\pi$, $\alpha_2' = \alpha_2 + 0.001\pi$ for $(\hat{\rho}')$.



parameter input
QTM
quantum dynamics
$\{\alpha_m\}$
$\hat{H}$
$\{\sigma_i^{(S)}, \sigma_j^{(1)}\}$



(a)
(d)
(b)
(c)
σ_3
σ_2



(a)
(b)
D^2
n
A
B