Left-modular Elements

Shu-Chung Liu
Institute of Mathematics
Academia Sinica, Nankang
Taipei, Taiwan, R.O.C.
liularry@math.sinica.edu.tw

Bruce E. Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824 U.S.A.
sagan@math.msu.edu

Dedicated to the memory of Gian-Carlo Rota
without whose work on Möbius functions
this paper might never have been written

February 1, 2008

Key Words: characteristic polynomial, factorization, lattice, left-modular, modular, semimodular, supersolvable

AMS subject classification (1991): Primary 06C10; Secondary 05A15, 06A07.

Proposed running head:

Left-modular elements

Send proofs to:

Bruce E. Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027

Tel.: 517-355-8329
FAX: 517-432-1562
Email: sagan@math.msu.edu

Abstract

Left-modularity [2] is a more general concept than modularity in lattice theory. In this paper, we give a characterization of left-modular elements and demonstrate two formulae for the characteristic polynomial of a lattice with such an element, one of which generalizes Stanley's Partial Factorization Theorem. Both formulae provide us with inductive proofs for Blass and Sagan's Total Factorization Theorem for LL lattices. The characteristic polynomials and the Möbius functions of non-crossing partition lattices and shuffle posets are computed as examples.

1 Left-modular elements

Throughout this paper L is a finite lattice where $\hat{0} = \hat{0}_L$ and $\hat{1} = \hat{1}_L$ are the minimal and maximal elements, respectively. We say that x is *covered* by y, and write $x \prec y$, if $x < y$ and there is no element $z \in L$ such that $x < z < y$.

We use $\wedge$ for the meet (greatest lower bound) and $\vee$ for the join (least upper bound) in L. Given any $x, y, z \in L$ with $z < y$, the *modular inequality*

$$z \vee (x \wedge y) \leq (z \vee x) \wedge y \tag{1}$$

is always true and equality holds whenever y or z is comparable to x. We say that x and y form a *modular pair* (x, y) if (1) is an equality for any $z < y$. Note that this relation is not symmetric, in general. Two kinds of elements are associated to the modular pair:

Definition 1.1 1. *An element x is called a* left-modular element *if (x, y) is a modular pair for every $y \in L$.*

2. *An element x is called a* modular element *if both (x, y) and (y, x) are modular pairs for every $y \in L$.*

In a semimodular lattice with rank function ρ, the pair (x, y) is modular if and only if $\rho(x \wedge y) + \rho(x \vee y) = \rho(x) + \rho(y)$ [1, p. 83]; so in this case the relation of being a modular pair is symmetric, and then there is no difference between modularity and left-modularity. However, there are examples such as the non-crossing partition lattices (see Sec. 3) and the Tamari lattices where the two concepts do not coincide.

Let L be a graded lattice of rank n with rank function ρ. Then the *characteristic polynomial* of L is defined by

$$\chi(L, t) = \sum_{x \in L} \mu(x) t^{n - \rho(x)}$$

where t is an indeterminate, $\mu : L \times L \to \mathbb{Z}$ is the Möbius function of L, and $\mu(x) = \mu(\hat{0}, x)$. There are two important factorization theorems for χ given by R. Stanley:

Theorem 1.2 (Partial Factorization Theorem [6]) *Let L be an atomic, semimodular lattice (i.e., a geometric lattice) of rank n. If x is a modular element of L, then*

$$\chi(L, t) = \chi([\hat{0}, x], t) \sum_{b \,:\, b \wedge x = \hat{0}} \mu(b) t^{n - \rho(x) - \rho(b)}. \blacksquare$$

Theorem 1.3 (Total Factorization Theorem [7]) *Let (L, Δ) be a supersolvable, semimodular lattice of rank n with $\Delta : \hat{0} = x_0 \prec x_1 \prec \ldots \prec x_n = \hat{1}$. Then*

$$\chi(L, t) = (t - a_1)(t - a_2) \cdots (t - a_n) \tag{2}$$

where a_i is the number of atoms of L that are below x_i but not below x_{i-1}. ■

Note that all elements in the maximal chain Δ of a supersolvable lattice are left-modular (see [6]). So the hypotheses of Theorem 1.3 imply that they are modular. In recent work [2], A. Blass and B. Sagan generalized the Total Factorization Theorem to LL lattices where the first "L" stands for the fact that the lattice has a maximal chain all of whose elements are all left-modular. The purpose of this paper is to generalize the Partial Factorization Theorem by replacing the modular element with a left-modular one and relaxing the hypotheses requiring that the lattice be atomic and semimodular. To do so, we will derive a general characterization of left-modular elements in this section. In the next section, we introduce a generalized rank function for a lattice which might not be graded in the usual sense, and then develop a general formula for the characteristic polynomial of a lattice with a left-modular element in Theorem 2.3. Under an extra rank-preserving hypothesis we obtain our generalization of the Partial Factorization Theorem (Theorem 2.6). In Sections 3 and 4, we calculate the characteristic polynomials and the Möbius functions of the non-crossing partition lattices and the shuffle posets by using these two formulae, respectively. The last section contains two inductive proofs for Blass and Sagan's Total Factorization Theorem for LL lattices using our two main theorems. Consequently, our factorization theorem generalizes the three others.

We say that y is a *complement* of x if $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. Stanley [6] showed that, in an atomic and semimodular lattice, x is modular if and only if no two complements of x are comparable. The next theorem provides an analog for left-modular elements.

Theorem 1.4 *Let x be an element of any lattice L. The following statements are equivalent:*

i. The element x is left-modular.

ii. For any y, $z \in L$ with $z < y$, we have $x \wedge z \neq x \wedge y$ or $x \vee z \neq x \vee y$.

iii. *For any y, $z \in L$ with $z \prec y$, we have $x \wedge z = x \wedge y$ or $x \vee z = x \vee y$ but not both.*

iv. *For every interval $[a, b]$ containing x, no two complements of x with respect to the sublattice $[a, b]$ are comparable.*

Proof. We will prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). The proof of (ii) $\Leftrightarrow$ (iv) is immediate.

First we make some preliminary observations. Suppose $z < y$. We claim that $x \vee y = x \vee z$ if and only if $y = (z \vee x) \wedge y$. The forward direction is trivial since $(x \vee y) \wedge y = y$. For the reverse, note that $y = (z \vee x) \wedge y$ implies $y \le x \vee z$. Now $z < y \le x \vee z$, and joining all sides with x gives $x \vee y = x \vee z$. Dually $x \wedge y = x \wedge z$ if and only if $z = z \vee (x \wedge y)$.

For any $z < y$ the inequalities

$$z \le z \vee (x \wedge y) \le (z \vee x) \wedge y \le y \tag{3}$$

are true by the modular inequality (1). Since $z \ne y$, at least one of the $\le$'s in (3) should be $<$. Therefore (i) $\Rightarrow$ (ii). If $z \prec y$, then exactly two of the $\le$'s should be $=$ and the remaining one must be $\prec$. Thus (ii) $\Rightarrow$ (iii).

To show (iii) $\Rightarrow$ (i), let us consider the contrapositive: assume that there are u, $v \in L$ with $u < v$ such that $u \vee (x \wedge v) < (u \vee x) \wedge v$. Given any y, $z \in [u \vee (x \wedge v), (u \vee x) \wedge v]$ with $z \prec y$, we have $y \le (u \vee x) \wedge v \le v$. This implies $u \vee (x \wedge y) \le u \vee (x \wedge v) \le z$, so that $x \wedge y \le z$. It follows that $x \wedge z = x \wedge y$. Similarly, we can get $x \vee z = x \vee y$. ■

The existence of a left-modular element in L implies that such elements are also present in certain sublattices as the next proposition shows.

Proposition 1.5 *Let x be a left-modular element in lattice L. Then for any $y \in L$*

1. *the meet $x \wedge y$ is a left-modular element in $[\hat{0}, y]$, and*
2. *the join $x \vee y$ is a left-modular element in $[y, \hat{1}]$.*

Proof. Let a, $b \in [\hat{0}, y]$ with $b < a$. By left-modularity of x, we have

$$\begin{aligned} b \vee ((x \wedge y) \wedge a) &= b \vee (x \wedge (y \wedge a)) = (b \vee x) \wedge (y \wedge a) \\ &= ((b \vee x) \wedge y) \wedge a = (b \vee (x \wedge y)) \wedge a. \end{aligned}$$

So $x \wedge y$ is a left-modular element in $[\hat{0}, y]$. The proof for join is similar. ■

2 The characteristic polynomial

We begin with a general lemma.

Lemma 2.1 *Let L be a lattice with an arbitrary function $r : L \to \mathbb{R}$ and let $n \in \mathbb{R}$. If $x \in L$ is a left-modular element, then*

$$\sum_{y \in L} \mu(y) t^{n-r(y)} = \sum_{b \wedge x = \hat{0}} \mu(b) \sum_{y \in [b, b \vee x]} \mu(b, y) t^{n-r(y)}.$$

Proof. We will mimic Stanley's proof in [6]. By Crapo's Complementation Theorem [3], for any given $a \in [\hat{0}, y]$

$$\mu(y) = \sum_{a', a''} \mu(\hat{0}, a') \zeta(a', a'') \mu(a'', y),$$

where a' and a'' are complements of a in $[\hat{0}, y]$, and ζ is the zeta function defined by $\zeta(u, v) = 1$ if $u \leq v$ and $\zeta(u, v) = 0$ otherwise. Let us choose $a = x \wedge y$. The element a is left-modular in $[\hat{0}, y]$ by Proposition 1.5. But no two complements of a in $[\hat{0}, y]$ are comparable by Theorem 1.4. Thus

$$\mu(y) = \sum_{b} \mu(\hat{0}, b) \mu(b, y), \tag{4}$$

where the sum is over all complements b of a in $[\hat{0}, y]$, i.e., over all b satisfying $b \leq y$, $b \wedge (x \wedge y) = \hat{0}$ and $b \vee (x \wedge y) = y$. Since x is left-modular, it is equivalent to say that the sum in (4) is over all $b \in L$ satisfying $b \wedge x = \hat{0}$ and $y \in [b, b \vee x]$. Thus we have

$$\begin{aligned}
\sum_{y \in L} \mu(y) t^{n-r(y)} &= \sum_{y \in L} \sum_{\substack{b \wedge x = \hat{0} \\ y \in [b, b \vee x]}} \mu(\hat{0}, b) \mu(b, y) t^{n-r(y)} \\
&= \sum_{b \wedge x = \hat{0}} \mu(b) \sum_{y \in [b, b \vee x]} \mu(b, y) t^{n-r(y)}. \blacksquare
\end{aligned}$$

Obviously, the previous lemma is true for the ordinary rank function if L is graded. To apply this result to more general lattices we make the following definition.

Definition 2.2 *A* generalized rank function *of a lattice L is a function $\rho : \{(x,y) \in L \times L \mid x \leq y\} \to \mathbb{R}$ such that for any $a \leq b \leq c$*

$$\rho(a,c) = \rho(a,b) + \rho(b,c).$$

In this case, we say L is generalized graded *by ρ.*

For short we write $\rho(x) = \rho(\hat{0}, x)$. Conversely, if we take any function $\rho : L \to \mathbb{R}$ such that $\rho(\hat{0}) = 0$, then we can easily construct a generalized rank function, namely $\rho(x,y) = \rho(y) - \rho(x)$. So the ordinary rank function is a special case.

If L is generalized graded by ρ, we now define a generalized characteristic polynomial of L by

$$\chi(L,t) = \sum_{x \in L} \mu(x) t^{\rho(x,\hat{1})} = \sum_{x \in L} \mu(x) t^{\rho(\hat{1}) - \rho(x)}. \tag{5}$$

Note that χ will depend on which generalized rank function we pick. Since the restriction of a generalized rank function to an interval $[a,b]$ still satisfies Definition 2.2 with $L = [a,b]$, the characteristic polynomial of the interval is defined in the same manner.

The following theorem, which follows easily from Lemma 2.1, is one of our main results. In it, the *support* of μ is defined by

$$H(L) = \{x \in L \mid \mu(x) \neq 0\}.$$

Theorem 2.3 *Let L be generalized graded by ρ. If $x \in L$ is a left-modular element, then*

$$\chi(L,t) = \sum_{\substack{b \in H(L) \\ b \wedge x = \hat{0}}} \left[\mu(b) t^{\rho(\hat{1}) - \rho(b \vee x)} \chi([b, b \vee x], t) \right]. \blacksquare \tag{6}$$

In the sum (6), the term $\chi([b, b \vee x], t)$ depends on b. To get a factorization formula, we will remove the dependency by applying certain restrictions so that $\chi([b, b \vee x], t) = \chi([\hat{0}, x], t)$ for all b in the sum.

First, we will obtain a general condition under which two lattices have the same characteristic polynomial. In the following discussion, let L and L' be lattices and let $\tau : L \to L'$ be any map. For convenience, we also denote $\hat{0} = \hat{0}_L$, $\hat{0}' = \hat{0}_{L'}$ and similarly for $\hat{1}$, $\hat{1}'$, μ, μ', etc.

We say τ is a *join-preserving* map if

$$\tau(u \vee v) = \tau(u) \vee \tau(v)$$

for any u, $v \in L$. Note that from this definition τ is also order-preserving since

$$x \leq y \;\Rightarrow\; y = x \vee y \;\Rightarrow\; \tau(y) = \tau(x \vee y) = \tau(x) \vee \tau(y) \;\Rightarrow\; \tau(x) \leq \tau(y).$$

If τ is join-preserving, then given any $x' \in \tau(L)$, we claim that the subset $\tau^{-1}(x')$ has a unique maximal element in L. Suppose that $\tau(u) = \tau(v) = x'$ for some u, $v \in L$. We have $\tau(u \vee v) = \tau(u) \vee \tau(v) = x'$. Thus $u \vee v \in \tau^{-1}(x')$ and the claim follows.

If, in addition, τ is surjective then we can define a map $\sigma : L' \to L$ by

$$\sigma(x') = \text{ the maximal element of } \tau^{-1}(x'). \tag{7}$$

The map σ must also be order preserving. To see this, suppose $x' \leq y'$ in L' and consider $x = \sigma(x'), y = \sigma(y')$. Then

$$\tau(x \vee y) = \tau(x) \vee \tau(y) = x' \vee y' = y'.$$

So $x \vee y \in \tau^{-1}(y')$ which forces $x \vee y \leq y$ by definition of σ. Thus $x \leq y$ as desired.

Lemma 2.4 *Using the previous notation, suppose that τ is surjective and join-preserving and that σ satisfies $\sigma(\hat{0}') = \hat{0}$. Then for any $x' \in L'$ we have*

$$\mu'(x') = \sum_{y \in \tau^{-1}(x')} \mu(y).$$

Proof. This is trivial when $x' = \hat{0}'$. Let $x = \sigma(x')$. From the assumptions on τ and σ it is easy to see that

$$[\hat{0}, x] = \biguplus_{y' \in [\hat{0}', x']} \tau^{-1}(y'). \tag{8}$$

Now, by surjectivity of τ and induction, we get

$$\mu'(x') = -\sum_{y' < x'} \mu'(y') = -\sum_{\substack{y \in \tau^{-1}(y') \\ y' < x'}} \mu(y) = \sum_{y \in \tau^{-1}(x')} \mu(y). \blacksquare$$

Let L and L' be generalized graded by ρ and ρ', respectively. We say an order-preserving map $\tau : L \to L'$ is *rank-preserving* on a subset $S \subseteq L$ if $\rho(x, y) = \rho'(\tau(x), \tau(y))$ for any x, $y \in S$, $x \leq y$.

Lemma 2.5 *If, in addition to the hypotheses of Lemma 2.4, the map τ is rank-preserving on $H(L) \cup \{\hat{1}\}$ then*

$$\chi(L, t) = \chi(L', t).$$

Proof. From (8) in the proof of Lemma 2.4, we know $L = \biguplus_{x' \in L'} \tau^{-1}(x')$. Then by Lemma 2.4 and the rank-preserving nature of τ, we have

$$\begin{aligned}
\chi(L', t) &= \sum_{x' \in L'} \mu'(x') t^{\rho'(x', \hat{1}')} \\
&= \sum_{x' \in L'} \sum_{y \in \tau^{-1}(x')} \mu(y) t^{\rho'(x', \hat{1}')} \\
&= \sum_{x' \in L'} \sum_{y \in \tau^{-1}(x') \cap H(L)} \mu(y) t^{\rho'(\tau(y), \tau(\hat{1}))} \\
&= \sum_{y \in H(L)} \mu(y) t^{\rho(y, \hat{1})} \\
&= \chi(L, t). \blacksquare
\end{aligned}$$

It is easy to generalize the previous lemma to arbitrary posets as long as the map σ is well defined. However, we know of no application of the result in this level of generality.

Returning to our factorization theorem, we still need one more tool. For any given a, b in a lattice, we define

$$\sigma_a : [b, a \vee b] \to [a \wedge b, a] \quad \text{by} \quad \sigma_a(u) = u \wedge a,$$

$$\tau_b : [a \wedge b, a] \to [b, a \vee b] \quad \text{by} \quad \tau_b(v) = v \vee b.$$

The map τ_b is the one we need to achieve $\chi([b, b \vee x], t) = \chi([\hat{0}, x], t)$. In the following, we write $H(x, y)$ for $H([x, y])$ which is the support of μ defined on the sublattice $[x, y]$. We can now prove our second main result.

Theorem 2.6 *Let L be generalized graded by ρ and let $x \in L$ be an left-modular element. If the map τ_b is rank-preserving on $H(\hat{0}, x) \cup \{x\}$ for every $b \in H(L)$ satisfying $b \wedge x = \hat{0}$. Then*

$$\chi(L, t) = \chi([\hat{0}, x], t) \sum_{\substack{b \in H(L) \\ b \wedge x = \hat{0}}} \mu(b) t^{\rho(\hat{1}) - \rho(x) - \rho(b)}. \tag{9}$$

Proof. First, we will show that $\chi([b, b\vee x], t) = \chi([\hat{0}, x], t)$ for any $b \in H(L)$ with $b \wedge x = \hat{0}$ by verifying the hypotheses of Lemma 2.5. By left-modularity of x, we have

$$\tau_b \sigma_x(y) = b \vee (x \wedge y) = (b \vee x) \wedge y = y \tag{10}$$

for any $y \in [b, b \vee x]$. So τ_b is surjective. And it is easy to check that τ_b is join-preserving. As for σ_x, we must check that it satisfies the definition (7). Given $z \in \tau_b^{-1}(y)$ we have $y = \tau_b(z) = z \vee b$. So by the modular inequality (1) we get

$$\sigma_x(y) = y \wedge x = (z \vee b) \wedge x \geq z \vee (b \wedge x) \geq z.$$

Since this is true for any such z, we have $\sigma_x(y) \geq \max \tau_b^{-1}(y)$. But equation (10) implies $\sigma_x(y) \in \tau_b^{-1}(y)$, so we have equality. Finally $\hat{0}_{[b, b\vee x]} = b$ so $\sigma_x(b) = b \wedge x = \hat{0}$ as desired.

Now we need only worry about the exponent on t in Theorem 2.3. But since τ_b is rank-preserving on $H(\hat{0}, x) \cup \{x\}$, we get

$$\rho(b \vee x) = \rho(\hat{0}, b) + \rho(b, b \vee x) = \rho(\hat{0}, b) + \rho(\hat{0}, x) = \rho(b) + \rho(x). \blacksquare$$

Here we state a corollary which relaxes the hypothesis in Stanley's Partial Factorization Theorem.

Corollary 2.7 *Equation (9) holds when L is a semimodular lattice (graded by the ordinary rank function) with a modular element x.*

Proof. To apply Theorem 2.6, it suffices to show that $\rho(\hat{0}, z) = \rho(b, z\vee b)$ for every $z \in [\hat{0}, x]$. Since (b, x) is a modular pair, we have $(z\vee b)\wedge x = z\vee(b\wedge x) = z \vee \hat{0} = z$. By Proposition 1.5, $z = (z \vee b) \wedge x$ is left-modular in $[\hat{0}, z \vee b]$, so (z, b) is a modular pair in this lattice. Thus $\rho(z \wedge b) + \rho(z \vee b) = \rho(z) + \rho(b)$, because $[\hat{0}, z \vee b]$ is a semimodular lattice. Since $z \wedge b = \hat{0}$ we are done. $\blacksquare$

We take the divisor lattice D_n as an example. It is semimodular, but not atomic in general, so Stanley's theorem does not apply. However, Corollary 2.7 can be used for any $x \in D_n$, since all elements are modular.

We will now present a couple of applications of the previous results in the following two sections.

3 Non-crossing Partition Lattices

The non-crossing partition lattice was first studied by Kreweras [5] who showed its Möbius function is related to the Catalan numbers. By using

NBB sets (see Sec. 5 for the definition), Blass and Sagan [2] combinatorially explained this fact. In this section we will calculate the characteristic polynomial for a non-crossing partition lattice and then offer another explanation for the value of its Möbius function.

If it causes no confusion, we will not explicitly write out any blocks of a partition that are singletons. Let $n \geq 1$. We say that a partition $\pi \vdash [n]$ is *non-crossing* if there do not exist two distinct blocks B, C of π with $i,\, k \in B$ and $j,\, l \in C$ such that $i < j < k < l$. Otherwise π is *crossing*.

Another way to view non-crossing partitions will be useful. Let $G = (V, E)$ be a graph with vertex set $V = [n]$ and edge set E. We say that G is *non-crossing* if, when the vertices are arranged in their natural order clockwise around a circle and the edges are drawn as straight line segments, no two edges of G cross geometrically. Given a partition π we can form a graph G_π by representing each block $B = \{i_1 < i_2 < \ldots < i_l\}$ by a cycle with edges $i_1i_2, i_2i_3, \ldots, i_li_1$. (If $|B| = 1$ or 2 then B is represented by an isolated vertex or edge, respectively.) Then it is easy to see that π is non-crossing as a partition if and only if G_π is non-crossing as a graph.

The set of non-crossing partitions of $[n]$, denoted by NC_n, forms a meet-sublattice of partition lattice Π_n with the same rank function. However unlike Π_n, the non-crossing partition lattice is not semimodular in general, since if $\pi = 13$ and $\sigma = 24$ then $\pi \wedge \sigma = \hat{0}$ and $\pi \vee \sigma = 1234$. So we have

$$\rho(\pi) + \rho(\sigma) = 2 < 3 = \rho(\pi \wedge \sigma) + \rho(\pi \vee \sigma).$$

The Π_n-join $\pi \vee \sigma = 13/24$ also explains why NC_n is not a sublattice of Π_n.

Let $n \geq 2$ and $\pi = 12\ldots(n-1)$. It is well-known [7] that π is modular in Π_n and so left-modular there. Given any $\alpha,\, \beta \in NC_n$ with $\alpha < \beta$ and both incomparable to π. It is clear that $\alpha \vee \pi = \beta \vee \pi = \hat{1}$ in Π_n as well as in NC_n. By Theorem 1.4 we get $\alpha \wedge \pi < \beta \wedge \pi$ in Π_n. Since NC_n is a meet-sublattice of Π_n, this inequality for the two meets still holds in NC_n. This fact implies that π is left-modular in NC_n. In general, π is not modular in NC_n. If $n \geq 4$, let $\sigma = 2n$ and $\phi = 1(n-1)/23\ldots(n-2)$. Clearly $\phi < \pi$, $\pi \wedge \sigma = \phi \wedge \sigma = \hat{0}$ and $\pi \vee \sigma = \phi \vee \sigma = \hat{1}$ in NC_n, so that (σ, π) is not a modular pair.

Proposition 3.1 *The characteristic polynomial of the non-crossing partition lattice NC_n satisfies*

$$\chi(NC_n, t) = t\ \chi(NC_{n-1}, t) - \sum_{i=1}^{n-1} \chi(NC_i, t)\chi(NC_{n-i}, t)$$

with the initial condition $\chi(NC_1, t) = 1$.

Proof. The initial condition is trivial. Let $n \geq 2$ and $\pi = 12\ldots(n-1)$. We will apply Theorem 2.3. Note that $b \wedge \pi = \hat{0}$ if and only if any two numbers of $[n-1]$ are in different blocks of b, so either $b = \hat{0}$ or $b = mn$ with $1 \leq m \leq n-1$.

If $b = \hat{0}$, then $\chi([b, b \vee \pi], t) = \chi([\hat{0}, \pi], t) = \chi(NC_{n-1}, t)$. Thus we get the first term of the formula. Now let $b = mn$. It is clear that $b \vee \pi = \hat{1}$, so we need to consider the sublattice $[b, \hat{1}]$. Given any $\omega \in [b, \hat{1}]$, the edge mn (which may not be in $E(G_\omega)$) geometrically separates the graph G_ω into two parts, $G_{\omega,1}$ and $G_{\omega,2}$, which are induced by vertex sets $\{1, 2, \ldots, m, n\}$ and $\{m, m+1, \ldots, n-1, n\}$, respectively. By contracting the vertices m and n in both $G_{\omega,1}$ and $G_{\omega,2}$, we get two non-crossing graphs $\bar{G}_{\omega,1}$ and $\bar{G}_{\omega,2}$. It is easy to check that the map $f : [b, \hat{1}] \to NC_m \times NC_{n-m}$ defined by $f(G_\omega) = (\bar{G}_{\omega,1}, \bar{G}_{\omega,2})$ is an isomorphism between these two lattices. Therefore

$$\chi([b, b \vee \pi], t) = \chi(NC_m, t)\chi(NC_{n-m}, t),$$

and the proof is complete. ■

For any $\omega = B_1/B_2/\ldots/B_k \in NC_n$, the interval $[\hat{0}, \omega] \cong \prod_i NC_{|B_i|}$. Hence to compute the Möbius function of NC_n, it suffices to do this only for $\hat{1}$. By Proposition 3.1 we have the recurrence relation

$$\begin{aligned}
\mu(NC_n) &= \chi(NC_n, 0) \\
&= -\sum_{i=1}^{n-1} \chi(NC_i, 0)\chi(NC_{n-i}, 0) \\
&= -\sum_{i=1}^{n-1} \mu(NC_i)\mu(NC_{n-i})
\end{aligned}$$

with the initial condition $\mu(NC_1) = 1$. Recall that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ satisfy the recurrence relation

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$$

with the initial condition $C_0 = 1$. Therefore, by induction, we obtain Krew-eras' result that

$$\mu(NC_n) = (-1)^{n-1} C_{n-1}.$$

4 Shuffle Posets

The poset of shuffles was introduced by Greene [4], and he obtained a formula for its characteristic polynomial

$$\chi(\mathcal{W}_{m,n}, t) = (t-1)^{m+n} \sum_{i \geq 0} \binom{m}{i} \binom{n}{i} \frac{1}{(1-t)^i}.$$

In this section we will derive an equivalent formula by using Theorem 2.6. Before doing this, we need to recall some definitions and results of Greene. Let $\mathcal{A}$ be a set, called the *alphabet of letters*. A *word* over $\mathcal{A}$ is a sequence $\mathbf{u} = u_1 u_2 \ldots u_n$ of distinct letters of $\mathcal{A}$. We will sometimes also use $\mathbf{u}$ to stand for the set of letters in the word, depending upon the context. A *subword* of $\mathbf{u}$ is $\mathbf{w} = u_{i_1} \ldots u_{i_l}$ where $i_1 < \ldots < i_l$. If $\mathbf{u}$, $\mathbf{v}$ are any two words then the *restriction* of $\mathbf{u}$ to $\mathbf{v}$ is the subword $\mathbf{u_v}$ of $\mathbf{u}$ whose letters are exactly those of $\mathbf{u} \cap \mathbf{v}$. A *shuffle* of $\mathbf{u}$ and $\mathbf{v}$ is any word $\mathbf{s}$ such that $\mathbf{s} = \mathbf{u} \uplus \mathbf{v}$ as sets and $\mathbf{s_u} = \mathbf{u}$, $\mathbf{s_v} = \mathbf{v}$ as words.

Given nonnegative integers m and n, fix disjoint words $\mathbf{x} = x_1 \ldots x_m$ and $\mathbf{y} = y_1 \ldots y_n$. The *poset of shuffles* $\mathcal{W}_{m,n}$ consists all shuffles $\mathbf{w}$ of a subword of $\mathbf{x}$ with a subword of $\mathbf{y}$ while the partial order is that $\mathbf{v} \leq \mathbf{w}$ if $\mathbf{v_x} \supseteq \mathbf{w_x}$, $\mathbf{v_y} \subseteq \mathbf{w_y}$ as sets and $\mathbf{v_w} = \mathbf{w_v}$ as words. The covering relation is more intuitive: $\mathbf{v} \prec \mathbf{w}$ if $\mathbf{w}$ can be obtained from $\mathbf{v}$ by either adding a single y_i or deleting a single x_j. It is easy to see that $\mathcal{W}_{m,n}$ has $\hat{0} = \mathbf{x}$, $\hat{1} = \mathbf{y}$, and is graded by the rank function

$$\rho(\mathbf{w}) = (m - |\mathbf{w_x}|) + |\mathbf{w_y}|.$$

For example, $\mathcal{W}_{2,1}$ is shown in Figure 1 where $\mathbf{x} = de$ and $\mathbf{y} = D$.

Every shuffle poset is actually a lattice. To describe the join operation in $\mathcal{W}_{m,n}$, Greene defined crossed letters as follows. Given $\mathbf{u}$, $\mathbf{v} \in \mathcal{W}_{m,n}$ then $x \in \mathbf{u} \cap \mathbf{v} \cap \mathbf{x}$ is *crossed* in $\mathbf{u}$ and $\mathbf{v}$ if there exist letters y_i, $y_j \in \mathbf{y}$ with $i \leq j$ and x appears before y_i in one of the two words but after y_j in the other. For example, let $\mathbf{x} = def$ and $\mathbf{y} = DEF$. Then in the two shuffles $\mathbf{u} = dDEe$, $\mathbf{v} = Fdef$, the only crossed letter is d. The join of $\mathbf{u}$, $\mathbf{v}$ is then the unique word $\mathbf{w}$ greater than both $\mathbf{u}$, $\mathbf{v}$ such that

$$\begin{aligned} \mathbf{w_x} &= \{x \in \mathbf{u_x} \cap \mathbf{v_x} \mid x \text{ is not crossed}\} \\ \mathbf{w_y} &= \mathbf{u_y} \cup \mathbf{v_y}. \end{aligned}$$

In the previous example, $\mathbf{u} \vee \mathbf{v} = DEFe$. This join also shows that $\mathcal{W}_{m,n}$ is not semimodular in general, because $\rho(\mathbf{u}) + \rho(\mathbf{v}) = 3 + 1 < 5 = \rho(\mathbf{u} \vee \mathbf{v}) \leq$

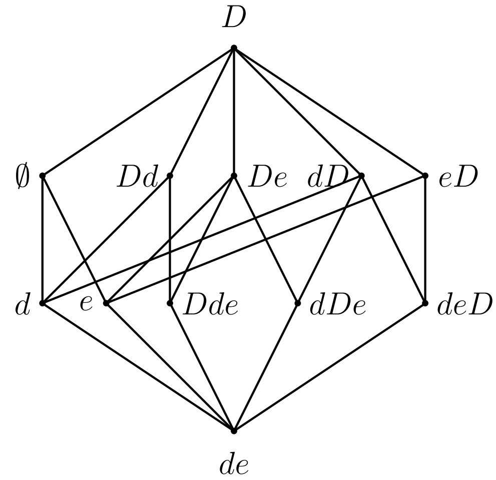

Figure 1: The lattice $\mathcal{W}_{2,1}$

$\rho(\mathbf{u} \vee \mathbf{v}) + \rho(\mathbf{u} \wedge \mathbf{v})$. Since $(\mathcal{W}_{n,m})^* = \mathcal{W}_{m,n}$, the meet operation in $\mathcal{W}_{m,n}$ is as same as the join operation in $(\mathcal{W}_{n,m})^*$. So to find the meet in the analogous way we need to consider those letter $y \in \mathbf{u} \cap \mathbf{v} \cap \mathbf{y}$ crossed in $\mathbf{u}$ and $\mathbf{v}$.

Greene also showed that subwords of $\mathbf{x}$ and subwords of $\mathbf{y}$ are modular elements of $\mathcal{W}_{m,n}$. In particular, the empty set $\emptyset$ is modular. Also note that $[\hat{0}, \emptyset] \cong B_m$. We now give our formula for the characteristic polynomial of $\mathcal{W}_{m,n}$.

Proposition 4.1 *The characteristic polynomial of the shuffle poset is*

$$\chi(\mathcal{W}_{m,n}, t) = (t-1)^m \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{m+i}{i} t^{n-i}. \tag{11}$$

Proof. Consider any $\mathbf{u}$ with $\mathbf{u} \wedge \emptyset = \hat{0}$. In general, if $\mathbf{u} \wedge \emptyset = \mathbf{w}$ then $\mathbf{w_x} = \mathbf{u_x} \cup \emptyset_{\mathbf{x}} = \mathbf{u_x}$. So $\mathbf{u} \wedge \emptyset = \hat{0}$ if and only if $\mathbf{x}$ is a subword of $\mathbf{u}$, i.e., the element $\mathbf{u}$ is a shuffle of $\mathbf{x}$ with a subword of $\mathbf{y}$. Furthermore, for any $\mathbf{v} \in [\hat{0}, \emptyset]$, there is no crossed letter x in $\mathbf{u}$ and $\mathbf{v}$ since $\mathbf{v_y} = \emptyset$. It follows that $(\mathbf{u} \vee \mathbf{v})_{\mathbf{x}} = \mathbf{u_x} \cap \mathbf{v_x} = \mathbf{v}$ and $(\mathbf{u} \vee \mathbf{v})_{\mathbf{y}} = \mathbf{u_y} \cup \mathbf{v_y} = \mathbf{u_y}$ as sets. Then we get

$$\begin{aligned} \rho(\mathbf{u} \vee \mathbf{v}) - \rho(\mathbf{u}) &= [(m - |\mathbf{v}|) + |\mathbf{u_y}|] - [(m-m) + |\mathbf{u_y}|] \\ &= m - |\mathbf{v}| \;=\; \rho(\mathbf{v}) - \rho(\hat{0}). \end{aligned}$$

Thus the map $\tau_{\mathbf{u}} : [\hat{0}, \emptyset] \to [\mathbf{u}, \emptyset \vee \mathbf{u}]$ is rank-preserving.

Since $[\hat{0}, \emptyset] \cong B_m$ we get, by Theorem 2.6,

$$\chi(\mathcal{W}_{m,n}, t) = (t-1)^m \sum_{\mathbf{u} \wedge \emptyset = \hat{0}} \mu(\mathbf{u}) t^{(m+n)-\rho(\mathbf{u})-m}.$$

It is easy to see that the interval $[\hat{0}, \mathbf{u}]$ is isomorphic to B_i where $i = |\mathbf{u}_\mathbf{y}|$. So $\mu(\mathbf{u}) = (-1)^{|\mathbf{u}_\mathbf{y}|} = (-1)^{\rho(\mathbf{u})}$. Now we conclude that

$$\begin{aligned} \chi(\mathcal{W}_{m,n}, t) &= (t-1)^m \sum_{i=0}^{n} \begin{bmatrix} \text{the number of ways to} \\ \text{shuffle } \mathbf{x} \text{ with } i \text{ letters of } \mathbf{y} \end{bmatrix} (-1)^i t^{n-i} \\ &= (t-1)^m \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{m+i}{i} t^{n-i}. \ \blacksquare \end{aligned}$$

To determine the Möbius function of $\mathcal{W}_{m,n}$, it suffices to compute $\mu(\hat{1})$ since for any $\mathbf{w} \in \mathcal{W}_{m,n}$ the interval $[\hat{0}, \mathbf{w}]$ is isomorphic to a product of $\mathcal{W}_{p,q}$'s for certain $p \leq m$ and $q \leq n$. Simply plugging $t = 0$ into formula (11) gives us the Möbius function $\mu(\mathcal{W}_{m,n})$.

Corollary 4.2 (Greene, [4]) *We have*

$$\mu(\mathcal{W}_{m,n}) = (-1)^{m+n} \binom{m+n}{n}. \ \blacksquare$$

5 NBB Sets and Factorization Theorems

Blass and Sagan [2] derived a Total Factorization Theorem for LL lattices which generalizes Theorem 1.3. Applying Theorem 2.3 and 2.6, respectively, we will offer two inductive proofs for their theorem. First of all, we would like to outline their work.

Given a lattice L, let $A = A(L)$ is the set of atoms of L. Let $\trianglelefteq$ be an arbitrary partial order on A. A nonempty set $D \subseteq A$ is *bounded below* or *BB* if, for every $d \in D$ there is an $a \in A$ such that

$$a \lhd d \qquad \text{and} \qquad a < \bigvee D.$$

A set $B \subseteq A$ is called *NBB* (*no bounded below* subset) if it does not contain any D which is bounded below. An NBB set is said to be a base for its join. One of the main results of Blass and Sagan's paper is the following theorem which is a simultaneous generalization of both Rota's NBC and Crosscut Theorems (for the crosscut $A(L)$).

Theorem 5.1 (Blass and Sagan, [2]) *Let L be a finite lattice and let $\trianglelefteq$ be any partial order on A. Then for all $x \in L$ we have*

$$\mu(x) = \sum_B (-1)^{|B|}$$

where the sum is over all NBB bases B of x. ■

Given an arbitrary lattice L, let $\Delta : \hat{0} = x_0 \prec x_1 \prec \ldots \prec x_n = \hat{1}$ be a maximal chain of L. The i^{th} *level* of A is defined by

$$A_i = \{a \in A \mid a \leq x_i \text{ but } a \not\leq x_{i-1}\},$$

and we partially order A by setting $a \lhd b$ if and only if $a \in A_i$ and $b \in A_j$ with $i < j$. We say a is in *lower level* than b or b is in *higher level* than a if $a \lhd b$. Note that the level A_i is an empty set if and only if x_i is not an atomic element. A pair (L, Δ) is said to satisfy the *level condition* if this partial order $\trianglelefteq$ of A has the following property.

$$\text{If } a \lhd b_1 \lhd b_2 \lhd \ldots \lhd b_k \text{ then } a \not\leq \bigvee_{i=1}^{k} b_i.$$

If all elements of Δ are left-modular, then we say (L, Δ) is a *left-modular* lattice. A pair (L, Δ) is called an *LL lattice* if it is left-modular and satisfies the level condition.

A generalized rank function $\rho : L \to \mathbb{N}$ is defined by

$$\rho(x) = \text{ number of } A_i \text{ containing atoms less than or equal to } x.$$

Note that, for any $x \in L$, we have $\rho(x) = \rho(\delta(x))$ where $\delta(x)$ is the maximum atomic element in $[\hat{0}, x]$. So $\rho(\hat{1})$ is not necessary equal to n, the length of Δ.

In the following we list several properties in [2] that we need.

(A) If a and b are distinct atoms from the same level A_i in a left-modular lattice, then $a \vee b$ is above some atom $c \in A_j$ with $j < i$.

(B) In an LL lattice, a set $B \subseteq A$ is NBB if and only if $|B \cap A_i| \leq 1$ for every i.

(C) Let B be an NBB set in an LL lattice. Then every atom $a \leq \bigvee B$ is in the same level as some element of B. In particular, any NBB base for x has exactly $\rho(x)$ atoms.

Blass and Sagan generalized Stanley's Total Factorization Theorem to LL lattices using their theory of NBB sets. Here we present two inductive proofs for their theorem. In the first proof we will apply Theorems 2.6 as well as the theory of NBB sets.

Theorem 5.2 (Blass and Sagan, [2]) *If (L, Δ) is an LL lattice then its characteristic polynomial factors as*

$$\chi(L,t) = \prod (t - |A_i|)$$

where the product is over all non-empty levels A_i.

Proof of Theorem 5.2 I. We will induct on n, the length of Δ. The theorem is trivial when $n \leq 1$. If $A_n = \emptyset$, then $\rho(x_n) = \rho(x_{n-1})$ and $\mu(x) = 0$ for $x \not\leq x_{n-1}$. Thus $\chi(L,t) = \chi([\hat{0}, x_{n-1}], t)$, so we are done by induction.

If $A_n \neq \emptyset$, consider $b \in H(L)$. Then, by Theorem 5.1, b must have an NBB base, say B. In addition, if $b \wedge x_{n-1} = \hat{0}$ then $B \subseteq A_n$ and also $|B \cap A_n| \leq 1$ by (B). So $b = \hat{0}$ or $b \in A_n$. Now it suffices to check that τ_b is rank-preserving on $H(\hat{0}, x_{n-1}) \cup \{x_{n-1}\}$ for every $b \in A_n$ since then we get $\chi(L,t) = \chi([\hat{0}, x_{n-1}], t)(t - |A_n|)$ by Theorem 2.6. Because $A_n \neq \emptyset$ and $\rho(b) = 1$, τ_b is rank-preserving on $\{x_{n-1}\}$. Given any $y \in H(\hat{0}, x_{n-1})$, suppose B be an NBB base for y. By (B), $B' = B \cup \{b\}$ is an NBB base for $\tau_b(y)$. Now $\rho(\tau_b(y)) = |B'| = |B| + 1 = \rho(y) + \rho(b)$ by (C). Hence $\rho(b, \tau_b(y)) = \rho(\tau_b(y)) - \rho(b) = \rho(y) = \rho(\hat{0}, y)$. ■

In a similar way, Corollary 2.7 provides us with an inductive proof for Theorem 1.3. Note that the lattice in Theorem 1.3 is graded, so $\rho(\hat{1})$ equals the length of Δ. Therefore the product (2) is over all levels A_i (including empty ones).

We will use Theorem 2.3 for the second proof. This demonstration sidesteps the machinery of NBB sets and reveals some properties of LL lattices in the process. To prepare, we need the following two lemmas.

Lemma 5.3 *If w is a left-modular element in L and $v \prec w$, then $v \vee u \preceq w \vee u$ for any $u \in L$.*

Proof. Suppose not and then there exists $s \in L$ such that $v \vee u < s < w \vee u$. Taking the join with w and using $v \vee w = w$, we get $w \vee (v \vee u) = w \vee s = w \vee (w \vee u)$. So we should have $w \wedge (v \vee u) < w \wedge s < w \wedge (w \vee u) = w$ by Theorem 1.4. Combining this with $v \leq w \wedge (v \vee u)$, we have a contradiction to $v \prec w$. ■

Lemma 5.4 *If (L, Δ) is an LL lattice with $\Delta : \hat{0} = x_0 \prec x_1 \prec \ldots \prec x_n = \hat{1}$ and $A_n \neq \emptyset$, then $([b, \hat{1}], \Delta')$ is also an LL lattice for any $b \in A_n$ where Δ' consists of the distinct elements of the multichain*

$$b = x'_0 \preceq x'_1 \preceq x'_2 \preceq \ldots \preceq x'_{n-2} \preceq x'_{n-1} = \hat{1}$$

where $x'_i = x_i \vee b, 0 \leq i \leq n-1$. Furthermore we have $|A_i| = |A'_i|$ for such i, where

$$A'_i = \{a \in A(b, \hat{1}) \mid a \leq x'_i \text{ but } a \not\leq x'_{i-1}\}.$$

Proof. By Lemma 5.3, the chain Δ' is indeed saturated. So Δ' is a left-modular maximal chain by Proposition 1.5.

Let $\tau(x) = \tau_b(x) = x \vee b$. This map is surjective (see the proof of Theorem 2.6) and order-preserving from $[\hat{0}, x_{n-1}]$ to $[b, \hat{1}]$. Also let $A = A(\hat{0}, x_{n-1})$ and $A' = A(b, \hat{1})$. First, We prove that the map $\tau : A \to A'$ is well-defined and bijective. Suppose that there is an $a \in A_i$ such that $b \prec x < \tau(a) = a \vee b$ for some x. By the level condition, any atom $c \leq a \vee b$ is in a level at least as high as a; furthermore, if $c \in A_i$ we must have $c = a$ because of (A). Since $x < a \vee b$ and $a \not\leq x$, any atom $d \leq x$ is in a higher level than a. It follows that $x_i \wedge x = \hat{0}$. Now $b \vee (x_i \wedge x) = b$ and $(b \vee x_i) \wedge x \geq (b \vee a) \wedge x = x$ contradicts the left-modularity of x_i. We conclude that $\tau : A \to A'$ is well-defined.

The restriction $\tau|_A$ is surjective since τ is surjective and order-preserving. To show injectivity of $\tau|_A$, let us suppose there are two distinct atoms u and v such that $\tau(u) = \tau(v)$. If u and v are from two different levels then this contradicts the level condition. If u and v are from the same level, by (A), there exists an atom c in a lower level such that $c \leq u \vee v \leq \tau(u) \vee \tau(v) = \tau(u)$, contradicting the level condition again.

Now let us prove $|A_i| = |A'_i|$. This is trivial for $i = 1$. Let $u \in A_i$ for some nonempty A_i with $2 \leq i \leq n-1$. It is clear that $\tau(u) \leq x'_i$. Suppose that $\tau(u) \leq x'_{i-1}$, i.e., $u \vee b \leq x_{i-1} \vee b$. By the level condition, $b \vee (x_{i-1} \wedge (u \vee b)) = b \vee \hat{0} = b$. But $(b \vee x_{i-1}) \wedge (u \vee b) = u \vee b > b$ contradicts the modularity of x_{i-1}. Thus $\tau(A_i) \subseteq A'_i$ and then the bijectivity of $\tau|_A$ implies that $|A_i| = |A'_i|$ for all $i \leq n-1$.

Since $\tau|_A$ is bijective and level-preserving, if $\tau(a) \leq \bigvee_{i=1}^{k} \tau(b_i)$ for some $\tau(a) \lhd \tau(b_1) \lhd \tau(b_2) \lhd \ldots \lhd \tau(b_k)$ in $[b, \hat{1}]$, then $a < a \vee b \leq (\bigvee_{i=1}^{k} b_i) \vee b$ with $a \lhd b_1 \lhd b_2 \lhd \ldots \lhd b_k \lhd b$ in L. Therefore $([b, \hat{1}], \Delta')$ satisfies the level condition. ∎

Proof of Theorem 5.2 II. We will induct on $n = \ell(\Delta)$. The cases $n \leq 1$ and $A_n = \emptyset$ are handled as before.

If $A_n \neq \emptyset$, consider $b \in H(L)$ with $b \wedge x_{n-1} = \hat{0}$. Then b is atomic and can only be above atoms in A_n. So by (A), b must be the join of at most one atom, i.e., either $b = \hat{0}$ or $b \in A_n$. Thus by Lemma 5.4 and induction we get, for any $b \in A_n$,

$$\chi([b, \hat{1}], t) = \prod_{i \leq n-1} (t - |A'_i|) = \prod_{i \leq n-1} (t - |A_i|) = \chi([\hat{0}, x_{n-1}], t)$$

where the product is over all non-empty A_i. Applying Theorem 2.3 gives $\chi(L, t) = \chi([\hat{0}, x_{n-1}], t)(t - |A_n|)$, so again we are done. ■

References

[1] G. Birkhoff, "Lattice Theory", Third Edition, American Mathematical Society, 1967.

[2] A. Blass and B. E. Sagan, Möbius Functions of Lattices, *Adv. in Math.* **127** (1997), no. 1, 94–123.

[3] H. Crapo, The Möbius function of a lattice, *J. Combin. Theory* **1** (1966), 126–131.

[4] C. Greene, Posets of Shuffles, *J. Combin. Theory Ser. A* **47** (1988), 191–206.

[5] G. Kreweras, Sur les partitions non-croisées d'un cycle, *Discrete Math.* **1** (1972), 333–350.

[6] R. P. Stanley, Modular Elements of Geometric Lattices, *Alg. Univ.* **1** (1971), 214–217.

[7] R. P. Stanley, Supersolvable Lattices, *Alg. Univ.* **2** (1972), 197–217.