A_∞-STRUCTURES ON AN ELLIPTIC CURVE

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Abstract. The main result of this paper is the proof of the "transversal part" of the homological mirror symmetry conjecture for an elliptic curve which states an equivalence of two A_∞-structures on the category of vector bundles on an elliptic curves. The proof is based on the study of A_∞-structures on the category of line bundles over an elliptic curve satisfying some natural restrictions (in particular, m_1 should be zero, m_2 should coincide with the usual composition). The key observation is that such a structure is uniquely determined up to homotopy by certain triple products.

Introduction

Let E be an elliptic curve over a field k. Let us denote by $\mathrm{Vect}(E)$ the category of algebraic vector bundles on E, where as space of morphisms from V_1 to V_2 we take the graded space $\mathrm{Hom}(V_1,V_2)\oplus\mathrm{Ext}^1(V_1,V_2)$ with the natural composition law. In this paper we study extensions of this (strictly associative) composition to A_∞-structures on $\mathrm{Vect}(E)$ (see section 1 for the definition). The motivation comes from the homological mirror symmetry for elliptic curves formulated by Kontsevich (see [9]) which provides two such extensions in the case $k=\mathbb{C}$ and states that they should be equivalent. We recall the definitions of these A_∞-structures in section 1.5. One of them is an A_∞-version of the derived category of vector bundles while another comes from a general construction in symplectic geometry due to Fukaya. Roughly speaking, one can associate to an indecomposable vector bundle on E a geodesic circle on the torus $\mathbb{R}^2/\mathbb{Z}^2$ with a local system on it. Then the second A_∞-structure is defined using generating series counting holomorphic maps from the disk bounding given geodesic circles.

In [16] we checked that the double products defined in this way coincide with the standard composition law on $\mathrm{Vect}(E)$. In this paper we use this together with some calculations of triple products (see [14]) to prove the essential part of the homological mirror conjecture for E. Namely, we construct a homotopy between *transversal* products given by these two A_∞-structures. This means that we are looking only at the products such that the corresponding geodesic circles form a transversal configuration. The advantage is that in this case the homotopy can be constructed in a canonical way. We leave to a future investigation more subtle points of defining non-transversal products in the Fukaya category and extending the above homotopy to the entire derived categories.

Note that the equality of double products and triple Massey products in A_∞-categories corresponding to a mirror dual pair (symplectic torus, abelian variety) was established by Fukaya in [3]. In the case of elliptic curves, as we show in the present paper, this is enough for (transversal part of) the homological mirror conjecture. For abelian varieties of higher dimensions, a version of this conjecture was recently proved by Kontsevich and Soibelman in [11]. [1] The main point of our paper is that in the case of elliptic curves we can formulate a result on A_∞-structures on the category $\mathcal{L}$ of line bundles on E which is valid over an arbitrary field k. More precisely, we axiomatize the notion of transversality and prove that if one imposes some natural restrictions on a transversal A_∞-structure on $\mathcal{L}$ (in particular $m_1=0$, m_2 is equal to the standard composition), then such a structure is uniquely determined (up to homotopy) by certain

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[1]The A_∞-equivalence established in [11] deals with certain full subcategories in symplectic and holomorphic A_∞-categories. In fact, both sides are slightly modified: Fukaya category is replaced by its degeneration, while on the holomorphic side the ground field is changed from $\mathbb{C}$ to $\mathbb{C}((q))$. Also, only transversal products are considered.

triple products. Namely, these are triple products which are invariant under any homotopy. We apply this result to two A_∞-structures arising in the homological mirror symmetry and then use the isogenies between elliptic curves (as in [16]) to construct the required homotopy on the category of vector bundles on E.

The natural framework for the generalization of our result which is valid over any field k should involve the notion of a triangulated A_∞-category (as sketched in [10]). Our result seems to imply that there exists a unique up to homotopy triangulated A_∞-structure on the derived category of an elliptic curve which is compatible with the standard products and with Serre duality (see section 1.3 for the definition of the latter compatibility). Indeed, the triple products appearing in our statement are univalued Massey products which are uniquely determined by the double products in the case when A_∞-structure is triangulated. One may hope that such a uniqueness of A_∞-structure on the derived category holds for other varieties (e.g. for abelian varieties of arbitrary dimension). The main reason why in the case of A_∞-structures on elliptic curve only triple products matter is the absense of non-trivial univalued well-defined k-tuple Massey products for $k > 3$ [2].

Conventions: We always work over a ground field k; we specialize to $k = \mathbb{C}$ when talking about homological mirror symmetry. To shorten the formulas sometimes we denote the tensor product of vector spaces V_1 and V_2 over k simply by V_1V_2 omitting the sign of the tensor product. We use the same abbreviation for tensor products of vector bundles. By a *bundle* we always mean an algebraic vector bundle (or a holomorphic vector bundle if $k = \mathbb{C}$). When working with A_∞-categories it is convenient to denote the n-tuple products of composable morphisms $a_1 : X_0 \to X_1$, $a_2 : X_1 \to X_2$, ..., $a_n : X_{n-1} \to X_n$ by $m_n(a_1, a_2, \dots, a_n)$. In particular, we denote the double composition by $m_2(a_1, a_2)$ which we often abbreviate to a_1a_2. This contradicts to the usual convention of going from right to left when considering composition in the usual categories. To avoid confusion we will use the notation $a_2 \circ a_1$ for the composition in the usual categories.

1. A_∞-structures and their homotopies

In the following definitions we use the sign convention of [4] which is different from the one in the original definition of [17].

1.1. **A_∞-algebras.** A ($\mathbb{Z}$-graded) A_∞-algebra is a $\mathbb{Z}$-graded vector space A equipped with linear maps $m_k : A^{\otimes k} \to A$ for $k \geq 1$ of degree $2 - k$ satisfying for every $n \geq 1$ the following A_∞-constraint Ax_n:

$$\sum_{k+l=n+1} \sum_{j=1}^{k} (-1)^{l(\tilde{a}_1+\ldots+\tilde{a}_{j-1})+j(l+1)} m_k(a_1, \dots, a_{j-1}, m_l(a_j, \dots, a_{j+l-1}), a_{j+l}, \dots, a_n) = 0$$

where $\tilde{a}_i = \deg(a_i) \mod (2)$. For example, Ax_1 says that $m_1^2 = 0$, Ax_2 gives the Leibnitz identity for m_1 and m_2, etc. One can consider m_n as components of a coderivation d of the coalgebra $T(sA)$ where s denote the suspension. The elements of $T(sA)$ are denoted traditionally as follows:

$$[a_1|a_2|\dots|a_k] = (sa_1) \otimes (sa_2) \otimes \dots (sa_k).$$

The coderivation d has a component $d_k : (sA)^{\otimes k} \to sA$ given by $s \circ m_k \circ (s^{-1})^{\otimes k}$, so that

$$d([a_1|\dots|a_n]) = \sum_{k+l=n+1} \sum_{j=1}^{k} (-1)^{\tilde{a}_1+\ldots+\tilde{a}_{j-1}+j-1+\mu(a_j,\dots,a_{j+l-1})} [a_1|\dots|a_{j-1}|m_l(a_j, \dots, a_{j+l-1})|a_{j+l}|\dots|a_n].$$

where for every collection of elements $(a_1, \dots, a_k)$ in A we denote

$$\mu(a_1, \dots, a_k) = (k-1)\tilde{a}_1 + (k-2)\tilde{a}_2 + \ldots + \tilde{a}_{k-1} + \frac{k(k-1)}{2}.$$

[2] In fact, there exist non-zero univalued quadruple Massey products on elliptic curve but they can be expressed via triple products.

The A_∞-constraints are equivalent to the condition $d^2 = 0$.

For a pair of A_∞-algebras (A, m^A) and (B, m^B) there is a natural notion of a A_∞-morphism from A to B. Namely, such a morphism consists of the data $(f_n, n \geq 1)$ where $f_n : A^{\otimes n} \to B$ is a linear map of degree $1-n$ such that

$$\sum_{1 \leq k_1 < k_2 < \ldots < k_i = n} (-1)^{\epsilon_L} m_i^B(f_{k_1}(a_1, \ldots, a_{k_1}), f_{k_2-k_1}(a_{k_1+1}, \ldots, a_{k_2}), \ldots, f_{n-k_{i-1}}(a_{k_{i-1}+1}, \ldots, a_n))$$

$$= \sum_{k+l=n+1} \sum_{j=1}^{k} (-1)^{\epsilon_R} f_k(a_1, \ldots, a_{j-1}, m_l^A(a_j, \ldots, a_{j+l-1}), a_{j+l}, \ldots, a_n)$$

where the signs ϵ_L and ϵ_R are defined as follows:

$$\epsilon_L = \mu(a_1, \ldots, a_{k_1}) + \mu(a_{k_1+1}, \ldots, a_{k_2}) + \ldots + \mu(a_{k_{i-1}+1}, \ldots, a_n) + \mu(f_{k_1}(a_1, \ldots, a_{k_1}), \ldots, f_{n-k_{i-1}}(a_{k_{i-1}+1}, \ldots, a_n)),$$

$$\epsilon_R = \widetilde{a}_1 + \ldots + \widetilde{a}_{j-1} + j - 1 + \mu(a_j, \ldots, a_{j+l-1}) + \mu(a_1, \ldots, a_{j-1}, m_l^A(a_j, \ldots, a_{j+l-1}), a_{j+l}, \ldots, a_n).$$

Again one can consider (f_n) as components of a coalgebra homomorphism $F : T(sA) \to T(sB)$, so that the above equation is equivalent to

$$F \circ d^A = d^B \circ F,$$

where d_A (resp. d^B) is the coderivation on A (resp. B) defined by m^A (resp. m^B). In particular, there is a natural composition of A_∞-morphisms defined as follows:

$$(f \circ g)_n(a_1, \ldots, a_n) = \sum_{1 \leq k_1 < k_2 < \ldots < k_i = n} (-1)^{\epsilon} f_i(g_{k_1}(a_1, \ldots, a_{k_1}), g_{k_2-k_1}(a_{k_1+1}, \ldots, a_{k_2}), \ldots, g_{n-k_{i-1}}(a_{k_{i-1}+1}, \ldots, a_n))$$

where

$$\epsilon = \mu(a_1, \ldots, a_{k_1}) + \mu(a_{k_1+1}, \ldots, a_{k_2}) + \ldots + \mu(a_{k_{i-1}+1}, \ldots, a_n) + \mu(g_{k_1}(a_1, \ldots, a_{k_1}), \ldots, g_{n-k_{i-1}}(a_{k_{i-1}+1}, \ldots, a_n)).$$

In the case when B and A have the same underlying spaces and $f_1 = \mathrm{id}$ we will call the data $f = (f_n, n \geq 2)$ a *homotopy* between two A_∞-structures $m = m^A$ and $m' = m^B$ on the same space. Note that for homotopic m and m' we necessarily have $m_1 = m_1'$. If f is a homotopy between m and m', g is a homotopy between m' and m'' then $g \circ f$ is a homotopy between m and m''.

Lemma 1.1. *Let $m = (m_n)$ be an A_∞-structure on A, $(f_n : A^{\otimes n} \to A, n \geq 2)$ be an arbitrary family of maps,* $\deg f_n = 1 - n$. *Then there exists a unique A_∞-structure m' on A such that $f = (f_n)$ (where $f_1 = \mathrm{id}$) is a homotopy between m and m'.*

Proof. This follows immediately from the fact that the coalgebra homomorphism $T(sA) \to T(sA)$ defined by (f_n) is an isomophism. □

We denote the A_∞-structure m' constructed in the above lemma by $m + \delta f$ (note that it depends non-linearly on f).

1.2. **A_∞-categories.** The definition of an A_∞-category is similar to that of an A_∞-algebra (see [1], [8]). Namely, an A_∞-category $\mathcal{C}$ consists of a class of objects $\mathrm{Ob}\,\mathcal{C}$, for every pair of objects X and X' a graded space of morphisms $\mathrm{Hom}(X, X')$, and a collection of linear maps (compositions)

$$m_k : \mathrm{Hom}(X_0, X_1) \otimes \mathrm{Hom}(X_1, X_2) \otimes \ldots \otimes \mathrm{Hom}^*(X_{k-1}, X_k) \to \mathrm{Hom}(X_0, X_k)$$

of degree $2 - k$ for all $k \geq 1$. The associativity constraint is that these compositions define a structure of A_∞-algebra on $\oplus_{ij} \mathrm{Hom}(X_i, X_j)$ for every collection $X_0, \ldots, X_n \in \mathrm{Ob}\,\mathcal{C}$.

An A_∞-functor (see [2], [8]) $\phi : \mathcal{C} \to \mathcal{C}'$ between A_∞-categories consists of a map $\phi : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}'$ and of a collection of linear maps

$$f_k : \mathrm{Hom}_{\mathcal{C}}(X_0, X_1) \otimes \mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \otimes \ldots \otimes \mathrm{Hom}_{\mathcal{C}}(X_{k-1}, X_k) \to \mathrm{Hom}_{\mathcal{C}'}(\phi(X_0), \phi(X_k))$$

of degree $1-k$ for $k \geq 1$, which define A_∞-morphisms $\oplus_{ij} \mathrm{Hom}_{\mathcal{C}}(X_i, X_j) \to \oplus_{ij} \mathrm{Hom}_{\mathcal{C}'}(\phi(X_i), \phi(X_j))$.

Now assume that we are given two structures of A_∞-category with the same class of objects $\mathcal{C}$ and with the same morphism spaces. Let $m = (m_n)$ and $m' = m'_n$ be the collections of the corresponding composition maps. A homotopy between m and m' is an A_∞-functor $\phi : (\mathcal{C}, m) \to (\mathcal{C}, m')$ such that the corresponding map on objects is identity and such that f_1 is the identity map on morphisms. The analogue of lemma 1.1 is valid in this situation.

In the case when $m_1 = 0$ for an A_∞-category $\mathcal{C}$ the products m_2 define a structure of the usual category on $\mathcal{C}$ (without units). If we have two such A_∞-categories $\mathcal{C}$ and $\mathcal{C}'$ and a functor $\phi_0 : \mathcal{C} \to \mathcal{C}'$ between them considered as usual categories then we say that ϕ_0 is *strictly compatible* with A_∞-structures if it extends to an A_∞-functor with $f_k = 0$ for $k > 1$.

Let X be an object of an A_∞-category which has $m_1 = 0$ equipped with a decomposition $X = X_1 \oplus X_2$ into a direct sum. By definition (here we deal with the usual category structure without units) this means that we have functorial in Y isomorphisms

$$\mathrm{Hom}(X, Y) \simeq \mathrm{Hom}(X_1, Y) \oplus \mathrm{Hom}(X_2, Y)$$

and

$$\mathrm{Hom}(Y, X) \simeq \mathrm{Hom}(Y, X_1) \oplus \mathrm{Hom}(Y, X_2).$$

We say that the decomposition $X = X_1 \oplus X_2$ is strictly compatibile with an A_∞-structure if every composition m_n involving the spaces $\mathrm{Hom}(X, Y)$ or $\mathrm{Hom}(Y, X)$ is a direct sum of the corresponding compositions with the spaces $\mathrm{Hom}(X_i, Y)$ and $\mathrm{Hom}(Y, X_i)$.

1.3. **Cyclic A_∞-structures.** We will consider a special class of A_∞-algebras, namely, those equipped with a cyclic symmetry.

Definition 1.2. Let A be an A_∞-algebra equipped with a bilinear form $b : A \otimes A \to k$. We will call A *cyclic* if for every $n \geq 1$ the following identity is satisfied:

$$b(m_n(a_1, \ldots, a_n), a_{n+1}) = (-1)^{n(\tilde{a}_1+1)} b(a_1, m_n(a_2, \ldots, a_{n+1})). \tag{1.1}$$

Remark. Assume in addition that b satisfies the following symmetry:

$$b(a_1, a_2) = (-1)^{\widetilde{a_1}\widetilde{a_2}} b(a_2, a_1).$$

Then (1.1) can be rewritten as follows:

$$b(m_n(a_1, \ldots, a_n), a_{n+1}) = (-1)^{n+\widetilde{a_1}(\widetilde{a_2}+\ldots+\widetilde{a_{n+1}})} b(m_n(a_2, \ldots, a_{n+1}), a_1).$$

Using b we can define a linear functional ξ on $T(sA)$ by setting $\xi = b \circ (s^{-1})^{\otimes 2}$ on $(sA)^{\otimes 2}$ while $\xi = 0$ on $(sA)^{\otimes n}$ for $n \neq 2$. Thus, we have

$$\xi([a_1|a_2]) = (-1)^{\widetilde{a}_1+1} b(a_1, a_2),$$

Then the equation (1.1) is equivalent to the condition $\xi \circ d = 0$ where d is the coderivation defined by (m_n).

The collection of maps $f = (f_n : A^{\otimes n} \to A, n \geq 1)$, $\deg f_n = 1-n$, $f_1 = \mathrm{id}$ is called a *cyclic homotopy* if

$$\sum_{k+l=n} (-1)^{(l+1)(\tilde{a}_1+\ldots+\tilde{a}_k)+nk} b(f_k(a_1, \ldots, a_k), f_l(a_{k+1}, \ldots, a_n)) = 0 \tag{1.2}$$

for $n \geq 3$. This is equivalent to the condition $\xi \circ F = \xi$ where $F : T(sA) \to T(sA)$ is the coalgebra homomorphism defined by (f_n). Let m be a cyclic A_∞-structure, f be a cyclic homotopy. Then the A_∞-structure $m + \delta f$ is cyclic. If f and g are cyclic homotopies then $f \circ g$ is also cyclic.

Remark. Assume that $f_k = 0$ unless $k = 1$ or $k = n$ for some $n \geq 2$. Then f is a cyclic homotopy if and only if

$$b(f_n(a_1, \dots , a_n), a_{n+1}) = (-1)^{(n+1)\tilde{a}_1+n} b(a_1, f_n(a_2, \dots , a_{n+1}))$$

and

$$b(f_n(a_1, \dots , a_n), f_n(a_{n+1}, \dots , a_{2n})) = 0.$$

The definition of cyclic A_∞-categories follows the same pattern. We assume that there is a bilinear form

$$b : \mathrm{Hom}(X, Y) \otimes \mathrm{Hom}(Y, X) \to k$$

for every pair of objects (X, Y). Then an A_∞-category is called cyclic if the identity (1.1) is satisfied whenever $a_1 \in \mathrm{Hom}(X_1, X_2), \dots , a_n \in \mathrm{Hom}(X_n, X_{n+1})$, $a_{n+1} \in \mathrm{Hom}(X_{n+1}, X_1)$. Similarly we define cyclic homotopy between two cyclic A_∞-structures with the same objects and morphism spaces (and the same bilinear form b).

1.4. **Transversal A_∞-structures.** Assume that we are given a class of objects and a notion of transversality for pairs of objects. We will call an n-tuple of objects $(X_1, \dots , X_n)$ *transversal* if for every $1 \leq i < j \leq n$ the pair (X_i, X_j) is transversal. Then the structure of *transversal* (cyclic) A_∞-category on this class of objects consists of the following data. For every pair of transversal objects (X, Y) a graded space of morphisms $\mathrm{Hom}(X, Y)$ is given. For every transversal collection $(X_0, \dots , X_n)$, $n \geq 1$ we have linear maps

$$m_n : \mathrm{Hom}(X_0, X_1)\, \mathrm{Hom}(X_1, X_2) \dots \mathrm{Hom}(X_{n-1}, X_n) \to \mathrm{Hom}(X_0, X_n)$$

of degree $2 - n$ such that the axioms Ax_n and the identity (1.1) are satisfied whenever the objects involved in it form a transversal collection. Similarly we define a notion of homotopy between transversal A_∞-structures and the cyclic analogues of these notions.

The motivating example is that of Fukaya category (see [1]) where objects are Lagrangain submanifolds in a symplectic manifold with some additional structure. Then we have the standard notion of transversality for pairs of Lagrangians. However, notice that the notion of transversality for n-tuples we use is weaker than the standard one: we just require every pair of them to intersect transversally but, for example, we allow three Lagrangians to intersect in one point.

1.5. **Two A_∞-structures on $\mathrm{Vect}(E)$.** The first A_∞-structure (or rather a class of equivalent structures) can be defined on the category $\mathrm{Vect}(M)$ where M is a variety over k or a complex manifold as follows. Let us choose some functorial acyclic resolution $V \to R^{\cdot}(V)$ for every vector bundle V on M such that for every pair of bundles there are functorial morphisms

$$R^{\cdot}(V_1) \otimes R^{\cdot}(V_2) \to R^{\cdot}(V_1 \otimes V_2)$$

inducing the identity map on $V_1 \otimes V_2$ and satisfying the natural associativity condition (e.g. one can take Cech complexes of acyclic covering or in the case $k = \mathbb{C}$ Dolbeault complexes). Then we can define a dg-category whose objects are vector bundles with $\mathrm{Hom}(V_1, V_2) = R^{\cdot}(V_1^\vee \otimes V_2)$. By homotopic invariance of the notion of A_∞-algebra there exists an equivalent A_∞-category structure on $\mathrm{Vect}(M)$ with morphisms $\mathrm{Hom}^*(V_1, V_2) = \oplus_i \mathrm{Ext}^i(V_1, V_2)$ which has $m_1 = 0$ (see [5],[6],[7],[12],[13]).

We will use a particular representative of this class of A_∞-structures in the case when M is a compact complex manifold equipped with a hermitian metric. This A_∞-structure appears naturally on the category $\mathrm{Vect}^h(M)$ of holomorphic vector bundles equipped with a hermitian metric. Starting with the dg-category given by Dolbeault complexes one can use metrics to write an explicit formula for higher compositions involving some Hodge theory operators (see [15]). We will denote this A_∞-structure by $m^H = (m^H_n)$.

Note that since m_2^H is the standard composition (while $m_1 = 0$) the choices of hermitian metrics on bundles are not really important. More precisely, by the standard argument in the homotopy theory the objects (V, h) and (V, h'), where h and h' are different metrics on the same bundle V, are equivalent objects of this A_∞-category. By the definition (that appeared in [10]), this means that there exists an A_∞-functor from the category with two isomorphic objects $O_1 \simeq O_2$ and no other non-trivial morphisms to our A_∞-category, that sends O_1 to (V, h) and O_2 to (V, h').

Assume in addition that ω_M is trivialized. Then the Serre duality gives a non-degenerate pairing

$$\operatorname{Hom}^*(V_1, V_2) \otimes \operatorname{Hom}^*(V_2, V_1) \to \mathbb{C}.$$

The main feature of our particular choice of an A_∞-structure is the cyclic symmetry (1.1) of m_n^H with respect to the Serre duality (see [15]). Note also that the higher products m_n^H are compatible with Massey products when the latter are well-defined.

To define the second A_∞-structure on $\operatorname{Vect} E$ (or rather, transversal A_∞-structure) let us recall the definition of the Fukaya A_∞-category of the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ with the (complexified) symplectic form $-2\pi i\tau dx \wedge dy$ where τ is an element of the upper half-plane. We give here a very concrete version of the general definition which can be found in [1],[9],[16]. The objects of this category are pairs $(\overline{L}, A)$ where $\overline{L} = p(L)$ is the image of a non-vertical line L with rational slope under the natural projection $p : \mathbb{R}^2 \to T$ (a *geodesic circle*), $A : V \to V$ is an operator on a finite dimensional complex vector space V with real eigenvalues [3]. We call a pair of objects $(\overline{L_1}, A_1)$ and $(\overline{L_2}, A_2)$ transversal if $\overline{L_1}$ and $\overline{L_2}$ are different. For such a pair the morphism space is

$$\operatorname{Hom}((\overline{L_1}, A_1), (\overline{L_2}, A_2)) = \operatorname{Hom}(V_1, V_2) \otimes \operatorname{Hom}(\overline{L_1}, \overline{L_2})$$

where

$$\operatorname{Hom}(\overline{L_1}, \overline{L_2}) = \oplus_{P \in \overline{L_1} \cap \overline{L_2}} \mathbb{C}[P]$$

($[P]$ is a basis vector attached to a point P). Note that there is a natural pairing

$$\operatorname{Hom}((\overline{L_1}, A_1), (\overline{L_2}, A_2)) \otimes \operatorname{Hom}((\overline{L_2}, A_2), (\overline{L_1}, A_1)) \to \mathbb{C} \tag{1.3}$$

induced by the natural duality between $\operatorname{Hom}(V_1, V_2)$ and $\operatorname{Hom}(V_2, V_1)$ and by the self-duality of $\operatorname{Hom}(\overline{L_1}, \overline{L_2}) = \operatorname{Hom}(\overline{L_2}, \overline{L_1})$ (such that the basis $([P])$ is autodual). Let λ_i be the slope of the line L_i ($i = 1, 2$). Then $\operatorname{Hom}((\overline{L_1}, A_1), (\overline{L_2}, A_2)) \neq 0$ only if $\lambda_1 \neq \lambda_2$. This space has grading 0 if $\lambda_1 < \lambda_2$ and grading 1 if $\lambda_1 > \lambda_2$. By definition the differential m_1 is zero. The compositions m_k for $k \geq 2$ are defined as follows. Let $(\overline{L_i}, A_i)$, $i = 0, \ldots, k$ be objects of the Fukaya categories such that the corresponding circles are pairwise different. Below it will be convenient to identify the set of indices $[0, k]$ with $\mathbb{Z}/(k+1)\mathbb{Z}$. For every $i \in \mathbb{Z}/(k+1)\mathbb{Z}$ let $d_i \in [0, 1]$ be the grading of $\operatorname{Hom}((\overline{L_i}, A_i), (L_{i+1}, A_{i+1}))$. The composition

$$m_k^F : \operatorname{Hom}((\overline{L_0}, A_0), (\overline{L_1}, A_1)) \otimes \ldots \otimes \operatorname{Hom}((\overline{L_{k-1}}, A_{k-1}), (\overline{L_k}, A_k)) \to \operatorname{Hom}((\overline{L_0}, A_0), (\overline{L_k}, A_k))$$

is non-zero only if $\sum_{i=0}^k d_i = k-1$. Let $P_{i,i+1}$ be some intersection points of $\overline{L_i}$ and $\overline{L_{i+1}}$ ($i = 0, \ldots, k-1$). For every $i = 0, \ldots, k-1$ let $M_{i,i+1}$ be an element of $\operatorname{Hom}(V_i, V_{i+1})$. Then

$$\begin{aligned} &m_k^F(M_{0,1}[P_{0,1}], M_{1,2}[P_{1,2}], \ldots, M_{k-1,k}[P_{k-1,k}]) = \\ &\sum_{P_{0,k},\Delta} \pm \exp(2\pi i\tau \cdot \int_\Delta dx \wedge dy) \exp(2\pi i(x(p_k) - x(p_{k-1}))A_k) \circ M_{k-1,k} \circ \exp(2\pi i(x(p_{k-1}) - x(p_{k-2}))A_{k-1}) \\ &\ldots \circ M_{1,2} \circ \exp(2\pi i(x(p_1) - x(p_0))A_1) \circ M_{0,1} \circ \exp(2\pi i(x(p_0) - x(p_k))A_0)[P_{0,k}] \end{aligned}$$

where the sum is taken over points of intersections $P_{0,k}$ of $\overline{L_0}$ with $\overline{L_k}$ and over all $(k+1)$-gons Δ (considered up to translation by $\mathbb{Z}^2$) with vertices $p_i \equiv P_{i,i+1} \mod \mathbb{Z}^2$, $i \in \mathbb{Z}/(k+1)\mathbb{Z}$, such that the edge $[p_{i-1}, p_i]$ belongs to $p^{-1}(\overline{L_i})$ (the restriction on degrees d_i implies that Δ is convex). We also require that the path formed by the edges $[p_0, p_1], [p_1, p_2], \ldots, [p_k, p_0]$ goes in the clockwise direction. The sign

[3] The slight difference with [16] is that we don't attach to L an integer and don't consider vertical lines.

in the RHS is given by the following rule. If k is even then all signs are "plus". If k odd then the sign is equal to the sign of $x(p_0) - x(p_k)$ (recall that we do not allow vertical lines).

It is not difficult to check that $m^F = (m_k^F)$ is a (transversal) cyclic A_∞-category with respect to the pairing (1.3). This A_∞-structure is strictly compatible with decomposition of an operator A into a direct sum of operators. The main theorem of [16] identifies the corresponding usual category given by m_2^F with a full subcategory of $\mathrm{Vect}(E)$ where $E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ (which contains all indecomposable bundles). In order to get all vector bundles one has to modify the Fukaya category by adding formally direct sums. We extend the A_∞-structure to this larger category using strict compatibility with direct sums. Thus, we get a transversal A_∞-structure (which we still denote m^F) on $\mathrm{Vect}(E)$. The construction of [16] identifies the pairing (1.3) with the Serre duality (for some trivialization of ω_E), so the obtained A_∞-structure is cyclic with respect to it.

We will remind some details of the correspondence between vector bundles on E and objects of the Fukaya category later. Let us only mention here that the slope of a line corresponding to an indecomposable bundle V is equal to the slope of V (the ratio of the degree and the rank). Stable bundles correspond to objects $(\overline{L}, A)$ where $A \in \mathbb{R}$ is a real number (considered as an operator on a one-dimensional space).

2. Transversal A_∞-structures on the category of line bundles over an elliptic curve

2.1. **Transversality and admissibility.** Let E be an elliptic curve over a field k. Let $\mathcal{L}$ be the full subcategory in $\mathrm{Vect}(E)$ consisting of line bundles. One can consider extensions of the (strictly associative) composition m_2 on $\mathcal{L}$ to A_∞-structures. The following definition gives some natural restrictions one can impose on such an extension. Let us fix a trivialization of ω_E. Then the Serre duality gives a non-degenerate pairing

$$\mathrm{Hom}^*(V_1, V_2) \otimes \mathrm{Hom}^*(V_2, V_1) \to k.$$

Definition 2.1. Let us call a cyclic (with respect to the Serre duality) A_∞-structure m on the category $\mathcal{L}$ *admissible* if $m_1 = 0$, m_2 is the standard composition, and the functor of tensor multiplication by a line bundle is strictly compatible with m.

Note that if $m_1 = 0$ then for any A_∞-structure m' which is homotopic to m one has $m_2' = m_2$. So it makes sense to try to classify admissible A_∞-structures on $\mathcal{L}$ up to cyclic homotopy, strictly compatible with tensor multiplication by any line bundle. We refer to such homotopies as *admissible* ones.

We also define an admissible *transversal* A_∞-structure on $\mathcal{L}$ by similar restrictions provided that we have some notion of transversality for pairs of line bundles. We assume that such a notion is given and that it has the following properties:
(i) (L, M) is transversal if and only if (M, L) is transversal;
(ii) (L, M) is transversal if and only if (L^{-1}, M^{-1}) is transversal;
(iii) (L_1, L_2) is transversal if and only if (L_1M, L_2M) is transversal;
(iv) for every finite collection of line bundle $(L_1, \dots, L_n)$ and every integer d there exists an infinite number of isomorphism classes of line bundles L of degree d such that L and L^2 are transversal to all L_i;
(v) if (L, M) is transversal then $L \not\simeq M$.

For example, assume that $E(k)$ is infinite (this is necessary for the property (iv) to hold). Then one can call (L, M) transversal if $L \not\simeq M$. Another example arises from the correspondence between line bundles and objects of Fukaya category defined in [16]. In this example the complex parameter describing an isomorphism class of a line bundle splits into two real parameters: one describes the position of the corresponding geodesic circle and another specifies the connection on it. Then the pair (L, M) is transversal if the first real parameter takes different values at L and M (see section 3.2 for details).

The data of an admissible transversal A_∞-structure are encoded in the sequence of maps

$$m_n : H^{i_1}(L_1)H^{i_2}(L_2)\dots H^{i_n}(L_n) \to H^{i_1+i_2+\dots+i_n+2-n}(L_1L_2\dots L_n)$$

for line bundles (L_i) such that the collection $(\mathcal{O}, L_1, L_1L_2, \dots, L_1\dots L_n)$ is transversal.

2.2. **Construction of the homotopy.**

Theorem 2.2. *Let m and m' be admissible transversal A_∞-structures on the category of line bundles on E. Assume that for every triple of line bundles (L_1, M, L_2) where* $\deg(L_1) = \deg(L_2) = 1$, $\deg(M) = -1$, *such that $(\mathcal{O}, L_1, L_1M, L_1ML_2)$ is transversal, the maps*

$$H^0(L_1) \otimes H^1(M) \otimes H^0(L_2) \to H^0(L_1L_2M) \tag{2.1}$$

given by m_3 and m'_3 coincide. Then there exist a unique admissible homotopy between m and m'.

The following lemma is the main ingredient of the proof.

Lemma 2.3. *Let L be a line bundle of degree ≥ 3 on E, $S \subset \mathrm{Pic}(E)$ be a subset such that for every d and every isomorphism classes $[L_1], [L_2] \in \mathrm{Pic}(E)$ there exists an infinite number of $[M] \in S$ such that* $\deg(M) = d$, *$2[M] \in S$, $[L_1] + [M] \in S$ and $[L_2] - [M] \in S$. Then the following sequence is exact:*

$$\oplus_{L_1L_2L_3=L,[L_3]\in S,[L_2L_3]\in S}\, H^0(L_1)H^0(L_2)H^0(L_3) \xrightarrow{\alpha} \oplus_{L_1L_2=L,[L_2]\in S} H^0(L_1)H^0(L_2) \xrightarrow{\beta} H^0(L) \to 0 \tag{2.2}$$

where L_i denote line bundles of positive degrees, the map α sends $s_1 \otimes s_2 \otimes s_3$ to $s_1s_2 \otimes s_3 - s_1 \otimes s_2s_3$, β sends $s_1 \otimes s_2$ to s_1s_2.

Proof. Clearly, β is surjective. Thus, it suffices to prove the following statement. Assume that for every pair of line bundles of positive degree (L_1, L_2) such that $[L_2] \in S$ and $L_1L_2 \simeq L$ we are given a linear map

$$b_{L_1,L_2} : H^0(L_1)H^0(L_2) \to k$$

such that for every triple (L_1, L_2, L_3) such that $\deg(L_i) > 0$, $i = 1, 2, 3$, $L_1L_2L_3 \simeq L$, $[L_3] \in S$, $[L_2L_3] \in S$, one has

$$b_{L_1,L_2L_3}(s_1, s_2s_3) = b_{L_1L_2,L_3}(s_1s_2, s_3)$$

where $s_i \in H^0(L_i)$, $i = 1, 2, 3$. Then there exists a functional ϕ on $H^0(L)$ such that for every (L_1, L_2) (with $[L_2] \in S$)

$$b_{L_1,L_2}(s_1, s_2) = \phi(s_1s_2). \tag{2.3}$$

We will consider separately several cases.

(i) $\deg(L) = 3$. Let $p_1, p_2 \in E$ be a pair of distinct points such that $[\mathcal{O}(p_i)] \in S$, $i = 1, 2$, and $[\mathcal{O}(p_1 + p_2)] \in S$. Let $s_{p_i} \in H^0(\mathcal{O}(p_i))$, $i = 1, 2$, be non-zero sections. Then we have the following exact sequence:

$$0 \to H^0(L(-p_1 - p_2)) \xrightarrow{\alpha'} H^0(L(-p_1)) \oplus H^0(L(-p_2)) \xrightarrow{\beta'} H^0(L) \to 0$$

where $\alpha'(s) = (ss_{p_2}, -ss_{p_1})$, $\beta'(t_1, t_2) = t_1s_{p_1} + t_2s_{p_2}$. Let us define a functional $\widetilde{\phi}$ on $H^0(L(-p_1)) \oplus H^0(L(-p_2))$ by the formula

$$\widetilde{\phi}(t_1, t_2) = b_{L(-p_1),\mathcal{O}(p_1)}(t_1, s_{p_1}) + b_{L(-p_2),\mathcal{O}(p_2)}(t_2, s_{p_2}).$$

Note that $\widetilde{\phi}$ vanishes on the image of α'. Indeed, we have

$$b(ss_{p_2}, s_{p_1}) = b(s, s_{p_2}s_{p_1}) = b(s, s_{p_1}s_{p_2}) = b(ss_{p_1}, s_{p_2}).$$

Therefore, there exists a functional ϕ on $H^0(L)$ such that $\widetilde{\phi} = \phi \circ \beta'$. We are going to show that this functional is the one we are looking for.

Let L_1 and L_2 be line bundles of degrees 1 and 2 respectively such that $L_1L_2 \simeq L$, $[L_2] \in S$. Assume in addition that $L_2 \not\simeq \mathcal{O}(p_1 + p_2)$. Then we claim that

$$b_{L_1,L_2}(s, t) = \phi(st)$$

for any $s \in H^0(L_1)$, $t \in H^0(L_2)$. Indeed, the space $H^0(L_2)$ is a direct sum of subspaces $H^0(L_2(-p_1))s_{p_1}$ and $H^0(L_2(-p_2))s_{p_2}$. Thus, it suffices to prove that $b_{L_1,L_2}(s,t) = \phi(st)$ for t in any of these subspaces. For example, let $t = t's_{p_1}$, where $t' \in H^0(L_2(-p_1))$. Then we have,

$$b(s,t) = b(s, t's_{p_1}) = b(st', s_{p_1}) = \phi(st's_{p_1})$$

as required.

Now we claim that if M_1 and M_2 are arbitrary line bundles of degrees 2 and 1 respectively such that $M_1M_2 \simeq L$ and $[M_2] \in S$, then $b_{M_1,M_2}(s,t) = \phi(st)$ for $s \in H^0(M_1)$, $t \in H^0(M_2)$. Indeed, let us choose points $q_1, q_2 \in E$ such that $M_1 \not\simeq \mathcal{O}(q_1+q_2)$, $M_2(q_i) \not\simeq \mathcal{O}(p_1+p_2)$ and $[M_2(q_i)] \in S$ for $i = 1,2$. Then $H^0(M_1)$ is a direct sum of $H^0(M_1(-q_1))H^0(\mathcal{O}(q_1))$ and $H^0(M_1(-q_2))H^0(\mathcal{O}(q_2))$, so we can assume that s is in one of these subspaces. For example, assume that $s = s's_{q_1}$ where $s_{q_1} \in H^0(\mathcal{O}(q_1))$, $s' \in H^0(M_1(-q_1))$. Then we have

$$b(s,t) = b(s's_{q_1}, t) = b(s', s_{q_1}t).$$

Applying the previous part of the proof to $L_1 = M_1(-q_1)$, $L_2 = M_2(q_1)$ we obtain that

$$b(s', s_{q_1}t) = \phi(s's_{q_1}t) = \phi(st)$$

as required.

Finally, a similar argument shows that for arbitrary line bundles L_1 and L_2 of degrees 1 and 2 one has $b_{L_1,L_2}(s,t) = \phi(st)$ where $s \in H^0(L_1)$, $t \in H^0(L_2)$.

(ii) $\deg(L) = 4$. Let us choose a pair of line bundles L_1 and L_2 both of degree 2 such that $L_1 \not\simeq L_2$, $[L_2] \in S$, and $L_1L_2 \simeq L$. Then the product map

$$H^0(L_1) \otimes H^0(L_2) \to H^0(L)$$

is an isomorphism, so we can define ϕ by setting

$$\phi(s_1s_2) = b_{L_1,L_2}(s_1,s_2)$$

where $s_1 \in H^0(L_1)$, $s_2 \in H^0(L_2)$.

Let L_1', L_2' be line bundles of degree 2 such that $L_1'L_2' \simeq L$ and $[L_2'] \in S$. Let $p, q \in E$ be a pair of points such that $\mathcal{O}(p+q) \simeq L_1$, $[L_2'(-q)] \in S$. Then we claim that the equality

$$b_{L_1',L_2'}(s,t) = \phi(st) \tag{2.4}$$

holds whenever $s \in H^0(L_1'(-p))$, $t \in H^0(L_2'(-q))$. Indeed, assume $s = s's_p$, $t = t's_q$ where $s_p \in H^0(\mathcal{O}(p))$, $s_q \in H^0(\mathcal{O}(q))$, $s' \in H^0(L_1'(-p))$, $t' \in H^0(L_2'(-q))$. Then

$$b(s,t) = b(s's_p, t's_q) = b(s's_ps_q, t') = b(s_ps_q, s't') = \phi(s_ps_qs't') = \phi(st).$$

Now let $p_1, p_2 \in E$ be a pair of distinct points such that $L_1' \simeq \mathcal{O}(p_1+p_2)$ and for $q_1, q_2 \in E$ defined by $\mathcal{O}(p_1+q_1) \simeq \mathcal{O}(p_2+q_2) \simeq L_1$ one has $[L_2'(-q_1)] \in S$, $[L_2'(-q_2)] \in S$. Note that we have $L_1'(q_1+q_2) \simeq L_1^2 \not\simeq L$ since $L_2 \not\simeq L_1$. Hence, $\mathcal{O}(q_1+q_2) \not\simeq L_2'$ and $H^0(L_2')$ has a basis (t_1,t_2) such that t_1 vanishes at q_1 and t_2 vanishes at q_2. Therefore, if s is a section of L_1' vanishing at p_1 and p_2 then by the previous part of the proof we have

$$b(s,t_i) = \phi(st_i)$$

for $i = 1,2$. Repeating this argument for another pair of points (p_1,p_2) as above we get a similar statement for a section of L_1' linearly independent from s. Thus, we conclude that (2.4) holds for all $s \in H^0(L_1')$, $t \in H^0(L_2')$.

Now let M_1 be a line bundle of degree 1, M_2 be a line bundle of degree 3 such that $M_1M_2 \simeq L$, $[M_2] \in S$. Let $s_1 \in H^0(M_1)$, $s_2 \in H^0(M_2)$, p be a point in the divisor of s_2. Then assuming that $[M_2(-p)] \in S$ we can write $s_2 = s_ps_2'$ where $s_p \in H^0(\mathcal{O}(p))$, $s_2' \in H^0(M_2(-p))$ and

$$b(s_1,s_2) = b(s_1, s_ps_2') = b(s_1s_p, s_2') = \phi(s_1s_ps_2') = \phi(s_1s_2).$$

Since $H^0(M_2)$ is spanned by $H^0(M_2(-p))$ and $H^0(M_2(-p'))$ for two distinct points $p, p' \in E$, this proves that $b_{M_1,M_2}(s_1, s_2) = \phi(s_1 s_2)$ for all s_1 and s_2. The case when $\deg(M_1) = 3$, $\deg(M_2) = 1$ is completely analogous.
(iii) $\deg(L) = d \geq 5$. Let us fix a line bundle L_2 of degree 2 such that $[L_2] \in S$ and $[L_2^2] \in S$. Then there is an exact sequence

$$0 \to L_2^{-1} \to H^0(L_2) \otimes \mathcal{O} \to L_2 \to 0,$$

which induces for every line bundle M of degree ≥ 3 an exact sequence

$$0 \to H^0(ML_2^{-1}) \to H^0(M)H^0(L_2) \to H^0(ML_2) \to 0.$$

Let $L_1 = LL_1^{-1}$. Consider the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & \oplus H^0(\mathcal{O}(p))H^0(L_1L_2^{-1}(-p)) & \to & \oplus H^0(\mathcal{O}(p))H^0(L_1(-p))H^0(L_2) & \xrightarrow{\alpha_2} & \oplus H^0(\mathcal{O}(p))H^0(L(-p)) & \to 0 \\ & \Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle \alpha_1} & & \Big\downarrow & \\ 0 \to & H^0(L_1L_2^{-1}) & \to & H^0(L_1)H^0(L_2) & \xrightarrow{\beta} & H^0(L) & \to 0 \end{array} \tag{2.5}$$

where the direct sums in first row are taken over all $p \in E$ such that $[L(-p)] \in S$. Notice that γ is surjective. Indeed, if $d \geq 6$ this is clear, while for $d = 5$ we have to check that for the unique point p such that $\mathcal{O}(p) \simeq L_1L_2^{-1}$ one has $[L(-p)] \in S$. But this follows from our assumptions of L_2, since $L(-p) \simeq L_2^2$ for such p. He have $b_{L_1,L_2} \circ \alpha_1 = \sum_p b_{\mathcal{O}(p),L(-p)} \circ \alpha_2$. From this by an easy diagram chasing (using the surjectivity of γ) we obtain that b_{L_1,L_2} vanishes on the kernel of β, hence, there exists a functional ϕ on $H^0(L)$ such that $b_{L_1,L_2} = \phi \circ \beta$. It follows that for any $p \in E$ such that $[L(-p)] \in S$ one has

$$b_{\mathcal{O}(p),L(-p)}(s,t) = \phi(st)$$

for $s \in H^0(\mathcal{O}(p))$, $t \in H^0(L(-p))$. Indeed, we can assume that $t = t_1t_2$ with $t_1 \in H^0(L_1(-p))$, $t_2 \in L_2$, in which case

$$b(s,t) = b(s, t_1t_2) = b(st_1, t_2) = \phi(st).$$

Now we can deduce (2.3) in the general case using the same argument as in the end of case (ii). □

Remark. It is easy to see from the proof that our assumptions on the set $S \in \mathrm{Pic}(E)$ can be weakened. Let us denote by S_d the subset of elements of S of degree d. Then it suffices to require that: (1) for any $[L] \in \mathrm{Pic}(E)$ and any d one has $|S_d \cap (S + [L])| > 4$, $|S_d \cap ([L] - S)| > 5$; (2) there exists $[L] \in S_2$ such that $2[L] \in S$.

Proof of theorem 2.2. Let us prove the existence first. Clearly we can replace m by $m + \delta f$ where $f = (f_n, n \geq 2)$ is an admissible homotopy. Therefore, we can argue by induction: for every $n \geq 3$, assuming that $m_k = m'_k$ for $k < n$ we will construct an admissible homotopy f^n such that $f^n_k = 0$ for $k < n - 1$ and $(m + \delta f^n)_n = m'_n$ (this implies that $(m + \delta f^n)_k = m'_k$ for all $k \leq n$).

Using the cyclic symmetry we can reduce various types of non-zero transversal n-tuple products to the following two types:
(i)

$$m_n : H^0(L_1)H^1(M_1)\dots H^1(M_i)H^0(L_2)H^1(M_{i+1})\dots H^1(M_{n-2}) \to H^0(L_1L_2M_1\dots M_{n-2})$$

where $1 \leq i \leq n - 2$,
(ii)

$$m_n : H^0(L_1) \otimes H^0(L_2) \otimes H^1(M_1) \otimes \dots \otimes H^1(M_{n-2}) \to H^0(L_1L_2M_1\dots M_{n-2}).$$

Let us call $w = \deg(L_1) + \deg(L_2)$ the *weight* of the corresponding n-tuple product type. Note that we have $w \geq 2$. The first observation is that any n-tuple product of type (i) of weight > 2 can be expressed via

k-tuple products with $k < n$ and via n-tuple products of smaller weight. Indeed, if $\deg(L_1) + \deg(L_2) > 2$ then either $\deg(L_1) > 1$ or $\deg(L_2) > 1$. Assume for example that $\deg(L_1) > 1$. Then $H^0(L_1)$ is spanned by various products $s_p s$ where $s_p \in \mathcal{O}(p)$, $s \in L_1(-p)$, a point $p \in E$ is such that the collection

$$(\mathcal{O}, \mathcal{O}(p), L_1, L_1M_1, \dots, L_1M_1 \dots M_i, L_1L_2M_1 \dots M_i, L_1L_2M_1 \dots M_{i+1}, \dots, L_1L_2M_1 \dots M_{n-2})$$

is transversal. Now for any collection of elements $e_j \in H^1(M_j)$, $j = 1, \dots, n-2$, $t \in H^0(L_2)$ and any $1 \le j < n-2$ we have

$$\begin{aligned} &m_n(s_p s, e_1, \dots, e_i, t, e_{i+1}, \dots, e_{n-2}) = m_n(s_p, se_1, \dots, e_i, t, e_{i+1}, \dots, e_{n-2}) \pm \\ &m_n(s_p, s, e_1, \dots, e_i t, e_{i+1}, \dots, e_{n-2}) \pm m_n(s_p, s, e_1, \dots, e_i, te_{i+1}, \dots, e_{n-2}) \pm \\ &s_p m_n(s, e_1, \dots, e_i, t, e_{i+1}, \dots) + \dots \end{aligned}$$

where the unwritten terms contain only m_k with $k < n$, while the weights of three n-tuple products in the RHS are smaller than w. If $j = n-2$ then there is an additional term $m_n(s_p, s, e_1, \dots, e_{n-2})t$ which doesn't affect our argument. Similarly one considers the case when $\deg(L_2) > 1$. On the other hand, the only non-zero transversal products of type (i) and weight 2 are those of type (2.1). As we'll see below this will allow us to restrict our attention to products of type (ii).

To construct the homotopy f^n we again apply induction. Namely, assuming that $m_n = m'_n$ for all products (of types (i) and (ii)) of weight $< w$ (and $m_k = m'_k$ for $k < n$) we will construct a homotopy $f^{n,w}$ such that $(m + \delta f^{n,w})_n = m'_n$ for all products of weight w and such that the only non-zero component $f^{n,w}$ (other than $f_1^{n,w} = \mathrm{id}$) reduces by cyclic symmetry to the following type:

$$f_{n-1}^{n,w} : H^0(L) \otimes H^1(M_1) \otimes \dots \otimes H^1(M_{n-2}) \to H^0(LM_1 \dots M_{n-2})$$

where $\deg(L) = w$. Note that $f^{n,w}$ is automatically cyclic. Indeed, any non-zero value of $f^{n,w}$ is an element of $H^i(M)$ where the degree of M is either $-w$ or d, such that $0 < d < w$. On the other hand, by definition of Serre duality $b(H^i(M), H^j(M')) = 0$ unless $\deg(M) + \deg(M') = 0$. It follows that one has

$$b(f_{n-1}^{n,w}(a_1, \dots, a_{n-1}), f_{n-1}^{n,w}(a_n, \dots, a_{2n-2})) = 0,$$

so $f^{n,w}$ is cyclic. By the above observation it will be sufficient to check the relation $(m + \delta f^{n,w})_n = m'_n$ only for products of type (ii) (and weight w).

Assume first that $w = 2$. Then we necessarily have $n = 3$. Let us fix line bundles L and M, $\deg(L) = 2$, $\deg(M) = 1$, such that the triple $(\mathcal{O}, L, LM)$ is transversal. We want to construct a map

$$f_2^{3,2} : H^0(L) \otimes H^1(M) \to H^0(LM)$$

such that for every pair of line bundles L_1, L_2 of degree 1, where $L_1L_2 \simeq L$ and the quadruple $(\mathcal{O}, L_1, L, LM)$ is transversal, the map

$$m'_3 - m_3 : H^0(L_1)H^0(L_2)H^1(M) \to H^0(LM)$$

is a composition of the product map $H^0(L_1)H^0(L_2) \to H^0(L)$ with $-f_2^{3,2}$.

Let us fix line bundles M' and L' such that $\deg(M') = -2$, $\deg(L') = 1$, $M'L' \simeq M$ and the quadruple $(\mathcal{O}, L, LM', LM)$ is transversal. Let $e \in H^1(M)$ be a non-zero element. Then $e = e's'$ for some $e' \in H^1(M')$, $s' \in H^0(L')$. Now for every line bundles L_1 and L_2 such that $L_1L_2 \simeq L$, where the quintuple $(\mathcal{O}, L_1, L, LM', LM)$ is transversal, and every $s_1 \in H^0(L_1)$ and $s_2 \in H^0(L_2)$ we have

$$m_3(s_1, s_2, e) = m_3(s_1, s_2, e's') = m_3(s_1, s_2e', s') - m_3(s_1s_2, e', s')$$

and the similar equality holds for m'_3. Note that we have

$$m_3(s_1, s_2e', s') = m'_3(s_1, s_2e', s')$$

by the assumption of the theorem. Therefore,

$$(m'_3 - m_3)(s_1, s_2, e) = -(m'_3 - m_3)(s_1s_2, e', s'). \tag{2.6}$$

Let us define the linear map

$$f_{e',s'} : H^0(L) \otimes H^1(M) \to H^0(LM)$$

by the formula $f_{e',s'}(s,e) = (m_3' - m_3)(s,e',s')$. We claim that $f_{e',s'}$ doesn't depend on a choice of (M',L') and e',s' such that $e's' = e$. Indeed, $H^0(L)$ is generated by sections of the form $s = s_1 s_2$ where s_1 and s_2 are as above and the equality (2.6) shows that for such sections $f_{e',s'}(s,e)$ doesn't depend on (e',s'). Thus, we can set $f_2^{3,2} = f_{e',s'}$. Now the same equality shows that for any line bundles L_1, L_2 such that $\deg(L_i) = 1$, $L_1 L_2 \simeq L$ and the quadruple $(\mathcal{O}, L_1, L, LM)$ is transversal one has

$$(m_3' - m_3)(s_1, s_2, e) = -f_2^{3,2}(s_1 s_2, e).$$

Now assume that $w \geq 3$. Let us fix line bundles $M_1, \dots, M_{n-2}$ and elements $e_i \in H^1(M_i)$ for $i = 1, \dots, n-2$. Let us also fix a line bundle L of degree w, such that the collection

$$(\mathcal{O}, L, LM_1, \dots, LM_1 \dots M_{n-2})$$

is transversal. Then for every pair of line bundles L_1 and L_2 of positive degree such that $L_1 L_2 \simeq L$ and the collection $(\mathcal{O}, L_2, L_2 M_1, \dots, L_2 M_1 \dots M_{n-2})$ is transversal, consider the map

$$b_{L_1,L_2} : H^0(L_1)H^0(L_2) \to H^0(L_1 L_2 M_1 \dots M_{n-2}) : (s_1, s_2) \mapsto (m_n' - m_n)(s_1, s_2, e_1, \dots, e_{n-2}).$$

We claim that these maps satisfy the condition

$$b(s_1 s_2, s_3) = b(s_1, s_2 s_3)$$

for any sections $s_i \in L_i$, $i = 1,2,3$, where $L_1 L_2 L_3 \simeq L$, $\deg(L_i) > 0$, the collection

$$(\mathcal{O}, L_2, L_2 L_3, L_2 L_3 M_1, \dots, L_2 L_3 M_1 \dots M_{n-2})$$

is transversal. Indeed, the constraint Ax_n implies that

$$m_n(s_1 s_2, s_3, e_1, \dots, e_{n-2}) - m_n(s_1, s_2 s_3, e_1, \dots, e_{n-2})$$

is a linear combination of terms either involving only m_k with $k < n$ or involving products m_n of weight $< w$. The same is true for m', so our claim follows from the induction assumptions on m and m'. Therefore, we can apply Lemma 2.3 to the line bundle L and the set of isomorphism classes

$$S = \{[M] : \ (\mathcal{O}, M, MM_1, \dots, MM_1 \dots M_{n-2}) \text{ is transversal}\}.$$

We conclude that there exists a linear map

$$f_{e_1,\dots,e_{n-2}} : H^0(L) \to H^0(LM_1 \dots M_{n-2})$$

satisfying

$$m_n'(s_1, s_2, e_1, \dots, e_{n-2}) - m_n(s_1, s_2, e_1, \dots, e_{n-2}) = (-1)^n f_{e_1,\dots,e_{n-2}}(s_1 s_2, e_1, \dots, e_{n-2}).$$

One can see from this defining property that the map

$$f_{n-1}^{n,w} : H^0(L)H^1(M_1) \dots H^1(M_{n-2}) \to H^0(LM_1 \dots M_{n-2}) : s \otimes \otimes e_1 \dots \otimes e_{n-2} \to f_{e_1,\dots,e_{n-2}}(s)$$

is linear and gives the required homotopy.

The proof of uniqueness is also achieved by induction. It suffices to check that an admissible transversal homotopy $f = (f_n)$ from m to m such that $f_k = 0$ for $2 \leq k < n$ has also $f_n = 0$. By cyclic symmetry it suffices to consider the maps

$$f_n : H^0(L)H^1(M_1) \dots H^1(M_{n-1}) \to H^0(LM_1 \dots M_{n-1})$$

where $(\mathcal{O}, L, LM_1, \dots, LM_1 \dots M_{n-1})$ is transversal. Now we use the induction in degree of L. If $\deg(L) = 1$ then such a map is automatically zero. If $\deg(L) > 1$ then it suffices to consider elements of $H^0(L)$ of the form ss_p where $s_p \in H^0(\mathcal{O}(p))$, $s \in H^0(L(-p))$ (where $\mathcal{O}(p)$ is transversal to all the relevant bundles). Then we can use the identity for f_n and the induction assumption to prove the desired vanishing. $\square$

2.3. **An identity between triple products.** Assume that we are given a transversal admissible A_∞-structure on the category of line bundles on E. Let (L_1, M, L_2) be a triple of line bundles such that $\deg(L_1) = \deg(L_2) = n > 0$, $\deg(M) = -n$, and the collection $(\mathcal{O}, L_1, L_1M, L_1ML_2)$ is transversal. Then the triple products

$$m_3 : H^0(L_1)H^1(M)H^0(L_2) \to H^0(L_1ML_2)$$

are invariant under any homotopy. However, in theorem 2.2 only such triple products with $n = 1$ appear. The reason is that one can express all triple products as above in terms of those with $n = 1$. This is done by induction in n using the identity below.

Assume that $L_i = L'_iL''_i$ for $i = 1, 2$, where $\deg(L'_i) = n'$, $\deg(L''_i) = n''$ for some positive integers n', n'' such that $n = n' + n''$. Assume also that the collection $(\mathcal{O}, L'_1, L_1, L_1M, L_1ML'_2, L_1ML_2)$ is transversal.

Proposition 2.4. *One has the following identity*

$$m_3(s'_1s''_1, e, s'_2s''_2) = m_3(s'_1, s''_1e, s'_2)s''_2 + s'_1m_3(s''_1, es'_2, s''_2)$$

where $s'_i \in H^0(L'_i)$*,* $s''_i \in H^0(L''_i)$*,* $e \in H^1(M)$*.*

Proof. Applying the A_∞-constraint Ax_3 we get

$$m_3(s'_1s''_1, e, s_2) = m_3(s'_1, s''_1e, s'_2s''_2) + s'_1m_3(s''_1, e, s'_2s''_2).$$

Applying Ax_3 again we obtain the following expressions for the terms in the RHS:

$$m_3(s'_1, s''_1e, s'_2s''_2) = m_3(s'_1, s''_1e, s'_2)s''_2 + s'_1m_3(s''_1e, s'_2, s''_2),$$

$$m_3(s''_1, e, s'_2s''_2) = m_3(s''_1, es'_2, s''_2) - m_3(s''_1e, s'_2, s''_2).$$

Substituting these expressions in the above equality we get the result. □

3. Application to homological mirror symmetry

3.1. **Adding unipotent bundles.** By a unipotent bundle we mean a vector bundle which has a filtration by subbundles such that the associated graded bundle is trivial. Let $\mathcal{LU} = \mathcal{LU}(E)$ be the full subcategory in $\mathrm{Vect}(E)$ consisting of bundles of the form LU, where L is a line bundle, U is a unipotent bundle. Note that a decomposition of LU into a tensor product of a line bundle and a unipotent bundle is unique up to an isomorphism.

Assume that we are given a notion of transversality for pairs of line bundles. We can extend it to the category $\mathcal{LU}$ by calling a pair $(LU, L'U')$ transversal if and only if (L, L') is transversal. Then we define an admissible transversal A_∞-structure on $\mathcal{LU}$ as a transversal A_∞-structure on $\mathcal{LU}$ which is cyclic with respect to Serre duality and is strictly compatible with tensor multiplication by a line bundle, has $m_1 = 0$ and m_2 equal to the standard product.

One defines a notion of admissible homotopy between admissible A_∞-structures on $\mathcal{LU}$ similarly to the case of the category $\mathcal{L}$.

The proof of the following theorem is very similar to that of theorem 2.2 so we omit it.

Theorem 3.1. *Let m and m' be admissible transversal A_∞-structures on the category $\mathcal{LU}$. Assume that for every triple of line bundles (L_1, M, L_2) such that $\deg(L_1) = \deg(L_2) = 1$, $\deg(M) = -1$ and such that $(\mathcal{O}, L_1, L_1M, L_1ML_2)$ is transversal, and for every quadruple of unipotent bundles U_0, U_1, U_2 and U_3 the maps*

$$\mathrm{Hom}(U_0, L_1U_1) \otimes \mathrm{Ext}^1(L_1U_1, L_1MU_2) \otimes \mathrm{Hom}(L_1MU_2, L_1ML_2U_3) \to \mathrm{Hom}(U_0, L_1ML_2U_3) \tag{3.1}$$

given by m_3 and m'_3 coincide. Then there exist a unique admissible homotopy between m and m'.

3.2. **Connection with the Fukaya category.** Let $\tau \in \mathbb{C}$ be an element in the upper half-plane, $E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, be the corresponding elliptic curve. Then as shown in [16] the (usual) category $\mathcal{LU}$ is equivalent to the subcategory in the Fukaya category (with compositions m_2^F) consisting of objects $(\overline{L}, \lambda \cdot \mathrm{Id} + N)$ where $\overline{L}$ has an integer slope, $\lambda \in \mathbb{R}$, N is a nilpotent operator.

Let $L(0)$ be the line bundle on E such that the theta-function

$$\theta(z) = \theta(z, \tau) = \sum_{n \in \mathbb{Z}} \exp(\pi i \tau n^2 + 2\pi i n z)$$

is the pull-back of a section of L. So $L(0) \simeq \mathcal{O}_E(z_0)$ where $z_0 = \frac{\tau+1}{2} \mod (\mathbb{Z} + \mathbb{Z}\tau)$. For every $u \in \mathbb{C}$ let us denote $L(u) = t_u^* L(0)$, where $t_u : E \to E$ is the translation by u. Then every line bundle of degree n is isomorphic to a line bundle of the form $L(0)^{\otimes(n-1)} \otimes L(u)$.

For a nilpotent operator $N : V \to V$ we denote by $\mathcal{V}_N$ the unipotent bundle on E, such that the sections of $\mathcal{V}_N$ correspond to V-valued functions on $\mathbb{C}$ satisfying the quasi-periodicity equations $f(z+1) = f(z)$, $f(z+\tau) = \exp(2\pi i N) f(z)$. Then every unipotent bundle is isomorphic to a bundle of the form $\mathcal{V}_N$.

The correspondence between bundles in $\mathcal{LU}$ and objects of the Fukaya category constructed in [16] associates to the bundle $\mathcal{V} = L(0)^{\otimes(n-1)} \otimes L(u) \otimes \mathcal{V}_N$ the object $O = (\overline{L}, -u_1 \mathrm{Id} + N)$, where $u = u_1 + \tau u_2$, $u_i \in \mathbb{R}$, $\overline{L} = \{(u_2 + x, (n-1)u_2 + nx), x \in \mathbb{R}/\mathbb{Z}\}$.

This correspondence extends to a functor from $\mathcal{LU}$ to the Fukaya category (with m_2^F as a composition) as follows. Let $\mathcal{V}' = L(0)^{\otimes(n'-1)} \otimes L(u') \otimes \mathcal{V}_{N'}$ be another bundle in $\mathcal{LU}$, where $n' \in \mathbb{Z}$, $u' = u_1' + \tau u_2' \in \mathbb{C}$, $N' : V' \to V'$ is a nilpotent operator. Let $O' = (\overline{L'}, -u_1' \mathrm{Id} + N')$ be the corresponding object in the Fukaya category. Note that O and O' are transversal if and only if either $n' \neq n$, or $n' = n$ and $u_2' - u_2 \notin \mathbb{Z}$. In the latter case $\mathrm{Hom}(\mathcal{V}, \mathcal{V}') = \mathrm{Hom}(O, O') = 0$ so we can assume that $n \neq n'$. Assume first that $n < n'$. Then $\mathrm{Hom}(O, O') = \mathrm{Hom}(V, V') \otimes \mathrm{Hom}(\overline{L}, \overline{L'})$ has degree zero. We can enumerate the points of intersection $\overline{L} \cap \overline{L'}$ by residues $k \in \mathbb{Z}/(n'-n)\mathbb{Z}$. Namely, this intersection consists of the points

$$P_k = \left(\frac{k + u_2' - u_2}{n' - n}, \frac{nk + nu_2' - n'u_2}{n' - n}\right)$$

where $k \in \mathbb{Z}/(n'-n)\mathbb{Z}$. On the other hand, we have

$$\mathrm{Hom}(\mathcal{V}, \mathcal{V}') = H^0(E, L(0)^{n'-n-1} \otimes L(u'-u) \otimes \mathcal{V}_{N'-N^*})$$

where we consider N^* and N' as operators on $V^* \otimes V'$ (acting trivially on one component). Note that if M is a line bundle on E of the form $L(0)^{\otimes(m-1)} \otimes L(u)$ where $m \neq 0$ and $N : V \to V$ is a nilpotent operator then there is a natural isomorphism between Dolbeault complexes of bundles $M \otimes V$ and $M \otimes_{\mathcal{O}} \mathcal{V}_N$. Indeed, using the trivialization of the pull-backs of M and $\mathcal{V}_N$ to $\mathbb{C}$ we can define the map from the Dolbeault complex of $M \otimes V$ to that of $M \otimes_{\mathcal{O}} \mathcal{V}_N$ by sending $\eta(z)$ to $\eta(z - N/m)$ where

$$(f \otimes v)(z - \frac{N}{m}) = \exp(-\partial_z \frac{N}{m})(f) \cdot v$$

In particular, we can identify $\mathrm{Hom}(\mathcal{V}, \mathcal{V}')$ with the space $\mathrm{Hom}(V, V') \otimes H^0(E, L(0)^{n'-n-1} \otimes L(u'-u))$. The space of global sections of the line bundle $L(0)^{n'-n-1} \otimes L(u'-u)$ has a natural basis of theta functions

$$\theta_k(z) = \sum_{m \in (n'-n)\mathbb{Z} + k} \exp\left(\frac{1}{n'-n}(\pi i \tau m^2 + 2\pi i m((n'-n)z + u' - u))\right)$$

where $k \in \mathbb{Z}/(n'-n)\mathbb{Z}$. Now we can identify $\mathrm{Hom}(\mathcal{V}, \mathcal{V}')$ with $\mathrm{Hom}(O, O')$ by sending $T \otimes [P_k]$ (where $T \in \mathrm{Hom}(V, V')$) to

$$\exp\left(\frac{1}{n'-n}(-\pi i \tau (u_2' - u_2)^2 \mathrm{Id} + 2\pi i (u_2' - u_2)(N' - N^* - (u_1' - u_1)\mathrm{Id}))\right) \cdot T \otimes \theta_k.$$

To construct similar identification in the case $n > n'$ we use Serre duality and its natural analogue on the Fukaya category to reduce to the case considered above. As shown in [16] this identification is compatible with compositions m_2. Using it we can consider m^F as a transversal A_∞-structure on $\mathcal{LU}$.

Furthermore, it is easy to see that m^F is admissible. The main point is that the functor of tensoring with a line bundle on $\mathcal{LU}$ corresponds to an automorphism of the Fukaya category given by some symplectic automorphism of the torus. As we will see in section 3.4 the assumptions of the theorem 3.1 are satisfied for the transversal A_∞-structures m^F and m^H on $\mathcal{LU}$. Hence, they are homotopic.

The equivalence of $\mathcal{LU}$ with a subcategory of the Fukaya category (with m_2^F as a composition) is extended to all bundles in [16] using the construction of vector bundles on E as push-forwards of objects in $\mathcal{LU}$ under isogenies. Below we consider the corresponding extension of equivalence between A_∞-structures.

3.3. **Equivalence.** Let $\tau \in \mathbb{C}$ be an element in the upper half-plane, $E = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, be the corresponding elliptic curve. We are going to prove that the two transversal A_∞-structures m^F and m^H on $\mathrm{Vect}(E)$ considered in section 1.5 are equivalent. More precisely, the definition of m^H requires us to work with bundles equipped with hermitian metrics. Since the different choices of metrics on a vector bundle really give equivalent objects of the A_∞-category $(\mathrm{Vect}^h(E), m^H)$ we can restrict to some preferred class of hermitian metrics (which we'll define below).

Note that both A_∞-structures are cyclic and are strictly compatible with tensor multiplication by a hermitian line bundle and with decompositions of bundles into orthogonal direct sums.

For every positive integer r we consider the elliptic curve $E_r = \mathbb{C}/\mathbb{Z} + \mathbb{Z}r\tau$. Then we have a natural isogeny $\pi^r : E_r \to E$ of degree r and for every $r|s$ an isogeny $\pi_r^s : E_s \to E_r$ such that $\pi^s = \pi^r \circ \pi_r^s$. We can consider two transversal A_∞-structures m^F and m^H on any of these elliptic curves. An important observation is that both m^F and m^H are strictly compatible with the functors of pull-back and push-forward with respect to isogenies π_r^s and π^r (these functors extend naturally to vector bundles with metric). For the structure m^H this is clear while for m^F this follows from the construction of equivalence in [16].

The idea of the proof is to use the decomposition of every bundle on elliptic curve into a direct sum $V = \oplus V_i U_i$ where (V_i) are pairwise non-isomorphic stable bundles, (U_i) are unipotent bundles. Then we want to use the fact that every stable bundle of rank r on E is the push-forward of a line bundle on E_r. Since our A_∞-structures are strictly compatible with isogenies we can derive the desired homotopy from theorem 3.1. More precisely, we need a slight modification of this theorem for the category of bundles with metrics: the assumption should be that the triple products (3.1) given by two A_∞-structures coincide for all choices of metrics on the bundles in question. To be able to apply this theorem in our case we will compute explicitly products m_3^F and m_3^H of the type (3.1) in section 3.4 and will see that they are equal (at this point it will be important to use a particular trivialization of ω_E on which the construction of equivalence in [16] depends). Then the uniqueness of the homotopies constructed in theorem 3.1 will imply that these homotopies are compatible with isogenies, hence, descend to a homotopy on the category $\mathrm{Vect}(E)$.

Let us call a vector bundle on E *almost stable* if it has form $V \otimes U$ where V is a stable bundle, U is a unipotent bundle (thus, every bundle on E is a direct sum of almost stable bundles). Let r be the rank of V. Then $V = \pi_*^r(L)$ for some line bundle L on E_r. Hence $V \otimes U = \pi_*^r(L \otimes (\pi^r)^* U)$. We call a hermitian metric on V *preferred* if it comes from a metric on $L \otimes (\pi^r)^* U$. Let $(V_1, \dots, V_{n+1})$ be a collection of almost stable bundles on E equipped with preferred metrics. From the strict compatibility of our A_∞-structures with isogenies we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^*(V_1,V_2)\dots\mathrm{Hom}^*(V_n,V_{n+1}) \arrow[r, "m_n"] \arrow[d, "(\pi^r)^*"] & \mathrm{Hom}^*(V_1,V_{n+1}) \arrow[d, "(\pi^r)^*"] \\
\mathrm{Hom}^*((\pi^r)^*V_1,(\pi^r)^*V_2)\dots\mathrm{Hom}((\pi^r)^*V_n,(\pi^r)^*V_{n+1}) \arrow[r, "m_n"] & \mathrm{Hom}^*((\pi^r)^*V_1,(\pi^r)^*V_{n+1})
\end{tikzcd} \tag{3.2}$$

where $m = m^F$ or $m = m^H$. Now if r is divisible by ranks of all bundles V_i then $(\pi^r)^*(V_i)$ is an orthogonal direct sum of bundles of the form LU where L is a line bundle, U is a unipotent bundle.

We will check in section 3.4 that the conditions of theorem 3.1 are satisfied for m^F and m^H. Therefore, we get a unique admissible homotopy f^r between these structures on $\mathcal{LU}(E^r)$ for every r. We can extend this homotopy to orthogonal direct sums of bundles in $\mathcal{LU}(E^r)$ in an obvious way. Note that for every isogeny of elliptic curves $\pi : E' \to E''$ we have a canonical splitting of the natural embedding $\mathcal{O}_{E''} \to \pi_* \mathcal{O}_{E'}$, hence for every pair of bundles (V_1, V_2) on E'' we get a canonical splitting

$$T(\pi) : \mathrm{Hom}(\pi^* V_1, \pi^* V_2) \to \mathrm{Hom}(V_1, V_2)$$

of the natural embedding $\pi^* : \mathrm{Hom}(V_1, V_2) \to \mathrm{Hom}(\pi^* V_1, \pi^* V_2)$. Now we claim that the homotopies f^r and f^s where $r|s$ are compatible in the following way: for any bundles $W_1 = L_1 U_1, \ldots, W_{n+1} = L_{n+1} U_{n+1}$ in $\mathcal{LU}(E^r)$ one has the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}^*(W_1, W_2) \ldots \mathrm{Hom}^*(W_n, W_{n+1}) \arrow[r, "f_n^r"] \arrow[d, "(\pi_r^s)^*"] & \mathrm{Hom}^*(W_1, W_{n+1}) \\
\mathrm{Hom}^*((\pi_r^s)^*(W_1), (\pi_r^s)^*(W_2)) \ldots \mathrm{Hom}((\pi_r^s)^*(W_n), (\pi_r^s)^*(W_{n+1})) \arrow[r, "m_n"] & \mathrm{Hom}^*((\pi_r^s)^*(W_1), (\pi_r^s)^*(W_{n+1})) \arrow[u, "T(\pi_r^s)"']
\end{tikzcd} \tag{3.3}$$

Indeed, the compatibility of m^F and m^H with the isogeny π_r^s implies that $T(\pi_r^s) \circ f^s \circ (\pi_r^s)^*$ is an admissible homotopy between m^F and m^H on $\mathcal{LU}(E^r)$, hence it coincides with f^r.

Now we define the homotopy f between m^F and m^H on the category of almost stable vector bundles on E with preferred metrics using commutativity of diagrams of the type (3.2). Namely, choosing r which is divisible by all ranks of bundles V_i we define the map

$$f_n : \mathrm{Hom}(V_1, V_2) \ldots \mathrm{Hom}(V_n, V_{n+1}) \to \mathrm{Hom}(V_1, V_{n+1})$$

by the formula $f_n = T(\pi^r) \circ f_n^r \circ (\pi^r)^*$. The compatibility (3.3) ensures that this definition doesn't depend on a choice r. Now to check that f is indeed a homotopy from m^F to m^H we choose r divisible by ranks of all the bundles involved and use the commutativity of (3.2).

Since every bundle V on E is a direct sum of almost stable bundles we have a class of preferred metrics on V coming from preferred metrics on almost stable bundles (so that the direct sum becomes orthogonal). We can extend the homotopy f to all bundles with preferred metrics in a natural way.

3.4. **Massey products.** It remains to compute explicitly the products m_3^F and m_3^H of the type (3.1). Let us trivialize ω_E in such a way that the Serre duality induces the pairing

$$b : \mathrm{Hom}(V_1, V_2) \otimes \mathrm{Ext}^1(V_2, V_1) \to \mathbb{C}$$

given by the formula

$$b(f, g d\overline{z}) = \int_E dz \wedge \mathrm{Tr}(f \circ g d\overline{z})$$

where $f \in \mathrm{Hom}(V_1, V_2)$, $g d\overline{z} \in \Omega^{0,1}(\mathrm{Hom}(V_2, V_1))$.

First let us compute m_3^H. We start with the case when all U_i are trivial of rank 1. Then we have to compute the product

$$m_3^H : H^0(L_1) H^1(M) H^0(L_2) \to H^0(L_1 M L_2)$$

where $L_1 M \not\simeq \mathcal{O}$, $L_2 M \not\simeq \mathcal{O}$. Using a translation on E we can assume without loss of generality that $M = L(0)^{-1}$. Let $L_1 = L(t)$, $L_2 = L(u)$ where $t, u \in \mathbb{C}$. Let z_1 and z_2 be the real components of the complex variable z defined by the equality $z = z_1 + \tau z_2$. The transversality condition means that

$t_2, u_2 \notin \mathbb{Z}$. We will compute the above product under the weaker assumption $t, u \notin \mathbb{Z} + \mathbb{Z}\tau$. It is easy to check that the $(0,1)$-form with values in $L(0)^{-1}$

$$\alpha(z) = \frac{i}{\sqrt{2\operatorname{Im}(\tau)}}\overline{\theta(z)}\exp(-2\pi\operatorname{Im}(\tau)(z_2^2))d\overline{z}$$

is a representative of the class in $H^1(L(0)^{-1})$ dual to the class in $H^0(L(0))$ given by $\theta(z)$. Now for every $u \in \mathbb{C}$, such that $u \notin \mathbb{Z} + \mathbb{Z}\tau$ there exists a unique section $h(z,u)$ of $L(0)^{-1}L(u)$ such that

$$\theta(z+u)\alpha(z) = \overline{\partial}h(z,u)$$

where $\overline{\partial} = \overline{\partial}_z$. Indeed, this follows from the fact that all the cohomologies of $L(0)^{-1}L(u)$ vanish. One can write an explicit formula for $h(z,u)$(see [15]):

$$h(z,u) = -\frac{1}{2\pi i}\sum_{m,n\in\mathbb{Z}}(-1)^{mn}\frac{\exp(-\frac{\pi}{2\operatorname{Im}(\tau)}(|\gamma|^2 + 2\overline{\gamma}u + u^2) + 2\pi i(mz_1 + (n-u)z_2))}{\gamma + u}$$

where $\gamma = m\tau - n$. Now we have

$$m_3^H(\theta(z+t), \alpha, \theta(z+u)) = h(z,t)\theta(z+u) - h(z,u)\theta(z+t).$$

As a function of z up to a constant factor this should be equal to $\theta(z+u+v)$, so we have

$$h(z,t)\theta(z+u) - h(z,u)\theta(z+t) = H(t,u)\theta(z+t+u) \tag{3.4}$$

for some meromorphic function H. We have $H(t,u) = -H(u,t)$. Also it is easy to see that the function $H(t,u)$ satisfies the following quasi-periodicity equations:

$$H(t+1,u) = H(t,u),$$

$$H(t+\tau,u) = \exp(2\pi iu)H(t,u).$$

The only poles of $H(t,u)$ are poles of order 1 along the divisors $t = \gamma$ and $u = \gamma$ where $\gamma \in \mathbb{Z} + \mathbb{Z}\tau$. It follows that $H(t,u)$ is equal up to a constant to the function

$$F(t,u) = \frac{\theta'(\frac{\tau+1}{2})\theta(t-u+\frac{\tau+1}{2})}{2\pi i\theta(t+\frac{\tau+1}{2})\theta(-u+\frac{\tau+1}{2})}.$$

Furthermore, comparing the residues at $t = 0$ we conclude that $H(t,u) = -F(t,u)$.

Now let us compute the product

$$m_3^H : H^0(L_1U_0^\vee U_1)H^1(MU_1^\vee U_2)H^0(L_2U_2^\vee U_3) \to H^0(L_1ML_2U_0^\vee U_3)$$

where U_i are unipotent bundles. As before we can take $M = L(0)^{-1}$, $L_1 = L(t)$, $L_2 = L(u)$. Let $U_i = \mathcal{V}_{N_i}$ where $N_i : V_i \to V_i$ are nilpotent operators. Then $U_i^*U_{i+1} \simeq \mathcal{V}_{N_{i+1}-N_i^*}$ where $N_{i+1} - N_i^*$ is an operator on $V_i^*V_{i+1}$. As in section 3.2 we use the isomorphisms between the Dolbeault complexes of bundles LV and $L\mathcal{V}_N$, where L is one of line bundles of degree 1 above, $N : V \to V$ is the corresponding nilpotent operator, sending $\eta(z)$ to $\eta(z-N)$. Similarly, we have an isomorphism between the Dolbeault complexes of $L(0)^{-1}V_1^*V_2$ and $L(0)^{-1}\mathcal{V}_{N_2-N_1^*}$ given by $\eta(z) \mapsto \eta(z+N_2-N_1^*)$. Let $v_{i,i+1} \in V_i^* \otimes V_{i+1}$ be some elements. Then we have

$$\begin{aligned}
&(\alpha(z+N_2-N_1^*)v_{1,2}) \circ (\theta(z+t-N_1+N_0^*))v_{0,1}) = \\
&\operatorname{Tr}_{V_1}(\overline{\partial}h(z+N_2-N_1^*, t-N_2+N_1^*-N_1+N_0^*))v_{0,1}v_{1,2}) = \\
&\operatorname{Tr}_{V_1}(\overline{\partial}h(z+N_2-N_1^*, t-N_2+N_0^*)v_{0,1}v_{1,2}),
\end{aligned}$$

since we can replace N_1^* by N_1 under the sign of Tr_{V_1}. Similarly, we get

$$(\theta(z+u-N_3+N_2^*))v_{2,3}) \circ (\alpha(z+N_2-N_1^*)v_{1,2}) = \operatorname{Tr}_{V_2}(\overline{\partial}h(z+N_2-N_1^*, u-N_3+N_1^*)v_{1,2}v_{2,3}).$$

Hence,

$$m_3^H(\theta(z+t-N_1+N_0^*))v_{0,1},\alpha(z+N_2-N_1^*)v_{1,2},\theta(z+u-N_3+N_2^*)v_{2,3}) =$$
$$\mathrm{Tr}_{V_1V_2}((\theta(z+u-N_3+N_2^*)h(z+N_2-N_1^*,t-N_2+N_0^*) -$$
$$h(z+N_2-N_1^*,u-N_3+N_1^*)\theta(z+t-N_1+N_0^*))v_{0,1}v_{1,2}v_{2,3}).$$

Making a substitution $z\mapsto z+N_2-N_1^*$, $t\mapsto t-N_2+N_0^*$, $u\mapsto u-N_3+N_1^*$ in the identity (3.4) and using the equality $H=-F$ we can rewrite the above formula as follows:

$$\begin{aligned}&m_3^H(\theta(z+t-N_1+N_0^*)v_{0,1},\alpha(z+N_2-N_1^*)v_{1,2},\theta(z+u-N_3+N_2^*)v_{2,3}) =\\ &\mathrm{Tr}_{V_1V_2}(F(t-N_2+N_0^*),u-N_3+N_1^*))\theta(z+t+u-N_3+N_0^*)v_{0,1}v_{1,2}v_{2,3}).\end{aligned} \tag{3.5}$$

Now let us compute the corresponding product m_3^F. The objects of the Fukaya category corresponding to our four bundles $U_0=\mathcal{V}_{N_0}$, $L_1U_1=L(t)\mathcal{V}_{N_1}$, $L_1MU_2=L(0)^{-1}L(t)\mathcal{V}_{N_2}$ and $L_1ML_2U_3=L(t+u)\mathcal{V}_{N_3}$ are $((x,0),N_0)$, $((x+t_2,x),-t_1+N_1)$, $((x,-t_2),-t_1+N_2)$ and $((x+t_2+u_2,x),-t_1-u_1+N_3)$, where $t=t_1+\tau t_2$, $u=u_1+\tau u_2$, $t_2,u_2\notin\mathbb{Z}$. Note that any two of these circles either don't intersect or intersect at a unique point. So we can identify morphisms between these objects with spaces $\mathrm{Hom}(V_0,V_1)$, $\mathrm{Hom}(V_1,V_2)$, etc. Now we have

$$-m_3^F(v_{0,1}[P_{0,1}],v_{1,2}[P_{1,2}],v_{2,3}[P_{2,3}]) = \mathrm{Tr}_{V_1V_2}\sum_{(m,n)\in\mathbb{Z}^2,(m-t_2)(n+u_2)>0}\mathrm{sign}(m-t_2)$$
$$\exp(2\pi i\tau(m-t_2)(n+u_2)+2\pi i(m-t_2)(-t_1+N_2-N_0^*)+2\pi i(n+u_2)(u_1-N_3+N_1^*))v_{0,1}v_{1,2}v_{2,3})[P_{0,3}]$$
$$=\mathrm{Tr}_{V_1V_2}(\sum\mathrm{sign}(m-t_2)\exp(2\pi i\tau mn+2\pi i m(u-N_3+N_1^*)+2\pi i n(-t+N_2-N_0^*))\cdot C\cdot v_{0,1}v_{1,2}v_{2,3})$$

where $C=\exp(-2\pi i\tau t_2u_2-2\pi i t_2(u_1-N_3+N_1^*)+2\pi i u_2(-t_1+N_2-N_0^*))$. At this point we need the following identity (which essentially coincides with the formula (2.3.4) of [14]):

$$\sum_{(m,n)\in\mathbb{Z}^2,(m-t_2)(n+u_2)>0}\mathrm{sign}(m-t_2)\exp(2\pi i\tau mn+2\pi i(mu-nt)) = F(t,u)$$

for arbitrary $t=t_1+\tau t_2$, $u=u_1+\tau u_2$ such that $t_2,u_2\in\mathbb{Z}$. This identity which is due to Kronecker can be proven as follows: first, one has to check that the left hand side extends to a meromorphic function of u and t with poles at the lattice points, then one has to compare its quasi-periodicity properties and residues at poles with those of F. Hence, we get

$$m_3^F(v_{0,1}[P_{0,1}],v_{1,2}[P_{1,2}],v_{2,3}[P_{2,3}]) = -\mathrm{Tr}_{V_1V_2}(F(t-N_2+N_0^*,u-N_3+N_1^*)\cdot C\cdot v_{0,1}v_{1,2}v_{2,3})[P_{0,3}]. \tag{3.6}$$

Now it easy to see that the exponential factors involved in the identification of morphisms in $\mathcal{LU}$ with morphisms in the Fukaya category (see section 3.2) kill the factor C and we get $m_3^H=m_3^F$ on the products of the type (3.1).

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