

# Perturbed Pöschl-Teller oscillators

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## Abstract

Within the framework of Lanczos-inspired perturbation theory wave functions and energies in the short-range potential  $V(x) = a\rho^2(x) + \lambda\rho^4(x)$  with  $\rho(x) = \operatorname{sech} \alpha x$  and a small coupling  $\lambda$  are shown obtainable in closed form.

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Pöschl-Teller [1] potential  $V^{(0)}(x) = -\mu(\mu+1)\text{sech}^2\alpha x$  resembles the celebrated harmonic oscillator. Within the so called shape invariant family [2] these two spatially symmetric exactly solvable potentials form a unique subset. In perturbation theory there seems to emerge one of the most significant differences between them. In contrast to an enormous interest in the various anharmonic forces [3] there exists virtually no analysis of a perturbed Pöschl-Teller model in the current literature. Partially, we intend to fill the gap. This letter is concerned with the most elementary quartic example

$$\left[ -\frac{d^2}{dx^2} - \frac{\mu(\mu+1)}{\cosh^2\alpha x} + \frac{4\lambda}{\cosh^4\alpha x} \right] \psi_{(PPT)}(x) = -\kappa_{(PPT)}^2 \psi_{(PPT)}(x). \quad (1)$$

The parity is conserved,  $\psi_{(PPT)}(-x) = (-1)^p \psi_{(PPT)}(x)$ ,  $p = 0, 1$ , and all the unperturbed  $\lambda = 0$  bound states are available in closed form. Unfortunately, after we re-scale  $\alpha \rightarrow 1$  for simplicity we immediately notice that the normalizable states form a mere finite set with  $\kappa_{(PPT)}^{(0)} = \mu - 2N - p = \kappa(N, p)$  and wave functions

$$\langle x | \psi_{2N+p}^{(0)} \rangle = \frac{\tanh^p x}{\cosh^{\kappa(N,p)} x} \cdot {}_2F_1\left(\mu - N + \frac{1}{2}, -N, 1 + \kappa(N, p), \frac{1}{\cosh^2 x}\right) \quad (2)$$

where  $0 \leq 2N + p < \mu$  [4]. This is the reason why these mutually orthogonal elementary functions cannot be used as an unperturbed basis.

A key to our present  $\lambda \neq 0$  construction will lie in the use of a non-orthogonal basis. We shall employ the ansatz

$$\psi_{(PPT)}(x) = \tanh^p x \sum_{n=0}^{\infty} \frac{c_n(\lambda)}{\cosh^{2n+\kappa_{(PPT)}} x} \quad (3)$$

with  $\kappa_{(PPT)} = \kappa(N, p) + 2\varepsilon(\lambda)$ . Its use is inspired by the general method of Lanczos [5] and its perturbative implementations [6]. The basis itself is taken from the particular terminating expansions (2) which define the unperturbed coefficients  $c_n^{(0)}$  such that  $c_N^{(0)} \neq 0$  and  $c_0^{(0)} = 1$  while  $c_{N+1}^{(0)} = c_{N+2}^{(0)} = \dots = 0$ .

As always in similar constructions one inserts eq. (3) in our differential eq. (1). This gives the current recurrence relations which may be written in the three-term form

$$\lambda c_{n-1} + \beta_n c_n + \alpha_{n+1} c_{n+1} = 0, \quad n = 0, 1, \dots \quad (4)$$

or as an infinite-dimensional linear algebraic problem

$$\begin{pmatrix} \beta_0 & \alpha_1 & & \\ \lambda & \beta_1 & \alpha_2 & \\ & \lambda & \beta_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \end{pmatrix} = 0. \quad (5)$$

In general, unfortunately, equation (5) *cannot* be treated as an infinite-dimensional limit of its truncated matrix subsystems due to their non-variational, power-series origin [7, 8]. Benefits brought by the tridiagonality concern only the wave functions. Their coefficients are defined by the closed determinantal formulae [9]

$$c_{n+1} \sim \det \begin{pmatrix} \beta_0 & \alpha_1 & & \\ \lambda & \beta_1 & \ddots & \\ & \ddots & \ddots & \lambda \alpha_n \\ & & \lambda & \beta_n \end{pmatrix}, \quad n = 0, 1, \dots \quad (6)$$

Once we postulate, in the spirit of the current perturbation theory,

$$c_n(\lambda) = c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} + \dots, \quad n = 1, 2, \dots, \quad (7)$$

we may generate the separate coefficients from eq. (6) by elementary algebra. This type of dependence of the wave functions upon the energy which is not known resembles the Brillouin-Wigner perturbation method able to treat the energy as an *external* parameter. In the similar vein the use of the variational energies has been recommended in the semi-analytic power-series context, e.g., by Hautot [8] and Tater [10]. The subtle problem of the determination of energies in the present perturbative example is to be settled in what follows.

For the sake of definiteness, perturbations with  $c_n(\lambda) = c_n^{(0)} + \lambda h_n(\lambda)$  will be normalized to  $h_0 = 0$ . In order to reduce the complexity of formulae we shall only pay attention to the ground state problem with quantum numbers  $p = 0$  and  $N = 0$ . This implies that for the sufficiently small perturbations our bound state exists at any non-negative  $\mu > 0$  [4]. The details of transition to  $p = 1$  and/or to  $N > 0$  [11] are not too interesting in the present context. For our choice of  $p = 0$  and

$N = 0$  the accepted normalizations simplify the very first recurrence relation. It reads  $\beta_0 + \lambda \alpha_1 h_1 = 0$ , defines the coefficient  $h_1 = \mathcal{O}(1)$  “to all orders” and suggests a change of the notation,  $\beta_0 = \lambda \gamma_0$ . As long as  $\gamma_0 = \mathcal{O}(1)$  we may recall our explicit  $\beta_0 \equiv \varepsilon(\varepsilon + \mu + 1/2)$  and infer that  $\varepsilon = \varepsilon(\lambda) = \mathcal{O}(\lambda)$ . Using a particularly convenient “strength-reparametrization”  $a = (2\mu + 1)/4$  we shall postulate that  $\varepsilon(\lambda) = \tau(\lambda) - a$  with some not yet known analytic function. In terms of its expansions, say,

$$\tau(\lambda) = a + b\lambda + c\lambda^2 + d\lambda^3 + f\lambda^4 + g\lambda^5 + O(\lambda^6) \quad (8)$$

our new “energy-parameters”  $a, b, \dots$  enter the matrix elements  $\alpha_n = n/2 - n^2 - 2n\tau$ ,  $\beta_0 = \tau^2 - a^2$  and  $\beta_n = n^2 + 2n\tau + \beta_0$ . For illustration we may insert our ansatz in  $\beta_0 = 2ab\lambda + (2ac + b^2)\lambda^2 + O(\lambda^3)$  giving  $\gamma_0 = 2ab + (2ac + b^2)\lambda + O(\lambda^2)$  etc.

Wave functions have already been specified by eq. (6). Apparently, there are no conditions imposed upon the energies. This is a paradox which we are going to explain now. In the first step we insert the coefficient  $h_1 = -\gamma_0/\alpha_1$  in the first, second and third row of recurrences (4) or (5). The former relation becomes an identity but the next one preserves a genuine three-component character,

$$2(2\varepsilon + \mu + 2)h_2 = 1 + (\varepsilon + 1)(\varepsilon + \mu + 3/2)h_1. \quad (9)$$

Up to a tiny perturbation the infinite rest with  $n = 1, 2, \dots$  has just a two-term form

$$(n + 2)(2\varepsilon + n + \mu + 2)h_{n+2} = (\varepsilon + n + 1)(\varepsilon + n + \mu + 3/2)h_{n+1} + \lambda h_n \quad (10)$$

giving immediately the asymptotic estimate valid for all  $|\lambda| \ll 1$ ,

$$h_n = h_n(\lambda) \sim \frac{\Gamma(\varepsilon + n)\Gamma(\varepsilon + n + \mu + 1/2)}{n!\Gamma(2\varepsilon + n + \mu + 1)} \sim n^{-3/2}, \quad n \gg 1. \quad (11)$$

This implies the convergence of  $\psi_{(PPT)}(x)$ , up to the central  $x = 0$ , rigorously.

The boundary of the circle of convergence coincides with the centre of the spatial symmetry. Hence, for our particular even-parity choice of  $p = 0$  the first derivative of the wave function must have a nodal zero there,

$$\partial_x \psi_{(PPT)}(x) \sim \sinh x \sum_{n=0}^{\infty} \frac{(2n + \kappa) h_n}{\cosh^{2n+\kappa+1} x} \rightarrow 0, \quad x \rightarrow 0. \quad (12)$$

As long as  $n h_n \sim n^{-1/2}$  the infinite sum itself is divergent in the origin. The whole expression (12) exhibits an  $0 \times \infty$  indeterminacy there. The conservation of parity must be imposed “by brute force” [11]. Vice versa, the strict validity of the boundary condition  $\partial_x \psi_{(PPT)}(0) = 0$  represents precisely the “seemingly lost” quantization condition.

In the nearest vicinity of  $x = 0$  the finite but very large value of the sum in eq. (12) is equal to a positive real number multiplied by the coefficient  $h_2$ . This observation follows from eq. (10) which gives, step-by-step,  $h_3(0) = 2(\mu + 5/2)/(3\mu + 9) \cdot h_2(0)$ ,  $h_4(0) = 3(\mu + 7/2)/(4\mu + 16) \cdot h_3(0)$  etc. In the interval  $\varepsilon > -(\mu + 3)/2$  all the multiplication factors remain positive. This guarantees that the parity is conserved, in a semi-infinite interval of  $\lambda$ , if and only if the coefficient  $h_2(\lambda)$  vanishes in the leading order approximation.

This is one of our most important observations. The required leading-order change of sign of  $h_2(\lambda)$  (i.e., relation  $h_2^{(1)} = 0$ ) may be interpreted as the consequence of the Sturm-Liouville oscillation theorems [12]. In their light, for an increasing or decreasing leading-order energy  $E = -\kappa^2 = -[2a - 1/2 + \mathcal{O}(\lambda)]^2$  the nodes of  $\partial_x \psi_{(PPT)}(x)$  would smoothly move along the real axis of  $x$ . In this sense the condition  $h_2^{(1)} = 0$  of coincidence of one of the nodes with the origin reflects the zero-order physical interpretation of the first coefficient  $a$  in our expansion (8) and forces us to postulate, consequently, the disappearance of all the subsequent leading-order coefficients,

$$h_2^{(1)} = h_3^{(1)} = \dots = 0. \quad (13)$$

Before writing it down as an explicit algebra let us first move up to the next order approximation. Due to the danger of a re-introduction of the asymmetry (or, in the other words, of a spike-shaped discontinuity in  $\psi_{(PPT)}(x)$  at  $x = 0$  [11]) in the higher orders of  $\lambda$  we just have to repeat the previous argumentation. In the second order we must first notice that the vanishing condition (13) changed also our recurrences (5). The role of the “last genuine three-term” relation (without a guarantee of a sign-preservation) moves from  $n = 1$  to  $n = 2$  in eq. (4). As long as our estimate (11) remains valid within its own error bound  $1 + \mathcal{O}(1/n)$  the role of an overall sign-

determining factor  $\lambda h_2(\lambda) = \mathcal{O}(\lambda^2)$  is taken over by the new norm  $\lambda h_3(\lambda) = \mathcal{O}(\lambda^2)$ . *Mutatis mutandis* we get the second-order condition  $h_3^{(2)} = h_4^{(2)} = \dots = 0$  and, in general,

$$c_n(\lambda) = \lambda h_n(\lambda) = \lambda^n h_n^{(n)} + \lambda^{n+1} h_n^{(n+1)} + \dots, \quad n = 1, 2, \dots \quad (14)$$

This is a perturbative generalization of the standard termination rules. It modifies our above naive expectations and incorporates correctly the physical definite-parity requirement in our original ansatz (3). In a more compact notation let us put  $c_0 = f_0(\lambda) = 1$  and  $\lambda h_n(\lambda) \equiv \lambda^n f_n(\lambda)$  with  $f_n(\lambda) = \mathcal{O}(\lambda^0)$ . This modifies our system of equations (5),

$$\begin{pmatrix} \beta_0 & \lambda \alpha_1 & & & \\ \lambda & \lambda \beta_1 & \lambda^2 \alpha_2 & & \\ & \lambda^2 & \lambda^2 \beta_2 & \lambda^3 \alpha_3 & \\ & & \lambda^3 & \lambda^3 \beta_3 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix} = 0. \quad (15)$$

All these equations may be solved by the closed determinantal formula again,

$$f_{n+1}(\lambda) = \frac{\Gamma(\mu + 1 + 2\lambda\eta)}{\Gamma(\mu + n + 2 + 2\lambda\eta)} \cdot \left[ \lambda^{-(n+1)(n+2)/2} \mathcal{M}_n(\lambda) \right], \quad (16)$$

$$\mathcal{M}_n(\lambda) = \det \begin{pmatrix} \beta_0(z) & \lambda \alpha_1(z) & & & \\ & \lambda & \lambda \beta_1(z) & \lambda^2 \alpha_2(z) & \\ & & \lambda^2 & \ddots & \ddots \\ & & & \ddots & \lambda^{n-1} \beta_{n-1} & \lambda^n \alpha_n \\ & & & & \lambda^n & \lambda^n \beta_n \end{pmatrix}.$$

It pre-factorizes the wave function coefficients in a certain less usual way. We may re-shuffle it slightly once more, in the more traditional Padé-approximation spirit, with  $f_0 = \gamma_{-1} = 1$  and

$$f_j = \frac{\gamma_{j-1}}{(-\alpha_1)(-\alpha_2) \dots (-\alpha_j)}, \quad j = 1, 2, \dots \quad (17)$$

This representation of the wave function coefficients is related to our final, “optimal”

recurrences

$$\begin{pmatrix} \beta_0 & -\lambda & & & \\ -\alpha_1 & \beta_1 & -\lambda & & \\ & -\alpha_2 & \beta_2 & -\lambda & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ \gamma_0 \\ \gamma_1 \\ \vdots \end{pmatrix} = 0 . \quad (18)$$

The proof of their equivalence to our previous equations is trivial. One has only to verify that the new notation is not inconsistent since the first row of eq. (18) just reproduces our old leading-order “change of the notation”  $\beta_0 = \lambda \gamma_0$ .

We are now prepared to compute the unknown auxiliary variables  $b, c, \dots$  and, in effect, the perturbed energies. Firstly we get rid of the “already known”  $\gamma_0$  in eq. (18). This only means that we omit the first line and abbreviate  $\hat{A}_2 = \alpha_2 \gamma_0$  and  $\hat{B}_1 = \beta_1 \gamma_0 - \alpha_1$  in the remaining equations

$$\begin{pmatrix} \hat{B}_1 & -\lambda & & & \\ -\hat{A}_2 & \beta_2 & -\lambda & & \\ & -\alpha_3 & \beta_3 & -\lambda & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ \gamma_1 \\ \gamma_2 \\ \vdots \end{pmatrix} = 0 . \quad (19)$$

Now we may parallel the above study where, retrospectively,  $\hat{B}_0 = \beta_0$ . In the other words we only have to require that  $\hat{B}_1 \equiv \lambda \gamma_1$  vanishes in the unperturbed limit (cf. the first line in eq. (19)). In contrast to its trivial  $j = 0$  predecessor the new  $j = 1$  situation imposes a constraint upon  $\hat{B}_1 = 2a + 1/2 + 2ab(1 + 2a) + \mathcal{O}(\lambda)$ . Its zero-order component must be zero. This determines the physical value of the first moment  $b = b(a) = \tau^{(1)}$ ,

$$\tau^{(1)} = -\frac{1 + 4a}{4a(1 + 2a)} . \quad (20)$$

As expected on variational grounds this result means a positive first order change in our energy  $E = -\kappa^2$  where  $\kappa = \kappa(\lambda) = [4\tau(\lambda) + 2\mu - 1]/4$ .

The whole  $j \rightarrow j + 1$  procedure can be, obviously, iterated. We arrive at the

equations

$$\begin{pmatrix} \hat{B}_j & -\lambda & & & \\ -\hat{A}_{j+1} & \beta_{j+1} & -\lambda & & \\ & -\alpha_{j+2} & \beta_{j+2} & -\lambda & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ \gamma_j \\ \gamma_{j+1} \\ \vdots \end{pmatrix} = 0 \quad (21)$$

with the elements  $\hat{A}_{j+1} = \alpha_{j+1}\gamma_{j-1}$  and  $\hat{B}_j = \beta_j\gamma_{j-1} - \alpha_j\gamma_{j-2}$ . For any  $j$ , as we have already shown, the first-line rule  $\hat{B}_j = \lambda\gamma_j = \mathcal{O}(\lambda)$  determines the value of the moment  $\tau^{(j)}$ . Beyond the formal choice of  $j = 0$  (confirming the consistency of  $a = \tau^{(0)} = \mu/2 + 1/4$ ) and of  $j = 1$  (giving  $b = \tau^{(1)}$ ) we must perform symbolic manipulations which become a task for the computer. Without its assistance one easily obtains just the next and quite compact  $j = 2$  formula

$$c = \tau^{(2)} = -1/32 \frac{(1+4a)(8a^2+8a^3+3a+1)}{(1+2a)^3 a^3 (1+a)}.$$

With the assistance of the MAPLE language [13] the evaluation of the next corrections remains entirely straightforward giving  $d$  or

$$\tau^{(3)} = -\frac{(1+4a)\mathcal{D}_7}{128a^5(1+2a)^5(1+a)^2(3+2a)}$$

with

$$\mathcal{D}_7 = 3 + 26a + 97a^2 + 168a^3 + 196a^4 + 288a^5 + 320a^6 + 128a^7$$

or  $f$  or rather

$$\tau^{(4)} = -\frac{(1+4a)\mathcal{D}_{12}}{2048a^7(1+2a)^7(1+a)^3(3+2a)^2(2+a)}$$

with

$$\mathcal{D}_{12} = 90 + 1335a + 8815a^2 + 32715a^3 + 69135a^4 + 54250a^5$$

$$-106568a^6 - 340152a^7 - 378096a^8 - 165184a^9 + 22272a^{10} + 43008a^{11} + 10240a^{12}$$

etc. These results inspire the general ansatz

$$\tau^{(K)} = -\frac{2^{-M(K)}(1+4a)\mathcal{D}_L}{a^{2K-1}(1+2a)^{2K-1}(a+1)^{K-1}(a+3/2)^{K-2}(a+2)^{K-3}\dots(a+K/2)}$$



re-confirmed by the explicit evaluation of the next-order correction  $g$ ,

$$\tau^{(5)} = -\frac{(1+4a)\mathcal{D}_{18}}{8192a^9(1+2a)^9(1+a)^4(3+2a)^3(2+a)^2(5+2a)}$$

with the rather complicated

$$\mathcal{D}_{18} = 3780 + 80892a + 794817a^2 + \dots + 458752a^{18}.$$

This did not disprove the obtrusive hypothesis that  $\mathcal{D}_L$  are polynomials in  $a$  with integer coefficients and with a growing degree  $L = L(K) = (K+1)(K+2)/2 - 3$ .

We may summarize our approach to eq. (1) as a scheme which can be easily generalized. Its use of a suitable non-orthogonal basis may find applications in different settings. In a way paralleling the Lanczos method (with a strong numerical flavor) the idea itself has already been tested on several “less solvable” semi-numerical examples [6, 11]. In contrast to them our present construction seems to exhibit a much closer similarity to the current perturbation constructions of anharmonic oscillators. Even in this comparison, our new example seems to hide several pleasant surprises (e.g., a possible non-zero radius  $\lambda_{max}$  of convergence) which could make it worth of a further study.

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