From random sets to continuous tensor products: answers to three questions of W. Arveson

Boris Tsirelson

Abstract

The set of zeros of a Brownian motion gives rise to a product system in the sense of William Arveson (that is, a continuous tensor product system of Hilbert spaces). Replacing the Brownian motion with a Bessel process we get a continuum of non-isomorphic product systems.

Introduction

"The term *product system* is a less tortured contraction of the phrase *continuous tensor product system of Hilbert spaces*" (Arveson [3, p. 6]). The theory of product systems, elaborated by W. Arveson in connection with E_0-semigroups and quantum fields (see [2], [3] and refs therein) suffers from lack of rich sources of examples. I propose such a source by combining A. Vershik's idea of a *measure type factorization* [9, Sect. 1c], my own idea of a *spectral type of a noise* [8, Sect. 2], and J. Warren's idea (private communication, Nov. 1999) of constructing a measure type factorization from a given random set. The new rich source of examples leads to rather simple answers to three questions of Arveson; see Sections 2,4,5 for the questions, and Theorems 2.1, 4.2 and 5.4 for the answers.

It is interesting to compare measure type factorizations with so-called *noises* (a less tortured substitute for such phrases as *homogeneous continuous tensor product system of probability spaces* or *stationary probability measure factorization*), see [9], [10], [7] and refs therein. Theory of noises is able to answer two out of the three questions of Arveson, however, the new approach makes it easier. I still do not know whether the third question (see Sect. 4) also has a noise-theoretic answer, or not.

1 The construction

Consider the standard Brownian motion $B(\cdot)$ in $\mathbb{R}$, and the random set

$$Z_{t,a} = \{ s \in [0,t] : B(s) = a \} \, ,$$

where $a, t \in (0,\infty)$ are parameters.[1] The set $Z_{t,a}$ may be treated as a random variable taking on values in the space $\mathcal{C}_t$ of all closed subsets of $[0,t]$.[2] There is a natural Borel σ-field $\mathcal{B}_t$ on $\mathcal{C}_t$, and $(\mathcal{C}_t, \mathcal{B}_t)$ is a standard Borel space. Moreover, $\mathcal{C}_t$ is a compact metric space w.r.t. the Hausdorff metric $\rho_t(C_1, C_2) = \inf\{ \varepsilon > 0 : C_1 \subset (C_2)_{+\varepsilon} \,\&\, C_2 \subset (C_1)_{+\varepsilon} \}$ (here $C_{+\varepsilon}$ means the ε-neighborhood of C), and $\mathcal{B}_t$ is the Borel σ-field of the metric space $(\mathcal{C}_t, \rho_t)$. Let $P_{t,a}$ be the law of the $\mathcal{C}_t$-valued random variable $Z_{t,a}$, then $(\mathcal{C}_t, \mathcal{B}_t, P_{t,a})$ is a probability space.

1.1. Lemma. $P_{t,a_1} \sim P_{t,a_2}$; that is, measures P_{t,a_1} and P_{t,a_2} are equivalent (= mutually absolutely continuous) for all $a_1, a_2 \in (0,\infty)$.

Proof. Consider the random time $T_a = \min\{ t \in [0,\infty) : B(t) = a \}$. The shifted set $Z_{\infty,a} - T_a$ is independent of T_a and distributed like $Z_{\infty,0}$. Thus, $P_{\infty,a}$ is a mix of shifted copies of $P_{\infty,0}$, weighted according to the law of T_a. However, laws of T_{a_1}, T_{a_2} are equivalent measures, therefore $P_{\infty,a_1} \sim P_{\infty,a_2}$, which implies $P_{t,a_1} \sim P_{t,a_2}$. $\square$

Denote by $\mathcal{P}_t$ the set of all probability measures on $(\mathcal{C}_t, \mathcal{B}_t)$ that are equivalent to $P_{t,a}$ for some (therefore, all) $a \in (0,\infty)$. The triple $(\mathcal{C}_t, \mathcal{B}_t, \mathcal{P}_t)$ is an example of a structure called *measure-type space.*

Denote by $P_{s,a} \otimes P_{t,a}$ the law of the random set $C_1 \cup (C_2 + s)$, where $C_1 \in \mathcal{C}_s$ is distributed $P_{s,a}$, and $C_2 \in \mathcal{C}_t$ is distributed $P_{t,a}$, and C_1, C_2 are independent; of course, $C_2 + s \subset [s,t]$ is the shifted C_2.

1.2. Lemma. $P_{s,a} \otimes P_{t,a} \sim P_{s+t,a}$ for all $s, t, a \in (0,\infty)$.

Proof. The conditional distribution of the set $(Z_{s+t,a} \cap [s, s+t]) - s$, given the set $Z_{s,a}$, is the mix (over x) of its conditional distributions, given $Z_{s,a}$ and $B(s) = x$. The latter conditional distribution, being equal to $P_{t,|a-x|}$, belongs to $\mathcal{P}_t$ (except for $x = a$, which case may be neglected). Therefore the former conditional distribution also belongs to $\mathcal{P}_t$. $\square$

[1] When writing $Z_{t,a}$ I always assume that $a, t \in (0,\infty)$ unless otherwise stated; the reservation applies when I write, say, $Z_{\infty,0}$.

[2] Also the empty set $\emptyset$ belongs to $\mathcal{C}_t$.

We cannot identify the Cartesian product $\mathcal{C}_s \times \mathcal{C}_t$ with $\mathcal{C}_{s+t}$, since natural maps $\mathcal{C}_{s+t} \to \mathcal{C}_s \times \mathcal{C}_t$ and $\mathcal{C}_s \times \mathcal{C}_t \to \mathcal{C}_{s+t}$ are not mutually inverse (in fact, both are non-invertible). However, $\mathcal{P}_{s+t}\{C : s \in C\} = 0$;[3] neglecting some sets of probability 0, we get

$$(\mathcal{C}_s, \mathcal{B}_s, \mathcal{P}_s) \otimes (\mathcal{C}_t, \mathcal{B}_t, \mathcal{P}_t) = (\mathcal{C}_{s+t}, \mathcal{B}_{s+t}, \mathcal{P}_{s+t}) \,, \tag{1.3}$$

or simply $\mathcal{P}_s \otimes \mathcal{P}_t = \mathcal{P}_{s+t}$ for $s, t \in (0, \infty)$.

In order to introduce Hilbert spaces $L_2(\mathcal{C}_t, \mathcal{B}_t, \mathcal{P}_t)$ note that Hilbert spaces $L_2(\mathcal{C}_t, \mathcal{B}_t, P_1)$ and $L_2(\mathcal{C}_t, \mathcal{B}_t, P_2)$ for $P_1, P_2 \in \mathcal{P}_t$ are in a natural unitary correspondence; namely, $\psi_1 \in L_2(\mathcal{C}_t, \mathcal{B}_t, P_1)$ corresponds to $\psi_2 \in L_2(\mathcal{C}_t, \mathcal{B}_t, P_2)$ if

$$\psi_2 = \sqrt{\frac{P_1}{P_2}} \psi_1 \,,$$

where $\frac{P_1}{P_2}$ is the Radon-Nikodym density. Define an element ψ of $L_2(\mathcal{C}_t, \mathcal{B}_t, \mathcal{P}_t)$ as a family $\psi = (\psi_P)_{P \in \mathcal{P}_t}$ satisfying $\psi_P \in L_2(\mathcal{C}_t, \mathcal{B}_t, P)$ and

$$\psi_{P_2} = \sqrt{\frac{P_1}{P_2}} \psi_{P_1} \quad \text{for all } P_1, P_2 \in \mathcal{P}_t \,.$$

Clearly, $L_2(\mathcal{C}_t, \mathcal{B}_t, \mathcal{P}_t)$ is a separable Hilbert space, naturally isomorphic to every $L_2(\mathcal{C}_t, \mathcal{B}_t, P)$, $P \in \mathcal{P}_t$.[4] Relation (1.3) gives

$$L_2(\mathcal{C}_s, \mathcal{B}_s, \mathcal{P}_s) \otimes L_2(\mathcal{C}_t, \mathcal{B}_t, \mathcal{P}_t) = L_2(\mathcal{C}_{s+t}, \mathcal{B}_{s+t}, \mathcal{P}_{s+t}) \tag{1.4}$$

in the sense that the two Hilbert spaces are *naturally* isomorphic.

However, (1.4) is only a part of requirements stipulated in the definition of a product system [3, Def. 1.4]. The point is that (1.4) holds for each (s, t) individually; nothing was said till now about measurability in s, t. In order to get a product system, we need a *measurable* unitary correspondence between spaces $L_2(\mathcal{C}_t, \mathcal{B}_t, \mathcal{P}_t)$ for different t, making the map implied by (1.4) jointly measurable. The correspondence need not be natural, but our case is especially nice, having a natural correspondence described below.

For every $\lambda \in (0, \infty)$ the random process $t \mapsto \sqrt{\lambda} B(t/\lambda)$ is a Brownian motion, again. Therefore the two random sets $\{s : B(s) = a\}$ and $\{s : \sqrt{\lambda} B(s/\lambda) = a\} = \lambda \cdot \{s : B(s) = a/\sqrt{\lambda}\}$ are identically distributed. It means that the "rescaling" map $R_\lambda : \mathcal{C}_1 \to \mathcal{C}_\lambda$, defined by $R_\lambda(C) = \lambda \cdot C$,

[3] I mean, of course, that $P\big(\{C \in \mathcal{C}_{s+t} : s \in C\}\big) = 0$ for some (therefore all) $P \in \mathcal{P}_{s+t}$.

[4] Intuitively we may think that $\sqrt{P} \psi_P = \psi$ for all $P \in \mathcal{P}_t$. See also [1].

sends $P_{1,a/\sqrt{\lambda}}$ to $P_{\lambda,a}$. Accordingly, it sends $\mathcal{P}_1$ to $\mathcal{P}_\lambda$. We define a unitary operator $\tilde{R}_t : L_2(\mathcal{C}_1, \mathcal{B}_1, \mathcal{P}_1) \to L_2(\mathcal{C}_t, \mathcal{B}_t, \mathcal{P}_t)$ by

$$(\tilde{R}_t\psi)_{R_t(P)}(R_t(C)) = \psi_P(C) \quad \text{for } P\text{-almost all } C \in \mathcal{C}_1,$$

for all $\psi \in L_2(\mathcal{C}_1, \mathcal{B}_1, \mathcal{P}_1)$ and $P \in \mathcal{P}_1$; of course, $R_t(P)$ is the R_t-image of P (denoted also by $P \circ R_t^{-1}$). The disjoint union $E = \cup_{t\in(0,\infty)} L_2(\mathcal{C}_t, \mathcal{B}_t, \mathcal{P}_t)$ (not a Hilbert space, of course) is now parametrized by the Cartesian product $(0,\infty) \times L_2(\mathcal{C}_1, \mathcal{B}_1, \mathcal{P}_1)$, namely, $(t,\psi) \in (0,\infty) \times L_2(\mathcal{C}_1, \mathcal{B}_1, \mathcal{P}_1)$ parametrizes $\tilde{R}_t(\psi) \in L_2(\mathcal{C}_t, \mathcal{B}_t, \mathcal{P}_t) \subset E$. We equip E with the Borel structure that corresponds to the natural Borel structure on $(0,\infty) \times L_2(\mathcal{C}_1, \mathcal{B}_1, \mathcal{P}_1)$. Linear operations and the scalar product are Borel measurable (on their domains) for trivial reasons. It remains to consider the multiplication $E \times E \to E$,

$$E \times E \supset H_s \times H_t \ni (\psi_1, \psi_2) \mapsto \psi_1 \otimes \psi_2 \in H_s \otimes H_t = H_{s+t} \subset E\,;$$

it must be Borel measurable.[5] In other words, we consider $\psi = \tilde{R}_{s+t}^{-1}\big(\tilde{R}_s(\psi_1)\otimes \tilde{R}_t(\psi_2)\big)$ as an H_1-valued function of four arguments $s,t \in (0,\infty)$, $\psi_1, \psi_2 \in H_1$; we have to check that the function is jointly Borel measurable. After substituting all relevant definition it boils down to $C = R_{s+t}^{-1}\big((R_sC_1)\cup(s+R_tC_2)\big)$ treated as a $\mathcal{C}_1$-valued function of four arguments $s,t \in (0,\infty)$, $C_1, C_2 \in \mathcal{C}_1$; the reader may check that the function is jointly Borel measurable. So, Hilbert spaces

$$H_t = L_2(\mathcal{C}_t, \mathcal{B}_t, \mathcal{P}_t)$$

form a product system.

2 Units

Every measure $P \in \mathcal{P}_t$ has an atom, since $\mathbb{P}\big(Z_{t,a} = \emptyset\big) > 0$; in fact, $\{\emptyset\}$ is the only atom of P.

For every $t \in (0,\infty)$ the space $H_t = L_2(\mathcal{C}_t, \mathcal{B}_t, \mathcal{P}_t)$ contains a special element v_t defined by

$$(v_t)_P(C) = \begin{cases} \frac{1}{\sqrt{P(\{\emptyset\})}} & \text{if } C = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $v_{s+t} = v_s \otimes v_t$ for all $s,t \in (0,\infty)$. Also, $\|v_t\| = 1$ for all t.

[5] I do not distinguish between $H_s \otimes H_t$ and H_{s+t} in the notation. A cautious reader may insert a notation for the natural unitary operator $H_s \otimes H_t \to H_{s+t}$.

A unit of a product system (H_t) is a family $(u_t)_{t\in(0,\infty)}$ such that $u_t \in H_t$ for all $t \in (0,\infty)$, and $u_s \otimes u_t = u_{s+t}$ for all $s,t \in (0,\infty)$, and the map $\mathbb{R} \ni t \mapsto u_t \in \cup_t H_t$ is measurable, and $u_t \ne 0$ for some t (which implies $u_t \ne 0$ for all t); see [2, p. 10], [3, Sect. 4].

The family (v_t) is a unit, since $\tilde{R}_t^{-1}(v_t)$ is measurable in t; in fact, it is constant, $\tilde{R}_t^{-1}(v_t) = v_1$.

If (u_t) is a unit (of a product system) then $(e^{i\lambda t}u_t)$ is also a unit for every $\lambda \in \mathbb{C}$. All these units may be called equivalent. Some product systems contain non-equivalent units. Some product systems contain no units at all. The trivial product system (consisting of one-dimensional Hilbert spaces) contains a unit, and all its units are equivalent. Arveson [2, p. 12] asked: is there a nontrivial product system that contains a unit but does not contain non-equivalent units? The product system constructed in Sect. 1 appears to be such an example; the question is answered by the following result. (Note however that the question is already answered by noise theory; I mean the system of [9, Sect. 5].)

2.1. Theorem. Every unit (u_t) is of the form $u_t = e^{i\lambda t}v_t$.

Proof. Every $\psi \in H_t$ determines a measure $|\psi|^2$ on $(\mathcal{C}_t, \mathcal{B}_t)$ by[6]

$$\frac{|\psi|^2}{P} = |\psi_P|^2 \quad \text{for some (therefore, all) } P \in \mathcal{P}_t\,. \tag{2.2}$$

Note that $|\psi_1 \otimes \psi_2|^2 = |\psi_1|^2 \otimes |\psi_2|^2$ whenever $\psi_1 \in H_s$, $\psi_2 \in H_t$. If (u_t) is a unit, then $|u_s|^2 \otimes |u_t|^2 = |u_{s+t}|^2$. We may assume that $\|u_t\| = 1$ for all t (since $(u_t/\|u_t\|)$ is a unit equivalent to (u_t), see [3, Th. 4.1]), then $|u_t|^2$ is a probability measure. Applying [3, Th. 4.1] again we get $\langle u_t, v_t\rangle = e^{\gamma t}$ for some $\gamma \in \mathbb{C}$. However, for every $\psi \in H_t$

$$\langle \psi, v_t\rangle = \int \psi_P \overline{(v_t)_P}\, dP = \psi_P(\emptyset) \frac{1}{\sqrt{P(\{\emptyset\})}} P(\{\emptyset\})\,,$$
$$|\langle \psi, v_t\rangle|^2 = |\psi_P(\emptyset)|^2 P(\{\emptyset\}) = |\psi|^2(\{\emptyset\})\,.$$

Applying it to $\psi = u_t$ we get $|u_t|^2(\{\emptyset\}) = e^{2\operatorname{Re}\gamma t}$. In combination with the property $|u_s|^2 \otimes |u_t|^2 = |u_{s+t}|^2$ it shows that $|u_t|^2$ is the law of the Poisson point process with intensity $(-2\operatorname{Re}\gamma)$ on $[0,t]$.[7] Thus, $|u_t|^2$ is concentrated on finite sets $C \in \mathcal{C}_t$. On the other hand, being absolutely continuous w.r.t.

[6] Do not confuse the *measure* $|\psi|^2$ with the *number* $\|\psi\|^2$, the squared norm; in fact, $\|\psi\|^2 = (|\psi|^2)(\mathcal{C}_t)$, the total mass.

[7] A simple way to check it: divide $(0,t)$ into n equal intervals; each of them is free of C (distributed $|u_t|^2$) with probability $e^{2\operatorname{Re}\gamma t/n}$, independently of others. Consider $n = 2, 4, 8, 16, \dots$

$\mathcal{P}_t$, the measure $|u_t|^2$ is concentrated on sets $C \in \mathcal{C}_t$ with no isolated points. Therefore $|u_t|^2$ is concentrated on $C = \emptyset$ only. It means that $\operatorname{Re}\gamma = 0$, that is, $\gamma = i\lambda$, $\lambda \in \mathbb{R}$. So, $\|u_t\| = 1$, $\|v_t\| = 1$ and $\langle u_t, v_t \rangle = e^{i\lambda t}$; therefore $u_t = e^{i\lambda t} v_t$. $\square$

3 Using Bessel processes

Introduce a parameter $\delta \in (0, 2)$ and consider the random set

$$Z_{t,a,\delta} = \{ s \in [0, t] : \mathrm{BES}_{\delta,a}(s) = 0 \} ,$$

and its law $P_{t,a,\delta}$; here $\mathrm{BES}_{\delta,a}(\cdot)$ is the Bessel process of dimension δ started at a (see [6, Chap. XI, Defs 1.1 and 1.9]). As before, $t, a \in (0, \infty)$. The law $P_{t,a,1}$ of $Z_{t,a,1}$ is equal to the law $P_{t,a}$ of $Z_{t,a}$ of Sect. 1, since $\mathrm{BES}_{1,a}$ is distributed like $|B(\cdot) + a|$. The structure of $Z_{\infty,0,\delta}$ was well-understood long ago;[8] especially, measures $P_{t,0,\delta_1}$ and $P_{t,0,\delta_2}$ for $\delta_1 \ne \delta_2$ are mutually singular. Measures P_{t,a,δ_1} and P_{t,a,δ_2} (where $a > 0$) are not singular because of a common atom ($Z_{t,a,\delta} = \emptyset$ with a positive probability).

Below, $\mu \ll \nu$ means that a measure μ is absolutely continuous w.r.t. a measure ν; $\mu \sim \nu$ means $\mu \ll \nu \,\&\, \nu \ll \mu$.

3.1. Lemma. (a) $P_{t,a_1,\delta} \sim P_{t,a_2,\delta}$;

(b) if $\delta_1 \ne \delta_2$, $\mu \ll P_{t,a,\delta_1}$ and $\mu \ll P_{t,a,\delta_2}$, then μ is concentrated on $\{\emptyset\}$.

Proof. Similarly to the proof of Lemma 1.1, consider the random time $T_a = \min\{ s \in [0, \infty) : \mathrm{BES}_{\delta,a}(s) = 0 \}$; $T_a \in (0, \infty)$ almost sure (since $\delta < 2$). The shifted set $Z_{\infty,a,\delta} - T_a$ is independent of T_a and distributed like $Z_{\infty,0,\delta}$. Statement (a) follows from the fact that laws of T_{a_1}, T_{a_2} are equivalent measures. Statement (b): μ is concentrated on sets that must have two different Hausdorff dimensions near each point; the only such set is $\emptyset$. $\square$

3.2. Lemma. $P_{s,a,\delta} \otimes P_{t,a,\delta} \sim P_{s+t,a,\delta}$.

The proof is quite similar to the proof of Lemma 1.2.

The Bessel process has the same scaling property as the Brownian motion: the process $t \mapsto \sqrt{\lambda}\, \mathrm{BES}_{\delta,a/\sqrt{\lambda}}(t/\lambda)$ has the law $P_{t,a,\delta}$ irrespective of $\lambda \in (0, \infty)$.

So, all properties of Brownian motion, used in Sect. 1, hold for Bessel processes. Generalizing the construction of Sect. 1 we get a product system $(H_{t,\delta})_{t \in (0,\infty)}$ for every $\delta \in (0, 2)$. The product system of Sect. 1 corresponds to $\delta = 1$.

[8] Namely, $Z_{\infty,0,\delta}$ is the closure of the range of a stable subordinator of index $1 - \delta/2$ (see [5, Example 6]); it is of Hausdorff dimension $1 - \delta/2$ near every point [4].

4 Continuum of non-isomorphic product systems

"At this point, we are not even certain of the *cardinality* of Σ! It is expected that Σ is uncountable, but this has not been proved." W. Arveson [2, p. 12].

An isomorphism between two product systems (H_t), (H'_t) is defined naturally as a family $(\theta_t)_{t\in(0,\infty)}$ of unitary operators $\theta_t : H_t \to H'_t$ such that, first, $\theta_{s+t}(\psi_1 \otimes \psi_2) = \theta_s(\psi_1) \otimes \theta_t(\psi_2)$ whenever $\psi_1 \in H_s$, $\psi_2 \in H_t$, and second, $\theta_t(\psi)$ is jointly measurable in t and ψ; see [3, p. 6]. Are there uncountably many non-isomorphic product systems? This question, asked by Arveson [2, p. 12], will be answered here in the positive by showing that product systems $(H_{t,\delta})$ for different δ are non-isomorphic.

Consider the projection operator (the index δ is suppressed)

$$Q_t : H_t \to H_t \,, \qquad (Q_t\psi)_P(C) = \begin{cases} \psi_P(C) & \text{if } C = \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

just the orthogonal projection onto the one-dimensional subspace corresponding to the atom of $\mathcal{P}_{t,\delta}$. Given $0 < r < s < t$, we introduce an operator $Q_{t,(r,s)} = Q_r \otimes \mathbf{1}_{s-r} \otimes Q_{t-s}$ on the space $H_t = H_r \otimes H_{s-r} \otimes H_{t-s}$; of course, $\mathbf{1}_{s-r}$ is the identical operator on H_{s-r}. Operators $Q_{t,E}$ are defined similarly for every elementary set (that is, a union of finitely many intervals) $E \subset (0,t)$.[9] Clearly,

$$(Q_{t,E}\psi)_P(C) = \begin{cases} \psi_P(C) & \text{if } C \subset E, \\ 0 & \text{otherwise.} \end{cases}$$

Note a relation to measures $|\psi|^2$ defined by (2.2):

$$\langle Q_{t,E}\psi, \psi\rangle = |\psi|^2(\{C \in \mathcal{C}_t : C \subset E\})\,. \tag{4.1}$$

4.2. Theorem. If $\delta_1 \ne \delta_2$ then product systems (H_{t,δ_1}), (H_{t,δ_2}) are non-isomorphic.

Proof. Assume the contrary: operators $\theta_t : H_{t,\delta_1} \to H_{t,\delta_2}$ are an isomorphism of the product systems. The system (H_{t,δ_1}) has a unit, and all its units are equivalent, which is Theorem 2.1 when $\delta_1 = 1$, and a (straightforward) generalization of Theorem 2.1 for arbitrary δ_1. The same for the other product

[9] For example, $Q_{t,(r,s)\cup(u,v)} = Q_r \otimes \mathbf{1}_{s-r} \otimes Q_{u-s} \otimes \mathbf{1}_{v-u} \otimes Q_{t-v}$ for $0 < r < s < u < v < t$; $Q_{t,(0,s)} = \mathbf{1}_s \otimes Q_{t-s}$; $Q_{t,(s,t)} = Q_s \otimes \mathbf{1}_{t-s}$; $Q_{t,(0,t)} = \mathbf{1}_t$; $Q_{t,\emptyset} = Q_t$.

system (H_{t,δ_2}). It follows that operators Q_t are preserved by isomorphisms; $Q_t\theta_t = \theta_t Q_t$ (that is, $Q_t^{(\delta_2)}\theta_t = \theta_t Q_t^{(\delta_1)}$). Tensor products of these operators are also preserved:

$$Q_{t,E}\theta_t = \theta_t Q_{t,E} \,.$$

In combination with 4.1 it gives for $\psi \in H_{t,\delta_1}$

$$|\psi|^2(A) = |\theta_t\psi|^2(A) \tag{4.3}$$

for every A of the form $A = A_E = \{C \in \mathcal{C}_t : C \subset E\}$ where E is an elementary set. However, $A_{E_1\cap E_2} = A_{E_1} \cap A_{E_2}$, and the σ-field generated by sets A_E is the whole $\mathcal{B}_t$. It follows (by Dynkin Class Theorem) that 4.3 holds for all $A \in \mathcal{B}_t$, that is,

$$|\psi|^2 = |\theta_t\psi|^2 \quad \text{for all } \psi \in H_{t,\delta_1} \,,$$

which contradicts to Lemma 3.1(b). $\square$

5 Asymmetry via countable random sets

The law $P_{t,a}$ of the random set $Z_{t,a}$ of Sect. 1 is asymmetric in the sense that $P_{t,a}$ is not invariant under the time reversal

$$\mathcal{C}_t \ni C \mapsto t - C \in \mathcal{C}_t$$

(of course, $t - C = \{t - s : s \in C\}$). However, the measure type $\mathcal{P}_t$ is symmetric; therefore the product system (H_t) is symmetric, which means existence of unitary operators $\theta_t : H_t \to H_t$ such that, first, $\theta_{s+t}(\psi_1 \otimes \psi_2) = \theta_t(\psi_2) \otimes \theta_s(\psi_1)$ whenever $\psi_1 \in H_s$, $\psi_2 \in H_t$, and second, $\theta_t(\psi)$ is jointly measurable in t and ψ; see [2, p. 12], [3, p. 6]. It was noted by Arveson [3, p. 6] that we do not know if an arbitrary product system is symmetric. Apparently, the first example of an asymmetric product system is "the noise made by a Poisson snake" of J. Warren [10]; there, asymmetry emerges from a random countable closed set that has points of accumulation from the left, but never from the right. A different, probably simpler way from such sets to asymmetric product systems is presented here.

Our first step toward a suitable countable random set is choosing a (non-random) set $S \subset [0,\infty)$ and a function $\lambda : S \times S \to [0,\infty)$ such that

(a) S is closed, countable, 1-periodic (that is, $s \in S \iff s+1 \in S$ for $s \in [0,\infty)$), totally ordered (that is, no strictly decreasing infinite sequences), $0 \in S$, and $S \cap (0,1)$ is infinite;[10]

[10] An example: $S = \{k - 2^{-l} : k,l = 1,2,3,\dots\} \cup \{0,1,2,\dots\}$; another example: $S = \{k - 2^{-l} - 2^{-l-m} : k,l,m = 1,2,3,\dots\} \cup \{k - 2^{-l} : k,l = 1,2,3,\dots\} \cup \{0,1,2,\dots\}$.

(b) $\lambda(s_1, s_2) > 0$ whenever $s_1, s_2 \in S$, $s_1 < s_2 \le s_1 + 1$; and $\lambda(s_1, s_2) = 0$ whenever $s_1, s_2 \in S$ do not satisfy $s_1 < s_2 \le s_1 + 1$;

(c) denoting by s_+ the least element of $S \cap (s, \infty)$ we have

$$\lambda(s, s_+) = \frac{1}{s_+ - s}\,, \qquad \sum_{s' \in S,\, s' > s_+} \lambda(s, s') \le 1$$

for all $s \in S$.

On the second step we construct a Markov process $\big(X(t)\big)_{t \in [0,\infty)}$ that jumps, from one point of S to another, according to the rate function $\lambda(\cdot, \cdot)$. Initially, $X(0) = 0$. We introduce independent random variables $\tau_{0,s}$ for $s \in S \cap (0, 1]$ such that $\mathbb{P}\big(\tau_s > t\big) = e^{-\lambda(0,s)t}$ for all $t \in [0, \infty)$. We have $\inf_s \tau_s > 0$, since $\sum_s \lambda(0, s) < \infty$. We let

$$X(t) = 0 \text{ for } t \in [0, T_1)\,, \qquad X(T_1) = s_1\,,$$

where random variables $T_1 \in (0, \infty)$ and $s_1 \in S$ are defined by

$$T_1 = \inf_s \tau_s = \tau_{s_1}\,.$$

The first transition of $X(\cdot)$ is constructed. Now we construct the second transition, $X(T_2-) = s_1$, $X(T_2) = s_2$ using rates $\lambda(s_1, s)$; and so on. It may happen (in fact, it happens almost always) that $\sup_k T_k = T_\infty < \infty$, and then (almost always) $X(T_k) \to s_\infty \in S$ (recall that S is closed). We let $X(T_\infty) = s_\infty$ and construct the next transition of $X(\cdot)$ using rates $\lambda(s_\infty, s)$. And so on, by a transfinite recursion over countable ordinals, until exhausting the time domain $[0, \infty)$. Almost surely, $X(t) \in S$ is well-defined for all $t \in [0, \infty)$, and $X(t) \to \infty$ for $t \to \infty$.

The last step is simple. We define the random set $Z_{\infty,0,S}$ as the closure of the set of all instants when $X(\cdot)$ jumps. That is, $Z_{\infty,0,S}$ is the set of all t such that $X(t - \varepsilon) < X(t + \varepsilon)$ for all $\varepsilon \in (0, t)$. Instead of starting at 0 we may start at another point $a \in S$, which leads to another process $X_a(\cdot)$ and random set $Z_{\infty,a,S}$; the law $P_{t,a,S}$ of $Z_{t,a,S} = Z_{\infty,a,S} \cap [0, t]$ is a probability measure on $(\mathcal{C}_t, \mathcal{B}_t)$.

5.1. Lemma. $P_{t,a_1,S} \sim P_{t,a_2,S}$ for all $a_1, a_2 \in S$.

Proof. (Similar to 1.1.) Consider the random time $T_a = \min Z_{a,S}$, just the instant of the first jump: $X_a(T_a-) = a$, $X_a(T_a) > a$. The conditional distribution of the shifted set (without the first point), $(Z_{\infty,a,S} - T_a) \setminus \{0\}$, given T_a and $X_a(T_a)$, is $P_{\infty, X_a(T_a), S}$. Thus, $P_{\infty,a,S}$ is a mix of shifted copies of $P_{\infty,b,S} \cup \{0\}$ for various $b \in S \cap (a, a + 1]$. However, $P_{\infty,b,S} = P_{\infty,b+1,S}$ for all $b \in S$. It remains to note that the joint law of T_{a_1} and $\big(X_{a_1}(T_{a_1}) \bmod 1\big)$ is equivalent to the joint law of T_{a_2} and $\big(X_{a_2}(T_{a_2}) \bmod 1\big)$. $\square$

We denote by $\mathcal{P}_{t,S}$ the set of all probability measures on $(\mathcal{C}_t, \mathcal{B}_t)$ that are equivalent to $P_{t,a,S}$ for some (therefore, all) $a \in S$.

5.2. Lemma. $P_{s,a,S} \otimes P_{t,a,S} \sim P_{s+t,a,S}$ for all $s, t \in (0, \infty)$, $a \in S$.

Proof. (Similar to 1.2.) The conditional distribution of the set $(Z_{s+t,a,S} \cap [s, s+t]) - s$, given the set $Z_{s,a,S}$, is the mix (over b) of its conditional distributions, given $Z_{s,a,S}$ and $X_a(s) = b$. The latter conditional distribution, being equal to $P_{t,b,S}$, belongs to $\mathcal{P}_{t,S}$. Therefore the former conditional distribution also belongs to $\mathcal{P}_{t,S}$. $\square$

Now we can construct the corresponding product system $(H_{t,S})_{t \in [0,\infty)}$ as before. Though, scaling invariance is absent; unlike Sect. 1, R_t does not send $\mathcal{P}_{1,S}$ to $\mathcal{P}_{t,S}$. We have no *natural* correspondence between spaces $L_2(\mathcal{C}_t, \mathcal{B}_t, \mathcal{P}_{t,S})$, but still, *some* Borel-measurable correspondence exists; I do not dwell on this technical issue.

A more important point: in contrast to previous sections, the product system $(H_{t,S})$ contains non-equivalent units (since the law of a Poisson point process on $(0, t)$ is absolutely continuous w.r.t. $\mathcal{P}_{t,S}$). Unlike Sect. 4, an isomorphism need not preserve projection operators Q_t and measures $|\psi|^2$, which prevents us from deriving asymmetry of the product system $(H_{t,S})$ just from asymmetry of measure types $\mathcal{P}_{t,S}$. Instead, we'll adapt some constructions of [7] (see (2.15) and (3.4) there).

As before, $Q_t : H_{t,S} \to H_{t,S}$ is the one-dimensional projection operator corresponding to the atom $\{\emptyset\}$ of $\mathcal{P}_{t,S}$ (you see, $\mathbb{P}\big(Z_{t,a,S} = \emptyset \big) > 0$). Introduce operators

$$U_{t,p,n} = \big((1-p) Q_{t/n} + p \mathbf{1}_{t/n} \big)^{\otimes n}$$

on $H_t = H_{t/n} \otimes \dots \otimes H_{t/n} = H_{t/n}^{\otimes n}$ (here $p \in (0, 1)$ is a parameter).[11] It is just multiplication by a function of $C \in \mathcal{C}_t$; the function counts intervals $(\frac{k}{n}, \frac{k+1}{n})$ that contain points of C, and returns p^m where m is the number of such intervals. For $n \to \infty$, operators $U_{t,p,n}$ converge (in the strong operator topology) to

$$U_{t,p} = \lim_{n \to \infty} U_{t,p,n} \, , \qquad (U_{t,p} \psi)_P (C) = p^{|C|} \psi_P (C) \, ,$$

just multiplication by $p^{|C|}$ where $|C|$ is the cardinality of C; naturally, $p^{|C|} = 0$ for infinite sets C. (In fact, $U_{t,p_1} U_{t,p_2} = U_{t,p_1 p_2}$.) The operator $U_{t,1-} = \lim_{p \to 1-} U_{t,p}$ is especially interesting:

$$(U_{t,1-} \psi)_P (C) = \begin{cases} \psi_P(C) & \text{if } C \text{ is finite,} \\ 0 & \text{otherwise.} \end{cases}$$

[11] Of course, $\mathbf{1}_t$ is the identical operator on $H_{t,S}$.

(In fact, $U_{t,1-}$ is the projection onto the stable (= linearizable) part of the product system [7, (2.15)], which is not used here.)

Operators $U_{t,p}$ correspond to a particular unit (or rather, equivalence class of units) of the product system $(H_{t,S})$. However, we may do the same for any given unit $u = (u_t)$. Namely,

$$Q_{t,u}\psi = \frac{\langle \psi, u_t \rangle}{\langle u_t, u_t \rangle} u_t \quad \text{for } \psi \in H_t \,;$$
$$U_{t,p,n,u} = \big((1-p) Q_{t/n,u} + p \mathbf{1}_{t/n} \big)^{\otimes n} \,;$$
$$U_{t,p,u} = \lim_{n\to\infty} U_{t,p,n,u} \,.$$

Existence of the limit is an easy matter, since operators $U_{t,p,n,u}$ for all n belong to a single commutative subalgebra. Even simpler, we may take $\lim_{n\to\infty} U_{t,p,2^n,u}$, the limit of a *decreasing* sequence of commuting operators.

5.3. Lemma. $U_{t,1-,u} = U_{t,1-}$ for all units u of the product system $(H_{t,S})$.

Proof. Let $u = (u_t)$ and $v = (v_t)$ be two units; we'll prove that $U_{t,1-,u} = U_{t,1-,v}$. Due to [3, Th. 4.1] we may assume that $\|u_t\| = 1$, $\|v_t\| = 1$ and $\langle u_t, v_t \rangle = e^{-\gamma t}$ for some $\gamma \in [0,\infty)$. An elementary calculation (on the plane spanned by u_t, v_t) gives[12]

$$\| Q_{t,u} - Q_{t,v} \| = \sqrt{1 - e^{-2\gamma t}} \,.$$

Opening brackets in $U_{t,p,n,u} = \big((1-p) Q_{t/n,u} + p \mathbf{1}_{t/n} \big)^{\otimes n}$ we get a sum of 2^n terms, each term being a tensor product of n factors. After rearranging the factors (which changes the term, of course, but does not change its norm), a term becomes simply $(1-p)^k p^{n-k} Q_{\frac{k}{n} t,u} \otimes \mathbf{1}_{\frac{n-k}{n} t}$. We see that

$$\| U_{t,p,n,u} - U_{t,p,n,v} \| \le \mathbb{E} \| Q_{\frac{k}{n} t,u} - Q_{\frac{k}{n} t,v} \| \,,$$

where the expectation is taken w.r.t. a random variable k having the binomial distribution $\operatorname{Bin}(n, 1-p)$. Using concavity of $\sqrt{1 - e^{-2\gamma t}}$ in t,

$$\mathbb{E} \| Q_{\frac{k}{n} t,u} - Q_{\frac{k}{n} t,v} \| = \mathbb{E} \sqrt{1 - e^{-2\gamma k t/n}} \le \sqrt{1 - e^{-2\gamma \mathbb{E} k t/n}} = \sqrt{1 - e^{-2\gamma t(1-p)}} \,,$$

therefore

$$\| U_{t,p,n,u} - U_{t,p,n,v} \| \le \sqrt{1 - e^{-2\gamma t(1-p)}} \quad \text{for all } n \,;$$
$$\| U_{t,p,u} - U_{t,p,v} \| \le \sqrt{1 - e^{-2\gamma t(1-p)}} \,;$$

so, $\| U_{t,1-,u} - U_{t,1-,v} \| = 0$. □

[12] It is not about product systems, just two vectors in a Hilbert space.

Informally, the distinction between empty and non-empty sets $C \in \mathcal{C}_t$ is relative (to a special unit) and non-invariant (under isomorphisms of product systems), while the distinction between finite and infinite sets $C \in \mathcal{C}_t$ is absolute, invariant.

For any $C \in \mathcal{C}_t$ denote by C' the set of all accumulation points of C; clearly, $C' \in \mathcal{C}_t$, and $C' = \emptyset$ if and only if C is finite. We proceed similarly to Sect. 4, but C' is used here instead of C. Given an elementary set $E \subset (0,t)$, we define operators $Q'_{t,E}$ by

$$\big(Q'_{t,E}\psi\big)_P(C) = \begin{cases} \psi_P(C) & \text{if } C' \subset E, \\ 0 & \text{otherwise.} \end{cases}$$

We do not worry about boundary points of E, since $\mathcal{P}_{t,S}$-almost all C avoid them. Operators $Q'_{t,E}$ are tensor products of operators $U_{s,1-}$. (For example, if $E = (r,s)$, $0 < r < s < t$, then $Q'_{t,E} = U_{r,1-} \otimes \mathbf{1}_{s-r} \otimes U_{t-s,1-}$.) By Lemma 5.3, every isomorphism preserves $U_{s,1-}$; therefore it preserves $Q'_{t,E}$. Given $\psi \in H_{t,S}$, we define a measure $|\psi|'^2$ on $(\mathcal{C}_t, \mathcal{B}_t)$ as the image of the measure $|\psi|^2$ (defined by (2.2)) under the map $\mathcal{C}_t \ni C \mapsto C' \in \mathcal{C}_t$. Similarly to (4.3) we see that $|\psi|'^2$ is preserved by isomorphisms (even though $|\psi|^2$ is not).

5.4. Theorem. If $S'' \ne \emptyset$ then the product system $(H_{t,S})$ is asymmetric.[13]

Proof. Assume the contrary: the product system is symmetric; $\theta_t : H_{t,S} \to H_{t,S}$, $\theta_{s+t}(\psi_1 \otimes \psi_2) = \theta_t(\psi_2) \otimes \theta_s(\psi_1)$ for $\psi_1 \in H_{s,S}$, $\psi_2 \in H_{t,S}$. Then

$$\theta_t Q'_{t,E} = Q'_{t,t-E} \theta_t \,.$$

It follows that

$$R_t\big(|\psi|'^2\big) = |\theta_t \psi|'^2 \quad \text{for } \psi \in H_{t,S}\,; \tag{5.5}$$

here $R_t(|\psi|'^2)$ is the image of the measure $|\psi|'^2$ under the time reversal $R_t : \mathcal{C}_t \to \mathcal{C}_t$, $R_t(C) = t - C$. However, for $\mathcal{P}_{t,S}$-almost all $C \in \mathcal{C}_t$, C is totally ordered, therefore C' is also totally ordered. Both measures, $|\psi|'^2$ and $|\theta_t\psi|'^2$, being absolutely continuous w.r.t. $\mathcal{P}_{t,S}$, are concentrated on totally ordered sets. In combination with (5.5) it means that they are concentrated on finite sets. So, $C'' = \emptyset$ for $\mathcal{P}_{t,S}$-almost all $C \in \mathcal{C}_t$.

The Markov process $X(\cdot)$ consists of "small jumps" $X(t) = \big(X(t-)\big)_+$ and "big jumps" $X(t) > \big(X(t-)\big)_+$.[14] The rate of big jumps never exceeds 1. The rate of small jumps results in the mean speed 1 in the sense that

[13] Of course, S'' means $(S')'$; recall examples of S on page 8.

[14] As before, s_+ is the least element of $S \cap (s, \infty)$.

$X(t) - t$ is a martingale between big jumps. There is a chance that $X(\cdot)$ increases by 1 (or more) by small jumps only (between big jumps). In such a case, $S'' \ne \emptyset$ implies $Z''_{t,a,S} \ne \emptyset$. So, $\{C \in \mathcal{C}_t : C'' \ne \emptyset\}$ is not $\mathcal{P}_{t,S}$-negligible, in contradiction to the previous paragraph. $\square$

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School of Mathematics, Tel Aviv Univ., Tel Aviv 69978, Israel
`tsirel@math.tau.ac.il`
`http://www.math.tau.ac.il/~tsirel/`