The Bisognano-Wichmann Theorem for Charged States and the Conformal Boundary of a Black Hole*

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Dedicated to Eyvind Wichmann on the occation of his seventieth birthday

This talk is based on results contained in the following references [11, 12, 6, 13]. It will concern a study of the incremental entropy of a quantum black hole, based on Operator Algebras methods.

1 Case of Rindler spacetime

Let $\mathcal{M} = \mathbb{R}^4$ be the Minkowski spacetime and $\mathcal{O} \subset \mathcal{M} \to \mathcal{A}(\mathcal{O})$ be a net of von Neumann algebras generated by a local Wightman field on $\mathcal{M}$. Let $W \subset \mathcal{M}$ be a wedge region; I may assume $W = \{x : x_1 > x_0\}$ as any other wedge is a Poincaré translated of this special one. The Bisognano-Wichmann theorem [2] describes the modular structure associated with $(\mathcal{A}(W), \Omega)$, with Ω the vacuum vector (which is cyclic and separating for $\mathcal{A}(W)$ by the Reeh-Schlieder theorem):

$$\Delta^{it} = U(\Lambda_W(-2\pi t)), \tag{1}$$

$$J = U(R)\Theta. \tag{2}$$

Here Δ and J are the Tomita-Takesaki modular operator and conjugation associated with $(\mathcal{A}(W), \Omega)$, Λ_W is the one-parameter group of pure Lorentz transformations

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in the x_1-direction, U is the unitary representation of the Poincaré group, R is the rotation of π around the x_1-axis and Θ is the PCT anti-unitary.

In particular, for any fixed $a > 0$, the rescaled pure Lorentz transformations Λ_W give rise to a one parameter automorphism group $\alpha_t = \mathrm{Ad}U(\Lambda_W(at))$ of $\mathcal{A}(W)$ satisfying the KMS thermal equilibrium condition with respect to $\omega|_{\mathcal{A}(W)}$ at inverse temperature $\beta = \frac{2\pi}{a}$, with $\omega = (\Omega, \cdot\Omega)$.

As is known the Rindler spacetime may be identified with W. The pure vacuum state ω is thus a (mixed) thermal state when restricted to the algebra of W. Now, if $K = -i\frac{\mathrm{d}}{\mathrm{d}t}U(\Lambda_W(t))|_{t=0}$, then $H = aK$ is the proper Hamiltonian for an uniformly accelerated observer in the x_1-direction with acceleration a. As noticed by Sewell [16], the Bisognano-Wichmann theorem is thus essentially equivalent to the Hawking-Unruh effect in the case of a Rindler black hole.

Suppose now one adds a short range charge (superselection sector) localized in the exterior of the black hole: according to DHR [3], this amounts to consider the net $\mathcal{A}$ in a different irreducible localized representation. I may assume Haag duality to hold and this representation to be realized by a localized endomorphism ρ of the C^*-algebra $\mathfrak{A}(\mathcal{M})$, where for an unbounded region $\mathcal{R}$, $\mathfrak{A}(\mathcal{R})$ denotes the C^*-algebra generated by the $\mathcal{A}(\mathcal{O})$'s with $\mathcal{O} \subset \mathcal{R}$ bounded. I may further choose the charge ρ to be localized in W and to have finite statistics.

Now, as ρ is transportable, it turns out that $\rho|_{\mathfrak{A}(W)}$ is normal and I denote by $\rho|_{\mathcal{A}(W)}$ its weakly continuous extension to $\mathcal{A}(W)$. With Φ_ρ the left inverse of ρ (the unique completely positive map $\Phi_\rho : \mathfrak{A}(\mathcal{M}) \to \mathfrak{A}(\mathcal{M})$ with $\Phi_\rho \cdot \rho = \mathrm{id}$), the state $\varphi_\rho = \omega \cdot \Phi_\rho|_{\mathcal{A}(W)}$ is clearly invariant under the automorphism group $\alpha^\rho_t = \mathrm{Ad}U_\rho(\Lambda_W(at))$ of $\mathcal{A}(W)$, where U_ρ is the unitary representation of $\mathcal{P}^\uparrow_+$ in the representation ρ.

The modular structure associated with φ_ρ is given by the following theorem [11].

Theorem 1.1.

(i) α^ρ *satisfies the KMS condition with respect to* φ_ρ *at the same inverse temperature* $\beta = \frac{2\pi}{a}$*;*

(ii) $\log \Delta_{\Omega,\xi_\rho} = -2\pi K_\rho + \log d(\rho) = -\beta H_\rho + \log d(\rho)$.

Here K_ρ is the infinitesimal generator of $U_\rho(\Lambda_W(\cdot))$, ξ_ρ is the vector representative of φ_ρ in the natural cone of $(\mathcal{A}(W), \Omega)$, $d(\rho)$ is the DHR statistical dimension of ρ [3] and Δ_{Ω,ξ_ρ} is the relative modular operator.

Now $H_\rho = aK_\rho$ is the proper Hamiltonian, in the representation ρ, for the uniformly accelerated observer, so in particular one sees by (i) that the Bisognano-Wichmann theorem extends with its Hawking-Unruh effect interpretation in the charged representation ρ. This is to be expected inasmuch as the addition of a single charge should not alter the temperature of a thermodynamical system.

Point (ii) shows however a new feature: $\log(e^{-\beta H_\rho}\Omega, \Omega)$ may be interpreted as the incremental partition function between the states ω and φ_ρ on $\mathcal{A}(W)$. The incremental free energy is then given by

$$F(\omega|\varphi_\rho) = -\beta^{-1} \log(e^{-\beta H_\rho}\Omega, \Omega) ,$$

and indeed it turns out that the thermodynamical relation $\mathrm{d}F = \mathrm{d}E - T\mathrm{d}S$ holds, namely

$$F(\omega|\varphi_\rho) = \varphi_\rho(H_\rho) - \beta^{-1}S(\omega|\varphi_\rho) \ ,$$

where S is the Araki's relative entropy.

As the statistical dimension takes only integer values by the DHR theorem [3], one has as a corollary that the values of the incremental free energy are quantized:

Corollary 1.2. *The possible values of the incremental free energy are*

$$F(\omega|\varphi_\rho) = -\beta^{-1}\log n, \ \ n = 1, 2, 3, \dots$$

The index-statistics theorem [10] relates $d(\rho)$ to the square root of a Jones index [9]; this and other, see [11], leads to the formula

$$F(\omega|\varphi_\rho) = -\frac{1}{2}\beta^{-1}S_c(\rho),$$

where $S_c(\rho) = d(\rho)^2$ is the conditional entropy of ρ.

Notice now that the above discussion is subject to several restrictions. I have indeed focused on a particular spacetime, the Rindler wedge W, and considered endomorphisms of $\mathfrak{A}(W)$ that are restriction of endomorphisms of $\mathfrak{A}(\mathcal{M})$.

Moreover I have analyzed thermodynamical variables associated with the exterior of the black hole, without a direct relation to, say, the entropy of the black hole.

In the following I shall generalize this discussion leading to a certain clarification of the above aspects.

2 Case of a globally hyperbolic spacetime

I shall now consider more realistic black holes spacetimes, namely globally hyperbolic spacetimes a with bifurcate Killing horizon.

By considering charges localizable on the horizon, we shall get quantum numbers for the increment of the entropy of the black hole itself, rather than of the outside region.

A main point here is that the restriction of the net $\mathcal{A}$ of local observable von Neumann algebras to each of the two horizon components $\mathfrak{h}_+$ and $\mathfrak{h}_-$ gives rise a conformal net on S^1, a general fact that is obtained by applying Wiesbrock's characterization of conformal nets [18], as already discussed in [12, 6].

This is analogous to the holography on the anti-de Sitter spacetime that independently appeared in the Maldacena-Witten conjecture [14, 19], proved by Rehren [15]. Yet their context differs inasmuch as the anti-de Sitter spacetime is not globally hyperbolic and the holography there is a peculiarity of that spacetime, rather than a general phenomenon.

Finally, I shall deal with general KMS states, besides the Hartle-Hawking temperature state. In this context, a non-zero chemical potential [1] can appear, see [11].

The extension of the DHR analysis of superselection sectors [3] to a quantum field theory on a curved globally hyperbolic spacetime has been pursued in [6] and I refer to this paper the necessary background material. I however recall here the construction of conformal symmetries for the observable algebras on the horizon of the black hole.

Let $\mathcal{V}$ be a $d+1$ dimensional globally hyperbolic spacetime with a bifurcate Killing horizon. A typical example is given by the Schwarschild-Kruskal manifold that, by Birkoff theorem, is the only spherically symmetric solution of the Einstein-Hilbert equation; one might first focus on this specific example, as the more general case is treated similarly. I denote by $\mathfrak{h}_+$ and $\mathfrak{h}_-$ the two codimension 2 submanifolds that constitute the horizon $\mathfrak{h} = \mathfrak{h}_+ \cup \mathfrak{h}_-$. I assume that the horizon splits $\mathcal{V}$ in four connected components, the future, the past and the "left and right wedges" that I denote by $\mathcal{R}$ and $\mathcal{L}$ (in the Minkowski spacetime $\mathcal{R} = W$ and $\mathcal{L} = W'$).

Let $\kappa = \kappa(\mathcal{V})$ be the surface gravity, namely, denoting by χ the Killing vector field, the equation on $\mathfrak{h}$

$$\nabla g(\chi, \chi) = -2\kappa\chi \ , \tag{3}$$

with g the metric tensor, defines a function κ on $\mathfrak{h}$, that is actually constant on $\mathfrak{h}$ [17]. If $\mathcal{V}$ is the Schwarschild-Kruskal manifold, then

$$\kappa(\mathcal{V}) = \frac{1}{4M},$$

where M is the mass of the black hole. In this case $\mathcal{R}$ is the exterior of the Schwarschild black hole.

Our spacetime is $\mathcal{R}$ and $\mathcal{V}$ is to be regarded as a completion of $\mathcal{R}$.

Let $\mathcal{A}(\mathcal{O})$ be the von Neumann algebra on a Hilbert space $\mathcal{H}$ of the observables localized in the bounded diamond $\mathcal{O} \subset \mathcal{R}$. I make the assumptions of Haag duality, properly infinitness of $\mathcal{A}(\mathcal{O})$, Borchers property B, see [7]. The Killing flow Λ_t of $\mathcal{V}$ gives rise to a one parameter group of automorphisms α of the quasi-local C^*-algebra $\mathfrak{A}(\mathcal{R})$, since $\mathcal{R} \subset \mathcal{V}$ is a Λ-invariant region.

I now consider a locally normal α-invariant state φ on $\mathfrak{A}(\mathcal{R})$, that restricts to a KMS state at inverse temperature β on the horizon algebra, as I will explain. This is clearly the case of φ is KMS on all $\mathfrak{A}(\mathcal{R})$.

The case where φ is the Hartle-Hawking state is of particular interest; in this case the temperature

$$\beta^{-1} = \frac{\kappa}{2\pi}$$

is related to the geometry of the spacetime.

For convenience, I shall assume that the net $\mathfrak{A}$ is already in the GNS representation of φ, hence φ is represented by a cyclic vector ξ. Let's denote by $\mathcal{R}_a$ the wedge $\mathcal{R}$

“shifted by” $a \in \mathbb{R}$ along, say, $\mathfrak{h}_+$ (see [6]). If $I = (a, b)$ is a bounded interval of $\mathbb{R}_+$, I set

$$\mathcal{C}(I) = \mathfrak{A}(\mathcal{R}_a)'' \cap \mathfrak{A}(\mathcal{R}_b)', \quad 0 < a < b \ .$$

One obtains in this way a net of von Neumann algebras on the intervals of $(0, \infty)$, where the Killing automorphism group α acts covariantly by rescaled dilations.

I can now state my assumption: $\varphi|_{\mathfrak{C}(0,\infty)}$ is a KMS state with respect to α at inverse temperature $\beta > 0$. Here $\mathfrak{C}(0, \infty)$ is the C^*-algebra generated by all $\mathcal{C}(a, b), b > a > 0$ (for the Schwarzschild spacetime cf. [16]).

It follows that the restriction of the net to the black hole horizon $\mathfrak{h}_+$ has many more symmetries than on global spacetime.

Theorem 2.1. ([6]). *The Hilbert space $\mathcal{H}_0 = \overline{\mathcal{C}(I)\xi}$ is independent of the bounded open interval I.*

The net $\mathcal{C}$ extends to a conformal net $\mathbb{R}$ of von Neumann algebras acting on $\mathcal{H}_0$, where the Killing flow corresponds to the rescaled dilations.

This theorem says in particular that one may compactify $\mathbb{R}$ to the circle S^1, extend the definition of $\mathcal{C}(I)$ for all proper intervals $I \subset S^1$, find a unitary positive energy representation of the Möbius group $\mathrm{PSL}(2, \mathbb{R})$ acting covariantly on $\mathcal{C}$, so that the rescaled dilation subgroup is the Killing automorphism group.

If ξ is cyclic for $\mathfrak{C}$ on $\mathcal{H}$, namely $\mathcal{H}_0 = \mathcal{H}$ (as is true for the free field in Rindler case, see [6]), then $\mathcal{C}$ automatically satisfies Haag duality on $\mathbb{R}$. Otherwise one would pass to the dual net $\mathcal{C}^d$ of $\mathcal{C}$, which is is automatically conformal and strongly additive [5]. In the following I assume that $\mathcal{C}$ is strongly additive.

2.1 Charges localizable on the horizon

I now consider an irreducible endomorphism ρ of $\mathfrak{A}(\mathcal{R})$ with finite dimension that is localizable in an interval (a, b), $b > a > 0$, of $\mathfrak{h}_+$, namely ρ acts trivially on $\mathfrak{A}(\mathcal{R}_b)$ and on $\mathcal{C}(I)$ if $\bar{I} \subset (0, a)$, therefore ρ restricts to a localized endomorphism of $\mathfrak{C}(0, \infty)$.

This last requirement is necessary to extend ρ to a normal endomorphism of $\pi_\varphi(\mathfrak{C}(0, \infty))''$, a result obtained by conformal methods [13].

Theorem 2.2. *With the above assumptions, if ρ is localized in an interval of $\mathfrak{h}_+$, then $d(\rho|_{\mathfrak{C}(0,\infty)})$ has a normal extension to the weak closure $\mathfrak{C}(0, \infty)''$ with dimension $d(\rho)$.*

Let σ be another irreducible endomorphism of $\mathfrak{A}$ localized in $(a, b) \subset \mathfrak{h}_+$ and denote by φ_ρ and φ_σ the thermal states for the Killing automorphism group in the representation ρ. As shown in [11], $\varphi_\rho = \varphi \cdot \Phi_\rho$, where Φ_ρ is the left inverse of ρ, and similarly for σ.

The increment of the free energy between the thermal equilibrium states φ_ρ and φ_σ is expressed as in [11] by

$$F(\varphi_\rho|\varphi_\sigma) = \varphi_\rho(H_{\rho\bar{\sigma}}) - \beta^{-1} S(\varphi_\rho|\varphi_\sigma)$$

Here S is the Araki relative entropy and $H_{\rho\bar{\sigma}}$ is the Hamiltonian on $\mathcal{H}_0$ corresponding to the charge ρ and the charge conjugate to $\bar{\sigma}$ localized in $(-\infty, 0)$ as in [11] (by [4], a transportable localized endomorphism ρ of $\mathcal{C}$ with finite dimension is Möbius covariant). In particular, if σ is the identity representation, then $H_{\rho\bar{\sigma}} = H_\rho$ is the Killing Hamiltonian in the representation ρ.

By the analysis in [13] one can write formulas in the case of two different KMS states φ_ρ and φ_σ. In particular

$$F(\varphi_\sigma|\varphi_\rho) = \frac{1}{2}\beta^{-1}(S_c(\sigma) - S_c(\rho)) + \mu(\varphi_\sigma|\varphi_\rho), \tag{4}$$

where $\mu(\varphi_\sigma|\varphi_\rho)$ is the chemical potential, cf. [1].

The above formula gives a canonical splitting for the incremental free energy. The first term is symmetric under charge conjugation, the second term is anti-symmetric. The quantization of the possible values of the symmetric part goes through similarly as in Corollary 1.2.

3 Quantum index theorem

The above discussion is part of a general aim concerning a quantum index theorem. This point of view is discussed in [13].

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