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EQUIVARIANT KASPAROV THEORY AND GENERALIZED HOMOMORPHISMS

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Abstract. Let G be a locally compact group. We describe elements of $KK^G(A,B)$ by equivariant homomorphisms, following Cuntz's treatment in the non-equivariant case. This yields another proof for the universal property of KK^G: It is the universal split exact stable homotopy functor.

To describe a Kasparov triple $(\mathcal{E},\phi,F)$ for A,B by an equivariant homomorphism, we have to arrange for the Fredholm operator F to be equivariant. This can be done if A is of the form $\mathbb{K}(L^2G)\otimes A'$ and more generally if the group action on A is proper in the sense of Exel and Rieffel.

1. Introduction

In this article, we carry over the description of Kasparov theory in terms of generalized homomorphisms to the equivariant case. Let us first recall the well-known situation for Kasparov theory without group actions.

The existence and associativity of the Kasparov product mean that we can define a category KK whose objects are the separable C*-algebras and whose morphisms from A to B are the elements of $KK(A,B)$. In [5] and [6], Cuntz relates elements of $KK(A,B)$ with trivially graded C*-algebras A and B to ordinary $*$-homomorphisms. He defines a certain ideal $\mathrm{q}A$ in the free product $A*A$ and constructs a natural bijection between $KK(A,B)$ and the set $[\mathrm{q}A,\mathbb{K}\otimes B]$ of homotopy classes of $*$-homomorphisms $\mathrm{q}A\to\mathbb{K}\otimes B$, where $\mathbb{K}$ denotes the algebra of compact operators on a separable Hilbert space. Skandalis [20] remarks that we also have $KK(A,B)\cong[\mathbb{K}\otimes\mathrm{q}A,\mathbb{K}\otimes\mathrm{q}B]$. The Kasparov product becomes simply the composition of $*$-homomorphisms in this picture. Cuntz's description of $KK(A,B)$ is used by Higson [12] to characterize Kasparov theory by a universal property: The canonical functor from separable C*-algebras to KK is the universal split exact stable homotopy functor.

For graded C*-algebras, Haag [10] describes $KK(A,B)$ in a similar way as the set of homotopy classes of grading preserving $*$-homomorphisms from χA to $\hat{\mathbb{K}}\mathbin{\hat\otimes} B$ for a suitable graded C*-algebra χA. He shows that $KK(A,B)\cong KK^{\mathbb{Z}_2}(\hat S\mathbin{\hat\otimes} A,B)$, where $KK^{\mathbb{Z}_2}$ is the $\mathbb{Z}_2$-equivariant Kasparov theory for trivially graded algebras and $\hat S$ is $\mathrm{C}_0(\mathbb{R})$ graded by reflection at the origin. Furthermore, Haag identifies the Kasparov product for graded C*-algebras in this setting [9].

It is straightforward to carry over these results to KK^G for a *compact* group G. However, new ideas are necessary if G is merely locally compact. The only result in that generality I am aware of is due to Thomsen [21]. He shows that KK^G can still be characterized as the universal split exact stable homotopy functor. However, he does not obtain a description of KK^G by equivariant $*$-homomorphisms.

Let A and B be G-C*-algebras. Let $\mathbb{K}$ be the algebra of compact operators on the direct sum of infinitely many copies of L^2G. We would like to associate a

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G-equivariant $*$-homomorphism $\mathrm{q}A \to \mathbb{K} \otimes B$ (or $\chi A \to \hat{\mathbb{K}} \hat{\otimes} B$ in the graded case) to a Kasparov triple $(\mathcal{E}, \phi, F)$ for A, B. This may be impossible for two reasons: The operator F need not be G-equivariant, and there may be no G-equivariant embedding $\mathcal{E} \subset L^2(G,B)^\infty$. It is surprisingly easy to overcome these problems: We just have to replace A by $\mathbb{K}(L^2G) \otimes A$. If the map $\phi\colon \mathbb{K}(L^2G) \otimes A \to \mathbb{L}(\mathcal{E})$ is *essential* in the sense that $\phi(\mathbb{K}(L^2G) \otimes A) \cdot \mathcal{E}$ is dense in $\mathcal{E}$, then $\mathcal{E} = L^2(G, \mathcal{E}')$ for some Hilbert B, G-module $\mathcal{E}'$. Hence $\mathcal{E}$ can be embedded in $L^2(G,B)^\infty$. Moreover, the additional copy of L^2G gives us enough freedom to replace F by a *compact perturbation* F' that is G-equivariant (Lemma 3.1).

Once we have that F is G-equivariant and $\mathcal{E} \subset L^2(G,B)^\infty$, we can proceed as in the non-equivariant case. We show that we get the same KK^G-groups if we restrict to Kasparov triples and homotopies $(\mathcal{E}, \phi, F)$ with a G-equivariant symmetry F and $\mathcal{E} \subset L^2(G,B)^\infty$ (Proposition 3.4). *Symmetry* means that $F = F^*$ and $F^2 = 1$. This yields a bijection between $KK^G(\mathbb{K}(L^2G) \otimes A, B)$ and the set of homotopy classes of G-equivariant $*$-homomorphisms $\mathrm{q}(\mathbb{K}(L^2G) \otimes A) \to \mathbb{K} \otimes B$ (Proposition 5.4). In addition, we obtain an analogous statement for graded algebras and show that we may tensor $\mathrm{q}(\mathbb{K}(L^2G) \otimes A)$ with $\mathbb{K}(\mathcal{H})$ for any G-Hilbert space $\mathcal{H}$.

The universal property of KK^G for trivially graded separable G-C*-algebras is an immediate consequence of this description of KK^G because $F(\mathrm{q}A) \cong F(A)$ for any split exact stable homotopy functor F. For graded algebras, we prove that

$$KK^G(A,B) \cong KK^{G \times \mathbb{Z}_2}(\hat{S} \hat{\otimes} A, B),$$

where KK^G and $KK^{G \times \mathbb{Z}_2}$ denote the Kasparov theories for graded G-C*-algebras and trivially graded $G \times \mathbb{Z}_2$-C*-algebras, respectively. We describe the Kasparov product in this setting.

In addition, we show that we can obtain $KK^G(A,B)$ using only Kasparov triples $(\mathcal{E}, \phi, F)$ with G-equivariant F and $\mathcal{E} \subset L^2(G,B)^\infty$ if A is *proper* in the sense of Exel [7] and Rieffel [18]. This notion of properness is quite general and covers both algebras of the form $\mathbb{K}(L^2G) \otimes A$ and the proper algebras of [8].

Our key result concerning proper group actions is that a countably generated Hilbert A, G-module $\mathcal{E}$ satisfies $\mathcal{E} \oplus L^2(G,A)^\infty \cong L^2(G,A)^\infty$ if and only if $\mathbb{K}(\mathcal{E})$ is a proper G-C*-algebra. This is not surprising in view of Rieffel's treatment of square-integrable representations of groups on Hilbert space [18].

2. Notation and Conventions

For the convenience of the reader, we recall the definitions of Hilbert modules, Kasparov triples, and connections. Moreover, we fix some notation.

Let G be a locally compact, σ-compact topological group. Let dg be a left invariant Haar measure on G and let $L^2G = L^2(G, dg)$. The *left regular representation* of G, defined by $\lambda_g(f)(g') := f(g^{-1}g')$ for $f \in L^2G$ and $g, g' \in G$, is a strongly continuous unitary representation of G on L^2G. We always equip the C*-algebra $\mathbb{K}(L^2G)$ of compact operators on L^2G with the G-action induced by λ. That is, $\lambda_g(T) = \lambda_g \circ T \circ \lambda_g^{-1}$ for all $g \in G$, $T \in \mathbb{K}(L^2G)$.

Let $\mathbb{Z}_2$ be the group with two elements and let $G_2 := G \times \mathbb{Z}_2$. A *$\mathbb{Z}_2$-graded G-C*-algebra* or, briefly, G_2-C*-algebra is a C*-algebra with a strongly continuous action of G_2. Recall that a grading is nothing but a $\mathbb{Z}_2$-action. We always write α_g and β_g for the actions of $g \in G_2$ on the G_2-C*-algebras A and B, respectively.

Let $\mathbb{M}_2$ and $\hat{\mathbb{M}}_2$ be the algebra of 2×2-matrices with the trivial grading and with the off-diagonal grading, respectively. That is, the off-diagonal terms in $\hat{\mathbb{M}}_2$ are odd. Let $\mathbb{C}\mathrm{l}_1$ be the first Clifford algebra, that is, the universal C*-algebra generated by an odd symmetry.

2.1. **Hilbert modules.** Let B be a G_2-C*-algebras. A $\mathbb{Z}_2$*-graded Hilbert* B,G*-module* or, briefly, *Hilbert* B,G_2*-module* is a Hilbert B-module $\mathcal{E}$ with B-valued inner product $\langle\llcorner\!\lrcorner \mid \llcorner\!\lrcorner\rangle_B$ that is equipped with a strongly continuous linear action γ_g of G_2 satisfying $\gamma_g(\xi\cdot b) = \gamma_g(\xi)\beta_g(b)$ and $\beta_g(\langle\xi \mid \eta\rangle_B) = \langle\gamma_g\xi \mid \gamma_g\eta\rangle_B$ for all $g \in G_2$, $\xi,\eta \in \mathcal{E}$, $b \in B$. We call $\mathcal{E}$ *full* iff the linear span of $\langle\mathcal{E} \mid \mathcal{E}\rangle_B$ is dense in B.

We write $\mathbb{L}(\mathcal{E})$ and $\mathbb{K}(\mathcal{E})$ for the C*-algebras of *adjointable* and *compact* operators on $\mathcal{E}$. The latter is generated by the *rank one operators* $|\xi\rangle\langle\eta|$ defined by

$$|\xi\rangle\langle\eta|(\zeta) := \xi\cdot\langle\eta \mid \zeta\rangle_B \qquad \text{for all } \xi,\eta,\zeta \in \mathcal{E}.$$

We always endow $\mathbb{L}(\mathcal{E})$ with the induced G_2-action, $\gamma_g(T) := \gamma_g \circ T \circ \gamma_g^{-1}$. This action is strongly continuous on $\mathbb{K}(\mathcal{E})$ but usually not on $\mathbb{L}(\mathcal{E})$. We call $T \in \mathbb{L}(\mathcal{E})$ G*-continuous* iff the map $g \mapsto \gamma_g(T)$ is norm continuous.

We denote graded and ungraded spatial *tensor products* of $\mathbb{Z}_2$-graded C*-algebras and Hilbert modules by $\hat{\otimes}$ and $\otimes$, respectively. If A is trivially graded, then there is no difference between $A \hat{\otimes} B$ and $A \otimes B$.

2.1.1. *Standard Hilbert modules.* Let $\ell^2(\mathbb{N})$ be the separable Hilbert space with trivial G_2-action. Let $\ell^2(\mathbb{Z}_2\mathbb{N})$ be the graded Hilbert space $\ell^2(\mathbb{N})^{\mathrm{even}} \oplus \ell^2(\mathbb{N})^{\mathrm{odd}}$. Let $L^2(G\mathbb{N}) := L^2G \otimes \ell^2(\mathbb{N})$ and $L^2(G_2\mathbb{N}) := L^2G \otimes \ell^2(\mathbb{Z}_2\mathbb{N})$. We abbreviate $\mathbb{K}(G) := \mathbb{K}(L^2G)$, $\mathbb{K}(\mathbb{N}) := \mathbb{K}\big(\ell^2(\mathbb{N})\big)$, etc. Moreover, we write $\mathbb{K}(\cdots)A$ instead of $\mathbb{K}(\cdots)\hat{\otimes}A \cong \mathbb{K}(\cdots)\otimes A$. Let $\mathcal{E}$ be a Hilbert B,G_2-module. Define $\mathcal{E}^\infty := \ell^2(\mathbb{N})\hat{\otimes}\mathcal{E}$, $L^2(G,\mathcal{E}) := L^2G \hat{\otimes} \mathcal{E}$, and $L^2(G_2,\mathcal{E}) := L^2(G\times\mathbb{Z}_2) \hat{\otimes} \mathcal{E}$. Let

$$\hat{\mathcal{H}}_B := L^2(G_2,B)^\infty \cong L^2(G, B\oplus B^{\mathrm{op}})^\infty, \qquad \mathcal{H}_B := L^2(G,B)^\infty.$$

The Hilbert module $\mathcal{H}_B$ is important only if B is trivially graded.

2.1.2. *Isometric embeddings.* Let B be a G_2-C*-algebra and let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert B,G_2-modules. A map $\iota\colon \mathcal{E} \to \mathcal{F}$ is called an *isometric embedding* iff it is a linear, G_2-equivariant B-module map and satisfies $\langle\iota(\xi) \mid \iota(\eta)\rangle_B = \langle\xi \mid \eta\rangle_B$ for all $\xi,\eta \in \mathcal{E}$. Hence ι is injective and $\iota(\mathcal{E}) \subset \mathcal{F}$ is a closed G_2-invariant B-submodule. We do not require ι to be adjointable. This happens iff $\iota(\mathcal{E})$ is *complementable*, that is, $\iota(\mathcal{E}) \oplus \iota(\mathcal{E})^\perp = \mathcal{F}$. We write $\mathcal{E} \subset \mathcal{F}$ iff there is an isometric embedding $\mathcal{E} \to \mathcal{F}$.

2.2. **Hilbert bimodules.** Let A and B be G_2-C*-algebras. A $\mathbb{Z}_2$*-graded Hilbert* A,B,G*-bimodule* or, briefly, *Hilbert* A,B,G_2*-bimodule* is a Hilbert B,G_2-module $\mathcal{E}$ with a G_2-equivariant $*$-homomorphism $\phi\colon A \to \mathbb{L}(\mathcal{E})$. We often use module notation for the action of A on $\mathcal{E}$, writing $a\xi$ instead of $\phi(a)(\xi)$. The equivariance of ϕ means that $\gamma_g(a\xi) = \alpha_g(a)\gamma_g(\xi)$ for all $g \in G_2$, $a \in A$, $\xi \in \mathcal{E}$.

Let $A\cdot\mathcal{E} \subset \mathcal{E}$ be the subset of all elements of the form $a\xi$ with $a \in A$, $\xi \in \mathcal{E}$. The Cohen-Hewitt factorization theorem [3], [11] implies that $A\cdot\mathcal{E}$ is a closed linear subspace. We call $\mathcal{E}$ *essential* iff $A\cdot\mathcal{E} = \mathcal{E}$. Let $\mathcal{M}(A)$ be the *multiplier algebra* of A. If $\mathcal{E}$ is essential, then there is a unique extension of ϕ to a G_2-equivariant $*$-homomorphism $\phi\colon \mathcal{M}(A) \to \mathbb{L}(\mathcal{E})$. The extension is defined by $\phi(m)(a\cdot\xi) = (m\cdot a)\cdot\xi$ for all $m \in \mathcal{M}(A)$, $a \in A$, $\xi \in \mathcal{E}$.

If $\mathcal{E}_1$ is a Hilbert A,B,G_2-module and $\mathcal{E}_2$ is a Hilbert B,C,G_2-module, then the tensor product $\mathcal{E}_1 \hat{\otimes}_B \mathcal{E}_2$ over B is defined as in [15]. It is a Hilbert A,C,G_2-bimodule. If B acts on $\mathcal{E}_2$ via $\phi\colon B \to \mathbb{L}(\mathcal{E}_2)$, then we also use the more precise notation $\mathcal{E}_1 \hat{\otimes}_\phi \mathcal{E}_2$ of [2].

2.3. **Imprimitivity bimodules.** Let A and B be G_2-C*-algebras. A Hilbert A,B,G_2-bimodule $(\mathcal{E},\phi)$ is called an *imprimitivity bimodule* iff it is full and ϕ is an isomorphism onto $\mathbb{K}(\mathcal{E})$ [19]. We call A and B *Morita-Rieffel equivalent* iff there is an imprimitivity bimodule for them. This is an equivalence relation. Especially, if $\mathcal{E}$ is an imprimitivity bimodule for A,B,G_2, then there is a *dual* imprimitivity bimodule $\mathcal{E}^*$ for B,A,G_2. It satisfies $\mathcal{E}^* \hat{\otimes}_A \mathcal{E} \cong B$ as Hilbert B,B,G_2-bimodules and

$\mathcal{E} \hat{\otimes}_B \mathcal{E}^* \cong A$ as Hilbert A, A, G_2-bimodules. A concrete model for $\mathcal{E}^*$ is $\mathbb{K}(\mathcal{E}, B)$. The algebras $\mathbb{K}(\mathcal{E}) \cong A$ and $\mathbb{K}(B) \cong B$ operate on $\mathbb{K}(\mathcal{E}, B)$ by composition. The $\mathbb{K}(\mathcal{E})$-valued inner product is defined by $\langle T_1 \mid T_2\rangle := T_1^* T_2$ for all $T_1, T_2 \in \mathbb{K}(\mathcal{E}, B)$.

2.4. **Kasparov triples.** Let A and B be σ-unital G_2-C*-algebras. A *Kasparov triple for* A, B is a triple $(\mathcal{E}, \phi, F)$, where $(\mathcal{E}, \phi)$ is a countably generated Hilbert A, B, G_2-bimodule and $F \in \mathbb{L}(\mathcal{E})$ is odd with respect to the grading and satisfies

$$(1) \qquad [F, \phi(a)],\ (1 - F^2)\phi(a),\ (F - F^*)\phi(a),\ (\gamma_g(F) - F)\phi(a) \in \mathbb{K}(\mathcal{E})$$

for all $a \in A$, $g \in G$. The expression $[F, \phi(a)]$ in (1) is a *graded commutator*. In the following, all commutators will be graded. The Kasparov triple is called *degenerate* iff all the terms in (1) are zero.

Thomsen [22] shows that (1) implies that the operators $F \cdot \phi(a)$ are G-continuous for all $a \in A$. Hence this additional requirement of Kasparov [15] is redundant.

Two Kasparov triples $(\mathcal{E}_t, \phi_t, F_t)$, $t = 0, 1$, are *unitarily equivalent* iff there is a G_2-equivariant unitary $U\colon \mathcal{E}_0 \to \mathcal{E}_1$ with $\phi_1(a)U = U\phi_0(a)$ for all $a \in A$ and $F_1 U = U F_0$. Up to unitary equivalence, Kasparov triples are functorial for G_2-equivariant $*$-homomorphisms in both variables. If $f\colon B_1 \to B_2$ is a G_2-equivariant $*$-homomorphism and $(\mathcal{E}, \phi, F)$ is a Kasparov triple for A, B_1, then

$$f_*(\mathcal{E}, \phi, F) := (\mathcal{E} \hat{\otimes}_f B_2, \phi \hat{\otimes} 1, F \hat{\otimes} 1).$$

Let $B[0,1] := C([0,1]; B)$ with the pointwise action of G_2 and let $\mathrm{ev}_t\colon B[0,1] \to B$ be the evaluation homomorphism at $t \in [0,1]$. A *homotopy* between two Kasparov triples T_0 and T_1 is a Kasparov triple $\bar{T} = (\bar{\mathcal{E}}, \bar{\phi}, \bar{F})$ for $A, B[0,1]$ such that $\bar{T}|_t := (\mathrm{ev}_t)_*(\bar{\mathcal{E}}, \bar{\phi}, \bar{F})$ is unitarily equivalent to T_t for $t = 0, 1$. The *Kasparov group* $KK^G(A, B)$ is defined as the set of homotopy classes of Kasparov triples for A, B.

Let $(\mathcal{E}, \phi, F)$ be a Kasparov triple for A, B. We call $F' \in \mathbb{L}(\mathcal{E})$ a *compact perturbation* of F iff

$$(F' - F)\phi(a) \in \mathbb{K}(\mathcal{E}) \quad \text{and} \quad \phi(a)(F' - F) \in \mathbb{K}(\mathcal{E}) \qquad \text{for all } a \in A.$$

If F' is a compact perturbation of F, then $(\mathcal{E}, \phi, F')$ is a Kasparov triple as well. The triples $(\mathcal{E}, \phi, F)$ and $(\mathcal{E}, \phi, F')$ are *operator homotopic* via the obvious path $F_t := (1-t)F + tF'$, and therefore also homotopic.

2.5. **Connections.** Let $\mathcal{E}_1$ be a Hilbert A, G_2-module and let $\mathcal{E}_2$ be a Hilbert A, B, G_2-bimodule. Let $\mathcal{E}_{12} := \mathcal{E}_1 \hat{\otimes}_A \mathcal{E}_2$. For $\xi \in \mathcal{E}_1$, define an adjointable operator $T_\xi\colon \mathcal{E}_2 \to \mathcal{E}_{12}$ by $T_\xi(\eta) := \xi \hat{\otimes} \eta$ and $T_\xi^*(\eta \hat{\otimes} \zeta) := \langle \xi \mid \eta\rangle_A \cdot \zeta$. For $\xi \in \mathcal{E}_1$, $F_2 \in \mathbb{L}(\mathcal{E}_2)$, and $F_{12} \in \mathbb{L}(\mathcal{E}_{12})$, define adjointable operators on $\mathcal{E}_2 \oplus \mathcal{E}_{12}$ by

$$\tilde{T}_\xi := \begin{pmatrix} 0 & T_\xi^* \\ T_\xi & 0 \end{pmatrix} \qquad \text{and} \qquad F_2 \oplus F_{12} := \begin{pmatrix} F_2 & 0 \\ 0 & F_{12} \end{pmatrix}.$$

The operator F_{12} is called an F_2-*connection* iff $[F_2 \oplus F_{12}, \tilde{T}_\xi] \in \mathbb{K}(\mathcal{E}_2 \oplus \mathcal{E}_{12})$ for all $\xi \in \mathcal{E}_1$. Assume that F_2 and F_{12} are odd and self-adjoint and denote the grading automorphism on $\mathcal{E}_1$ by τ. Then F_{12} is an F_2-connection iff

$$F_{12} T_\xi - T_{\tau\xi} F_2 \in \mathbb{K}(\mathcal{E}_2, \mathcal{E}_{12}) \qquad \text{for all } \xi \in \mathcal{E}_1.$$

We will freely use the standard properties of connections [2, 18.3].

3. Equivariant connections and special Kasparov triples

Let A and B be σ-unital G_2-C*-algebras and let $\mathcal{H}$ be a separable G_2-Hilbert space. A Kasparov triple $(\mathcal{E}, \phi, F)$ for A, B is called $\mathcal{H}$-*special* iff

(i) F is a G-equivariant symmetry; and
(ii) $\mathcal{H} \hat{\otimes} \mathcal{E} \subset \hat{\mathcal{H}}_B$.

An $\mathcal{H}$-*special homotopy* is given by an $\mathcal{H}$-special Kasparov triple for $A, B[0,1]$. We let $KK^G_{s,\mathcal{H}}(A,B)$ be the set of $\mathcal{H}$-special Kasparov triples modulo $\mathcal{H}$-special homotopy. If $\mathcal{H} = \mathbb{C}$, we omit the $\mathcal{H}$ and talk about *special triples*, *special homotopies*, and $KK^G_s(A,B)$. We are mostly interested in the cases $\mathcal{H} = \mathbb{C}$ and $\mathcal{H} = L^2(G_2\mathbb{N})$. In the latter case, the condition $\mathcal{H} \hat{\otimes} \mathcal{E} \subset \hat{\mathcal{H}}_B$ becomes tautological. The additional flexibility of choosing $\mathcal{H}$ is useful in connection with Proposition 5.4. A special triple is automatically $\mathcal{H}$-special because $\mathcal{H} \hat{\otimes} \hat{\mathcal{H}}_B \cong \hat{\mathcal{H}}_B$. Hence there are canonical maps $KK^G_s(A,B) \to KK^G_{s,\mathcal{H}} \to KK^G(A,B)$. Usually, these maps fail to be isomorphisms. For instance, if the unit element of $KK^G(\mathbb{C},\mathbb{C})$ comes from an element of $KK^G_s(\mathbb{C},\mathbb{C})$, then G has to be compact.

In this section, we show that $KK^G_s(A,B) \cong KK^G_{s,\mathcal{H}}(A,B) \cong KK^G(A,B)$ if A has the property AE that is defined below. We verify that algebras of the form $\mathbb{K}(L^2G)A$ have this property. In Section 8, we will see that proper algebras have property AE as well.

Lemma 3.1. *Let A and B be σ-unital G_2-C^*-algebras. Let $(\mathcal{E}, \phi, F)$ be an essential Kasparov triple for A, B. Let $\mathcal{E}' := L^2(G,A) \hat{\otimes}_\phi \mathcal{E} \cong L^2(G,\mathcal{E})$.*

There is a G-equivariant F-connection F' on $\mathcal{E}'$. Even more, we can achieve that F' is a G-equivariant self-adjoint contraction.

Proof. Let $\mathrm{C_c}(G,\mathcal{E})$ be the space of continuous functions $G \to \mathcal{E}$ with compact support. The inner product $\langle f_1 \mid f_2\rangle_B := \int_G \langle f_1(g) \mid f_2(g)\rangle_B \, dg$ turns $\mathrm{C_c}(G,\mathcal{E})$ into a pre-Hilbert B-module. Its completion is $L^2(G,\mathcal{E})$. We have $L^2(G,A) \hat{\otimes}_\phi \mathcal{E} \cong L^2(G,\mathcal{E})$ because ϕ is essential. We may assume that F is a self-adjoint contraction by [2, 17.4.3]. Define $F'\colon \mathrm{C_c}(G,\mathcal{E}) \to \mathrm{C_c}(G,\mathcal{E})$ by

$$(F'f)(g) = \gamma_g(F)f(g) = \gamma_g\big(F\gamma_g^{-1}f(g)\big) \qquad \text{for all } g \in G,\ f \in \mathrm{C_c}(G,\mathcal{E}).$$

It is straightforward to check that F' is G-equivariant and odd and extends to a self-adjoint contraction $F'\colon L^2(G,\mathcal{E}) \to L^2(G,\mathcal{E})$.

We claim that F' is an F-connection. Denote the grading automorphisms on A and $L^2(G,A)$ by τ. We have to check that $K := T_\xi F - F' T_{\tau\xi} \in \mathbb{K}(\mathcal{E},\mathcal{E}')$ for all $\xi \in L^2(G,A)$. We may restrict to ξ of the form $\xi(g) = f(g)a$ with $f \in \mathrm{C_c}(G)$, $a \in A$, because such elements span a dense subspace of $L^2(G,A)$. We have

$$(K\eta)(g) = f(g)\phi(a)F\eta - f(g)\gamma_g(F)\phi\tau(a)\eta = K_g(\eta)$$

for all $\eta \in \mathcal{E}$, where

$$K_g := f(g)\phi(a)F - f(g)\gamma_g(F)\phi\tau(a) = f(g)[\phi(a),F] + f(g)\big(F - \gamma_g(F)\big)\phi\tau(a).$$

Since $(\mathcal{E},\phi,F)$ is a Kasparov triple and f has compact support, K_g is a norm continuous compactly supported function $G \to \mathbb{K}(\mathcal{E})$. Using a partition of unity, we can approximate the function $g \mapsto K_g$ uniformly by finite sums of functions $g \mapsto \psi(g)T$ with $\psi \in \mathrm{C_c}(G)$, $T \in \mathbb{K}(\mathcal{E})$. Approximating T by sums of finite rank operators, we can approximate K in norm by finite sums of operators of the form $\eta \mapsto \psi \hat{\otimes} |\xi\rangle\langle\zeta|(\eta)$. Hence $K \in \mathbb{K}(\mathcal{E},\mathcal{E}')$, so that F' is an F-connection. □

We say that a G_2-C*-algebra A has *property AE* iff: For all σ-unital G_2-C*-algebras B and all *essential* Kasparov triples $(\mathcal{E},\phi,F)$ for A, B, there is a G-equivariant compact perturbation F' of F and there is an isometric embedding $\mathcal{E} \subset \hat{\mathcal{H}}_B$.

The letters AE are an abbreviation for "automatic equivariance".

Proposition 3.2. *Let A and B be σ-unital G_2-C^*-algebras and let $(\mathcal{E},\phi,F)$ be an* essential *Kasparov triple for $\mathbb{K}(G)A, B$. Then we can find a G-equivariant compact perturbation of F and an isomorphism*

$$\mathcal{E} \oplus \hat{\mathcal{H}}_B \cong \hat{\mathcal{H}}_B$$

of Hilbert B, G_2-modules.

Thus $\mathbb{K}(G)A := \mathbb{K}(L^2G) \hat{\otimes} A$ has property AE.

Proof. Let

$$\psi\colon \mathbb{K}(G)A \xrightarrow{\cong} \mathbb{K}\big(L^2(G,A)\big)$$

be the canonical isomorphism. Thus $(L^2(G,A),\psi)$ is an imprimitivity bimodule. Let $(L^2(G,A)^*,\psi^*)$ be the corresponding dual imprimitivity bimodule. That is, $L^2(G,A)^*$ is a Hilbert $\mathbb{K}(G)A, G$-module and ψ^* is an isomorphism between A and $\mathbb{K}(L^2(G,A)^*)$ such that

$$L^2(G,A) \mathbin{\hat{\otimes}_{\psi^*}} L^2(G,A)^* \cong \mathbb{K}(G)A$$

as Hilbert $\mathbb{K}(G)A, \mathbb{K}(G)A, G_2$-bimodules. Let

$$\mathcal{E}_0 := L^2(G,A)^* \mathbin{\hat{\otimes}_\phi} \mathcal{E}, \qquad \phi_0 := \psi^* \mathbin{\hat{\otimes}} 1\colon A \to \mathbb{L}(\mathcal{E}_0).$$

Let $F_0 \in \mathbb{L}(\mathcal{E}_0)$ be an F-connection. Then $(\mathcal{E}_0,\phi_0,F_0)$ is an essential Kasparov triple for A, B. It is a Kasparov product of $(L^2(G,A)^*,\psi^*,0)$ and $(\mathcal{E},\phi,F)$.

Since ϕ is essential, we have $\mathbb{K}(G)A \mathbin{\hat{\otimes}_\phi} \mathcal{E} \cong \mathcal{E}$ and hence

$$\mathcal{E} \cong L^2(G,A) \mathbin{\hat{\otimes}_{\psi^*}} L^2(G,A)^* \mathbin{\hat{\otimes}_\phi} \mathcal{E} \cong L^2(G,A) \mathbin{\hat{\otimes}_{\phi_0}} \mathcal{E}_0$$

as Hilbert B, G_2-modules. We have $\phi = \psi \mathbin{\hat{\otimes}} 1\colon \mathbb{K}(G)A \to \mathbb{L}(L^2(G,A) \mathbin{\hat{\otimes}_{\phi_0}} \mathcal{E}_0)$.

By Lemma 3.1, there is a G-equivariant F_0-connection F' on $\mathcal{E}$. The operator F' is an F-connection on $\mathbb{K}(G)A \mathbin{\hat{\otimes}_\phi} \mathcal{E}$ by [2, 18.3.4.f]. Thus $F - F'$ is a 0-connection. This means that F' is a compact perturbation of F by [2, 18.3.2.c]. As a result, F' is a G-equivariant compact perturbation of F.

The equivariant stabilization theorem [17, Theorem 2.5] for the compact group $\mathbb{Z}_2$ yields $\mathcal{E}_0 \oplus (B \oplus B^{\mathrm{op}})^\infty \cong (B \oplus B^{\mathrm{op}})^\infty$ as $\mathbb{Z}_2$-graded Hilbert B-modules. Hence

$$\mathcal{E} \oplus \hat{\mathcal{H}}_B \cong L^2(G, \mathcal{E}_0 \oplus (B \oplus B^{\mathrm{op}})^\infty) \cong L^2(G, (B \oplus B^{\mathrm{op}})^\infty) = \hat{\mathcal{H}}_B$$

as Hilbert B, G_2-modules by [17, Lemma 2.3]. Thus $\mathcal{E} \subset \hat{\mathcal{H}}_B$. □

It is a well-known fact that any Kasparov triple is homotopic to an essential triple [2, 18.3.6]. We need a more explicit construction of the homotopy.

Lemma 3.3. *Let A and B be σ-unital G_2-C^*-algebras. Let $(\mathcal{E},\phi,F)$ be a Kasparov triple for A, B. Let $\mathcal{E}_{\mathrm{es}} := \phi(A)\cdot\mathcal{E} \cong A \mathbin{\hat{\otimes}_\phi} \mathcal{E}$ and define $\phi_{\mathrm{es}}\colon A \to \mathbb{L}(\mathcal{E}_{\mathrm{es}})$ by $\phi_{\mathrm{es}}(a) = a \mathbin{\hat{\otimes}_\phi} \mathrm{id}_{\mathcal{E}}$ for all $a \in A$. Let F_{es} be an F-connection on $\mathcal{E}_{\mathrm{es}}$.*

Then $(\mathcal{E}_{\mathrm{es}},\phi_{\mathrm{es}},F_{\mathrm{es}})$ is a Kasparov triple.

There is a canonical homotopy $(\bar{\mathcal{E}},\bar{\phi},\bar{F})$ between $(\mathcal{E},\phi,F)$ and $(\mathcal{E}_{\mathrm{es}},\phi_{\mathrm{es}},F_{\mathrm{es}})$. We have $\bar{\mathcal{E}} \subset (\mathcal{E}\oplus\mathcal{E})[0,1]$. The operator $\bar{F}$ is a G-equivariant self-adjoint contraction if both F and F_{es} are G-equivariant self-adjoint contractions.

Proof. Define maps $\phi_{11}\colon A \to \mathbb{L}(\mathcal{E})$, $\phi_{12}\colon A \to \mathbb{L}(\mathcal{E}_{\mathrm{es}},\mathcal{E})$, $\phi_{21}\colon A \to \mathbb{L}(\mathcal{E},\mathcal{E}_{\mathrm{es}})$, $\phi_{22}\colon A \to \mathbb{L}(\mathcal{E}_{\mathrm{es}})$ by $\phi_{ij}(a)\xi := \phi(a)\xi$ for all ξ in the appropriate source $\mathcal{E}$ or $\mathcal{E}_{\mathrm{es}}$. These maps combine to a G_2-equivariant $*$-homomorphism

$$\phi_* := \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} : \mathbb{M}_2(A) \to \begin{pmatrix} \mathbb{L}(\mathcal{E},\mathcal{E}) & \mathbb{L}(\mathcal{E}_{\mathrm{es}},\mathcal{E}) \\ \mathbb{L}(\mathcal{E},\mathcal{E}_{\mathrm{es}}) & \mathbb{L}(\mathcal{E}_{\mathrm{es}},\mathcal{E}_{\mathrm{es}}) \end{pmatrix} = \mathbb{L}(\mathcal{E}\oplus\mathcal{E}_{\mathrm{es}}).$$

We claim that $T := (\mathcal{E}\oplus\mathcal{E}_{\mathrm{es}}, \phi_*, F\oplus F_{\mathrm{es}})$ is a Kasparov triple for $\mathbb{M}_2(A)$ and B. We have $\phi_{11} = \phi$ and $\phi_{22} = \phi_{\mathrm{es}}$. If $a \in A$, then $\phi_{21}(a)$ and $\phi_{12}(a^*)$ are the operators named T_a and T_a^* in the definition of a connection in Section 2.5. Hence $[F\oplus F_{\mathrm{es}}, \phi_*(x)] \in \mathbb{K}(\mathcal{E}\oplus\mathcal{E}_{\mathrm{es}})$ if x is off-diagonal. Using $A\cdot A = A$, we can extend this to arbitrary $x \in \mathbb{M}_2(A)$. The other conditions for a Kasparov triple like $(1-(F\oplus F_{\mathrm{es}})^2)\phi_*(x) \in \mathbb{K}(\mathcal{E}\oplus\mathcal{E}_{\mathrm{es}})$ follow easily from the standard properties of connections [2, 18.3.4] if x is diagonal. We can extend this to off-diagonal x using once again that $A\cdot A = A$. Hence T is a Kasparov triple as asserted.

Let $\iota_t\colon A \to \mathbb{M}_2(A)[0,1]$ be the rotation homotopy

$$\iota_t(a) := \begin{pmatrix} (1-t^2)a & t\sqrt{1-t^2}a \\ t\sqrt{1-t^2}a & t^2a \end{pmatrix}.$$

We have

$$(\iota_0)_*(T) = (\mathcal{E}, \phi, F) \oplus (\mathcal{E}_{\mathrm{es}}, 0, F_{\mathrm{es}}) \quad \text{and} \quad (\iota_1)_*(T) = (\mathcal{E}, 0, F) \oplus (\mathcal{E}_{\mathrm{es}}, \phi_{\mathrm{es}}, F_{\mathrm{es}}).$$

Thus up to degenerate triples $(\mathcal{E} \oplus \mathcal{E}_{\mathrm{es}}, \phi_* \circ \iota_t, F \oplus F_{\mathrm{es}})$ is a homotopy between $(\mathcal{E}, \phi, F)$ and $(\mathcal{E}_{\mathrm{es}}, \phi_{\mathrm{es}}, F_{\mathrm{es}})$. Using also the canonical homotopy between a degenerate triple and zero [2, 17.2.3], we obtain an explicit homotopy $(\bar{\mathcal{E}}, \bar{\phi}, \bar{F})$ between $(\mathcal{E}, \phi, F)$ and $(\mathcal{E}_{\mathrm{es}}, \phi_{\mathrm{es}}, F_{\mathrm{es}})$. Clearly, $\bar{\mathcal{E}}$ and $\bar{F}$ have the desired properties. $\square$

Proposition 3.4. *Let A and B be σ-unital G_2-C^*-algebras and let $\mathcal{H}$ be a separable G-Hilbert space. Assume that A has property AE. Then the canonical maps $KK^G_s(A,B) \to KK^G_{s,\mathcal{H}}(A,B) \to KK^G(A,B)$ are bijective.*

That is, any Kasparov triple for A, B is homotopic to a special triple; if two $\mathcal{H}$-special triples are homotopic, then there is an $\mathcal{H}$-special homotopy between them.

Proof. Let $(\mathcal{E}, \phi, F)$ be a Kasparov triple for A, B. We may replace the operator F_{es} in Lemma 3.3 by an arbitrary compact perturbation [2, 18.3.2.c]. Hence we may select a connection F_{es} that is a G-equivariant self-adjoint contraction by property AE and [2, 17.4.2–3]. A standard trick [2, 17.6] allows us replace F_{es} by a symmetry. First add the degenerate triple $(\mathcal{E}_{\mathrm{es}}^{\mathrm{op}}, 0, -F_{\mathrm{es}})$. The operator

$$\tilde{F} := \begin{pmatrix} F_{\mathrm{es}} & \sqrt{1-F_{\mathrm{es}}^2} \\ \sqrt{1-F_{\mathrm{es}}^2} & -F_{\mathrm{es}} \end{pmatrix} \in \mathbb{L}(\mathcal{E}_{\mathrm{es}} \oplus \mathcal{E}_{\mathrm{es}}^{\mathrm{op}})$$

is a G-equivariant symmetry and a compact perturbation of $F_{\mathrm{es}} \oplus -F_{\mathrm{es}}$. Property AE implies that $\mathcal{E}_{\mathrm{es}} \oplus \mathcal{E}_{\mathrm{es}}^{\mathrm{op}} \subset \hat{\mathcal{H}}_B \oplus \hat{\mathcal{H}}_B^{\mathrm{op}} \cong \hat{\mathcal{H}}_B$. Thus

$$\Psi(\mathcal{E}, \phi, F) := (\mathcal{E}_{\mathrm{es}} \oplus \mathcal{E}_{\mathrm{es}}^{\mathrm{op}}, \phi_{\mathrm{es}} \oplus 0, \tilde{F})$$

is a special Kasparov triple that is homotopic to $(\mathcal{E}_{\mathrm{es}}, \phi_{\mathrm{es}}, F_{\mathrm{es}})$ and hence to $(\mathcal{E}, \phi, F)$ by Lemma 3.3. The Kasparov triple $\Psi(\mathcal{E}, \phi, F)$ is not quite well-defined because we have to choose a G-equivariant connection F_{es}. Since F_{es} is determined uniquely up to a compact perturbation, $\Psi(\mathcal{E}, \phi, F)$ is well-defined up to special homotopy.

We have $\Psi \circ \Psi(\mathcal{E}, \phi, F) = \Psi(\mathcal{E}, \phi, F)$ because the essential part of $\Psi(\mathcal{E}, \phi, F)$ is equal to $(\mathcal{E}_{\mathrm{es}}, \phi_{\mathrm{es}}, F_{\mathrm{es}})$. Assume that two special Kasparov triples of the form $\Psi(T_0)$ and $\Psi(T_1)$ are homotopic. If we apply Ψ to a homotopy between them, we obtain a special homotopy between representatives of $\Psi \circ \Psi(T_j) = \Psi(T_j)$, $j = 0, 1$. Hence if Kasparov triples of the form $\Psi(T)$ are homotopic, then they are specially homotopic and a fortiori $\mathcal{H}$-specially homotopy.

The proof will be finished if we show that if $T = (\mathcal{E}, \phi, F)$ is an $\mathcal{H}$-special Kasparov triple, then there is an $\mathcal{H}$-special homotopy between T and $\Psi(T)$.

By Lemma 3.3, there is a homotopy $(\bar{\mathcal{E}}, \bar{\phi}, \bar{F})$ between T and $(\mathcal{E}_{\mathrm{es}}, \phi_{\mathrm{es}}, F_{\mathrm{es}})$ such that $\bar{F}$ is a G-equivariant self-adjoint contraction and $\bar{\mathcal{E}} \subset (\mathcal{E} \oplus \mathcal{E})[0,1]$. Thus $\mathcal{H} \mathbin{\hat{\otimes}} \bar{\mathcal{E}} \subset \hat{\mathcal{H}}_B[0,1] = \hat{\mathcal{H}}_{B[0,1]}$ because $\mathcal{H} \mathbin{\hat{\otimes}} \mathcal{E} \subset \hat{\mathcal{H}}_B$. Replacing $\bar{F}$ by a symmetry as above, we obtain an $\mathcal{H}$-special homotopy between $T \oplus (\mathcal{E}^{\mathrm{op}}, 0, -F)$ and $\Psi(T)$. The canonical homotopy between T and $T \oplus (\mathcal{E}^{\mathrm{op}}, 0, -F)$ [2, 17.2.3] is $\mathcal{H}$-special as well. $\square$

4. Isometric embeddings of Hilbert modules

In this section, we provide some techniques to deal with not necessarily adjointable embeddings of Hilbert modules. Although the group action does not create any additional difficulty here, we give complete proofs because the corresponding arguments in [5], [6], and [10] are rather sketchy.

Let B be a G_2-C*-algebra and let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert B,G_2-modules. Let $\iota\colon \mathcal{E} \to \mathcal{F}$ be an isometric embedding as defined in Section 2.1.2. Let

$$\begin{aligned}\mathbb{L}_{\mathcal{F}}(\mathcal{E}) &:= \{T \in \mathbb{L}(\mathcal{F}) \mid T(\mathcal{F}) \subset \iota(\mathcal{E}) \text{ and } T^*(\mathcal{F}) \subset \iota(\mathcal{E})\},\\ \mathbb{K}_{\mathcal{F}}(\mathcal{E}) &:= \mathbb{L}_{\mathcal{F}}(\mathcal{E}) \cap \mathbb{K}(\mathcal{F}).\end{aligned}$$

Clearly, $\mathbb{L}_{\mathcal{F}}(\mathcal{E})$ and $\mathbb{K}_{\mathcal{F}}(\mathcal{E})$ are C*-subalgebras of $\mathbb{L}(\mathcal{F})$. If $T_1, T_2 \in \mathbb{L}_{\mathcal{F}}(\mathcal{E})$, then $T_1\mathbb{L}(\mathcal{F})T_2 \subset \mathbb{L}_{\mathcal{F}}(\mathcal{E})$. Thus $\mathbb{L}_{\mathcal{F}}(\mathcal{E})$ and $\mathbb{K}_{\mathcal{F}}(\mathcal{E})$ are hereditary subalgebras of $\mathbb{L}(\mathcal{F})$.

Lemma 4.1. *For $T \in \mathbb{L}_{\mathcal{F}}(\mathcal{E})$, define $\rho(T)\colon \mathcal{E} \to \mathcal{E}$ by $\rho(T)(\xi) := \iota^{-1}(T\iota\xi)$ for all $\xi \in \mathcal{E}$. This yields a G_2-equivariant isometric $*$-homomorphism $\rho\colon \mathbb{L}_{\mathcal{F}}(\mathcal{E}) \to \mathbb{L}(\mathcal{E})$. Its restriction to $\mathbb{K}_{\mathcal{F}}(\mathcal{E})$ is an isomorphism onto $\mathbb{K}(\mathcal{E})$.*

Let $\mathbb{K}(\iota)\colon \mathbb{K}(\mathcal{E}) \to \mathbb{K}_{\mathcal{F}}(\mathcal{E}) \subset \mathbb{K}(\mathcal{F})$ be the inverse of $\rho|_{\mathbb{K}_{\mathcal{F}}(\mathcal{E})}$. Then $\mathbb{K}(\iota)$ is the unique $*$-homomorphism satisfying

$$\mathbb{K}(\iota)(|\xi\rangle\langle\eta|) = |\iota\xi\rangle\langle\iota\eta| \qquad \text{for all } \xi, \eta \in \mathcal{E}. \tag{2}$$

Proof. Clearly, $\rho(T)$ is adjointable for all $T \in \mathbb{L}_{\mathcal{F}}(\mathcal{E})$, with adjoint $\rho(T^*)$. Thus ρ is a $*$-homomorphism $\mathbb{L}_{\mathcal{F}}(\mathcal{E}) \to \mathbb{L}(\mathcal{E})$. If $\rho(T) = 0$, then T vanishes on $\iota(\mathcal{E}) \supset \operatorname{Ran} T^*$, so that $T \circ T^* = 0$ and hence $T = 0$. Thus ρ is isometric. Since ρ is natural, it is G_2-equivariant. If $\xi, \eta \in \mathcal{E}$, then $|\iota\xi\rangle\langle\iota\eta| \in \mathbb{K}_{\mathcal{F}}(\mathcal{E})$ and $\rho(|\iota\xi\rangle\langle\iota\eta|) = |\xi\rangle\langle\eta|$. Thus $\rho\bigl(\mathbb{K}_{\mathcal{F}}(\mathcal{E})\bigr)$ contains $\mathbb{K}(\mathcal{E})$ and $\mathbb{K}(\iota)$ satisfies (2).

It remains to show $\rho\bigl(\mathbb{K}_{\mathcal{F}}(\mathcal{E})\bigr) \subset \mathbb{K}(\mathcal{E})$. It suffices to verify $\rho(TT^*) \in \mathbb{K}(\mathcal{E})$ for all $T \in \mathbb{K}_{\mathcal{F}}(\mathcal{E})$. Evidently, $\rho(T|\xi\rangle\langle\eta|T^*) = \rho(|T\xi\rangle\langle T\eta|)$ is a rank one operator for all $\xi, \eta \in \mathcal{F}$ because $T\xi, T\eta \in \iota(\mathcal{E})$. Therefore, $\rho(TuT^*) \in \mathbb{K}(\mathcal{E})$ for all $u \in \mathbb{K}(\mathcal{F})$. If we let u run through an approximate unit for $\mathbb{K}(\mathcal{F})$, we get $\rho(TT^*) \in \mathbb{K}(\mathcal{E})$. □

By the way, if $\rho(T) = 1$, then $T^*T\colon \mathcal{F} \to \iota(\mathcal{E})$ is a projection onto $\iota(\mathcal{E})$, so that $\iota(\mathcal{E})$ is complementable. Hence ρ is surjective iff $\iota(\mathcal{E})$ is complementable. As an immediate consequence, we obtain the following result of Combes and Zettl [4].

Corollary 4.2. *Let B be a C*-algebra and $\mathcal{F}$ a Hilbert B-module. Let $H \subset \mathbb{K}(\mathcal{F})$ be a hereditary subalgebra. Then $H = \mathbb{K}_{\mathcal{F}}(H \cdot \mathcal{F}) \cong \mathbb{K}(H \cdot \mathcal{F})$.*

Thus the hereditary subalgebras of $\mathbb{K}(\mathcal{F})$ correspond bijectively to the not necessarily complementable Hilbert submodules of $\mathcal{F}$.

Proof. Since H is hereditary, $|\xi\rangle\langle\eta| \in H$ for all $\xi, \eta \in H \cdot \mathcal{F}$. By Lemma 4.1, these operators generate $\mathbb{K}_{\mathcal{F}}(H\cdot\mathcal{F})$. Thus $\mathbb{K}_{\mathcal{F}}(H\cdot\mathcal{F}) \subset H$. Obviously, $H \subset \mathbb{K}_{\mathcal{F}}(H\cdot\mathcal{F})$. □

Two isometric embeddings $\iota_0, \iota_1\colon \mathcal{E} \to \mathcal{F}$ are *homotopic* iff they can be connected by a continuous path of isometric embeddings $\iota_t\colon \mathcal{E} \to \mathcal{F}$, $t \in [0,1]$. Such a path ι_t gives rise to an isometric embedding $h\colon \mathcal{E}[0,1] \to \mathcal{F}[0,1]$, $(hf)(t) = \iota_t\bigl(f(t)\bigr)$. The embedding h induces a map $\mathbb{K}(h)\colon \mathbb{K}(\mathcal{E})[0,1] \to \mathbb{K}(\mathcal{F})[0,1]$ by Lemma 4.1. Composing it with the inclusion $\mathbb{K}(\mathcal{E}) \to \mathbb{K}(\mathcal{E})[0,1]$ by constant functions, we obtain a G_2-equivariant homotopy between $\mathbb{K}(\iota_0)$ and $\mathbb{K}(\iota_1)$. As a result, homotopic isometric embeddings $\mathcal{E} \to \mathcal{F}$ induce homotopic $*$-homomorphisms $\mathbb{K}(\mathcal{E}) \to \mathbb{K}(\mathcal{F})$.

Lemma 4.3. *Let B be a G_2-C*-algebra and let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert B,G_2-modules. Then any two isometric embeddings $\mathcal{E} \to \mathcal{F}^\infty$ are homotopic.*

Proof. Let $\iota_0, \iota_1\colon \mathcal{E} \to \mathcal{F}^\infty$ be two isometric embeddings. It is well-known that $\mathcal{F}^\infty \oplus \mathcal{F}^\infty \cong \mathcal{F}^\infty$ as Hilbert B,G_2-modules, and that the inclusions of the direct summands $j_0, j_1\colon \mathcal{F}^\infty \to \mathcal{F}^\infty$ are homotopic to the identity map. These homotopies may be chosen G_2-equivariant. Hence ι_0 is homotopic to $\iota_0' := j_0 \circ \iota_0$ and ι_1 is homotopic to $\iota_1' := j_1 \circ \iota_1$. By construction, ι_0' and ι_1' have orthogonal ranges, that is, $\langle \iota_0'(\xi) \mid \iota_1'(\eta)\rangle_B = 0$ for all $\xi, \eta \in \mathcal{E}$. Hence $\iota_t' := \sqrt{1-t^2}\iota_0' + t\iota_1'\colon \mathcal{E} \to \mathcal{F}^\infty$ is an isometric embedding for all $t \in [0,1]$. Thus ι_0' and ι_1' are homotopic. □

The following lemma generalizes the observation of Skandalis [20] that a degenerate Kasparov triple is homotopic to zero. It is also related to [5, Lemma 5.1].

Lemma 4.4. *Let $(\mathcal{E}, \phi, F)$ be a Kasparov triple for A, B. Let E be the C^*-subalgebra of $\mathbb{L}(\mathcal{E})$ generated by $\phi(A)$ and the operators $\gamma_g(F)$ for $g \in G$. Let $J \triangleleft E$ be the smallest G-invariant ideal containing the operators*

$$[F, \phi(a)],\ (1 - F^2)\phi(a),\ (F - F^*)\phi(a),\ (\gamma_g(F) - F)\phi(a)$$

for all $a \in A$, $g \in G$. These are precisely the operators in (1) *whose compactness (or vanishing) is required for a (degenerate) Kasparov triple. Let $\mathcal{E}' := J \cdot \mathcal{E}$.*

Then $\mathcal{E}' \subset \mathcal{E}$ is a closed, G_2-invariant submodule and $E(\mathcal{E}') \subset \mathcal{E}'$. Hence restriction to $\mathcal{E}'$ yields a well-defined G_2-equivariant $*$-homomorphism $\rho\colon E \to \mathbb{L}(\mathcal{E}')$. Let $F' := \rho(F)$, $\phi' := \rho \circ \phi$. Then $(\mathcal{E}', \phi', F')$ is a Kasparov triple for A, B. There is a canonical homotopy $(\bar{\mathcal{E}}, \bar\phi, \bar F)$ between $(\mathcal{E}, \phi, F)$ and $(\mathcal{E}', \phi', F')$.

If $(\mathcal{E}, \phi, F)$ is an $\mathcal{H}$-special Kasparov triple, then $(\mathcal{E}', \phi', F')$ and $(\bar{\mathcal{E}}, \bar\phi, \bar F)$ are $\mathcal{H}$-special Kasparov triples as well.

Proof. Since $J \triangleleft E$ is an ideal, $E(\mathcal{E}') \subset \mathcal{E}'$. If $T \in \mathbb{L}(\mathcal{E})$ satisfies $T(\mathcal{E}') \subset \mathcal{E}'$ and $T^*(\mathcal{E}') \subset \mathcal{E}'$, then the restriction of T to $\mathcal{E}'$ is an adjointable operator $\rho(T)\colon \mathcal{E}' \to \mathcal{E}'$. This yields the desired map $\rho\colon E \to \mathbb{L}(\mathcal{E}')$. Since $(\mathcal{E}, \phi, F)$ is a Kasparov triple, $J \subset \mathbb{K}(\mathcal{E})$. We have defined $\mathcal{E}'$ so that even $J \subset \mathbb{K}_{\mathcal{E}}(\mathcal{E}')$. Hence $\rho(J) \subset \mathbb{K}(\mathcal{E}')$ by Lemma 4.1. This means that $(\mathcal{E}', \phi', F')$ is a Kasparov triple.

Let $\bar{\mathcal{E}}$ be the Hilbert $B[0,1], G_2$-module $\{f \in \mathcal{E}[0,1] \mid f(1) \in \mathcal{E}'\}$. Define $\bar F \in \mathbb{L}(\bar{\mathcal{E}})$ and $\bar\phi\colon A \to \mathbb{L}(\bar{\mathcal{E}})$ by $(\bar F f)(t) := F f(t)$ and $(\bar\phi(a) f)(t) := \phi(a) f(t)$ for all $a \in A$, $f \in \bar{\mathcal{E}}$, $t \in [0,1]$. An argument similar to the proof that $(\mathcal{E}', \phi', F')$ is a Kasparov triple shows that $(\bar{\mathcal{E}}, \bar\phi, \bar F)$ is a Kasparov triple for $A, B[0,1]$. It provides the desired homotopy between $(\mathcal{E}, \phi, F)$ and $(\mathcal{E}', \phi', F')$.

Clearly, $(\mathcal{E}', \phi', F')$ and $(\bar{\mathcal{E}}, \bar\phi, \bar F)$ are $\mathcal{H}$-special if $(\mathcal{E}, \phi, F)$ is $\mathcal{H}$-special. □

5. Some universal algebra

In this section, we recall the definitions and some elementary properties of the algebras $\mathrm{q}A$ and χA introduced by Cuntz [6] and Haag [10]. We examine their relationship to special Kasparov triples and utilize this to describe $KK^G(A, B)$ as a set of homotopy classes of equivariant homomorphisms.

5.1. The algebras χA, $\mathrm{X}A$, and $\mathcal{X}A$.

Let A be a C*-algebra. Define $\mathcal{X}A$ as the universal (unital) C*-algebra generated by A and a symmetry [10]. That is, we have a $*$-homomorphism $j_A\colon A \to \mathcal{X}A$ and a symmetry $F_A \in \mathcal{X}A$ such that for all triples (B, ϕ, F) consisting of a unital C*-algebra B, a $*$-homomorphism $\phi\colon A \to B$, and a symmetry $F \in B$, there is a unique unital $*$-homomorphism $(\phi, F)_*\colon \mathcal{X}A \to B$ satisfying $(\phi, F)_* \circ j_A = \phi$ and $(\phi, F)_*(F_A) = F$.

The construction of $\mathcal{X}A$ is clearly functorial. Hence if A is a G-C*-algebra, then there is an induced action of G on $\mathcal{X}A$. This action is uniquely determined by the requirement that j_A be G-equivariant and F_A be G-invariant. Since non-commutative polynomials in $j_A(a)$, $a \in A$, and F_A are dense in $\mathcal{X}A$, the G-action on $\mathcal{X}A$ is strongly continuous. If A is graded, then we endow $\mathcal{X}A$ with the unique grading τ for which j_A is equivariant and F_A is *odd*, that is, $\tau(F_A) = -F_A$.

If $\phi\colon A \to B$ is a G_2-equivariant $*$-homomorphism, then the induced map $\mathcal{X}\phi\colon \mathcal{X}A \to \mathcal{X}B$ is a G_2-equivariant $*$-homomorphism as well.

Let $\chi A \triangleleft \mathcal{X}A$ be the ideal generated by the *graded* commutators $[j_A(a), F]$ with $a \in A$. The ideal χA is G_2-invariant and essential. Thus $\mathcal{X}A \subset \mathcal{M}(\chi A)$. The quotient $\mathcal{X}A/\chi A$ is the universal unital C*-algebra generated by A and a symmetry that graded commutes with A. Thus $\mathcal{X}A/\chi A \cong \mathrm{Cl}_1 \hat\otimes A^+$, where A^+ is the C*-algebra obtained by adjoining a unit to A, with $A^+/A = \mathbb{C}$. Let $\mathrm{X}A \triangleleft \mathcal{X}A$

be the ideal generated by $j_A(A)$. It follows that $\mathrm{X}A/\chi A \cong \mathbb{Cl}_1 \hat{\otimes} A$, so that we have a canonical extension of G_2-C*-algebras

$$\chi A \rightarrowtail \mathrm{X}A \twoheadrightarrow \mathbb{Cl}_1 \hat{\otimes} A. \tag{3}$$

It is shown in the proof of [10, Theorem 3.6] that this extension has a natural—hence G_2-equivariant—completely positive section. Roughly speaking, $\mathrm{X}A$ is the universal C*-algebra generated by A and a symmetry in the *multiplier algebra* $\mathcal{M}(\mathrm{X}A)$.

Let A and B be G_2-C*-algebras. There is a canonical map $\mathrm{X}(A \hat{\otimes} B) \to \mathrm{X}A \hat{\otimes} B$ that restricts to a map $\chi(A \hat{\otimes} B) \to \chi A \hat{\otimes} B$. It is defined by the homomorphism $j_A \hat{\otimes} \mathrm{id}_B \colon A \hat{\otimes} B \to \mathrm{X}A \hat{\otimes} B$ and the symmetry $F_A \hat{\otimes} 1 \in \mathcal{M}(\mathrm{X}A \hat{\otimes} B)$. For $B = \mathrm{C}([0,1])$, we obtain that X and χ are *homotopy functors*. That is, if $f_0, f_1 \colon A \to A'$ are homotopic, then $\chi f_0, \chi f_1 \colon \chi A \to \chi A'$ are homotopic as well. For $A = \mathbb{C}$, we obtain canonical maps $\chi B \to (\chi\mathbb{C}) \hat{\otimes} B$ and $\mathrm{X}B \to (\mathrm{X}\mathbb{C}) \hat{\otimes} B$. Our next goal is to show that these maps are KK-equivalences. We follow arguments in the proof of [10, Proposition 3.8] in the non-equivariant case.

Proposition 5.1. *Let A be a G_2-C*-algebra. Then the canonical $*$-homomorphism $\mathrm{id} \hat{\otimes} j_A \colon \mathbb{K}(\mathbb{Z}_2\mathbb{N})A \to \mathbb{K}(\mathbb{Z}_2\mathbb{N})\mathrm{X}A$ is a homotopy equivalence.*

Proof. We call a map of the form $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)$ an *upper left corner embedding*. We will exhibit a canonical homomorphism $f \colon \mathrm{X}A \to \hat{\mathbb{M}}_2 A$ such that $f \circ j_A$ and $(\mathrm{id}_{\hat{\mathbb{M}}_2} \hat{\otimes} j_A) \circ f$ are both homotopic to the upper left corner embeddings $A \to \hat{\mathbb{M}}_2 A$ and $\mathrm{X}A \to \hat{\mathbb{M}}_2 \mathrm{X}A$. It follows that $\mathrm{id}_{\mathbb{K}(\mathbb{Z}_2\mathbb{N})} \hat{\otimes} f$ is a homotopy inverse for $\mathrm{id} \hat{\otimes} j_A$.

The homomorphism f is defined by requiring $f \circ j_A$ to be the upper left corner embedding and $f(F_A)$ to be the *standard symmetry*

$$S := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right). \tag{4}$$

By definition, $f \circ j_A \colon A \to \hat{\mathbb{M}}_2 A$ is equal to the upper left corner embedding. The symmetries S and $F' := F_A \oplus -F_A$ in $\hat{\mathbb{M}}_2 \mathcal{M}(\mathrm{X}A)$ anti-commute. Hence $t \mapsto \sqrt{1-t^2} \cdot S + tF'$ is a path of G-equivariant symmetries in $\hat{\mathbb{M}}_2 \mathcal{M}(\mathrm{X}A)$ connecting them. This path yields a homotopy between $(\mathrm{id}_{\hat{\mathbb{M}}_2} \hat{\otimes} \mathrm{X}A) \circ f \colon \mathrm{X}A \to \hat{\mathbb{M}}_2 \mathrm{X}A$ and the upper left corner embedding $\mathrm{X}A \to \hat{\mathbb{M}}_2 \mathrm{X}A$. ∎

Hence the canonical map $\mathrm{X}A \to (\mathrm{X}\mathbb{C}) \hat{\otimes} A$ is invertible in $KK^{G_2}(\mathrm{X}A, (\mathrm{X}\mathbb{C}) \hat{\otimes} A)$.

Proposition 5.2. *Let A be a separable G_2-C*-algebra. Then the canonical map $\chi A \to (\chi\mathbb{C}) \hat{\otimes} A$ is invertible in $KK^{G_2}(\chi A, (\chi\mathbb{C}) \hat{\otimes} A)$.*

Proof. This canonical map is part of a morphism of extensions from $\chi A \rightarrowtail \mathrm{X}A \twoheadrightarrow \mathbb{Cl}_1 \hat{\otimes} A$ to $(\chi\mathbb{C}) \hat{\otimes} A \to (\mathrm{X}\mathbb{C}) \hat{\otimes} A \twoheadrightarrow \mathbb{Cl}_1 \hat{\otimes} A$, where the map $\mathrm{X}A \to (\mathrm{X}\mathbb{C}) \hat{\otimes} A$ is a KK-equivalence by Proposition 5.1 and the map $\mathbb{Cl}_1 \hat{\otimes} A \to \mathbb{Cl}_1 \hat{\otimes} A$ is the identity map. Since the two extensions have completely positive G_2-equivariant sections, the long exact sequences in KK-theory are available. The Five Lemma yields that the map $\chi A \to (\chi\mathbb{C}) \hat{\otimes} A$ is a KK-equivalence as well. ∎

5.2. **The algebras $\mathrm{q}A$ and $\mathrm{Q}A$.** Let $\mathrm{Q}A := A * A$ be the free product of two copies of A [6]. Thus there are two $*$-homomorphisms $\iota_A^+, \iota_A^- \colon A \to \mathrm{Q}A$ such that for any triple (B, ϕ^+, ϕ^-) consisting of a C*-algebra B and a pair of $*$-homomorphisms $\phi^+, \phi^- \colon A \to B$, there is a unique $*$-homomorphism $\phi^+ * \phi^- \colon \mathrm{Q}A \to B$ satisfying $(\phi^+ * \phi^-) \circ \iota_A^\pm = \phi^\pm$.

Let $\mathrm{q}A \triangleleft \mathrm{Q}A$ be the ideal that is generated by the differences $\iota^+(a) - \iota^-(a)$ with $a \in A$. Alternatively, we can describe $\mathrm{q}A$ as the kernel of the homomorphism $\mathrm{id}_A * \mathrm{id}_A \colon \mathrm{Q}A \to A$. Thus we obtain an extension of C*-algebras $\mathrm{q}A \rightarrowtail \mathrm{Q}A \twoheadrightarrow A$.

The $*$-homomorphisms $\iota_A^{\pm}\colon A \to \mathrm{Q}A$ are sections for $\mathrm{id}_A * \mathrm{id}_A$. There is a natural $*$-homomorphism $\pi_A := (\mathrm{id}_A * 0)|_{\mathrm{q}A}\colon \mathrm{q}A \to A$.

If A is a G_2-C*-algebra, then there is a unique strongly continuous G_2-action on $\mathrm{Q}A$ for which the $*$-homomorphisms $\iota_A^{\pm}$ are G_2-equivariant. The ideal $\mathrm{q}A$ is G_2-invariant. The maps $\iota_A^{\pm}$, π_A, and $\mathrm{id}_A * \mathrm{id}_A$ above are G_2-equivariant. The functor $A \mapsto \mathrm{q}A$ is a homotopy functor.

Proposition 5.3. *Let A and B be G_2-C*-algebras.*

Let $\iota_1\colon A \to A \oplus B$ and $\iota_2\colon B \to A \oplus B$ be the standard inclusions.

The homomorphism $\mathrm{id}_{\mathbb{K}(\mathbb{N})} \otimes (\iota_1 * \iota_2)\colon \mathbb{K}(\mathbb{N})(A*B) \to \mathbb{K}(\mathbb{N})(A\oplus B)$ is a homotopy equivalence. In particular, $\mathbb{K}(\mathbb{N})\mathrm{Q}A$ is homotopy equivalent to $\mathbb{K}(\mathbb{N})(A \oplus A)$.

Proof. The stable homotopy inverse for $\iota_1 * \iota_2$ is the map $f\colon A \oplus B \to \mathbb{M}_2(A * B)$,

$$f(a,b) := \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \qquad \text{for } a \in A,\ b \in B.$$

The compositions $f \circ (\iota_1 * \iota_2)$ and $\big(\mathrm{id}_{\mathbb{M}_2} \otimes (\iota_1 * \iota_2)\big) \circ f$ are homotopic to the upper left corner embeddings $A * B \to \mathbb{M}_2(A * B)$ and $A \oplus B \to \mathbb{M}_2(A \oplus B)$ in a natural way. Roughly speaking, the homotopies leave a fixed and rotate b to the upper left corner. Consequently, $\mathrm{id}_{\mathbb{K}(\mathbb{N})} \otimes (\iota_1 * \iota_2)$ is a homotopy equivalence. The occurring homotopies are natural and therefore G_2-equivariant. ∎

5.3. **Universal algebras and Kasparov triples.**

Proposition 5.4. *Let A and B be σ-unital G_2-C*-algebras and let $\mathcal{H}$ be a separable G_2-Hilbert space. There are natural bijections*

$$KK_s^G(A,B) \cong [\chi A, \mathbb{K}(G_2\mathbb{N})B], \qquad KK_{s,\mathcal{H}}^G(A,B) \cong [\mathbb{K}(\mathcal{H})\chi A, \mathbb{K}(G_2\mathbb{N})B].$$

If A, B, and $\mathcal{H}$ are trivially graded, then there are natural bijections

$$KK_s^G(A,B) \cong [\mathrm{q}A, \mathbb{K}(G\mathbb{N})B], \qquad KK_{s,\mathcal{H}}^G(A,B) \cong [\mathbb{K}(\mathcal{H})\mathrm{q}A, \mathbb{K}(G\mathbb{N})B].$$

All the sets $KK_s^G(A,B)$, $[\chi A, \mathbb{K}(G_2\mathbb{N})B]$, etc., in the proposition are functorial for G_2-equivariant $*$-homomorphisms $A \to A'$, $B' \to B$. Naturality means that the isomorphisms are compatible with this functoriality, so that we have isomorphisms of functors, not just of sets.

Proof. Since special Kasparov triples are nothing but $\mathbb{C}$-special Kasparov triples, it suffices to prove the assertions about $KK_{s,\mathcal{H}}^G$. Let $T := (\mathcal{E}, \phi, F)$ be an $\mathcal{H}$-special Kasparov triple. The pair (ϕ, F) defines a G_2-equivariant $*$-homomorphism $(\phi,F)_*\colon \mathcal{X}A \to \mathbb{L}(\mathcal{E})$ whose restriction to χA has values in $\mathbb{K}(\mathcal{E})$. Hence we get a map $(\phi,F)_*^{\mathcal{H}} := \mathrm{id}_{\mathbb{K}(\mathcal{H})} \hat{\otimes} (\phi,F)_*\colon \mathbb{K}(\mathcal{H})\chi A \to \mathbb{K}(\mathcal{H})\mathbb{K}(\mathcal{E}) \cong \mathbb{K}(\mathcal{H} \hat{\otimes} \mathcal{E})$. Since the Kasparov triple T is $\mathcal{H}$-special, there is an isometric embedding $\iota\colon \mathcal{H} \hat{\otimes} \mathcal{E} \to \hat{\mathcal{H}}_B$. Let $\Psi(T)\colon \chi A \to \mathbb{K}(G_2\mathbb{N})B$ be the composition $\mathbb{K}(\iota) \circ (\phi,F)_*^{\mathcal{H}}$.

The homomorphism $\Psi(T)$ is determined uniquely up to homotopy by Lemma 4.3. Since we can apply Ψ to $\mathcal{H}$-special homotopies as well, it descends to a map on homotopy classes $\Psi\colon KK_{s,\mathcal{H}}^G(A,B) \to [\mathbb{K}(\mathcal{H})\chi A, \mathbb{K}(G_2\mathbb{N})B]$. It is straightforward to verify that Ψ is natural. That is, if $f\colon A' \to A$ and $g\colon B \to B'$ are G_2-equivariant $*$-homomorphisms, and $T \in KK_{s,\mathcal{H}}^G(A,B)$, then $\Psi\big(f^*(T)\big) = \Psi(T) \circ (\mathrm{id}_{\mathbb{K}(\mathcal{H})} \hat{\otimes} \chi f)$ and $\Psi\big(g_*(T)\big) = (\mathrm{id}_{\mathbb{K}(G_2\mathbb{N})} \hat{\otimes} g) \circ \Psi(T)$—even if g is not essential.

Conversely, let $f\colon \mathbb{K}(\mathcal{H})\chi A \to \mathbb{K}(G_2\mathbb{N})B \cong \mathbb{K}(\hat{\mathcal{H}}_B)$ be a G_2-equivariant $*$-homomorphism. Let $\mathcal{E}_1 = f(\mathbb{K}(\mathcal{H})\chi A) \cdot \hat{\mathcal{H}}_B \subset \hat{\mathcal{H}}_B$ and let $\iota\colon \mathcal{E}_1 \to \hat{\mathcal{H}}_B$ be the inclusion mapping. By construction, $f(\mathbb{K}(\mathcal{H})\chi A) \subset \mathbb{K}_{\hat{\mathcal{H}}_B}(\mathcal{E}_1)$. Hence Lemma 4.1 yields $f = \mathbb{K}(\iota) \circ f_1$ for a G_2-equivariant essential $*$-homomorphism $f_1\colon \mathbb{K}(\mathcal{H})\chi A \to \mathbb{K}(\mathcal{E}_1)$.

We claim that $\mathcal{E}_1 \cong \mathcal{H} \hat{\otimes} \mathcal{E}_2$ and $f_1 \cong \mathrm{id}_{\mathbb{K}(\mathcal{H})} \hat{\otimes} f_2$ for a Hilbert B,G-module $\mathcal{E}_2$ and an essential G-equivariant $*$-homomorphism $f_2\colon \chi A \to \mathbb{K}(\mathcal{E}_2)$. This is trivial

if $\mathcal{H} = \mathbb{C}$. Consider the dual $(\mathcal{H}^* \mathbin{\hat\otimes} \chi A, \psi^*)$ of the $\mathbb{K}(\mathcal{H})\chi A, \chi A, G_2$-imprimitivity bimodule $\mathcal{H} \mathbin{\hat\otimes} \chi A$. Thus $(\mathcal{H} \mathbin{\hat\otimes} \chi A) \mathbin{\hat\otimes}_{\psi^*} (\mathcal{H}^* \mathbin{\hat\otimes} \chi A) \cong \mathbb{K}(\mathcal{H})\chi A$. Let

$$\mathcal{E}_2 := (\mathcal{H}^* \mathbin{\hat\otimes} \chi A) \mathbin{\hat\otimes}_{f_1} \mathcal{E}_1 \qquad \text{and} \qquad f_2 := \psi^* \mathbin{\hat\otimes} 1.$$

Since ψ^* is essential, so is f_2. Since f_1 is essential as well, we have $\mathcal{H} \mathbin{\hat\otimes} \mathcal{E}_2 \cong (\mathcal{H} \mathbin{\hat\otimes} \chi A) \mathbin{\hat\otimes}_{f_2} \mathcal{E}_2 \cong \mathcal{E}_1$. Under this isomorphism, f_1 corresponds to $\mathrm{id}_{\mathbb{K}(\mathcal{H})} \mathbin{\hat\otimes} f_2$. Since $f_1(\mathbb{K}(\mathcal{H})\chi A) \subset \mathbb{K}(\mathcal{E}_1)$, it follows that $f_2(\chi A) \subset \mathbb{K}(\mathcal{E}_2)$.

We may extend f_2 to $\mathcal{X}A \subset \mathcal{M}(\chi A)$. By the universal property of $\mathcal{X}A$, this extension is of the form $(\phi, F)_*\colon \mathcal{X}A \to \mathbb{K}(\mathcal{E}_2)$ for some G_2-equivariant $*$-homomorphism $\phi\colon A \to \mathbb{L}(\mathcal{E}_2)$ and some G-invariant symmetry $F \in \mathbb{L}(\mathcal{E}_2)$. The triple $\Psi^{-1}(f) := (\mathcal{E}_2, \phi, F)$ is a Kasparov triple because $(\phi, F)_*(\chi A) \subset \mathbb{K}(\mathcal{E}_2)$. It is $\mathcal{H}$-special because F is a G-equivariant symmetry and $\mathcal{H} \mathbin{\hat\otimes} \mathcal{E}_2 \cong \mathcal{E}_1 \subset \hat{\mathcal{H}}_B$. Evidently, Ψ^{-1} descends to a map $[\mathbb{K}(\mathcal{H})\chi A, \mathbb{K}(G_2\mathbb{N})B] \to KK^G_{s,\mathcal{H}}(A, B)$. By construction,

$$\mathbb{K}(\iota) \circ (\mathrm{id}_{\mathbb{K}(\mathcal{H})} \mathbin{\hat\otimes} (\phi, F)_*) = \mathbb{K}(\iota) \circ (\mathrm{id}_{\mathbb{K}(\mathcal{H})} \mathbin{\hat\otimes} f_2) = \mathbb{K}(\iota) \circ f_1 = f.$$

That is, $\Psi \circ \Psi^{-1}$ is the identity map on $[\mathbb{K}(\mathcal{H})\chi A, \mathbb{K}(G_2\mathbb{N})B]$.

Let $(\mathcal{E}, \phi, F)$ be an $\mathcal{H}$-special Kasparov triple. Going through the above constructions, we find that $\Psi^{-1} \circ \Psi(\mathcal{E}, \phi, F)$ is the Kasparov triple that is called $(\mathcal{E}', \phi', F')$ in Lemma 4.4. Therefore, $[\Psi^{-1} \circ \Psi(\mathcal{E}, \phi, F)] = [(\mathcal{E}, \phi, F)]$ in $KK^G_{s,\mathcal{H}}(A, B)$. The proof of the isomorphism $KK^G_{s,\mathcal{H}}(A, B) \cong [\mathbb{K}(\mathcal{H})\chi A, \mathbb{K}(G_2\mathbb{N})B]$ is finished.

Suppose now that A, B, and $\mathcal{H}$ are trivially graded. Let $(\mathcal{E}, \phi, F)$ be an $\mathcal{H}$-special Kasparov triple for A, B. The even and odd part $\mathcal{E}^+$ and $\mathcal{E}^-$ of $\mathcal{E}$ are Hilbert B, G-modules as well. We may use F to identify $\mathcal{E}^+ \cong \mathcal{E}^-$. Then F becomes the standard symmetry $S \in \mathbb{L}(\mathcal{E}^+ \oplus \mathcal{E}^+)$ of (4). Since A is trivially graded, we have $\phi = \phi_+ \oplus \phi_-$ for certain $*$-homomorphisms $\phi^\pm\colon A \to \mathbb{L}(\mathcal{E}^+)$. The condition $[F, \phi(a)] \in \mathbb{K}(\mathcal{E})$ becomes $\phi_+(a) - \phi_-(a) \in \mathbb{K}(\mathcal{E}^+)$ for all $a \in A$. Thus $\mathcal{H}$-special Kasparov triples correspond bijectively to G-equivariant $*$-homomorphisms $f\colon \mathrm{Q}A \to \mathbb{L}(\mathcal{E})$ with $f(\mathrm{q}A) \subset \mathbb{K}(\mathcal{E})$ and $\mathcal{H} \otimes \mathcal{E} \subset \mathcal{H}_A$. Copying the argument above with $\mathrm{q}A \triangleleft \mathrm{Q}A$ instead of $\chi A \triangleleft \mathcal{X}A$, we obtain the desired bijection $KK^G_{s,\mathcal{H}}(A, B) \cong [\mathbb{K}(\mathcal{H})\mathrm{q}A, \mathbb{K}(G\mathbb{N})B]$ if A and B are trivially graded. $\square$

If $K = \mathbb{K}(G\mathbb{N})$ or $K = \mathbb{K}(G_2\mathbb{N})$, let $[A, B]_K$ be the set of homotopy classes of G_2-equivariant $*$-homomorphisms from $K \mathbin{\hat\otimes} A$ to $K \mathbin{\hat\otimes} B$. Let

$$\chi_s A := \chi(\mathbb{K}(G_2\mathbb{N})A) \qquad \text{and} \qquad \mathrm{q}_s A := \mathrm{q}(\mathbb{K}(G\mathbb{N})A).$$

Theorem 5.5. *Let G be a locally compact, σ-compact topological group. Let A and B be σ-unital G_2-C*-algebras. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable G_2-Hilbert spaces. There are natural bijections*

$$KK^G(A, B) \cong [\mathbb{K}(\mathcal{H}_1)\, \chi(\mathbb{K}(L^2G \mathbin{\hat\otimes} \mathcal{H}_2)A),\, \mathbb{K}(G_2\mathbb{N})B] \cong [\chi_s A, B]_{\mathbb{K}(G_2\mathbb{N})}.$$

If A, B, $\mathcal{H}_1$, and $\mathcal{H}_2$ are trivially graded, then there are natural bijections

$$KK^G(A, B) \cong [\mathbb{K}(\mathcal{H}_1)\, \mathrm{q}(\mathbb{K}(L^2G \otimes \mathcal{H}_2)A),\, \mathbb{K}(G\mathbb{N})B] \cong [\mathrm{q}_s A, B]_{\mathbb{K}(G\mathbb{N})}.$$

The sets $KK^G(A, B)$, etc., occurring in the Theorem are functorial for equivariant $*$-homomorphisms $A' \to A$, $B \to B'$. The naturality of the isomorphisms means that they are compatible with this functoriality.

Proof. Since Morita-Rieffel equivalent G_2-C*-algebras are KK^G-equivalent, there are natural isomorphisms

$$KK^G(A, B) \cong KK^G(\mathbb{K}(G)A, B) \cong KK^G(\mathbb{K}(G)\mathbb{K}(\mathcal{H}_2)A, B).$$

By Proposition 3.2, $\mathbb{K}(G)A$ has the property AE. Hence Proposition 3.4 yields $KK^G(\mathbb{K}(G)A, B) \cong KK^G_s(\mathbb{K}(G)A, B) \cong KK^G_{s,\mathcal{H}_1}(\mathbb{K}(G)A, B)$. A similar statement

holds for $\mathbb{K}(L^2G \hat{\otimes} \mathcal{H}_2)A$ instead of $\mathbb{K}(G)A$. Therefore, Proposition 5.4 yields the assertions. □

6. The universal property of equivariant Kasparov theory

In this section, we formulate and establish the universal property of equivariant Kasparov theory for trivially graded separable G-C*-algebras.

Let G-C* be the category of separable G-C*-algebras with G-equivariant $*$-homomorphisms as morphisms. Let $[G\text{-C}^*]_s$ be the $\mathbb{K}(G\mathbb{N})$-stable homotopy category, whose objects are the separable G-C*-algebras and whose set of morphisms from A to B is $[A,B]_s := [A,B]_{\mathbb{K}(G\mathbb{N})}$. Let KK^G be the category whose objects are the separable G-C*-algebras and whose set of morphisms from A to B is $KK^G(A,B)$. The Kasparov product yields the composition of morphisms in KK^G. We rely on Kasparov's work [15] and assume that the Kasparov product exists and is associative. We do not attempt an alternative definition of the Kasparov product as in [6]. It is clear that KK^G is an additive category. There are obvious functors $G\text{-C}^* \to [G\text{-C}^*]_s$ and $G\text{-C}^* \to KK^G$.

Let $\mathfrak{C}$ be a category. A functor $F\colon G\text{-C}^* \to \mathfrak{C}$ is called a *homotopy functor* iff $F(f_0) = F(f_1)$ whenever f_0 and f_1 are G-equivariantly homotopic.

A functor $F\colon G\text{-C}^* \to \mathfrak{C}$ is called *stable* iff the map $F(\mathbb{K}(\mathcal{H})A) \to F(\mathbb{K}(\mathcal{H}\oplus\mathcal{H}')A)$ induced by the inclusion $\mathcal{H} \subset \mathcal{H}\oplus\mathcal{H}'$ is an isomorphism for all separable G-Hilbert spaces $\mathcal{H},\mathcal{H}'$ and all separable G-C*-algebras A.

Proposition 6.1. *The functor $G\text{-C}^* \to [G\text{-C}^*]_s$ is a stable homotopy functor. A functor $F\colon G\text{-C}^* \to \mathfrak{C}$ is a stable homotopy functor iff it can be factored through the functor $G\text{-C}^* \to [G\text{-C}^*]_s$. This factorization is automatically unique.*

In other words, $[G\text{-C}^*]_s$ is the universal stable homotopy functor.

Proof. It is left to the reader to check that the canonical functor $G\text{-C}^* \to [G\text{-C}^*]_s$ is a stable homotopy functor. Thus any functor $G\text{-C}^* \to \mathfrak{C}$ that factors through it is a stable homotopy functor as well.

Conversely, let $F\colon G\text{-C}^* \to \mathfrak{C}$ be a stable homotopy functor. Let $\mathcal{H} = \mathbb{C}\oplus L^2(G\mathbb{N})$ and let $j_1^A\colon A \to \mathbb{K}(\mathcal{H})A$ and $j_2^A\colon \mathbb{K}(G\mathbb{N})A \to \mathbb{K}(\mathcal{H})A$ be the canonical inclusions. Since F is stable, $F(j_1^A)$ and $F(j_2^A)$ are isomorphisms. Thus $\sigma_A := F(j_2^A)^{-1}\circ F(j_1^A)$ is a natural isomorphism $F(A) \xrightarrow{\cong} F(\mathbb{K}(G\mathbb{N})A)$. Define

$$F_*\colon [A,B]_s \to \mathrm{Mor}_{\mathfrak{C}}\big(F(A),F(B)\big), \qquad F_*[\phi] := \sigma_B^{-1}\circ F(\phi)\circ\sigma_A.$$

It is left to the reader to check that this defines a functor $F_*\colon [G\text{-C}^*]_s \to \mathfrak{C}$ that extends F and that the functor F_* is determined uniquely. □

Remark 6.2. A homotopy functor $F\colon G\text{-C}^* \to \mathfrak{C}$ is stable iff $F(A) \cong F(\mathbb{K}(G\mathbb{N})A)$ naturally. The proof of Proposition 6.1 shows that a natural isomorphism $F(A) \cong F(\mathbb{K}(G\mathbb{N})A)$ allows us to factor F through $[G\text{-C}^*]_s$. Our definition of a stable homotopy functor is equivalent to the definitions in [8] and in [21].

A functor $F\colon G\text{-C}^* \to \mathfrak{C}$ into an additive category $\mathfrak{C}$ is called *split exact* iff $\big(F(i),F(s)\big)\colon F(A)\oplus F(C) \to F(B)$ is an isomorphism for all extensions

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of G-C*-algebras that split by a G-equivariant $*$-homomorphism $s\colon C \to B$.

Proposition 6.3. *The canonical functor $G\text{-C}^* \to KK^G$ is a split exact stable homotopy functor.*

Proof. Clearly, KK^G is a stable homotopy functor. Split exactness is a straightforward consequence of the associativity of the Kasparov product. The argument in [6, Proposition 2.1] carries over without change. □

Since $A \mapsto \mathrm{q}A$ is a homotopy functor, $A \mapsto \mathrm{q}_s A$ descends to a functor from $[G\text{-C}^*]_s$ to itself. The map $\pi_{\mathbb{K}(G\mathbb{N})A} \colon \mathrm{q}_s A \to \mathbb{K}(G\mathbb{N})A$ gives rise to a natural morphism $\pi^s_A \in [\mathrm{q}_s A, A]_s$.

Lemma 6.4. *Let $F\colon G\text{-}C^* \to \mathfrak{C}$ be a split exact stable homotopy functor and let $F_*\colon [G\text{-}C^*]_s \to \mathfrak{C}$ be the unique extension of F. Then $F_*(\pi^s_A)$ is invertible for all A.*

Proof. Split exactness applied to the extension $A \rightarrowtail A \oplus B \twoheadrightarrow B$ yields that the canonical map $F(A \oplus B) \to F(A) \oplus F(B)$ is an isomorphism. That is, F is *additive*. Proposition 5.3 yields $F(A * B) \cong F(A) \oplus F(B)$. Split exactness applied to the extension $\mathrm{q}A \rightarrowtail \mathrm{Q}A \twoheadrightarrow A$ implies that $F(\pi_A)\colon F(\mathrm{q}A) \to F(A)$ is an isomorphism for all A. This implies that $F_*(\pi^s_A)$ is an isomorphism as well. □

By Proposition 6.3 and Proposition 6.1, the canonical functor $G\text{-C}^* \to KK^G$ factors through a functor $\natural\colon [G\text{-C}^*]_s \to KK^G$. Lemma 6.4 implies that $\natural(\pi^s_A) \in KK^G(\mathrm{q}_s A, A)$ is invertible for all A.

Theorem 6.5. *Let A and B be separable G-C^*-algebras. The map*

$$[\mathrm{q}_s A, \mathrm{q}_s B]_s \to KK^G(A, B), \qquad f \mapsto \natural(\pi^s_B) \circ \natural(f) \circ \natural(\pi^s_A)^{-1},$$

is a natural isomorphism. Hence the Kasparov product on KK^G corresponds to the composition of homomorphisms.

Proof. Since π^s_B induces an isomorphism $KK^G(A, \mathrm{q}_s B) \cong KK^G(A, B)$, it suffices to verify that the isomorphism $[\mathrm{q}_s A, B]_s \to KK^G(A, B)$ of Theorem 5.5 is given by $f \mapsto \natural(f) \circ \natural(\pi^s_A)^{-1}$. By naturality, it suffices to check this for the identity map in $[\mathrm{q}_s A, \mathrm{q}_s A]_s$. Composing with the invertible element π^s_A, we can reduce the theorem to the following claim: The isomorphism of Theorem 5.5 maps $\pi^s_A \in [\mathrm{q}_s A, A]_s$ to the unit in $KK^G(A, A)$, represented by the Kasparov triple $(A, \mathrm{id}_A, 0)$. The proof of this claim is made somewhat messy by stabilizations, but otherwise straightforward. Therefore, we omit it. □

Theorem 6.6. *The functor $G\text{-}C^* \to KK^G$ is the universal split exact stable homotopy functor in the following sense. An additive functor $F\colon G\text{-}C^* \to \mathfrak{C}$ into an additive category $\mathfrak{C}$ can be extended to a functor $F_*\colon KK^G \to \mathfrak{C}$ iff it is a split exact stable homotopy functor. The extension is necessarily unique.*

Proof. Let $F\colon G\text{-C}^* \to \mathfrak{C}$ be a split exact stable homotopy functor. By Proposition 6.1, we may assume that F is a functor $F\colon [G\text{-C}^*]_s \to \mathfrak{C}$. Split exactness implies that $F(\pi^s_A)$ is an isomorphism for all A. If $f \in [\mathrm{q}_s A, \mathrm{q}_s B]_s$, define $F_*(f) := F(\pi^B_s) \circ F(f) \circ F(\pi^A_s)^{-1}$. By Theorem 6.5, this yields a functor $KK^G \to \mathfrak{C}$. Evidently, this is the unique functor extending F. It is clear that any additive functor that factors through KK^G is a split exact stable homotopy functor. □

7. The case of graded algebras

Following Haag [10], we write $\mathrm{Ex}^G(A, B) := KK^{G_2}(A, B)$ for the G_2-equivariant KK-theory for trivially graded algebras. We show $KK^G(A, B) \cong \mathrm{Ex}^G(\hat{S} \hat{\otimes} A, B)$ and describe the Kasparov product in KK^G in terms of the product in Ex^G.

We redefine KK^G to be the category whose objects are the $\mathbb{Z}_2$-graded separable G-C*-algebras and whose set of morphisms from A to B is $KK^G(A, B)$. Let G_2-C* be the category of separable G_2-C*-algebras and let $[G_2\text{-C}^*]_s$ be the $\mathbb{K}(G_2\mathbb{N})$-stable homotopy category, as defined in the previous section. We redefine $\mathrm{q}_s A := \mathrm{q}(\mathbb{K}(G_2\mathbb{N})A)$, so that $\mathrm{Ex}^G(A, B) \cong [\mathrm{q}_s A, B]_s$ by Theorem 5.5.

The canonical functor $G_2\text{-C}^* \to KK^G$ is still a split exact stable homotopy functor. By Theorem 6.6, we may extend it to a functor $\alpha\colon \mathrm{Ex}^G(A, B) \to KK^G(A, B)$.

The functor α can be computed as follows. As in [10, p. 15], the $*$-homomorphism $\iota^+ \oplus \iota^- \colon A \to \hat{\mathbb{M}}_2(\mathrm{Q}A)$ and the symmetry S of (4) yield a canonical map

$$\alpha_0 := (\iota^+ \oplus \iota^-, S)_* \colon \chi A \to \hat{\mathbb{M}}_2(\mathrm{q}A)$$

We view α_0 as an element of $[\chi A, \mathrm{q}A]_s$. Replacing A by $\mathbb{K}(G_2\mathbb{N})A$, we obtain $\alpha_0 \in [\chi_s A, \mathrm{q}_s A]_s \cong KK^G(A, \mathrm{q}_s A)$ by Theorem 5.5.

Lemma 7.1. *$\alpha_0 \in KK^G(A, \mathrm{q}_s A)$ is the inverse of $\pi_A^s \in [\mathrm{q}_s A, A]_s$.*

Proof. Lemma 6.4 implies that the image of π_A^s in $KK^G(\mathrm{q}_s A, A)$ is invertible. It remains to prove that $(\pi_A^s)_*(\alpha_0)$ is the identity element of $KK^G(A, A)$.

We may suppose $\mathbb{K}(G_2\mathbb{N})A \cong A$, so that we may omit the stabilizations and work with the map $\alpha_0 \colon \chi A \to \hat{\mathbb{M}}_2(\mathrm{q}A)$. It corresponds to the Kasparov triple $(\mathrm{q}A \oplus (\mathrm{q}A)^{\mathrm{op}}, \iota^+ \oplus \iota^-, S)$. Since $\pi_A \circ \iota^+ = \mathrm{id}_A$, $\pi_A \circ \iota^- = 0$, we have

$$(\pi_A)_*(\alpha_0) = (A \oplus A^{\mathrm{op}}, \mathrm{id}_A \oplus 0, S).$$

The right hand side represents the identity element of $KK^G(A, A)$. □

Corollary 7.2. *Let A and B be separable G_2-C*-algebras. Using the isomorphisms of Theorem 5.5, we obtain a map*

$$[\mathrm{q}_s A, B]_s \cong \mathrm{Ex}^G(A, B) \xrightarrow{\alpha} KK^G(A, B) \cong [\chi_s A, B]_s.$$

This map is equal to composition with $[\alpha_0] \in [\chi_s A, \mathrm{q}_s A]_s$.

Proof. Let $f \in [\mathrm{q}_s A, B]_s$, then the image of f in $KK^G(A, B)$ is $f_*(\pi_A^s)^{-1} = f_*[\alpha_0]$. This is mapped to $f \circ [\alpha_0] \in [\chi_s A, B]_s$. □

There is a canonical Kasparov triple $(\chi A, j_A, F_A)$ for $A, \chi A$. Replacing A by $\mathbb{K}(G_2\mathbb{N})A$, we obtain a canonical element $i_A \in KK^G(A, \chi_s A)$. The isomorphism $KK^G(A, \chi_s A) \to [\chi_s A, \chi_s A]_s$ maps i_A to the identity map. The naturality of the isomorphism $[\chi_s A, B] \to KK^G(A, B)$ of Theorem 5.5 implies that it maps $f \mapsto f_*(i_A)$ for all $f \in [\chi_s A, B]_s$.

Lemma 7.3. *Let A and B be separable G_2-C*-algebras. The canonical map*

$$\mathrm{Ex}^G(\chi_s A, B) \xrightarrow{\alpha} KK^G(\chi_s A, B) \xrightarrow{i_A^*} KK^G(A, B)$$

is an isomorphism.

Proof. Theorem 5.5 yields canonical isomorphisms $\mathrm{Ex}^G(\chi_s A, B) \cong [\mathrm{q}_s \chi_s A, B]_s$ and $KK^G(A, B) \cong [\chi_s A, B]_s$. We are going to show that $\pi := \pi_{\chi_s A}^s \colon \mathrm{q}_s \chi_s A \to \chi_s A$ is invertible in $[G_2\text{-C}^*]_s$. Therefore, $[\mathrm{q}_s \chi_s A, B]_s \cong [\chi_s A, B]_s$. It is straightforward to show that the corresponding isomorphism $\mathrm{Ex}^G(\chi_s A, B) \cong KK^G(A, B)$ is equal to the map in the statement Lemma 7.3.

The homotopy inverse for π is constructed as a Kasparov product. Let $i = i_A \in KK^G(A, \chi_s A)$ be as above. Let $j \in KK^G(\chi_s A, \mathrm{q}_s \chi_s A)$ be the inverse of π. Let $h \in KK^G(A, \mathrm{q}_s \chi_s A) \cong [\chi_s A, \mathrm{q}_s \chi_s A]_s$ be the Kasparov product of i and j. The associativity of the Kasparov product implies $\pi \circ h = i$ in $KK^G(A, \chi_s A) = [\chi_s A, \chi_s A]_s$. Since π is invertible in Ex^G, composition with π is an isomorphism

$$\mathrm{Ex}^G(\chi_s A, \mathrm{q}_s \chi_s A) \xrightarrow{\cong} \mathrm{Ex}^G(\chi_s A, \chi_s A).$$

Hence the equality $\pi \circ h \circ \pi = \pi$ in $[\mathrm{q}_s \chi_s A, \chi_s A]_s$ implies $h \circ \pi = \mathrm{id}$ in $[\mathrm{q}_s \chi_s A, \mathrm{q}_s \chi_s A]_s$. Thus h is inverse to π in $[G_2\text{-C}^*]_s$. □

Let $\hat{S}$ be the algebra $\mathrm{C}_0(\mathbb{R})$ graded by $\tau f(x) = f(-x)$ for all $x \in \mathbb{R}$, $f \in \mathrm{C}_0(\mathbb{R})$ and with trivial G-action. It is shown in the proof of [10, Proposition 3.8] that $\chi\mathbb{C} \cong \hat{\mathbb{M}}_2\hat{S}$, so that $\hat{S}$ and $\chi\mathbb{C}$ are Morita-Rieffel equivalent. Together with Proposition 5.2, we obtain a canonical isomorphism in $\mathrm{Ex}^G(\chi_s A, \hat{S} \hat{\otimes} A)$.

Let $e \in KK^G(\mathbb{C}, \hat{S}) \cong [\chi_s\mathbb{C}, \hat{S}]_s$ be represented by the isomorphism $\chi\mathbb{C} \to \hat{\mathbb{M}}_2\hat{S}$. It is easy to verify that e is homotopic to the Kasparov triple $(\hat{S}, 1, x/\sqrt{1+x^2})$, where $1\colon \mathbb{C} \to \mathbb{L}(\hat{S}) \cong \mathrm{C}_\mathrm{b}(\mathbb{R})$ is the unique unital $*$-homomorphism and $x/\sqrt{1+x^2}$ denotes the bounded function $x \mapsto x/\sqrt{1+x^2}$ on $\mathbb{R}$.

Theorem 7.4. *Let G be a locally compact σ-compact topological group and let A and B be separable G_2-C*-algebras. The composition*

$$\sigma\colon \mathrm{Ex}^G(\hat{S} \hat{\otimes} A, B) \xrightarrow{\alpha} KK^G(\hat{S} \hat{\otimes} A, B) \xrightarrow{(e \hat{\otimes} \mathrm{id}_A)^*} KK^G(A, B)$$

is an isomorphism. Here $(e \hat{\otimes} \mathrm{id}_A)^*$ denotes the Kasparov product with the exterior product $e \hat{\otimes} \mathrm{id}_A \in KK^G(A, \hat{S} \hat{\otimes} A)$.

Proof. The isomorphism $KK^G(A, \chi_s A) \cong KK^G(A, (\chi\mathbb{C}) \hat{\otimes} A)$ induced by the canonical map $\chi_s A \to (\chi\mathbb{C}) \hat{\otimes} \mathbb{K}(G_2\mathbb{N})A$ maps i_A to the exterior product $i_\mathbb{C} \hat{\otimes} \mathrm{id}_A$. Hence the isomorphism $KK^G(A, \chi_s A) \to KK^G(A, \hat{S} \hat{\otimes} A)$ maps i_A to $e \hat{\otimes} \mathrm{id}_A$. If we compose the isomorphism $\mathrm{Ex}^G(\chi_s A, B) \to KK^G(A, B)$ of Lemma 7.3 with the the isomorphism $\mathrm{Ex}^G(\hat{S} \hat{\otimes} A, B) \to \mathrm{Ex}^G(\chi_s A, B)$ induced by the Ex^G-equivalence $\hat{S} \hat{\otimes} A \to \chi_s A$, we obtain that σ is an isomorphism. □

We have to compute the exterior product $e \hat{\otimes} e \in KK^G(\mathbb{C}, \hat{S} \hat{\otimes} \hat{S})$. Since the G-action on $\chi\mathbb{C}$ and $\hat{S}$ is trivial, we may forget about the G-actions. Therefore, we briefly resort to the case of trivial G. Theorem 5.5 implies

$$KK(\mathbb{C}, B) = [\chi\mathbb{C}, \mathbb{K}(\mathbb{Z}_2\mathbb{N})B] \cong [\hat{S}, \mathbb{K}(\mathbb{Z}_2\mathbb{N})B].$$

We claim that $e \hat{\otimes} e \in KK(\mathbb{C}, \hat{S} \hat{\otimes} \hat{S})$ belongs to the homomorphism $\hat{S} \to \hat{S} \hat{\otimes} \hat{S}$ that is called l by Haag [9, p. 87] and Δ by Higson and Kasparov [13].

To verify this elementary claim, it is convenient to describe $KK(A, B)$ by unbounded operators following Baaj and Julg [1] because in this picture exterior products are straightforward to compute. The unbounded picture of $KK(\mathbb{C}, B)$ is also nicely related to the isomorphism $KK(\mathbb{C}, B) \cong [\hat{S}, \mathbb{K}(\mathbb{Z}_2\mathbb{N})B]$. The essential, grading preserving $*$-homomorphisms $\hat{S} \to \mathbb{L}(\mathcal{E})$ correspond bijectively to odd, self-adjoint, possibly unbounded multipliers of $\mathcal{E}$ via $f \mapsto f(\mathrm{id}_\mathbb{R})$ for $f\colon \hat{S} \to \mathbb{L}(\mathcal{E})$. Since e belongs to the unbounded multiplier $\mathrm{id}_\mathbb{R}$ of $\hat{S}$, the exterior product $e \hat{\otimes} e$ belongs to the unbounded multiplier $\mathrm{id}_\mathbb{R} \hat{\otimes} 1 + 1 \hat{\otimes} \mathrm{id}_\mathbb{R}$ of $\hat{S} \hat{\otimes} \hat{S}$. Thus $e \hat{\otimes} e$ is represented by the map Δ of [13]. It is easy to check that the concrete formula for l in [10] yields nothing but Δ.

Theorem 7.5. *Let A, B, and C be G_2-C*-algebras and let $x \in \mathrm{Ex}^G(\hat{S} \hat{\otimes} A, B)$, $y \in \mathrm{Ex}^G(\hat{S} \hat{\otimes} B, C)$. The Kasparov product of $\sigma(y) \in KK^G(B, C)$ and $\sigma(x) \in KK^G(A, B)$ is mapped by σ^{-1} to the composition*

$$\hat{S} \hat{\otimes} A \xrightarrow{\Delta \hat{\otimes} \mathrm{id}_A} \hat{S} \hat{\otimes} \hat{S} \hat{\otimes} A \xrightarrow{\mathrm{id}_{\hat{S}} \hat{\otimes} y} \hat{S} \hat{\otimes} B \xrightarrow{x} C$$

in Ex^G.

Proof. Recall the definition of σ in Theorem 7.4 and that α is multiplicative. Moreover, it is easy to check that α is compatible with exterior products, so that

$\alpha(\mathrm{id}_{\hat{S}} \hat{\otimes} x) \cong \mathrm{id}_{\hat{S}} \hat{\otimes} \alpha(x)$. Hence we compute

$$\begin{aligned}\sigma(y) \circ \sigma(x) &= \alpha(y) \circ (e \hat{\otimes} \mathrm{id}_B) \circ \alpha(x) \circ (e \hat{\otimes} \mathrm{id}_A)\\ &= \alpha(y) \circ \big(e \hat{\otimes} \alpha(x)\big) \circ (e \hat{\otimes} \mathrm{id}_A) = \alpha(y) \circ \big(\mathrm{id}_{\hat{S}} \hat{\otimes} \alpha(x)\big) \circ (e \hat{\otimes} \mathrm{id}_{\hat{S}\hat{\otimes}A}) \circ (e \hat{\otimes} \mathrm{id}_A)\\ &= \alpha(y) \circ \alpha(\mathrm{id}_{\hat{S}} \hat{\otimes} x) \circ (e \hat{\otimes} e \hat{\otimes} \mathrm{id}_A) = \sigma\big(y \circ (\mathrm{id}_{\hat{S}} \hat{\otimes} x) \circ (\Delta \hat{\otimes} \mathrm{id}_A)\big).\end{aligned}$$

We used that the Kasparov product is compatible with exterior products. □

8. Proper actions and square-integrable Hilbert modules

Exel [7] and Rieffel [18] define the concept of a proper action of a locally compact group on a C*-algebra. Furthermore, Rieffel relates proper G-actions on the algebra $\mathbb{K}(\mathcal{H})$ to square-integrable representations of G. It is very illuminating to consider also square-integrable group actions on Hilbert modules. The main result is that a countably generated Hilbert A, G-module is square-integrable iff it is a direct summand of $\mathcal{H}_A$. We conclude that proper algebras have property AE.

Concerning questions of properness, we may ignore gradings whenever convenient. Since the group $\mathbb{Z}_2$ is compact, a G_2-C*-algebra is proper iff it is proper as a G-C*-algebra.

Let A be a G-C*-algebra and let $\mathcal{E}$ be a Hilbert A, G-module. We denote the G-actions on A and $\mathcal{E}$ by α and γ, respectively. We frequently view A as a right Hilbert A, G-module. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets of G such that K_{n+1} is a neighborhood of K_n for all n and $G = \bigcup K_n$. Let $(\kappa_n)_{n\in\mathbb{N}}$ be an increasing sequence of functions $\kappa_n\colon G \to [0,1]$ with $\kappa_n|_{K_n} = 1$ and $\kappa_n|_{G\setminus K_{n+1}} = 0$.

A continuous function $f\colon G \to A$ is called *square-integrable* iff the sequence $\int_G f^*(g)f(g)\kappa_n(g)\,dg$ is a norm Cauchy sequence in A. Equivalently, the sequence of integrals $\int_{K_n} f^*(g)f(g)\,dg$ is norm convergent. Observe that these sequences are increasing sequences of positive elements and that the notion of square-integrability does not depend on the choice of the sets K_n or the functions κ_n.

It is easy to check that f is square-integrable iff the sequence $(f \cdot \kappa_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the norm $\|h\| := \|\int_G h^*(g)h(g)\,dg\|^{1/2}$ on $\mathrm{C_c}(G, A)$. Since the completion of $\mathrm{C_c}(G, A)$ with respect to this norm is precisely $L^2(G, A)$, we can view square-integrable continuous functions as elements of $L^2(G, A)$.

If $\xi, \eta \in \mathcal{E}$, then we define the *coefficient function* $c_{\xi\eta}\colon G \to A$ by

$$c_{\xi\eta}(g) := \langle \gamma_g(\xi) \mid \eta\rangle_A \qquad \text{for all } g \in G.$$

In the special case $\mathcal{E} = A$, we have $c_{ab}(g) := \alpha_g(a)^* b$.

We call $\xi \in \mathcal{E}$ *square-integrable* iff the function $c_{\xi\eta}$ is square-integrable for all $\eta \in \mathcal{E}$. The Hilbert module $\mathcal{E}$ is called *square-integrable* iff the set of square-integrable elements is dense in $\mathcal{E}$. A G-C*-algebra A is called *proper* iff it is square-integrable as a right Hilbert A, G-module. Let $A_+ \subset A$ be the cone of positive elements. We call $a \in A_+$ *integrable* iff $a^{1/2}$ is square-integrable.

By definition, $a \in A_+$ is integrable iff the integrals $\int_{K_n} b^*\alpha_g(a)b\,dg$ form a Cauchy sequence with respect to the norm topology for all $b \in A$. Moreover, $a \in A$ is square-integrable iff aa^* is integrable. Hence A is proper iff the set of integrable elements is dense in A_+. The above definition of properness is equivalent to Rieffel's definition in [18] and thus also to Exel's definition in [7].

Lemma 8.1. *Let $\mathcal{E}$ be a Hilbert A, G-module and let $\xi, \eta, \zeta \in \mathcal{E}$.*

(i) If ξ is square-integrable, then the map

$$\Gamma_\xi\colon \mathcal{E} \to L^2(G, A), \qquad \Gamma_\xi(\eta) := c_{\xi\eta},$$

is adjointable. The adjoint $\Gamma_\xi^*\colon L^2(G,A)\to\mathcal{E}$ *satisfies*

$$\Gamma_\xi^*(f) := \int_G \gamma_g(\xi)\cdot f(g)\,dg \qquad \text{for all } f\in \mathrm{C_c}(G,A). \tag{5}$$

(ii) The operators Γ_ξ *and* Γ_ξ^* *are* G*-equivariant.*

(iii) The closure of the range of Γ_ξ^* *is the smallest* G*-invariant Hilbert submodule of* $\mathcal{E}$ *containing* ξ*. In particular,* ξ *is contained in the closure of* $\operatorname{Ran}\Gamma_\xi^*$*.*

(iv) If ξ *and* ζ *are square-integrable, then the sequence*

$$\int_G \gamma_g(\xi)\cdot\langle\gamma_g(\zeta)\mid\eta\rangle_A\,\kappa_n(g)\,dg, \qquad n\in\mathbb{N},$$

in $\mathcal{E}$ *is norm convergent. Its limit is* $\Gamma_\xi^*\Gamma_\zeta(\eta)$*.*

(v) ξ *is square-integrable iff* $|\xi\rangle\langle\xi|\in\mathbb{K}(\mathcal{E})$ *is integrable.*

Proof. The Banach-Steinhaus theorem yields that Γ_ξ is bounded. We can define an operator $\Gamma_\xi^*\colon \mathrm{C_c}(G,A)\to\mathcal{E}$ by (5). For $f\in\mathrm{C_c}(G,A)$, we compute

$$\langle\Gamma_\xi^*(f)\mid\eta\rangle_A = \int_G\langle\gamma_g(\xi)f(g)\mid\eta\rangle_A\,dg = \int_G f(g)^*\cdot c_{\xi\eta}(g)\,dg = \langle f\mid\Gamma_\xi(\eta)\rangle_A. \tag{6}$$

Hence $\|\langle\Gamma_\xi^*(f)\mid\eta\rangle_A\|\le\|f\|_{L^2(G,A)}\|\eta\|\cdot\|\Gamma_\xi\|$. Since η is arbitrary, it follows that $\|\Gamma_\xi^*(f)\|\le\|f\|_{L^2(G,A)}\cdot\|\Gamma_\xi\|$. Thus we may extend Γ_ξ^* to $L^2(G,A)$. Equation (6) shows that Γ_ξ^* is adjoint to Γ_ξ.

Straightforward computations show that Γ_ξ and Γ_ξ^* are G-equivariant.

Assertion (iii) follows easily once we know that ξ is contained in the closed range of Γ_ξ^*. Choose $\epsilon>0$. There is $u\in A$ with $0\le u\le 1$ and $\|\xi\cdot u-\xi\|\le\epsilon/2$. There is a compact neighborhood U of $1\in G$ with $\|\gamma_g(\xi)-\xi\|<\epsilon/2$ for all $g\in U$. Let $f\colon G\to\mathbb{R}_+$ be a continuous function with support U and $\int_G f(g)\,dg=1$. Then $\|\Gamma_\xi^*(f\otimes u)-\xi\|\le\epsilon$. Hence ξ is contained in the closure of $\operatorname{Ran}\Gamma_\xi^*$.

We compute

$$\Gamma_\xi^*\Gamma_\zeta(\eta)=\Gamma_\xi^*(c_{\zeta\eta})=\lim_{n\to\infty}\Gamma_\xi^*(c_{\zeta\eta}\kappa_n)=\lim_{n\to\infty}\int_G\gamma_g(\xi)\langle\gamma_g(\zeta)\mid\eta\rangle_A\,\kappa_n(g)\,dg. \tag{7}$$

The boundedness of Γ_ξ^* implies that the sequence is norm convergent.

Equation (7) implies that the sequence

$$I_n := \int_G|\gamma_g(\xi)\rangle\langle\gamma_g(\xi)|\,\kappa_n(g)\,dg = \int_G\gamma_g(|\xi\rangle\langle\xi|)\,\kappa_n(g)\,dg\in\mathbb{K}(\mathcal{E})$$

is bounded and converges strongly (that is, pointwise on $\mathcal{E}$) towards $\Gamma_\xi^*\Gamma_\xi$. Therefore, the sequences $(I_n\cdot T)$ and $(T\cdot I_n)$ converge in norm for all $T\in\mathbb{K}(\mathcal{E})$. This means that $|\xi\rangle\langle\xi|\in\mathbb{K}(\mathcal{E})$ is integrable. Conversely, if $|\xi\rangle\langle\xi|\in\mathbb{K}(\mathcal{E})$ is integrable, then the sequence $\langle\eta\mid I_n(\eta)\rangle_A$ is norm convergent for all $\eta\in\mathbb{K}(\mathcal{E})\cdot\mathcal{E}=\mathcal{E}$. Since

$$\langle\eta\mid I_n(\eta)\rangle_A=\int_G\langle\gamma_g(\xi)\mid\eta\rangle_A^*\langle\gamma_g(\xi)\mid\eta\rangle_A\,\kappa_n(g)\,dg=\int_G c_{\xi\eta}(g)^*c_{\xi\eta}(g)\,\kappa_n(g)\,dg,$$

this means that ξ is square-integrable. □

Remark 8.2. If $\Gamma_\xi^*\colon \mathrm{C_c}(G,A)\to\mathcal{E}$ extends to an *adjointable* map $L^2(G,A)\to\mathcal{E}$, then ξ is square-integrable.

The map Γ_ξ^* extends to a bounded operator $L^2(G,A)\to\mathcal{E}$ if and only if the function $g\mapsto c_{\xi\eta}(g)^*c_{\xi\eta}(g)$ is *order-integrable* in the sense of Rieffel [18] for all $\eta\in\mathcal{E}$. Hence it may happen that Γ_ξ^* extends to a bounded operator on $L^2(G,A)$ that is not adjointable.

Proposition 8.3. *Let* A *be a* G*-*C^**-algebra and let* $\mathcal{E}$ *be a Hilbert* A,G*-module. Then* $\mathcal{E}$ *is square-integrable iff* $\mathbb{K}(\mathcal{E})$ *is proper.*

Proof. If $\mathcal{E}$ is square-integrable, then the linear span of the integrable elements is dense in $\mathbb{K}(\mathcal{E})$ by Lemma 8.1.(v). Therefore, $\mathbb{K}(\mathcal{E})$ is proper. Conversely, assume that $\mathbb{K}(\mathcal{E})$ is proper. Let $T \in \mathbb{K}(\mathcal{E})_+$ be square-integrable. If $\xi \in \mathcal{E}$, then $|T\xi\rangle\langle T\xi| = T|\xi\rangle\langle\xi|T^* \leq \|\xi\|^2 TT^*$ is integrable because TT^* is integrable and the set of integrable elements is a hereditary cone in $\mathbb{K}(\mathcal{E})_+$ [18]. Hence $T\xi \in \mathcal{E}$ is square-integrable by Lemma 8.1.(v). Since $\mathbb{K}(\mathcal{E})$ is proper, the set of elements of $\mathcal{E}$ of the form $T\xi$ with square-integrable $T \in \mathbb{K}(\mathcal{E})$ is dense in $\mathcal{E}$. Thus $\mathcal{E}$ is square-integrable. □

Proposition 8.4. *Let A and B be G-C^*-algebras, let $\mathcal{E}$ be a Hilbert B, G-module, and let $\phi\colon A \to \mathbb{L}(\mathcal{E})$ be an essential G-equivariant $*$-homomorphism.*

If A is proper, then $\mathcal{E}$ is square-integrable.

Proof. Identify $\mathbb{L}(\mathcal{E}) \cong \mathcal{M}\big(\mathbb{K}(\mathcal{E})\big)$. By [18, Theorem 5.3], we conclude that $\mathbb{K}(\mathcal{E})$ is proper. Thus $\mathcal{E}$ is square-integrable by Proposition 8.3. □

Theorem 8.5. *Let A be a G-C^*-algebra and let $\mathcal{E}$ be a countably generated Hilbert A, G-module. Then the following assertions are equivalent:*

(i) $\mathcal{E}$ *is square-integrable;*
(ii) $\mathbb{K}(\mathcal{E})$ *is proper;*
(iii) there is a G-equivariant unitary $\mathcal{E} \oplus \mathcal{H}_A \cong \mathcal{H}_A$;
(iv) $\mathcal{E}$ *is a direct summand of* $\mathcal{H}_A$.

Proof. Proposition 8.3 asserts that (i) and (ii) are equivalent. It is trivial that (iii) implies (iv). It remains to show that (iv) implies (i) and that (i) implies (iii).

We prove that (iv) implies (i). It is straightforward to show that $\mathbb{K}(L^2G)$ is a proper G-C*-algebra. Equivalently, L^2G is a square-integrable G-Hilbert space. Let $\mathcal{F}$ be an arbitrary Hilbert A, G-module. The canonical $*$-homomorphism $\mathbb{K}(L^2G) \to \mathbb{L}(L^2G \otimes \mathcal{F})$, $T \mapsto T \otimes 1$, is essential and G-equivariant. Hence $L^2(G, \mathcal{F})$ is square-integrable by Proposition 8.4. Especially, $\mathcal{H}_A$ is square-integrable. A direct summand of a square-integrable Hilbert module is square-integrable as well because the projection onto the direct summand maps square-integrable elements to square-integrable elements. Hence any direct summand of $\mathcal{H}_A$ is square-integrable. That is, (iv) implies (i).

The proof that (i) implies (iii) is very similar to the proof of the stabilization theorem by Mingo and Phillips [17]. Suppose that $\mathcal{E}$ is square-integrable. Hence there is a sequence $(\xi_n)_{n\in\mathbb{N}}$ of square-integrable elements of $\mathcal{E}$, whose linear span is dense in $\mathcal{E}$. Let $\Gamma_n := \Gamma_{\xi_n}$ be as in Lemma 8.1. We may assume that $\|\Gamma_n\| \leq 1$ for all $n \in \mathbb{N}$ and that each ξ_n is repeated infinitely often. An element of $\mathcal{H}_A$ can be viewed as a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \in L^2(G, A)$. We formally write $\sum f_n\delta_n$ for this sequence. Define an adjointable operator $T\colon \mathcal{H}_A \to \mathcal{E} \oplus \mathcal{H}_A$ by

$$T\left(\sum_{n=1}^{\infty} f_n\delta_n\right) := \sum_{n=1}^{\infty} 2^{-n}\Gamma_n^*(f_n) \oplus \sum_{n=1}^{\infty} 4^{-n} f_n\delta_n,$$

$$T^*|_{\mathcal{E}}(\eta) := \sum_{n=1}^{\infty} 2^{-n}\Gamma_n(\eta)\delta_n, \qquad T^*|_{\mathcal{H}_A}\left(\sum_{n=1}^{\infty} f_n\delta_n\right) := \left(\sum_{n=1}^{\infty} 4^{-n} f_n\delta_n\right).$$

Lemma 8.1.(ii) implies that T is G-equivariant. Evidently, T^* has dense range.

We claim that T has dense range as well. Let $\mathcal{F} \subset \mathcal{E} \oplus \mathcal{H}_A$ be the closure of the range of T. Let $f \in L^2(G, A)$. Since each Γ_n^* is repeated infinitely often, we have $\Gamma_n^*(f) \oplus 2^{-k} f\delta_k \in \operatorname{Ran} T$ for infinitely many $k \in \mathbb{N}$. Hence $\Gamma_n^*(f) \oplus 0 \in \mathcal{F}$ for all $f \in L^2(G, A)$. By Lemma 8.1.(iii), this implies $\xi_n \oplus 0 \in \mathcal{F}$ for all n and hence $\mathcal{E} \subset \mathcal{F}$. Finally, we get $0 \oplus f\delta_n \in \mathcal{F}$ for all $f \in L^2(G, A)$ and thus $\mathcal{F} = \mathcal{E} \oplus \mathcal{H}_A$.

Since both T and T^* have dense range, the composition T^*T has dense range. Thus $|T| := (T^*T)^{1/2}$ has dense range because $|T|(\mathcal{E}) \supset |T|(|T|\mathcal{E}) = T^*T(\mathcal{E})$. Since $\langle |T|\eta \mid |T|\eta\rangle_A = \langle T^*T\eta \mid \eta\rangle_A = \langle T\eta \mid T\eta\rangle_A$, the formula $U(|T|\eta) := T\eta$ well-defines an isometry U from $\operatorname{Ran}|T|$ onto $\operatorname{Ran} T$. Extending U continuously, we obtain the desired unitary $U\colon \mathcal{H}_A \to \mathcal{E} \oplus \mathcal{H}_A$. Since T is G-equivariant, so is U. ☐

Thomsen [22] calls a G-C*-algebra A *K-proper* iff $\mathcal{E} \oplus \mathcal{H}_A \cong \mathcal{H}_A$ for all Hilbert A, G-bimodules $\mathcal{E}$. Theorem 8.5 implies that A is K-proper in Thomsen's sense iff all G-C*-algebras that are Morita-Rieffel equivalent to A are proper in our sense. For instance, the algebra $\mathbb{K}(G)$ is not K-proper, unless G is compact, because it is Morita-Rieffel equivalent to the improper G-C*-algebra $\mathbb{C}$.

Proposition 8.6. *All σ-unital proper G_2-C*-algebras have property AE.*

Proof. Let A and B be σ-unital G_2-C*-algebras and let $(\mathcal{E}, \phi, F)$ be an essential Kasparov triple for A, B. Suppose that A is proper. Proposition 8.4 implies that $\mathcal{E}$ is square-integrable. Hence $\mathcal{E} \subset \hat{\mathcal{H}}_B$ by Theorem 8.5.

Let $\hat{\mathcal{H}}_A^*$ be the imprimitivity bimodule dual to $\hat{\mathcal{H}}_A$. As remarked in Section 2.3, we have $\hat{\mathcal{H}}_A^* = \mathbb{K}(\hat{\mathcal{H}}_A, A)$ with a canonical Hilbert $A, \mathbb{K}(\hat{\mathcal{H}}_A), G_2$-bimodule structure. Since A is proper and σ-unital, Theorem 8.5 yields a G_2-equivariant, adjointable isometry $T\colon A \to \hat{\mathcal{H}}_A$. Composition with T gives rise to an adjointable isometry $T_*\colon \mathbb{K}(\hat{\mathcal{H}}_A, A) \to \mathbb{K}(\hat{\mathcal{H}}_A, \hat{\mathcal{H}}_A)$. Thus $\hat{\mathcal{H}}_A^*$ is a direct summand in $\mathbb{K}(\hat{\mathcal{H}}_A)$.

Lemma 3.1 yields a G-equivariant F-connection $\bar{F}$ on $L^2(G_2, \mathcal{E})^\infty = \hat{\mathcal{H}}_A \hat{\otimes}_\phi \mathcal{E}$. If we view $\bar{F}$ as an operator on $\mathbb{K}(\hat{\mathcal{H}}_A) \hat{\otimes}_{\mathbb{K}(\hat{\mathcal{H}}_A)} L^2(G_2, \mathcal{E})^\infty \cong L^2(G_2, \mathcal{E})^\infty$, then we obtain an $\bar{F}$-connection. Hence the compression

$$F' := (T_* \hat{\otimes}_{\mathbb{K}(\hat{\mathcal{H}}_A)} 1)^* \cdot \bar{F} \cdot (T_* \hat{\otimes}_{\mathbb{K}(\hat{\mathcal{H}}_A)} 1)$$

of $\bar{F}$ to $\hat{\mathcal{H}}_A^* \hat{\otimes}_{\mathbb{K}(\hat{\mathcal{H}}_A)} \hat{\mathcal{H}}_A \hat{\otimes}_\phi \mathcal{E} \cong A \hat{\otimes}_\phi \mathcal{E} \cong \mathcal{E}$ is an $\bar{F}$-connection as well. By [2, 18.3.4.f], F' is an F-connection. Another F-connection is F itself. Therefore, F' is a compact perturbation of F. Since $\bar{F}$ and T are G-equivariant, so is F'. ☐

Theorem 8.7. *Let A and B be σ-unital G_2-C*-algebras. If A is proper, then*

$$KK^G(A, B) \cong KK^G_s(A, B) \cong [\chi A, \mathbb{K}(G_2\mathbb{N})B] \cong [\chi A, B]_{\mathbb{K}(G_2\mathbb{N})}.$$

If A is proper and A and B are trivially graded, then

$$KK^G(A, B) \cong KK^G_s(A, B) \cong [\mathrm{q}A, \mathbb{K}(G\mathbb{N})B] \cong [\mathrm{q}A, B]_{\mathbb{K}(G\mathbb{N})}.$$

Proof. By Proposition 8.6, A has property AE. Hence the assertions follow from Proposition 3.4 and Proposition 5.4. ☐

9. Other equivariant theories

The arguments above generalize to other more general versions of equivariant Kasparov theory. First of all, we did not even care to specify whether we work with complex or real C*-algebras: The theory above goes through in both cases. In the real case, we only have to interpret $\mathbb{C}$ as the algebra $\mathbb{R}$ of real numbers everywhere above. Hence KK^G is the universal split exact stable homotopy functor also for real C*-algebras. We may also treat "real" C*-algebras as in [14].

Our results carry over to Kasparov's functor $\mathcal{R}KK^G$ with some obvious changes. We first define the functors $\mathcal{R}KK^G$ and RKK^G.

Let G be, as usual, a locally compact σ-compact topological group and let X be a locally compact σ-compact G-space. An *$X \rtimes G_2$-C*-algebra* is a G_2-C*-algebra A equipped with a G_2-equivariant *essential* $*$-homomorphism from $\mathrm{C}_0(X)$ into the *center* of $\mathcal{M}(A)$. Let A, B be $X \rtimes G_2$-C*-algebras. A $*$-homomorphism $\phi\colon A \to$

$\mathcal{M}(B)$ is called $X \rtimes G_2$*-equivariant* iff it is G_2-equivariant and in addition satisfies $\phi(f \cdot a) = f \cdot \phi(a)$ for all $f \in \mathrm{C}_0(X)$, $a \in A$.

If B is an $X \rtimes G_2$-C*-algebra and $\mathcal{E}$ is a Hilbert B, G-module, then $\mathbb{K}(\mathcal{E})$ is an $X \rtimes G_2$-C*-algebra as well. The homomorphism $\mathrm{C}_0(X) \to \mathcal{M}\big(\mathbb{K}(\mathcal{E})\big) = \mathbb{L}(\mathcal{E})$ is defined by $f \cdot (\xi \cdot b) := \xi \cdot (f \cdot b)$ for all $f \in \mathrm{C}_0(X)$, $\xi \in \mathcal{E}$, $b \in B$. By the Cohen-Hewitt factorization theorem, all elements of $\mathcal{E}$ are of the form $\xi \cdot b$ for suitable $\xi \in \mathcal{E}$, $b \in B$. Computing the inner products $\langle f \cdot (\xi \cdot b) \mid \eta \cdot c\rangle$ shows that $f \cdot (\xi \cdot b)$ is well-defined and defines a $*$-homomorphism from $\mathrm{C}_0(X)$ into the center of $\mathbb{L}(\mathcal{E})$.

Kasparov [15] defines the functor $\mathcal{R}KK^G(X; A, B)$ for $X \rtimes G_2$-C*-algebras using Kasparov triples $(\mathcal{E}, \phi, F)$ for A, B with the additional assumption that $\phi\colon A \to \mathbb{L}(\mathcal{E})$ be $X \rtimes G_2$-equivariant. If A and B are just G_2-C*-algebras, then he puts $RKK^G(X; A, B) := \mathcal{R}KK^G\big(X; \mathrm{C}_0(X, A), \mathrm{C}_0(X, B)\big)$. Hence RKK^G is a special case of $\mathcal{R}KK^G$.

The presence of the central homomorphism $\mathrm{C}_0(X) \to \mathbb{L}(\mathcal{E})$ creates no problems in Sections 3 and 4. In Section 5, we have to modify the definitions of the universal algebras $\mathcal{X}A$ and $\mathrm{Q}A$ because, as they are defined there, they are not $X \rtimes G_2$-C*-algebras. Thus we replace them by algebras with analogous universal properties in the category of $X \rtimes G_2$-C*-algebras. This amounts to dividing out the relations $\iota^{\pm}(fa)\iota^{\pm}(b) = \iota^{\pm}(a)\iota^{\pm}(fb)$ for all $a, b \in A$, $f \in \mathrm{C}_0(X)$ in $\mathrm{q}A$ and the relation $j_A(fa)F_A j_A(b) = j_A(a)F_A j_A(fb)$ for all $a, b \in A$, $f \in \mathrm{C}_0(X)$ in χA. The quotients by the ideals generated by these relations carry a canonical $X \rtimes G_2$-C*-algebra structure. Once this modification is made, the results of Sections 5, 6, and 7 carry over without change. Especially, $\mathcal{R}KK^G(X; A, B)$ is the universal split exact stable homotopy functor for trivially graded separable $X \rtimes G$-C*-algebras. Theorem 8.7 remains valid as well.

The functor $\mathcal{R}KK^G(X; A, B)$ is a special case of the equivariant Kasparov theory for groupoids developed in [16]. I expect that the arguments above carry over to the case of locally compact groupoids with Haar system. However, I have not checked the details. Some work has to be done to carry over the proof of Lemma 3.1. Moreover, to carry over the results of Section 8, one first has to define properness in the sense of Rieffel for actions of groupoids.

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