

# Polynomial and Rational Solutions of Holonomic Systems

Toshinori Oaku, Nobuki Takayama and Harrison Tsai

January 9, 1999

## 1 Introduction

Polynomial and rational solutions for linear ordinary differential equations can be obtained by algorithmic methods. For instance, the maple package **DEtools** provides efficient functions **polysols** and **ratsols** to find polynomial and rational solutions for a given linear ordinary differential equation with rational function coefficients.

A natural analogue of the notion of linear ordinary differential equation in the several variable case is the notion of holonomic system. A holonomic system is a system of linear partial differential equations whose characteristic variety is middle dimensional.

Chyzak [4] gave an algorithm to find the rational solutions of holonomic systems by using elimination in the ring of differential operators with rational function coefficients combined with Abramov's algorithm for rational solutions of ordinary differential equations with parameters. To the authors, solving holonomic systems is analogous to solving systems of algebraic equations of zero-dimensional ideals. Under this analogy, the method of Chyzak corresponds to the elimination method for solving systems of algebraic equations.

The aim of this paper is to give two new algorithms, which are elimination free, to find polynomial and rational solutions for a given holonomic system associated to a set of linear differential operators in the Weyl algebra

$$D = \mathbf{k}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$$

where  $\mathbf{k}$  is a subfield of  $\mathbf{C}$ .

Polynomial and rational solutions can be obtained, if they exist, by using an exhaustive search. For instance, when  $f = 0$  is the singular locus of a holonomic system  $M = D/I$ , any rational solution has the form  $g/f^r$ . If we have upper bounds for the degree of the polynomial  $g$  and for  $r$ , then we can construct all rational solutions by solving linear equations satisfied by the coefficients of  $g$ . Alternatively, if we know the dimension of rational solutions, then we can obtain all rational solutions by increasing the degree of  $g$  and  $r$ . Hence, the problem reduces to finding effective bounds for these numbers.

In sections 2 and 3, we give algorithms for upper bounds on the degree of  $g$  and on  $r$ . The main techniques we use are Gröbner deformations in  $D$  as introduced in the book [11] and the  $b$ -function for  $D/I$  and  $f$ .

In section 4, we give an algorithm to evaluate the dimension of polynomial and rational solutions. Our approach is an analog in  $D$  of a question studied by Singer [12], who gave an algorithm to compute  $\text{Hom}_R(M, N)$  for left  $R := \mathbf{k}(x_1)\langle\partial_1\rangle$ -modules  $M$  and  $N$  and studied its relation to factorizations of ordinary differential operators. The theory of  $D$ -modules translates our problem on polynomial and rational solutions to constructions in the ring of differential operators  $D$ . For example, the  $\mathbf{k}$ -vector space

$$\text{Hom}_D(D/I, \mathbf{k}[\mathbf{x}]) \simeq H^{-n}(\Omega \otimes_D^L \mathbf{D}(D/I))$$

is the space of the polynomial solutions of the left ideal  $I$ . Here,  $\Omega$  is the module of the top dimensional differential forms and  $\mathbf{D}$  is the dualizing functor. See, e.g., the book of Björk [3] on this translation. We evaluate the dimension of the right hand side by recent developments of computational algebra such as construction of free resolutions in the ring  $D$  and restrictions of  $D$ -modules [8], [9], [14], [17]. Our method also allows us to evaluate the dimension of solutions to a holonomic system inside any holonomic module. For instance, we can find the dimension of the delta function solutions to  $I$ .

Throughout the paper, we refer to the book [11] for fundamental facts on the algorithmic treatment of  $D$ . Also, the algorithms which appear in the paper have been implemented in either `kan` [13] or `Macaulay 2` [5].

We deeply thank Dan Grayson, Anton Leykin, and Mike Stillman, who helped to implement  $D$ -modules in `Macaulay 2`, and Frédéric Chyzak and Michael Singer for discussions on rational solutions.

## 2 Polynomial solutions by Gröbner deformations

How can we obtain all polynomial solutions for ordinary differential equations? One method is to compute the indicial polynomial at infinity, find an upper bound on the degrees of polynomial solutions, and determine the coefficients of polynomials. The analogous method works for holonomic systems by using Gröbner deformations. For  $\ell \in D$  and the weight vector  $w \in \mathbf{R}^n$ , we denote by  $\text{in}_{(-w, w)}(\ell)$  the initial term of  $\ell$  with respect to the weight  $(-w, w)$  (see, e.g., [11, §1.1]). The following proposition follows from the definition of  $\text{in}_{(-w, w)}(\ell)$ .

**Proposition 2.1** *Suppose that  $f(x_1, \dots, x_n)$  is a polynomial solution of  $I = D \cdot \{\ell_1, \dots, \ell_m\}$ . Take  $w \in \mathbf{Z}^n$ . Then  $f(t^{w_1}x_1, \dots, t^{w_n}x_n)$  can be expanded as a polynomial in  $t$  as*

$$f_w(\mathbf{x})t^p + O(t^{p+1}).$$

*Then we have*

$$\text{in}_{(-w, w)}(\ell_i) \bullet f_w = 0.$$

The initial ideal  $\text{in}_{(-w,w)}(I)$  is sometimes called the Gröbner deformation of  $I$  with respect to  $(-w, w)$ .

**Theorem 2.2** [2] *There exist only finitely many Gröbner deformations.*

The Newton polytope of a polynomial solution  $f$  is defined as the convex hull of the exponent vectors of  $f$ . For generic  $w$ ,  $f_w$  is a monomial  $cx^a$  and the point  $a$  is a vertex of the Newton polytope of  $f$ .

Let  $R = \mathbf{k}(x_1, \dots, x_n)\langle \partial_1, \dots, \partial_n \rangle$  and  $\theta_i = x_i \partial_i$ . Since  $a \in \mathbf{Z}^n$  belongs to the zero set of the indicial ideal

$$\widetilde{\text{in}_{(-w,w)}(I)} = R \cdot \text{in}_{(-w,w)}(I) \cap \mathbf{k}[\theta_1, \dots, \theta_n],$$

we can construct a polytope that contains the Newton polytopes of the polynomial solutions by taking the convex hull of all the non-negative integral roots of all possible indicial ideals.

It is not necessary to find all Gröbner deformations to obtain polynomial solutions. Let  $b(s)$  be the generator of  $\text{in}_{(-w,w)}(I) \cap \mathbf{k}[s]$ ,  $s = \sum_{i=1}^n w_i \theta_i$ . The polynomial  $b(s)$  is called the  $b$ -function of  $I$  with respect to  $(-w, w)$ . The next proposition follows from the definition of  $b(s)$ .

**Proposition 2.3** *Let  $w$  be a strictly negative weight vector. In other words, we assume that  $w_i < 0$  for all  $i$ . Consider the  $b$ -function  $b(s)$  of  $I$  with respect to  $(-w, w)$  and let  $-k_1$  be the smallest integer root of  $b(s) = 0$ . The polynomial solutions of  $I$  have the form*

$$\sum_{p_i \geq 0, p \cdot w \leq k_1} c_p x^p. \quad (1)$$

**Algorithm 2.4** (Finding the polynomial solutions by a Gröbner deformation)

INPUT: a holonomic left ideal  $I$ .

OUTPUT: the polynomial solutions of  $I$ .

1. Take a strictly negative weight vector  $w$ , compute the Gröbner deformation  $\text{in}_{(-w,w)}(I)$ , and compute the smallest non-positive integer root  $-k_1$  of the  $b$ -function with respect to  $(-w, w)$ . See, e.g., [11, Alg. 5.15] for these procedures.
2. If we do not have such a root, then there is no polynomial solution other than 0.
3. If there is a minimal integer root, then determine the coefficients  $c_p$  of (1) by solving linear equations for the coefficients.

**Example 2.5** The following system of differential equations of two variables is called the Appell differential equation  $F_1(a, b, b', c)$  [1]:

$$\begin{aligned} &\theta_x(\theta_x + \theta_y + c - 1) - x(\theta_x + \theta_y + a)(\theta_x + b), \\ &\theta_y(\theta_x + \theta_y + c - 1) - y(\theta_x + \theta_y + a)(\theta_y + b'), \\ &(x - y)\partial_x\partial_y - b'\partial_x + b\partial_y \end{aligned}$$

where  $a, b, b', c$  are complex parameters. Let us demonstrate how Algorithm 2.4 works for the system of parameter values  $(a, b, b', c) = (2, -3, -2, 5)$ . First, we choose a strictly negative weight vector  $w = (-1, -2)$  and compute the  $b$ -function  $b(s)$ ,  $s = -\theta_x - 2\theta_y$ , which is the generator of the principal ideal in  ${}_{(-w, w)}(I) \cap \mathbf{Q}[-\theta_x - 2\theta_y]$ . We can use the V-homogenization or the homogenized Weyl algebra to get the generator (see, e.g., [11, §1.2]). Second, we need to find the integer roots of the  $b$ -function  $b(s) = 0$ . In our example, these are

$$-7, 0, 4.$$

From Proposition 2.1, the highest  $(-w)$ -degree monomial  $cx^py^q$  in a polynomial solution gives rise to an integer solution  $w_1p + w_2q = -p - 2q$  of the  $b$ -function. Hence, the polynomial solutions are of the form

$$f = \sum_{p, q \geq 0, p+2q \leq 7} c_{pq} x^p y^q.$$

Finally, we determine the coefficients  $c_{pq}$  by applying the differential operators to  $f$  and putting the results to 0. In our example, we have only one polynomial solution

$$\begin{aligned} &(-\frac{1}{21}y^2 + \frac{1}{7}y - \frac{4}{35})x^3 + (\frac{3}{14}y^2 - \frac{24}{35}y + \frac{3}{5})x^2 \\ &+ (-\frac{12}{35}y^2 + \frac{6}{5}y - \frac{6}{5})x + \frac{1}{5}y^2 - \frac{4}{5}y + 1. \end{aligned}$$

### 3 Rational solutions by Gröbner deformations

The singular locus of a  $D$ -ideal  $I$  is defined to be the projection of the characteristic variety of  $I$  minus the zero section from the cotangent bundle to the coordinate base space. In other words, it is the zero set

$$\text{Sing}(I) = V(\langle \text{in}_{(0, e)}(I) : (\xi_1, \dots, \xi_n)^\infty \rangle \cap \mathbf{k}[x_1, \dots, x_n]).$$

Any rational solution to  $I$  has its poles contained inside the singular locus. Thus if  $f(\mathbf{x})$  defines the codimension 1 component of  $\text{Sing}(I)$ , we may limit our search for rational solutions to  $\mathbf{k}[\mathbf{x}][\frac{1}{f}]$ .

We will present a method to obtain an upper bound of the order of the poles along  $f = 0$  for each rational solution. For this purpose we use the notion of the

$b$ -function for  $f$  and a section  $u$  of a holonomic system, which was introduced by Kashiwara [7]: Let  $\mathcal{D}$  be the sheaf of algebraic differential operators on  $X = \mathbf{C}^n$ . For a holonomic  $\mathcal{D}$ -module  $\mathcal{M} = \mathcal{D}/\mathcal{D}I$  and a polynomial  $f$ , consider the tensor product

$$\mathcal{N} = \mathcal{O}[f^{-1}, s]f^s \otimes_{\mathcal{O}_X} \mathcal{M}. \quad (2)$$

This  $\mathcal{N}$  has a structure of a left  $\mathcal{D}$ -module via the Leibnitz rule. Let  $u$  be a section of  $\mathcal{M}$ . Then the  $b$ -function for  $f$  and  $u$  (or for  $f^s u$ ) at  $p \in \mathbf{C}^n$  is the minimum degree monic polynomial  $0 \neq b(s) \in \mathbf{C}[s]$  such that

$$b(s)f^s \otimes u \in \mathcal{D}[s](f^{s+1} \otimes u) \quad (3)$$

holds in  $\mathcal{N}$  at  $p$  (i.e., as a germ of  $\mathcal{N}$  at  $p$ ). This  $b$ -function depends on the point  $p$ . As a function of  $p$ , there is a stratification of  $\mathbf{C}^n$  for which the  $b$ -function does not change on each strata (see e.g. [8] for an algorithmic proof of this fact). In the definitions (2) and (3) for  $b$ -function, if we replace  $\mathcal{O}$  by the polynomial ring  $\mathbf{k}[\mathbf{x}]$ ,  $\mathcal{D}$  by the Weyl algebra  $D$ , and  $\mathcal{M}$  by a holonomic  $D$ -module  $M = D/I$ , then we obtain the global  $b$ -function for  $f$  and  $u$ . It is the least common multiple of  $b$ -functions at every point.

**Theorem 3.1** *Let  $u$  be the residue class of 1 in  $\mathcal{D}/\mathcal{D}I$ , and let  $b(s)$  be the  $b$ -function for  $f$  and  $u$  at a point  $p \in \mathbf{C}^n$  where  $f(p) = 0$ . Assume that  $I$  admits an analytic solution of the form  $gf^r$  around  $p$ , where  $r \in \mathbf{C}$ ,  $g$  is a holomorphic function on a neighborhood of  $p$ , and  $g(p) \neq 0$ . Then  $s + r + 1$  divides  $b(s)$ .*

**Proof:** Let  $\mathcal{D}^{\text{an}}$  and  $\mathcal{O}^{\text{an}}$  be respectively the sheaf of analytic differential operators and the sheaf of holomorphic functions on  $\mathbf{C}^n$ . We may define the analytic  $b$ -function by replacing  $\mathcal{O}$  by  $\mathcal{O}^{\text{an}}$ ,  $\mathcal{D}$  by  $\mathcal{D}^{\text{an}}$ , and  $\mathcal{M}$  by a  $\mathcal{D}^{\text{an}}$ -module  $\mathcal{M}^{\text{an}}$  in the definitions (2) and (3). Since the  $b$ -function is an analytic invariant and the analytic and the algebraic  $b$ -functions coincide (see e.g. [8, §8]), we may work in the analytic category. We do this to consider solutions  $gf^r$  where  $g$  is holomorphic at  $p$ . If we only wish to consider solutions  $gf^r$  where  $g$  is a polynomial, then we may work in the algebraic category.

In general, given a map of left  $\mathcal{D}^{\text{an}}$ -modules  $\phi : \mathcal{M}_1^{\text{an}} \rightarrow \mathcal{M}_2^{\text{an}}$  and a section  $u$  of  $\mathcal{M}_1^{\text{an}}$ , the  $b$ -function for  $f^s u$  at a point  $p$  is divisible by the  $b$ -function for  $f^s \phi(u)$  at  $p$ . We apply this basic fact to the following map  $\varphi$ . Let  $J^{\text{an}}$  be the annihilating ideal of  $gf^r$  in  $\mathcal{D}^{\text{an}}$ . Since  $J^{\text{an}} \supseteq I^{\text{an}} := \mathcal{D}^{\text{an}}I$  and  $g(p) \neq 0$ , we have a left  $\mathcal{D}^{\text{an}}$ -homomorphism

$$\varphi : \mathcal{D}^{\text{an}}/I^{\text{an}} \longrightarrow \mathcal{D}^{\text{an}}gf^r = \mathcal{D}^{\text{an}}f^r \hookrightarrow \mathcal{O}^{\text{an}}[f^{-1}]f^r$$

which sends  $u$  to  $gf^r$ . This map extends to a left  $\mathcal{D}^{\text{an}}[s]$ -homomorphism

$$\begin{aligned} 1 \otimes \varphi : \mathcal{O}^{\text{an}}[f^{-1}, s]f^s \otimes_{\mathcal{O}^{\text{an}}} \mathcal{D}^{\text{an}}/I^{\text{an}} \\ \longrightarrow \mathcal{O}^{\text{an}}[f^{-1}, s]f^s \otimes_{\mathcal{O}^{\text{an}}} \mathcal{O}^{\text{an}}[f^{-1}]f^r = \mathcal{O}^{\text{an}}[f^{-1}, s]f^{s+r} \end{aligned}$$

which sends  $f^s \otimes u$  to  $gf^{s+r}$ . By the definition of  $b(s)$ , there exists a germ  $P(s)$  of  $\mathcal{D}[s]$  at  $p$  such that

$$(P(s)f - b(s))(f^s \otimes u) = 0.$$

Since  $1 \otimes \varphi$  is a left  $\mathcal{D}^{\text{an}}$ -homomorphism, applying it to the above equation gives the equation  $(P(s)f - b(s))(gf^{s+r}) = 0$ , or in other words,

$$g^{-1}P(s)gf^{s+r+1} = b(s)f^{s+r}.$$

Thus, we see that the Bernstein-Sato polynomial  $b_f(s)$  of  $f$  at  $p$  divides  $b(s-r)$ . Note that  $s+1$  divides  $b_f(s)$  since  $f(p) = 0$  (cf. [6]). In conclusion, we have proved that  $s+1$  divides  $b(s-r)$ . This completes the proof.  $\square$

By virtue of the above theorem, we can obtain upper bounds by computing the  $b$ -function for  $f^s u$  at a smooth point of each irreducible component of the singular locus of  $I$ . From now on, let us also take  $f \in \mathbf{k}[\mathbf{x}]$  to be a square-free polynomial defining the codimension one component of the singular locus, and let  $f = f_1 \cdots f_m$  be its irreducible decomposition in  $\mathbf{k}[\mathbf{x}]$ .

**Theorem 3.2** *Let  $b_i(s)$  be the  $b$ -function for  $f_i^s u$  at a generic point of  $f_i = 0$ . Denote by  $r_i$  the maximum integer root of  $b_i(s) = 0$ . Then any rational solution (if any) to  $I$  can be written in the form  $gf_1^{-r_1-1} \cdots f_m^{-r_m-1}$  with a polynomial  $g \in \mathbf{C}[\mathbf{x}]$ . If some  $b_i(s)$  has no integral root, then there exist no rational solutions to  $I$  other than zero.*

**Proof:** An arbitrary rational solution to  $M$  is written in the form  $gf_1^{-\nu_1} \cdots f_m^{-\nu_m}$  with integers  $\nu_1, \dots, \nu_m$  and  $g \in \mathbf{C}[\mathbf{x}]$ . Since the space of the rational solutions with coefficients in  $\mathbf{C}$  is spanned by those with coefficients in  $\mathbf{k}$ , we may assume  $g \in \mathbf{k}[\mathbf{x}]$ , and  $f$  and  $g$  are relatively prime in  $\mathbf{k}[\mathbf{x}]$ . Let  $p$  be a generic point of  $f_i = 0$ . We may assume that  $f_i$  is smooth at  $p$ ,  $g(p) \neq 0$ , and  $f_j(p) \neq 0$  for  $j \neq i$ . It follows from Theorem 3.1 that  $b_i(\nu_i - 1) = 0$ . This implies  $\nu_i \leq r_i + 1$ .  $\square$

Since  $b$ -functions divide the global  $b$ -function, an upper bound can also be obtained from the global  $b$ -function.

**Corollary 3.3** *Let  $b_i(s)$  be the global  $b$ -function for  $f_i^s u$ , and denote by  $r_i$  the maximum integer root of  $b_i(s) = 0$ . Then any rational solution (if any) to  $I$  can be written in the form  $gf_1^{-r_1-1} \cdots f_m^{-r_m-1}$  with a polynomial  $g \in \mathbf{C}[\mathbf{x}]$ .*

We mention the corollary since the algorithm to compute global  $b$ -functions is simpler than the algorithm to compute  $b$ -functions. However, the  $b$ -function offers finer information. For instance, the well-known example  $f = x^2 + y^2 + z^2 + w^2$  has Bernstein-Sato polynomial  $(s+1)(s+2)$  coming from the functional equation,  $\frac{1}{4}(\partial_x^2 + \partial_y^2 + \partial_z^2 + \partial_w^2) \cdot f^{s+1} = (s+1)(s+2)f^s$ . Now consider the module  $M = D \cdot f^{-1}$  and let  $u$  be the section of  $f^{-1}$ . The global  $b$ -function for  $f^s u$  is  $s(s+1)$ , and hence Corollary 3.3 implies that rational solutions of  $M$  all have the form  $gf^{-1}$  or  $gf^0$ , where  $g$  is a polynomial not divisible by  $f$ . On the other hand, the Bernstein-Sato polynomial of  $f$  at any nonsingular point  $p$  of  $f = 0$  (i.e. except for the origin) is  $s+1$ . It follows that the  $b$ -function for  $f^s u$  equals  $s$  at the generic point of  $f = 0$  and hence Theorem 3.2 implies that all rational solutions actually have the form  $gf^{-1}$ .

An algorithm to compute the  $b$ -function and the global  $b$ -function for  $f^s u$  was first given in [8] based upon tensor product computation, which is slow and memory intensive. Shortly thereafter, Walther introduced in [16] a more efficient method to compute the global  $b$ -function for  $f^s u$ . Both methods give the global  $b$ -function exactly, under the condition that  $I$  is  $f$ -saturated. Otherwise, we get a multiple of the global  $b$ -function. Similarly, the method of [8] gives the  $b$ -function exactly if  $I$  is  $f$ -saturated and additionally a certain primary decomposition in  $\mathbf{C}[\mathbf{x}]$  is known. If primary decomposition is only available in  $\mathbf{k}[\mathbf{x}]$ , we again get a multiple of the  $b$ -function.

Let us now describe an algorithm to compute the  $b$ -function for  $f^s u$  at a generic point of  $f = 0$  by combining the method of [16] and the primary decomposition as was used in [8].

**Algorithm 3.4** (Computing an upper bound of the  $b$ -function at a generic point)

INPUT: a finite set  $G_0$  of generators of a holonomic  $D$ -ideal  $I$  and an irreducible polynomial  $f \in \mathbf{k}[\mathbf{x}]$ .

OUTPUT:  $b'(s) \in \mathbf{k}[s]$ , which is a multiple of the  $b$ -function  $b(s)$  for  $f^s u$  at a generic point of  $f = 0$ , where  $u$  is the residue class of 1 in  $D/I$ .

1. Introducing a new variable  $t$ , put  $\vartheta_i = \partial_i + (\partial f / \partial x_i) \partial_t$ . Let  $\tilde{I}$  be the left ideal of  $D_{n+1}$ , the Weyl algebra on the variables  $x_1, \dots, x_n, t$ , that is generated by

$$\{P(\mathbf{x}, \vartheta_1, \dots, \vartheta_n) \mid P(\mathbf{x}, \partial_1, \dots, \partial_n) \in G_0\} \cup \{t - f(\mathbf{x})\}.$$

2. Let  $G_1$  be a finite set of generators of the left ideal  $\text{in}_{(-1, 0, \dots, 0; 1, 0, \dots, 0)}(\tilde{I})$  of  $D_{n+1}$ . Here,  $-1$  is the weight for  $t$  and  $1$  is the weight for  $\partial_t$ .
3. Rewrite each element  $P$  of  $G_1$  in the form

$$P = \partial_t^\mu P'(t \partial_t, \mathbf{x}, \partial_1, \dots, \partial_n) \quad \text{or} \quad P = t^\mu P'(t \partial_t, \mathbf{x}, \partial_1, \dots, \partial_n)$$

with a non-negative integer  $\mu$ , and define  $\psi(P)$  by,

$$\psi(P) := t^\mu \partial_t^\mu P' = t \partial_t \cdots (t \partial_t - \mu + 1) P'(t \partial_t, \mathbf{x}, \partial_1, \dots, \partial_n)$$

$$\text{or} \quad \psi(P) := \partial_t^\mu t^\mu P' = (t \partial_t + 1) \cdots (t \partial_t + \mu) P'(t \partial_t, \mathbf{x}, \partial_1, \dots, \partial_n).$$

Put

$$G_2 := \{\psi(P)(-s - 1, \mathbf{x}, \partial_1, \dots, \partial_n) \mid P \in G_1\}.$$

4. Compute the elimination ideal  $J := \mathbf{k}[s, \mathbf{x}] \cap D[s]G_2$ . (The global  $b$ -function can be obtained at this stage by computing the monic generator of the ideal  $J \cap \mathbf{k}[s]$ .)
5. Compute a primary decomposition of  $J$  in  $\mathbf{k}[s, \mathbf{x}]$  as

$$J = Q_1 \cap \cdots \cap Q_\nu.$$

6. For each  $i = 1, \dots, \nu$ , compute  $Q_{ix} := Q_i \cap \mathbf{k}[\mathbf{x}]$ , which is a primary ideal of  $\mathbf{k}[\mathbf{x}]$ .
7. Let  $b'(s)$  be the monic generator of the ideal

$$\bigcap \{Q_i \cap \mathbf{k}[s] \mid \sqrt{Q_{ix}} \subset \mathbf{k}[\mathbf{x}]f\}$$

of  $\mathbf{k}[s]$ . (Note that  $\sqrt{Q_{ix}} \subset \mathbf{k}[\mathbf{x}]f$  implies that  $\sqrt{Q_{ix}}$  equals  $\mathbf{k}[\mathbf{x}]f$  or  $\{0\}$ .)

**Theorem 3.5** *In the above algorithm, the polynomial  $b'(s)$  is precisely the  $b$ -function for  $f^s u$  at a generic point of  $f = 0$  if  $I$  is  $f$ -saturated (i.e.,  $I : f^\infty = I$ ) and each  $\mathbf{C}[s, \mathbf{x}]Q_i$  remains primary in  $\mathbf{C}[s, \mathbf{x}]$ . Otherwise, the polynomial  $b'(s)$  is a multiple of the  $b$ -function for  $f^s u$  at a generic point of  $f = 0$ .*

**Proof:** Using essentially the same method as the proof of Lemma 4.1 in [16], we can prove that  $\widetilde{I}$  is precisely the annihilator ideal for  $\delta(t - f(\mathbf{x})) \otimes u$  in

$$\widetilde{M} := (D_{n+1}\delta(t - f(\mathbf{x}))) \otimes_{\mathbf{C}[\mathbf{x}]} D/I,$$

where  $\delta(t - f(\mathbf{x}))$  denotes the residue class of  $(t - f(\mathbf{x}))^{-1}$  in  $\mathbf{k}[\mathbf{x}, (t - f(\mathbf{x}))^{-1}]/\mathbf{k}[\mathbf{x}]$ . Let  $b_t(s)$  be the indicial polynomial for  $\delta(t - f(\mathbf{x})) \otimes u$  along  $t = 0$  at a point  $(0, p)$  with  $f(p) = 0$ . Then by Theorem 6.14 of [8], the  $b$ -function  $b(s)$  for  $f^s u$  at  $p$  divides out, and if  $I$  is  $f$ -saturated, coincides with  $b_t(-s - 1)$ .

It follows from the definition that  $b_t(-s - 1)$  is a generator of the ideal  $\mathcal{O}_p[s]J \cap \mathbf{C}[s]$  of  $\mathbf{C}[s]$ , where  $\mathcal{O}_p$  denotes the stalk of  $\mathcal{O}$  at  $p$ . If  $\mathbf{C}[s, \mathbf{x}]Q_i$  are primary in  $\mathbf{C}[s, \mathbf{x}]$ ,  $b'(s)$  generates the above ideal in view of Theorem 4.7 of [8] (cf. also Lemma 4.4 of [9]). In general, although  $Q_i$  is primary in  $\mathbf{k}[s, \mathbf{x}]$ , the extension  $\mathbf{C}[s, \mathbf{x}]Q_i$  is no longer primary in  $\mathbf{C}[s, \mathbf{x}]$  and admits a primary decomposition

$$\mathbf{C}[s, \mathbf{x}]Q_i = Q_{i1} \cap \dots \cap Q_{i\mu_i}.$$

In this case,  $b'(s)$  is the least common multiple of the generators of the ideals  $\mathcal{O}_p[s]Q_{ij} \cap \mathbf{C}[s]$  for  $j = 1, \dots, \mu_i$ , while  $b_t(-s - 1)$  is the generator of  $\mathcal{O}_p[s]Q_{ij} \cap \mathbf{C}[s]$  for some  $j$  (such that  $p$  belongs to the zero set of  $Q_{ij} \cap \mathbf{C}[\mathbf{x}]$  which is also the zero set of a factor of  $f$ ). This completes the proof.  $\square$

**Remark 3.6** In the notation in the above proof, the linear factors of  $b'(s)$  and those of  $b_t(-s - 1)$  coincide. In particular the set of integer roots of  $b'(s) = 0$  is the same as that of  $b_t(-s - 1) = 0$ . In fact, this follows from the fact that the linear factors of  $b'(s)$  in  $\mathbf{k}[s]$  are invariant under the action of the Galois group of  $\overline{\mathbf{k}}$  over  $\mathbf{k}$ .

**Remark 3.7**  $I$  is  $f$ -saturated if and only if the  $-1$ -th cohomology group of the restriction of  $\widetilde{M}$  in the proof of Proposition 3.5 to  $t = 0$  vanishes (see Theorem 6.4 and Proposition 6.13 of [8]), which is computable by Algorithm 5.10 of [8] or by Algorithm 5.4 of [9].



**Remark 3.8** The  $f$ -saturation of  $I$ , which is the ideal  $D[\frac{1}{f}] \cdot I \cap D$ , may be computed by using the localization algorithm of [10] (if  $I$  is specializable along  $f$ ) or by using the less efficient algorithm of [14] (if  $I$  is general). By replacing  $I$  with its  $f$ -saturation, we may then compute the local  $b$ -function exactly. However, since saturation is often an expensive algorithm, we avoid making this replacement in practice.

Once we have determined the integers  $r_1, \dots, r_m$  of Theorem 3.2, we can use Gröbner deformations to obtain the rational solutions. Put  $k_i = r_i + 1$ . Then by virtue of Theorem 3.2, we have only to determine rational solutions of the form  $gf_1^{-k_1} \dots f_m^{-k_m}$  for some polynomial  $g$ . This amounts to computing polynomial solutions of some twisted ideal  $I_{(k_1, \dots, k_m)}$  of  $I$ . Namely, consider how  $f\partial_i$  acts on an element  $gf_1^{-k_1} \dots f_m^{-k_m}$ :

$$f\partial_i \bullet (gf_1^{-k_1} \dots f_m^{-k_m}) = \left( f \frac{\partial g}{\partial x_i} - \sum_{j=1}^m k_j \frac{f}{f_j} \frac{\partial f_j}{\partial x_i} g \right) f_1^{-k_1} \dots f_m^{-k_m}.$$

In other words,  $f\partial_i$  acts on the numerator  $g$  as the differential operator

$$L_i = f\partial_i - \sum_{j=1}^m k_j \frac{f}{f_j} \frac{\partial f_j}{\partial x_i}. \quad (4)$$

Thus, if we multiply a set of generators  $\{g_1, \dots, g_m\}$  of  $I$  by sufficiently high powers  $f^{m_i}$  such that  $f^{m_i}g_i \in \mathbf{k}\langle x_1, \dots, x_n, f\partial_1, \dots, f\partial_n \rangle$ , and if  $I'$  is the ideal obtained from  $\{f^{m_1}g_1, \dots, f^{m_r}g_r\}$  by substituting  $L_i$  for  $f\partial_i$ , then the rational solutions of  $I$  have the polynomial solutions of  $I'$  as numerators. Thus, it only remains to compute the polynomial solutions of  $I'$ . However, since  $I'$  might not be holonomic, we cannot apply Algorithm 2.4 just yet.

Hence let us define  $I_{(k_1, \dots, k_m)} := \mathbf{k}(\mathbf{x})\langle \partial \rangle \cdot I' \cap D$ , which is the Weyl closure of  $I'$  and whose polynomial solutions are the same as  $I$ . The advantage of  $I_{(k_1, \dots, k_m)}$  is that it is indeed holonomic, which follows from a theorem of Kashiwara. Namely, note that since  $I$  is of finite rank,  $I'$  remains of finite rank because an element in  $I$  of the form  $(g_i(\mathbf{x})\partial_i^{N_i} + \text{lower order elements})$  will be sent to an element in  $I'$  of the form  $(f^{M_i}g_i(\mathbf{x})\partial_i^{N_i} + \text{lower order elements})$ . Now let  $h(\mathbf{x})$  be any polynomial vanishing on the singular locus of  $I'$ . Then the non-holonomic locus of  $I'$  is contained inside the zero set of  $h(\mathbf{x})$  regarded as a function on the cotangent bundle, and a theorem due to Kashiwara [7] states that the ideal  $D[h^{-1}] \cdot I' \cap D$  is holonomic. Furthermore, an argument in [14] shows that the Weyl closure of  $I'$  also equals  $D[h^{-1}] \cdot I' \cap D$ . Summing up, we arrive at the following algorithm.

**Algorithm 3.9** (Computing the rational solutions of a holonomic ideal)

INPUT: generators of a holonomic  $D$ -ideal  $I$ .

OUTPUT: A basis of the rational solutions  $h \in \mathbf{k}(\mathbf{x})$  of  $I \bullet h = 0$ .

1. Compute a polynomial  $f$  defining the codimension 1 component of  $\text{Sing}(I)$ .

2. Compute the irreducible decomposition  $f = f_1 \cdots f_m$  in  $\mathbf{k}[\mathbf{x}]$ .
3. For each  $i = 1, \dots, m$ , compute the output  $b'(s)$  of Algorithm 3.4 with  $I$  and  $f_i$  as input. Let  $r_i$  be the maximum integer root of  $b'(s) = 0$  and put  $k_i = r_i + 1$ . If  $b'(s)$  has no integral root for some  $i$ , then there exists no rational solution other than zero.
4. Compute the twisted ideal  $I_{(k_1, \dots, k_m)}$  as follows. First, form the ideal  $I'$  described in the paragraphs preceding the algorithm. Second, compute any polynomial  $h(\mathbf{x})$  vanishing on the singular locus of  $I'$ . Third, compute the localization  $(D/I')[h^{-1}]$  using the algorithm in [10]. Then the ideal  $I_{(k_1, \dots, k_m)}$  is the kernel of  $D \rightarrow D/I' \rightarrow (D/I')[h^{-1}]$ .
5. Compute a basis  $\{g_1, \dots, g_k\}$  of the polynomial solutions of  $I_{(k_1, \dots, k_m)}$  using Algorithm 2.4.
6. Output:  $\{g_1 f_1^{-k_1} \cdots f_m^{-k_m}, \dots, g_k f_1^{-k_1} \cdots f_m^{-k_m}\}$ , a basis of the rational solutions of  $I$ .

**Example 3.10** Let  $I$  be the left ideal generated by

$$L_1 = \theta_x(\theta_x + \theta_y) - x(\theta_x + \theta_y + 3)(\theta_x - 1)$$

and

$$L_2 = \theta_y(\theta_x + \theta_y) - y(\theta_x + \theta_y + 3)(\theta_y + 1).$$

The Appell function  $F_1(3, -1, 1, 1; x, y)$  is a solution of this system. The singular locus of  $I$  is  $xy(x - y)(1 - x)(1 - y) = 0$ . We can compute the local indicial polynomial of  $u$ , the modulo class of 1 in  $D_2/I$ , along  $x = 0$  directly by the algorithm of [8, Section 4]: It is  $s(s - 1)$  on  $\{(0, y) \mid y \neq 0\}$ , and  $s(s - 1)^2$  at  $(0, 0)$ . In the same way, the indicial polynomial of  $u$  along  $y = 0$  is  $s(s + 1)$  on  $\{(x, 0) \mid x \neq 0\}$ , and  $s(s + 1)(s - 1)$  at  $(0, 0)$ .

Now let us compute the  $b$ -function for  $(1 - y)^s u$ . The local indicial polynomial of  $\delta(t + y - 1) \otimes u$  along  $t = 0$  is  $s(s + 3)$  at any point of  $t = 0$ . Hence the  $b$ -function for  $(1 - y)^s u$  divides  $(s + 1)(s - 2)$ . In the same way, the local indicial polynomial of  $\delta(t + x - 1) \otimes u$  along  $t = 0$  is  $s(s + 1)$  at any point of  $t = 0$ . Finally, the indicial polynomial of  $\delta(t - x + y) \otimes u$  is  $s(s - 1)$  on  $\{(x, x) \mid x \neq 0\}$ , and  $s(s - 1)(s - 2)$  at  $(0, 0)$ .

Therefore, we conclude that any rational solution to  $I$ , if it exists, can be written in the form  $g(x, y)y^{-1}(1 - x)^{-1}(1 - y)^{-3}$  with a polynomial  $g$ . Now we may compute the twisted ideal  $I_{(0, 1, 0, 1, 3)}$ , where  $f_1 = x, f_2 = y, f_3 = x - y, f_4 = x - 1, f_5 = y - 1$ , and  $f$  is the product. Multiplying by  $f^2$ , we get the expressions,

$$\begin{aligned} f^2 L_1 &= (x^2 - x^3)(f \partial_x)^2 + x((1 - 3x)f - (1 - x)y \frac{\partial f}{\partial y} - (1 - x)x \frac{\partial f}{\partial x})(f \partial_x) + \\ &\quad x(1 - x)y(f \partial_y)(f \partial_x) + xyf(f \partial_y) + 3xf^2 \\ f^2 L_2 &= (y^2 - y^3)(f \partial_y)^2 + y((1 - 5y)f - (1 - y)x \frac{\partial f}{\partial x} - (1 - y)y \frac{\partial f}{\partial y})(f \partial_y) + \\ &\quad y(1 - y)x(f \partial_x)(f \partial_y) - yxf(f \partial_x) - 3yf^2, \end{aligned}$$

and we set  $T_1$  and  $T_2$  to be the operators obtained from the substitution (4). We remark that the ideal generated by  $T_1$  and  $T_2$  is neither holonomic nor specializable with respect to the weight vector  $(1, 1, -1, -1)$ , so it is difficult to apply Gröbner deformations to it just yet.

The twisted ideal  $I_{(0,1,0,1,3)}$  is the Weyl closure of the ideal generated by  $T_1$  and  $T_2$ . At the moment, this Weyl closure is computationally too intensive to compute. However, we are able to compute a partial closure by noting that  $T_1$  is divisible by  $g = f_1^3 f_2 f_3^2 f_4 f_5$  and  $T_2$  is divisible by  $h = f_1^2 f_2^2 f_3^2 f_4 f_5$ . Now the ideal  $J$  generated by

$$\begin{aligned} \frac{1}{g}T_1 &= (-x^3y + x^3 + 2x^2y - 2x^2 - xy + x)\partial_x^2 + \\ &\quad (-x^2y^2 + x^2y + 2xy^2 - 2xy - y^2 + y)\partial_x\partial_y + \\ &\quad (3x^2y - 6xy + 3y)\partial_x + (2xy^2 - 2xy - 2y^2 + 2y)\partial_y + \\ &\quad (-4xy - 2x + 4y + 2) \\ \frac{1}{h}T_2 &= (-xy^4 + 2xy^3 + y^4 - xy^2 - 2y^3 + y^2)\partial_y^2 + \\ &\quad (-x^2y^3 + 2x^2y^2 + xy^3 - x^2y - 2xy^2 + xy)\partial_x\partial_y + \\ &\quad (3x^2y^2 - 4x^2y - 3xy^2 + x^2 + 4xy - x)\partial_x + \\ &\quad (4xy^3 - 6xy^2 - 3y^3 + 2xy + 4y^2 - y)\partial_y + \\ &\quad (-6xy^2 + 8xy + 3y^2 - 2x - 4y + 1), \end{aligned}$$

is indeed holonomic, hence we may apply Algorithm 2.4. We find that the  $b$ -function with respect to the weight  $w = (-1, -1)$  is  $(s+5)(s+2)^2(s+1)s(s-2)(s-3)^2$ , which implies that a polynomial solution to  $J$  must have degree less than or equal to 5. We find that  $J$  has 2 polynomial solutions, so that  $I$  has 2 rational solutions,  $(xy^2 - 3xy + 3x - 1)/(y-1)^3$  and  $(x-y)/y(y-1)^3$ .

## 4 Solutions by duality

For holonomic  $M$  and  $N$ , it is well known [7] that

$$\mathrm{Ext}_D^i(M, N) \simeq H^{i-n}(\Omega \otimes_D^L (\mathbf{D}(M) \otimes_{\mathbf{k}[x]}^L N)), \quad (5)$$

where  $\Omega := (D/\{x_1, \dots, x_n\} \cdot D)$ , and  $\mathbf{D}(M)$  is the holonomic dual,

$$\mathbf{D}(M) := \mathrm{Hom}_{\mathbf{k}[x]}(\Omega, \mathrm{Ext}_D^n(M, D)).$$

The spaces  $\mathrm{Ext}_D^i(M, N)$  are finite-dimensional  $\mathbf{k}$ -vector spaces and correspond to the solutions of  $M$  in  $N$  when  $i = 0$ . For example, if  $N = \mathbf{k}[\mathbf{x}]$ , then we obtain the polynomial solutions of  $M$ , whereas if  $N = D/D \cdot \{x_1, \dots, x_n\}$ , then we obtain the delta function solutions of  $M$  with support at the origin.

In this section, we explain how (5) can be used to compute the dimensions of  $\mathrm{Ext}_D^i(M, N)$ . We first discuss how to compute the holonomic dual, next discuss the special cases  $N = \mathbf{C}[\mathbf{x}]$  and  $N = \mathbf{C}[\mathbf{x}][\frac{1}{\tau}]$ , and last discuss the general case of holonomic  $N$ . A method to extend these algorithms to compute an explicit basis of  $\mathrm{Hom}_D(M, N)$  and  $\mathrm{Ext}_D^i(M, N)$  is the subject of the forthcoming paper [15].

**Notation:** Let us explain the notation we will use to write maps of left or right  $D$ -modules. As usual, maps between finitely generated modules will be

represented by matrices, but some care has to be given to the order in which elements are multiplied due to the noncommutativity of  $D$ .

Given an  $r \times s$  matrix  $A = [a_{ij}]$  with entries in  $D$ , we get a map of free left  $D$ -modules,

$$D^r \xrightarrow{\cdot A} D^s \quad [g_1, \dots, g_r] \mapsto [g_1, \dots, g_r] \cdot A,$$

where  $D^r$  and  $D^s$  are regarded as modules of row vectors, and the map is matrix multiplication. Under this convention, the composition of maps  $D^r \xrightarrow{\cdot A} D^s$  and  $D^s \xrightarrow{\cdot B} D^t$  is the map  $D^r \xrightarrow{\cdot AB} D^t$  where  $AB$  is usual matrix multiplication. In general, suppose  $M$  and  $N$  are left  $D$ -modules with presentations  $D^r/M_0$  and  $D^s/N_0$ . Then the matrix  $A$  induces a left  $D$ -module map between  $M$  and  $N$ , denoted  $(D^r/M_0) \xrightarrow{\cdot A} (D^s/N_0)$ , precisely when  $\vec{g} \cdot A \in N_0$  for all row vectors  $\vec{g} \in M_0$ . Conversely, any map of left  $D$ -modules between  $M$  and  $N$  can be represented by some matrix  $A$  in the manner above.

Now let us discuss maps of right  $D$ -modules. The  $r \times s$  matrix  $A$  also defines a map of right  $D$ -modules in the opposite direction,

$$(D^s)^T \xrightarrow{\cdot A} (D^r)^T \quad [h_1, \dots, h_s]^T \mapsto A \cdot [h_1, \dots, h_s]^T,$$

where the superscript- $T$  means to regard the free modules  $(D^s)^T$  and  $(D^r)^T$  as consisting of column vectors. This map is equivalent to the map obtained by applying  $\text{Hom}_D(-, D)$  to  $D^r \xrightarrow{\cdot A} D^s$ , thus  $(D^s)^T$  may be regarded as the dual module  $\text{Hom}_D(D^s, D)$ . We will suppress the superscript- $T$  when the context is clear. As before, the matrix  $A$  induces a right  $D$ -module map between right  $D$ -modules  $N' = (D^s)^T/N'_0$  and  $M' = (D^r)^T/M'_0$  when  $A \cdot \vec{g} \in M'_0$  for all column vectors  $\vec{g} \in N'_0$ . We denote the map by  $(D^s)^T/N'_0 \xrightarrow{\cdot A} (D^r)^T/M'_0$ .

**Left-right correspondence and  $\Omega$ :** As is well-known, a standard use for  $\Omega$  is to establish a correspondence between the categories of left and right  $D$ -modules. The correspondence can be expressed through the adjoint operator  $\tau$ , which is the algebra involution

$$\tau : D \longrightarrow D \quad x^\alpha \partial^\beta \mapsto (-\partial)^\beta x^\alpha.$$

Namely, given a left  $D$ -module  $M \simeq D^r/M_0$ , the corresponding right  $D$ -module is  $\Omega \otimes_{\mathbf{k}[\mathbf{x}]} M \simeq D^r/\tau(M_0)$ . Conversely, given a right  $D$ -module  $N \simeq D^s/N_0$ , the corresponding left  $D$ -module is  $\text{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N) \simeq D^s/\tau(N_0)$ . Similarly, given a homomorphism of left  $D$ -modules  $\phi : (D^r/M_0) \rightarrow (D^s/N_0)$  defined by left multiplication by the  $r \times s$  matrix  $A = [a_{ij}]$ , the corresponding homomorphism of right  $D$ -modules  $\tau(\phi) : (D^r/\tau(M_0)) \rightarrow (D^s/\tau(N_0))$  is defined by right multiplication by the  $s \times r$  matrix  $\tau(A) := [\tau(a_{ij})]^T$ .

Let us explain details of the above correspondence for the non-specialist. Given a left  $D$ -module  $M$ , there is a corresponding right  $D$ -module  $\Omega \otimes_{\mathbf{k}[\mathbf{x}]} M$  where the structure is given by extending the actions,

$$(w \otimes m)f = wf \otimes m \quad (w \otimes m)\xi = w\xi \otimes m - w \otimes \xi m$$

for  $f \in \mathbf{k}[\mathbf{x}]$  and  $\xi \in \text{Der}(\mathbf{k}[\mathbf{x}])$ . Given a presentation  $D^r/M_0$  for  $M$  with generators denoted  $\{e_i\}_{i=1}^r$ , then in  $\Omega \otimes_{\mathbf{k}[\mathbf{x}]} M$  we have

$$(1 \otimes e_i)x^\alpha \partial^\beta = (1 \otimes x^\alpha e_i)\partial^\beta = 1 \otimes (-\partial)^\beta x^\alpha e_i = 1 \otimes \tau(x^\alpha \partial^\beta)e_i.$$

It follows that  $\Omega \otimes_{\mathbf{k}[\mathbf{x}]} M$  is generated by  $\{1 \otimes e_i\}_{i=1}^r$  and gets the presentation  $D_n^r/\tau(M_0)$ .

Conversely, given a right  $D$ -module  $N$ , there is a corresponding left  $D$ -module  $\text{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N)$  where the structure is given by extending the action,

$$(f\varphi)(w) = \varphi(w)f \quad (\xi\varphi)(w) = \varphi(w\xi) - \varphi(w)\xi$$

for  $\varphi \in \text{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N)$ ,  $w \in \Omega$ ,  $f \in \mathbf{k}[\mathbf{x}]$ , and  $\xi \in \text{Der}(\mathbf{k}[\mathbf{x}])$ . A morphism  $\varphi \in \text{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N)$  can be identified with its image  $\varphi(1) \in N$ . Since

$$\begin{aligned} (x^\alpha \partial^\beta \varphi)(1) &= (x^\alpha (\partial^\beta \varphi))(1) = (\partial^\beta \varphi)(1)x^\alpha = \varphi(1)(-\partial)^\beta x^\alpha \\ &= \varphi(1)\tau(x^\alpha \partial^\beta), \end{aligned}$$

the morphism  $x^\alpha \partial^\beta \varphi$  gets identified with  $\varphi(1)\tau(x^\alpha \partial^\beta)$ . In particular, given a presentation  $D^s/N_0$  of  $N$ , then  $\text{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N)$  is generated as a left  $D$ -module by the morphisms  $\{\varphi_i\}_{i=1}^s$  such that  $\varphi_i(1) = e_i$ . By the computation above, a relation  $\sum_i e_i g_i = 0$  in  $N$  corresponds to a relation  $\sum_i \tau(g_i)\varphi_i$  in  $\text{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N)$  because  $(\sum_i \tau(g_i)\varphi_i)(1) = \sum_i e_i \tau(\tau(g_i)) = \sum_i e_i g_i$ . It follows that  $\text{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N)$  is generated by  $\{\varphi_i\}_{i=1}^s$  and gets the presentation

$$\text{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, N) \simeq D_n^s/\tau(N_0). \quad (6)$$

## 4.1 Holonomic dual

Let us discuss how  $\mathbf{D}(M)$  can be computed.

**Algorithm 4.1** [Computing the holonomic dual]

INPUT:  $D^{r_0}/D \cdot \{\vec{g}_1, \dots, \vec{g}_{r_1}\}$ , a presentation of a holonomic left  $D$ -module  $M$ .

OUTPUT: The holonomic dual  $\mathbf{D}(M)$ .

1. Compute the first  $n+1$  steps of any free resolution of  $M$ . Let the  $n$ -th part of the resolution be  $D^p \xrightarrow{\cdot P} D^q \xrightarrow{\cdot Q} D^r$ .
2. Dualize and apply the adjoint operator (recall if  $P = [p_{ij}]$ , then  $\tau(P) = [\tau(p_{ij})]^T$ ) to get  $D^p \xleftarrow{\cdot \tau(P)} D^q \xleftarrow{\cdot \tau(Q)} D^r$ .
3. Return  $\ker(\cdot \tau(P))/\text{Im}(\cdot \tau(Q))$ .

**Proof:** Let the first  $n+1$  steps of a free resolution of  $M$  be denoted,

$$F^\bullet : D^{r_{n+1}} \xrightarrow{\cdot P} D^{r_n} \xrightarrow{\cdot Q} D^{r_{n-1}} \rightarrow \dots \rightarrow D^{r_0} \rightarrow 0.$$

Applying  $\text{Hom}_D(D, -)$  yields a complex of right  $D$ -modules,

$$\text{Hom}_D(D, F^\bullet) : (D^{r_{n+1}})^T \xleftarrow{P^\cdot} (D^{r_n})^T \xleftarrow{Q^\cdot} (D^{r_{n-1}})^T \leftarrow \dots \leftarrow (D^{r_0})^T \leftarrow 0,$$

and by definition,

$$\text{Ext}_D^n(M, D) \simeq \frac{\ker(D^{r_{n+1}} \xleftarrow{P^\cdot} D^{r_n})}{\text{Im}(D^{r_n} \xleftarrow{Q^\cdot} D^{r_{n-1}})}.$$

Since  $\mathbf{D}(M) = \text{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, \text{Ext}_D^n(M, D))$ , it only remains to determine the effect of applying  $\text{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, -)$ . Using the equation (6), if  $\{\vec{L}_1, \dots, \vec{L}_k\}$  are generators of  $K = \ker(D^{r_{n+1}} \xleftarrow{P^\cdot} D^{r_n})$ , and  $\sum_i \vec{L}_i g_i \in I = \text{Im}(D^{r_n} \xleftarrow{Q^\cdot} D^{r_{n-1}})$  is a relation, then the corresponding relation  $\sum_i \tau(g_i) \varphi_i$  in  $\text{Hom}_{\mathbf{k}[\mathbf{x}]}(\Omega, \text{Ext}_D^n(M, D))$  can be realized as the relation  $\sum_i \tau(\vec{L}_i g_i) = \tau(g_i) \tau(\vec{L}_i) \in \tau(I)$ . It follows that

$$\mathbf{D}(M) \simeq \frac{\ker(D^{r_{n+1}} \xleftarrow{\tau(P)} D^{r_n})}{\text{Im}(D^{r_n} \xleftarrow{\tau(Q)} D^{r_{n-1}})},$$

which is the output of step 3.  $\square$

**Example 4.2** The Appell differential equation  $F_1(2, -3, -2, 5)$  of Example 2.5 has the resolution  $0 \rightarrow D^1 \xrightarrow{Q_1} D^2 \xrightarrow{Q_0} D^1 \rightarrow 0$ , where

$$Q_0 = \begin{bmatrix} (\theta_x - 3)\partial_y - (\theta_y - 2)\partial_x \\ (y^2 - y)(\partial_x \partial_y + \partial_y^2) - 2(y + x)\partial_x + 4y\partial_y + 2\partial_x - 8\partial_y - 4 \end{bmatrix}^T$$

$$Q_1 = \begin{bmatrix} (y^2 - y)(\partial_x \partial_y + \partial_y^2) - 2x\partial_x + 6y\partial_y + \partial_x - 9\partial_y \\ -(\theta_x - 3)\partial_y + (\theta_y - 1)\partial_x \end{bmatrix}$$

The holonomic dual  $\mathbf{D}(F_1(2, -3, -2, 5))$  is the cokernel of  $\tau(Q_1)$  and is the Appell differential equation  $F_1(-1, 4, 2, -3)$ .

## 4.2 Polynomial and rational solutions by duality

When  $N = \mathbf{k}[\mathbf{x}]$ , the isomorphism (5) specializes to

$$\text{Ext}_D^i(M, \mathbf{k}[\mathbf{x}]) \simeq H^{n-i}(\Omega \otimes_D^L \mathbf{D}(M)). \quad (7)$$

The right hand side is equivalently the  $(n - i)$ -th integration of  $\mathbf{D}(M)$  to the origin. An algorithm to compute integration is given in [9]. Using it, we can evaluate the dimensions of  $\text{Ext}_D^i(M, \mathbf{k}[\mathbf{x}])$  and in particular  $\text{Hom}_D(M, \mathbf{k}[\mathbf{x}])$ .

**Algorithm 4.3** [Evaluating dimensions of polynomial solution spaces]

INPUT: a holonomic left  $D$ -module  $M$ .

OUTPUT: dimensions of  $\text{Ext}_D^i(M, \mathbf{k}[\mathbf{x}])$ .

1. Compute the dual  $\mathbf{D}(M)$  using Algorithm 4.1
2. Compute the integrations of  $\mathbf{D}(M)$  to the origin using the algorithm in [9]. They are finite dimensional vector spaces.
3. Return the dimensions.

The dimensions of rational solution spaces can be evaluated in a similar way. When  $N = \mathbf{k}[\mathbf{x}][\frac{1}{f}]$ , the isomorphism (5) specializes to

$$\mathrm{Ext}_D^i(M, \mathbf{k}[\mathbf{x}][1/f]) \simeq H^{n-i}(\Omega \otimes_D^L \mathbf{D}(M)[1/f]). \quad (8)$$

The right hand side is now equivalently the  $(n-i)$ -th integration of  $\mathbf{D}(M)[\frac{1}{f}]$  to the origin. An algorithm to compute localization is given in [10]. Using it and the integration algorithm, we can evaluate the dimensions of  $\mathrm{Ext}_D^i(M, \mathbf{k}[\mathbf{x}][\frac{1}{f}])$  and  $\mathrm{Hom}_D(M, \mathbf{k}[\mathbf{x}][\frac{1}{f}])$ . To get the dimension of all rational solutions, take  $f$  to be any polynomial vanishing on the singular locus.

We summarize how to compute the integration of a module  $N$  to the origin according to [9] in a slightly more general way. The generalization sometimes gives a more efficient strategy than [9]. We need to recall some definitions. To any strictly positive  $w \in \mathbf{Z}_{>0}^n$ , we get an integration filtration  $F_w$  of  $D$  defined by  $F_w^i(D) = \mathrm{Span}_k\{x^\alpha \partial^\beta | w \cdot \alpha - w \cdot \beta \leq i\}$ . More generally, for  $\vec{m} \in \mathbf{Z}^r$ , we also get a shifted filtration  $F_w[\vec{m}]$  of the free module  $D^r$  defined by  $F_w^i[\vec{m}](D^r) = \mathrm{Span}_k\{x^\alpha \partial^\beta e_j | w \cdot \alpha - w \cdot \beta - m_j \leq i\}$ . We will often write  $D^r[\vec{m}]$  for the free module  $D^r$  equipped with the shifted filtration  $F_w[\vec{m}]$  when the context is clear. The filtrations  $F_w[\vec{m}]$  induce filtrations on subquotients of  $D^r$  in the natural way. Now we may say the steps of the integration algorithm. First, compute a  $(w, -w)$ -strict free resolution  $G^\bullet$  of  $N$  of length  $n+1$ . This is a resolution of  $N$  by free modules  $D^{r_j}[\vec{m}_j]$  with the property that the differentials preserve the filtration and moreover induce a resolution on the associated graded level. Second, compute the integration  $b$ -function of  $N$  with respect to  $(w, -w)$ , and find its minimal and maximal integral roots  $k_0$  and  $k_1$ . The integration  $b$ -function is the monic polynomial  $b(s)$  of least degree satisfying  $b(\sum_i w_i \partial_i x_i) \cdot F^0(N) \subset F^{-1}(N)$ . Third, compute the cohomology of the complex  $F_{-k_0}(\Omega \otimes_D G^\bullet)/F_{-k_1-1}(\Omega \otimes_D G^\bullet)$ , which is a complex of finite-dimensional vector spaces. The dimensions of the cohomology groups are equal to the dimensions of the integration modules of  $N$ .

**Example 4.4** Let us evaluate the dimension of polynomial solutions to the Appell differential equation  $M = F_1(2, -3, -2, 5)$  of Example 2.5. Choose the weight vector  $w = (1, 2)$ . The resolution of Example 4.2, after dualizing, applying the adjoint operator, and shifting,

$$0 \rightarrow D^1[0] \xrightarrow{\cdot \tau(Q_0)} D^2[-1, 1] \xrightarrow{\cdot \tau(Q_1)} D^1[0] \rightarrow 0,$$

preserves filtrations but does not induce a resolution on the associated graded level. On the other hand, if we adjust the resolution to

$$G^\bullet : 0 \rightarrow D^1[1] \xrightarrow{\cdot P_0} D^2[0, 1] \xrightarrow{\cdot P_1} D^1[0] \rightarrow 0,$$

where

$$P_0 = \begin{bmatrix} -(\theta_x + 5)\partial_y + (\theta_y + 2)\partial_x \\ (x^2 - x)(\partial_x^2 + \partial_x\partial_y) + 4x\partial_x + 2(3x + 2y)\partial_y + 4\partial_x - 5\partial_y - 2 \end{bmatrix}^T$$

$$P_1 = \begin{bmatrix} (x^2 - x)(\partial_x^2 + \partial_x\partial_y) + 2x\partial_x + 4(x + y)\partial_y + 5\partial_x - 4\partial_y - 4 \\ (\theta_x + 4)\partial_y - (\theta_y + 2)\partial_x \end{bmatrix},$$

then we do obtain a  $(w, -w)$ -strict resolution of  $\mathbf{D}(M) = F_1(-1, 4, 2, -3)$ .

The integration  $b$ -function with respect to  $(w, -w)$  is  $(s + 5)(s - 2)(s - 5)$ , hence the integration complex for  $\mathbf{D}(M)$  is quasi-isomorphic to the truncated complex  $F_w^5(\Omega \otimes_D G^\bullet)/F_w^{-6}(\Omega \otimes_D G^\bullet)$ , which is a complex of finite-dimensional vector spaces with dimensions,

$$0 \rightarrow \mathbf{Q}^{16} \xrightarrow{\cdot P_0} \mathbf{Q}^{28} \xrightarrow{\cdot P_1} \mathbf{Q}^{12} \rightarrow 0.$$

For instance,  $F_w^5(\Omega[0])$  consists of the 12 monomials,

$$\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, x^4, xy^3, x^5\},$$

and so on. Note that  $\tau(P_1)$  is a  $(w, -w)$ -Gröbner basis of  $F_1(2, -3, -2, 5)$  and hence for this case, the duality method essentially coincides with the Gröbner deformation method of Section 2 at the level of  $\text{Hom}_D(M, \mathbf{k}[\mathbf{x}])$ . The above computations were made in Macaulay 2, where we get the output,

```
i1 : PolyExt(M)
o1 = HashTable{ 0 => QQ^1 }
              1 => QQ^2
              2 => QQ^1
```

Here, the output  $i \Rightarrow \text{QQ}^j$  means that  $\dim \text{Ext}_D^i(M, \mathbf{k}[\mathbf{x}]) = j$ .

**Example 4.5** Let us now evaluate the dimension of rational solutions to  $M = F_1(2, -3, -2, 5)$ . The singular locus is  $xy(x - y)(x - 1)(y - 1)$ . We will search for solutions in  $\mathbf{k}[x, y][\frac{1}{x}]$  first. From Example 4.2,  $\mathbf{D}(M)$  has the presentation  $D/\tau(Q_1)$ . Let  $u$  be the section corresponding to the residue class of  $\bar{1}$  in this presentation. Then the localization  $\mathbf{D}(M)[\frac{1}{x}]$  is generated by  $u \otimes \frac{1}{x^\tau}$  and gets the presentation  $D/J$ , where

$$J = D \cdot \left\{ \begin{array}{l} (\theta_x\theta_y + \theta_y^2 + 8\theta_y + 2\theta_x + 12) - (\theta_x + \theta_y + 4)\partial_y \\ (\theta_x\theta_y + 2\theta_x + 7\theta_y + 14) - (\theta_x + 10)x\partial_y \end{array} \right\}.$$

The natural localization map can be written as  $\varphi : D/\tau(Q_1) \rightarrow D/J$ , where  $\varphi(1) = x^7$ . Choose the integration weight vector  $w = (1, 2)$ . Then  $\mathbf{D}(M)[\frac{1}{x}]$  has a  $(w, -w)$ -strict resolution

$$G^\bullet : 0 \rightarrow D^1[-1] \xrightarrow{\cdot [v_1, v_2]^T} D^2[0, -1] \xrightarrow{\cdot [u_1, u_2]^T} D^1[0] \rightarrow 0$$



where

$$\begin{aligned}
u_1 &= -x^2\partial_x\partial_y + xy\partial_x\partial_y + 2x\partial_x - 11x\partial_y + 7y\partial_y + 14 \\
u_2 &= x^3\partial_x^2 + x^3\partial_x\partial_y - x^2\partial_x^2 - x^2\partial_x\partial_y + 16x^2\partial_x + 11x^2\partial_y + \\
&\quad 4xy\partial_y - 9x\partial_x - 11x\partial_y + 52x - 7 \\
v_1 &= x^3\partial_x^2 + x^3\partial_x\partial_y - x^2\partial_x^2 - x^2\partial_x\partial_y + 16x^2\partial_x + 12x^2\partial_y + \\
&\quad 4xy\partial_y - 8x\partial_x - 11x\partial_y + 52x - 6 \\
v_2 &= x^2\partial_x\partial_y - xy\partial_x\partial_y - 2x\partial_x + 11x\partial_y - 6y\partial_y - 12
\end{aligned}$$

The integration  $b$ -function is  $(s+12)(s+5)(s+2)$ , hence we want the cohomology of the complex  $F_w^{12}(\Omega \otimes_D G^\bullet)/F_w^1(\Omega \otimes_D G^\bullet)$ , which has the shape,

$$0 \rightarrow \mathbf{Q}^{41} \longrightarrow \mathbf{Q}^{86} \longrightarrow \mathbf{Q}^{45} \rightarrow 0.$$

By evaluating the dimensions of the cohomology groups in Macaulay 2, we find

```

i2      : RatlExt(M,x)
o2      = HashTable{ 0 => qq^2 }
                  1 => qq^5
                  2 => qq^3

```

Since we already computed a polynomial solution, this means there is one rational solution with pole along  $x$ . Similarly, we get the exact same dimensions for  $\text{Ext}_D^i(M, \mathbf{k}[x, y][\frac{1}{y}])$ , which means that there is also one rational solution with pole along  $y$ . The rank of the system is 3, therefore we have found all the solutions. We could also compute,

```

i3      : RatlExt(M,f)
o3      = HashTable{ 0 => qq^1 }
                  1 => qq^3
                  2 => qq^2

```

where  $f$  is any of the polynomials  $x - y$ ,  $x - 1$ , or  $y - 1$ . As expected, there are no rational solutions with poles along  $x - y$ ,  $x - 1$ , or  $y - 1$ , but in all cases there are new  $\text{Ext}^1$  and  $\text{Ext}^2$ . We have not computed  $\text{Ext}$  with respect to any products of poles since it is computationally too intensive for now.

Once we have evaluated the dimension of the solution spaces, we can compute the solutions by a brute force method.

1. For a given holonomic system  $M$ , compute its singular locus. Let  $f$  be a polynomial such that  $f = 0$  contains the singular locus.
2. Evaluate the dimension  $d$  of the rational solutions by the homological duality method.
3. Try to find rational solutions of the form  $r(x)/f^k$ ,  $\text{degree}(r) = p$ . Increase  $p + k$  until we find  $d$  linearly independent solutions.

### 4.3 Holonomic solutions by duality

The isomorphism (5) can also be expressed (see e.g. [3]) as

$$\mathrm{Ext}_D^i(M, N) \simeq \mathrm{Tor}_{n-i}^D(\mathrm{Ext}_D^n(M, D), N).$$

To compute the right-hand-side, it is also well known that for all  $M'$ , and in particular  $M' = \mathrm{Ext}_D^n(M, D)$ ,

$$\mathrm{Tor}_{n-i}^D(M', N) \simeq H^i(K^\bullet((M' \otimes_{\mathbf{k}[\mathbf{x}]} \Omega^{-1}) \widehat{\otimes} N; \{x_i - y_i, \partial_i + \delta_i\}_{i=1}^n)),$$

where  $K^\bullet$  denotes the Koszul complex and  $\widehat{\otimes}$  denotes the external tensor product into the category of  $D_{2n} = \mathbf{k}\langle x_1, y_1, \dots, x_n, y_n, \partial_1, \delta_1, \dots, \partial_n, \delta_n \rangle$ -modules. Combining these isomorphisms leads to

$$\mathrm{Ext}_D^i(M, N) \simeq H^i(K^\bullet((\mathbf{D}(M) \widehat{\otimes} N; \{x_i - y_i, \partial_i + \delta_i\}_{i=1}^n)))$$

By an automorphism of  $D$ , we can transform  $\{x_i - y_i, \partial_i + \delta_i\}_{i=1}^n$  into  $\{x_i, y_i\}_{i=1}^n$ , for which the Koszul complex computes the derived restriction to the origin.

**Algorithm 4.6** [Evaluating dimensions of holonomic solution spaces]

INPUT: holonomic left  $D$ -modules  $M$  and  $N$

OUTPUT: dimensions of  $\mathrm{Ext}_D^i(M, N)$ .

1. Compute the dual  $\mathbf{D}(M)$  using Algorithm 4.1
2. Form the  $D_{2n}$ -module  $\mathbf{D}(M) \otimes_k N$  and apply the change of coordinates  $\eta : D_{2n} \rightarrow D_{2n}$  where  $\eta$  maps,
$$\begin{aligned} x_i &\mapsto \frac{1}{2}x_i - \delta_i, & \partial_i &\mapsto \frac{1}{2}y_i + \partial_i, \\ y_i &\mapsto -\frac{1}{2}x_i - \delta_i, & \delta_i &\mapsto \frac{1}{2}y_i - \partial_i \end{aligned}.$$
3. Compute the restrictions of  $\eta(\mathbf{D}(M) \otimes_k N)$  to the origin using the algorithm in [9]. They are finite dimensional vector spaces.
4. Return the dimensions.

**Example 4.7** Let  $M = F_1(2, -3, -2, 5)$  be the Appell differential equation of example 2.5, and let  $N = \mathbf{k}[x, y][\frac{1}{x}]/\mathbf{k}[x, y]$ . It has presentation  $D/D \cdot \{x, \partial_y\}$ , where the generator 1 corresponds to  $\frac{1}{x}$ . Using the above algorithm, we compute

```
i4 : DExt(M,N)
o4 = HashTable{ 0 => QQ^1 }
           1 => QQ^3
           2 => QQ^2
```

Similarly, let  $N = \mathbf{k}[\partial_x, \partial_y] \simeq D/D \cdot \{x, y\}$ , the module of the delta functions with the support  $(0, 0)$ . Then we compute

```
i5 : DExt(M,N)
o5 = HashTable{ 0 => QQ^0 }
           1 => QQ^1
           2 => QQ^2
```

As before, once we know the dimension of  $\text{Hom}_D(M, N)$ , we can compute the solutions of  $M$  in  $N$  by a brute force method.

1. For given holonomic systems  $M$  and  $N$ , evaluate the dimension  $d$  of  $\text{Hom}_D(M, N)$  by the homological duality method.
2. Filter  $N$  by finite-dimensional vector spaces  $F^i(N)$  and search for solutions in  $F_i(N)$  for increasing  $i$  until  $d$  linearly independent solutions are found.

For instance in step 2, if  $N = D/J$ , then we can use the induced Bernstein filtration  $B$  where  $B^i(D/J)$  consists of residues of elements  $L \in D$  whose total degree is less than or equal to  $i$ .

## References

- [1] Appell, P., Kampé de Fériet, J. (1926): *Fonctions Hypergéométrique et Hyperphériques — Polynômes d'Hermite*. Gauthier-Villars, Paris.
- [2] Assi, A., Castro-Jiménez, F.J., Granger, M. (1998): The standard fan of a  $D$ -module. To appear in Journal of Pure and Applied algebra.
- [3] Björk, J. (1979): Rings of Differential Operators. North-Holland, Amsterdam.
- [4] Chyzak, F. (1997): An Extension of Zeilberger's Fast Algorithm to General Holonomic Functions, Formal Power Series and Algebraic Combinatorics, 9th Conference, Universitt Wien, pp. 172-183.
- [5] Grayson, D., Stillman, M. (1999): Macaulay 2: a computer algebra system for algebraic geometry, Version 0.8.56, <http://www.math.uiuc.edu/Macaulay2>
- [6] Kashiwara, M. (1976):  $B$ -functions and holonomic systems. *Inventiones mathematicae* **38**, 33–53.
- [7] Kashiwara, M. (1978): On the holonomic systems of linear partial differential equations II. *Inventiones mathematicae* **49**, 121–135.
- [8] Oaku, T. (1997): Algorithms for  $b$ -functions, restrictions, and algebraic local cohomology groups of  $D$ -modules. *Advances in Applied Mathematics*, **19**, 61-105.
- [9] Oaku, T., Takayama, N. (1998): Algorithms for  $D$ -modules — Restrictions, tensor product, localization and algebraic local cohomology groups, math.AG/9805006. To appear in Journal of Pure and Applied Algebra.
- [10] Oaku, T., Takayama, N., Walther, U. (1999): A localization algorithm for  $D$ -modules. To appear in Journal of Symbolic Computation.

- [11] Saito, M., Sturmfels, B. and Takayama, N. (1999): *Gröbner Deformations of Hypergeometric Differential Equations*. Algorithms and Computation in Mathematics, Vol. 6, Springer.
- [12] Singer, M. (1996): Testing reducibility of linear differential operators: a group theoretic perspective. *Applicable Algebra in Engineering, Communication and Computing*, **7**, 77–104.
- [13] Takayama, N. *Kan: A system for computation in algebraic analysis*, 1991, 1994, 1999. <ftp.math.kobe-u.ac.jp>
- [14] Tsai, H. (1999): Weyl closure, torsion, and local cohomology of  $D$ -modules, preprint.
- [15] Tsai, H., Walther, U. (1999): Computing homomorphisms between holonomic  $D$ -modules, preprint.
- [16] Walther, U. (1999): Algorithmic computation of local cohomology modules and the local cohomological dimension of algebraic varieties, *Journal of Pure and Applied Algebra*, **139**, 303–321
- [17] Walther, U. (1999): Algorithmic computation of deRham cohomology of complements of complex affine varieties. To appear in *Journal of Symbolic Computation*.

Toshinori Oaku, [oaku@twcu.ac.jp](mailto:oaku@twcu.ac.jp)  
 Department of Mathematics, Tokyo Woman's Christian University  
 2-6-1 Zempukuji, Suginami-ku, Tokyo 167-8585, Japan

Nobuki Takayama, [takayama@math.kobe-u.ac.jp](mailto:takayama@math.kobe-u.ac.jp)  
 Department of Mathematics, Kobe University  
 Kobe 657-8501, Japan

Harrison Tsai, [htsai@math.berkeley.edu](mailto:htsai@math.berkeley.edu)  
 Department of Mathematics, University of California, Berkeley  
 Berkeley, CA 94720-3840, USA