Asymptotic Dynamics of Ripples *

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February 8, 2008

Abstract

A new nonlinear equation governing asymptotic dynamics of ripples is derived by using a short wave perturbative expansion on a generalized version of the Green-Naghdi system. It admits peakon solutions with amplitude, velocity and width in interrelation and static compacton solutions with amplitude and width in interrelation. Short wave pattern formation is shown to result from a balance between linear dispersion and nonlinearity.

1 Introduction

Since the classical works of Boussinesq [1] and Korteweg and de Vries[2], nonlinear evolution of long waves of small amplitude in shallow fluids has been widely studied. The asymptotic dynamics (in space and time) is by now well understood and represented by a large number of model equations, among which the different versions of the Boussinesq equations [3], the Korteweg-de Vries (KdV), the Benjamin-Bona-Mahoney-Peregrini (BBMP) [4], and the Camassa-Holm [5] equations.

On the contrary, *short waves* have been studied very little and only a few results are known on their asymptotic dynamics. The main purpose of this paper is to study nonlinear short surface waves in fluids, the *ripples*, shown to build up as a result of superposition of two surface motions: an oscillatory flow and a laminar flow. The oscillatory flow corresponds to mechanical perturbations which propagate like a wave. The laminar flow may be created in many ways: by the action of an external wind, or in a two-layer liquid where the upper one displaces particles belonging to the upper surface of the lower fluid, or by an external electric field if the surface particles are charged, etc...

Here we study an ideal fluid (inviscid, incompressible and without surface tension) in which surface displacement can be achieved through the action of a steady external wind directed parallel to the water surface. A surface wind on a lake which produces surface flow is an ideal physical environment conducive to the above mentioned phenomenon.

2   Model equation for ripples propagation

Let's consider $u(x,t)$ which represents at time t an unidirectional surface wave propagation in the x direction of a fluid medium which is involved in a flow on a large scale. This large scale flow is a superposed motion of the surface which moves under the action of the wind with a velocity c_0 in relation to the bulk. Using perturbative methods we will show that ripples can propagate and obey the nonlinear equation

$$u_{xt} = -\frac{3g}{hc_0}u - uu_{xx} + (u_x)^2 \tag{2.1}$$

where subscripts denote partial derivatives, h is the unperturbed initial depth and g is the acceleration of gravity. The equation (2.1) has several types of interesting solutions. The peakon solution is (using the formula $(1 - \partial_{zz})e^{-|z|} = 2\delta(z)$, [5])

$$u = -\alpha\lambda^2 \exp(-|\frac{x + \alpha\lambda^2 t}{\lambda}|), \quad \alpha = -\frac{3g}{hc_0}, \tag{2.2}$$

where the width λ is a free parameter. Unlike the peakon of Camassa-Holm equation, the amplitude $(-\alpha\lambda^2)$, the velocity $(\alpha\lambda^2)$ and the width (λ) are interrelated.

The static compacton solution is

$$u = -8\alpha\lambda^2 \cos^2(\frac{x}{4\lambda}), \quad |\frac{x}{\lambda}| \leq 2\pi, \tag{2.3}$$

and $u = 0$ otherwise. Unlike the compacton solution recently introduced and investigated by Roseneau and Hyman [6], this solution presents a dependence between width and amplitude. Solutions (2.2) is a coherent struture analogous to the solitary wave of KdV. Moreover, contrary to KdV, (2.1) possess a plane monochromatic wave solution of arbitrary amplitude A

$$u = A \exp i(kx - \Omega t) \tag{2.4}$$

with the dispersion relation $\Omega(k) = 3g/khc_0$ identical to the one of the linearized system [7]. Moreover this dispersion relation is a function of k which has a good behavior in the short wave limit.

Despite the fact that solution (2.3) is unnatural in the physics under consideration, it shows that (2.1) is an adequate mathematical tool to modelize statics patterns in nature.

The underlying mechanism responsible for structural stability of solutions of the Camassa-Holm equation or the KdV equation with nonlinear dispersion is the balance between nonlinear dispersion, nonlinear convection and nonlinearity. Indeed these equations are nonlinear evolution equations without linear dispersion (the plane wave is not a solution of the linear associated evolution equations). However the exibited solutions of (2.1) come from the balance between linear dispersion and two nonlinear terms (in the case of (2.2)) and only nonlinearity (in the case of (2.3)). An essential point, which remains to be proved by numerical or analytical methods is: *under what conditions do the solution (2.2) dominates the initial value problem of equation (2.1)?*. Another important open problem is the *structural stability* of (2.2) and (2.3) . Here we only show that the plane wave (2.4) is unstable.

3 Physical context: modified Green-Naghdi system of equations.

We consider an inviscid, incompressible, homogeneous fluid with density σ. Let the particles of this continuum medium be identified by a fixed rectangular Cartesian system of center O and axes $(x_1, x_2, x_3) = (x, z, y)$ with Oz the upward vertical direction. We assume symmetry in y and we will only consider a sheet of fluid in the xz plane. This fluid sheet is moving in a domain with a rigid botton at $z = 0$ and an upper free surface at $z = \phi(x, t)$. The vector velocity is $\vec{v} = (v_1, v_2) = (u, w)$ for $0 \leq z < \phi(x, t)$. Thanks to homogenity and incompressibility the continuity equation reduces to

$$\vec{\nabla} \cdot \vec{v} = u_x(x, z, t) + w_z(x, z, t) = 0, \ \ 0 \leq z \leq \phi(x, t), \tag{3.1}$$

where $\vec{\nabla} = (\partial_x, \partial_z)$. The Euler equations of motions (law of conservation of momentun) of a fluid under gravity g and for $0 \leq z \leq \phi(x, t)$ are

$$\sigma \dot{u}(x, z, t) = -p_x^*(x, z, t), \tag{3.2}$$

$$\sigma \dot{w}(x, z, t) = -p_z^*(x, z, t) - g\sigma, \tag{3.3}$$

where $p^*(x, z, t)$ is the pressure and a superposed dot denotes the material derivative for x, z fixed.

We complete now the fundamental equations of continuity and momentun conservation with appropiate kinematic and dynamics boundaries conditions. Let $S(x, z, t)$ be the interface between the inviscid fluid sheet and the air (external medium). We represented $S(x, z, t)$ by the (classical) equation

$$S(x, z, t) = \phi(x, t) - z = 0. \tag{3.4}$$

The kinematic condition is that the normal velocity of the surface $S(x, z, t)$ must be equal to the velocity of the fluid sheet normal to the surface. The normal

velocity of the surface is [8]

$$-\frac{S_t}{\| \vec{\nabla} S \|},\tag{3.5}$$

while the velocity on the surface $z = \phi(x,t)$ is

$$\vec{v} = (u + c_0, w), \quad z = \phi(x,t) \ ,\tag{3.6}$$

whose normal component is

$$\vec{v} \cdot \frac{\vec{\nabla} S}{\| \vec{\nabla} S \|} \ .\tag{3.7}$$

From (3.5), (3.6) and (3.7) the kinematic conditions read

$$\phi_t + u\phi_x - w + c_0\phi_x = 0, \quad z = \phi.\tag{3.8}$$

The equation (3.6) leading to (3.8) lies at the heart of our approach. It points out that an external agent - wind in our case - drives the particles belonging to $S(x,z,t)$. The motion is uniform, of value c_0 and in the x direction only. So $S(x,z,t)$ is a surface of discontinuity for the u component of $\vec{v}$, which experiences finite jumps of value c_0 in $z = \phi$. This wind induced motion of the fluid's surface is found in particular in the hydrodynamics of lakes [9]. From a theoretical point of view this phenomenon was studied in the ray tracing theory by Lighthill in [10] where an equation analogous to (3.8) was derived.

At the surface of the sheet $z = \phi$, there is a constant normal pressure p_0. At the bed $z = 0$, there is an unknown pressure $p^*(x,0,t)$ and the normal fluid velocity is zero: $w = 0$. Several types of approximations can be used in order to solve this wave problem. Here we adopt an approximation in the velocity field introduced by Green, Laws and Naghdi in [11] and based on the monumental work of Naghdi [12]. We assume that u is independent of z. This is equivalent to considering the vertival component w as a linear function of z. This simple and realistic assumption enables us to satisfy exactly the equation of incompressibility and the boundary condition at the bed. Hence $u = u(x,t)$ and from (3.1) we have

$$w = z\xi(x,t) \ , \quad \xi(x,t) = -u_x(x,t).\tag{3.9}$$

Now we integrate (3.2) in the variable z from $z = 0$ to $z = \phi$ (the integration is granted by the Riemann's condition of integrability). The result is

$$\sigma(u_t + uu_x)\phi = -p_x,\tag{3.10}$$

where we use $\dot{u} = u_t + uu_x$ and

$$p(x,t) = \int_0^{\phi(x,t)} p^*(x,z,t)dz - p_0\phi(x,t).\tag{3.11}$$

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Next, we multiply equation (3.3) by z and integrate from $z = 0$ to $z = \phi$ which yields

$$\sigma(\xi^2 + \dot{\xi})\frac{\phi^3}{3} + \sigma g\frac{\phi^2}{2} = p. \tag{3.12}$$

The pressure p can be eliminated using (3.12) in (3.10) and, with the help of (3.9) we eventually obtain

$$u_t + uu_x + g\phi_x = \quad \phi\phi_x(u_{xt} + uu_{xx} - (u_x)^2) + $$
$$\frac{\phi^3}{3}(u_{xt} + uu_{xx} - (u_x)^2)_x \tag{3.13}$$

The remaining upper boundary condition on ϕ reads using (3.9)

$$\phi_t + (\phi u)_x + c_0\phi_x = 0. \tag{3.14}$$

With $c_0 = 0$, the system (3.13) (3.14) is the Green-Naghdi system of equations [11]. The inclusion of the term $c_0\phi_x$ drastically changes its dynamics: c_0 cannot be eliminated neither by a Galilean transformation nor by a rescaling of u or ϕ. These extended Green-Naghdi equations represent the nonlinear interaction between two separate forms of motion: a wave motion associated with the elastic response of the fluid to a perturbation, a surface (uniform) motion generated by an external agent.

4 Nonlinear dynamics of ripples.

Let us consider now asymptotic nonlinear dynamics of ripples in (3.13) and (3.14). Ripples are, from a geometrical point of view, *nearly local objects* and we are looking for their *large time behavior*. So we need to introduce two appropiate variables: one space variable ζ describing a local and asymptotic pattern and a time variable τ measuring asymptotic time dynamics. These asymptotic are worked out by means of the change of variables

$$\zeta = \frac{1}{\epsilon}x , \quad \tau = \epsilon t. \tag{4.1}$$

where the small parameter ϵ is related to the size of the wavelength: $\ell = 2\pi/k \sim \epsilon$.

Such variables cannot always be defined as shown in [15], but exist if the linear dispersion relation $\Omega(k)$ can be expanded as a Laurent series with a simple pole for $k \to \infty$, that is

$$\Omega(k) = Ak + \frac{B}{k} + \frac{C}{k^3} + ... \quad k \to \infty.$$

where A, B, C, ..., are constants. Consequently the phase and group velocities remain bounded in the short wave limit $k \to \infty$. This is the case here because the

dispersion relation for (3.13) (3.14), obtained for $\phi(x,t) = h + a \exp i[kx - \Omega(k)t]$ and $u(x,t) = b \exp i[kx - \Omega(k)t]$, has the asymptotic expansion

$$\Omega(k) = -(\frac{3g}{c_0 h})\frac{1}{k} + \frac{3}{c_0 h^2} \, (\frac{3g}{h} + \frac{3g^2}{c_0^2})\frac{1}{k^3} + \mathcal{O}(\frac{1}{k^5}). \tag{4.2}$$

Such an expansion justifies the change of variables (4.1) and it is essential in revealing in (3.13) (3.14) the asymptotic short wave dynamics [15]. We are looking now for nonlinear dynamics of ripples of small amplitude represented by the expansions

$$\phi = h + \epsilon^2(H_0 + \epsilon^2 H_2 + \epsilon^4 H_4 + ...) , \tag{4.3}$$

$$u = \epsilon^2(U_0 + \epsilon^2 U_2 + \epsilon^4 U_4 + ...) , \tag{4.4}$$

which from (3.13) and (3.14) give at lower order in ϵ:

$$h(U_0)_\varsigma + c_0(H_0)_\varsigma = 0 , \tag{4.5}$$

$$g(H_0)_\varsigma = \frac{h^2}{3}\{(U_0)_{\tau\varsigma} + U_0(U_0)_{\varsigma\varsigma} - (U_0)_\varsigma^2\}_\varsigma . \tag{4.6}$$

Using (4.5) in (4.6), going back to the laboratory fields (ϕ, u) and co-ordinates (x,t) and integrating once we eventually arrive at equation (2.1).

5 Benjamin-Feir instability of the plane wave.

For a plane wave, the nonlinears terms in (2.1) give rise to harmonics of the fundamental. Assume that a disturbance is present consisting of modes with sideband frequences and wavenumbers close to the fundamental. We can have interaction between harmonics and these sideband modes. This interaction is likely to produce a resonant phenomenon manifesting itself by the modulation of the plane wave solution. The exponencial growth in time of the modulation, originating from synchronous resonance between harmonics and sideband modes, leads to the Benjamin-Feir instability[13]. A formal solutions can be given via an asymptotic expansion conducing to the nonlinear Schodinger equation (NLS)[14]. The particular interest of NLS is the existence of a general and simple criterion enable to detect stability or unstability of the monochromatic wave train. Let us seek for a solution of (2.1) under the form of a Fourier expansion in harmonics of the fundamental $\exp i(kx - \Omega t)$ and where the Fourier components are developed in a Taylor serie in powers of a small parameter γ mesuring the amplitude of the fundamental

$$u = \sum_{l=-p}^{l=p} \sum_{p=1}^{\infty} \exp il(kx - \omega t)\gamma^p u_l^p(\xi, \tau) \tag{5.1}$$

In (5.1), $u_{-l}^p = u_l^{*p}$ ("star" denotes complex conjugation) and ξ and τ are slow variables introduced through the stretching $\xi = \gamma(x - vt)$ and $\tau = \gamma^2 t$

and where v will be determined as a solvability condition. The expansions (5.1) includes fast local oscillations through the dependence on the harmonics and slow variation (modulation) in amplitude taken into account by the ξ, τ dependence of u_l^p. Introducing now this expansion and the slow variables in (2.1) we may proceed to collect and solve different order γ and l. We obtain: $u_0^1 = u_2^2 = u_2^3 = u_3^3 = 0$, $v = \gamma/k^2$, $u_1^1 = A(\xi, \tau) \neq 0$ and $u_0^2 = -4k^3/\alpha$. At order $\gamma = 3$, $l = 1$ we obtain NLS for $A(\xi, \tau)$

$$iA_\tau + \frac{\alpha}{k^3}A_{\xi\xi} + \frac{4k^3}{\alpha}A|A|^2 = 0. \tag{5.2}$$

The nature of solutions of NLS depends drastically of the sign of the product between the coeficient of $A_{\xi\xi}$ and that of $A|A|^2$. In this case the product of α/k^3 by $4k^3/\alpha$ is positive, and according to a well known stability criterion (see for exemple [3]) the plane wave solution (2.4) is unstable.

6 Conclusion and final coments on short waves.

We have shown that ripples result from balance between linear dispersion and nonlinearity as in the case of long waves. However the physical model under consideration represents an ideal fluid because viscosity has been neglected, and, since dissipative phenomena take place at small scales, viscosity must affect asymptotic dynamics of ripples.

The same approach can be used to study a flow of multiple layers. Such a flow occurs in many practical applications (e.g multilayer coating) in which the short wave behavior is of primordial importance.

Note that, while the original Green-Naghdi system is Galilean invariant, this invariance is lost in the Camassa-Holm equation derived by Hamiltonian methods. In our case the perturbative theory conserves Galilean invariance and consequently equation (2.1) inherits this property from modified Green-Naghdi.

In nonlinear dispersive systems, short wave dynamics does not occur as naturally as long wave dynamics. It is interesting to point out that some intermediate *long wave* shallow water models behave well, paradoxically, in the short wave limit. In these intermediate models a second asymptotic limit is always possible, in general of long wave type, leading then to the ubiquitous KdV equation. In some cases however, there exists a second asymptotic limit of short wave type, leading to new nonlinear evolution equations. This is the case for BBMP, the integrable Camassa-Holm equation, and one of the Boussinesq system. The short wave limit of BBMP reads [15]: $u_{xt} = u - 3u^2$, where u has the same meaning as in (2.1). Its solitary wave solution comes from the balance between dispersion and nonlinearity as in KdV. Explicit solutions, blow-up and nonlinear instabilities are studied in [16]. The Camassa-Holm equation has as short wave limit, an integrable equation which belongs to the Harry-Dym hierarchy (κ is a constant): $u_{xt} = \kappa u - \frac{1}{2}(u_x)^2 - uu_{xx}$. Finally the short wave limit of the

Boussinesq system results as : $u_{xtt} = u_t - uu_x$. The linear limit for all these cases is the equation $u_{xt} = au$ (a constant) or else

$$u_t - a \int_{-\infty}^{x} u dx = 0. \qquad (6.1)$$

This linear nonlocal equation for u corresponds to the linear local wave equation $u_t - au_x = 0$ that appears in the long wave case. Note that the short wave limits of BBMP, Camassa-Holm and Boussinesq system are not evolution equations *stricto sensu*, rather being integro-differential equations. Thus the nonlocality of nonlinear evolution equations for short waves appears to reflect the basic non locality of (6.1).

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