Anomalous Scaling in Passive Scalar Advection and Lagrangian Shape Dynamics

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The problem of anomalous scaling in passive scalar advection, especially with δ-correlated velocity field (the Kraichnan model) has attracted a lot of interest since the exponents can be computed analytically in certain limiting cases. In this paper we focus, rather than on the evaluation of the exponents, on elucidating the *physical mechanism* responsible for the anomaly. We show that the anomalous exponents ζ_n stem from the Lagrangian dynamics of shapes which characterize configurations of n points in space. Using the shape-to-shape transition probability, we define an operator whose eigenvalues determine the anomalous exponents for all n, in all the sectors of the SO(3) symmetry group.

I. INTRODUCTION

In the lecture at the IUTAM symposium the work of our group on the consequences of anisotropy on the universal statistics of turbulence has been reviewed. This material is available in print, and the interested reader can find it in [1–6]. A short review is available in the proceedings of "Dynamics Days Asia" [7]. In this paper we review some recent work aimed at understanding the *physical mechanism* responsible for the anomalous exponents that characterize the statistics of passive scalars advected by turbulent velocity fields. We will consider isotropic advecting velocity fields, but will allow anisotropy in the forcing of the passive scalar. In such case the statistical objects like structure functions and correlation functions are not isotropic. Instead, they are composed of an isotropic and non-isotropic parts. We overcome this complication by characterizing these functions in terms of the SO(3) irreducible representations. Any such function can be written as a linear combination of parts which belong to a given irreducible representation of SO(3). We will show that each part is characterized by a set of *universal* scaling exponents. The weight of each part however will turn out to be non-universal, set by the boundary conditions.

The SO(3) classification will appear to be natural once we focus on the physics of Lagrangian trajectories in the flow. We will see that one can offer a satisfactory understanding of the physics of anomalous scaling by connecting the the statistics of the passive scalar to the Lagrangian trajectories. This connection provides a very clear understanding of the physical origin of the anomalous exponents, relating them to the dynamics in the space of *shapes* of groups of Lagrangian particles.

II. THE KRAICHNAN MODEL OF PASSIVE SCALAR ADVECTION

The model of passive scalar advection with rapidly decorrelating velocity field was introduced by R.H. Kraichnan [8] already in 1968. In recent years [9–14] it was shown to be a fruitful case model for understanding multiscaling in the statistical description of turbulent fields. The basic dynamical equation in this model is for a scalar field $T(\boldsymbol{r}, t)$ advected by a random velocity field $\boldsymbol{u}(\boldsymbol{r}, t)$:

$$\left[\partial_t - \kappa_0 \nabla^2 + \boldsymbol{u}(\boldsymbol{r}, t) \cdot \boldsymbol{\nabla}\right] T(\boldsymbol{r}, t) = f(\boldsymbol{r}, t) \ . \tag{2.1}$$

In this equation $f(\boldsymbol{r}, t)$ is the forcing and κ_0 is the molecular diffusivity. In Kraichnan's model the advecting field $\boldsymbol{u}(\boldsymbol{r}, t)$ as well as the forcing field $f(\boldsymbol{r}, t)$ are taken to be Gaussian, time and space homogeneous, and delta-correlated in time:

$$\left\langle (u^\alpha(\boldsymbol{r}, t) - u^\alpha(\boldsymbol{r}', t))(u^\beta(\boldsymbol{r}, t') - u^\beta(\boldsymbol{r}', t')) \right\rangle_{\boldsymbol{u}} = h^{\alpha\beta}(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t') \ , \tag{2.2}$$

where the "eddy-diffusivity" tensor $h^{\alpha\beta}(\boldsymbol{r})$ is defined by

$$h^{\alpha\beta}(\boldsymbol{r}) = \left(\frac{r}{\Lambda}\right)^\xi \left(\delta^{\alpha\beta} - \frac{\xi}{d - 1 + \xi}\frac{r^\alpha r^\beta}{r^2}\right) \ , \quad \eta \ll r \ll \Lambda \ . \tag{2.3}$$

Here η and Λ are the inner and outer scale for the velocity fields, and the coefficients are chosen such that $\partial_\alpha h^{\alpha\beta} = 0$. The averaging $\langle\ldots\rangle_{\boldsymbol{u}}$ is done with respect to the realizations of the velocity field.

The forcing f is also taken white in time and Gaussian:

$$\langle f(\boldsymbol{r},t)f(\boldsymbol{r}',t')\rangle_f = \Xi(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t') \ . \tag{2.4}$$

Here the average is done with respect to realizations of the forcing. The forcing is taken to act only on the large scales, of the order of L (with a compact support in Fourier space). This means that the function $\Xi(r)$ is nearly constant for $r \ll L$ but is decaying rapidly for $r > L$.

¿From the point of view of the statistical theory one is interested mostly in the scaling exponents characterizing the structure functions

$$S_{2n}(\boldsymbol{r}_1,\boldsymbol{r}_2) = \left\langle (T(\boldsymbol{r}_1,t) - T(\boldsymbol{r}_2,t))^{2n}\right\rangle_{\boldsymbol{u},f} \ . \tag{2.5}$$

For isotropic forcing one expects S_{2n} to depend only on the distance $R \equiv |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ such that in the scaling regime

$$S_{2n}(R) \propto R^{\zeta_{2n}} \propto [S_2(R)]^n \left(\frac{L}{R}\right)^{\delta_{2n}} \ . \tag{2.6}$$

In this equation we introduced the "normal" $(n\zeta_2)$ and the anomalous (δ_{2n}) parts of the scaling exponents $\zeta_{2n} = n\zeta_2 - \delta_{2n}$. The first part can be obtained from dimensional considerations, but the anomalous part cannot be guessed from simple arguments.

When the forcing is anisotropic, the structure functions depend on the vector distance $\boldsymbol{R} = \boldsymbol{r}_1 - \boldsymbol{r}_2$. In this case we can represent them in terms of spherical harmonics,

$$S_{2n}(\boldsymbol{R}) = \sum_{\ell,m} a_{\ell,m}(R)Y_{\ell,m}(\hat{\boldsymbol{R}}) \ , \tag{2.7}$$

where $\hat{\boldsymbol{R}} \equiv \boldsymbol{R}/R$. This is a case in which the statistical object is a scalar function of one vector, and the appropriate irreducible representation of the SO(3) symmetry group are obvious. We are going to explain in the next section that the coefficients $a_{\ell,m}(R)$ are expected to scale with a universal leading scaling exponent $\zeta_{2n}^{(\ell)}$. The exponent will turn out to be ℓ dependent but not m dependent.

Theoretically it is natural to consider correlation functions rather than structure functions. The $2n$-order correlation functions are defined as

$$F_{2n}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_{2n}) \equiv \langle T(\boldsymbol{r}_1)T(\boldsymbol{r}_2)\ldots T(\boldsymbol{r}_{2n})\rangle_{\boldsymbol{u},f} \ . \tag{2.8}$$

For separations $r_{ij} \to 0$ the correlation functions converges to $\left\langle T^{2n}\right\rangle_{\boldsymbol{f}}$, whereas for $r_{ij} \to L$ decorrelation leads to convergence to $\langle T\rangle_{\boldsymbol{u},f}^n$. For all $r_{ij} \approx O(r) \ll L$ one expects a behaviour according to

$$F_{2n}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_{2n}) = L^{n(2-\xi)}(c_0 + \cdots + c_k(r/L)^{\zeta_{2n}}\tilde{F}_{2n}(\tilde{\boldsymbol{r}}_1,\ldots,\tilde{\boldsymbol{r}}_{2n}) + \cdots) \ , \tag{2.9}$$

where $\tilde{F}_{2n}$ is a scaling function depending on $\tilde{\boldsymbol{r}}_i$ which denote a set of dimensionless coordinates describing the configuration of the $2n$ points. The exponents and scaling functions are expected to be universal, but not the c coefficients, which depend on the details of forcing.

It has been shown [11] that the anomalous exponents ζ_{2n} can be obtained by solving for the zero modes of the exact differential equations which are satisfied by F_{2n}. The equations for the zero modes read

$$\left[-\kappa\sum_i \nabla_i^2 + \hat{\mathcal{B}}_{2n}\right]\mathcal{F}_{2n}(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_{2n}) = 0 \ . \tag{2.10}$$

The operator $\hat{\mathcal{B}}_{2n} \equiv \sum_{i>j}^{2n}\hat{\mathcal{B}}_{ij}$, and $\hat{\mathcal{B}}_{ij}$ are defined by

$$\hat{\mathcal{B}}_{ij} \equiv \hat{\mathcal{B}}(\boldsymbol{r}_i,\boldsymbol{r}_j) = h^{\alpha\beta}(\boldsymbol{r}_i-\boldsymbol{r}_j)\partial^2/\partial r_i^\alpha\partial r_j^\beta \ . \tag{2.11}$$

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III. LAGRANGIAN TRAJECTORIES, CORRELATION FUNCTIONS AND SHAPE DYNAMICS

An elegant approach to the correlation functions is furnished by Lagrangian dynamics [7,15–18]. In this formalism one recognizes that the actual value of the scalar at position $\boldsymbol{r}$ at time t is determined by the action of the forcing along the Lagrangian trajectory from $t = -\infty$ to t:

$$T(\boldsymbol{r}_0, t_0) = \int_{-\infty}^{t_0} dt \, \langle f(\boldsymbol{r}(t), t) \rangle_{\boldsymbol{\eta}} \, , \tag{3.1}$$

with the trajectory $\boldsymbol{r}(t)$ obeying

$$\boldsymbol{r}(t_0) = \boldsymbol{r}_0 \, ,$$
$$\partial_t \boldsymbol{r}(t) = \boldsymbol{u}(\boldsymbol{r}(t), t) + \sqrt{2\kappa}\boldsymbol{\eta}(t) \, , \tag{3.2}$$

and $\boldsymbol{\eta}$ is a vector of zero-mean independent Gaussian white random variables, $\langle \eta^\alpha(t) \eta^\beta(t') \rangle = \delta^{\alpha\beta} \delta(t - t')$. With this in mind, we can rewrite F_{2n} by substituting each factor of $T(\boldsymbol{r}_i)$ by its representation (3.1). Performing the averages over the random forces, we end up with

$$F_{2n}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{2n}, t_0) = \Big\langle \int_{-\infty}^{t_0} dt_1 \cdots dt_n \Big[\Xi(\boldsymbol{r}_1(t_1) - \boldsymbol{r}_2(t_1)) \cdots \tag{3.3}$$

$$\times \Xi(\boldsymbol{r}_{2n-1}(t_n) - \boldsymbol{r}_{2n}(t_n)) + \text{permutations} \Big] \Big\rangle_{\boldsymbol{u}, \{\boldsymbol{\eta}_i\}} \, , \tag{3.4}$$

To understand the averaging procedure recall that each of the trajectories $\boldsymbol{r}_i$ obeys an equation of the form (3.2), where $\boldsymbol{u}$ as well as $\{\boldsymbol{\eta}_i\}_{i=1}^{2n}$ are independent stochastic variables whose correlations are given above. Alternatively, we refer the reader to section II of [18], where the above analysis is carried out in detail.

In considering Lagrangian trajectories of *groups* of particles, we should note that every initial configuration is characterized by a *center of mass*, say $\boldsymbol{R}$, a *scale* s (say the radius of gyration of the cluster of particles) and a *shape* $\boldsymbol{Z}$. In "shape" we mean here all the degrees of freedom other than the scale and $\boldsymbol{R}$: as many angles as are needed to fully determine a shape, in addition to the Euler angles that fix the shape orientation with respect to a chosen frame of coordinates. Thus a group of $2n$ positions $\{\boldsymbol{r}_i\}$ will be sometime denoted below as $\{\boldsymbol{R}, s, \boldsymbol{Z}\}$.

One component in the evolution of an initial configuration is a rescaling of all the distances which increase on the average like t^{1/ζ_2}; this rescaling is analogous to Richardson diffusion. The exponent ζ_2 which determines the scale increase is also the characteristic exponent of the second order structure function [8]. This has been related to the exponent ξ of (2.3) according to $\zeta_2 = 2 - \xi$. After factoring out this overall expansion we are left with a normalized 'shape'. It is the evolution of this shape that determines the anomalous exponents.

Consider a final shape $\boldsymbol{Z}_0$ with an overall scale s_0 which is realized at $t = 0$. This shape has evolved during negative times. We fix a scale $s > s_0$ and examine the shape when the configuration reaches the scale s for the last time before reaching the scale s_0. Since the trajectories are random the shape $\boldsymbol{Z}$ which is realized at this time is taken from a distribution $\gamma(\boldsymbol{Z}; \boldsymbol{Z}_0, s \to s_0)$. As long as the advecting velocity field is scale invariant, this distribution can depend only on the ratio s/s_0.

Next, we use the shape-to-shape transition probability to define an operator $\hat{\gamma}(s/s_0)$ on the space of functions $\Psi(\boldsymbol{Z})$ according to

$$[\hat{\boldsymbol{\gamma}}(s/s_0)\Psi](Z_0) = \int d\boldsymbol{Z} \gamma(\boldsymbol{Z}; \boldsymbol{Z}_0, s \to s_0) \Psi(\boldsymbol{Z}) \tag{3.5}$$

We will be interested in the eigenfunction and eigenvalues of this operator. This operator has two important properties. First, for an isotropic statistics of the velocity field the operator is isotropic. This means that this operator commutes with all rotation operators on the space of functions $\Psi(\boldsymbol{Z})$. In other words, if $\mathcal{O}_\Lambda$ is the rotation operator that takes the function $\Psi(\boldsymbol{Z})$ to the new function $\Psi(\Lambda^{-1}\boldsymbol{Z})$, then

$$\mathcal{O}_\Lambda \hat{\gamma} = \hat{\gamma} \mathcal{O}_\Lambda \, . \tag{3.6}$$

This property follows from the obvious symmetry of the Kernel $\gamma(\boldsymbol{Z}; \boldsymbol{Z}_0, s \to s_0)$ to rotating $\boldsymbol{Z}$ and $\boldsymbol{Z}_0$ simultaneously. Accordingly the eigenfunctions of $\hat{\gamma}$ can be classified according to the irreducible representations of SO(3) symmetry group. We will denote these eigenfunctions as $B_{q\ell m}(\boldsymbol{Z})$. Here $\ell = 0, 1, 2, \ldots$, $m = -\ell, -\ell+1, \ldots \ell$ and q stands for a running index if there is more than one representation with the same ℓ, m. The fact that the $B_{q\ell m}(\boldsymbol{Z})$ are classified according to the irreducible representations of SO(3) in manifested in the action of the rotation operators upon them:

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$$\mathcal{O}_\Lambda B_{q\ell m} = \sum_{m'} D^{(\ell)}_{m'm}(\Lambda) B_{q\ell m'} \tag{3.7}$$

where $D^{(\ell)}_{m'm}(\Lambda)$ is the SO(3) $\ell \times \ell$ irreducible matrix representation.

The second important property of $\hat{\gamma}$ follows from the δ-correlation in time of the velocity field. Physically this means that the future trajectories of n particles are statistically independent of their trajectories in the past. Mathematically, it implies for the kernel that

$$\gamma(\boldsymbol{Z}; \boldsymbol{Z}_0, s \to s_0) = \int d\boldsymbol{Z}_1 \gamma(\boldsymbol{Z}; \boldsymbol{Z}_1, s \to s_1) \gamma(\boldsymbol{Z}_1; \boldsymbol{Z}_0, s_1 \to s_0) , \quad s > s_1 > s_0 \tag{3.8}$$

and in turn, for the operator, that

$$\hat{\gamma}(s/s_0) = \hat{\gamma}(s/s_1)\hat{\gamma}(s_1/s_0) . \tag{3.9}$$

Accordingly, by a successive application of $\hat{\gamma}(s/s_0)$ to an arbitrary eigenfunction, we get that the eigenvalues of $\hat{\gamma}$ have to be of the form $\alpha_{q,\ell} = (s/s_0)^{\zeta_{2n}^{(q,\ell)}}$:

$$(\frac{s}{s_0})^{\zeta_{2n}^{(q,\ell)}} B_{q\ell m}(\boldsymbol{Z}_0) = \int d\boldsymbol{Z}\gamma(\boldsymbol{Z}; \boldsymbol{Z}_0, s \to s_0) B_{q\ell m}(\boldsymbol{Z}) \tag{3.10}$$

Notice that the eigenvalues are not a function of m. This follows from Schur's lemmas [19], but can be also explained from the fact that the rotation operator mixes the different m's (3.7): Take an eigenfunction $B_{q\ell m}(\boldsymbol{Z})$, and act on it once with the operator $\mathcal{O}_\Lambda\hat{\gamma}(s/s_0)$ and once with the operator $\hat{\gamma}(s/s_0)\mathcal{O}_\Lambda$. By virtue of (3.6) we should get that same result, but this is only possible if all the eigenfunctions with the same ℓ *and* the same q share the same eigenvalue.

To proceed we want to introduce into the averaging process in (3.4) by averaging over Lagrangian trajectories of the $2n$ particles. This will allow us to connect the shape dynamics to the statistical objects. To this aim consider any set of Lagrangian trajectories that started at $t = -\infty$ and end up at time $t = 0$ in a configuration characterized by a scale s_0 and center of mass $\boldsymbol{R}_0 = 0$. A full measure of these have evolved through the scale L or larger. Accordingly they must have passed, during their evolution from time $t = -\infty$ through a configuration of scale $s > s_0$ at least once. Denote now

$$\mu_{2n}(t, R, \boldsymbol{Z}; s \to s_0, \boldsymbol{Z}_0)dtd\boldsymbol{R}d\boldsymbol{Z} \tag{3.11}$$

as the probability that this set of $2n$ trajectories crossed the scale s for the last time before reaching $s_0, \boldsymbol{Z}_0$, between t and $t + dt$, with a center of mass between $\boldsymbol{R}$ and $\boldsymbol{R} + d\boldsymbol{R}$ and with a shape between $\boldsymbol{Z}$ and $\boldsymbol{Z} + d\boldsymbol{Z}$.

In terms of this probability we can rewrite Eq.(3.4) (displaying, for clarity, $\boldsymbol{R}_0 = 0$ and $t = 0$) as

$$F_{2n}(\boldsymbol{R}_0 = 0, s_0, \boldsymbol{Z}_0, t = 0) = \int d\boldsymbol{Z} \int_{-\infty}^0 dt \int d\boldsymbol{R}\mu_{2n}(t, R, \boldsymbol{Z}; s \to s_0, \boldsymbol{Z}_0)$$

$$\times \left\langle \int_{-\infty}^0 dt_1 \cdots dt_n \Big[\Xi(\boldsymbol{r}_1(t_1) - \boldsymbol{r}_2(t_1)) \cdots \Xi(\boldsymbol{r}_{2n-1}(t_n) - \boldsymbol{r}_{2n}(t_n)) + \mathrm{perms}\Big] \Big| (s; \boldsymbol{R}, \boldsymbol{Z}, t) \right\rangle_{\boldsymbol{u}, \boldsymbol{\eta}_i} . \tag{3.12}$$

The meaning of the conditional averaging is an averaging over all the realizations of the velocity field and the random $\boldsymbol{\eta}_i$ for which Lagrangian trajectories that ended up at time $t = 0$ in $\boldsymbol{R} = 0, s_0, \boldsymbol{Z}_0$ passed through $\boldsymbol{R}, s, \boldsymbol{Z}$ at time t.

Next, the time integrations in Eq.(3.12) are split to the interval $[-\infty, t]$ and $[t, 0]$ giving rise to 2^n different contributions:

$$\int_{-\infty}^t dt_1 \cdots \int_{-\infty}^t dt_n + \int_t^0 dt_1 \int_{-\infty}^t dt_2 \cdots \int_{-\infty}^t dt_n + \ldots \tag{3.13}$$

Consider first the contribution with n integrals in the domain $[-\infty, t]$. It follows from the delta-correlation in time of the velocity field, that we can write

$$\left\langle \int_{-\infty}^t dt_1 \cdots dt_n \Big[\Xi(\boldsymbol{r}_1(t_1) - \boldsymbol{r}_2(t_1)) \cdots \Xi(\boldsymbol{r}_{2n-1}(t_n) - \boldsymbol{r}_{2n}(t_n)) + \mathrm{perms}\Big] \Big| (s; \boldsymbol{R}, \boldsymbol{Z}, t) \right\rangle_{\boldsymbol{u}, \boldsymbol{\eta}_i}$$

$$= \left\langle \int_{-\infty}^t dt_1 \cdots dt_n \Big[\Xi(\boldsymbol{r}_1(t_1) - \boldsymbol{r}_2(t_1)) \cdots \Xi(\boldsymbol{r}_{2n-1}(t_n) - \boldsymbol{r}_{2n}(t_n)) + \mathrm{perms}\Big] \right\rangle_{\boldsymbol{u}, \boldsymbol{\eta}_i}$$

$$= F_{2n}(\boldsymbol{R}, s, \boldsymbol{Z}, t) = F_{2n}(s, \boldsymbol{Z}) . \tag{3.14}$$

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The last equality follows from translational invariance in space-time. Accordingly the contribution with n integrals in the domain $[-\infty, t]$ can be written as

$$\int d\boldsymbol{Z} F_{2n}(s, \boldsymbol{Z}) \int_{-\infty}^{0} dt \int d\boldsymbol{R} \ \mu_{2n}(t, R, \boldsymbol{Z}; s \to s_0, \boldsymbol{Z}_0) \ . \tag{3.15}$$

We identify the shape-to-shape transition probability:

$$\gamma(\boldsymbol{Z}; \boldsymbol{Z}_0, s \to s_0) = \int_{-\infty}^{0} dt \int d\boldsymbol{R} \ \mu_{2n}(t, R, \boldsymbol{Z}; s \to s_0, \boldsymbol{Z}_0) \ . \tag{3.16}$$

Finally, putting all this added wisdom back in Eq.(3.12) we end up with

$$F_{2n}(s_0, \boldsymbol{Z}_0) = I + \int d\boldsymbol{Z} \gamma(\boldsymbol{Z}; \boldsymbol{Z}_0, s \to s_0) F_{2n}(s, \boldsymbol{Z}) \ . \tag{3.17}$$

Here I represents all the contributions with one or more time integrals in the domain $[t, 0]$. The key point now is that only the term with n integrals in the domain $[-\infty, t]$ contains information about the evolution of $2n$ Lagrangian trajectories that probed the forcing scale L. Accordingly, the term denoted by I cannot contain information about the leading anomalous scaling exponent belonging to F_{2n}, but only of lower order exponents. The anomalous scaling dependence of the LHS of Eq.(3.17) has to cancel against the integral containing F_{2n} without the intervention of I.

Representing now

$$F_{2n}(s_0, \boldsymbol{Z}_0) = \sum_{q\ell m} a_{q,\ell m}(s_0) B_{q\ell m}(\boldsymbol{Z}_0) \ ,$$

$$F_{2n}(s, \boldsymbol{Z}) = \sum_{q\ell m} a_{q,\ell m}(s) B_{q\ell m}(\boldsymbol{Z}) \ ,$$

$$I = \sum_{q\ell m} I_{q\ell m} B_{q\ell m}(\boldsymbol{Z}_0) \tag{3.18}$$

and substituting on both sides of Eq.(3.17) and using Eq.(3.10) we find, due to the linear independence of the eigenfunctions $B_{q\ell m}$

$$a_{q,\ell m}(s_0) = I_{q\ell m} + \left(\frac{s}{s_0}\right)^{\zeta_{2n}^{(q,\ell)}} a_{q,\ell m}(s) \tag{3.19}$$

To leading order the contribution of $I_{q\ell m}$ is neglected, leading to the conclusion that *the spectrum of anomalous exponents of the correlation functions is determined by the eigenvalues of the shape-to-shape transition probability operator.* Calculations show that the leading exponent in the isotropic sector is always smaller than the leading exponents in all other sectors. This gap between the leading exponent in the isotropic sector to the rest of the exponents determines the rate of decay of anisotropy upon decreasing the scale of observation.

IV. CONCLUDING REMARKS

The derivation presented above has used explicitly the properties of the advecting field, in particular the δ-correlation in time. Accordingly it cannot be immediately generalized to more generic situations in which there exist time correlations. Nevertheless we find it pleasing that at least in the present case we can trace the physical origin of the exponents anomaly, and connect it to the underlying dynamics. In more generic cases the mechanisms may be more complicated, but one should still keep the lesson in mind - higher order correlation functions depend on many coordinates, and these define a configuration in space. The scaling properties of such functions may very well depend on how such configurations are reached by the dynamics. Focusing on static objects like structure functions of one variable may be insufficient for the understanding of the physics of anomalous scaling.

ACKNOWLEDGMENTS

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