THE BIANCHI–DARBOUX TRANSFORM OF L-ISOTHERMIC SURFACES

Emilio Musso and Lorenzo Nicolodi

Abstract. We study an analogue of the classical Bäcklund transformation for L-isothermic surfaces in Laguerre geometry, the Bianchi–Darboux transformation. We show how to construct the Bianchi–Darboux transforms of an L-isothermic surface by solving an integrable linear differential system. We then establish a permutability theorem for iterated Bianchi–Darboux transforms.

1. Introduction.

Certain types of integrable non-linear PDEs (soliton equations) arise in differential geometry as compatibility conditions for the linear equations obeyed by frames adapted to surfaces in higher dimensional manifolds. In a number of situations, the construction of new solutions of the arising PDE relies on the existence of Bäcklund type transformations for the surfaces and on their permutability properties. Well-known examples include pseudo-spherical surfaces, affine minimal surfaces, and isothermic surfaces in Möbius geometry [CT],[BHPP],[CGS],[TT]. The loop group approach to soliton theory, recently developed by Terng–Uhlenbeck and others [TU], explains the uniformity of results in several of these examples.

This paper concentrates on L-isothermic surfaces. An immersion $f : M \to \mathbb{R}^3$ is called L-isothermic if about each point of M there exist curvature line coordinates (x, y) which are conformal with respect to the third fundamental form of f. L-isothermic surfaces are invariant under a group of contact transformations, the Laguerre group, which is isomorphic to the identity component of the isometry group of Minkowski 4-space. This group acts transitively on the set of oriented 2-spheres and points as well as on the set of oriented planes of $\mathbb{R}^3$. We first became interested in L-isothermic surfaces when studying the (infinitesimal) deformation problem for surfaces under the Laguerre group. Namely, L-isothermic surfaces are the only surfaces allowing

1991 *Mathematics Subject Classification.* 53A40, 53A05.

Key words and phrases. Transformations of surfaces, Laguerre geometry, L-isothermic surfaces, Bäcklund transformation, Bianchi–Darboux transformation, Permutability theorem.

Partially supported by MURST Project "Proprietà geometriche delle varietà reali e complesse".

1-parameter families of second order deformations [MN1],[MN2]. Such families correspond to the solutions of the non-linear fourth-order equation $\Delta(w_{xy}w^{-1}) = 0$ which in turn is equivalent to the Gauss-Codazzi equations and arises as the integrability condition of a linear differential system containing a free parameter [MN3].

The theory of L-isothermic surfaces presents many analogies with that of isothermic surfaces in Möbius geometry. The latter has received much attention in recent studies [Bu],[BHPP],[He],[HP],[HMN], especially in relation with the general theory of curved flats in pseudo-Riemannian symmetric spaces [FP]. Curved flats arise in 1-parameter families as solutions to a certain integrable system expressed in the form of a zero-curvature equation with spectral parameter. This approach to the study of isothermic surfaces provides a uniform framework for understanding their theory of transformations and deformation. The notion of isothermic submanifolds have also been defined in other contexts; for instance, a general theory of isothermic submanifolds in symmetric R-spaces have been recently developed in [BPP].

In this paper, we study a geometric transformation for L-isothermic surfaces by realizing them as enveloping surfaces of a 2-sphere congruence. The Laguerre invariant conditions that the congruence preserve curvature lines and that the third fundamental forms of the two envelopes at corresponding points be conformal yield that both surfaces are L-isothermic. This is the content of Theorem 1. The resulting congruence is an analogue of the Darboux congruence occurring in Möbius geometry and its two envelopes are said to form a Bianchi–Darboux pair. A Bianchi–Darboux pair gives rise to a curved flat in the Grassmannian $\tilde{G}_{1,1}(\mathbb{R}^4_1)$ of oriented 2-planes of signature $(1,1)$ in $\mathbb{R}^4_1$; conversely, a curved flat in this Grassmannian only determines the normals of a Bianchi–Darboux pair. Moreover, the spectral deformation of curved flats amounts to second order deformation as in the conformal situation. In Section 5, we prove the existence of Bianchi–Darboux transforms by explicitly constructing new L-isothermic surfaces from a given one; the construction requires solving an integrable linear differential system. This furnishes a geometrical method of deriving new solutions of the defining PDE $\Delta(w_{xy}w^{-1}) = 0$ from any given one. Finally, in Section 6, we establish a permutability theorem for the Bianchi–Darboux transformation.

Acknowledgements. It is a pleasure to thank Fran Burstall and Udo Hertrich-Jeromin for many interesting and instructive discussions concerning isothermic surfaces and their transformation theory. Thanks also to the referee for his comments and questions.

2. L-isothermic surfaces.

Let $f : M \to \mathbb{R}^3$ be an immersion of a surface M in Euclidean space with no parabolic points oriented by a field of unit normals $n : M \to S^2$. Consider on M the unique complex structure compatible with the given orientation and the conformal structure defined by the third fundamental form $\mathrm{III} = \mathrm{d}n \cdot \mathrm{d}n$. (Here $\cdot$ denotes the Euclidean inner product.) Accordingly, the second fundamental form II decomposes into bidegrees:

$$\mathrm{II} = \mathrm{II}^{(2,0)} + \mathrm{II}^{(1,1)} + \mathrm{II}^{(0,2)}.$$

$\mathrm{II}^{(2,0)}$ is a globally defined $(2,0)$ symmetric bilinear form on M which plays the role of the usual Hopf differential for the pair of quadratic forms III, II. We refer to $\mathrm{II}^{(2,0)}$ simply as the Hopf differential.

Definition. A Riemann surface M equipped with a holomorphic quadratic differential q is called a *polarized surface*. An immersion $f : (M, q) \to \mathbb{R}^3$ is called *L-isothermic* if $\mathrm{II}^{(2,0)} = \mu q$, for a real-valued smooth function $\mu : M \to \mathbb{R}$.

Near any point $p \in M$ where $q_{|p} \neq 0$ there exists local complex coordinate $z = x + iy : \Omega \subset M \to \mathbb{C}$ such that $q_{|\Omega} = \mathrm{d}z^2$ and $\mathrm{III}_{|\Omega} = e^{2u}\mathrm{d}z\mathrm{d}\overline{z}$. Then (x, y) are curvature line coordinates which are conformal for the third fundamental form of f. (x, y) will be called *conformal principal coordinates* for f[1].

Examples of L-isothermic surfaces include surfaces of revolution, molding surfaces [BCG], surfaces with plane lines of curvature in both systems [MN4], and the class of Weingarten surfaces on which $aH + bK = 0$, for constants a, b with $a \neq 0$, where H and $K \neq 0$ denote the mean and Gaussian curvatures, respectively. The last example follows as an application of Hopf's classical argument: H/K which is $H(\mathrm{III}, \mathrm{II}) := (1/2)\mathrm{tr}_{\mathrm{III}}\mathrm{II}$ is constant if and only if $\mathrm{II}^{(2,0)}$ is holomorphic [H].

3. The geometry of L-isothermic surfaces.

A pair of real quadratic forms III and II on M such that: III is positive definite, the intrinsic curvature $K(\mathrm{III}) \equiv 1$, and such that II satisfies the Codazzi equations with respect to the metric III, defines, up to contact tansformations, an immersion $F = (f, n) : M \to \mathbb{R}^3 \times S^2$ in the space of contact elements of $\mathbb{R}^3$ satisfying the *contact condition* $\mathrm{d}f \cdot n = 0$. A map satisfying this condition is called a Legendre immersion. The *Euclidean projection* $f : M \to \mathbb{R}^3$ need not be an immersion nor even have constant rank.

Let $\Lambda = \mathbb{R}^3 \times S^2$ denote the space of contact elementsof $\mathbb{R}^3$. Now each contact element $(x, n) \in \Lambda$ corresponds to a null line in Minkowski 4-space $\mathbb{R}^4_1$ with its Lorentz scalar product $\langle\ ,\ \rangle = (u^1)^2 + (u^2)^2 + (u^3)^2 - (u^4)^2$ via

$$(x, n) \mapsto [x, n] := \left\{ {}^t(x - tn, t), t \in \mathbb{R} \right\} \subset \mathbb{R}^4_1.$$

Under this identification, the identity component $L = \mathbb{R}^4 \rtimes SO_0(3, 1)$ of the isometry group of $\mathbb{R}^4_1$ acts transitively on Λ and preserves the contact condition. The action of L permutes Legendre immersions: let $F = (f, n)$ be a Legendre immersion and, for $p \in M$, consider the null line $[f(p), n(p)]$. For each $A \in L$, $A \cdot [f(p), n(p)] \in \Lambda$ and intersects $\mathbb{R}^3 = \{{}^t(x, t) \in \mathbb{R}^4_1 : t = 0\}$ in the unique point $f'(p)$. If ${}^t(n', -1)$ denotes

[1] The umbilic points of the immersion are precisely the zeros of the Hopf diffrential. The idea here is that not all umbilic points prohibit conformal principal coordinates. If an umbilic is caused by a zero in the polarization, then it is a *bad* umbilic, where no conformal principal coordinates may be found; instead, umbilic points caused by zeros in μ will not cause problems.

the null direction of $A \cdot [f(p), n(p)]$, then $AF := (f', n')$ defines another Legendre immersion. Notice that the action does not preserve the Euclidean fibration $\Lambda \to \mathbb{R}^3$.

Remark. A standard model for Laguerre geometry is obtained by identifying $\mathbb{R}^4_1$ with the set $\mathcal{S}$ of oriented spheres (including point spheres) of $\mathbb{R}^3$ by

$$\Sigma : \mathcal{S} \to \mathbb{R}^4_1, \quad \sigma_r(p) \mapsto {}^t(p, r),$$

where $\sigma_r(p)$ denotes a sphere with center p and signed radius r. Note that if $r > 0$ (resp. $r < 0$) then ${}^t(p, r)$ is the vertex of the backward (resp. forward) pointing light-cone which intersects $\mathbb{R}^3$ in exactly the sphere $\sigma_r(p)$. Two spheres $\sigma_r(p)$ and $\sigma'_r(p)$ are in oriented contact if and only if ${}^t(p - p', r - r')$ is a null vector. Thus each contact element (x, n) determines a null line of 2-spheres all of which are in oriented contact at x with normal n. Following the classical terminology, a *Laguerre transformation* is a contact transformation of Λ induced by an element in the group L. In terms of $\mathbb{R}^3$, a Laguerre transformation takes oriented planes in $\mathbb{R}^3$ to oriented planes, and oriented spheres to oriented spheres. In this context, two immersions $f, f' : M \to \mathbb{R}^3$ are said to be *Laguerre equivalent* if there exists $A \in L$ such that their respective Legendrian lifts F, F' satisfy $AF = F'$. (For more details about Laguerre geometry we refer to [Bl],[C],[MN1].)

The theory of L-isothermic surfaces belongs in the Laguerre geometry:

(Laguerre invariance). *Let $F = (f, n)$ be the Legendrian lift of an L-isothermic surfaces f and $A : \Lambda \to \Lambda$ be a Laguerre contact transformation. Then the Euclidean projection of $AF = (f', n')$ is L-isothermic also.*

Proof. Let $\mathcal{L}^+$ be the positive light-cone in $\mathbb{R}^4_1$. The projective light-cone $\mathbb{P}(\mathcal{L}^+)$ identifies with the conformal 2-sphere S^2 and the projection $\mathcal{L}^+ \to \mathbb{P}(\mathcal{L}^+)$ is a principal $\mathbb{R}^+$–bundle which is trivial. For each $A \in L$, $\mathrm{d}A \in SO_0(3, 1)$, which preserves the light-cone $\mathcal{L}^+$ and descends to an orientation-preserving conformal diffeomorphism $\tilde{A}$ of S^2. This implies that $n' = \tilde{A}(n)$. Thus, the conformal class of III and then the conformal class of II (mod III) are Laguerre invariant. $\square$

3.1 Notations: moving frames in Laguerre geometry.

Here and in the following we shall consider $\mathbb{R}^4_1$ with linear coordinates $x^1, \dots, x^4$ such that $\langle \, , \, \rangle = -2x^1x^4 + (x^2)^2 + (x^3)^2$; an orientation for which $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^4 \neq 0$; and a time-orientation given by $x^1 + x^4 > 0$. By a *Laguerre frame* is meant a position vector $a_0 \in \mathbb{R}^4_1$ and an oriented basis a_1, a_2, a_3, a_4 of $\mathbb{R}^4_1$ such that $(a_0; a_1, a_2, a_3, a_4) \in L$. Up to the choice of a reference frame, the manifold of Laguerre frames may be identified with the group L. For any $A = (a_0, a) \in L$, a_i, $i = 1, 2, 3, 4$, denote the column vectors of the matrix a. Geometrically, the null directions a_1, a_4 represent oriented planes which are in oriented contact with the oriented sphere represented by a_0. By $[a_0, a_1], [a_0, a_4]$ we denote the lines in $\mathbb{R}^4_1$ through a_0 with null directions a_1 and a_4, respectively.

The transitive action of L on Λ defines a principal L_0-bundle

$$\pi_L : L \to \Lambda = L/L_0, A \mapsto [a_0, a_1].$$

Definition. A *Laguerre frame field* in Λ is a local smooth section $A = (a_0, a)$ of π_L. A (local) *Laguerre framing along a Legendre immersion* $F : M \to \Lambda$ is a smooth map $A : \mathcal{U} \to L$ defined on an open subset $\mathcal{U} \subset M$, such that $\pi_L A = [a_0, a_1] = F$.

Remark (Principal frames). Let $F : (M, q) \to \Lambda$ be a Legendrian immersion. Then F can be equipped with a Laguerre framing $A = (a_0; a_1, a_2, a_3, a_4) : M \to L$ such that $F(p) = [a_0, a_1](p)$ and $\mathrm{span}\{a_2, a_3\}_{|p} = \mathrm{d}F_{|p}(T_pM)$, for any $p \in M$. Let $\alpha = A^{-1}\mathrm{d}A$ be its connection form. Then $\alpha_0^4 = 0$, $\alpha_1^2 \wedge \alpha_1^3 \neq 0$. Differentiating $\alpha_0^4 = 0$, it follows that the quadratic form $\alpha_0^2\alpha_1^2 + \alpha_0^3\alpha_1^3$ is symmetric and then diagonalizable. So we may assume that $\alpha_0^2 \wedge \alpha_1^2 = \alpha_0^3 \wedge \alpha_1^3 = 0$. We call such an A a *principal framing* along F. An easy calculation using the Maurer–Cartan equation of L yields

$$\mathrm{d}(\alpha_1^2 + i\alpha_1^3) = (\alpha_1^1 - *\alpha_2^3) \wedge (\alpha_1^2 + i\alpha_1^3),$$

from which follows that F *is L-isothermic if and only if* $d(\alpha_1^1 - *\alpha_2^3) = 0$.

4. Sphere congruences and the Bianchi–Darboux transformation.

In classical surface theory, a sphere congruence is an immersion $S : M \to \mathbb{R}^4_1$ of a surface M into the space $\mathbb{R}^4_1$ of all oriented 2-spheres (including points) of $\mathbb{R}^3$. A Legendre immersion $F = (f, n)$ is said to envelope the sphere congruence S if for each $p \in M$, the sphere $\Sigma^{-1}(S(p))$ is in oriented contact with the Legendre surface at $F(p)$. If S is a space-like immersion, i.e., the induced metric on M is positive definite, then there exist two enveloping surfaces [Bl]. It follows that the spheres of the congruence are the common tangent spheres of the two envelopes. Accordingly, there results a mapping between the enveloping surfaces such that the sphere congruence consists of the spheres passing through the points on the envelopes.

Definition. A space-like sphere congruence is called *Ribaucour* if the resulting mapping between the two envelopes preserves curvature lines. A Ribaucour sphere congruence is called *Darboux*[2] if the resulting mapping is conformal with respect to the third fundamental forms induced on M by the two envelopes F, $\hat{F}$.

Let $S : M \to \mathbb{R}^4_1$ be a space-like congruence enveloping F and $\hat{F}$; we can consider an adapted Laguerre frame $A = (S; a_1, \ldots, a_4) : M \to L$ such that $F = [S, a_1]$,

[2] In the context of Möbius geometry a sphere congruence is an immersion of a surface into the Lorentzian sphere S^4_1 interpreted as the space of all oriented spheres (excluding points) and oriented planes in $\mathbb{R}^3$. A sphere congruence is then called Ribaucour if the curvature lines on the two envelopes correspond; a Ribaucour sphere congruence is Darboux if the correspondence between the two envelopes is conformal with respect to the first fundamental forms of the envelopes, see for example [BHPP] and the literature therein.

$\hat{F} = [S, a_4]$, a_2, a_3 are tangential, and a_1, a_4 are normal null directions. The connection form $\alpha = (\alpha_0, \alpha') = A^{-1}\mathrm{d}A$ of A is then

$$\left[\begin{pmatrix} 0 \\ \alpha_0^2 \\ \alpha_0^3 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha_1^1 & \alpha_2^1 & \alpha_3^1 & 0 \\ \alpha_1^2 & 0 & -\alpha_2^3 & \alpha_2^1 \\ \alpha_1^3 & \alpha_2^3 & 0 & \alpha_3^1 \\ 0 & \alpha_1^2 & \alpha_1^3 & -\alpha_1^1 \end{pmatrix}\right].$$

Observe that $\alpha_0^2 \wedge \alpha_0^3 \neq 0$ on M. The Maurer–Cartan equation $\mathrm{d}\alpha + \alpha \wedge \alpha = 0$ yields the Ricci equations:

$$0 = \alpha_0^2 \wedge \alpha_2^1 + \alpha_0^3 \wedge \alpha_3^1, \tag{1}$$

$$0 = \alpha_0^2 \wedge \alpha_1^2 + \alpha_0^3 \wedge \alpha_1^3, \tag{2}$$

$$d\alpha_1^1 + \alpha_2^1 \wedge \alpha_1^2 + \alpha_3^1 \wedge \alpha_1^3 = 0, \tag{3}$$

the Gauss equation:

$$\mathrm{d}\alpha_2^3 = \alpha_2^1 \wedge \alpha_1^3 + \alpha_1^2 \wedge \alpha_3^1, \tag{4}$$

and the Codazzi equations:

$$\mathrm{d}\alpha_0^2 = \alpha_2^3 \wedge \alpha_0^2, \quad \mathrm{d}\alpha_0^3 = -\alpha_2^3 \wedge \alpha_0^2 \tag{5}$$

$$\mathrm{d}\alpha_1^2 = \alpha_1^1 \wedge \alpha_1^2 + \alpha_1^2 \wedge \alpha_2^3, \quad \mathrm{d}\alpha_1^3 = \alpha_1^1 \wedge \alpha_1^3 + \alpha_1^2 \wedge \alpha_2^3 \tag{6}$$

$$\mathrm{d}\alpha_2^1 = \alpha_2^1 \wedge \alpha_1^1 + \alpha_2^3 \wedge \alpha_3^1, \quad \mathrm{d}\alpha_3^1 = \alpha_3^1 \wedge \alpha_1^1 - \alpha_2^3 \wedge \alpha_2^1. \tag{7}$$

We thus can state:

Theorem 1. *Let $S : M \to \mathbb{R}_1^4$ be a flat space-like immersion with flat normal bundle. Then S is a Darboux sphere congruence and both its enveloping surfaces – which have opposite orientations – are L-isothermic.*

Proof. Assume S induces a flat space-like metric on M, i.e., $\mathrm{d}\alpha_1^1 = 0$, and has flat normal bundle, i.e., $\mathrm{d}\alpha_2^3 = 0$. According to (2), the second fundamental form $\alpha_0^2\alpha_1^2 + \alpha_0^3\alpha_1^3$ of S in the normal direction a_1 is diagonalizable. So we may choose A such that

$$\alpha_0^2 = h_2\alpha_1^2, \quad \alpha_0^3 = h_3\alpha_1^3, \quad h_2 - h_3 \neq 0. \tag{8}$$

In particular, A becomes a principal framing along F. We can now write $\alpha_2^1 = a_{11}\alpha_1^2 + a_{12}\alpha_1^3$, $\alpha_3^1 = a_{21}\alpha_1^2 + a_{22}\alpha_1^3$ for smooth functions a_{ij}. From equations (3) and (4) we obtain $a_{12} = a_{21}$, $a_{11} = -a_{22}$. Further substituting into (1) and (2) yields

$$a_{12}(h_2 - h_3) = 0,$$

and hence $a_{12} = 0$. Therefore $\alpha_2^1 = a_{11}\alpha_1^2$ and $\alpha_3^1 = -a_{11}\alpha_1^3$. This implies, in particular, that the correspondence induced by S on the two envelopes preserves curvature

lines and that $\langle \mathrm{d}a_1, \mathrm{d}a_1\rangle \propto \langle \mathrm{d}a_4, \mathrm{d}a_4\rangle$, that is, S is a Darboux sphere congruence. Also, this tells us that the two envelopes have opposite orientations.

As for the isothermic property of the envelopes, by the Codazzi equations (7),

$$\mathrm{d}a_{11} \wedge \alpha_1^2 = -2a_{11}(\alpha_1^1 \wedge \alpha_1^2 + \alpha_2^3 \wedge \alpha_1^3)$$
$$\mathrm{d}a_{11} \wedge \alpha_1^3 = -2a_{11}(\alpha_1^1 \wedge \alpha_1^3 - \alpha_2^3 \wedge \alpha_1^2),$$

and from these $2(\alpha_1^1 - *\alpha_2^3) = -\mathrm{d}\log|a_{11}|$, which is the condition for F being L-isothermic according to the remark in the previous section.

Concerning the second envelope $\hat{F}$, $B = (b_0; b_1, b_2, b_3, b_4) := (a_0; a_4, a_2, -a_3, a_1)$ defines a frame along $\hat{F}$. Its connection form is computed to be

$$\beta = B^{-1}\mathrm{d}B = \left[\begin{pmatrix} 0 \\ \alpha_0^2 \\ -\alpha_0^3 \\ 0 \end{pmatrix}, \begin{pmatrix} -\alpha_1^1 & \alpha_1^2 & -\alpha_1^3 & 0 \\ \alpha_2^1 & 0 & \alpha_2^3 & \alpha_1^2 \\ -\alpha_3^1 & -\alpha_2^3 & 0 & -\alpha_1^3 \\ 0 & \alpha_2^1 & -\alpha_3^1 & \alpha_1^1 \end{pmatrix}\right].$$

B is then a principal framing along $\hat{F}$, and the one form $\beta_1^1 - *\beta_2^3$ is closed. So, also $\hat{F}$ is L-isothermic. $\Box$

If S is a Darboux congruence, then S is either as in Theorem 1, or the normals a_1 and a_4 of the two envelopes are Möbius equivalent in S^2, that is F and $\hat{F}$ are Laguerre equivalent and have the same orientations. In the latter case the congruence S belongs to a fixed sphere complex [Bl].

Up to this degenerate situations, we can state:

Theorem 2. *A map $S : M \to \mathbb{R}_1^4$ defines a nondegenerate Darboux sphere congruence if and only if it is a flat space-like immersion with flat normal bundle.*

Definition. The L-isothermic surfaces enveloping a nondegenerate Darboux congruence are called *Bianchi–Darboux transforms* of each other. They form a Bianchi–Darboux pair.

Remark. Let $F = (f, n)$, $\hat{F} = (\hat{f}, \hat{n})$ be a Bianchi–Darboux pair enveloped by $S : M \to \mathbb{R}_1^4$. Next, let $\mathfrak{so}(3,1) = \mathfrak{k} \oplus \mathfrak{m}$, $\mathfrak{k} = \mathfrak{so}(2) \times \mathfrak{so}(1,1)$ be a symmetric decomposition of the Lie algebra of $SO_0(3,1)$. Then, according to (3) and (4), the flatness of both the induced metric and the normal bundle of S is expressed by

$$\alpha'_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}} = 0, \tag{9}$$

where $\alpha'_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-part of α'. This condition expresses the fact that the map $(n, \hat{n}) : M \to N = S^2 \times S^2 \setminus \Delta$ into the manifold of ordered pairs of distinct points of S^2 is a curved flat, see [FP],[BHPP]. N may be viewed as the Grassmannian $\tilde{G}_{1,1}(\mathbb{R}_1^4)$ of oriented 2-planes of signature $(1,1)$ in $\mathbb{R}_1^4$ — a pseudo-Riemannian symmetric space with invariant metric of signature $(2,2)$. Note that the converse is not true, that is, from a curved flat in N we can only construct the normals of a Bianchi–Darboux pair.

Definition. A map $A = (a_0, a) : M \to L$ such that $\alpha = (\alpha_0, \alpha') = A^{-1}dA$ satisfies (9) and $\alpha_0^1 = \alpha_0^4 = 0$ is called a *curved flat framing* and $a_0 : M \to \mathbb{R}^4_1$ the associated sphere congruence. Note that, up to a gauge change, such a frame can be chosen so that $\alpha_1^1 = 0$.

The above discussion yields:

Corollary 1. *The sphere congruence $a_0 : M \to \mathbb{R}^4_1$ of a curved flat framing $A = (a_0, a) : M \to L$ is a Darboux sphere congruence enveloped by the two L-isothermic immersions $F = [a_0, a_1]$ and $\hat{F} = [a_0, a_4]$. Conversely, a (nondegenerate) Darboux congruence S defines a curved flat framing $(S; a_1, a_2, a_3, a_4) : M \to L$, where $\mathrm{d}S(TM) = \mathrm{span}\{a_2, a_3\}$ and a_1, a_4 generate the null subbundles of the normal bundle of S.*

In the next section we shall discuss the existence of Bianchi–Darboux transforms.

5. Integrability: construction of the Bianchi–Darboux transform.

Let $F : M \to \Lambda$ be L-isothermic, and consider a principal frame $A = (a_0; a_1, a_2, a_3, a_4)$ along F, $F = [a_0, a_1]$. Let α denote its connection form and ρ be a smooth positive function such that

$$2(\alpha_1^1 - *\alpha_2^3) = -\mathrm{d}\log\rho.$$

Define

$$\alpha_\rho = (\alpha_0, \alpha'_\rho) := \left[\begin{pmatrix} \alpha_0^1 \\ \alpha_0^2 \\ \alpha_0^3 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha_1^1 & \alpha_2^1 + \rho\alpha_1^2 & \alpha_3^1 - \rho\alpha_1^3 & 0 \\ \alpha_1^2 & 0 & -\alpha_2^3 & \alpha_2^1 + \rho\alpha_1^2 \\ \alpha_1^3 & \alpha_2^3 & 0 & \alpha_3^1 - \rho\alpha_1^3 \\ 0 & \alpha_1^2 & \alpha_1^3 & -\alpha_1^1 \end{pmatrix}\right].$$

Note that $0 = \mathrm{d}\alpha_\rho + \alpha_\rho \wedge \alpha_\rho$.

Consider a solution $v = {}^t(v^1, \ldots, v^4) : M \to \mathcal{L}^+ \subset \mathbb{R}^4_1$ of the completely integrable linear system

$$\mathrm{d}v = -\alpha'_\rho v.$$

By a direct computation:

Lemma. *The one form* $\gamma := v^2\alpha_0^2 + v^3\alpha_0^3 - v^4\alpha_0^1$ *is closed.*

Next put $s := \frac{r}{v_4}$, where[3] $dr = \gamma$, and consider the gauged frame $\tilde{A} = Ag(s, v)$, where[4]

$$g(s, v) = \left[\begin{pmatrix} s \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/v^4 & v^2/v^4 & v^3/v^4 & v^1 \\ 0 & 1 & 0 & v^2 \\ 0 & 0 & 1 & v^3 \\ 0 & 0 & 0 & v^4 \end{pmatrix}\right].$$

[3] As it will be clear from the following discussion, the function r, which locally integrates γ, gives essentially the signed radius of the Darboux sphere congruence we are going to construct (cf. Section 5.1).

[4] Observe that the group of all elements $g(s, v)$, $s \in \mathbb{R}$,$v \in \mathcal{L}^+$, is diffeomorphic to the structure group of the fibration $\mathcal{P}(F) \to M$ of principal frames along F.

Observe that $[\tilde{a_0}, \tilde{a_1}] = [a_0, a_1] = F$. The connection form $\tilde{A}^{-1}\mathrm{d}\tilde{A}$ takes the form

$$\left[\begin{pmatrix} 0 \\ \alpha_0^2 + s\alpha_1^2 \\ \alpha_0^3 + s\alpha_1^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & -\rho v^4\alpha_1^2 & \rho v^4 \alpha_1^3 & 0 \\ \frac{1}{v^4}\alpha_1^2 & 0 & -\alpha_2^3 + \frac{1}{v^4}(v^3\alpha_1^2 - v^2\alpha_1^3) & -\rho v^4\alpha_1^2 \\ \frac{1}{v^4}\alpha_1^3 & \alpha_2^3 - \frac{1}{v^4}(v^3\alpha_1^2 - v^2\alpha_1^3) & 0 & \rho v^4\alpha_1^3 \\ 0 & \alpha_1^2 & \alpha_1^3 & 0 \end{pmatrix}\right].$$

Thus $\tilde{A}$ is a curved flat framing and $\tilde{a}_0 : M \to \mathbb{R}^4_1$ is a Darboux congruence which envelopes the Bianchi–Darboux pair F and $\hat{F} = [\tilde{a_0}, \tilde{a_4}]$.

We thus have proved:

Theorem 3. *Let $F : M \to \Lambda$ be an L-isothermic immersion and A be any principal frame along F. Then any solution $v : M \to \mathcal{L}^+$ of the linear system*

$$\mathrm{d}v = -\alpha'_\rho v$$

defines an L-isothermic immersion

$$\hat{F} := [a_0 + sa_1, v^1a_1 + v^2a_2 + v^3a_3 + v^4a_4] : M \to \Lambda$$

which is a Bianchi-Darboux transform of F.

Remark. The space of principal frames $\mathcal{P}(F)$ along a Legendrian immersion F can be viewed as a 6-dimensional integral submanifold of the exterior differential system $\omega_0^4 = 0$, $\omega_0^2 \wedge \omega_1^2 = \omega_0^3 \wedge \omega_1^3 = 0$ on L with independence condition $\omega_1^2 \wedge \omega_1^3 \neq 0$, where ω denotes the Maurer–Cartan form of L. Proving the existence of Bianchi–Darboux transforms for an L-isothermic F can be reduced to checking the Frobenius integrability condition for the Pfaffian system on $\mathcal{P}(F) \times \mathbb{R}$ given by:

$$\omega_0^1 = 0, \quad \omega_2^1 - a\omega_1^2 = 0, \quad \omega_3^1 + a\omega_1^3 = 0, \quad \mathrm{d}a + 2a(\omega_1^1 - *\omega_2^3) = 0.$$

5.1 The Bianchi-Darboux transform in Euclidean terms.

Let $f : (M, q) \to \mathbb{R}^3$ be an L-isothermic immersion with normal n and conformal principal coordinate $z = x + iy$. If we identify $\mathbb{E}(3)$ with the subgroup of L consisting of all $A \in L$ fixing the time-like vector $\epsilon_1 + \epsilon_4$ ($\epsilon_1, \ldots, \epsilon_4$ the canonical basis of $\mathbb{R}^4_1$), the Euclidean framing $(f; t_1, t_2, t_3) : M \to \mathbb{E}(3)$ defined by $t_1 = n$, $t_2 = \frac{f_x}{\|f_x\|}$, $t_3 = \frac{f_y}{\|f_y\|}$ corresponds to the Laguerre framing

$$E = (e_0; e) = \left(\begin{pmatrix} \frac{f^1}{\sqrt{2}} \\ f^2 \\ f^3 \\ \frac{-f^1}{\sqrt{2}} \end{pmatrix}; \begin{pmatrix} \frac{1+t_1^1}{2} & \frac{t_2^1}{\sqrt{2}} & \frac{t_3^1}{\sqrt{2}} & \frac{1-t_1^1}{2} \\ \frac{t_1^2}{\sqrt{2}} & t_2^2 & t_3^2 & \frac{-t_1^2}{\sqrt{2}} \\ \frac{t_1^3}{\sqrt{2}} & t_2^3 & t_3^3 & \frac{-t_1^3}{\sqrt{2}} \\ \frac{1-t_1^1}{2} & \frac{-t_2^1}{\sqrt{2}} & \frac{-t_3^1}{\sqrt{2}} & \frac{1+t_1^1}{2} \end{pmatrix}\right),$$

whose connection form can be written as

$$\left[\begin{pmatrix} 0 \\ h_1\mathrm{d}x \\ h_2\mathrm{d}y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{e^u}{\sqrt{2}}\mathrm{d}x & -\frac{e^u}{\sqrt{2}}\mathrm{d}y & 0 \\ \frac{e^u}{\sqrt{2}}\mathrm{d}x & 0 & u_y\mathrm{d}x - u_x\mathrm{d}y & -\frac{e^u}{\sqrt{2}}\mathrm{d}x \\ \frac{e^u}{\sqrt{2}}\mathrm{d}y & -u_y\mathrm{d}x + u_x\mathrm{d}y & 0 & -\frac{e^u}{\sqrt{2}}\mathrm{d}y \\ 0 & \frac{e^u}{\sqrt{2}}\mathrm{d}x & \frac{e^u}{\sqrt{2}}\mathrm{d}y & 0 \end{pmatrix}\right]$$

for u, h_1, and h_2 smooth functions with $h_1h_2 \neq 0$ at each point.

In this setting $\rho = me^{-2u}$, for a constant m, and the linear system becomes

$$\begin{cases} \mathrm{d}v^1 = \frac{e^u - me^{-u}}{\sqrt{2}}\mathrm{d}x v^2 + \frac{e^u + me^{-u}}{\sqrt{2}}\mathrm{d}y v^3 \\ \mathrm{d}v^2 = -\frac{e^u}{\sqrt{2}}\mathrm{d}x v^1 - (u_y\mathrm{d}x - u_x\mathrm{d}y)v^3 + \frac{e^u - me^{-u}}{\sqrt{2}}\mathrm{d}x v^4 \\ \mathrm{d}v^3 = -\frac{e^u}{\sqrt{2}}\mathrm{d}y v^1 + (u_y\mathrm{d}x - u_x\mathrm{d}y)v^2 + \frac{e^u + me^{-u}}{\sqrt{2}}\mathrm{d}y v^4 \\ \mathrm{d}v^4 = -\frac{e^u}{\sqrt{2}}\mathrm{d}x v^1 - \frac{e^u}{\sqrt{2}}\mathrm{d}y v^2 \end{cases} . \tag{10}$$

Let ${}^t(v^1, v^2, v^3, v^4) : M \to \mathcal{L}^+$ be a solution to (10) and let r be a smooth function that locally integrates the closed 1-form $\gamma = v^2h_1\mathrm{d}x + v^3h_2\mathrm{d}y$; we will refer to the functions r, v^1, v^2, v^3, v^4 as *transforming functions*. We are in a position to state:

Theorem 4. *Let $f : M \to \mathbb{R}^3$ be an L-isothermic immersion and let r, v^1, v^2, v^3, v^4 be a set of transforming functions. Then*

$$\hat{f} = f + \frac{\sqrt{2}r}{v^1 + v^4}\frac{f_x \times f_y}{\|f_x \times f_y\|} - \frac{rv^2}{v^4(v^1 + v^4)}\frac{f_x}{\|f_x\|} - \frac{rv^3}{v^4(v^1 + v^4)}\frac{f_y}{\|f_y\|}.$$

is a Bianchi–Darboux transform of f. All Bianchi–Darboux transforms of f arise this way.

Proof. Let $F = [e_0, e_1]$ be the Legendrian lift of f. According to Theorem 3, its Bianchi–Darboux transform $\hat{F} = [e_0 + \frac{r}{v^4}e_1, v^1e_1 + v^2e_2 + v^3e_3 + v^4e_4]$, that is

$$\hat{F} = e_0 + \frac{r}{v^4}e_1 + \mu(v^1e_1 + v^2e_2 + v^3e_3 + v^4e_4),$$

for a smooth function $\mu : M \to \mathbb{R}$. Now $\hat{F}$ takes values in $\mathbb{E}(3)$ if and only if $\langle \hat{F}, \epsilon_1 + \epsilon_4\rangle = 0$ if and only if $\mu = -r/v^4(v^1 + v^4)$. Substituting and using the above realization of $\mathbb{R}^3$ in $\mathbb{R}^4_1$, we obtain the required expression for $\hat{f}$. $\square$

6. Superposition and permutability.

Let $A^{(1)}, A^{(2)} : M \to L$ be two curved flat framings having the same L-isothermic map $F = [a_0^{(1)}, a_1^{(1)}] = [a_0^{(2)}, a_1^{(2)}] : M \to \Lambda$ as first envelope, and with second envelopes $F^{(1)}$ and $F^{(2)}$, respectively. Let $\alpha^{(1)}$, $\alpha^{(2)}$ be the corresponding connection forms.

The deformed forms associated with $\alpha^{(1)}$ are given by

$$\alpha_\lambda^{(1)} = \left[\begin{pmatrix} 0 \\ \alpha_0^2 \\ \alpha_0^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & \lambda\alpha_2^1 & \lambda\alpha_3^1 & 0 \\ \alpha_1^2 & 0 & -\alpha_2^3 & \lambda\alpha_2^1 \\ \alpha_1^3 & \alpha_2^3 & 0 & \lambda\alpha_3^1 \\ 0 & \alpha_1^2 & \alpha_1^3 & 0 \end{pmatrix}\right].$$

for some constant $\lambda \in \mathbb{R}$. Similarly for $\alpha^{(2)}$.

According to the discussion in Section 4, the framings $A^{(1)}, A^{(2)}$ are related by a gauge change

$$A^{(2)} = A^{(1)} g(s, v), \tag{11}$$

where $v = {}^t(v^1, \dots, v^4) : M \to \mathcal{L}^+$ is a solution to the integrable linear system

$$\mathrm{d}v = -{\alpha_\lambda^{(1)}}' v, \tag{12}$$

for some λ, and $s = \frac{r}{v^4}$ with $\mathrm{d}r - v^2\alpha_0^2 - v^3\alpha_0^3 = 0$. Next, define the mapping $g_\lambda^-(s, v) : M \to L$ by

$$g_\lambda^-(s, v) = \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\lambda r}{v^1} \end{pmatrix}, \begin{pmatrix} \lambda^{-1}v^1 & 0 & 0 & 0 \\ v^2 & 1 & 0 & 0 \\ v^3 & 0 & 1 & 0 \\ \lambda v^4 & \frac{\lambda v^2}{v^1} & \frac{\lambda v^3}{v^1} & \frac{\lambda}{v^1} \end{pmatrix}\right].$$

Consider $\bar{A}^{(1)} := A^{(1)} g_\lambda^-(s, v)$. The corresponding connection form $\bar{\alpha}^{(1)} = (\theta, \eta)$ takes the form

$${}^t\theta = \left(0,\ \alpha_0^2 + \frac{\lambda r}{v^1}\alpha_2^1,\ \alpha_0^3 + \frac{\lambda r}{v^1}\alpha_3^1,\ 0\right),$$

$$\eta = \begin{pmatrix} 0 & \frac{\lambda}{v^1}\alpha_2^1 & \frac{\lambda}{v^1}\alpha_3^1 & 0 \\ v^1(\frac{1}{\lambda} - 1)\alpha_1^2 & 0 & -\alpha_2^3 + \frac{\lambda v^3}{v^1}\alpha_2^1 - \frac{\lambda v^2}{v^1}\alpha_3^1 & \frac{\lambda}{v^1}\alpha_2^1 \\ v^1(\frac{1}{\lambda} - 1)\alpha_1^3 & -\alpha_2^3 + \frac{\lambda v^3}{v^1}\alpha_2^1 - \frac{\lambda v^2}{v^1}\alpha_3^1 & 0 & \frac{\lambda}{v^1}\alpha_3^1 \\ 0 & v^1(\frac{1}{\lambda} - 1)\alpha_1^2 & v^1(\frac{1}{\lambda} - 1)\alpha_1^3 & 0 \end{pmatrix}.$$

We then have

Lemma. $\bar{A}^{(1)} := A^{(1)} g_\lambda^-(s,v)$ *is a curved flat framing such that* $[\bar{a}_0^{(1)}, \bar{a}_4^{(1)}] = F^{(1)}$.

Let $F' = [\bar{a}_0^{(1)}, \bar{a}_1^{(1)}]$ be the first envelope of the congruence $\bar{a}_0^{(1)}$. We say that the L-isothermic map F' is the *superposition* of $F^{(1)}$ and $F^{(2)}$ and write

$$F' = F^{(1)} *_F F^{(2)}.$$

Theorem 5 (Permutability Theorem). *If an L-isothermic immersion F has two Bianchi–Darboux transforms $F^{(1)}$ and $F^{(2)}$, then there is another L-isothermic immersion F^* which is a Bianchi–Darboux transform of $F^{(1)}$ and $F^{(2)}$ and is such that*

$$F^* = F^{(1)} *_F F^{(2)} = F^{(2)} *_F F^{(1)}.$$

Proof. Write

$$A^{(1)} = A^{(2)} g(s,v)^{-1} = A^{(2)} g(\hat{s}, \hat{v}),$$

where

$$\hat{s} = -s, \quad \hat{v} = {}^t(v^1, -\frac{v^2}{v^4}, -\frac{v^3}{v^4}, \frac{1}{v^4}). \tag{13}$$

By a direct calculation it is verified that, if $v : \mathcal{U} \to \mathcal{L}^+$ is a solution to (12), then $\hat{v}$ is a solution of

$$\mathrm{d}\hat{v} = -\alpha_\mu^{(2)\prime} \hat{v},$$

where μ is given by

$$\mu = \lambda(\mu - 1). \tag{14}$$

Thus, $F^{(2)} *_F F^{(1)}$ is the L-isothermic immersion represented by the first envelope corresponding to $\bar{A}^{(2)} = A^{(2)} g_\mu^-(\hat{s}, \hat{v})$. It is now easily seen that, if $\hat{v}$, $\hat{s}$, and μ are related to v, s, and λ as in (13) and (14), then $[\bar{a}_0^{(2)}, \bar{a}_1^{(2)}] = [\bar{a}_0^{(1)}, \bar{a}_1^{(1)}]$. The situation is visualized in the following diagram:

$$\begin{tikzcd}
F \arrow[r, "A^{(2)}"] \arrow[d, "A^{(1)}"] & F^{(2)} \arrow[d, "\bar{A}^{(2)}"] \\
F^{(1)} \arrow[r, "\bar{A}^{(1)}"] & F^{(1)} *_F F^{(2)}
\end{tikzcd}.$$

□

References

[B] L. Bianchi, *Lezioni di geometria differenziale*, Zanichelli, Bologna, 1927.

[Bl] W. Blaschke, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie*, B. 3, bearbeitet von G. Thomsen, J. Springer, Berlin, 1929.

[BCG] R.L. Bryant, S.S. Chern, P.A. Griffiths, *Exterior differential systems*, Proceedings of 1980 Beijing DD-Symposium, vol. 1, pp. 219–338, Gordon and Breach, New York, 1982.

[Bu] F. Burstall, *Isothermic surfaces: conformal geometry, Clifford algebras and integrable systems*, International Press (to appear).

[BHPP] F. Burstall, U. Hertrich-Jeromin, F. Pedit, U. Pinkall, *Curved flats and isothermic surfaces*, Math. Z. **225** (1997), 199–209.

[BPP] F. Burstall, F. Pedit, U. Pinkall, *Isothermic submanifolds of symmetric R-spaces*, in preparation.

[C] T.E. Cecil, *Lie sphere geometry: with applications to submanifolds*, Springer-Verlag, New York, 1992.

[CT] S.S. Chern, C.L. Terng, *An analogue of Bäcklund's theorem in affine geometry*, Rocky Mountain J. Math. **10** (1980), 105–124.

[CGS] J. Cieśliński, P. Goldstein, A. Sym, *Isothermic surfaces in* $\mathbf{E}^3$ *as soliton surfaces*, Physics Letters **A 205** (1995), 37–43.

[Ci] J. Cieśliński, *The Darboux–Bianchi transformation for isothermic surfaces. Classical versus the soliton approach.*, Differential Geom. Appl. **2** (1997), 1–28.

[FP] D. Ferus, F. Pedit, *Curved flats in symmetric spaces*, Manuscripta Math. **91** (1996), 445–454.

[He] U. Hertrich-Jeromin, *Supplement on curved flats in the space of point pairs and isothermic surfaces: a quaternionic calculus*, Documenta Math. J. DMV **2** (1997), 335–350.

[HP] U. Hertrich-Jeromin, F. Pedit, *Remarks on the Darboux transform of isothermich surfaces*, Documenta Math. J. DMV **2** (1997), 313–333.

[HMN] U. Hertrich-Jeromin, E. Musso, L. Nicolodi, *Möbius geometry of surfaces of constant mean curvature 1 in hyperbolic space*, e-print math.DG/9810157.

[H] H. Hopf, *Differential Geometry in the Large*, Lecture Notes in Mathematics, 1000, Springer-Verlag, 1983.

[MN1] E. Musso, L. Nicolodi, *A variational problem for surfaces in Laguerre geometry*, Trans. Amer. Math. Soc. **348** (1996), 4321–4337.

[MN2] ———, *Isothermal surfaces in Laguerre geometry*, Boll. Un. Mat. Ital. (7) II-B, Suppl. fasc. 2 (1997), 125–144.

[MN3] ———, *On the equation defining isothermic surfaces in Laguerre geometry*, New Developments in Differential Geometry, Budapest 1996 (J. Szenthe, ed.), Kluwer Academic Publishers, Dordrecht, The Netherlands, 1998, pp. 285–294.

[MN4] ———, *Laguerre geometry of surfaces with plane lines of curvature*, Abh. Math. Sem. Univ. Hamburg **69** (1999), 123–138.

[TT] K. Tenenblat, C.L. Terng, *Bäcklund's theorem for* n*-dimensional submanifolds of* $\mathbb{R}^{2n-1}$, Ann. Math. **111** (1980), 477–490.

[TU] C.L. Terng, K. Uhlenbeck, *Bäcklund's transformations and loop group actions*, e-print math.DG/9805074.

Dipartimento di Matematica Pura ed Applicata, Università di L'Aquila, via Vetoio, I-67010 Coppito (L'Aquila), Italy

E-mail address: `musso@univaq.it`

Dipartimento di Matematica "G. Castelnuovo", Università di Roma "La Sapienza", piazzale A. Moro 2, I-00185 Roma, Italy

E-mail address: `nicolodi@mat.uniroma1.it`