Incidence algebras of simplicial complexes

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Abstract

With any locally finite partially ordered set $\mathcal{K}$ its incidence algebra $\Omega(\mathcal{K})$ is associated. We shall consider algebras over fields with characteristic zero. In this case there is a correspondence $\mathcal{K} \leftrightarrow \Omega(\mathcal{K})$ such that the poset $\mathcal{K}$ can be reconstructed from its incidence algebra up to an isomorphism — due to Stanley theorem. In the meantime, a monotone mapping between two posets in general induces no homomorphism of their incidence algebras.

In this paper I show that if the class of posets is confined to simplicial complexes then their incidence algebras acquire the structure of differential moduli and the correspondence $\mathcal{K} \leftrightarrow \Omega$ is a contravariant functor.

Introduction

This paper, being motivated by physical problems, brings together the issues which traditionally belong to 'disjoint' areas of mathematics: combinatorics and differential moduli and, besides that, uses the notation from quantum mechanics.

It is shown that simplicial complexes resemble differential manifolds from the algebraic point if view, namely, their incidence algebras are similar to algebras of exterior differential forms on manifolds: they are graded, and possess an analog of Cartan differential.

To make the paper self-consistent, I begin it with an outline of basic definitions and results.

Dirac notation. Let $\mathcal{H}$ be a finite-dimensional linear space with a basis labelled by an index set $\mathcal{K}$, and $\mathcal{H}^*$ be its dual. Write down the elements of $\mathcal{H}$ as

$$h \in \mathcal{H} \;\Leftrightarrow\; h = \sum_{P \in \mathcal{K}} c_P |P\rangle$$

Since the dimension of $\mathcal{H}$ is finite, the same index set $\mathcal{K}$ is used to label the dual basis in the space $\mathcal{H}^*$

$$h^* \in \mathcal{H}^* \;\Leftrightarrow\; h^* = \sum_{P\in\mathcal{K}} c_P\langle P|$$

such that

$$\langle P \mid Q\rangle = \delta_{PQ} = \begin{cases} 1, & \text{if } P = Q \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The elements of $\mathcal{H}^*$ are called bra-vectors and the elements of $\mathcal{H}$ are ket-vectors (the terms derived from splitting the word 'bracket').

Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator and $A^* : \mathcal{H}^* \to \mathcal{H}^*$ be its adjoint. In the dual bases we have for any $P, Q \in \mathcal{K}$:

$$\big(A^*(\langle P|)\big)|Q\rangle = \langle P|\big(A(|Q\rangle)\big) \tag{2}$$

therefore with no confusion we can use the same symbol, say, a for both A and its adjoint A^*:

$$\begin{array}{rccc} A: & |Q\rangle & \mapsto & a|Q\rangle \\ A^*: & \langle P| & \mapsto & \langle P|a \end{array}$$

and the identity (2) reads

$$(\langle P|a)|Q\rangle = \langle P|(a|Q\rangle) = \langle P|a|Q\rangle = a_{pq}$$

Then A can be written down as

$$A = \sum_{P,Q\in\mathcal{K}} a_{PQ}|P\rangle\langle Q|$$

and the product is calculated in accordance with (1):

$$A \cdot B = \left(\sum_{P,Q\in\mathcal{K}} a_{PQ}|P\rangle\langle Q|\right) \cdot \left(\sum_{R,S\in\mathcal{K}} b_{RS}|R\rangle\langle S|\right) = \sum_{P,Q,S\in\mathcal{K}} a_{PQ}b_{QS}|P\rangle\langle S| \tag{3}$$

This notation was introduced by P.A.M.Dirac [2] for state vectors in quantum mechanics.

Incidence algebras. The Dirac notation turns out to be natural for incidence algebras. Let $\mathcal{K}$ be an arbitrary finite poset. Denote by $\mathcal{H}$ its linear span

$$\mathcal{H} = \mathrm{span}\{|P\rangle : P \in \mathcal{K}\} = \left\{ \sum_{P\in\mathcal{K}} c_P |P\rangle \right\}$$

with the coefficients taken from a field with characteristic zero.

Definition 1. *The* incidence algebra *of a poset* $\mathcal{K}$ *is the following linear span*

$$\Omega = \Omega(\mathcal{K}) = \mathrm{span}\{|P\rangle\langle Q| : P, Q \in \mathcal{K} \text{ and } P \le Q\} \tag{4}$$

and the product defined on the basic elements according to (4):

$$|P\rangle\langle Q| \cdot |R\rangle\langle S| = |P\rangle\langle Q \mid R\rangle\langle S| = \delta_{QR}|P\rangle\langle S|$$

This definition of product is correct due to the transitivity of partial orders:

$$|P\rangle\langle Q|,\, |Q\rangle\langle S| \in \Omega \quad \Rightarrow \quad P \le Q \text{ and } Q \le S \quad \Rightarrow \quad P \le S \quad \Rightarrow \quad |P\rangle\langle S| \in \Omega$$

Incidence algebras were introduced by Rota [5]. It was proved by Stanley [6] that a poset $\mathcal{K}$ can be reconstructed from its incidence algebra $\Omega(\mathcal{K})$ up to a poset isomorphism.

Meanwhile, poset homomorphisms (namely, monotone mappings) induce no homomorphism of incidence algebras. However, and this is the contents of this paper, the situation drastically changes when the class of posets is restricted to simplicial complexes.

The category $\mathfrak{SC}$ of simplicial complexes. For the sake of self-consistency I give a brief account of the standard theory, mainly to introduce the notation. Let V be a non-empty finite set, call the elements of V *vertices*.

Definition 2. *A collection $\mathcal{K}$ of non-empty subsets of V is called (abstract)* simplicial complex *with the set of vertices V whenever*

- $\forall v \in V \quad \{v\} \in \mathcal{K}$
- $\forall P \in \mathcal{K}, \forall Q \subseteq V \quad Q \subseteq P \Rightarrow Q \in \mathcal{K}$

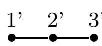
Evidently, $\mathcal{K}$ is a poset with respect to set inclusion. The elements of $\mathcal{K}$ are called *simplices*.

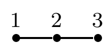
Definition 3. *Let $\mathcal{K}$, $\mathcal{K}'$ be two simplicial complexes with the sets of vertices V, V', respectively. A mapping $\pi : \mathcal{K}' \to \mathcal{K}$ is called simplicial if*

- *vertices are mapped on vertices:* $\quad V'\pi \subseteq V$
- $\pi\big|_{V'}$ *completely determines π on the whole $\mathcal{K}'$:* $\quad \forall P' \in \mathcal{K}' \quad P'\pi = \cup_{v' \in P'}\{v'\}\pi$

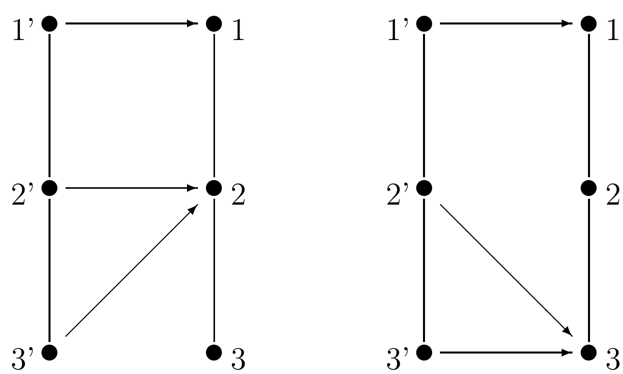
Remark. Sometimes (in particular, in the rest of this paper) the notion of simplicial mapping is referred to a mapping π between vertices, then the second condition reads:

$$\{v'_0, \ldots, v'_n\} \in \mathcal{K}' \quad \Rightarrow \quad \{v'_0\pi, \ldots, v'_n\pi\} \in \mathcal{K}' \tag{5}$$

For instance, let $\mathcal{K}' =$ 

and $\mathcal{K} =$ 

and let π_1, π_2 be



$\pi_1 : \mathcal{K}' \to \mathcal{K}$ $\qquad$ $\pi_2 : \mathcal{K}' \to \mathcal{K}$

then π_1 is simplicial and π_2 is not (since $\{1', 2'\}\pi_2 = \{1, 3\} \notin \mathcal{K}$).

Denote by $\mathfrak{SC}$ the category whose objects are simplicial complexes and whose arrows are simplicial mappings

$$\mathfrak{SC} = (\texttt{simplicial complexes}, \texttt{simplicial mappings})$$

It follows immediately from the definition that simplicial mappings are monotone with respect to set inclusion and that $\mathfrak{SC}$ is (not a full) sub-category of the category $\mathfrak{POSET} = (\texttt{posets}, \texttt{monotone mappings})$

Dirac notation for homological operations. Fix an enumeration of the vertices of $\mathcal{K}$, then for any simplex $P = \{v_0, \dots, v_n\} \in \mathcal{K}$ and any its vertex v_i the incidence coefficient is defined

$$\epsilon_{v_i P} = (-1)^i \tag{6}$$

A *face* P_v of a simplex P is its subset $P \setminus \{v\}$, we write

$$P_v = P - v;\ P = p_v + v \tag{7}$$

The *dimension* of a simplex P is the number of its vertices minus one:

$$\dim P = \operatorname{card} P - 1 \tag{8}$$

Denote by $\mathcal{K}^n$ the n-skeleton of $\mathcal{K}$ — the set of its simplices of dimension n

$$\mathcal{K}^n = \{P \in \mathcal{K} :\ \dim P = n\}$$

and consider the linear spans

$$\mathcal{H}^n = \operatorname{span} \mathcal{K}^n = \left\{ \sum_{P \in \mathcal{K}^n} c_P |P\rangle \right\}$$

Definition 4. *The* border operator $\delta : \mathcal{H}^n \to \mathcal{H}^{n-1}$ *acts as*

$$\delta|P\rangle = \sum_{v \in P} \epsilon_{vP} |P_v\rangle \tag{9}$$

and then extends to the space $\mathcal{H}$ *(assuming* $\delta|v\rangle = 0$, $\forall v \in V$*):*

$$\mathcal{H} = \oplus \mathcal{H}^n = \operatorname{span} \mathcal{K}$$

It is proved that δ^2 is always zero. Due to Dirac notation the same symbol δ is used to denote its adjoint, called *coborder operator* acting from $\mathcal{H}^{*n}$ to $\mathcal{H}^{*n+1}$, so

$$\forall P \in \mathcal{K}^n \qquad \left\{ \begin{array}{lcl} \langle P|\delta & \in & \mathcal{H}^{*n+1} \\ \delta|P\rangle & \in & \mathcal{H}^{n-1} \end{array} \right. \tag{10}$$

Finite-dimensional differential moduli. Recall the basic definitions concerning finite-dimensional analogs of moduli of exterior differential forms. Let $\mathcal{A}$ be a semisimple finite-dimensional commutative algebra.

Definition 5. *A differential module $\mathcal{D}$ over a basic algebra $\mathcal{A}$ is a triple*

$$\mathcal{D} = (\Omega, \mathcal{A}, \boldsymbol{d})$$

where Ω is a graded algebra

$$\Omega = \Omega^0 \oplus \Omega^1 \oplus \cdots, \qquad \Omega^0 = \mathcal{A} \tag{11}$$

equipped with the Kähler differential $\boldsymbol{d} : \Omega^n \to \Omega_{n+1}$ such that for any $\omega^r \in \Omega^r$, $\omega^s \in \Omega^s$

$$\begin{array}{l} \boldsymbol{d}^2 = 0 \\ \boldsymbol{d}(\omega^r \cdot \omega^s) = \boldsymbol{d}\omega^r \cdot \omega^s + (-1)^r \omega^r \cdot \boldsymbol{d}\omega^s \end{array} \tag{12}$$

The second equality is called *graded Leibniz rule*.

Universal differential envelope. Given an algebra $\mathcal{A}$, any differential module over it can be obtained as a quotient of a universal object $\Omega_u = \Omega_u(\mathcal{A})$, called universal differential envelope of $\mathcal{A}$ over appropriate differential ideal $\mathfrak{I}$. Recall the necessary definitions (for details the Reader is referred to [3]).

Consider the tensor product $\mathcal{A} \otimes \mathcal{A}$ and define the operator $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ as follows:

$$m(a \otimes b) = a \cdot b$$

Then consider its kernel

$$\Omega_u^1 = \ker m$$

define by induction

$$\Omega_u^{n+1} = \Omega_u^n \oplus_{\mathcal{A}} \Omega_u^1$$

and form the sum

$$\Omega_u = \bigoplus_{n=0}^{\infty} \Omega_u^n \tag{13}$$

Define the Kähler differential $\mathtt{d}_u$ first on $\Omega_u^0 = \mathcal{A}$:

$$\mathtt{d}_u a = \mathbf{1} \otimes a - a \otimes \mathbf{1}$$

and then extend it by induction to higher degrees using the identities (12), for instance

$$\mathtt{d}_u(a\mathtt{d}_u b) = \mathtt{d}_u a \cdot \mathtt{d}_u b \tag{14}$$

Definition 6. *The universal differential envelope $\Omega_u = \Omega_u(\mathcal{A})$ of the algebra $\mathcal{A}$ is the differential module $(\Omega_u, \mathcal{A}, \mathtt{d}_u)$.*

1 Differential structure of incidence algebras

In this section we introduce a particular representation for universal differential envelopes of finite-dimensional algeras called *stories semantics*. Let $\mathcal{K}$ be an arbitrary (yet structureless) finite set, denote by $\mathcal{A}$ the algebra of all complex-valued functions on $\mathcal{K}$.

Stories semantics for universal differential envelope. Call the elements of $\mathcal{K}$ statements, and consider first all possible sequences of statements of finite length. A *homogeneous n-story* is a sequence $\prec P_0, \dots, P_n \succ$ whose no neighbor statements are the same. Denote by Ω_S^n the linear span of all homogeneous n-stories:

$$\Omega_S^n = \mathrm{span}\bigl\{\prec P_0, \dots, P_n \succ \,:\; \forall i = 1, \dots, n \quad P_{i-1} \neq P_i\bigr\} \tag{15}$$

and define the product of two stories as follows:

$$\prec P_0, \dots, P_n \succ \cdot \prec Q_0, \dots, Q_m \succ = \begin{cases} \prec P_0, \dots, P_n Q_1, \dots, Q_m \succ, & \text{if } P_n = Q_0 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

which, being extended by linearity to the direct sum $\bigoplus_{n=0}^{\infty} \Omega_S^n$ makes it graded algebra, call it *stories algebra*.

Now let us write down the explicit form of the Kähler differential. For $\prec P\succ \in \mathcal{A}$ we have:

$$\mathtt{d}_u \prec P\succ = \sum_{Q\neq P} (\prec QP\succ - \prec PQ\succ)$$

and for arbitrary $\prec P_0, \ldots, P_n\succ \in \Omega_u^n$

$$\begin{aligned}\mathtt{d}_u \prec P_0, \ldots, P_n\succ = \sum_{Q:\, Q\neq P_0} \prec QP_0, \ldots, P_n\succ + \\ + \sum_{k=1}^{n} (-1)^k \sum_{Q: P_{k-1}\neq Q\neq P_k} \prec P_0, \ldots, P_{k-1}QP_k, \ldots, P_n\succ + \\ + (-1)^{n+1} \sum_{Q: P_n\neq Q} \prec P_0, \ldots, P_nQ\succ \end{aligned} \quad (17)$$

Lemma 1. *The universal differential envelope of a finite-dimensional semisimple algebra $\mathcal{A}$ is isomorhic (as graded algebra) to the stories algebra:*

$$\Omega_u = \bigoplus_{n=0}^{\infty} \Omega_S^n$$

Simplicial differential ideals. Let us specify the structure of the set $\mathcal{K}$ assuming it to be a simplicial complex. Our goal is to find the explicit form of the differential ideal $\mathfrak{I}$ giving rise to its incidence algebra $\Omega(\mathcal{K})$. To gather it, first let us classify the stories in a way taking into account that $\mathcal{K}$ is a simplicial complex.

Definition 7. *A story $\prec P_0, \ldots, P_n\succ$ is called* fair *whenever $P_{i-1} = P_i - v$ — see (7). Otherwise the story $\prec P_0, \ldots, P_n\succ$ is* unfair*.*

Form the linear span $\mathfrak{I}_N = \oplus_{n=0}^{\infty} \mathfrak{I}_N^n$, where

$$\mathfrak{I}_N^n = \mathrm{span}\{\mathtt{unfair}\ n\mathtt{-stories}\} = \mathrm{span}\{\prec P_0, \ldots, P_n\succ :\ \neg(\forall i\ P_{i-1} = P_i - v)\} \quad (18)$$

Fix an enumeration of vertices of $\mathcal{K}$ and with each fair story w associate a number $\epsilon_w = \pm 1$ as follows:

$$\epsilon_w = \prod_{i=1}^{n} \epsilon_{v_i P_i}$$

where $\epsilon_{v_i P_i}$ is the incidence coefficient (6). Let P_0, P_n be two simplices such that $P_0 \subseteq P_n$, and $\dim P_0 - \dim P_n$ is exactly n. For any two fair n-stories $w = \prec P_0 P_1, \ldots, P_{n-1} P_n \succ$ and $w' = \prec P_0 P_1', \ldots, P_{n-1}' P_n \succ$ having common initial and final statements form the difference $\epsilon_w w - \epsilon_{w'} w'$, and consider the linear hull of all such differences

$$\mathfrak{I}_S^n = \operatorname*{span}_w \{\epsilon_w w - \epsilon_{w'} w' :\ w, w' \text{ as described above}\}$$

Take for each n the sum

$$\mathfrak{I}^n = \mathfrak{I}_N^n \oplus \mathfrak{I}_S^n$$

and form the following graded linear space

$$\mathfrak{I} = \bigoplus_{n=1}^{\infty} \mathfrak{I}^n \tag{19}$$

Lemma 2. *$\mathfrak{I}$ is a differential ideal in Ω_u.*

Proof. Evidently $\mathfrak{I}_N = \oplus_n \mathfrak{I}_N^n$ is an ideal in Ω_u. Besides that, a product of an element of $\mathfrak{I}_S^n$ and a fair m-story is always in $\mathfrak{I}_S^{m+n}$, therefore $\mathfrak{I}$ is an ideal. It remains to prove that the ideal $\mathfrak{I}$ is differential: $\mathtt{d}_u(\mathfrak{I}^n) \in \mathfrak{I}^{n+1}$.

First let $w \in \mathfrak{I}_N^n$, consider $\mathtt{d}_u w$ as the sum (17). The 1st and the 3rd summands of (17) are always in $\mathfrak{I}_N^{n+1}$, the same for any term from the middle sum of (17) with the only possible exception when $w = \prec P_0, \ldots, P_i P_{i+1}, \ldots, P_n \succ$ such that both $\prec P_0, \ldots, P_i \succ$ and $\prec P_{i+1}, \ldots, P_n \succ$ are fair stories, while $P_i = P_{i-1} - u - v$. Then

$$\mathtt{d}_u w = \nu + (-1)^{i+1}(\rho + \rho')$$

where ν is the sum of elements of (17) from $\mathfrak{I}_N^{n+1}$, and ρ, ρ' are the following fair stories:

$$\begin{array}{rcl} \rho &=& \prec P_0, \ldots, P_i, P_{i+1} - u, P_{i+1}, \ldots, P_n \succ \\ \rho' &=& \prec P_0, \ldots, P_i, P_{i+1} - v, P_{i+1}, \ldots, P_n \succ \end{array}$$

for which $\epsilon_\rho = -\epsilon_{\rho'}$, therefore $\rho + \rho' \in \mathfrak{I}_S^{n+1}$ and

$$\mathtt{d}_u \mathfrak{I}_N^n \subseteq \mathfrak{I}_N^{n+1} \oplus \mathfrak{I}_S^{n+1}$$

Now let $w = \epsilon_\rho \rho - \epsilon_{\rho'} \rho' \in \mathfrak{I}_S^n$. That is,

$$\rho = \prec P_0, P_1, \dots, P_{n-1}, P_n \succ, \quad \rho' = \prec P_0, P'_1, \dots, P'_{n-1}, P_n \succ$$

Consider the three sums (17) for $\mathtt{d}_u w$. All the terms in the 2nd sum (17) will be in $\mathfrak{I}_N^{n+1}$. The terms from the first sum will also belong to $\mathfrak{I}_N^{n+1}$ with the only exception — the terms of the form $k = \epsilon_\tau \tau - \epsilon_{\tau'} \tau'$, where

$$\tau = \prec P_0 - v, P_0 \succ \rho, \quad \tau' = \prec P_0 - v, P_0 \succ \rho'$$

but for them $\epsilon_\tau = \epsilon_{vP_0} \epsilon_\rho$ and $\epsilon_{\tau'} = \epsilon_{vP_0} \epsilon'_\rho$, therefore $\epsilon_\tau \epsilon_{\tau'} = \epsilon_\rho \epsilon_{\rho'}$ and $k \in \mathfrak{I}_S^{n+1}$, so $\mathtt{d}_u \mathfrak{I}_S^n \subseteq \mathfrak{I}_N^n \oplus \mathfrak{I}_S^n$. This completes the proof:

$$\mathtt{d}_u(\mathfrak{I}_N^n \oplus \mathfrak{I}_S^n) \subseteq \mathfrak{I}_N^{n+1} \oplus \mathfrak{I}_S^{n+1}$$

$\square$

Theorem 3. *The quotient $\Omega_u/\mathfrak{I}$ and the incidence algebra $\Omega(\mathcal{K})$ are isomorphic algebras.*

Proof. Consider the mapping $\sigma : \Omega_u \to \Omega$ defined on any story $w = \prec P_0, \dots, P_n \succ \in \Omega_u^n$ as

$$\sigma(w) = \begin{cases} \epsilon_w |P_0\rangle\langle P_n|, & \text{if } w \text{ is a fair story} \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

then $\ker \sigma = \mathfrak{I}_N \oplus \mathfrak{I}_S = \mathfrak{I}$, so $\Omega \simeq \Omega_u/\mathfrak{I}$ as graded linear spaces. To verify that σ preserves products, let $w = \prec P_0, \dots, P_m \succ$ and $w' = \prec Q_0, \dots, Q_n \succ$ be two fair stories. If $P_m \neq Q_0$ everything is trivial. Suppose $P_m = Q_0$, then $ww' = \prec P_0, \dots, P_m Q_1, \dots, Q_n \succ$ and

$$\sigma(ww') = \epsilon_{(ww')} |P_0\rangle\langle Q_n| = \epsilon_w \epsilon_{w'} |P_0\rangle\langle P_m||Q_0\rangle\langle Q_n| = \sigma(w)\sigma(w')$$

$\square$

The induced differential structure on $\Omega(\mathcal{K})$. Having a projection of the universal differential envelope Ω_u onto the incidence algebra $\Omega(\mathcal{K})$ we first conclude that $\Omega(\mathcal{K})$ necessarily has the structure of a differential module over $\mathcal{A}$, namely, that induced by the projection σ onto the quotient. Now, having the expression (20) for σ, let us explicitly calculate the form of the differential $\mathtt{d}$ on $\Omega(\mathcal{K})$ according to the formula

$$\mathtt{d} = \sigma^{-1} \circ \mathtt{d}_u \circ \sigma \tag{21}$$

Theorem 4. *The differential in the incidence algebra $\Omega(\mathcal{K})$ of a simplicial complex $\mathcal{K}$ has the following form. Let $|P\rangle\langle Q| \in \Omega^n$, then*

$$\mathtt{d}|P\rangle\langle Q| = |\delta P\rangle\langle Q| - (-1)^n |P\rangle\langle Q\delta| \tag{22}$$

where δ is the symbol (10) for both border and coborder operations.

Proof. Begin with $\Omega^0 = \mathcal{A}$. In this case $\sigma^{-1} = \mathrm{id}\big|_{\mathcal{A}}$, therefore

$$\mathtt{d}|P\rangle\langle P| = \sigma(\mathtt{d}_u \prec P \succ) = \sum_{v \in P} \epsilon_{vP} |P - v\rangle\langle P| - \sum_{u: P+u \in \mathcal{K}} \epsilon_{u(P+u)} |P\rangle\langle P + u|$$

According to (9), the first term is $|\delta P\rangle\langle P|$ while the second is $|P\rangle\langle P\delta|$, so

$$\mathtt{d}\,|P\rangle\langle P| = |\delta P\rangle\langle P| - |P\rangle\langle P\delta|$$

Now let $|P\rangle\langle Q| \in \Omega^1$, then $P = Q - v$ for some $v \in V$ and

$$|P\rangle\langle Q| = |P\rangle\langle P| \cdot \epsilon_{vQ}\, \mathtt{d}(|Q\rangle\langle Q|)$$

therefore it follows from (14) $\epsilon_{vQ} = \langle P|\delta|Q\rangle$ that

$$\begin{aligned}\mathtt{d}|P\rangle\langle Q| = \mathtt{d}|P\rangle\langle P| \cdot \epsilon_{vQ} \mathtt{d}(|Q\rangle\langle Q|) = (|\delta P\rangle\langle P| - |P\rangle\langle P\delta|)\epsilon_{vQ}(|\delta Q\rangle\langle Q| - |Q\rangle\langle Q\delta|) = \\ \epsilon_{vQ}(|\delta P\rangle\langle P|\delta|Q\rangle\langle Q| - |Q\rangle\langle P|\delta|Q\rangle\langle Q\delta|) = |\delta P\rangle\langle Q| + |P\rangle\langle Q\delta|\end{aligned}$$

For higher degrees the formula (22) is proved by induction. First note that $\mathtt{d}$ enjoys the Leibniz rule as both σ, σ' in (21) preserve products. Then represent any $|P\rangle\langle Q| \in \Omega^n$ as a product

$$|P\rangle\langle Q| = |P\rangle\langle Q - v| \cdot |Q - v\rangle\langle Q| \in \Omega^{n-1} \cdot \Omega^n$$

and carry out a routine calculation.

□

Summary of this section. It was established that the incidence algebra $\Omega(\mathcal{K})$ of any simplicial complex $\mathcal{K}$ is a differential module over the algebra $\mathcal{A}$ of all functions on $\mathcal{K}$ (see Definition 5). The algebra $\Omega(\mathcal{K})$ was represented as a quotient of the universal differential envelope $\Omega_u(\mathcal{A})$ over the simplicial differential ideal $\mathfrak{I}$ (19). The form of the Kähler differential for $\Omega(\mathcal{K})$ is given in (22).

2 Functorial properties

As it was already mentioned, an arbitrary monotone mapping $\pi : \mathcal{K}' \to \mathcal{K}$ between two posets $\mathcal{K}$ and $\mathcal{K}'$ produces no homomorphism of their incidence algebras. However, if we consider the category $\mathfrak{SC}$ of simplicial complexes, the situation becomes completely different. In this section I show that that the correspondence $\mathcal{K} \mapsto \Omega(\mathcal{K})$ is a contravariant functor from the category $\mathfrak{SC}$ to the category $\mathfrak{DM}$ of differential moduli over commutative algebras.

The category $\mathfrak{DM}$. The objects of the category $\mathfrak{DM}$ are differential moduli over semisimple commutative algebras — see Definition 5. The morphisms of $\mathfrak{DM}$ are *differentiable mappings.*

Definition 8. *Let $\mathcal{D} = (\Omega, \mathcal{A}, \boldsymbol{d})$, $\mathcal{D}' = (\Omega', \mathcal{A}', \boldsymbol{d}')$ be two differential moduli. A mapping $\phi : \Omega \to \Omega'$ is called* differentiable *iff*

$$\begin{array}{ll} \phi & \text{is a homomorphism of graded algebras} \\ \phi \circ \boldsymbol{d}' = \boldsymbol{d} \circ \phi & \end{array} \tag{23}$$

Lemma 5. *Any differentiable mapping is completely defined by its values on the basic algebra $\mathcal{A}$.*

Proof. By induction, let $w \in \Omega^1$, then $w = \sum_i a_i \,\mathtt{d} b_i$ with $a_i, b_i \in \mathcal{A}$. Then $\phi(w) = \sum \phi(a_i)\phi(\mathtt{d} b_i) = \sum \phi(a_i)\mathtt{d}'\phi(b_i)$. When $w^{n+1} \in \Omega^{n+1}$, it reads $w^{n+1} = \sum_i a_i \,\mathtt{d} w_i^n$ with $a_i \in \mathcal{A}$ and $w_i^n \in \Omega^n$. Applying of ϕ and using the second property (23) completes the proof. $\square$

So, the notion of differentiable mappings can be referred to homomorphisms $\phi : \mathcal{A} \to \mathcal{A}'$ between basic algebras. Let $\mathcal{D} = (\Omega, \mathcal{A}, \mathtt{d})$, $\mathcal{D}' = (\Omega', \mathcal{A}', \mathtt{d}')$ be two differential moduli. A mapping $\phi : \mathcal{A} \to \mathcal{A}'$ completely determines both the homomorhisms $\phi : \Omega \to \Omega$ and $\overline{\phi} : \Omega_u \to \Omega_u$. Then ϕ is differentiable if and only if the following holds:

$$\overline{\phi}(\mathfrak{I}) \subseteq \mathfrak{I}' \tag{24}$$

Let the basic algebras $\mathcal{A}, \mathcal{A}'$ are represented by functions on the sets $\mathcal{K}$, $\mathcal{K}'$, respectively. Then the dual mapping $\pi : \mathcal{K}' \to \mathcal{K}$ determines ϕ completely, and the explicit form of the mapping $\overline{\phi}$ is the following: for any $\prec P_0, \ldots, P_n \succ \in \Omega_u^n$

$$\phi \prec P_0, \ldots, P_n \succ \; = \sum_{P_i' :\, \forall i\, P_i'\pi = p_i} \prec P_0', \ldots, P_n' \succ \tag{25}$$

Functorial properties. To prove that the correspondence $\mathcal{K} \mapsto \Omega(\mathcal{K})$ is a functor we have to provide a correspondence between the arrows of the categories. As stated above, any (set-theoretical) mapping $\pi : \mathcal{K}' \to \mathcal{K}$ gives rise to a homomorphism (25) $\phi = \pi^*$ of the basic algebras.

Theorem 6. *Let $\mathcal{K}$, $\mathcal{K}'$ be two simplicial complexes, and let a mapping $\pi : \mathcal{K}' \to \mathcal{K}$ be simplicial. Then its dual $\phi = \pi^*$ is differentiable.*

Proof. By virtue of (24) it suffices to prove that simplicial ideals are mapped into simplicial ideals in the target algebra. Let $W' = \prec P_0', \ldots, P_n' \succ$ be a fair story. Let us prove that its image

$$W'\pi = \prec P_0'\pi, \ldots, P_n'\pi \succ$$

being a sequence of elements of $\mathcal{K}$ is either a fair story or not a story. $P_i' = P_{i-1}' + v_i$ for any $i = 1, \ldots, n$, and $\forall i \neq j\; v_i \neq v_j$. Denote $P_i = P_i'\pi$ and $v_i = v_i'\pi$ (recall that $V'\pi \subseteq V$). For each i we have exactly two possibilities: either $v_i \in P_{i-1}$ or $v_i \notin P_{i-1}$ (and then according to (5) $P_i = P_{i-1} + v_i \in \mathcal{K}$). If $v_i \in P_i$ for some for some i then $W'\pi$ contains $P_i = P_{i-1}$ (and therefore is not a story) otherwise $\forall i\; P_i = P_{i-1} + v_i$, so W is a fair story. So,

$$\{\texttt{fair stories}\}\pi \subseteq \{\texttt{fair stories}\} \tag{26}$$

Return to simplicial ideals. Let $\prec P_0, \ldots, P_n \succ$ be an unfair story, then $\phi(\prec P_0, \ldots, P_n \succ)$ is according to (25) a sum of unfair stories (otherwise (26) is violated).

Now let $i = \epsilon_w - \epsilon_{\tilde{w}} \in \mathfrak{I}_S$ such that $P_n = P_0 + v_1 + \cdots + v_n$ in w and $P_n = P_0 + v_{j_1} + \cdots + v_{j_n}$ in $\tilde{w}$. Let $w' = P_0', \ldots, P_n'$ be a fair story in $\mathcal{K}'$ such that $w'\pi = w$, then necessarily

- $P_n' = P_0' + v_1' + \cdots + v_n'$
- all v_j' are disjoint
- $\forall j\; v_j'\pi = v_j$ — there is 1–1 correspondence

Make a new fair story $\tilde{w}'$ from w' performing the same permutation of vertices as that making $\tilde{w}$ from w, and we obtain for it $\tilde{w}'\pi = \tilde{w}$ and $\epsilon_{\tilde{w}'} = \epsilon_{w'}$. So, any fair preimage of $i \in \mathfrak{I}_S$ will be in $\mathfrak{I}'_S$

$\square$

Summary of this section. The correspondence $\mathcal{K} \mapsto \Omega(\mathcal{K})$ is proved to be a contravariant functor from the category $\mathfrak{SC} = \{\mathtt{simplicial\ complexes}, \mathtt{simplicial\ mappings}\}$ into the category $\mathfrak{DM}$ of differential moduli over finite-dimensional semisimple commutative algebras.

3 Concluding remarks

It was shown that the incidence algebras of simplicial complexes possess the structure of differential moduli, and it was established that, contrary to posets of general form, the correspondence between simplicial complexes and their incidence algebras is a contravariant functor.

It occurs that simplicial complexes possess the natural structure of discrete differential manifolds — finite sets equiped with differential calculi — see, e.g. [1] for a review.

Aside of purely mathematical context, the presented results have physical application being a basis for discrete approximations of spacetime structure [4]. When simplicial complexes are treated as coarse-grained spacetime patterns, our results enable the possibility of linking more and more refined approximations between each other.

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