Derived categories for the working mathematician

R. P. Thomas

ABSTRACT. It is becoming increasingly difficult for geometers and even physicists to avoid papers containing phrases like "triangulated category", not to mention derived functors. I will give some motivation for such things from algebraic geometry, and show how the concepts are already familiar from topology. This gives a natural and simple way to look at cohomology and other scary concepts in homological algebra like Ext, Tor, hypercohomology and spectral sequences.

1. Introduction

I should begin by apologising for the title of this talk [**Ma**]; while the title is intended for mathematicians the talk itself is aimed at least as much at physicists. Kontsevich's mirror symmetry programme [**K**] and the mathematical description of D-branes have brought triangulated categories into mainstream string theory and geometry (there are now even papers in which the reference [Ha] means [RD] rather than [AG]). Since they have such a fearsome reputation (probably mainly due to the references being in French), and since I will need them for my second talk, as presumably will other speakers, I wanted to motivate this beautiful piece of homological algebra. This motivation is of course what guided the creators of derived categories (principally Verdier, or as he is traditionally known in this context, Grothendieck's student Verdier) but, equally of course, is not written down. For the technical details of the theory the working mathematician should consult the excellent [**GM**].

The main idea of derived categories is simple: work with complexes rather than their (co)homology. We will take simple examples from algebraic geometry to demonstrate why one might want to do this, then examples from algebraic topology to show that the ideas and structure are already familiar. (The link between the two subjects is sheaves, but we shall not pursue this.) This also gives us a way of using pictures and techniques from topology in algebraic geometry. Section 1 begins with chain complexes in algebraic geometry and topology, and why they are preferable to (co)homology. Thus we would like to consider the natural invariant of a space/sheaf/etc. to be a complex rather than its homology, so to do this we must identify which complexes should be considered isomorphic – this leads to the notion of quasi-isomorphism that is the subject of Section 2. Setting up

---

the general (abstract) theory is where categories creep in, in Section 3, leading to cones, triangles, and triangulated categories in Section 4. Finally Section 5 shows how more classical homological algebra, in particular derived functors, fit into this framework; this makes them more transparent and leads to easier proofs and explanations of standard results.

Acknowledgements. I learnt a lot of homological algebra from Brian Conrad, Mikhail Khovanov and Paul Seidel. I would like to thank members of the CALF (junior **C**ambridge-**O**xford-**W**arwick) algebraic geometry seminar for enduring a trial run of a preliminary version of this talk, and the organisers of the 1999 Harvard Winter School on Mirror Symmetry for inviting me to take part.

2. Chain complexes

Chain complexes in topology. Chain complexes are familiar in algebraic topology. A simplicial complex X (such as a triangulation of a manifold) gives rise to two chain complexes. Letting C_i be the free abelian group generated by the i-simplices in the space, and C^i its dual,

$$(2.1) \qquad C_i = \mathbb{Z}\{i\text{-simplices}\}, \qquad C^i = \mathrm{Hom}\,(C_i, \mathbb{Z}),$$

we have the complexes

$$C_\bullet = \ldots \to C_i \xrightarrow{\partial_i} C_{i-1} \to \ldots, \qquad C^\bullet = \ldots \to C^{i-1} \xrightarrow{d^i} C^i \to \ldots.$$

Here ∂ is the boundary operator taking a simplex to its boundary, d is its adjoint, and $\partial^2 = 0 = d^2$. Thus we can form the homology and cohomology groups

$$H_i(X) = \frac{\ker \partial_i}{\mathrm{im}\ \partial_{i+1}}, \qquad H^i(X) = \frac{\ker d^{i+1}}{\mathrm{im}\ d^i}.$$

Simplicial maps $X \to Y$ induce chain maps (that is, they commute with the (co)boundary operators) on the corresponding chain complexes, inducing maps on the (co)homology groups.

However, the (co)homology H_* (H^*) contains less information than the complexes $C_\bullet$, $C^\bullet$, for instance Massey products. There exist spaces with the same homology H_* but different homotopy type, making H_* a limited invariant of homotopy type. But $C_\bullet$ is a very powerful "invariant", at least for simply connected spaces (to which we shall confine ourselves in this talk), due to the Whitehead theorem. This states that the underlying topological spaces $|X|$, $|Y|$ of simplicial complexes X and Y (both simply connected) are homotopy equivalent if and only if there are maps of simplicial complexes

$$(2.2) \qquad \begin{array}{ccc} & Z & \\ & \swarrow \quad \searrow & \\ X & & Y \end{array} \qquad \text{inducing chain maps} \qquad \begin{array}{ccc} & C_\bullet^Z & \\ & \swarrow \quad \searrow & \\ C_\bullet^X & & C_\bullet^Y \end{array}$$

inducing isomorphisms on homology $H_*(X) \xleftarrow{\sim} H_*(Z) \xrightarrow{\sim} H_*(Y)$. (The reason for the appearance of Z is that we may need to subdivide X – thus giving a Z with a simplicial approximation $Z \to X$ to the identity map $|Z| \xrightarrow{\sim} |X|$ – before we can get a simplicial approximation $Z \to Y$ to a given homotopy equivalence $|X| \to |Y|$.)

Thus $C_\bullet$ contains as much information as we could hope for, and we would like to think of it as an invariant of the topological space $|X|$. We shall see how to do this later; for now we can think of it as an invariant of the triangulation (simplicial

complex).

Another place in topology where the advantage of using complexes rather than their homology is already familiar is the construction of the dual, cohomology, theory as above. Namely we *do not* define H^* to be the dual of H_* in the sense that

$$H^* \neq \mathrm{Hom}\,(H_*, \mathbb{Z}).$$

This is because $\mathrm{Hom}\,(\ .\ , \mathbb{Z})$ destroys torsion information and its square is not the identity (the double dual in this sense is not the original H_*). Instead we take $\mathrm{Hom}\,(\ .\ , \mathbb{Z})$ at the level of chain complexes as in (2.1). Then no information is lost, and the double dual is the original complex.

For instance applying $\mathrm{Hom}\,(\ .\ , \mathbb{Z})$ to the complex

$$\mathbb{Z} \xrightarrow{\ 2\ } \mathbb{Z} \qquad H_1 = 0,\ H_0 = \mathbb{Z}/2,$$

gives the dual complex

$$\mathbb{Z} \xleftarrow{\ 2\ } \mathbb{Z} \qquad H^1 = \mathbb{Z}/2,\ H^0 = 0.$$

Thus the torsion $\mathbb{Z}/2$ has not been lost on dualising (as it would have been using $\mathrm{Hom}\,(H_*, \mathbb{Z})$), it has just moved in degree.

So we have seen two examples of what will be the theme of this talk, namely

$$\boxed{\text{Complexes good, (co)homology bad}}$$

Chain complexes in algebraic geometry. It might seem strange to consider chain complexes in algebraic geometry at all. The main place they arise is in resolutions of sheaves, which we describe now. Coherent sheaves often arise naturally as kernels or cokernels of maps, i.e. as the cohomology of the 2-term complex made from the map. For instance if $D \subset X$ is a divisor in a smooth algebraic variety X, it corresponds to a line bundle L, and a section $s \in H^0(L)$ vanishing on D (we shall confuse L and its sheaf of sections L). This gives us the standard exact sequence

(2.3) $$0 \to L^{-1} \xrightarrow{s} \mathcal{O}_X \to \mathcal{O}_D \to 0,$$

where $\mathcal{O}_X$ is the structure sheaf of X (sections of the trivial line bundle on X) and $\mathcal{O}_D$ is the structure sheaf of D pushed forward to X (extending by zero – this is a torsion sheaf concentrated on D). Thus $\mathcal{O}_D$ is the cohomology of the complex $\{L^{-1} \xrightarrow{s} \mathcal{O}_X\}$. (Notice this is the cohomology of a complex of sheaves, and as such is a sheaf, and should not be confused with sheaf cohomology, which is vector-space-valued.)

Similarly if $Z \subset X$ is a codimension r subvariety, the transverse zero locus of a regular section $s \in H^0(E)$ of a rank r vector bundle E, the exact sequence (Koszul complex)

$$0 \to \Lambda^r E^* \to \Lambda^{r-1} E^* \to \ldots \to E^* \to \mathcal{O}_X,$$

where each arrow is given by interior product with s, has cokernel $\mathcal{O}_Z$ by inspection. Thus $\mathcal{O}_Z$ is the cohomology of this complex.

We have actually gained something here: we have replaced nasty torsion sheaves $\mathcal{O}_D$, $\mathcal{O}_Z$ by nicer, locally free sheaves (i.e. vector bundles) on X. In general one

can consider such *resolutions*, replacing arbitrary sheaves $\mathcal{F}$ by complexes $F^\bullet$ of sheaves that are "nicer" in some way,

$$F^\bullet \to \mathcal{F} \qquad \text{or} \qquad \mathcal{F} \to F^\bullet,$$

and now use the complex $F^\bullet$ instead of its less manageable cohomology $\mathcal{F}$. We think of the nasty looking curly $\mathcal{F}$ being made up from the nicer straight Fs as $F^1 - F^2 + F^3 - \ldots$, where the sense in which we subtract the F^is is given by the maps between them in the resolution, and the F^i are the building blocks: the generators of $\mathcal{F}$ form F^1, the relations F^2, relations amongst the relations F^3, and so on.

Of course the resolution may not arise naturally in general and we must pick one; this is directly analogous to picking a (non-canonical) triangulation of a topological space as we shall see in Section 3, and is the problem that derived categories resolve. The sense in which the sheaves $F^\bullet$ are "nicer" will be tackled in general in Section 6, but the following is a simple example.

Intersection theory via sheaves. We concentrate on the easiest case of intersecting two divisors D_1, D_2 in a smooth complex surface X. These correspond to line bundles L_i and sections $s_i \in H^0(L_i)$ with zero locus D_i. The D_i have homology classes in $H_2(X)$ that we may intersect, or dually we can consider $c_1(L_i) \in H^2(X)$. In terms of sheaf theory the first corresponds to tensoring structure sheaves: if D_1, D_2 intersect transversely then we have $\mathcal{O}_{D_1 \cap D_2} = \mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2}$. The second corresponds to using the resolution (2.3) for $\mathcal{O}_{D_1}$ and tensoring *that* with $\mathcal{O}_{D_2}$, as the following table shows:

Data	Divisor D_i	$s_i \in H^0(L_i)$ such that $s_i^{-1}(0) = D_i$
Alg geom	$\mathcal{O}_{D_i}$	$\{L_i^{-1} \xrightarrow{s_i} \mathcal{O}_X\}$
Topology	$[D_i] \in H_2(X)$	$c_1(L_i) \in H^2(X)$
Intersection	$[D_1] . [D_2]$	$c_1(L_1) \cup c_1(L_2) = \langle c_1(L_1), D_2 \rangle = c_1(L_1|_{D_2})$
Transverse case	$\mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2}$ $= \mathcal{O}_{D_1 \cap D_2}$	$c_1(L_1|_{D_2}) = (s_1|_{D_2})^{-1}(0)$, i.e. restrict L_1 to D_2 and take zeros of its section $s_1|_{D_2}$, i.e. take cokernel of $\{L_1^{-1}|_{D_2} \xrightarrow{s_1|_{D_2}} \mathcal{O}_{D_2}\}$
Non transverse case; e.g. $D_1 = D_2 = D$	$\mathcal{O}_D \otimes \mathcal{O}_D$ $= \mathcal{O}_D$	$\{L^{-1}|_D \xrightarrow{s|_D=0} \mathcal{O}_D\}$. So we still see L on D, just with the section $= 0$.

So restricting L_1 to D_2 corresponds to tensoring $\{L_1^{-1} \to \mathcal{O}_X\}$ with $\mathcal{O}_{D_2}$, and in the transverse case the cokernel of this is just $\mathcal{O}_{D_1 \cap D_2}$. Thus the way to pass from the right hand column to the left is to take cokernels; this corresponds to the fact that tensoring with $\mathcal{O}_{D_2}$ is *right exact*: tensoring the exact sequence (2.3) with $\mathcal{O}_{D_2}$ gives a sequence in which the final three arrows are still exact.

What the table shows is that while tensoring $\mathcal{O}_{D_1}$ with $\mathcal{O}_{D_2}$ gives the correct answer for D_1 and D_2 transverse, in the non-transverse case it does not ($\mathcal{O}_D$ is the

structure sheaf of $D \cap D$, but not of the correct topological intersection of D with itself). However tensoring instead $\{L_1^{-1} \to \mathcal{O}_X\}$ with $\mathcal{O}_{D_2}$ always gives the right answer: taking the cokernel of the resulting complex gives the same $\mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2}$ as before, but in the non-transverse case we get more – we still have the line bundle $L_1|_{D_2}$ and so the intersection information (we need only take its first Chern class); by changing the section $L_1^{-1}|_{D_2} \xrightarrow{s_1|_{D_2}} \mathcal{O}_{D_2}$ from one which may be identically zero on (components of) D_2 to a transverse section we get the correct intersection. More generally we take the divisor of $L_1|_{D_2}$ on D_2. If D_1 can be moved to be transverse to D_2 this gives the same answer; if $L_1|_{D_2}$ has no sections this cannot be achieved but our method still gives the correct intersection product, *inside* the scheme-theoretic intersection $D_1 \cap D_2$ as well.

The moral is that

$$\mathcal{O}_D \otimes \mathcal{O}_D \qquad \text{``should be''} \qquad \{L^{-1} \xrightarrow{s} \mathcal{O}_X\} \otimes \mathcal{O}_D = \{L^{-1}|_D \xrightarrow{0} \mathcal{O}_D\}.$$

This still has cokernel $\mathcal{O}_D \otimes \mathcal{O}_D = \mathcal{O}_D$, but now (because the intersection was not transverse) has kernel too, namely $L^{-1}|_D$, containing all the intersection information. The difference of the first Chern classes of these sheaves on D is precisely the self intersection $D \cdot D$.

So just as the dual of H_*, in the case of simplicial complexes in Section 2, "should be" given by applying $\mathrm{Hom}(\,.\,, \mathbb{Z})$ not to H_* but to the chain complex, here we apply $\otimes \mathcal{O}_D$ to the complex of locally free (and so better behaved) sheaves rather than $\mathcal{O}_D$. This is the prototype of a derived functor which will be dealt with systematically in Section 6. The kernel $L^{-1}|_D$ above will be the first derived functor $\mathrm{Tor}_1(\mathcal{O}_D, \mathcal{O}_D)$ of $\otimes \mathcal{O}_D$.

Thus we see an example where it is beneficial to consider complexes of sheaves rather than just single (cohomology) sheaves. Having applied $\otimes \mathcal{O}_{D_2}$ we get a genuine complex, potentially with cohomology in more than one degree, i.e. it is not simply the resolution of a single sheaf. But again we should not now pass to its cohomology, as we may want to intersect with further cycles, by tensoring the complex with some $\mathcal{O}_{D_3}$ if X is higher dimensional, for instance. Complexes good, cohomology bad, after all.

So having motivated considering *all* complexes (rather than just resolutions of sheaves) we will now set about working with them, forgetting all about derived functors until Section 6. Replacing sheaves by complexes of which they are the cohomology, i.e. by resolutions, we come across the problem mentioned earlier: how do we pick a resolution functorially ? Again there is an analogous issue in topology.

3. Quasi-isomorphisms

Quasi-isomorphisms in topology. We saw in Section 2 that a good invariant of a (simply connected) topological space $|X|$ underlying a simplicial complex X is the simplicial chain complex $C_\bullet^X$ which determines the homotopy type of $|X|$ by the Whitehead theorem. The problem is functoriality: how to pick a triangulation of $|X|$ canonically, to pass from the topological space to a complex. The standard mathematical trick is to consider all at once on an equal footing, for instance by making them all isomorphic. As described in Section 2, different triangulations of a space (yielding different chain complexes $C_\bullet$, $D_\bullet$) may have no simplicial map between them, but taking finer subdivisions and using simplicial approximations

we can find a third chain complex $E_\bullet$ fitting into the diagram

$$
\begin{array}{ccc}
 & E_\bullet & \\
\swarrow & & \searrow \\
C_\bullet & & D_\bullet
\end{array}
$$

where both maps are *quasi-isomorphisms* – they are chain maps inducing isomorphisms on homology. We extend this to be an equivalence relation; thus two chain complexes are quasi-isomorphic if they can be related by a sequence of quasi-isomorphisms of the above type. So quasi-isomorphic complexes have the same homology, but the converse does *not* hold; for instance the complexes

$$
\mathbb{C}[x,y]^{\oplus 2} \xrightarrow{(x,y)} \mathbb{C}[x,y] \qquad \text{and} \qquad \mathbb{C}[x,y] \xrightarrow{0} \mathbb{C}
$$

have the same homology but are not quasi-isomorphic. Quasi-isomorphism is exactly the equivalence relation we want on complexes: by the Whitehead theorem (2.2), $|X|$ and $|Y|$ are homotopy equivalent if and only if there is a Z inducing a quasi-isomorphism

(3.1)
$$
\begin{array}{ccc}
 & C_\bullet^Z & \\
\swarrow & & \searrow \\
C_\bullet^X & \dashrightarrow & C_\bullet^Y .
\end{array}
$$

Thus we would like to think of quasi-isomorphic complexes as isomorphic. Though there may not be a map between them (as there may be no simplicial map between two different triangulations of the same space) we pretend there is by putting one in by hand, putting in the dotted arrow in (3.1) even if it does not exist as a genuine map. We consider the complexes to be isomorphic since the underlying spaces are, after all.

In particular homotopy equivalences are quasi-isomorphisms. Algebraically this means that if there is an $s : C_\bullet \to D_{\bullet+1}$ such that $f - g = \partial_D \circ s + s \circ \partial_D$ for two chains maps f, $g : C_\bullet \to D_\bullet$, then f and g induce the same map on homology.

Quasi-isomorphisms in algebraic geometry. The cokernel map (2.3)

$$
\{L^{-1} \to \mathcal{O}_X\} \longrightarrow \mathcal{O}_D
$$

is of course a quasi-isomorphism (it induces an isomorphism on cohomology). Similarly any resolution $F^\bullet \to \mathcal{F}$ (or $\mathcal{F} \to F^\bullet$) gives a quasi-isomorphism $\{F^\bullet\} \to \mathcal{F}$ (or $\mathcal{F} \to \{F^\bullet\}$). As in the topological case we want to consider these as equalities (more strictly, isomorphisms) $\mathcal{F} \cong \{F^\bullet\}$, to make choice of a resolution functorial. So again we need to introduce inverse arrows $\{F^\bullet\} \dashleftarrow \mathcal{F}$ (or $\mathcal{F} \dashleftarrow \{F^\bullet\}$), which is what we turn to now.

4. Category theory

In general we would like to replace *objects* (e.g. sheaves, vector bundles, abelian groups, modules, etc.) by complexes of objects that are *"better behaved"* for some particular *operation* such as $\otimes A$, $\mathrm{Hom}\,(\ .\ , A)$, $\mathrm{Hom}\,(A,\ .\ )$, $\Gamma(\ .\ )$, ... (better behaved in the sense that *no information is lost* when the operation is applied to such objects). Sometimes there is a canonical such complex, in general we want to make its choice *natural*.

The words in italics are meant to be suggestive of category theory; abelian categories are the natural setting for the general theory, with the above words being replaced by, respectively, objects of a category, acyclic objects, functor, exact,

functorial. We shall now define such things; those scared of the word category should think of the (category of) sheaves on a complex manifold.

DEFINITION 4.1. An additive category is a category $\mathcal{A}$ such that
- Each set of morphisms $\mathrm{Hom}\,(A, B)$ forms an abelian group.
- Composition of morphisms distributes over the addition of morphisms given by the abelian group structure, i.e. $f \circ (g+h) = f \circ g + f \circ h$ and $(f+g) \circ h = f \circ h + g \circ h$.
- There exist products (direct sums) $A \times B$ of any two objects A, B satisfying the usual universal properties (see e.g. [**Ma**]).
- There exists a zero object 0 such that $\mathrm{Hom}\,(0,0)$ is the zero group (i.e. just the identity morphism). Thus $\mathrm{Hom}\,(0, A) = 0 = \mathrm{Hom}\,(A, 0)$ for all A, and the unique zero morphism between any two objects is the one that factors through the zero object.

An abelian category is an additive category that also satisfies
- All morphisms have kernels and cokernels such that monics (morphisms with zero kernel) are the kernels of their cokernel maps and epis (zero cokernel) are the cokernels of their kernels.

The kernels and cokernels mentioned above are defined in a general category via obvious universal properties (e.g. a kernel of $f : A \to B$ is a $K \to A$ through which any $g : C \to A$ factors uniquely if and only if $f \circ g = 0$). We refer to [**Ma**] for such things; we shall only be concerned with concrete categories (those whose objects are sets) and then kernels and cokernels will be the usual ones, and the final axiom will be immediate.

So in an abelian category we can talk about exact sequences and *chain complexes*, and cohomology of complexes. Additive functors between abelian categories are *exact* (respectively left or right exact) if they preserve exact sequences (respectively short exact sequences $0 \to A \to B \to C$ or $A \to B \to C \to 0$).

DEFINITION 4.2. The bounded derived category $D^b(\mathcal{A})$ of an abelian category $\mathcal{A}$ has as objects bounded (i.e. finite length) $\mathcal{A}$-chain complexes, and morphisms given by chain maps with quasi-isomorphisms inverted as follows ([**GM**] III 2.2). We introduce morphisms f for every chain map between complexes $f : X_f \to Y_f$, and $g^{-1} : Y_g \to X_g$ for every quasi-isomorphism $g : X_g \xrightarrow{\sim} Y_g$. Then form all products of these morphisms such that the range of one is the domain of the next. Finally identify any combination $f_1 f_2$ with the composition $f_1 \circ f_2$, and gg^{-1} and $g^{-1}g$ with the relevant identity maps id_{Y_g} and id_{X_g}.

REMARK 4.3. Similarly one can define the unbounded derived category, and the categories $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ of bounded below and above complexes respectively. We shall use $D(\mathcal{A})$ to mean one of the four such derived categories.

Morphisms in $D(\mathcal{A})$ are represented by roofs

$$
\begin{array}{ccc}
& Z^\bullet & \\
{}^s \swarrow & & \searrow {}^g \\
X^\bullet \dashrightarrow{\ \ f\ \ } & & Y^\bullet,
\end{array}
$$

where s is a quasi-isomorphism, and we set $f = gs^{-1}$ (it is clear any morphism is a composition of such roofs; that one will suffice is proved in [**GM**] III 2.8).

This "localisation" procedure of inverting quasi-isomorphisms has some remarkable properties (for instance we shall see that homotopic maps are identified with each other to give the same morphism in the derived category). Although they give $D(\mathcal{A})$ the structure of an additive category, we will see it does not have kernels or cokernels. As ever we go back to the topology to see why not, what they are replaced by, and what the structure of $D(\mathcal{A})$ is (since it is not that of an abelian category).

5. Cones and triangles

When working with topological spaces (or simplicial or cell complexes) up to homotopy there is no notion of kernel or cokernel. In fact the standard cylinder construction shows that any map $f : X \to Y$ is homotopic to an inclusion $X \to \mathrm{cyl}\,(f) = Y \sqcup (X \times [0,1])/f(x) \sim (x,1)$, while the path space construction shows it is also homotopic to a fibration.

For some *fixed* maps $f : X \to Y$, rather then equivalence classes of homotopic maps, we can make sense of the kernel (the fibre of f if it is a fibration) or cokernel (Y/X, the space with the image of X collapsed to a point, if f is a cofibration, which means an inclusion for our purposes). In general of course neither makes sense, but there is something which acts as both, namely cones and the Dold-Puppe construction.

The cone C_f on a map $f : X \to Y$ is the space formed from $Y \sqcup (X \times [0,1])$ by identifying $X \times \{1\}$ with its image $f(X) \subset Y$, and collapsing $X \times \{0\}$ to a point. It fits into the sequence of maps
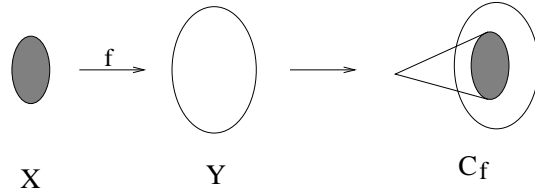
It is clear that this can act as a cokernel, in that if $X \overset{f}{\hookrightarrow} Y$ is an inclusion, then C_f above is clearly homotopy equivalent to Y/X. In fact we can now iterate this process, forming the cone on the natural inclusion $i : Y \to C_f$ to give the sequence:
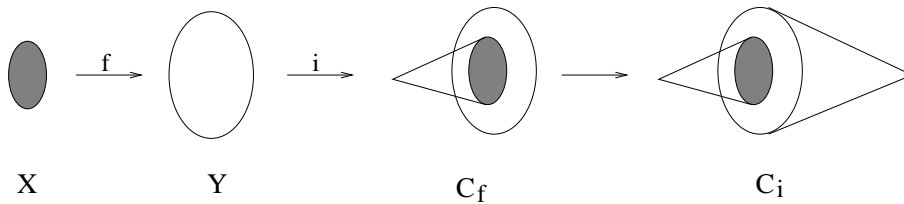
By retracting Y to the right hand apex of C_i in the last term of the above diagram, we see that C_i is homotopic to ΣX, the suspension of X. Thus, up to homotopy, we get a sequence

(5.1) $X \longrightarrow Y \longrightarrow Y/X \longrightarrow \Sigma X \longrightarrow \dots$

Taking the ith cohomology H_i of each term, and using the suspension isomorphism $H_i(\Sigma X) \cong H_{i-1}(X)$ gives a sequence

(5.2) $\qquad H_i(X) \to H_i(Y) \to H_i(Y, X) \to H_{i-1}(X) \to H_{i-1}(Y) \to \dots$

which is just the long exact sequence associated to the pair $X \subset Y$ (it is an interesting exercise to check that the map $H_i(Y/X) \to H_i(\Sigma X)$ induced above is indeed the boundary map $H_i(Y, X) \xrightarrow{\partial_*} H_{i-1}(X)$).

Up to homotopy we can make (5.1) into a sequence of simplicial maps, so that taking the associated chain complexes we get a lifting of the long exact sequence of homology (5.2) to the level of complexes. Of course it exists for all maps f, not just inclusions, with Y/X replaced by C_f.

For instance if f is a fibration, C_f acts as the "kernel" or fibre of the map. The most extreme case is $f : X \to$ point, which gives $C_f = \Sigma X$, the suspension of the fibre X, which to homology is X shifted in degree by 1.

So C_f acts as a combination of both cokernel and kernel, and as such gives an easy proof of the Whitehead theorem we have been quoting: i.e. if $f : X \to Y$ is a map inducing an isomorphism of homology groups of simply connected spaces then the sequence

$$H_i(X) \to H_i(Y) \to H_i(C_f) \to H_{i-1}(X) \to H_{i-1}(Y) \to \dots$$

shows that $H_*(C_f) = 0$. Thus, by the Hurewicz theorem, the homotopy groups $\pi_*(C_f)$ are zero too, making C_f homotopy equivalent to a point by the more famous Whitehead theorem. It is an easy consequence of this that f is a homotopy equivalence.

The final thing we note from the topology is that if X and Y are simplicial complexes, with $f : X \to Y$ a simplicial map, then the cone C_f is naturally a simplicial complex with i-simplices being those in Y, plus the cones on $(i-1)$-simplices in X. A little thought shows that this makes the corresponding (cohomology, for convenience) complex

$$C_X^\bullet\,[\,1\,] \oplus C_Y^\bullet \quad \text{with differential} \quad d_{C_f} = \begin{pmatrix} d_X\,[\,1\,] & 0 \\ f & d_Y \end{pmatrix},$$

where $[\,n\,]$ means shift a complex n places left.

Thus we can define the cone C_f on any map of chain complexes $f : A^\bullet \to B^\bullet$ in an abelian category $\mathcal{A}$ by the above formula, replacing $C_X^\bullet$ by $A^\bullet$ and $C_Y^\bullet$ by $B^\bullet$. If $A^\bullet = A$ and $B^\bullet = B$ are chain complexes concentrated in degree zero then C_f is the complex $\{A \xrightarrow{f} B\}$. This has zeroth cohomology $h^0(C_f) = \ker f$, and $h^1(C_f) = \operatorname{coker} f$, so combines the two (in different degrees). In general it is just the total complex of $A^\bullet \to B^\bullet$.

There is an obvious map $i : B^\bullet \to C_f$, and C_i is, as could be guessed from above, quasi-isomorphic to $A^\bullet\,[\,1\,]$. So what we get in a derived category is not kernels or cokernels, but "exact triangles"

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet\,[\,1\,].$$

(For a proof see ([**GM**] III 3.5) – we should really modify B, replacing it by the quasi-isomorphic $\operatorname{cyl}(f)$, but once quasi-isomorphisms are inverted it does not matter.)

Thus we have long exact sequences rather than short exact ones; taking ith cohomology h^i of the above gives the standard long exact sequence

$$h^i(A^\bullet) \to h^i(B^\bullet) \to h^i(C^\bullet) \to h^{i+1}(A^\bullet) \to \dots$$

Thus $D(A)$ is not an abelian category, it is an example of a *triangulated category*. This is an additive category with a functor T (often denoted $[\,1\,]$) and a set of *distinguished triangles* satisfying a list of axioms. We refer to ([**GM**] IV 1.1) for the precise details, but the triangles include, for all objects X of the category,

$$X \xrightarrow{\text{id}} X \to 0 \to X\,[\,1\,],$$

and any morphism $f : X \to Y$ can be completed to a distinguished triangle

$$X \to Y \to C \to X\,[\,1\,].$$

There is also a derived analogue of the 5-lemma, and a compatibility of triangles known as the octahedral lemma, which is pretty unTeXable, as you might imagine.

Why are we abstracting again? It is because this structure that is present in $D(\mathcal{A})$, for an abelian category $\mathcal{A}$, is also present in topology (as we have been hinting all along) without any underlying abelian category. I.e. the topology described throughout this talk might lead one to suspect that some category of nice topological spaces should have just this structure, and we have certainly not constructed it as a derived category of any abelian category.

In fact we are not quite there yet, the reason being that the translation functor of suspension $T = \Sigma$ is not invertible on spaces. If it did exist it is clear it would be the loop space functor Ω, as Ω and Σ are adjoints by a standard argument[1]. In particular $\Sigma X \simeq \Omega^{-1}X$ and we would be able to deloop spaces, making them infinite loop spaces.

So it is not surprising to find that *spectra* (essentially infinite loop spaces, plus all their loopings and deloopings to keep track of homotopy in negative degrees) form a triangulated category in the way we have been describing (though there are many technicalities we have bypassed, due to the spaces involved not being simplicial complexes).

The final thing to note from the topology is that homotopic maps $X \to Y$ get identified when we invert quasi-isomorphisms. By composing with the homotopy $X \times [0,1] \to Y$ it is clear it is enough to show the two inclusions ι_0, ι_1 of X into $X \times [0,1]$ (as $X \times \{0\}$ and $X \times \{1\}$ respectively) are identified. But they are both right inverses for the projection $p : X \times [0,1] \to X$: $p\iota_i = \text{id}$. As they are also isomorphisms on homology, when we invert quasi-isomorphisms they both become identified with p^{-1}.

Algebraically, in $D(\mathcal{A})$, we can mimic the (dual, cohomology) proof by defining the cylinder of a complex in the way dictated by the obvious product cell complex structure on the cylinder of a simplicial complex. Namely consider, for any chain complex $A^\bullet$,

$$\text{cyl}\,(A^\bullet) = A^\bullet \oplus A^\bullet\,[\,1\,] \oplus A^\bullet, \quad \text{with differential} \quad d = \begin{pmatrix} d_A & 0 & 0 \\ -\text{id} & d_A\,[\,1\,] & \text{id} \\ 0 & 0 & d_A \end{pmatrix}.$$

---

[1] If we denote homotopy classes of maps from X to Y by $[X, Y]$, then applying the invertible Σ to $[X, \Sigma^{-1}Y]$ gives an isomorphism $[X, \Sigma^{-1}Y] \cong [\Sigma X, Y] \cong [X, \Omega Y]$. Choosing $X = \Sigma^{-1}Y$ and $X = \Omega Y$ gives canonical maps $\Sigma^{-1}Y \leftrightarrows \Omega Y$ that induce isomorphisms on homotopy groups (as is seen by choosing $X = S^n$). Thus $\Sigma^{-1}Y$ is, up to weak homotopy equivalence, ΩY.

Set ι_1^*, ι_2^* to be the projections to the first and third factors respectively, with p^* the sum of the corresponding two inclusions. These are all chain maps, and in fact quasi-isomorphisms, with $\iota_i^* p^* = \mathrm{id}$. Thus ι_1^*, ι_2^* are identified with $(p^*)^{-1}$ in $D(\mathcal{A})$ and so also with each other.

6. Derived functors

Now we can go back to (derived) functors and replace arbitrary complexes by resolutions of objects which are suited to the functor concerned. We deal with left exact functors; right exact functors are similar.

DEFINITION 6.1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. A class of objects $\mathcal{R} \subset \mathcal{A}$ is *adapted* ([**GM**] III 6.3) to a left exact functor $F : \mathcal{A} \to \mathcal{B}$ if

- $\mathcal{R}$ is stable under direct sums,
- F applied to an acyclic complex in $\mathcal{R}$ (that is, a complex with vanishing cohomology) is acyclic, and
- any $A \in \mathcal{A}$ injects $0 \to A \to R$ into some $R \in \mathcal{R}$.

Let $K^+(\mathcal{R})$ be the category of bounded below chain complexes in $\mathcal{R}$ with morphisms homotopy equivalence classes of chain maps. Then [**GM**] inverting quasi-isomorphisms in $K^+(\mathcal{R})$ gives a category equivalent to $D^+(\mathcal{A})$.

The final statement is a fancy way of saying we have finally reached our goal – we can functorially replace (i.e. resolve) any $\mathcal{A}$-complex by a quasi-isomorphic $\mathcal{R}$-complex using the conditions of the definition. Examples of such $\mathcal{R} \subset \mathcal{A}$ include {injective sheaves} $\subset$ {quasi-coherent sheaves}, or for right exact functors, {projectives modules over a ring} $\subset$ {all modules}, {locally free sheaves over an affine variety or scheme} $\subset$ {coherent sheaves}, {flat sheaves} $\subset$ {quasi-coherent sheaves} if F is tensoring with a sheaf, etc.

So instead of applying F to arbitrary complexes, we apply it only to $\mathcal{R}$-complexes.

Thus we define the *right derived functor* $\mathbf{R}F$ of F to be the composition $D^+(\mathcal{A}) \to K^+(\mathcal{R})/\mathrm{q.\,i.} \xrightarrow{F} D^+(\mathcal{B})$. (The more classical right derived functors $R^i F$ are the cohomology $R^i F = h^i(\mathbf{R}F)$, but taking cohomology is bad, we now know.) We are glossing over some details here; really $\mathbf{R}F$ should be defined by some universal property ([**GM**] III 6.11) to make it independent of $\mathcal{R}$ up to canonical isomorphism. Similarly we can define the left derived functor $\mathbf{L}F$ of a right exact F.

This gives an *exact functor* $\mathbf{R}F : D^+(\mathcal{A}) \to D^+(\mathcal{B})$, i.e. it takes exact triangles to exact triangles. In particular taking cohomology gives the classical long exact sequence of derived functors of the form

$$\ldots \to R^i F(A) \to R^i F(B) \to R^i F(C) \to R^{i+1} F(A) \to \ldots$$

For instance the right derived functor of the left exact global sections functor $\Gamma : \{\,\mathrm{Sheaves}\,\} \to \{\,\mathrm{Vector\ spaces}\,\}$ is just sheaf cohomology $\mathbf{R}\Gamma = \mathbf{R}H^\bullet$, with cohomology (of the complex of vector spaces) the standard sheaf cohomology H^i. Then the above sequence becomes the long exact sequence in cohomology.

There are two main advantages of this approach. Firstly that we have managed to make the *complex* $\mathbf{R}F(A)$, rather than its less powerful cohomology $R^i F(A)$, into an invariant of A, unique up to quasi-isomorphism. Secondly, the derived functor has simply become the original functor applied to complexes (though not arbitrary

ones, they have to be in $\mathcal{R}$). This gives easier and more conceptual proofs for results about derived functors that usually require complicated double complex, spectral-sequence type arguments. We give some examples in sheaf theory, skating over a few technical conditions (issues about boundedness of complexes and resolutions that are certainly not a problem on a smooth projective variety; for precise statements see [**HRD**]):

EXAMPLE 6.2. Tensor product of sheaves is symmetric, thus its derived functor $\overset{\mathbf{L}}{\otimes}$, and its homology Tor_i, are symmetric: $\mathrm{Tor}_i(A, B) \cong \mathrm{Tor}_i(B, A)$. Here we simply tensor complexes of flat sheaves (for instance, locally free sheaves).

EXAMPLE 6.3. $\mathbf{R}\mathcal{H}om$ of sheaves, being just local $\mathcal{H}om$ on complexes, can be defined by resolving either the first variable by locally frees or the second by injectives.

EXAMPLE 6.4. Under some mild conditions, $\mathbf{R}(F \circ G) \cong \mathbf{R}F \circ \mathbf{R}G$. If we take cohomology before applying $\mathbf{R}F$ we get an approximation to $\mathbf{R}(F \circ G)$, and the Grothendieck spectral sequence $R^i F(R^j G) \Rightarrow R^{i+j}(F \circ G)$.

For instance for a morphism $p : X \to Y$ the equality $\Gamma_X = \Gamma_Y \circ p_*$ yields $\mathbf{R}\Gamma_X \cong \mathbf{R}\Gamma_Y \circ \mathbf{R}p_*$, and so the Leray spectral sequence $H_X^{i+j} \Leftarrow H_Y^i(R^j p_*)$.

Similarly $\mathrm{Hom} = \Gamma \circ \mathcal{H}om$ yields $\mathbf{R}\mathrm{Hom} \cong \mathbf{R}H^\bullet(\mathbf{R}\mathcal{H}om)$ and the local-to-global spectral sequence $\mathrm{Ext}^{i+j} \Leftarrow H^i(\mathcal{E}xt^j)$.

Hypercohomology $\mathbb{H}$ is given by the derived functor of global sections of complexes (so it is nothing but normal sheaf cohomology for us). Applying the above to $F = \Gamma$ and $G = \mathrm{id}$ we get the hypercohomology spectral sequence $H^i(h^j(A^\bullet)) \Rightarrow \mathbb{H}^{i+j}(A^\bullet)$.

EXAMPLE 6.5. For A a sheaf or complex of sheaves, denote by $A^\vee$ the dual complex $\mathbf{R}\mathcal{H}om(A, \mathcal{O})$. Then $\mathbf{R}\mathcal{H}om(A, B) \cong B \overset{\mathbf{L}}{\otimes} A^\vee$.

EXAMPLE 6.6. We now consider an example on a real n-manifold, namely the DeRham theorem. Let $A^i(\mathbb{R})$ denote the sheaf of C^∞ i-forms on a manifold X, with d the exterior derivative. Then the Poincaré lemma gives a quasi-isomorphism (resolution of the constant sheaf $\mathbb{R}$)

$$\mathbb{R} \to \{A^0(\mathbb{R}) \xrightarrow{d} A^1(\mathbb{R}) \xrightarrow{d} \ldots \xrightarrow{d} A^n(\mathbb{R})\}.$$

Because of the existence of partitions of unity the $A^i(\mathbb{R})$ sheaves are acyclic for the global sections functor Γ, i.e. they have no higher sheaf cohomology (they are what is known as *fine* sheaves). Thus it is easy to see that we may apply Γ to the resolution to obtain a complex isomorphic to the derived functor $\mathbf{R}\Gamma(\mathbb{R})$ of the constant sheaf $\mathbb{R}$, i.e. its sheaf cohomology. This yields

$$\Omega^0(\mathbb{R}) \xrightarrow{d} \Omega^1(\mathbb{R}) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(\mathbb{R}),$$

which means that the DeRham complex computes the real cohomology $H^*(X; \mathbb{R})$ of the manifold X.

Similarly taking cohomology of the Dolbeault complex or resolution

$$\mathbb{C} \to \{\mathcal{O} \xrightarrow{\partial} \Omega^{1,0} \xrightarrow{\partial} \Omega^{2,0} \xrightarrow{\partial} \ldots \xrightarrow{\partial} \Omega^{n,0}\},$$

on a complex manifold X, gives a spectral sequence relating $H^*(X;\mathbb{C})$ to the Hodge groups $\bigoplus_{i,j} H^i(\Omega^{j,0}) = \bigoplus_{i,j} H^{i,j}(X)$ (the spectral sequence famously degenerates for a Kähler manifold).

Hopefully this talk has shown derived and triangulated categories to be natural objects, just categories of complexes with quasi-isomorphisms made into isomorphisms, which we can think of via topological pictures (modulo some technical details). Unfortunately this is a bit of a disservice to anyone who comes to work in some $D(\mathcal{A})$, where it is important to chase all quasi-isomorphisms, compatibilities, etc. The main problem is that while cones are defined up to isomorphism in triangulated categories, choosing them functorially is not possible. This leads many to think that there should be some more refined concept still to be worked out.

References

[GM]   S. Gelfand and Yu. Manin, *Methods of Homological Algebra*, Springer, 1996.

[HRD] R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics 20, Springer-Verlag, 1966.

[K]   M. Kontsevich, *Homological Algebra of Mirror Symmetry*, International Congress of Mathematicians, Zürich 1994. Birkhäuser, 1995.

[Ma]   S. MacLane, *Categories for the working mathematician*, Graduate texts in mathematics 5, Springer-Verlag, 1971.

MATHEMATICAL INSTITUTE, 24–29 ST GILES', OXFORD OX1 3LB. UK
E-mail address: thomas@maths.ox.ac.uk