(2+0)-DIMENSIONAL INTEGRABLE EQUATIONS AND EXACT SOLUTIONS

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ABSTRACT

We propose a nonlinear σ-model in a curved space as a general integrable elliptic model. We construct its exact solutions and obtain energy estimates near the critical point. We consider the Pohlmeyer transformation in Euclidean space and investigate the gauge equivalence conditions for abroad class of elliptic equations. We develop the inverse scattering transform method for the sinh-Gordon equation and evaluate its exact and asymptotic solutions

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1. INTODUCTION.

This work is devoted to studying a broad class of problems involving two-dimensional nonlinear integrable elliptic equations. These equations, which describe stationary processes in nonlinear media, usually, govern the spatial distributions of physical fields whose value are fixed at the boundary of the domain under consideration. The resulting boundary-value problems have a characteristic feature: as the "observer" penetrates deeper into the medium (which means considering the asymptotic behavior as $r^2 = x^2 + y^2 \to \infty$), the field in question become weak, and the equations become the classical linear equations of mathematical physics. For this reason, the inverse scattering transform method (ISTM) applied to the elliptic equations (as well as the hyperbolic ones) is a nonlinear analogue of the Fourier transform method, a tool often applied to linear problems. This analogy allows one, in particular, to interpret the discrete-spectrum solutions of nonlinear elliptic equations as "solitonlike" excitations (vortices, or defects) in nonlinear condensed media. These excitations possess some properties of solitons and also some special properties of their own.

In [1-4], the boundary-value problems on the half-plane were posed for two-dimensional elliptic equations, the direct and the inverse scattering problems were investigated for the corresponding operators of the associated linear problems, and the exact solutions, the conservation laws, and the trace identities were also found. In addition, the boundary-value problems for the elliptic versions of the two-dimensional Heisenberg ferromagnet (of the $O(3)$ σ-model) were shown to be gauge equivalent to the sinh-Gordon equation [4]-[5]. These results have thus allowed us to extend the methods used with the hyperbolic equations to the elliptic case, which has numerous important physical applications.

At the same time, a currently important problem consists in constructing more-realistic models that would still be integrable on the one hand and would account for various features of the physical systems on the other hand. Solving one such problem was begun in [6], where a σ-model in a curved space was proposed (a magnet with a variable nominal magnetization). We show in what follows that this model (more precisely, its integrable version) is the most general of the presently known physical models that are described by elliptic equations and possess Lax representations.

In this paper, which is an expanded version of [6, 1] and a revised version of [7], our aim is to obtain exact solutions of this model, to investigate its properties, and also to find a chain of transitions leading from this model to those that have been known and investigated before. For one such model (sinh-Gordon), we use the ISTM to construct exact solutions and give their physical interpretation.

The paper is organized as follows. In Sec.2, we consider a σ-model in a curved space, find its exact solutions, and investigate some thermodynamic properties. In Sec.3, we propose a Euclidean version of the Pohlmeyer transformation and investigate the question of the gauge equivalence (GE) of a broad family of boundary-value problems. In Sec.4, we develop the ISTM for a model of charged particles on a half-plane, find a series of its exact solutions, and also find the conservation laws and trace identities.

2. NONLINEAR σ-MODEL IN CURVED SPACE.

As is well known, the condition that the absolute value of the magnetic moment be constant is an essential assumption in the phenomenological description of weakly excited states of magnets. With this assumption, the evolution of system states amounts to the rotation of the magnetization vector. A similar statement is true for magnet models described by the Landau-Lifshits equation. Such models provide a physically adequate picture far from the Curie point. For moderate temperatures, however, one can expect a spatial variation of the absolute (nominal) magnetization of the sample material. This hypothesis allows one to investigate the magnet near the critical point (but outside the fluctuational region) using the phenomenological approach and to describe a smooth spatial transition to the paramagnetic phase.

We consider the two-dimensional isotropic Heisenberg ferromagnet, i.e., the two-dimensional stationary Landau-Lifshits equation

$$\mathbf{M} \wedge \triangle \mathbf{M} = 0, \quad \mathbf{M}^2 = \alpha(\mathbf{r}), \tag{2.1}$$

where $\mathbf{M} = \mathbf{M}(x, y) = (M_1, M_2, M_3)$, $\mathbf{r} = (x, y) \in R^2_+$, and $\alpha(\mathbf{r})$ is the square of the nominal magnetization, which is an arbitrary function of $\mathbf{r}$, in general.

The equation

$$\alpha \triangle \mathbf{M} + (\mathbf{M}_x^2 + \mathbf{M}_y^2)\mathbf{M} = \frac{1}{2}\mathbf{M} \triangle \alpha \tag{2.2}$$

is easily obtained from (2.1). We set

$$M = \sum_{i=1} M_i \sigma_i,$$

where σ_i, $i = 1, 2, 3$, are the Pauli matrices. In terms of the matrix M, Eq. (2.2) is

$$(\alpha M_z M^{-1})_{\bar{z}} + (\alpha M_{\bar{z}} M^{-1})_z = \alpha_{z\bar{z}} I, \tag{2.3}$$

where $z = x + iy$ and I is the 2×2 unit matrix.

Equation (2.2) (or the equivalent equation (2.3)) describes an inhomogeneous magnet, a generalization of the two-dimensional version of the stationary Landau-Lifshits magnet. Since the question of the integrability of these equations remains unanswered in the general case of an arbitrary function $\alpha(\mathbf{r})$), we consider the simplest version, where Eq. (2.3) is integrable, obtained by setting

$$4\triangle\alpha = \alpha_{z\bar{z}} = 0. \tag{2.4}$$

Then (2.3) implies that a two-dimensional isotropic magnet whose squared saturation moment is a harmonic function of the coordinates is described by a nonlinear σ-model in a curved space.

We note an interesting analogy with gravity theory. Equation (2.3) under condition (2.4) has the same form as the gravity equations with two commuting Killing vectors for the part of the metric g containing the off-diagonal terms [8-11]. The difference is that we have $M = M^*$ (where $*$ denotes the Hermitian conjugation) and $\det M = -\alpha$ here and $g = g^T$ and $\det g = -\alpha^2$ in the gravity case.

We set

$$S = M/\sqrt{\alpha}, \quad S = \sum_{i=1}^{3} S_i \sigma_i.$$

Therefore, $\mathbf{S}^2 = 1$, i.e. $\mathbf{S}$ is the normalized magnetization vector. In accordance with (2.3) and (2.4), we now have

$$(\alpha S_z S)_{\bar{z}} + (\alpha S_{\bar{z}} S)_z = 0, \tag{2.5}$$

which is the equation of a nonlinear $O(3)$ σ-model in a curved space. If $\alpha = 1$, Eq. (2.5) becomes the equation of the standard $O(3)$ σ-model (the two-dimensional isotropic Heisenberg ferromagnet), for which the boundary-value problem was solved in [3]-[4]. We also note that the above-mentioned case of gravity reduces to a similar $O(2,1)$ σ-model in a curved space and that dimensional reduction of the axially symmetrical self-duality equations in the Yang-Mills theory also leads to a similar model. The analogue of S is then given by $g^* g$, where $g \in SL(N,R)$ [12].

We first consider some simple automodel solutions of (2.5). We take the ansatz

$$S = \begin{pmatrix} \cos\chi & \sin\chi \exp(-i\Phi) \\ \sin\chi \exp(i\Phi) & -\cos\chi \end{pmatrix}, \tag{2.6}$$

where $\chi = \chi(\alpha,\beta), \Phi = \Phi(\alpha,\beta)$ is a real-value function, and β is the harmonic function that is conjugate to α in the sense of the Cauchy-Riemann conditions. Substituting (2.6) in (2.5) produces the system of equations

$$\begin{gathered} (\Phi_{\alpha\alpha} + \alpha^{-1}\Phi_\alpha + \Phi_{\beta\beta})\sin\chi + \ 2(\Phi_\alpha \chi_\alpha + \Phi_\beta \chi_\beta)\cos\chi = 0, \\ 2(\chi_{\alpha\alpha} + \alpha^{-1}\chi_\alpha + \chi_{\beta\beta}) = \ (\Phi_\alpha^2 + \Phi_\beta^2)\sin 2\chi \end{gathered} \tag{2.7}$$

For $\chi = 0$, we have $S = \sigma_3$. For $\chi = \pi/2$, the function Φ satisfies the Laplace equation; it then follows that

$$\Phi(\alpha,\beta) = \int dR(s) Z_0(is\alpha) e^{is\beta},$$

where $Z_0(x)$ is the Bessel function (J_0 or N_0) and $dR(s)$ is a spectral measure. Finally, for $\Phi = k\beta$ with a free parameter k, the function $\chi = \chi(\alpha)$ satisfies the equation

$$\chi_{\alpha\alpha} + \alpha^{-1}\chi_\alpha = \frac{k^2}{2}\sin 2\chi,$$

which reduces to the third Painlevé equation.

System (2.7) is analogous to the equations that arise in gravity theory [13]; however, one of those equations resembles the Liouville equation rather than the sine-Gordon equation as is the case here.

Equation (2.5) is the compatibility condition for the linear 2×2 matrix system

$$(\partial_z - \frac{\alpha S_z S}{\varrho + \alpha})\Psi = 0, \quad (\partial_{\bar{z}} + \frac{\alpha S_{\bar{z}} S}{\varrho - \alpha})\Psi = 0, \tag{2.8}$$

where

$$\varrho = \varrho(\alpha, \beta, u) = i\beta - u + \sqrt{(u-\gamma)\ (u+\bar{\gamma})},\ \gamma(z) = \alpha + i\beta, \tag{2.9}$$

γ is the function that is analytic within the sample, and $u \in C$ is a "hidden" spectral parameter that does not depend on the coordinate. Thus, Equation (2.5) can be solved by a general version of ISTM. This version, however, has not yet been fully described in view of the technical complications (some progress was recently made in [14] for the gravity theory equations). Therefore, we use the dressing method for deriving exact solutions. Because the strategy in this case is essentially the same as in [15], we give only parts of the calculations that allow us to obtain the final answer.

To system (2.8), we add the reduction conditions and the formula for reconstructing the potential (magnetization):

$$\Psi^{-1}(u) = \Psi^*(-\bar{u}),\ \ S\Psi(u) = \Psi(\tau(u))J,\ \ S = \Psi(\infty^+)C,$$

$$JJ^* = J^2 = I,\ \ CC^* = I,\ \ J = C\Psi(\infty^-), \tag{2.10}$$

where τ is the transposition of the sheets of the Riemann surface Γ of the square root entering (2.9) and the signs + and - pertain to the upper and the lower sheet, with the upper sheet being fixed by the condition that $\varrho \to 0$ as $u \to \infty^+$. Further, the 2×2 matrices C and J are introduced to select different versions of the background solutions (the classical vacuum). In addition, the involution conditions

$$\varrho(u) = -\bar{\varrho}(-\bar{u}),\ \ \varrho(\tau(u) = \alpha^2/\varrho(u) \tag{2.11}$$

hold. Relations (2.9)-(2.11) allow us to write the answer. We seek the solution of (2.8) in the form

$$\Psi(u) = \chi(u)\Psi^0(u), \tag{2.12}$$

where $\Psi^0(u)$ is the bare solution of (2.8) corresponding to the vacuum S^0 and the matrix $\chi(u)$ is to be determined. We further assume that the matrices $\chi(u)$ and $\chi^{-1}(u)$ are meromorphic functions on the Riemann surface Γ and have a finite number of poles (those of the χ matrix occurring at the points $u = v_i$),

$$\chi(u) = I + \sum_i \frac{|m_i> \otimes <q_i|}{\varrho(u) - \mu_i},\ \ \chi^{-1}(u) = I + \sum_i \frac{|p_i> \otimes <l_i|}{\varrho(u) - \nu_i}, \tag{2.13}$$

where $\mu_i = \varrho(v_i)$, ν_i are some complex numbers specified below, and the standard Dirac notation for vectors is used.

Obviously, if χ has a pole at $u = v_i$, then it has a pole on the second sheet of Γ at $u = \tilde{v}_i = \tau(v_i)$. It follows from the condition $\chi(u)\chi^{-1}\ (u) = I$ and Eq. (2.13) that

$$|m_i> = \sum_j |p_j> (N^{-1})_{ji},\ \ <l_i| = -\sum_j (N^{-1})_{ij} <q_j|,\ \ N_{ij} = \frac{<q_i|p_j>}{\mu_i - \nu_j}.$$

Equations (2.8), (2.12)-(2.13) allow us to find

$$<q_i| = <d_i|(\Psi^0(v_i))^{-1},\ \ |p_i> = \Psi^0(-\bar{v}_i)|c_i>,$$

where $< d_i|$ and $|c_i >$ are arbitrary bra and ket vectors. The first of the reductions in (2.10) gives $< q_i| = |p_i >^*$ and $\nu_i = -\bar{\mu}_i$, and the second one implies that $< c_i|J =< \tilde{c}|$. Here, the vector $< c_i|$ corresponds to the pole contribution coming from $u = v_i$ and $< \tilde{c}_i|$ to the contribution of $u = \tilde{v}_i = \tau(v_i)$.

The relations obtained allow us to write the "N-soliton" solution of (2.5); for simplicity, we limit ourselves to the "one-soliton" solution

$$S = S_0 + \frac{< q|S^0|q >}{D(|\mu|^2 + \alpha^2)} \left[S^0|q > \otimes < q|S_0 + \frac{|\mu|^2}{\alpha^2}|q > \otimes < q| \right] -$$

$$(2.14)$$

$$-\frac{< q|q >}{\alpha^2 D(\mu + \bar{\mu})} \left[\bar{\mu}|q > \otimes < q|S^0 + \mu S^0|q > \otimes < q| \right].$$

Here, $\mu = \varrho(v) = \mu_R + i\mu_I$, $|q >= \Psi^0(-\bar{v})|c >$, $< q| = |q >^*$, $|c >\in C^2$ is an arbitrary constant vector, and

$$D = \frac{|\mu|^2}{\alpha^2} \left[\frac{< q|q >^2}{(\mu + \bar{\mu})^2} - \frac{\alpha^2 < q|S^0|q >^2}{(|\mu|^2 + \alpha^2)^2} \right].$$

Equation (2.14) allows one to construct the "$N + 1$-soliton" solution starting with an "N-soliton" back-ground.

We next analyze the explicit form of the solution obtained. The simplest choice for the background, $S^0 = \sigma_3$, corresponds to $C = I$, $J = \sigma_3$, $\Psi^0 = \sigma_3$. Setting $< c| = (1, \bar{c})$ (without losing generality), we have from (2.14) that

$$S_+ = S_1 + iS_2 = -\frac{c}{\alpha^2 D} \left[(1 - |c|^2)\frac{|\mu|^2 - \alpha^2}{|\mu|^2 + \alpha^2} + (1 + |c|^2)\frac{\mu - \bar{\mu}}{\mu + \bar{\mu}} \right], \; S_3 = 1 - \frac{2|c|^2}{\alpha^2 D}, \quad (2.15)$$

where

$$D = \frac{|\mu|^2}{\alpha^2} \left[\frac{(1 + |c|^2)^2}{(\mu + \bar{\mu})^2} - \frac{\alpha^2(1 - |c|^2)^2}{(|\mu|^2 + \alpha^2)^2} \right].$$

When $c = -i\exp(2i\Phi)$, Eqs. (2.15) are simplified considerably and become

$$S_+ = -\sin 2\theta e^{2i\Phi}, \quad S_3 = -\cos 2\theta, \quad (2.16)$$

where $\theta = \arg\mu$. This reproduces ansatz (2.6) with $\Phi = const$ and $\chi = \pi + 2\theta$. Moreover, it can be shown that χ satisfies the second equation in (2.7) with a vanishing right-hand side. For a purely imaginary v, it follows from (2.11) and (2.16) that the excitation over the $S_3^0 = 1$ vacuum is absent. This is true also in the general case described by Eq. (2.14). We also note that solutions (2.15) and (2.16) have no nontrivial analogues in the case of a constant saturation moment.

Another possible choice of the bare solution, $S^0 = \exp(-ik\beta\sigma_3)\,\sigma_1$, where k is a real constant, leads to the equations $C = J = \sigma_1$. In this case, system (2.8) can be explicitly integrated as

$$\Psi^0(\alpha, \beta, u) = e^{k\sigma_3(\varrho(\alpha,\beta,u) - 2i\beta)/2},$$

where ϱ is defined in (2.9). Again setting $< c| = (1, \bar{c})$, we can generalize the formulas describing magneticovortex excitations [16] to the inhomogeneous case under consideration:

$$S_+ = e^{ik\beta} + \frac{4|c|^2|\mu|}{\alpha D} e^{ik\beta} \left[\frac{\cos(\eta + i\sigma)\cos\eta}{|\mu|^2+\alpha^2} - \frac{\cosh\xi}{2\alpha\mu_R}\ \cosh(\xi - i\theta) \right],$$

$$(2.17)$$

$$S_3 = -\frac{2|c|^2}{\alpha^2 D} \left[\frac{|\mu|^2-\alpha^2}{|\mu|^2+\alpha^2}\ \sinh\xi\cos\eta - i\frac{\mu-\bar{\mu}}{\mu+\bar{\mu}}\cosh\xi\sin\eta \right],$$

where

$$D = \frac{4|\mu|^2|c|^2}{4\mu_R^2} \left[\frac{\cosh^2\xi}{4\mu_R^2} - \frac{\alpha^2\cos^2\eta}{(|\mu|^2+\alpha^2)^2} \right].$$

In these formulas, $\xi = k\mu_R + \ln|c|$, $\eta = k(\mu_I - \beta) - \arg c$, $\theta = \arg\mu$, $\sigma = \ln(|\mu|/\alpha)$. The formulas for S_+ and S_3 give the magnetization distribution characterized by local variation scale $L^{-1} \sim |k \bigtriangledown \mu_R|$ and the wave number $K \sim |k \bigtriangledown (\mu_I - \beta)|$, which correspond to the spatial modulation of the magnetic structure of the sample. For $KL \gg 1$, relations (2.7) describe an inhomogeneous spiral magnet that can produce multiple harmonics of the fundamental frequencies.

The above "solitons" allow us to estimate the energy functional evaluated on these solutions. To do so, we assume that the function α depends on the temperature in addition to the spatial variables: $\alpha = \alpha(\mathbf{r}, T)$. The exact general form of this dependence can be found by considering the system on a lattice [17,18], which is beyond the scope of this paper. We therefore merely estimate the energy and the heat capacity near the critical point.

According to the Landau theory [19], the functions α and β behave as follows near the Curie point T_c:

$$\alpha(r,T) \approx f_0(r)t, \quad \beta(r,T) \approx g_0(r)t, \tag{2.18}$$

where $t = (T_c - T)/T_c$. The second estimate in (2.18) follows from the first one and fact that the quantities α and β (as well as f_0 and g_0) are related by the Cauchy-Riemann conditions. Taking α and β to be small and $t \to 0^+$, we see from (2.9) that

$$\mu = -\frac{\alpha^2}{2v}(1 + \frac{i\beta}{v}) + O(|\gamma|^4) \qquad (2.19).$$

The system energy is defined as

$$E = \frac{1}{2}\int_\Omega dxdy(\bigtriangledown\mathbf{M})^2 = \frac{1}{2}\int_\Omega dxdy[\frac{(\bigtriangledown\alpha)^2}{4\alpha} + \alpha(\bigtriangledown\mathbf{S})^2], \tag{2.20}$$

where Ω is the domain occupied by the sample. This relation can be rewritten as

$$E = \int_\Omega d^2r[\alpha Tr(S_z S_{\bar{z}}) + \frac{\alpha_z\alpha_{\bar{z}}}{2\alpha}] = E_1 + E_0. \tag{2.21}$$

Using

$$d^2r = dxdy = -\frac{1}{2i}d\gamma\wedge d\bar{\gamma}\frac{1}{\gamma_z\ \bar{\gamma}_{\bar{z}}}, \quad d\alpha\wedge d\beta = -\frac{1}{2i}d\gamma\wedge d\bar{\gamma},$$

we obtain

$$E_0 = \frac{1}{8}\int_{\Omega'}\frac{d\alpha d\beta}{\alpha}, \quad E_1 = \int_{\Omega'}\frac{d\alpha d\beta}{\gamma_z\ \bar{\gamma}_{\bar{z}}}\ Tr(S_z S_{\bar{z}}), \tag{2.22}$$

where Ω' is the domain in the plane of the (α, β) variables obtained via the conformal mapping $\gamma(z)$ of Ω from the (x, y) plane. It follows from (2.22) that

$$E_0 = \frac{1}{8}(\beta_2 - \beta_1)\ln\frac{\alpha_2}{\alpha_1} \sim \ (T_c - T)\ln(T_c - T), \tag{2.23}$$

from which, in particular, we can find the contribution to the heat capacity as

$$c_0 = \frac{\partial E}{\partial T} \sim \ln[e(T_c - T)]. \tag{2.24}$$

Estimating E_1 and $c_1 = \partial E_1/\partial T$ requires knowing the explicit form of the solution for S. We first consider the simplest solution (2.16). We then find

$$E_1 = 8\int_{\Omega'} \frac{d\alpha d\beta}{\gamma_z\ \bar{\gamma}_{\bar{z}}}\ \alpha\theta_z\theta_{\bar{z}} = \ 8\int_{\Omega'} d\alpha d\beta\alpha\theta_\gamma\theta_{\bar{\gamma}}, \tag{2.25}$$

with θ_γ and $\theta_{\bar{\gamma}}$ satisfying the relations

$$\theta_\gamma = \frac{1}{2i}(\ln(\frac{\varrho}{\bar{\varrho}}))_\gamma, \quad \theta_{\bar{\gamma}} = \frac{1}{2i}(\ln(\frac{\varrho}{\bar{\varrho}}))_{\ \bar{\gamma}} \tag{2.26}$$

Using the deformation method developed in [20], we can show that

$$\left(\frac{4\alpha}{\mu - \alpha}\right)_\gamma = \frac{1}{\mu - \alpha} + \frac{1}{\mu + \alpha}, \quad \left(\frac{4\alpha}{\mu + \alpha}\right)_{\bar{\gamma}} = \frac{1}{\mu - \alpha} + \frac{1}{\mu + \alpha}, \tag{2.27}$$

from which, setting $\mu = \varrho$, we obtain

$$\theta_\gamma = \frac{1}{2i}\frac{\varrho + \bar{\varrho}}{(\varrho + \alpha)\ (\bar{\varrho} - \alpha)}, \quad \theta_{\bar{\gamma}} = -\ \frac{1}{2i}\frac{\varrho + \bar{\varrho}}{(\varrho - \alpha)\ (\bar{\varrho} + \alpha)}.$$

Thus, in accordance with (2.25), we have

$$E_1 = 2\int_{\Omega'} d\alpha d\beta\ \frac{\alpha(\varrho + \bar{\varrho})^2}{|\varrho^2 - \alpha^2|^2}. \tag{2.28}$$

Using (2.18) and (2.19) and taking the size of Ω' to be finite, we finally obtain the following relations satisfied near the Curie point:

$$E_0 \approx t\ln t, \quad c_0 \approx \ln t, \quad E_1 \approx t^3, \quad c_1 \approx t^2. \tag{2.29}$$

Thus, the background heat capacity satisfies the law characteristic of the two-dimensional Ising model [17, 18] with the critical index equal to zero, and the energy and the heat capacity of the excited state over the vacuum satisfy powerlike laws.

For convenience in calculating the energy of more complicated solutions, we rewrite Eq. (2.21) as

$$E = \int_{\Omega'} d\alpha d\beta\,[\frac{\alpha}{\gamma_z\ \bar{\gamma}_{\bar{z}}}\ (2S_{3z}S_{3\bar{z}} + |S_{+z}|^2 + |S_{+\bar{z}}|^2) + \frac{1}{8\alpha}]. \tag{2.30}$$

We consider solution (2.15), setting $|\mu| = \alpha\tan\Phi$ and $\Phi \in [0, \pi/2)$. Then,

$$S_+ = -\frac{c}{\alpha^2 D}[-B\cos 2\Phi + iA\tan\theta], \quad S_3 = 1 - \frac{2|c|^2}{\alpha^2 D}, \tag{2.31}$$

where

$$D = \frac{1}{4}\frac{A^2}{\alpha^2\cos^2\theta} - B^2\sin^2\Phi, \quad D = \bar{D},$$

$\theta = \arg\mu,\ A = 1 + |c|^2,\ B = 2 - A$. Using the deformation method, we obtain

$$\Phi_z = \frac{\cos^2\Phi}{\alpha}\bar{\gamma}_z\left[|\varrho|^2\ \frac{\bar{\varrho} - \varrho - 2\alpha}{(\varrho+\alpha)(\bar{\varrho}-\alpha)} - \ \frac{1}{2}\tan\Phi\right],$$

$$\Phi_{\bar{z}} = \frac{\cos^2\Phi}{\alpha}\bar{\gamma}_{\bar{z}}\left[|\varrho|^2\ \frac{\bar{\varrho} - \varrho - 2\alpha}{(\varrho-\alpha)(\bar{\varrho}+\alpha)} - \ \frac{1}{2}\tan\Phi\right]. \tag{2.32}$$

Taking (2.31) and (2.32) into account, we obtain

$$E_0,\ E_1 \approx t\ln t, \quad c_0,\ c_1 \approx \ln t, \tag{2.33}$$

which means that the background energy and the heat capacity of (2.15) give the same contributions as the corresponding quantities evaluated for the states with excitations.

We consider solution (2.17). Using the same notation as in the previous example, we write it as

$$S_+ = e^{ik\beta} + \frac{2|c|^2}{\alpha^2 D}e^{ik\beta}[\cos(\eta + \ i\sigma)\sin 2\Phi\cos\eta - \frac{\cosh\xi}{\cos\theta}\cosh(\xi - \ i\theta)],$$

$$S_3 = -\frac{2|c|^2}{\alpha^2 D}[-\cos 2\Phi\sinh\xi\cos\eta + \tan\theta\ \cosh\xi\sin\eta].$$

Evaluating the corresponding derivatives and estimating the integrals as before, we again obtain

$$E_0,\ E_1 \approx t\ln t; \quad c_0,\ c_1 \approx \ln t. \tag{2.34}$$

The approach described here contains, therefore, three main ingredients: the integrability of the model, the dressing method, and the deformation method. Because of its universality, the approach can be used with other interesting applications of (integrable) two-dimensional models.

To conclude this section, we show how the transition to the homogeneous case can be accomplished in the framework of the inhomogeneous magnet model. We consider system (2.8) and expression (2.9), which contains the "hidden" spectral parameter u. Setting $\lambda = \varrho/\alpha$ and letting α tend to unity, we obtain

$$\varrho = i\beta - u + \sqrt{(u-1)(u+1-i\beta)} = \lambda. \tag{2.35}$$

Because $\gamma(z,\bar{z}) = \alpha(z,\bar{z}) + i\beta(z,\bar{z})$ with $z = x + iy$ is an analytic function, the Cauchy-Riemann conditions imply that $\beta \to \beta_0$ as $\alpha \to 1$, where β_0 is a constant which we set equal to zero. Then we see from (2.35) that $u = -1/2(\ \lambda + 1/\lambda)$, and in the language of spectral parameters, therefore, the correspondence between the homogeneous and the inhomogeneous case is given by the "mirrored" Zhukovskii function.

3. The Pohlmeyer transformation and gauge equivalence of two-dimensional integrable boundary-value problems

We consider the nonlinear $O(3)\sigma$-model on the half-plane $R^2_+ = \{(x,y) : -\infty < x < +\infty, y \geq 0\}$:

$$\mathbf{S}_{z\bar{z}} + (\mathbf{S}_z\mathbf{S}_{\bar{z}})\mathbf{S} = 0, \tag{3.1}$$

where $\mathbf{S}(x,y) = (S_1, S_2, S_3)$, $|\mathbf{S}|^2 = 1$, and $\mathbf{S}_z = (1/2)(\mathbf{S}_x - i\mathbf{S}_y)$. Model (3.1) coincides with the elliptic version of the **n**-field model [21] and is one of the versions of the chiral field model. We assume that $\mathbf{S}(x,0)$ and $\mathbf{S}_y(x,0)$ are given smooth functions such that (unless otherwise stated)

$$\mathbf{S}(x,0) \to (\cos 2x, \sin 2x, 0), \quad |x| \to \infty, \tag{3.2}$$

i.e., $\mathbf{S}$ belongs to the class of functions describing spiral structures [22]. Following [23], we also assume that the relations

$$\mathbf{S}_z^2 = \mathbf{S}_{\bar{z}}^2 = 1, \tag{3.3}$$

hold on R^2_+. These relations then imply the constraints

$$\mathbf{S}_x^2 - \mathbf{S}_y^2 = 4, \quad \mathbf{S}_x\mathbf{S}_y = 0, \tag{3.4}$$

which are obviously compatible with (3.1).

Setting $f(z,\bar{z}) = \mathbf{S_z}\mathbf{S_{\bar{z}}}$, we perform the Pohlmeyer transformation assuming that

$$\mathbf{S}_{zz} = c_1\mathbf{S} + c_2\mathbf{S}_z + c_3\mathbf{S}_{\bar{z}},$$

where $c_i = c_i(z,\bar{z}), i = 1,3$, are some complex-valued functions. It can be easily verified that $c_1 = -1,\ c_2 = -ff_z/(1-f^2)$, and $c_3 = f_z/(1-f^2)$. This implies the equation

$$f_{z\bar{z}} = 1 - f^2 - \frac{ff_zf_{\bar{z}}}{1-f^2}. \tag{3.5}$$

This equation is integrable, because it is the compatibility condition of the linear matrix system

$$\Psi_z = A\Psi, \quad \Psi_{\bar{z}} = B\Psi, \tag{3.6}$$

where $\Psi = \Psi(z,\bar{z}),\ A = A(z,\bar{z}),\ B = B(z,\bar{z}) \in Mat(2,C)$, and the matrices A and B have the forms

$$A = \frac{1}{2}i\lambda\sigma_3 + \frac{\partial f}{2\sqrt{f^2-1}}\sigma_1, \quad B = \frac{1}{2\lambda}if\sigma_3 + \frac{1}{2\lambda}\sqrt{f^2-1}\sigma_2 \tag{3.7}$$

Here, $\lambda \in C$ is the spectral parameter, and $|f| > 1$.

In what follows, we use the notion of the "phase" space of a nonlinear elliptic equation, which can be introduced by a natural analogy with the hyperbolic case [21]. We do this in the example described by Eqs. (3.1) and (3.5).

The generating functional of the model given by (3.1) and (3.2) is

$$F^{(\mathbf{S})} = \int_{-\infty}^{+\infty} dx \int_0^{\infty} dy \ \left[\frac{1}{2}(\mathbf{S}_x^2 + \mathbf{S}_y^2) + \frac{\nu_1}{2}(\mathbf{S}^2 - 1) + \ \nu_2\mathbf{S}\mathbf{S}_y - 2\right], \tag{3.8}$$

where $\nu_j, j = 1,2$, are the Lagrange multipliers. The phase space $\mathcal{M}^{(\mathbf{S})}$ consists of the smooth functions $S_i(x,0)$ and $S_{iy}(x,0)$, $i = 1,3$, and the analogues the Hamilton equations are

$$\mathbf{S}_y = \{H^{(\mathbf{S})}, \mathbf{S}\}, \quad (\mathbf{S}_y)_y = \{H^{(\mathbf{S})}, \mathbf{S}_y\}, \tag{3.9}$$

with the "Hamiltonian" of the model given by

$$H^{(\mathbf{S})} = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}(\mathbf{S}_x^2 - \mathbf{S}_y^2) + 2\right]. \tag{3.10}$$

The Poisson structures of the phase space is defined by the nonvanishing fundamental brackets with the constraints involved in (3.8) duly taken into account [21]:

$$\{S_{ay}(x), S_b(x')\} = [\delta_{ab} - S_a(x)S_b(x)]\delta(x-x'),$$
$$\{S_{ay}(x), S_{by}(x')\} = -[S_{ay}(x)S_b(x) - S_{by}(x)S_a(x)]\delta(x-x').$$

For arbitrary smooth functionals F and G, we then have

$$\{F,G\} = \int dx \Bigg\{ \left[\frac{\delta F}{\delta \mathbf{S}_y}\frac{\delta G}{\delta \mathbf{S}} - \frac{\delta F}{\delta \mathbf{S}}\frac{\delta G}{\delta(\delta_y \mathbf{S})}\right] +$$
$$+\left(\frac{\delta G}{\delta \mathbf{S}_y}\mathbf{S}\right)\left[\left(\frac{\delta F}{\delta \mathbf{S}}\,\mathbf{S}\right) - \left(\frac{\delta F}{\delta \mathbf{S}_y}\,\mathbf{S}_y\right)\right] + \left(\frac{\delta F}{\delta \mathbf{S}_y}\mathbf{S}\right)\left[\left(\frac{\delta G}{\delta \mathbf{S}_y}\mathbf{S}_y\right) - \left(\frac{\delta G}{\delta \mathbf{S}}\mathbf{S}\right)\right]\Bigg\}.$$

We now consider Eq. (3.5) assuming that $f(x,0)$ and $f_y(x,0)$ are given smooth real-valued functions such that $f(x,0) \to 1$ and $f_x(x,0),\ f_y(x,0) \to 0$, as $|x| \to \infty$, with the second term in (3.5) vanishing at infinity. The generating functional of this equation is

$$F^{(f)} = \int_{-\infty}^{+\infty} dx \int_0^{+\infty} \left[\frac{1}{2}\,\frac{f_x^2 + f_y^2}{f^2-1} + 4(1-f)\right].$$

The "phase" space $\mathcal{M}^{(f)}$ is spanned by the functions $q = \ln|f + \sqrt{f^2-1}|$ and $p = f_y/\sqrt{|f^2-1|}$, $|f| > 1$, and the equations of "motion" are

$$q_y = \{H^{(f)}, q\}, \quad q_{yy} = \{H^{(f)}, q_y\},$$

where the "Hamiltonian" is given by

$$H^{(f)} = \int_{\infty}^{\infty} dx \left[\frac{1}{2}p^2 - \frac{1}{2}\frac{(f_x^2)}{f^2-1} - 4(1-f)\right],$$

and the Poisson structure of the "phase" space is defined by the fundamental brackets $\{p(x), q(x')\} = \delta(x-x')$.[2]

Equation (3.5) is useful because it generates a broad family of integrable elliptic equations. In what follows, we assume (unless otherwise stated) that $u = u(z,\bar{z})$ is a smooth real-valued function defined on R_+^2 and, in addition, that

$$u(x,y) \to 0, \quad |x| \to 0. \tag{3.11}$$

[2]We stress that it is not our aim here to construct the Hamiltonian formalism for elliptic equations. In this case, unfortunately, we do not have "intuitively obvious" variables of the angle-action type, and all the analogies with the hyperbolic variables become rather conventional.

Setting $f(z,\bar{z}) = \cosh u$, we obtain the equation

$$\triangle u = -4 \sinh u \tag{3.12}$$

from (3.5). Equation (3.12) emerges in the theory of two-dimensional Boltzmann-Poisson systems at negative temperatures [24] and also in some reductions of the Chern-Simons model [25]. It has a geometric meaning, describing the embedding of a negative-curvature surface in three-dimensional Euclidean space [26].

Using the GE of the problems given by (3.1), (3.2) and by (3.11), (3.12), which was proved in [4],[5], and also using Eq.(3.4) and the definition of f, we obtain

$$\mathbf{S}_x^2 = 4\cosh^2\frac{u}{2}, \quad \mathbf{S}_y^2 = 4\sinh^2\frac{u}{2}. \tag{3.13}$$

These equations, which relate the "potentials" of the two boundary-value problems, imply, in particular, that the GE does not impose any constraints on the corresponding "phase" spaces (provided, of course, that $f > 1$).

We list some other possible choices of the functions $f(z,\bar{z})$ and the corresponding equations that lead to interesting, important physical applications,

$$f(z,\bar{z}) = -\cosh u, \;\; \partial\bar{\partial}u = \sinh u, \tag{3.14}$$

$$f(z,\bar{z}) = e^{\pm u}, \;\; \partial\bar{\partial}u = -2\sinh u \pm \frac{\partial u \bar{\partial} u}{e^{\pm 2u} - 1}, \tag{3.15}$$

$$f(z,\bar{z}) = -e^{\pm u}, \;\; \partial\bar{\partial}u = 2\sinh u \pm \frac{\partial u \bar{\partial} u}{e^{\pm 2u} - 1}, \tag{3.16}$$

$$f(z,\bar{z}) = \mp\cos u, \;\; \partial\bar{\partial}u = \pm\sin u, \tag{3.17}$$

$$f(z,\bar{z}) = \pm\sin u, \;\; \partial\bar{\partial}u = \pm\cos u. \tag{3.18}$$

We consider the question of the GE of the problems given by (3.1) and (3.11), (3.17). For this purpose, we note that Eq. (3.1) can be rewritten as

$$c_{ikl} S_k \partial\bar{\partial} S_l = 0,$$

where $c_{ikl} = \epsilon_{ikl}$, with ϵ_{ikl} being the totally antisymmetric tensor in R^3. Therefore, Eq. (3.12), as well as Eqs. (3.14)-(3.16) corresponds to the model of a compact-manifold magnet. This situation changes for Eqs. (3.16)-(3.18), and the sought for GE is achieved on a noncompact manifold of matrices $S \in SU(1,1)$. We set

$$S = \begin{pmatrix} iS^3 & S^1 - iS^2 \\ S^1 + iS^2 & -iS^3 \end{pmatrix}. \tag{3.19}$$

The condition $\det S = -1$ and Eq. (3.19) imply that

$$(S^1)^2 + (S^2)^2 - (S^3)^2 = 1, \quad S^2 = I.$$

Geometrically, this means that the end of the vector $\mathbf{S} = (S^1, S^2, S^3)$ moves over the surface of the one-sheet hyperboloid (in contrast to the unit sphere in the "compact" case).

The matrix S can be expanded with respect to the basis elements of the Lie algebra $su(1,1)$ as $S = S^i\pi_i$, where $\pi_\alpha\pi_\beta = \eta_{\alpha\beta} + \ ic^{\gamma}_{\alpha\beta}\pi_\gamma$, $\alpha, \beta, \gamma = 1, 2, 3$, $\eta = \{\eta_{\alpha\beta}\} = diag(1, 1, -1)$ is the Killing metric, $c^{\gamma}_{\alpha\ \beta}$ are the structure constants of $su(1,1)$, and the commutation relations are $[\pi_\alpha, \pi_\beta] = 2ic^{\gamma}_{\alpha\beta}\ \pi_\gamma$.

If Eq. (3.17) is now taken with the + sign, it follows from (3.4) and from the expression for f that

$$(S^1_x)^2 + (S^2_x)^2 - (S^3_x)^2 = 4\sin^2\frac{u}{2},$$
$$(S^1_y)^2 + (S^2_y)^2 - (S^3_y)^2 = -4\cos^2\frac{u}{2},$$

and the requirement for conservation of the asymptotic behavior implies that the GE to (3.1), (3.2) is achieved under the condition that $u(x,0) \to \pi(mod 2\pi)$. Then, the constraints on the space $\mathcal{M}^{(\mathbf{S})}$ of the corresponding $O(1,2)$ σ-model take the form of the system of differential inequalities

$$0 \le (S^1_x)^2 + (S^2_x)^2 - (S^3_x)^2 \le 4,$$
$$-4 \le (S^1_y)^2 + (S^2_y)^2 - (S^3_y)^2 \le 0.$$

The GE conditions for the other equations in the above list are derived in essentially the same way.

Equations (3.15) and (3.16) seem new. It would be interesting to find applications of these equations, which could be considered the Boltzmann-Poisson equations (for positive temperatures in the case of (3.16) and negative ones in the case of (3.15)) generalized to the case of a quantum-classical system, and also to quantize them. We do not rule out other possible applications of these equations in physics and geometry (a certain regular method to obtain $(2+0)$-dimensional equations in the differential geometry of surface is given in [27])[3].

The elliptic version of the relativistic field theory model proposed in [28] takes the following form in the one-component case:

$$\partial\bar{\partial}w = -w(1-w^2) - \frac{w\partial w\bar{\partial}w}{1-w^2}, \quad w = \bar{w} \tag{3.20}$$

Equation (3.20) can be naturally called the elliptic Getmanov model (a similar equation for a complex-valued function was considered in [29]). It is known that in the hyperbolic version, this model is gauge equivalent to the sine-Gordon equation in Minkowski space [28]. The solution of (3.20) can be easily related to the solution of Eq. (3.5) as $f = 1-2w^2$. This allows us, in particular, to obtain from $F^{(f)}$ the density of the generating functional for (3.20) as

$$\hat{F}^{(G)} = 8(\partial w\ \bar{\partial}w/(w^2-1) + w^2).$$

Properly speaking, the instances of GE established above between a number of models on the half-plane and the model described by (3.1) and (3.2) require a more rigorous (mathematical) proof, which should include the demonstration of transitions of these models to the σ-model. Clearly, this demonstration can be done in each particular case

[3]Exact solutions of Eq. (3.16) were constructed in [7] using the Darboux transformation method; the relation of these equations to the Bitsadze equation was also considered there.

using the appropriate associated linear problem and constructing the gauge transformation matrix.

4. Exact solutions of the sinh-Gordon equation

We consider a plasma consisting of electrons and singly charged ions. We assume that the electron and ion temperatures are the same and each plasma component has relaxed to the Boltzmann distribution. In this case, the dimensionless potential of the electric field $u = e\Phi/T$ satisfies the Boltzmann-Poisson equation [30]

$$\triangle u = 4 \sinh u, \tag{4.1}$$

where $\triangle$ is the two-dimensional Laplace operator written in dimensionless variables, $\mathbf{r} = \mathbf{R}/2r_D$, e is the electron charge, T is the temperature, Φ is the potential, $\mathbf{R}$ is the radius vector, $r_D = T/(8\pi e n_0)^{1/2}$ is the Debye radius, and $n_0 = n_{0e} = n_{0i}$ are the electron and ion concentrations.

Equation (4.1) also emerges in other physical problems: in the theory of strong electrolytes and conducting films [31], in the $O(2,1)$ σ-models [32], in the calculation of the electrostatic contribution to the DNA molecule free energy [33], and so on. The same equation is obtained by the dimensional reduction (from four to two dimensions) of the self-duality equations for the $SU(2)$-valued Yang-Mills fields under a special choice of the ansatz [34]. It was assumed in [34] that a spontaneous gauge-symmetry breaking occurs: part of the potential components tend to constant matrices as $r \to \infty$ with half of the them playing the role of Higgs bosons thereby leading to the massive field theory corresponding to (4.1).

We also note that there are sufficiently rigorous procedures to derive (4.1) from the Bogoliubov-Born-Green-Kirkwood-Ivon chain of equations [35].

We assume that (4.1) is defined on the half-plane R^2_+, and that

$$u(x,y) \to 0, \quad r = \sqrt{x^2+y^2} \to \infty, \tag{4.2}$$

where $u = u(x,y)$ is a sufficiently smooth real-valued function.

It is useful to consider first the linear version of (4.1) and (4.2) that corresponds to the Debye-Hückel approximation. In this case, the solution obtained in the Fourier-transform parametrization is

$$\begin{gathered} u(x,y) = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda} b_B(\lambda)\; e^{ik(\lambda)x - l(\lambda)y}\;, \\ b_B(\lambda) = \frac{1}{4}\int dx e^{-ik(\lambda)x}\left[l(\lambda)u(x,0) - u_y(x,0)\right], \end{gathered} \tag{4.3}$$

where $k(\lambda) = \lambda - \lambda^{-1}$ and $l(\lambda) = \lambda + \lambda^{-1}$. It is assumed that the surface charge density at the boundary $u_y(x,0)$ decreases faster than $exp(-2|x|)$ as $|x| \to \infty$. Then $b_B(\lambda)$ can be analytically continued from the real axis to the domain in the plane of the complex variable $\lambda = \lambda_R + i\lambda_I$ that is bounded by the strophoid curve $\lambda_R^2(2-\lambda_I) - \lambda_I(\lambda_I-1)^2 = 0$ and that includes the unit semicircle in the right half-plane. In this domain, $b_B(\lambda) = b_B(1/\bar{\lambda})$ because u is real.

Solution (4.3) occurs under the boundary conditions

$$b_B(\lambda) = 0, \quad \lambda = \bar{\lambda} < 0, \tag{4.4}$$

$$u(x,0) + \frac{1}{\pi}\int dx' K_0(2|x-x'|)u_y(x',0) = 0, \tag{4.5}$$

which follow from the requirement that u be bounded as$r \to \infty$. In (4.5), $K_0(x)$ is the Macdonald function. Equation (4.5) is the Fourier transform of (4.4). The asymptotic form of (4.3) as $r \to \infty$ found using the saddle-point method is

$$u(\mathbf{r},\alpha) \simeq (\pi r)^{-\frac{1}{2}} b_B[iexp(-i\alpha)]exp(-2r), \tag{4.6}$$

where $x = r\cos\alpha$ and $y = r\sin\alpha$. Expression (4.6) describes the effect of the linear Debye screening.

We return to the original problem formulated in (4.1) and (4.2) and assume that the given functions $u(x,0)$ and $u_y(x,0)$ cannot be arbitrary in a given class of functions (for example, in the Schwartz class) and are constrained by some condition (analogous to (4.5)) to be formulated in what follows.

The auxiliary linear system of equations corresponding to (4.1) is

$$\Psi_x = \left[(i\frac{\lambda}{2} + \frac{\cosh u}{2i\lambda})\sigma_3 - \frac{u_z}{2}\sigma_2 - \frac{\sinh u}{2\lambda}\sigma_1 \right]\Psi, \tag{4.7}$$

$$\Psi_y = \left[-(\frac{\lambda}{2} + \frac{\cosh u}{2\lambda})\sigma_3 - i\frac{u_z}{2}\sigma_2 - \frac{\sinh u}{2i\lambda}\sigma_1 \right]\Psi. \tag{4.8}$$

We introduce the matrix of Jost solutions (4.7), which are determined by the asymptotic formulas

$$\Psi^{\pm} = (\Psi_1^{\pm}, \Psi_2^{\mp}) = e^{ikx\sigma_3/2}(1 + o(1)), \ x \to \pm\infty. \tag{4.9}$$

We set

$$\Psi^+ = \Psi^- T(\lambda), \ \ T(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}, \ \lambda = \bar{\lambda}, \tag{4.10}$$

where $T(\lambda)$ is the transition matrix. It has the properties following from (4.7) and (4.10)

$$T(-\lambda) = \sigma_2 T(\lambda)\sigma_2, \quad \bar{T}(\lambda) = T(\frac{1}{\bar{\lambda}}), \tag{4.11}$$

and also the unimodularity property $\det T(\lambda) = 1$. From (4.11), we have

$$\bar{a}(\lambda) = a(\frac{1}{\bar{\lambda}}), \quad \bar{b}(\lambda) = b(\frac{1}{\bar{\lambda}}), \tag{4.12}$$

where the second involution takes place, strictly speaking, for $\lambda = \bar{\lambda}$. It, however, admits an analytic continuation into some domain in the complex plane (we do not need more detailed information about that domain in what follows). We set $\phi^{\pm} = \Psi^{\pm}e^{-ikx\sigma_3/2} = (\phi_1^{\pm}, \phi_2^{\mp})$. Then the Volterra integral equations for the direct problem, which are equivalent to (4.7), become

$$\phi_j^{\pm}(x,\lambda) = e_j + \int dx' g_j^{\pm}(x-x',\lambda)\ Q(x',\lambda)\phi_j^{\pm}(x',\lambda), \tag{4.13}$$

where $j = 1, 2,\ e_1 = (1, 0)^T,\ e_2 = (0, 1)^T$, the Q matrix is

$$Q(x,\lambda) = \begin{pmatrix} \frac{\cosh u - 1}{2i\lambda} & -\frac{\sinh u}{2\lambda} + \frac{iu_z}{2} \\ -\frac{\sinh u}{2\lambda} - \frac{iu_z}{2} & -\frac{\cosh u - 1}{2i\lambda} \end{pmatrix},$$

and $g_j^{\pm}$ are the bare Green's functions

$$g_1(x,\lambda) = \mp\theta(\mp x)\ diag(1, e^{-ikx}),$$

$$g_2(x,\lambda) = \pm\theta(\pm x)\ diag(e^{ikx}, 1) \tag{4.14}$$

It follows from (4.13) that

$$\phi_1^{+},\ \phi_2^{-} \in H(\Omega^{+}),\ \ \phi_1^{-},\ \phi_2^{+}\ \in H(\Omega^{-}), \tag{4.15}$$

where $H(\Omega)$ is the class of functions that are analytic in the domain Ω, and $\Omega^{+} = \{\lambda : Im\lambda > 0\}$.

Taking (4.10) and (4.13) into account, we have the integral representations for the scattering data $a(\lambda)\ \ b(\lambda)$,

$$a(\lambda) = 1 - \int dx < e_1^T,\ Q(x,\lambda)\phi_1^{+}(x,\lambda) >, \tag{4.16}$$

$$b(\lambda) = \int dx e^{-ik(\lambda)x} < e_1^T,\ Q(x,\lambda)\phi_2^{+}(x,\ \lambda) >, \tag{4.17}$$

where $<,>$ denotes the scalar product of vectors in C^2. In the standard way, Eq. (4.8) implies the relations that describe the spectral data evolution,

$$a(y,\lambda) = a(y,0), \quad b(y,\lambda) = b(0,\lambda)e^{-l(\lambda)y}, \tag{4.18}$$

from which it can be easily seen that the condition that $b(y,\lambda)$ be finite for any $y > 0$ implies

$$b(\lambda) = 0,\ \ \lambda = \bar{\lambda} < 0, \tag{4.19}$$

and a gap thus appears in the spectrum of the associated linear problem (see also [3, 36]. Turning to (4.17), we see that a condition similar to (4.5) emerges in terms of the functions $u(x,0)$ and $u_y(x,0)$.

We consider in more detail the properties of the coefficient $a(\lambda)$. It follows from (4.16) that $a(\lambda) = 1 + O(1/\lambda)$ as $|\lambda| \to \infty$ and $a(\lambda) \in\ H(\Omega^{+})$ and the unimodularity of $T(\lambda)$ and Eq. (4.19) imply that $a(\lambda)a(-\lambda) = 1$. Therefore, using (4.16), we conclude that $a(0) = 1$. Assuming now that there exist simple zeros of the coefficient $a(\lambda), a(\lambda_n) = 0,\ Im\lambda_n > 0,\ n = 1, ...N$, and recalling the condition $u = \bar{u}$ together with Eq. (4.12), we obtain one more representation for $a(\lambda)$,

$$a(\lambda) = \prod_{n=1}^{2N_1} \frac{\lambda - \lambda_n}{\lambda +\ \lambda_n} \prod_{m=2N_1+1}^{2N_1+N_2} \frac{(\lambda - \lambda_m)(\lambda -\ \frac{1}{\lambda_m})}{(\lambda + \lambda_m)(\lambda +\ \frac{1}{\lambda_m})}, \tag{4.20}$$

where $2N_1+2N_2=N$ and the zeros $\lambda_n,\ 1\leq n\leq 2N_1$, belong to the unit circle (the analogues of kinks for the hyperbolic version of the sin-Gordon equation [21]). The zeros λ_m and $1/\bar\lambda_m,\ 2N_1+1\leq m\leq 2N_1+N_2$ constitute an inversion with respect to the unit circle (the analogues of breathers [21]). It is obvious that there is no topological charge in the model, in contrast to the equation $\triangle u=\sin u$ (see [37, 38]).

It follows from the first relation in (4.18) that $\ln a(\lambda)$, in particular, can be considered a generating functional of the integrals of "motion". The standard procedure brings the system of equations for the elements of the column ϕ_1^+ to the Riccati equation

$$F_x+Q_{12}F^2=-ik(\lambda)F+(Q_{22}-Q_{11})F+Q_{21}, \tag{4.21}$$

where $F=\phi_{21}^+/\phi_{11}^+$. Further, we obtain

$$\ln a(\lambda)=-\int_{\infty}^{\infty}dx[Q_{11}+Q_{12}F]. \tag{4.22}$$

Setting

$$F=\sum_{k=0}^{\infty}F_{-k}\lambda^k,\ |\lambda|\to 0,\quad F=\sum_{k=1}^{\infty}F_k\lambda^{-k},\ |\lambda|\to\infty,$$

$$R=Q_{11}+Q_{12}F=\sum_{k=0}^{\infty}R_{-k}\lambda^k,\quad |\lambda|\to 0,\quad R=\sum_{k=1}^{\infty}R_k\lambda^{-k},\ |\lambda|\to\infty,$$

we see from (4.21) that

$$R_{-1}=i[(\cosh u-1)/2+u_z^2/2]+iu_x^2/2-\ i\partial_x^2\ln(\cosh u/2)-(\cosh u-1)/2\partial_x(u_z/\sinh u),$$

$$R_0=\partial_x\ln\cosh u/2,\quad R_1=-i[(\cosh u-1)/2+u_z^2/2),\quad R_2=-(1/8)\partial_x(u_z^2),$$

and so on. Since the left-hand side of (4.22) does not depend on y, the rigth-hand side can be evaluated at $y=0$, which gives an infinite number of conservation laws.

Now let

$$\ln a(\lambda)=\sum_{s=1}^{\infty}I_s\lambda^{-s},\ |\lambda|\to\infty$$

and

$$\ln a(\lambda)=\sum_{s=0}^{\infty}\ I_{-s}\lambda^s,\ |\lambda|\to 0.$$

It follows from (4.12) that $I_s=\bar I_{-s},\ I_{2s}=0$ and $I_0=0$. Using (4.20)-(4.22), we arrive at the trace identity

$$\frac{2}{2s-1}[\sum_{m=1}^{2N1}\lambda_m^{2s-1}+\sum_{n=2N_1+1}^{2N_1+N_2}(\ \lambda_n^{2s-1}+\bar\lambda_n^{-2s+1})]=\int dxR_{2s-1}(x), \tag{4.23}$$

where $s=1,2,....$ The functions R_{2s} are either equal to zero or given by total derivatives in x, their integrals vanishing in view of (4.2).

We now solve the inverse problem. The standard argument employing the analytic properties of solutions $\phi^{\pm}_{1,2}$ leads to the Riemann problem of reconstructing a piecewise-analytic vector function from its jump on the boundary ($\lambda = \bar{\lambda}$):

$$\frac{\phi_2^+(x,y,\lambda)}{a(\lambda)} = \phi_2^-(x,y,\lambda) + \ r(y,\lambda)e^{ik(\lambda)x}\phi_1^+(x,y,\lambda), \tag{4.24}$$

where $r(y,\lambda) = b(y,\lambda)/a(\lambda) = b(0,\lambda)/a(\lambda)e^{-l(\lambda)y} = r(0,\lambda)e^{-l(\lambda)y}$ is the reflection coefficient. After some calculations, we finally obtain the system of singular integral inverse scattering equations

$$\phi_1^+ = e_1 + i\sum_{n=1}^{N}\frac{m_n(y)e^{ik(\lambda_n)x}}{\lambda +\ \lambda_n}\sigma_2\phi_{1n}^+(x,y) - \int_0^\infty \frac{d\mu}{2\pi}\,\frac{r(y,\mu)e^{ik(\mu)x}}{\mu+\lambda+i0}\sigma_2\phi_1^+(\mu), \tag{4.25}$$

$$\phi_{1m}^+ = e_1 + i\sum_{n=1}^{N}\frac{m_n(y)e^{ik(\lambda_n)x}}{\lambda_m +\ \lambda_n}\sigma_2\phi_{1n}^+(x,y) - \int_0^\infty \frac{d\mu}{2\pi}\,\frac{r(y,\mu)e^{ik(\mu)x}}{\mu+\lambda_m}\sigma_2\phi_1^+(\mu), \tag{4.26}$$

where $N = 2(N_1{+}N_2)$ and $m_n = b_n/a'(\lambda_n)$ are the discrete spectrum transition coefficients such that $m_n = -\lambda_n^2\bar{m}_n$ for $1 \le n \le 2N_1$ and $m_n = -\bar{\lambda}_n^2 m_{n+N_2}$ for $2N_1+1 \le n \le 2N_1+N_2$. In Eqs. (4.25) and (4.26), further, $\phi_1^+(x,y,\lambda)$, and $\phi_{1n}^+(x,y)$ are the respective eigenfunctions of the continuous and the discrete spectra.

The formulas for reconstructing the potential should be added to the system given by (4.25) and (4.26). Setting

$$\phi^+(x,y,\lambda) = \sum_{k=0}^{\infty}\phi_k^+(x,y)\lambda^k, \quad |\lambda| \to 0,$$

we insert this expansion into the matrix equation as ϕ^+ and compare it with the corresponding expansion (4.7). We thus obtain one of the formulas for reconstructing the potential,

$$i\sinh\frac{u}{2} = \sum_{n=1}^{N}\frac{\phi_{11n}^+e^{ik(\lambda_n)x}}{\lambda_n}m_n(y) - \int_0^\infty \frac{d\mu}{2\pi i}\,\frac{r(y,\mu)}{\mu}\ \phi_{11}^+(x,y,\mu)e^{ik(\mu)x}. \tag{4.27}$$

A similar expansion for $|\lambda| \to \infty$ yields

$$\frac{u_x - iu_y}{4} = \sum_{i=1}^{N}\phi_{11n}^+e^{ik(\lambda_n)x}m_n(y) - \ \int_0^\infty \frac{d\mu}{2\pi i}r(y,\mu)\phi_{11}^+(x,y,\mu)e^{ik(\mu)x}. \tag{4.28}$$

Relations (4.25)-(4.28) allow us to construct the simplest solutions of Eq. (4.1). We first consider the "no-soliton" case $a(\lambda) = 1$. Then system (4.25), in which $r \to \infty$, can be easily iterated, the first iteration giving $\phi_{11}^+(r,\lambda) = 1 + O(1/r)$. Using (4.27) and also the fact that $u(r,\alpha) \to 0$ as $r \to \infty$ for $\alpha \in [0,\pi]$, we obtain

$$u(r,\alpha) \cong \int_0^\infty \frac{d\mu}{\pi}\frac{b(0,\mu)}{\mu}\ e^{rf(\mu)}, \tag{4.29}$$

where $f(\mu) = ik(\mu)\cos\alpha - l(\mu)\sin\alpha$. Assuming that $b(\mu)$ can be analytically continued into the required domain in the complex plane, we obtain

$$u(r,\alpha) \cong \frac{b(0, ie^{-i\alpha})}{\sqrt{\pi\ r}} e^{-2r}[1 + O(\frac{e^{-2r}}{r})], \quad r \to \infty \tag{4.30}$$

Comparing this with (4.6), we see that the only difference in the two expressions is the replacement $b_B \to b$ and that (4.30) becomes (4.6) with sufficiently weak fields. This effect can be called a quasi-linear Debye screening: the scale of the screening is the same as in the linear case, and the influence of strong fields amounts to renormalization of the preexponential factor - the effective charge of the boundary - which contributes to the asymptotic behavior of the potential.

We now set $b(\lambda) = 0,\ 0 \le \lambda \le \infty$, which means that we consider the case of reflectionless potentials. The system of inverse scattering transform equations then reduces to a system of linear algebraic equations. Recalling (4.28), we obtain the "N-soliton" solution

$$u(x,y) = 2\ln\frac{\Delta^+}{\Delta^-}, \quad \Delta^{\pm} = \det(\delta_{mn} \pm \frac{im_n(y)}{\lambda_m + \lambda_n}\ \exp i\frac{x}{2}[k(\lambda_n) - k(\lambda_m)]). \tag{4.31}$$

where $1 \le m, n \le N$. One of the simplest solutions (the analogue of a breather) corresponds to setting $N_1 = 0,\ N_2 = 1,\ \lambda_1 = 1/\bar{\lambda}_2 = \exp(\gamma + i\theta)$, and $m_1(0) = \lambda_1^2 m_2(0) = \exp(\delta + i\rho)$. It follows from (4.31) that

$$u(x,y) = 2\ln\frac{1+h}{1-h}, \quad h = \frac{\sin[2r\sinh\gamma\cos(\theta+\alpha) + \rho - \theta)]\coth\gamma}{\cosh[2r\cosh\gamma\sin(\theta+\alpha) + \gamma - \delta - \ln\frac{\tanh\gamma}{2}]}. \tag{4.32}$$

In the general case where $(\theta + \alpha \ne 0, \pi)$, this solution describes a charge-density wave, which falls off as $r \to \infty$ according to the law

$$u \sim \exp(-2r\cosh\gamma|\sin(\theta+\alpha)|. \tag{4.33}$$

The parameters involved in (4.32) can be related to the values of the field at the boundary $y = 0$ by the trace identities (4.23), which now become

$$\cosh\gamma\cos\theta = -\frac{1}{32}\int dx u_x(x,0)u_y(x,0), \tag{4.34}$$

$$\cosh\gamma\sin\theta = -\frac{1}{32}\int dx[4(\cosh u(x,0) - 1) + \ \frac{1}{2}(u_x^2(x,0) - u_y^2(x,0))]. \tag{4.35}$$

As can be seen from (4.32), the field does not fall off for $\theta + \alpha = \pi$, which means that a long-range order occurs in that direction, i.e., there is a coherent structure describing the distribution of the electric field and plasma component densities with a considerable charge separation. We also note that for $h = \pm 1$, solution (4.32) becomes singular (we do not consider this case here).

Another simple solution (the analogue of a kink) that follows from (4.31) is obtained by taking $N_1 = 1,\ N_2 = 0,\ \lambda_n = e^{i\theta_n}, n = 1, 2,\ m_{1,2}(0) = i\ e^{\delta_{1,2} + i\theta_{1,2}}$:

$$h = \frac{\cosh[2r\sin(\frac{\theta_1-\theta_2}{2})\cos(\frac{\theta_1+\theta_2}{2} + \alpha) - \frac{\delta_1-\delta_2}{2}]\cot\frac{\theta_1+\theta_2}{2}}{\sinh[2r\cos(\frac{\theta_1-\theta_2}{2})\sin(\frac{\theta_1+\theta_2}{2} +\ \alpha) - \frac{\delta_1+\delta_2}{2} - \ln(\frac{1}{2}\tan\frac{\theta_1+\ \theta_2}{2})]}. \tag{4.36}$$

This case corresponds to the nonlinear Debye screening. As $r \to \infty$, we have

$$h \sim e^{-2r\sin(\theta_1+\alpha)+2\delta_1} + e^{-2r\sin(\theta_2+\alpha)+\ 2\delta_2},\ 0 < \frac{\theta_1+\theta_2}{2} + \alpha < \pi. \tag{4.37}$$

The term with the smallest coefficient in front of $2r$ survives in the exponent of (4.37). Because this coefficient is always less than one, the field falls off more slowly under the nonlinear Debye screening than in the quasi-linear case, and, moreover, the fall-off is anisotropic. The parameters characterizing the fall-off (4.37) can be related to the boundary values of the fields using trace formulas (4.34) and (4.35) if we replace $\cosh\gamma$ there by $\cos\frac{\theta_1-\theta_2}{2}$, and also θ by $\frac{\theta_1+\theta_2}{2}$.

The slower fall-off of the potential under the nonlinear Debye screening can be explained by the pressure of the electric field, which "pushes aside" the electrons and ions in the polarization cloud surrounding the boundary. The collective nonlinear phenomena observed here - the threadlike "radiographic examination" of the plasma with the electric field and its slower fall-off law - seem to take place only in the two-dimensional case. The field between the piontlike charges then decreases more slowly than in three dimensions, which explains the appearance of long-range correlations.

We now consider several thermodynamic relations and also those involving energy. The full energy of the medium (using dimensionless quantities) is

$$E = \int dxdy[(\nabla u)^2 + 4u\sinh u]. \tag{4.38}$$

In the linear case, this relation can be represent as

$$E = \int dxdy[(\nabla u)^2 + 4u^2] = E_f^{(x)} + E_f^{(y)} + E_p. \tag{4.39}$$

Using (4.3) and the relation

$$\int dxe^{i[k(\lambda)+k(\mu)]x} = (2\pi/(\lambda\ l(\lambda)))\delta(\mu - 1/\lambda),$$

we evaluate $E_f^{(x)}, E_f^{(y)}$ and E_p with the result

$$E_f^{(x)} = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}|\ b_B(\lambda)\ |^2(\frac{k(\lambda)}{l(\lambda)})^2, \quad E_f^{(y)} = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}|b_B(\lambda)|^2, \tag{4.40}$$

where

$$E = \frac{2}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}|b_B(\lambda)|^2. \tag{4.41}$$

Restricting ourselves to the "no-soliton" sector in the nonlinear case, we can exactly evaluate the quantities analogous to those found above only for the sum $E_f^{(x)} + E_f^{(y)}$ and only for the continuous spectrum (the integrals diverge in the other cases). From (4.28), we see that

$$\frac{u_x - iu_y}{4} = -\int_0^\infty \frac{d\mu}{2\pi i\mu} r(y,\mu)\ \chi_{11}^+(x,y,\mu),$$

where we introduce the function $\chi_1^\pm(x,y,\lambda) = \phi_1^\pm(x,y,\lambda)e^{ik(\lambda)x}$. It now follows that

$$\int dxdy(u_x^2 + u_y^2) = 4\int_0^\infty \frac{d\mu d\lambda}{\pi\mu\ \lambda} r(0,\mu)\bar{r}(0,\lambda)\int_0^\infty dye^{-[l(\lambda)+\ l(\mu)]x}\int dx\chi_{11}^+(x,\mu)\bar{\chi}_{11}^+(x,\lambda). \tag{4.42}$$

To evaluate the integral, therefore, we must know the orthogonality relations for the functions $\chi^+_{11}(x,\lambda)$. The method for obtaining these is to write the integral equations for $\chi^+_{11}(x,\lambda)$ and $\bar{\chi}^+_{11}(x,\lambda)$ and then multiply them using the formulas

$$\int dx\theta(\xi - x)e^{-i[k(\lambda)-k(\mu)](x-\xi)} = \frac{\pi\mu\delta(\ \lambda-\mu)}{l(\mu)} - iP\frac{1}{k(\lambda)-k(\mu)},$$

$$\int dx\theta(\xi_1 - x)\theta(\xi_2 - x)\ e^{-i[k(\lambda)-k(\mu)]x - ik(\lambda)\xi_1 + ik(\mu)\xi_2} = (\ \frac{\pi\mu\delta(\ \lambda-\mu)}{l(\mu)} - iP\frac{1}{k(\lambda)-k(\mu)})\Gamma_0,$$

where P denotes the principal value and $\Gamma_0 = e^{-ik(\mu)\ \xi_1 + ik(\mu)\ \xi_2}$, $\xi_1 > \xi_2$; $e^{ik(\lambda)\xi_2 - ik(\lambda)\xi_1}$, $\xi_2 > \xi_1$; 1, $\xi_1 = \xi_2$. After some calculations, we find

$$\int dx\chi^+_{11}(x,\mu)\bar{\chi}^+_{11}(x,\lambda) = \ \frac{\pi\mu\delta(\lambda-\ \mu)}{l(\mu)}(|a(\mu)|^2+1) - iP\frac{a(\lambda)\bar{a}(\mu)-1}{k(\lambda)-\ k(\mu)},$$

and finally

$$\int dxdy(u_x^2+u_y^2) = \frac{1}{2\pi}\int_0^\infty \frac{d\mu}{\mu}\ |b(\mu)|^2 \left(1+\frac{1}{|a(\mu)|^2}\right)\left(\frac{k^2(\mu)}{l^2(\mu)}+1\right). \tag{4.43}$$

In the linear limit as $u \to 0$ ($a(\mu) = 1$) and $b(\mu) \to b_B(\mu)$, this expression becomes equal to (4.41).

We consider the problem of the charge of the medium (as a whole). Its total value is

$$4\int_{R^2} dxdy \sinh u = \int_{R^2} dxdy\triangle u = -\int d\xi u_y(\xi,0). \tag{4.44}$$

Whenever the solution u is singular, a term given by the sum of integrals along the contour that cuts out the singularities should be added to the right-hand side of (4.44). Using the "breather" solution and evaluating this sum, we can show that this term vanishes and we have (4.44) again. This relation, therefore, shows that the electric neutrality condition should be understood such that the total charge of the medium and of the boundary vanishes.

The trace identities (4.34) and (4.35) also have a very clear physical interpretation. To show this, we introduce the Maxwell stress tensor of the electric field: $T_{\alpha\beta} = \hat{E}_\alpha\hat{E}_\gamma - (1/2)\delta_{\alpha\gamma}\hat{\mathbf{E}}^2$, where $\hat{\mathbf{E}} = (\hat{E}_x, \hat{E}_y)$ is the electric field vector. Because the field has a potential, we obtain

$$[curl\hat{\mathbf{E}}\ ,\ \hat{\mathbf{E}}]_\alpha = \partial_\gamma T_{\alpha\gamma} - 2T_0\ \partial_\alpha(\cosh\frac{e\Phi}{T} - 1), \tag{4.45}$$

where

$$T_{\xi\eta} = \Phi_\xi\Phi_\eta,\ \ T_{\eta\eta} = -\frac{1}{2}(\ \Phi_\xi^2 - \Phi_\eta^2), \tag{4.46}$$

$\xi = (\sqrt{T}/\alpha)x,\ \eta = (\sqrt{T}/\alpha)y$ and $\alpha = \sqrt{2\pi e^2 n_0}$. It follows from (4.45) that

$$\partial_\eta\left[T_{\eta\eta} - 2n_0T(\cosh\frac{e\Phi}{T} - 1)\right] = -\partial_\xi T_{\eta\xi}. \tag{4.47}$$

On the other hand, evaluating the force applied to the boundary from the medium, we obtain $\hat{F}_\xi = 0$, and

$$\hat{F}_\eta = 2en_0 \int_{R^2_+} d\xi d\eta \sinh \frac{e\Phi}{T} \Phi_\eta = -2Tn_0 \int d\xi (\cosh \frac{e\Phi}{T} - 1). \tag{4.48}$$

Comparing (4.35)-(4.48) with (4.34) and (4.35), we see that the "trace identities" involve the quantities that can be expressed through the Maxwell stress tensor and the force applied to the boundary.

We consider several properties of the parameter γ, entering (4.34) and (4.35), where we assume that $\gamma = \gamma(T, n_0)$. We note that the "breather" solution degenerates as $\gamma \to 0$, all the eigenvalues of the auxiliary linear problem becoming equal. This degeneration occur in two ways: () at $n_0 = n_c$, where T is arbitrary; (b) at $T = T_c$ where n_0 is arbitrary. Expanding the right-hand sides of these identities in the power series in $(n - n_c)$ or $(T - T_c)$, we find the behavior of γ near the critical parameters n_c and T_c.

In case a, it follows from (4.34) that $\gamma \sim \sqrt{2(n_c - n)/n_c}$, as $n \nearrow n_c$. This behavior of γ occurs under the additional condition, which follows from (4.35),

$$\int d\xi (\cosh \frac{e\Phi_c}{T} - 1) = 0, \quad \Phi_c = \Phi(\sqrt{\frac{2\pi e^2\ n_c}{T}}\xi, 0). \tag{4.49}$$

A similar analysis in case b results in $\gamma \sim \sqrt{2(T_c - T)/T}$ as $T \nearrow T_c$, with condition (4.49) replaced by

$$4 \int d\xi (\cosh \frac{e\Phi_c}{T_c} - 1) = \int d\xi \frac{e\Phi_c}{T_c}\ \sinh \frac{e\Phi_c}{T_c}, \quad \Phi_c = \Phi(\frac{\alpha\xi}{\sqrt{T_c}}, 0). \tag{4.50}$$

The degeneracy of the eigenvalues can probably be interpreted in this case as a certain "phase" transition in the nonlinear medium. This hypothesis, however, requires a further analysis.

Note also, that the equation

$$\triangle u = -4 \sinh u, \tag{4.51}$$

describing the plasmas at the negative temperatures, was solved in [39].

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