ANALYTIC TOPOLOGY
of groups, actions, strings and varietes

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Contents

Introduction

This paper is devoted to an application of Analysis to Topology. The latter is very broadly understood and includes geometric theory of finitely generated groups, group cohomology, Kazhdan groups, actions of groups on manifolds, superrigidity, fundamental groups of Kähler and quaternionic Kähler manifolds and conformal field theory. The motivation and philosophy which has led to the present research will be reflected upon in [Reznikov 7] and here we will merely say that we believe Analysis to be a major tool in studying finitely generated groups. An alternative look is provided by arithmetic method, notably by passing to a pro-p completion and using Galois cohomology. This will be described in [Reznikov 8].

Each of six chapters which constitute this paper opens with a short overview; a global picture is as follows. Chapter I and III treat analytic aspects of geometry of finitely generated groups. Given an immersion $M \hookrightarrow N$ of negatively curved manifolds (M compact) there is a boundary map $\partial\tilde{M} \to \partial\tilde{N}$, and it has remarkable regularity properties. Invoking the Thurston theory, we show that the actions A of pseudoAnosovs on $W_p^{1/p}(S^1)/const$ have striking properties from the viewpoint of functional analysis, namely,

$$\sum_{n\in\mathbb{Z}} \|A^n v\|^p < \infty$$

for some $v \neq 0$. We apply this to a classical problem: when a surface fibration is negatively curved and derive a strong necessary condition.

We then develop a theory of quantization for the mapping class group. A classical work on $\mathcal{D}iff^\infty(S^1)$ suggests a two-step quantization: first, obtaining a symplectic representation in $Sp(W_2^{1/2}(S^1)/const)$ with image in the restricted symplectic group [Pressley-Segal 1] and then using the Shale-Weil representation. The first step meets obstacles and the second step breaks down completely for the mapping class group: first, because $\mathcal{M}ap_{g,1}$ does not act smoothly on S^1, so it's unclear why it can be represented in $Sp(W_2^{1/2}(S^1)/const)$, second, if even it can (this happens to be the case), there is no way to show that the image lies in the restricted symplectic group (it almost certainly does not). The solution comes at the price of abandoning the classical scheme and developing a theory of a new object which we call bicohomology spaces $\mathcal{H}_{p,g}$. The mapping class group $\mathcal{M}ap_g$ act in $\mathcal{H}_{p,g}$ and the latter shows remarkable properties, like duality and existence of vacuum. The last property is translated into the fact that $H^1(\mathcal{M}ap_{g,1}, W_p^{1/p}(S^1)/const)$ is not zero. Finally, we find $\mathcal{M}ap_g$-equivariant maps of the space of all discrete representations of the surface group into $PSL_2(\mathbb{C})$ to our spaces $\mathcal{H}_{p,g}$.

Chapter II uses Analysis to study groups, acting on the circle (we need $\mathcal{D}iff^{1,\alpha}$ regularity, so $\mathcal{M}ap_{g,1}$ is not included). Our first main theorem says roughly that Kazhdan groups do not act on the circle. Very special cases of this result, for lattices in Lie groups, were recently found (see the references in Chapter II). The Hilbert transform, which played a major role in Chapter I, is crucial for the proof of this result as well. We then develop a theory of higher characteristic classes for subgroups of $\mathcal{D}iff^{1,\alpha}$ (the first being classically known as an integrated Godbillon-Vey class). All this classes vanish on $\mathcal{D}iff^{\infty}(S^1)$. It is safe to say that the less is smoothness, the more interesting is the geometry "of the circle".

Chapter III brings us back to asymptotic geometry of finitely generated groups. We propose, for a non-Kazhdan group, to study the asymptotic behaviour of unitary cocycles. We prove a general convexity result which shows that an embedding of G in the Hilbert space, given by a unitary cocycle, is "uniform". We then prove a growth estimate for unitary cocycles of a surface group, using very heavy machinery from complex analysis, adjusted for our situation. Similar result for cocycles in $H^1(G, l^p(G))$ has already been given in Chapter I.

Chapter IV studies symplectomorphism groups. There is a misterious similarity between groups acting on the circle and groups acting symplectically on a compact symplectic manifold. In parallel with the above mentioned result in Chapter II we show, roughly, that transformations of a Kazhdan group acting on a symplectic manifold must satisfy a partial differential equation. An example is $Sp(2n,\mathbb{Z})$ acting linearly on T^{2n} and, very probably, $\mathcal{M}ap_g$ acting on the space of stable bundles over a Riemann surface. (I don't know for sure if $\mathcal{M}ap_g$ is Kazhdan). In dimension 2 the result is very easy and was known before. We also introduce new charateristic classes for symplectomorphism groups, in addition to the two series of classes defined in our previous papers, and use them to express a volume of a negatively curved manifold through the Busemann function on the universal cover.

Chapter V studies volume-preserving actions. We introduce a new technique into the subject, that of (infinite-dimensional, non-positively curved) spaces of metrics. We define a invariant of an action which is an infinum of a displacement in the space of metrics and show that for an action of a Kazhdan group which does not fix a $\log L^2$-metric, this invariant is positive (a weak version of this result for the special case of lattices was known before). We then turn to a major open problem, that of non-linear superrigidity and prove what seems to be first serious breakthrough after many years of effort.

Chapter VI deals with fundamental groups of Kähler and quaternionic Kähler manifolds. The situation is exactly the opposite to the studied in Chapters II and IV, namely, these groups tend to be Kazhdan. We first extend our rationality theorem for secondary classes of flat bundles over projective varietes to the case of quasiprojective varietes, answering a question posed to us by P.Deligne. We then prove that a fundamental group of a compact quaternionic Kähler manifold is Kazhdan, therefore providing a very strong restriction on its topology. We also discuss polynomial growth of the group cohomology classes for Kähler groups, proved nontrivial in a previous paper.

The paper uses many different analytic techniques. Within each chapter, there is a certain coherence in the point of view adopted for study.

I started this project on a chilly evening of November, 1998 in an African café in Leipzig and finished it on a hot afternoon of July, 1999 in Jerusalem. The manuscript has been written up by August, 11, 1999; I would appreciate any mentioning of a possible overlap with any paper/preprint which appeared before this date. During the long time when the paper was being typed and then polished, I found a proof of several statements which had been conjectured in the paper, in particular a construction of a cocycle for the group of quasisymmetric homeomorphisms valued in $W_p^{1/p}(S^1)/const$, which was conjectured in Chapter I. The proofs will appear in a sequel to this paper.

Chapter 1

Analytic topology of negatively curved manifolds, quantum strings and mapping class groups

Chapter I opens with simple observations concerning the cohomology $H^1(G, l^\infty(G))$ for a finitely-generated group. If G is amenable we produce plenty of polynomial cohomology classes in $H^*(G, \mathbb{R})$ given by an explicit formula (Theorem 1.2.1). Then we prove a convexity theorem 1.2.2 saying that if there are Euclidean-type quasigeodesics in the Cayley graph of G, then $G/[G,G]$ is infinite.

We then review some standard facts on l^p-cohomology in sections 2–4. One defines an asymptotic invariant of a finitely generated group G, called a constant of coarse structure $\alpha(G)$, as an infinum of p, $1 \leq p \leq \infty$ such that $H^1(G, l^p(G)) \neq 0$. For all noncusp discrete groups of motions of complete manifolds of pinched negative curvature, $\alpha(G) < \infty$. For discrete subgroups G of $SO^+(1,n)$, $\alpha(G) \leq \delta(G)$, where δ is the exponent of the group. In section 4 we review function spaces. A classical result in weighted Sobolev spaces may be reformulated as an identification of the l^p-cohomology of cocompact real hyperbolic lattices: $H^1(G, l^p(G)) = W_p^{(n-1)/p}(S^{n-1})/const$

. It follows that $\alpha(G) = n - 1$.

In section 5 we prove a first result within a program to classify groups according to the cocycle growth. We show for surface groups, that if $L_g\mathcal{F} - \mathcal{F} \in l^p(G)$ for all $g \in G$, then $|\mathcal{F}(g)| \leq const \cdot [length(g)]^{1/p'}$. Here $\mathcal{F} : G \to \mathbb{R}$ is any function (Theorem 5.6). This result with no doubt generalizes to higher-dimensional cocompact lattices in simple Lie groups of rank one.

In section 6 we present a new theory for boundary maps of negatively curved spaces, associated with immersions of closed manifolds. The most striking is a partial regularity result (Theorem 6.1, part 4).

As is well known, the group of quasisymmetric ($n = 2$) or quasiconformal ($n \geq 3$) homeomorphisms of S^{n-1} act on $W_p^{\frac{n-1}{p}}(S^{n-1})$ for $p > n - 1$. The action of $\mathcal{G}_1$ on $W_2^{1/2}(S^1)/const$ is in fact symplectic. We give application to the regularity of quasisymmetric homeomorphisms (Theorem 8.2). In Corollary 9.2 we prove that the unitary representation of a subgroup G of $SO(1, 2)$ in $W_2^{1/2}(S^1)/const$ is an invariant of a component of the Teichmüller space $\mathbf{T}(G)$.

In Theorem 9.3 we show striking properties of invertible operators A in Banach spaces $W_p^{1/p}(S^1)/const,\ p > 2$, induced by quasiAnosov maps in $\mathcal{M}ap_{g,1}$, namely

$$\sum_{k\in\mathbb{Z}} \|A^k v\|^p < \infty$$

for some $0 \neq v$. In Theorem 9.5 we find a new inequality in topological Arakelov theory, based on the work of [Matsumoto-Morita 1]. In Theorem 9.6 we find very strong restrictions on a subgroup $G \subset \mathcal{M}ap_g$, such that an induced group extension $\widetilde{G}$:

$$1 \to \pi_1(\Sigma_g) \to \widetilde{G} \to G \to 1$$

is a fundamental group of a negatively curved compact manifold (this is a classical problem). In section 10 we extend the theory to the limit case $p = 1$, introducing an L^1-analogue of Zigmund spaces, which we call $\mathcal{L}_{k,\alpha}$.

In section 11 we start a new theory of secondary quantization of Teichmüller spaces. We introduce the bicohomology spaces $\mathcal{H}_{g,p}$ and show that $\mathcal{M}ap_g$ acts on these spaces. We show (difficult!) that $\mathcal{H}_{2,p}$ is an infinite-dimensional Hilbert space and there is a symmetric bilinear nondegenerate form of signature (∞, m) which is $\mathcal{M}ap_g$-invariant. What is the value of m, we don't know at the time of writing of this introduction (August,1999). So does the secondary quantization lead to ghosts? We provide a holomorphic realization in the space of L^2-holomorphic 2-forms on $\mathcal{H}^2 \times \mathcal{H}^2/G$ and $\mathcal{H}^2 \times \overline{\mathcal{H}^2}/G$ (Theorem 11.12). In section 12 we interpret $\mathcal{H}_{p,g}$ as operator spaces (proposition 12.2), and prove the existence of vacuum (Theorem 12.5). We prove that $H^1(\mathcal{M}ap_{g,1}, W_p^{1/p}(S^1)/const) \neq 0$ for $p \geq 2$. It still may be true that $\mathcal{M}ap_{g,1}$ is Kazhdan, because the action is not orthogonal.

In section 13 we construct $\mathcal{M}ap_g$-equivariant maps of the space of discrete representations of the surface group in $SO^+(1,3) = PSL_2(\mathbb{C})$ to our spaces $\mathcal{H}_{p,g}$ (Theorem 13.1). In Theorem 13.2 we summarize our knowledge of the functional-analytic structure coming from hyperbolic 3-manifolds which fiber over the circle.

1.1 Metric cohomology

1.1.1. Let G be a finitely generated group. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Let V be a locally convex topological $\mathbb{K}$-vector space which is a G-module, that is, there is a homomorphism $G \to Aut(V)$. If $\{g_i\}, \quad i = 1, \cdots, n$ is a finite set of generators of G, then the evaluation map $f \mapsto \{f(g_i)\}$ establishes an injective homomorphism $Z^1(G, V) \to \Pi_{i=1}^n V$ of the space of 1-cocycles of G in V. One calls the induced topology in $Z^1(G, V)$ the cocycle topology; it does not depend on the choice of generators. A coboundary map $V \to Z^1(G, V)$ may have an image $\overline{B^1(G, V)}$ which is not closed in $Z^1(G, V)$; the quotient $Z^1(G, V)/\overline{B^1(G, V)}$ is called *reduced* first cohomology space. One way to produce nontrivial cohomology classes is to consider limits of coboundaries, that is, elements of $\overline{B^1(G, V)}/B^1(G, V)$. That amounts to considering nets $\{v_\alpha \in V\}$ such that $g_i v_\alpha - v_\alpha \to l(g_i)$ for $i = 1, \cdots, n$. If V is a Banach space and G acts isometrically without invariant vectors, then $B^1(G, V)$ is closed in $Z^1(G, V)$ if and only if there are no almost invariant vectors, that is, sequences $v_j, \|v_j\| = 1$, such that $\|g_i v_j - v_j\| \to 0$ for all $i = 1, \cdots, n$. This statement is an immediate consequence of the Banach theorem and is called Guichardet's lemma [Guichardet 1]. So if there are almost invaiant vectors, then $H^1(G,V) \neq 0$, though the reduced cohomology $H^1_{red}(G,V)$ may be zero.

If V is Banach and G acts isometrically, let $l \in Z^1(G,V)$ be a cocycle. Then

$$\|l(g)\| \leq \max_{i=1}^{n} \|l(g_i)\| \cdot length(g)$$

, where $length(g)$ is the length of the element g in the word metric, induced by $\{g_i\}$. The proof is immediate by induction, using the cocycle equation $l(gh) = gl(h) + l(g)$.

Now let $V_j, \quad j = 1, \cdots, m$ be a collection of Banach spaces on which G acts isometrically and let $\varphi : \otimes_{j=1}^m V \to \mathbb{K}$ be a map continuous in a sense that $\varphi(\otimes_{j=1}^m v_j) \leq const \cdot \Pi_{j=1}^m \|v_j\|$. Let $l_j \in Z^1(G,V_j)$ and let $l \in Z^m(G,\mathbb{K})$ be the cup product $l(g_1,\cdots,g_m) = \varphi(\otimes_{j=1}^m l_j(g_j))$.

Lemma 1.1.— $l \in Z^m(G,\mathbb{K})$ is of polynomial growth, more precisely

$$|l(g_1,\cdots,g_m)| \leq const \cdot \Pi_{i=1}^m length(g_i).$$

Proof.— is immediate from the remarks made above.

A general definition of polynomial cohomology is to be found in [Connes-Moscovici 1]. As we will see, Lemma 1.1 is a very powerful tool for constructing cocycles of polynomial growth in concrete situations.

Proposition 1.1.— Let G be an infinite finitely generated group. Consider a left action of G on $l^\infty(G)$. Then $H^1(G,l^\infty(G)) \neq 0$. Moreover, $H^1(G,l^\infty_0(G)) \neq 0$.

Proof.— Let $\{g_i\}$ be a finite set of generators of G, and let $length(g)$ be a word length of an element g. Define a right-invariant word metric by $\rho(x,y) = length(xy^{-1})$. Let $x_0 \in G$ and let $F(x) = \rho(x_0,x)$. Obviously, F is unbounded. Now let $l(g) = L_gF - F$ where L_g is a left action on functions, that is, $l(g)(x) = F(g^{-1}x) - F(x)$. We find

$$|l(g)(x)| = |\rho(x_0,g^{-1}x) - \rho(x_0,x)| \leq |\rho(g^{-1}x,x)| = \rho(g^{-1},1).$$

So l is a cocycle of G in $l^\infty(G)$. If it were trivial, we would have a bounded function f such that $L_g F - F = L_g f - f$ that is, $F - f$ would be invariant, therefore constant, a contradiction. The second statement of the Proposition will be proved later in section 1.3.

1.1.2. Now let G be amenable. In this case we have a continuous map $\varphi : \Pi_{j=1}^m l^\infty(G) \to \mathbb{K}$ given by $(f_1, \cdots, f_m) \mapsto \int_G f_1 \cdots f_m$. By an integral we mean a left-invariant normalized mean of bounded functions. We obtain

Theorem 1.2.1.— Let G be a finitely generated amenable group, let $\rho_j,\ j = 1, \cdots, m$ be a collection of right-invariant word metrics on G. A formula

$$l(g_1, \cdots, g_m) = \int_G \Pi_{j=1}^m [\rho_j(x_0, g_j^{-1}x) - \rho_j(x_0, x)]$$

defines a real-valued m-cocycle on G of polynomial growth:

$$|l(g_1, \cdots, g_m)| \leq const \cdot \Pi_{j=1}^m length(g_j)$$

for any word length $length(\cdot)$.

Examples.— Let $G = \mathbb{Z}$. If we choose generators $\{-1, 1\}$, then $length(g) = |g|$, and

$$\rho(x_0, g^{-1}x) - \rho(x_0, x) = |x_0 - x + g| - |x_0 - x| \to \pm|g|$$

as $x \to \pm\infty$ and

$$\int_{\mathbb{Z}} (|x_0 - x + g| - |x_0 - x|) = 0.$$

However, if we choose generators $\{-1, 2\}$, then

$$length(g) = \begin{cases} |g|, & g \leq 0 \\ \frac{g}{2}, & g \geq 0 \text{ and even} \\ \frac{g+1}{2}, & g \geq 0 \text{ and odd} \end{cases}$$

Then

$$length(x_0 + g - x) - length(x_0 - x)$$

for $g > 0$ and even will have limits $\frac{g}{2}$ when $x \to -\infty$ and $-g$ when $x \to \infty$, so

$$\int_{\mathbb{Z}} [length(x_0 + g - x) - length(x_0 - x)] = -\frac{g}{4}.$$

So we obtain a cocycle $l : \mathbb{Z} \to \mathbb{R}$ given by $g \mapsto -\frac{g}{4}$. Now, if $G = \mathbb{Z}^k$, $k \geq 2$, let $\rho_j, j = 1, \cdots, k$ be a word metric defined by a set of generators

$$\{e_1^{\pm 1}, e_2^{\pm 1}, \cdots, e_j^{-1}, e_j^2, e_{j+1}^{\pm 1}, \cdots, e_k^{\pm 1}\}$$

where e_s is a generator of the s-th factor. If $1 \leq j_1 < j_2 < \cdots < j_m \leq k$ is a set of indices, then Theorem 2 provides a cocycle

$$l(g_1, \cdots, g_m) = \int_{\mathbb{Z}^k} \Pi_{r=1}^m [\rho_{j_r}(x_0, g_j^{-1} x) - \rho_{j_r}(x_0, x)].$$

If $\pi_i : \mathbb{Z}^k \to \mathbb{Z}$ is a projection to i-th factor, then $l(g_1, \cdots, g_m) = (-\frac{1}{4})^m \cdot \Pi_{r=1}^m \pi_{j_r}(g_r)$. It follows that classes of cocycles, given by Theorem 1.2.1, generate the real cohomology space of $\mathbb{Z}^k$.

Remark.— If G is amenable, ρ is a right-invariant word metrics and for some $x_0, g \in G$,

$$\int_G [\rho(x_0, g^{-1} x) - \rho(x_0, x)] \neq 0,$$

then $H_1(G, \mathbb{R}) \neq 0$ and in fact $g \notin [G, G]$ for all $s \neq 0$. This is a direct corollary of Theorem 1.2.1. A more interesting structure theorem is given below.

Theorem 1.2.2.— Let G be a finitely generated amenable group, ρ a right-invariant word metric. Let $g \in G$, assume a following convexity condition: there is some $C > 0$, such that for any $x \in G$ there exists $N \geq 0$ such that $\rho(g^k, g^{-1} x) - \rho(g^k, x) \geq C$ for $k \geq N$. Then $H_1(G, \mathbb{R}) \neq 0$ and moreover, $g^s \notin [G, G]$ for all $s \neq 0$.

Corollary 1.2.3.— Let G be a Heisenberg group $\{x, y, z | [x, y] = z, [x, z] = [y, z] = 1\}$. Then for any right-invariant word metric ρ, there exists $a \in G$ such that $\liminf_{k \to \infty} [\rho(z^k, z^{-1} a) - \rho(z^k, a)] \leq 0$.

Proof of the Corollary.— Since $z \in [G,G]$, the result follows from Theorem 1.2.2. Indeed G is nilpotent, therefore amenable.

Proof of the Theorem.— Consider a 1-cocycle $l(\gamma)(x) = \rho(x_0, \gamma^{-1}x) - \rho(x_0, x)$, $l \in Z^1(G, l^\infty(G))$. Set $x_0 = g^n$, so

$$l_n(g)(x) = \rho(g^n, g^{-1}x) - \rho(g^n, x).$$

If for any x and sufficiently big n, $\rho(g^n, g^{-1}x) - \rho(g^n, x) > C$ then a pointwise limit $\lim_{n\to\infty} l_n(g)(x)$ exists and is $\geq C$. Since $|l_n(z)(x)| \leq \rho(z^{-1}, 1)$, there is a subsequence n_k such that $l_{n_k}(z)$ converges pointwise for any z to a bounded function $l(z)$. One sees immediately that $l : G \to l^\infty(G)$ is a cocycle, so $z \mapsto \int_G l(z)$ is a homomorphism from G to $\mathbb{R}$. Since $l(g)(x) \geq C > 0$ for all x, $\int_G l(g) \geq C > 0$, so $H_1(G, \mathbb{R}) \neq 0$ and $g^s \notin [G,G]$, as desired.

1.1.3. Let $\varphi : R_+ \to R_+$ be a smooth function such that $\varphi(x) \to \infty$ as $x \to \infty$ and $\varphi'(x) \to 0$. Let G be a finitely generated group and let ρ be a right-invariant word metric. Consider $F(x) = \varphi(\rho(x_0, x))$ where $x_0 \in G$ is a fixed element. Since

$$\begin{aligned} |(L_gF - F)(x)| &= |F(g^{-1}x) - F(x)| \\ &= |\varphi(\rho(x_0, g^{-1}x)) - \varphi(\rho(x_0, x))| \\ &\leq \sup_{t\in I} |\varphi'(t)| \cdot |\rho(g^{-1}x, x)| \\ &\leq \sup_{t\in I} |\varphi'(t)| \rho(g^{-1}, 1) \end{aligned}$$

where $I = [\min(\rho(x_0, x), \rho(x_0, g^{-1}x)), \max(\rho(x_0, x), \rho(x_0, g^{-1}x))]$, we see that $L_gF - F \in l_0^\infty$. Therefore $H^1(G, l_0^\infty(G)) \neq 0$, because the cocycle $L_gF - F$ cannot be trivial as a cocycle valued in l_0^∞ (by the same reasons as in the proof of the first statement of Proposition 1.1) . The proof of Proposition 1.1 is now complete.

Notice that, since $\rho(u, v) = length(u \cdot v^{-1})$,

$$\rho(x_0, x) - length(g) \leq \rho(x_0, g^{-1}x) \leq \rho(x_0, x) + length(g),$$

so that

$$|(L_gF - F)(x)| \le \sup_{|t-\rho(x_0,x)|\le length(g)} |\varphi'(t)| \times \rho(g^{-1},1).$$

Remark.— Let $S(N) = \{g|length(g) = N\}$. If $S(N)/S(N-1) \to 1$ and $\sum_{k=1}^{N} S(k)/S(N) \to \infty$ as $N \to \infty$, then for $p > 1$ there is a radial function $F(x) = \varphi(\rho(x))$ such that $L_gF - F \in l^p(G)$ and the cocycle $l : G \to l^p(G)$ defined by $g \mapsto L_gF - F$ is not a coboundary. Note that G is automatically amenable. On the other hand, if $S(N) \sim e^{cN}$, then such radial function does not exist. This follows at once from Hardy's inequality. To produce classes in $H^1(G, l^p(G))$, one needs to use some more elaborate geometry than just distance function. In the next section we produce such classes for negatively curved groups/manifolds, using the visibility angles.

1.2 Constants of coarse structure for negatively curved groups

1.2.1. Throughout this section we assume that G is a finitely generated, non-amenable group, therefore $B^1(G, l^p(G))$ is closed in $Z^1(G, l^p(G))$ for $p \ge 1$.

Definition 2.1.— A number $\alpha(G) = \inf_{1\le p\le\infty}\{p|H^1(G, l^p(G)) \ne 0\}$ is called a constant of coarse structure of G.

Remark.— The definition makes sense since by Proposition 1.1, $H^1(G, l^\infty(G)) \ne 0$. We will need a proof of the following well-known fact (see, for example [Pansu 1]). The argument below is a slightly modified, from nonpositive curvature to negative curvature, version of a classical argument of [Mishchenko 1,2].

Proposition 2.1.— Let M^n be a complete Riemannian manifold of negative curvature, not a cusp, satisfying $K(M) \leq -1, Ric(M) \geq -(n-1)K$. Let $G = \pi_1(M)$. Then $\alpha(G) \leq (n-1)\sqrt{K}$.

Proof.— Let $q_0 \in \widetilde{M}$. Consider a map of G onto an orbit $\mathcal{O}$ of $q_0 : g \mapsto gq_0$; it is equivariant with respect to the left action of G on itself. Let $q \notin \mathcal{O}$ and let $v_q(s)$ be an outward pointing vector from q to s, that is, a unit vector in $T_s\widetilde{M}$, tangent to geodesic segment joining q and s. Consider for $x \in G$, $F(x) = v_q(xq_0)$. Notice that $F(x)$ takes values in $T_{xq_0}\widetilde{M}$. We can consider the restriction of $T\widetilde{M}$ on $\mathcal{O}$ as an equivariant vector bundle over $\mathcal{O}$. Pulling back to G, we obtain an left-equivariant vector bundle over G, equipped with an equivariant Euclidean structure. Then F is a section of this bundle. Now consider $(L_gF - F)(x)$. Since the action of G on sections is given by $L_gF(x) = g_*F(g^{-1}x)$, where g_* is the derivative map ($g_* : T_s\widetilde{M} \to T_{gs}\widetilde{M}$), we get $(L_gF - F)(x) = g_*F(g^{-1}x) - F(x) = g_*v_q(g^{-1}xq_0) - v_q(xq_0) = v_{gq}(xq_0) - v_q(xq_0)$. So $\|(L_gF - F)(x)\| = |2\sin\frac{1}{2}\sphericalangle(gq, xq_0, q)| \leq \sphericalangle(gq, xq_0, q)$.

Let $E|G$ be the equivariant Euclidean vector bundle considered above (the pullback of G of $T\widetilde{M}|\mathcal{O}$). Let $L^p(E)$ be a Banach space of L^p-sections of E. We claim $L_gF - F \in L^p(E)$ for $p > (n-1)\sqrt{K}$. Let $r(x) = dist_{\tilde{M}}(q_0, xq_0)$. For g, q_0, q fixed we have $\sphericalangle(gq, xq_0, q) \leq const_1 \cdot e^{-r(x)}$ by a standard comparison theorem, since $K(M) \leq -1$. On the other hand, for fixed $\delta > 0$, $\#(x|r - \delta \leq r(x) \leq r + \delta) \leq const_2 e^{(n-1)\sqrt{K}r}$ by the Bishop's theorem. Therefore $L_gF - F \in L^p(E)$ for $p > (n-1)\sqrt{K}$. Note we only need that G acts discretely in $\widetilde{M}$.

A map $l : G \to L^p(E)$ defined by $l(g) = L_gF - F$ is obviously a cocycle. If it were trivial, we would have an L^p-section $s \in L^p(E)$, such that $F - s$ is invariant. That means $g_*(v_q(g^{-1}xq_0) - s(g^{-1}x)) = v_q(xq_0) - s(x)$, or $v_{gq}(xq_0) - g_*s(g^{-1}x) = v_q(xq_0) - s(x)$. Notice that since $\|F(x)\| = 1$, $F - s$ is invariant and $\|s(g)\| \to 0$ as $length(g) \to \infty$, $\|(F - s)(x)\| = 1$ for all x. In particular, $w = v_q(xq_0) - s(x)$ has norm one. Fix x and let g vary. We get $\|v_{gq}(xq_0) - w\| = \|g_*s(g^{-1}x)\| \to 0$ as $length(g) \to \infty$. Let P_+, P_- be an attractive and repelling fixed points of g on the sphere at infinity of $\widetilde{M}$.

Let w_+, w_- be unit vectors in $T_{xq_0}(\widetilde{M})$, tangent to geodesics, joining xq_0 with P_+, P_-. Then $\|v_{g^nq}(xq_0) - w_\pm\| \to 0$ if $n \to \pm\infty$. It follows that $w_\pm =$

w. Therefore all elements of G are parabolic and have a common fixed point at infinity. So M is a cusp, a contradiction. So $H^1(G, l^p(E)) \neq 0$. However, $l^p(E)$ is equivariantly isometric to $l^P(G) \otimes T_{p_0}(\widetilde{M})$. So $H^1(G, l^p(E)) \simeq H^1(G, l^p(G)) \otimes T_{p_0}(\widetilde{M})$. We deduce that $H^1(G, l^p(G)) \neq 0$.

The estimate of the Proposition is sharp. We will see later that if G is a cocompact lattice in $SO^+(1,n)$, i.e. $K(M) = -1$, then $\alpha(G)$ is exactly $(n-1)$. Let now G be a discrete nonamenable subgroup of $SO^+(1,n)$, or, equivalently, $K(M) = -1$. Recall that the exponent $\delta(G)$ is defined by $\delta(G) = \inf\{\lambda | \sum_{g\in G} e^{-\lambda r(g)} < \infty\}$ where $r(g) = dist_{\widetilde{M}}(p_0, gp_0)$ for some fixed $p_0 \in \widetilde{M}$. If G is geometrically finite, then by a well-known theorem [Nicholls 1] $\delta(G)$ is equal to the Hausdorff dimension of the limit set $\dim(\Lambda(G)) \subset S^{n-1}$. Note that if $\Lambda(G) \neq S^{n-1}$, then $\dim \Lambda(G) < n-1$ by [Sullivan 1] and [Tukia 1].We now have

Proposition 2.2.— Let G be a discrete subgroup of $SO^+(1,n)$, not a cusp group. Then $\alpha(G) \leq \delta(G)$.

Proof.— The Proposition follows from the proof of the Proposition 2.1. Indeed, we only need that $\sum_{g\in G} e^{-pr(g)} < \infty$ to conclude that one has a cocycle $l : G \to l^p(G)$. It has been proven already that this cocycle is not a coboundary.

Remark.— The relation of the constant of coarse structure to "conformal dimension at infinity" is discussed in [Pansu 2].

1.3 Function spaces: an overview

For $s \geq 0$ an integer and fractional part of s are denoted $[s]$ and$\{s\}$ respectively. A Sobolev-Slobodečky space $W_p^s(\mathbb{R}^n)$,$(p > 1)$ consists of measurable locally integrable functions f on $\mathbb{R}^n$ such that $D^\alpha f \in L^p(R^n)$ for $|\alpha| \leq [s]$

and

$$\sum_{|\alpha|=[s]} \int\int \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+\{s\}p}}\, dxdy < \infty$$

A space of Bessel potentials H^s_p consists of functions f for which a Liouville-type operator

$$\mathcal{D}^s f = ((1+|\xi|^2)^{s/2} \hat{f}(\xi))^\wedge$$

satisfies $\mathcal{D}^s f \in L^p$. Warning: $H^s_p \neq W^s_p$ if s is not an integer. For $p = 2$ the condition is equivalent to

$$(1+\triangle)^{s/2} \hat{f} \in L^2(\mathbb{R}^n).$$

Here $f(x) \to \hat{f}(\xi)$ is the Fourier transform and $\triangle = -\sum \frac{\partial^2}{\partial x_i^2}$.

A space of BMO functions BMO($\mathbb{R}^n$) is defined as a space of functions f for which

$$\sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx < \infty,$$

where Q runs over all cubes in $\mathbb{R}^n$ and

$$f_Q = \frac{1}{|Q|} \int_Q f(x)\, dx,$$

$|Q| = \int_Q 1\, dx$. One has $W^{n/p}_p \subset BMO$ for all $1 < p < \infty$, and moreover $H^{n/p}_p \subset H^{n/p_1}_{p_1}$ for $1 < p < p_1 < \infty$ (this follows from Theorem 2.7.1 of [Triebel 1]. In some sense BMO is a limit of $H^{n/p}_p$ as $p \to \infty$.

If $f \in W^1_p$ the restriction of f on hyperplanes $\{x_n = \epsilon\} \subset R^n$ (where $(x_1, \cdots, x_n)$ are Euclidean coordinates) have both L^p and nontangential limits a.e. on $\mathbb{R}^{n-1} = \{x | x_n = 0\}$ and the limit function $f|_{\mathbb{R}^{n-1}}$, called trace of f, satisfies $f|_{\mathbb{R}^{n-1}} \in W^{1-1/p}_p$. By a nontangential limit we mean the following. Let $y \in \mathbb{R}^{n-1}$ and let C_δ be a Stolz angle centered at y, that is, a set $\{z, x_n | x_n \geq \delta \cdot |z|\}$ for $\delta > 0$. Then a function f defined in $\mathbb{R}^n_+ = \{x_n > 0\}$ has a nontangential limit $f(y)$ at y if

$$f(x) \underset{\substack{x \to y \\ x \in C_\delta}}{\longrightarrow} f(y)$$

for all δ. Note that the points in C_δ are within a bounded distance from any geodesic of a hyperbolic metric

$$\frac{\sum_{i=1}^n dx_i^2}{x_n^2},$$

which has y as a point at infinity. The trace theorem mentioned above may be found in [Triebel 1], section 2.7.2. Notice that functions in $W_2^1(\mathbb{R}^2)$ have traces in $W_2^{1/2}(\mathbb{R}^1)$.

Now let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. We define $W_p^s(\Omega)$ as a space of locally integrable functions with $D^\alpha f \in L^p$ for $|\alpha| \leq s$ and such that

$$\sum_{|\alpha|=[s]} \int\int \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+\{s\}p}} < \infty.$$

Equivalently, $W_p^s(\Omega)$ is a space of restrictions of function from $W_p^s(\mathbb{R}^n)$ on Ω. [Triebel 1, Chapter 3]. One also defines $H_p^s(\Omega)$ as a space of restrictions of $H_p^s(\mathbb{R}^n)$ on Ω. For a compact smooth manifold M without boundary (in particular, for the boundary $\partial\Omega$) one easily defines the spaces $W_p^s(M)$ and $H_p^s(M)$ [Triebel 1, Chapter 3] (H_p^s is $F_{p,2}^s$ in Triebel's notations).

If M is compact and g a Riemannian metric on M, let $\triangle_g$ be a corresponding Laplace-Beltrami operator. One can construct a space of Bessel potentials $(1+\triangle)^{-s/2}(L_p(M))$. It is known [Rempel-Schulze 1, Theorem 1, section 2.3.2.5], [Hörmander 1], that this space coincides with W_p^s (and not H_p^s). Warning: our W_p^s is called $H^{p,s}$ in [Rempel-Schulze 1] and in many other sources. In particular, $W_2^s(S^1)$ consists of functions $f = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$, such that $\sum |n|^{2s}|a_n|^2 < \infty$. We will see that $W_2^{1/2}(S^1)$ is especially important in topology.

If $f \in W_p^1(\Omega)$ then f has an L^p and nontangential limit a.e. on $\partial\Omega$ and $f|_{\partial\Omega} \in W_p^{1-1/p}(\partial\Omega)$. In particular, for a unit disc $D \subset \mathbb{R}^2$, and a function $f \in W_2^1(D)$, $f|_{S^1} \in W_2^{1/2}(S^1)$.

We will need trace theorems for weighted Sobolev-Lorentz spaces [Kudryavcev 1,2], [Vasharin 1], [Lions 1], [Lizorkin 1,2], [Uspenski 1]. Let Ω be as above and let $\rho(x) = dist(x, \partial\Omega)$. Consider $L_p^1(\Omega, \rho^\alpha)$ as a space of functions f such that $\int_\Omega |\nabla f|^p \cdot \rho^\alpha \, dx < \infty$. Then f has a nontangential limit a.e. on

$\partial\Omega$ and

1) $f|_{\partial\Omega} = 0$ if $\alpha \leq -1$
2) $f|_{\partial\Omega} \in W_p^{\frac{p-1-\alpha}{p}}(\partial\Omega), \quad \alpha > -1.$

Moreover,

$$\|f\|_{W_p^{\frac{p-1-\alpha}{p}}} \leq const \cdot \int_\Omega |\nabla f|^p \rho^\alpha \, dx$$

and for any $f \in W_p^{\frac{p-1-\alpha}{p}}(\partial\Omega)$ and harmonic h, $h|_{\partial\Omega} = f$, one has

$$\int_\Omega |\nabla h|^p \rho^\alpha \, dx \leq const \cdot \|f\|_{W_p^{\frac{p-1-\alpha}{p}}}.$$

1.4 l^p-cohomology of cocompact real hyperbolic lattices

The following result is an immediate corollary of the Poincaré inequality in hyperbolic space, which is equivalent to Hardy inequality, and the classical results on traces of functions in weighted Sobolev spaces, reviewed in the previous section. It first appeared in print, with a different proof, in [Pansu 1]. We include a proof here, as many parts of it will be used in the theory later.

Theorem 4.1, part 1.— Let $G \subset SO^+(1,n)$ be a cocompact (uniform) lattice. Then there is a G-equivariant isomorphism of Banach spaces

$$H^1(G, l^p(G)) \simeq W_p^{\frac{n-1}{p}}(S^{n-1})/const$$

for $p > n-1$. For $1 < p \leq n-1$, $H^1(G, l^p(G)) = 0$.

Corollary 4.1.— The constant of fine structure $\alpha(G)$ is equal $n-1$.

Remarks.
1)Theorem 4.1 is a first step in the program of linearization of 3-dimensional topology, which we will develop below in this chapter. A crucial fact is

that $W_2^{1/2}(S^1)$ admits a natural action of the extended mapping class group $\mathcal{M}ap_{g,1}$. This will be proved in section 7 below.
2)Let $\mathcal{H}^n$ be a hyperbolic space. Since $G = \pi_1(\mathcal{H}^n/G)$, by the work of [Goldštein-Kuzminov-Shvedov 1] we know that $H^1(G, l^p(G))$ equals L^p-cohomology of $\mathcal{H}^n$. So Theorem 4.1 computes the L^p-cohomology of the hyperbolic space.

Recall that any class l in $H^1(G, l^p(G))$ has a primitive function $\mathcal{F} : G \to \mathbb{R}$ defined up to a constant, such that $l(g) = L_g F - F$. This follows from the fact that a module of all functions $\mathbb{R}^G$ is coinduced from the trivial subgroup and therefore cohomologically trivial [Brown 1].

Theorem 4.1, part 2.— Let G be a cocompact lattice in $SO^+(1,n)$ and let $l \in H^1(G, l^p(G))$, let $\mathcal{F} : G \to \mathbb{R}$ be a primitive function for l (unique up to a constant). Let $\partial G \approx S^{n-1}$ be the boundary of G as a word-hyperbolic group. Then for almost all points $x \in \partial G$, $\mathcal{F}(g)$ has nontangential limits as $g \to x$, and the limit function $\mathcal{F}|_{S^{n-1}} \in W_p^{\frac{n-1}{p}}(S^{n-1})$.

Corollary 4.2.— If $1 < p < p_1 < \infty$, then a natural map $H^1(G, l^p(G)) \to H^1(G, l^{p_1}(G))$ is injective. In fact, for $n - 1 < p < p_1 < \infty$ one has a commutative diagram

$$\begin{array}{ccc} H^1(G, l^p(G)) & \xrightarrow{\sim} & W_p^{\frac{n-1}{p}}(S^{n-1})/const \\ \downarrow & & \downarrow \\ H^1(G, l^{p_1}(G)) & \xrightarrow{\sim} & W_{p_1}^{\frac{n-1}{p_1}}(S^{n-1})/const \end{array}$$

where the right vertical arrows exists by an embedding theorem of Sobolev-Slobodečki space [Triebel 1, 2.7.1].

Proof of the Corollary 4.2.— The commutative diagram is implied by the proof of the Theorem 4.1. The injectivity follows immediately.

Proof of the Theorem.— Though a shorter proof of part 1 of the Theorem can be given by using [Goldshtein-Kuzminov-Shvedov 1], in order to prove part 2 we need to make an isomorphism $H^1(G, l^p(G)) \simeq L^pH^1(\mathcal{H}^n)$ explicit. Here $L^pH^1(V)$ is the L^p-cohomology of a complete Riemannian manifold V.

Let l be a cocycle in $Z^1(G, l^p(G))$. We have then an affine isometric action $g \overset{\pi}{\mapsto} (v \mapsto L_g v + l(g))$ of G on $l^p(G)$. To it corresponds a smooth locally trivial affine Banach bundle over $M = \mathcal{H}^n/G : E = [\tilde{M} \times l^p(G)]/\text{diagonal action}$. By local triviality, smooth partition of unity and affine structure on fibers one constructs a smooth section s of this bundle. It can be interpreted as an equivalent smooth map $s : \widetilde{M} \to l^p(G)$, that is, $s(g^{-1}x) = L_g s(x) + l(g)$. We note that there is some sonstant $C > 0$ such that $\|\nabla s(x)\| < C$ for all $x \in \widetilde{M} \simeq \mathcal{H}^n$ ($\nabla s \in T^*_x\widetilde{M} \otimes l^p(G)$). This is because M is compact. Now let $\mathcal{F} \in R^G$ be a primitive for l, i.e. $l(g) = L_g\mathcal{F} - \mathcal{F}$. Put $\sigma(x) = s(x) + \mathcal{F}$: this is a function $\sigma : \widetilde{M} \to \mathbb{R}^G$ with the same derivative as s in the sense that for all $g \in G$, $\nabla\sigma_g = \nabla s_g$ where σ_g, s_g means g-th coordinate. Next, we claim that σ is invariant, i.e. $\sigma(g^{-1}x) = L_g\sigma(x)$. In fact, $l(g) = L_g\mathcal{F} - \mathcal{F}$, so $s(g^{-1}x) = L_g s(x) + L_g\mathcal{F} - \mathcal{F}$, so $\sigma(g^{-1}x) = L_g\sigma(x)$. So for $x \in \tilde{M}$ and $g, h \in G$ we have $\sigma(g^{-1}x)(h) = \sigma(x)(g^{-1}h)$. Let $f(x) = \sigma(x)(1)$, then $\sigma(x)(g) = f(gx)$. Since $\nabla\sigma(x) = \nabla s(x) \in l^p$ and is bounded in norm, we have for all $x \in \widetilde{M}$ that $\sum_{g\in G} |\nabla f(gx)|^p < C$. In particular,

$$\int_{\widetilde{M}} |\nabla f|^p = \int_{\widetilde{M}/G} \sum_{g\in G} |\nabla f(gx)|^p < C \cdot Vol(M).$$

In other words, $|\nabla f| \in L^p(\mathcal{H}^n)$. Now, we can use a Poincaré model for the hyperbolic space, that is, the unit ball $B^n \subset \mathbb{R}^n$ with a hyperbolic metric

$$g_h = \frac{g_e}{(1 - r^2)^2}.$$

If μ_e, μ_h denote a Euclidean and a hyperbolic measure respectively, $|\nabla f|_e, |\nabla f|_h$ denote a norm of a gradient of a function in the Euclidean and hyperbolic metric respectively, $\rho(z) = 1 - r(z)$ denote a Euclidean distance to the boundary $\partial B^n \approx S^{n-1}$, then

$$const_2 \cdot \rho^{p-n} \cdot |\nabla f|^p_e \cdot \mu_e \le |\nabla f|^p_h \cdot \mu_h \le const_1 \cdot \rho^{p-n} |\nabla f|^p_e \mu_e,$$

so we have $\int_{B^n} \rho^{p-n} |\nabla f|^p_e \mu_e < \infty$.

By a theorem of Kudryavcev-Vasharin-Lizorkin-Uspenski-Lions mentioned above, we find that $f|_{(1-\epsilon)S^{n-1}}$ has an L^p-limit $f|_{S^{n-1}}$ to which it converges nontangentially a.e. , and moreover $f|_{S^{n-1}} \in W_p^{\frac{n-1}{p}}(S^{n-1})$ if $p > n-1$ and $f|_{S^{n-1}} = const$ if $p \le n-1$. We claim that a map $l \mapsto f|_{S^{n-1}}$ is a well-defined bounded linear operator from $H^1(G, l^p(G))$ to $W_p^{\frac{n-1}{p}}$, $p > n-1$. First, we notice that since $s : \widetilde{M} \to l^p(G), \sigma(x) = s(x) + \mathcal{F}$ and $\sigma(x)(g) = f(gx)$, we have for almost all $x \in \widetilde{M}$, $f(gx) - \mathcal{F}(g) \in l^p(G)$ (as a function of g). In particular, $f(gx) - \mathcal{F}(g) \to 0$ as $length(g) \to \infty$. This proves that, identifying G with an orbit of x, $\mathcal{F}(g)$ has a nontangential limit a.e. on the boundary $\partial G \approx S^{n-1}$ and $\mathcal{F}|_{S^{n-1}} = f|_{S^{n-1}} \in W_p^{\frac{n-1}{p}}$. In particular, $f|_{S^{n-1}}$ mod constants does not depend on the choice of a section s. Since changing l by a coboundary leads to an isomorphc affine $l^p(G)$-bundle, $f|_{S^{n-1}}$mod constants depends only on the class $[l] \in H^1(G, l^p(G))$. So we get a well-defined operator $H^1(G, l^p(G)) \to W_p^{\frac{n-1}{p}}/const$. We claim it is bounded. An affine flat bundle E has been defined as $\widetilde{M} \underset{G}{\times} l^p(G)$, where G acts on $l^p(G)$ by $v \mapsto L_g v + l(g)$. It is enough to show, that there is a constant C, depending only on G but not on l, such that E possesses a Lipschitz section s with $\|\nabla s\| < C \cdot \|l\|$, where $\|l\| = \sup_i \|l(g_i)\|$ for a choice of generators $g_i, i = 1, \cdots, m$. We note that l effectively controls the monodromy of the flat connection in E. A construction of s mentioned above, that is, a choice of an open covering $\cup U_\alpha = M$, flat sections s_α over U_α, a partition of unity $\sum f_\alpha = 1$ with $supp f_\alpha \subset U_\alpha$, so that $s = \sum f_\alpha s_\alpha$, gives a bound of $|\nabla s|$ in terms of monodromy, as desired.

We note that by [Goldštein-Kuzminov-Shvedov 1], $H^1(G, l^p(G)) = L^p H^1(\mathcal{H}^n)$, so to any class in $H^1(G, l^p(G))$ we have associated a function f such that df is in L^p, or, equivalently, $\int_{\mathcal{H}^n} |\nabla f|_h^p \mu_h < \infty$. What we in fact did above was an explicit construction of this correspondence between l^p- and L^p-cohomology.

So far we have constructed a bounded operator $H^1(G, l^p(G)) \to W_p^{\frac{n-1}{p}}(S^{n-1})$, $p > n-1$. We wish to show that this operator is in fact an isomorphism of Banach spaces. To this end, we will need a Poincaré inequality in hyperbolic space.

Proposition 4.5 (Poincaré inequality in $\mathcal{H}^n$).— Let f be a locally integrable measurable function with $\int_{\mathcal{H}^n} |\nabla f|_h^p \, d\mu_h < \infty$. Then
1)If $p \leq n-1$, then $\int_{\mathcal{H}^n} |f-c|^p \, d\mu_h < \infty$ for some constant c;
2)If $p > n-1$ and $f|_{S^{n-1}}$ as an element of $W_p^{\frac{n-1}{p}}(S^{n-1})$ is zero, then $\int_{\mathcal{H}^n} |f|^p \, d\mu_h < \infty$.

Proof.— A much more general theorem is contained in [Strichartz 1].

We now claim that $H^1(G, l^p(G)) = 0$ for $p \leq n-1$. This in fact follows immediately from $H^1(G, l^p(G)) = L^p H^1(\mathcal{H}^n)$ [Goldštein-Kuzminov-Shvedov 1] and Proposition 4.5. Now, if $p > n-1$, then we claim that the operator $H^1(G, l^p(G)) \to W_p^{\frac{n-1}{p}}(S^{n-1})/const$ constructed above is injective. In fact, if $f|_{S^{n-1}} = 0$, then by Proposition 4.5, $f \in L^p(\mathcal{H}^n, \mu_h)$, so $\int_M \Sigma_g |f(gx)|^p \, d\mu_h < \infty$, so for almost all $x \in \tilde{M}$, $\sum_g |f(gx)|^p < \infty$. But $f(gx) - \mathcal{F}(g) \in l^p(G)$, so $\mathcal{F} \in l^p(G)$ and $[l] = 0$. Now, if $h \in W_p^{\frac{n-1}{p}}(S^{n-1})$, we denote by H its harmonic extension into B^n. Then [Uspenski 1], [Lizorkin 2], $\int_{B^n} \rho^{p-n} |\nabla H|^p \, d\mu_e < \|h\|_{W_p^{\frac{n-1}{p}}(S^{n-1})}$, so dH is an L^p 1-form on $\mathcal{H}^n$. This shows that the injective operator $H^1(G, l^p(G)) = L^p H^1(\mathcal{H}^n) \to W_p^{\frac{n-1}{p}}(S^{n-1})/const$ has a bounded right inverse, so it is an isomorphism by Banach theorem. This proves Theorem 4.1.

Corollary 4.5.— Let G be a cocompact lattice in $SO^+(1,n)$ and let $\mathcal{F} : G \to \mathbb{R}$ be such that $L_g\mathcal{F} - \mathcal{F} \in l^p(G)$, for all $g \in G \quad (p > n-1)$. Then the limit function $u = \mathcal{F}|_{S^{n-1}}$ belongs to $L^q(S^{n-1})$ for all $q > 1$. In fact,

$$\sup_{1<q<\infty} \left(\frac{n-1}{q} \right)^{1/q'} \|u\|_{L^q(S^{n-1})} < \infty$$

Moreover, u is in the linear hull of all functions f satisfying

$$\int_{S^{n-1}} exp(|f|^{p'}) < \infty.$$

Proof. is an immediate corollary of Theorem 4.1 and the properties of the Orlicz space $L_\infty(\log L)_{-a}$ and the fact that $W_p^{(n-1)/p}(S^{n-1}) \subset L_\infty(\log L)_{-a}(S^{n-1})$ for $a \geq 1/p'$, (see [Edmunds-Triebel 1]).

We will use this corollary in a sequel to this paper [Reznikov 10] in analyzing the local behaviour of the Cannon-Thurston Peano curves, corresponding to fibers of the hyperbolic 3-manifolds, fibered over the circle.

1.5 Growth of primitives for l^p-cocycles on the surface group

Theorem 5.1.— Let G be a cocompact lattice in $SO(2,1)$ and let $\mathcal{F} : G \to \mathbb{R}$ be such that for all $g \in G$, $L_g\mathcal{F} - \mathcal{F} \in l^p(G)$, $p > 1$. Then for any word length on G,

$$|\mathcal{F}(g)| \leq const \cdot [length(g)]^{1/p'}.$$

Proof follows from Theorem 4.1 and a following lemma.

Lemma 5.2.— Let u be a harmonic function in the unit disc such that $u|_{S^1} \in W_p^{1/p}(S^1)$. Then

$$|u(z)| \leq const \cdot [\log(1 - |z|)]^{1/p'}.$$

Proof of the lemma.— Here we only treat the case $p = 2$. The full proof will be given in Section 11. Let $u(e^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$. Since $(1 + \triangle)^{1/4}u \in L^2$, we have $\{|n|^{1/2}a_n\} \in l^2(\mathbb{Z})$, therefore for $|z| < 1 \quad (b_n = |a_n| + |a_{-n}|)$.

$$\begin{aligned} u(z) - a_0 \quad &\leq \textstyle\sum_{n>0}(|a_n| + |a_{-n}|)|z|^n \\ &= \textstyle\sum |n|^{1/2} b_n \cdot \frac{1}{|n|^{\frac{1}{2}}}|z|^n \\ &\leq (\textstyle\sum |n| b_n^2)^{1/2} \cdot (\sum \frac{1}{|n|}|z|^{2n})^{1/2} \\ &\leq const \cdot [\log(1 - |z|)]^{1/2}. \end{aligned}$$

1.6 Embedding of negatively curved manifolds and the boundaries of their universal covers

A problem of fundamental importance in topology is the following: let $M^m \overset{\varphi}{\hookrightarrow} N^n$ be a smooth π_1-injective embedding of manifolds of nonpositive curvature. Let $\widetilde{\varphi} : \widetilde{M} \to \widetilde{N}$ be a lift of $\widetilde{\varphi}$. Is there a limit map $S^{m-1} \approx \partial\widetilde{M} \overset{\partial\widetilde{\varphi}}{\to} \partial\widetilde{N} \approx S^{n-1}$ and if there is, how smooth it is? For instance, let N^3 be a compact hyperbolic 3-manifold, and M^2 be an incompressible embedded surface in N^3. Then there always exists a limit continuous map $S^1 \overset{\partial\widetilde{\varphi}}{\to} S^2$. Moreover, if M is not a virtual fiber of a fibration over the circle, then $\partial\widetilde{\varphi}(S^1)$ is a quasifuchsian Jordan curve. If M is a virtual fiber, then $\partial\widetilde{\varphi} : S^1 \to S^2$ is a Peano curve in a sense that its image fills S^2 [Cannon-Thurston 1]. This deep dichotomy follows from the result of [Bonahon 1]. We have a following very general theorem 9the embedding condition is superfluous but makes the proof more transparent):

Theorem 6.1.— Let $M^m \overset{\varphi}{\hookrightarrow} N^n$ be a smooth π_1-injective embedding of complete Riemannian manifolds, of pinched negative curvature. Suppose M is compact. Let $\widetilde{\varphi} : \widetilde{M} \to \widetilde{N}$ be a lift of $\widetilde{\varphi}$. Let $p_0 \in \widetilde{N}$ and $\pi : \widetilde{N}\backslash\{p_0\} \to S^{n-1}(T_{p_0}\widetilde{N})$ be a radial geodesic projection of $\tilde{N}\backslash\{p_0\}$ onto the unit tangent sphere. Identify $T_{p_0}\widetilde{N}$ with $\mathbb{R}^n$. Let $q_0 \in \widetilde{M}$. Then:

1) For almost all unit tangent vectors $v \in T_{q_0}(\widetilde{M})$, the restriction of $\pi\widetilde{\varphi}$ on a geodesic $\gamma(q_0, v)$ starting at q_0 with a tangent vector v has an L^1-derivative as a map $\widetilde{\varphi}|_{\gamma(q_0,v)} : \mathbb{R}_+ \to \mathbb{R}^n$.

2) For almost all v there exists a limit $\lim_{t\to\infty} \pi\widetilde{\varphi}[\gamma(q_0, v)(t)]$.

3) The resulting measurable map $\partial\widetilde{M} \approx S^{m-1} \overset{\partial\widetilde{\varphi}}{\to} S^{n-1} \approx \partial\widetilde{N}$ does not depend on the choice of p_0, q_0.

4) If both M,N are (real) hyperbolic, then for any $p > n-1$, $\partial\widetilde{\varphi}$ induces a bounded linear operator

$$\partial\widetilde{\varphi}_* : W_p^{\frac{n-1}{p}}(S^{n-1}) \to W_p^{\frac{m-1}{p}}(S^{m-1}).$$

5) If M is hyperbolic and $-K \leq K(N) \leq -1$, then for $p > (n-1)\sqrt{K}$, $\partial\widetilde{\varphi}$

induces a bounded linear operator

$$\partial\widetilde{\varphi}_* : C^\infty(S^{n-1}) \to W_p^{\frac{m-1}{p}}(S^{m-1})$$

for $p > (n-1)\sqrt{K}$.

Theorem 6.2.— Let N^3 be a compact oriented hyperbolic three-manifold, let $M^2 \overset{\varphi}{\hookrightarrow} N^3$ be an incompressible immersed surface, and let x_1, x_2, x_3 be Euclidean coordinates on $S^2 \approx \partial N^3$. Then
1) If $\partial\widetilde{\varphi}$ is quasifuchsian, then $x_i \circ \partial\widetilde{\varphi} : S^1 \to \mathbb{R}$ are in $W_p^{1/p}$ for $p \geq 2$.
2) If M^2 is a virtual fiber then $x_i \circ \partial\widetilde{\varphi} : S^1 \to \mathbb{R}$ are in $W_p^{1/p}$ for $p > 2$ (but probably not in $W_2^{1/2}$).

Proof of the Theorem 6.1.— We will assume $-k \leq K(M) \leq -1, -K \leq K(N) \leq -1$. For $x \in \widetilde{N}$ let $r(x) = \rho(p_0, x)$.

Lemma 6.3.— For $r_0 > 0$ and $r(x) > r_0$, $|\nabla\pi(x)| \leq const(r_0)e^{-r(x)}$, where we view π as a map $N\backslash\{p_0\} \to \mathbb{R}^n$.

Proof is an immediate application of the comparison theorem, mentioned above in the proof of Proposition 2.1.

Lemma 6.4.—

$$\int_{\widetilde{N}\backslash B(p_0,r_0)} |\nabla\pi(x)|^p < \infty \text{ for } p > (n-1)\sqrt{K}.$$

Proof repeats the argument in the proof of Proposition 2.1.

Now consider a tubular neighbourhood of M in N. There exists an embedding of $M \times I \to N$ where $I = [-1, 1]$. Moreover, the restriction of the metric g_N of N onto $M \times I$ is equivalent to the product metric $g_M + dx^2$ (we say two metrics are equivalent if each one is bounded above by another one times a constant). It follows that there is an embedding

$$\widetilde{M} \times I \xrightarrow{\Phi} \widetilde{N}$$

such that $g_{\widetilde{N}}|\widetilde{M}\times I$ is equivalent to $g_{\widetilde{M}}+dx^2$. Since φ is π_1-injective, for any $r_0>0$ there is $r_1>0$ such that if $\rho_M(q_0,z)>r_1$, then $\rho_N(p_0,\Phi(z,t))>r_0$ for $t\in[-1,1]$. It follows that

$$\int\int_{\widetilde{M}\backslash B(q_0,r_1)\times I}|\nabla\pi\circ\Phi|^p\ dVol(\widetilde{M})dt<\infty$$

Therefore for almost all $t_0\in I$,

$$\int_{\widetilde{M}\backslash B(q_0,r_1)}|\nabla(\pi\circ\Phi(z,t_0))|^p\ dVol(\widetilde{M})<\infty$$

Fix such t_0 and let $f=\pi\circ\Phi(z,t_0):\widetilde{M}\backslash B(q_0,r_1)\to\mathbb{R}^n$. We know that

$$\int_{\widetilde{M}\backslash B(q_0,r_1)}|\nabla f|^p\ dVol(\widetilde{M})<\infty.$$

Expressing the integral in polar coordinates and taking into account that $K(M)\leq -1$ we have

$$\int_{S^{m-1}(T_{q_0}\widetilde{M})}dv\int_{r_1}^{\infty}e^{(m-1)t}|\nabla f|^p\ dt<\infty.$$

In particular, for almost all $v\in S^{m-1}(T_{q_0}\widetilde{M})$,

$$\int_{r_1}^{\infty}e^{(m-1)t}|\frac{\partial f}{\partial t}|^p\ dt<\infty.$$

In other words, for such v, $|\frac{\partial f}{\partial t}|\cdot e^{\frac{(m-1)}{p}t}\in L^p[r_1,\infty]$, therefore $|\frac{\partial f}{\partial t}|\in L^1[r_1,\infty]$, since $e^{-\frac{m-1}{p}t}\in L^{p'}[r_1,\infty]$. This proves 1). The statements 2) and 3) follow directly.

Now suppose $K(M)=K(N)=-1$. Let $u\in W_p^{\frac{n-1}{p}}(S^{n-1})$, $p>n-1$. Then a harmonic extension g of u satisfies

$$\int_{\widetilde{N}}|\nabla g|^p<\infty$$

as we know from [Lizorkin 1], [Uspenski 1] and the proof of Theorem 5.1. By the argument above, there is a $t_0\in I$, such that the composite function $g\circ\Phi(z,t_0)$ satisfies

$$\int_{\widetilde{M}}|\nabla(g\circ\Phi(z,t_0)|^p<\infty$$

But then the trace $g \circ \Phi(z, t_0)|_{\partial \widetilde{M}}$ lies in $W_p^{\frac{m-1}{p}}(S^{m-1})$. This proves part 4) of Theorem 6.1. A proof of part 5) is identical. The Theorems 6.1 and 6.2,2) are proved. To prove Theorem 6.2, 1) we notice that a restriction of any function $u \in W_2^1(S^2)$ on a quasicircle belongs to the class $W_2^{\frac{1}{2}}(S^1)$. This follows immediately from the invariance of $W_2^1(S^2)$ under quasiconformal homeomorphisms, and a fact that functions from $W_2^1(B^2)$ have traces in $W_2^{\frac{1}{2}}(S^1)$(notice that the Dirichlet energy of a function of two variables is an invariant of the conformal class of a metric).

As the reader has noticed, we could assume $\pi_1(M) = \pi_1(N)$, so that $\pi_1(M)$ acts discretely in $\widetilde{N}$ and $N = \widetilde{N}/\pi_1(M)$. On the other hand, the proof does not use the fact that M is embedded, so the Theorem 6.1 stays true for (finite-to-one)immersions in N.

We will outline now, having in mind the applications in the sequel to this paper, how to study the limit maps from the point of view of ergodic theory. The results thus obtained are weaker then those proved above, but apply to non-discrete representations. Our treatment can be seen as a development of a vague remark of [Thurston 1, 6.4.4]. Let M^m be a compact hyperbolic mainfold, $\widetilde{N} = \mathcal{H}^n$ and $\rho : \pi_1(M) \to Iso(\widetilde{N})$ a discrete faithful representation. Let $N = \widetilde{N}/\rho(\pi_1(M))$. We would like to study a boundary map $\partial\widetilde{\varphi} : \widetilde{M} \to \widetilde{N}$ where φ is a smooth map $M \to N$, inducing ρ.

Lemma 6.5.— There exists a $\pi_1(M)$ equivariant measurable map ψ from $\partial\widetilde{M} = S^{m-1}$ to the space of probability measures on $\partial\widetilde{N} = S^{n-1}$.

Proof.— For any compact Riemannian manifold M, any compact metric space X and any representation $\rho : \pi_1(M) \to Homeo(X)$, there is a $\pi_1(M)$-equivariant harmonic function from $\widetilde{M}$ to the affine space of charges on X, taking values in probability measures. This simple fact is various degrees of generality has been shown in [Furstenberg 1], [L.Garnett 1], [Kaimanovich-Vershik 1]. If M is hyperbolic, then the Poisson boundary of $\widetilde{M}$ is $\partial\widetilde{M}$, and the result follows.

Now let $\psi_0 + \psi_c$ be the decomposition of ψ into atomic and non-atomic parts. Obviously, ψ_c is also $\pi_1(M)$-equivariant. We claim $\psi_c = 0$. First, $\int \psi_c$ is a $\pi_1(M)$-invariant function on $\partial\widetilde{M} = S^{n-1}$, whence a constant, since $\pi_1(M)$ acts on S^{n-1} ergodically. So if $\psi_c \neq 0$ we may assume ψ_c is a probability measure. Second, let G be a center of gravity map from the nonatomic measures on $\partial\widetilde{N}$ to N [Furstenberg 2]. Then $G \circ \phi_c$ is a $\pi_1(M)$-equivariant map from $\partial\widetilde{M}$ to N. In particular, $\rho(G \circ \psi_c(x), G \circ \psi_c(y))$ is a $\pi_1(M)$-invariant function on $\partial\widetilde{M} \times \partial\widetilde{M}$ whence a constant by [Hopf 1] and [Sullivan 3]. It follows easily that $G \circ \psi_c = \mathit{const}$ which is impossible since ρ is discrete. So $\psi_c = 0$.

We deduce that ψ is atomic, $\psi(z) = \sum_{i=1}^{\infty} m_i \delta(\psi_i(z)), m_1 \geq m_2 \geq \cdots$. Though $\psi_i(z) : \partial\widetilde{M} \to \partial\widetilde{N}$ are not uniquely defined, $m_i : \partial\widetilde{M} \to \mathbb{R}$ are. It follows that m_i are $\pi_1(M)$-invariant, whence constant. Since $\sum m_i = 1$, there is some i such that $m_{i+1} < m_1$. Choose first such i. Then $m_1 = \cdots = m_i$ and we get a measurable equivariant map

$$\partial\widetilde{M} = S^{m-1} \to \underbrace{S^{n-1} \times \cdots \times S^{n-1}}_{i} / S_i,$$

where S_i is the symmetric group in i letters.

So far we did not use the fact that ρ is discrete, but only that $\rho(\pi_1(M))$ does not have fixed points in $\widetilde{N} = \mathcal{H}^m$. So:

Propostion 6.6.— Let M^m be a compact hyperbolic manifold and let $\rho : \pi_1(M) \to SO^+(1,n)$ be such that $\rho(\pi_1(M))$ does not have fixed points in $\mathcal{H}^n$. Then there exists a $\pi_1(M)$-equivariant measurable map

$$S^{m-1} = \partial\widetilde{M} \xrightarrow{\psi} (\textit{subsets of cardinality } i \textit{ of } S^{n-1} = \partial\mathcal{H}^n)$$

for some $i \geq 1$.

Using cross-ratios and the ergodicity of the action of $\pi_1(M)$ on $\partial\widetilde{M} \times \partial\widetilde{M}$, one can easily show $i = 1$. Now to any $x \in \widetilde{M}$ one associates a Poisson measure μ_x on S^{m-1}. Its pushforward $\psi_* \mu_x$ is a probability measure on

S^{n-1}. The pushforward of a measure by a multivalued map is defined by

$$\int_{S^{n-1}} f \ d[\psi_* \mu] = \int_{S^{m-1}} \sum_{y \in \psi(x)} f(y) \ d\mu,$$

where $f \in C(S^{n-1})$.

Now under some natural conditions $\psi_* \mu_x$ does not have atoms and using the baricenter map G in $\mathcal{H}^n$ one can define $s(x) = G(\psi_* \mu_x)$. This can easily be shown to be continuous equivariant map $\widetilde{M} \xrightarrow{s} \mathcal{H}^n$, again under some natural assumption on ρ. The multivalued map ψ should be regarded as a weak radial limit of s, but we will not pursue this point any further.

1.7 The action of quasisymmetric and quasiconformal homeomorphisms on $W_p^{(n-1)/p}$

A well known result [Reimann 1] characterizes quasiconformal maps between domains D_1, D_2 in $\mathbb{R}^n, n > 2$ as those which induce an isomorphism of Banach spaces $BMO(D_1)$ and $BMO(D_2)$. We will see now that this result in case $D_1 = D_2 = \mathbb{R}^n$ is a limit as $p \to \infty$ of the following result which establishes a quasiconformal invariance of fractional Sobolev spaces $W_p^{n/p}$. Of special importance is the fact that the result holds for $n = 1$ and quasisymmetric homeomorphisms of S^1. The proof of the following lemma is "almost" contained in remarks made in [Pansu 1–3].

Lemma 7.1.— Let $\mathcal{G}_{n-1}, n \geq 2$ be a group of quasisymmetric $(n = 2)$ or quasiconformal $(n \geq 3)$ homeomorphisms of S^{n-1}. Then for any $p > 1 \quad (n = 2)$ or $p \geq n-1 \quad (n \geq 3)$, $\mathcal{G}_{n-1}$ leaves invariant a Sobolev-Slobodečki space $W_p^{n-1/p}(S^{n-1})$. For any $\Phi \in \mathcal{G}_{n-1}$, the corresponding map

$$\Phi_* : W_p^{n-1/p}(S^n) \to W_p^{n-1/p}(S^{n-1})$$

is an automorphism of the Banach space $W_p^{n-1/p}(S^{n-1})$.

Theorem 7.2.— There exists for any $n \geq 2$ a bounded antisymmetric polylinear map

$$\underbrace{W_n^{\frac{n-1}{n}}(S^{n-1})/const \times \cdots \times W_n^{\frac{n-1}{n}}(S^{n-1})/const}_{n} \to \mathbb{R}\ ,$$

defined on the smooth functions by $f_1, \cdots, f_n \to \int_{S^n} f_1\ df_2 \cdots df_n$, which is invariant under $\mathcal{G}_{n-1}$.

In particular, we have

Corollary 7.3.— There exists a representation

$$\mathcal{G}_1 \to Sp(W_2^{1/2}(S^1)/const),$$

defined by $\Phi(f) = f \circ \Phi^{-1}$.

Proof of the Lemma 7.1.— We will need a result, proved for $n = 2$ in [Ahlfors-Beurling 1] for $n = 3$ in [Carleson 1] and for $n \geq 4$ in [Tukia-Väisälä 1]:

Theorem.— Let $\phi : S^{n-1} \to S^{n-1}$ be quasisymmetric ($n = 2$) or quasiconformal ($n \geq 3$). Then there exists an extension $\widetilde{\phi}$ of ϕ as a homeomorphism of B^n, which is a quasiisometry of the hyperbolic metric:

$$const_2 \cdot g_h \leq \widetilde{\phi}_* g_h \leq const_1 \cdot g_h.$$

Now let $f \in W_p^{\frac{n-1}{p}}$ $(p > n-1)$. Let u be a harmonic function in B^n, extending f. We know that

$$\int |\nabla u|_h^p\ d\mu_h \leq const_3 \|f\|_{W_p^{\frac{n-1}{p}}(S^{n-1})}.$$

It follows that

$$\int |\nabla(u \circ \widetilde{\phi})|_h^p\ d\mu_h \leq const_4 \|f\|_{W_p^{\frac{n-1}{p}}} < \infty,$$

and by the trace theorem,

$$\|u \circ \widetilde{\phi}\|_{W_p^{\frac{n-1}{p}}} \leq const_5 \|f\|_{W_p^{\frac{n-1}{p}}},$$

which proves the theorem for $p > n-1$. For $p = n-1, n \geq 3$, the result is standard.

Proof of the Theorem 7.2.— Let $f_1, \cdots, f_n \in W_n^{\frac{n-1}{n}}(S^{n-1})$. Let u_i be a harmonic extension of f_i. The result follows at once from the formula

$$\int_{S^{n-1}} f_1 \, df_2 \cdots df_n = \int_{B^n} du_1 du_2 \cdots du_n$$

. Since $\int |\nabla u_i|_h^n \, du_h < \infty$, the integral $\int_{B^n} \, du_1 \cdots du_n$ is finite by Hölder inequality. The invariance is obvious.

Proof of the Corollary 7.3.— A formula $< f_1, f_2 >= \int_{S^1} f_1 \, df_2$ gives $W_2^{1/2}/const$ a structure of a symplectic Hilbert space. This means that a map

$$W_2^{1/2}/const \to (W_2^{1/2}/const)^*$$

given by $f \to < f, \cdot >$ is an isomorphism (not isometry) of Hilbert spaces. By $Sp(W_2^{1/2}/const)$ we mean a group of invertible bounded operators which leaves this symplectic form invariant. The result now follows from Lemma 7.1 and Theorem 7.2 .

1.8 Boundary values of quasiconformal maps and regularity of quasisymmetric homeomorphisms

Proposition 8.1.— Let ϕ be a quasiconformal map, defined in a neighborhood of the unit ball B^n. Then $\phi|_{S^n}$ as a map $S^n \to \mathbb{R}^n$ belongs to a class $W_n^{\frac{n-1}{n}+\delta}$ for some $\delta > 0$. In particular if $n = 2$ and $\phi(e^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$ then $\sum |n|^{1+\delta}|a_n|^2 < \infty$. If ϕ is just defined in B^n then for almost all $\alpha \in S^n$ there exists a limit $\lim_{r\to 1} \phi(rx)$ and $\phi|_{S^{n-1}} \in W^{\frac{n-1}{n}}$. In particular, for $n = 2$ and $\phi(e^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$, $\sum |n||a_n|^2 < \infty$.

Remark.— The last statement for conformal maps is the "Flachensatz".

Proof.— Since ϕ as a map $B^n \to \mathbb{R}^n$ belongs to W_n^1, the last statement follows immediatedly from the trace theorem. To prove the first, recall that ϕ is locally in $W^1_{n+\delta'}, \delta' > 0$ [Bojarski 1], [Gehring 2]. Therefore $\phi|_{S^{n-1}} \in W_n^{\frac{n-1}{n}+\delta}$, again by the trace theorem.

Theorem 8.2.— Let $\varphi : S^1 \to S^1$ be a quasisymmetric homeomorphism. Then as a map $S^1 \to \mathbb{R}^2, \varphi \in W_p^{1/p+\delta(p)}, \delta(p) > 0$, for all $p > 1$. If $\varphi(e^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$, then $\sum_{n\in\mathbb{Z}} |n|^{p'/p+\delta} |a_n|^{p'} < \infty$ for all $1 < p \leq 2$.

Proof.— Let $\Phi : D^2 \to D^2$ be a quasiisometry of the hyperbolic plane, extending φ. We know that Φ, Φ^{-1} are Hölder in Euclidean metric. Let f be a smooth function defined in a neighbourhood of D^2. Then for $p > 1$

$$\int_{D^2} |\nabla f|_h^p \rho_e^{-\epsilon}(x, \partial D^2) \cdot \ d\mu_h < \infty$$

for $\epsilon > 0$ small enough (one needs $\epsilon < p - 1$).

Since Φ is a quasiisometry for the hyperbolic metric and biHölder for the Euclidean metric, we have for $g = f \circ \Phi$:

$$\int_{D^2} |\nabla g|_h^p \rho_e^{-\beta}(y, \partial D^2) \ d\mu_h < \infty$$

for some $\beta > 0$. Rewriting in Euclidean terms, we have

$$\int_{D^2} |\nabla g|_e^p \cdot [\rho(y, \partial D^2)]^{p-\beta-2} < \infty,$$

therefore $g|_{S^1} \in W_p^{\frac{1}{p}+\delta}$ by the trace theorem for weighted Sobolev spaces. Letting f be an Euclidean coordinate function, we get $\varphi \in W_p^{\frac{1}{p}+\delta}$. The last statement follows from Young-Hausdorff theorem.

Remark 8.3.— It had been a famous problem in fifties if φ is absolutely continuous (that is, in W_1^1). Though the answer is well-known to be negative, we see that φ is as close to be absolutely continuous as one wishes. We will use Theorem 8.2 in a sequel to this paper to prove the existence of the vacuum vector for quantized moduli space for $p > 1$. We also notice that the argument above together with the proof of Theorem 6.1 shows the following: if $\varphi : M \to N$ is an π_1-injective immersion of hyperbolic manifolds, M compact, such that for $g \in \pi_1(M)$ and some fixed $z_0 \in \tilde{N}$

$$\rho(z_0, \varphi_*(g)z_0) \geq const \cdot length(g),$$

then $\partial\tilde{\varphi}$ is of class $W_p^{(m-1)/p+\delta}$ and therefore Hölder continuous. It is not enough, though, to prove a continuity if the Cannon-Thurston curve. See [Reznikov 10] for futher study.

1.9 Teichmüller spaces and quantization of the mapping class group, I

We denote $\mathcal{M}ap_g$ the mapping class group of genus g and $\mathcal{M}ap_{g,1}$ the extended mapping class group. If Σ^g is a closed oriented surface of genus g, $\Gamma_g = \pi_1(\Sigma^g)$ then $\mathcal{M}ap_{g,1} = Aut(\Gamma_g)$ and one has an exact sequence

$$1 \to \Gamma_g \to \mathcal{M}ap_{g,1} \to \mathcal{M}ap_g \to 1.$$

Proposition 9.1(Quantization of the moduli space)— For any $p > 1$ there exists a representation

$$\mathcal{M}ap_{g,1} \xrightarrow{\pi_p} Aut(W_p^{1/p}(S^1)/const)$$

given by the formula

$$\pi_p(\varphi)(f) = f \circ \Phi^{-1},$$

where Φ is a quasisymmetric homeomorphism of S^1, induced by φ and a choice of a hyperbolic structure in Σ^g. For $p = 2$ the representation

$$\pi_2 : \mathcal{M}ap_{g,1} \to Aut(W_2^{1/2}(S^1)/const)$$

is symplectic, that is, $\pi_2(\mathcal{M}ap_{g,1}) \subset Sp(W_2^{1/2}(S^1)/const)$.

Proof.— Fix a hyperbolic structure on Σ. Then by a classical theorem of Nielsen, one gets a representation $\mathcal{M}ap_{g,1} \to \mathcal{G}_1$. The theorem now follows from Theorem 7.1.

Now let $G \xrightarrow{\pi_0} PSL_2(\mathbb{R})$ be a Fuchsian group, possibly infinitely generated. We recall that a Teichmüller space $\mathbf{T}(G)$ is defined as follows : points of $\mathbf{T}(G)$ are discrete representation $G \xrightarrow{\pi} PSL_2(\mathbb{R})$, defined up to conjugation by an element of $PSL_2(\mathbb{R})$, which are quasiconformally conjugate to π_0, that is, there is a quasisymmetric homeomorphism Φ of S^1 such that $\pi = \Phi \circ \pi_0 \circ \Phi^{-1}$. Notice that this definition is equivalent to the standard one by a result of [Douady-Earle 1].

Corollary 9.2.— Let π_0, π be two discrete representation of a group G. Then if π lies in the Teichmüller space of π_0, then the unitary representations

$$G \overset{\pi_0}{\to} PSL_2(\mathbb{R}) \overset{\beta}{\to} U(W_2^{1/2}(S^1)/const)$$

and

$$G \overset{\pi}{\to} PSL_2(\mathbb{R}) \overset{\beta}{\to} U(W_2^{1/2}(S^1)/const)$$

are unitarily equivalent.

Remark 1.— The fact that $PSL_2(\mathbb{R})$ acts in $W_2^{1/2}(S^1)/const$ by unitary operators (with respect to the complex structure given by the Hilbert transform) is well-known [Nag 1]. In fact, this unitary representation belongs to the discrete series and may be realized in L^2 holomorphic 1-forms in B^2.

Proof.— Since $\pi = \Phi \circ \pi_0 \circ \Phi^{-1}$ and $\mathcal{G}_1$ act in $W_2^{1/2}(S^1)/const$, we get an invertible operator A such that $\beta \circ \pi = A\, \beta \circ \pi_0\, A^{-1}$. By polar decomposition $A = UP$ where P is positive self-adjoint, U is unitary, P commutes with $\beta \circ \pi_0$ and U intertwines $\beta \circ \pi_0$ and $\beta \circ \pi$, as desired.

The following special case is very important. Let $\pi_0 : G \to PSL_2(\mathbb{R})$ be a Fuchsian group corresponding to a Riemann surface of finite type (that is, a torsion-free lattice in $PSL_2(\mathbb{R})$). Let $\Sigma = \mathcal{H}^2/G$ and let $\varphi \in \mathcal{M}ap(\Sigma, x_0), x_0 \in \Sigma$. Let Φ be a quasisymmetric homeomorphism of S^1 which is the boundary value of a quasiconformal homeomorphism Ψ of (Σ, x_0), representing φ. Then

$$\pi_0 \circ \varphi^{-1} = \Phi \pi_0 \Phi^{-1}.$$

Let $A_\varphi : W_2^{1/2}/const \to W_2^{1/2}/const$ be an invertible operator, representing φ. Let $P_\varphi^2 = A_\varphi^* A_\varphi$.Then P_φ commutes with $\beta \circ \pi_0$. We obtained the following

Theorem 9.2.— Let $\pi_0 : G \to PSL_2(\mathbb{R})$ be a torsion-free lattice. Let $\Sigma = \mathcal{H}^2/G, x_0 \in \Sigma, \varphi \in \mathcal{M}ap(\Sigma, x_0)$, Ψ a quasiconformal homeomorphism inducing φ, Φ the trace of its lift to $\mathcal{H}^2$ on S^1, A_φ an invertible opreator in $W_2^{1/2}(S^1)/const$ given by $A_\varphi(f) = f \circ \Phi^{-1}$. Then a self-adjoint bounded operator

$$P_\varphi^2 = A_\varphi^* A_\varphi$$

commutes with $\beta \circ \pi_0$. If $P_\varphi^2 = \int \lambda \, dE_\lambda$ is the spectral decomposition then E_λ commute with $\beta \circ \pi_0$.

Remarks.
1) If G is cocompact, then we know that $W_2^{1/2}/const \approx H^1(G, l^2(G))$, so $W_2^{1/2}/const$ is a Hilbert module over the type II factor defined by G of dimension $\dim_G W_2^{1/2}/const = L^2 b_1(G) = 2g - 2$.
2) In practice, finding A_φ is difficult. The reason is that Φ is not a diffeomorphism, so the explicit formulae of Chapter 2 do not make sense. Moreover, Φ is given in a very implicit way as a boundary value of a quasiconformal map, defined by a quadratic differential on $\mathcal{H}^2$ which is G-invariant!

We will now show that for $p > 2$ the operator A_φ shows very unusual properties, from the point of view of functional analysis.

Theorem 9.3.— Let G be a fundamental group of a closed hyperbolic surface Σ^g. Let $\varphi \in \mathcal{M}ap(\Sigma, x_0)$ be such that its image in $\mathcal{M}ap(\Sigma)$ is pseudo-Anosov. Let A be the operator, representing φ in $W_p^{1/p}(S^1)$, $p > 2$. Then there is an element $0 \neq v \in W_p^{1/p}(S^1)$ such that

$$\sum_{k \in \mathbb{Z}} \|A_\varphi^k(v)\|^p < \infty.$$

Proof.— Let M be a mapping torus of Ψ, that is , $\mathbb{R} \times \Sigma/\mathbb{Z}$ where $1 \in \mathbb{Z}$ acts by $(t, x) \to (t+1, \Psi(x))$. Then M is hyperbolic [Thurston 2]. We will view

M as a fibration over a circle $\mathbb{R}/\mathbb{Z}$ with coordinate t, $0 \le t < 1$; the fiber over t will be called Σ_t. We can trivialize $M \xrightarrow{\psi} \mathbb{R}/\mathbb{Z}$ over $I = [0, 1/2]$ so that (t, x_0), $0 \le t \le 1/2$ will be a horizontal curve. Let g be the hyperbolic metric on M and g_0 be some hyperbolic metric on Σ, then g and $g_0 + dt^2$ are equivalent on $\Sigma \times [0, 1/2] \simeq \psi^{-1}([0, 1/2]) \subset M$. Lifting to $\widetilde{M} = \mathcal{H}^3$, we get a fibration $\mathcal{H}^3 \xrightarrow{\widetilde{\psi}} \mathbb{R}$ with $\widetilde{\psi}^{-1}(t) = \widetilde{\Sigma}_t$. Let $G \in PSL_2(\mathbb{C})$ be the mondromy element, corresponding to φ. Let $f : \mathcal{H}^3 \to \mathbb{R}$ be such that

$$\int_{\mathcal{H}^3} |\nabla f|^p \, d\mu_h < \infty.$$

We then have

$$\sum_{k \in \mathbb{Z}} \int_{G^k(\widetilde{\psi}^{-1}[0,1/2])} |\nabla f|^p \, d\mu_h \le \int_{\mathcal{H}^3} |\nabla f|^p \, d\mu_h < \infty;$$

on the other hand the left hand side is

$$\sum_{k \in \mathbb{Z}} \int_{\widetilde{\psi}^{-1}[0,1/2]} |\nabla (f \circ G^k)|^p \, d\mu_h \ge const \cdot \int_0^{\frac{1}{2}} dt \int_{\widetilde{\Sigma}_t} \sum_{k \in \mathbb{Z}} |\nabla (f \circ G^k)|^p \, dVol(g_0).$$

It follows that for some t_0,

$$\int_{\widetilde{\Sigma}_{t_0}} \sum_{k \in \mathbb{Z}} |\nabla (f \circ G^k)|^p \, dVol(g_0) < \infty.$$

Since g_0 is a hyperbolic metric, for any function F on $\widetilde{\Sigma}$

$$\int_{\widetilde{\Sigma}} |\nabla F|^p \, dVol(g_0) = const \cdot \|F|\partial \widetilde{\Sigma}\|^p_{W_p^{1/p}(S^1)/const},$$

actually, we may let the LHS be a definition of the norm in $W_p^{1/p}(S^1)/const$, making the constant equal one. So

$$\sum_{k \in \mathbb{Z}} \|f \circ G^k|\partial \widetilde{\Sigma}_{t_0}\|^p_{W_p^{1/p}(S^1)/const} < \infty.$$

We will now identify $f \circ G^k|\partial \widetilde{\Sigma}_{t_0}$. We have a boundary map

$$\partial \widetilde{\Sigma}_{t_0} = S^1 \xrightarrow{\alpha} S^2 = \partial \mathcal{H}^3.$$

We know that $G^k \circ \alpha = \alpha \circ \varphi^{-k}$, so $f \circ G^k = A_\varphi^k f$ and finally

$$\sum_{k \in \mathbb{Z}} \|A_\varphi^k (f|\partial\widetilde{\Sigma}_t)\|^p_{W_p^{1/p}(S^1)/const} < \infty.$$

Now, for any $u \in W_p^{2/p}(S^2)$ we can take f its harmonic extension. In particular, any smooth function u will do. Since $\alpha : S^1 \to S^2$ is continuous and nonconstant, we can take u such that $V = u \circ \alpha$ is nonconstant. Then

$$\sum_{k \in \mathbb{Z}} \|A_\varphi^k v\|^p_{W_p^{1/p}(S^1)/const} < \infty,$$

as desired.

We remark that $\sum_{k\in\mathbb{Z}} \|A_\varphi^k v\|^p < \infty$ will hold for all v which are in the image of the bounded operator

$$W_p^{2/p}(S^2) \to W_p^{1/p}(S^1),$$

induced by $\partial\widetilde{\Sigma} \to \partial\mathcal{H}^3$.

Corollary 9.4.— Suppose that the space of fixed vectors of A_φ acting in $W_p^{1/p}/const$ possesses a complementary invariant subspace W. Then the spectre of A_φ in W satisfies

$$\sigma(A_\varphi|W) \cap S^1 \neq \phi.$$

Proof.— Suppose the opposite, then $W = W_+ \oplus W_-$ such that $A_\varphi^k|W_+$ and $A_\varphi^{-k}|W_-$ are strict contractions for some $k > 0$. But then

$$\sum_{k \in \mathbb{Z}} \|A_\varphi^k v\|^p = \infty$$

for all $v \in W_p^{1/p}/const$.

We now turn to a generalization. Let $\widetilde{G} \subset \mathcal{M}ap_{g,1}$ be a subgroup, which containes $\pi_1(\Sigma_g)$, so that we have an extension

$$1 \to \pi_1(\Sigma_g) \to \widetilde{G} \to G \to 1.$$

Notice that $G \subset \mathcal{M}ap_g$. A well-known problem in hyperbolic topology is : when there exists a fibration

$$\begin{array}{ccc} \Sigma & \rightarrow & Q \\ & & \downarrow \\ & & T \end{array}$$

with $\pi_1(Q) = \widetilde{G}$ such that Q is a compact manifold of negative curvature. In case T is a closed surface, a corollary F.3 to the Theorem F.1 of [Reznikov 9] provided some necessary condition. This condition is unfortunately void, as we will show now.

Theorem 9.5.— Let $\Sigma^{g_1} \to Q \to \Sigma^{g_2}$ be a surface fibration over a surface (Σ^{g_i} are hyperbolic and oriented). Let Σ be a section of this fibration. Then

$$|\Sigma \cap \Sigma| \leq 2g_2 - 2.$$

Proof.— Let ξ be a vertical tangent bundle for Σ, $e(\xi)$ its Euler class, then $\Sigma \bigcap \Sigma = (e(\xi), [\Sigma])$. We have a natural homomorphism $\pi_1(Q) \to \mathcal{M}ap_{g_1,1}$ and a composite homomorphism

$$\pi_1(\Sigma) \to \pi_1(Q) \to \mathcal{M}ap_{g,1},$$

which we call φ. An inclusion $\mathcal{M}ap_{g_1,1} \to \mathcal{G}_1$ induces an Euler class ϵ in $H^2(\mathcal{M}ap_{g,1})$ coming from the action of $\mathcal{G}_1$ on S^1. By [Matsumoto-Morita 1], [Morita 2], $\varphi^{-1}\epsilon = e(\xi)$. Moreover, as is well known (and obvious) ϵ is a bounded class, in fact, for any homomorphism $\pi_1(\Sigma^g) \xrightarrow{\varphi} Homeo(S^1)$, $|(\varphi^*\epsilon, [\Sigma])| \leq 2g - 2$. This proves the theorem.

Remarks.

1) If the fibration $Q \to \Sigma^{g_2}$ is holomorphic and the action of $\pi_1(\Sigma^{g_2})$ on $H_1(\Sigma^{g_1}, \mathbb{R})$ is simple, then a famous inequality of Arakelov [Arakelov 1] reads $\Sigma \cap \Sigma < 0$ for all holomorphic sections. By Theorem 9.5,

$$-(2g_2 - 2) < \Sigma \cap \Sigma < 0.$$

We now have a following result, which seems to be a very strong restriction on G.

Theorem 9.6.— Let $1 \to \pi_1(\Sigma^g) \to \widetilde{G} \to G$ be an extension. Suppose $\widetilde{G}$ is a fundamental group of a compact manifold of negative curvature

$$-K \leq K(Q^n) \leq -1.$$

Then for $p > (n-1)\sqrt{K}$ there is a vector $const \neq v \in W_p^{1/p}(S^1)$, such that

$$\sum_{g \in G} \|A_g v\|^p_{W_p^{1/p}/const} < \infty \qquad (*).$$

Proof.— Since the proof is essentially identical to the proof of Theorem 9.3, we will only indicate the differences. Let $q_0 \in \widetilde{Q}$ and let $u : S^{n-1}(T_{q_0}\widetilde{Q}) \to \mathbb{R}$ be a smooth function. Composing with a geodesic projection $\widetilde{Q}\backslash\{0\} \to S^{n-1}(T_{q_0}\widetilde{Q})$ we arrive to a function $f : \widetilde{Q}\backslash B(q_0, r) \to \mathbb{R}$ with $\int_{\widetilde{Q}} |\nabla f|^p \, dVol < \infty$ for $p > (n-1)\sqrt{K}$. Since Σ is embedded in Q, one has a limit map $\partial\widetilde{\Sigma} = S^1 \to S^{n-1} = \widetilde{Q}$ be Theorem 6.1. Let $v = u \circ \alpha$, where we identified $\partial\widetilde{Q}$ and $S^{n-1}(T_{q_0}\widetilde{Q})$. Then $v \in W_p^{1/p}(S^1)$ by Theorem 6.2. As in Theorem 9.3 we have the inequality $(*)$. Finally, if $v = const$, for any choice of u, then α is almost everywhere a constant map, say to $z \in S^{n-1}$. Since α is equivariant, it follows that $\pi_1(\Sigma_g)$ stabilizes z. This is obviously impossible.

1.10 Spaces $\mathcal{L}^{(n-1)}_{k,\alpha}$ and cohomology with weights

In this section we will describe a limit form of Theorem 4.1 when $p = 1$, and discuss l^{n-1}-cohomology with weights of cocompact lattices in $SO^+(1, n)$.

Let G be a finitely generated group, $w : G \to R_+$ a function such that $w(g) \to \infty$ as $length(g) \to \infty$. Consider a space $l^p(G, w)$ defined by $f \in l^p(G, w)$ iff $\sum_g |f(g)|^p w^{-1}(g) < \infty$. Suppose $L_g w/w = O(1)$ for all $g \in G$, and the same for $R_g w/w$. Then $l^p(G, w)$ becomes a G-bimodule.

Example 1.— If $r(g) = length(g)$ then consider $w(g) = r^\alpha(g),\ \alpha > 0$ or $w(g) = r(g)^\alpha \log r(g) \log\log r(g) \cdots \underbrace{\log\log\cdots\log}_{k} r(g),\ \alpha > 0.$

2.— Consider $w(g)=e^{\alpha r(g)}, \alpha>0$.

Now let G be a cocompact lattice in $SO^+(1,n)$, We know by Theorem 5.1, that $H^1(G,l^p(G))\neq 0$ exactly for $p>n-1$. In particular, $H^1(G,l^{n-1}(G))=0$. However, by introducing of weights the situation is changed.

Theorem 10.1.— Let G be a cocompact lattice in $SO^+(1,n)$, then for any $k\geq 1$ and $\alpha>0$,

$$H^1(G,l^p(G,w))\neq 0$$

for $p=n-1$ and $w=r(g)\log r(g)\cdots(\underbrace{\log\log\cdots\log}_{k} r(g))^\alpha,\ \alpha>1, k\geq 1$.

Proof.— essentially repeats the argument of Proposition 2.1. Let $u: S^{n-1}\to\mathbb{R}$ be any smooth function and denote again by u its harmonic extension in B^n. We have $|\nabla u|_e< const$, therefore

$$|\nabla u|_h(z)< const\cdot\rho_e(z,S^{n-1})^{-1}$$

Let $\mathcal{F}(h)=u(h^{-1}z_0)$, then a direct computation shows that $L_g\mathcal{F}-\mathcal{F}\in l^{n-1}(G,w)$ and $\mathcal{F}-const\neq l^{n-1}(G,w)$ so $l(g)=L_g\mathcal{F}-\mathcal{F}$ is a nontrivial cocycle if u is one of the coordinate functions on S^{n-1}, as in Theorem 2.1.

We would like to compute $H^1(G,l^{n-1}(G,w))$. A construction of Theorem 4.1 produces from any class in $H^1(G,l^p(G,w))$ a function in $L^1_w(\mathcal{H}^n)$, where the latter space is defined as a space of locally integrable function f with distributional derivatives such that

$$\int_{\mathcal{H}^n}|\nabla f|^{n-1}\cdot w^{-1}(z)<\infty \qquad (*)$$

where $w(z)=\rho_h(z_0,z)\log\rho_h(z_0,z)\cdots(\underbrace{\log\log\cdots\log}_{k}\rho_h(z_0,z))^\alpha$.

Definition.— A space $\mathcal{L}^{(n-1)}_{k,\alpha}$ is defined as a Banach space of traces of $L^1_w(\mathcal{H}^n)$ on S^{n-1}. A norm in $\mathcal{L}^{(n-1)}_{k,\alpha}$ is defined as infinum of integrals $(*)$ taken over the set of all functions f with a given trace.

Remark.— The norm just defined depends on z_0. Therefore a natural action of $SO^+(1,n)$ in $\mathcal{L}^{(n-1)}_{k,\alpha}$ is not isometric.

We will describe $\mathcal{L}^1_{k,\alpha}$ as a Zygmund-type space. One can analogously describe $\mathcal{L}^{(n-1)}_{k,\alpha}$ for $n > 2$, of course, but we will not need it.

Theorem 10.2.— $\mathcal{L}^1_{k,\alpha}$ consists of all function $u : S^1 \to \mathbb{R}$ for which $(a > 0)$

$$\int_0^a dh \int_0^{2\pi} \frac{|u(x+h) - u(x)|}{h^2 \log h \cdots \underbrace{\log \cdots \log}_{k}{}^{\alpha} h} < \infty.$$

Proof is a word-to-word repetition of Uspenski's argument in [Uspenski 1]. One does not need to use Hardy's inequality, since $p = 1$.

Theorem 10.3.— $\mathcal{G}_1$ acts on $\mathcal{L}^1_{k,\alpha}$ by

$$A_\Phi u(x) = u \circ \Phi^{-1}.$$

Corollary 10.4.— If $\Phi : S^1 \to S^1$ is quasisymmetric, then as a function $S^1 \to \mathbb{R}^2$, $\phi \in \mathcal{L}^1_{k,\alpha}$.

We suggest the reader to compare this result to [Carleson 2] and [Gardiner-Sullivan 1]

Proof.— Let $\psi : B^2 \to B^2$ be a quasiisometry of the hyperbolic metric, extending Φ. If u satisfies $(*)$ then $u \circ \Phi^{-1}$ satisfies $(*)$ as well, whence the result.

Embedding $\mathcal{M}ap_{g,1} \subset \mathcal{G}_1$ we obtain a representation

$$\mathcal{M}ap_{g,1} \to Aut(\mathcal{L}^1_{k,\alpha}),$$

which is a limit case of Theorem 9.1.

1.11 Bicohomology and the secondary quantization of the moduli space

We will now introduce a very important notion of bicohomology spaces which to an extent linearize 3-dimensional topology.

Definition.— Let G be a finitely generated group. For $p > 1$ define

$$\mathcal{H}_p(G) = H^1(G_r, H^1(G_l, l^p(G)),$$

where r and l stand for the right and left action, respectively.

Proposition 11.1.— A group $Out(G)$ of outer automorphism of G acts naturally in $\mathcal{H}_p(G)$.

Proof.— By definition, $Out(G) = Aut(G)/(G/Z(G))$. Obviously $Aut(G)$ acts on $H^1(G_l, l^p(G))$ extending the right action of G, so $Aut(G)/(G/Z(G))$ will act on $H^1(G_r, (H^1(G_l, l^p(G)))$.

For a surface group $\pi_1(\Sigma_g)$ we write $\mathcal{H}_{p,g} = \mathcal{H}_p(G)$.

Theorem 11.2.— There exists a natural representation

$$\mathcal{M}ap_g \to Aut(\mathcal{H}_{p,g}).$$

Moreover, for $p > 1$, $\mathcal{H}_{p,g}$ is a nontrivial Banach space. For $p = 2$, $\mathcal{H}_{p,g}$ is an infinite-dimensional Hilbert space. There is a pairing

$$\mathcal{H}_{p,g} \times \mathcal{H}_{p',g} \to \mathbb{R}\ ,$$

which is $\mathcal{M}ap_g$-invariant. For $p = p' = 2$ this pairing is a nondegenerate symmetric bilinear form. One has therefore a representation

$$\mathcal{M}ap_g \xrightarrow{\psi} O(\infty, m),\ 0 \leq m \leq \infty,$$

which we call a secondary quantization of the moduli space of Riemann surfaces.

The proof of the theorem will occupy the rest of this section.

For a compact oriented manifold M let $\Omega^{1/p}$ be a space of measurable $1/p$-powers of densities such that for $\omega \in \Omega^{1/p}$

$$\int_M |\omega|^p < \infty.$$

Then $\Omega^{1/p}$ is Banach, and for $p = 2$, Hilbert. Let G be a finitely generated group acting in M.

Lemma 11.3.— Suppose that any element $g \in G$ has finitely many repelling points, say $x_1^-, \cdots, x_n^-$ and finitely many attractive points, say $x_1^+, \cdots, x_m^+$ such that for any set of neighbourhoods U_i^-, U_+^+ of $x_i^\pm$, there is N such that for $k \geq N$, $g^k(M \setminus \cup U_i^-) \subset \cup U_i^+$. Suppose there are $g_1, g_2, g_3, g_4 \in G$ such that $\cup U_{i,s}^- \cup U_{i,s}^+$ are disjoint for different $s = 1, 2, 3, 4$. Then the action of G in $\Omega^{1/p}$ does not have almost-invariant unit vectors.

Proof.— Suppose the opposite, then there is a sequence $\omega_j \in \Omega^{1/p}$, $\|\omega_j\| = 1$ and $\|g_s^k \omega_j - \omega_j\| \underset{j \to \infty}{\to} 0$ for all s, k. Choose $k_s, U_{i,s}^\pm$ such that

$$g_s^{k_s}(M \setminus \cup U_{i,s}^-) \subset \cup U_{i,s}^+$$

and $\cup U_{i,s}^-$(respectively $\cup U_{i,s}^+$) don't intersect for different i. Let ω be such that $\|\omega\| = 1$ and

$$\|g_s^{k_s}(\omega) - \omega\| < (2/3)^{1/p} - (1/3)^{1/p}.$$

For $E \subset M$ define $C(E, \omega) = \int_E |\omega|^p$. We claim that

$$C(M \setminus \cup U_{s,i}^- \setminus \cup U_{s,i}^+, \omega) < 2/3.$$

Suppose the opposite, then by the invariance of the density $|\omega|^p$,

$$\begin{aligned} &C(M \setminus \cup U_{s,i}^- \setminus \cup U_{s,i}^+, \omega \circ g_s^{k_s}) \leq \\ &= C(g_s^{k_s}(M \setminus \cup U_{s,i}^- \setminus \cup U_{s,i}^+), \omega) \leq \\ &\leq C(g_s^{k_s}(M \setminus \cup U_{s,i}^-), \omega) \leq 1/3. \end{aligned}$$

It follows that

$$[\int_{M\setminus\cup U^-_{s,i}\setminus\cup U^+_{s,i}}|\omega-\omega\circ g_s^{k_s}|^p]^{1/p}\geq$$
$$\geq|[\int_{M\setminus\cup U^-_{s,i}\setminus U^+_{s,i}}|\omega^p|]^{1/p}-[\int_{M\setminus\cup U^-_{s,i}\setminus\cup U^+_{s,i}}|\omega\circ g_s^{k_s}|^p]^{1/p}|\geq(2/3)^{1/p}-(1/3)^{1/p},$$

a contradiction.

So $C(\cup U^-_{s,i},\omega)+C(\cup U^+_{s,i},\omega)\geq 1/3$. Since $\cup U^{\pm}_{s,i}$ are disjoint for different s, we get

$$1\geq\sum_{s=1}^{4}C(\cup U^-_{s,i},\omega)+C(\cup U^+_{s,i},\omega)\geq 4/3,$$

a contradiction. This proves the lemma.

Corollary 11.4.— Let $G\subset SO^+(1,n)$ be a cocompact lattice. Then the natural isometric action of G in $W_p^{(n-1)/p}(S^{n-1})$ does not have almost-invariant vectors. In particular, $H^1(G,W_p^{(n-1)/p}(S^{n-1}))$ is Banach for $p>(n-1)$.

Proof.— For $u\in W_p^{(n-1)/p}(S^{n-1})/const$, let f be a harmonic extension of u so that

$$\|u\|=\int_{\mathcal{H}^n}|\nabla f|^p.$$

Since the energy density $|\nabla f|^p d\mu_h$ is invariant under isometries of $\mathcal{H}^n$, the proof of the Lemma 11.3 applies directly.

Corollary 11.5.— $\mathcal{H}_{p,g}$ is Banach (Hilbert for $p=2$).

Proof.— $H^1(G_l,l^p(G))=W_p^{1/p}(S^1)/const.$

We now describe the pairing

$$\mathcal{H}_{p,g}\times\mathcal{H}_{p',g}\to\mathbb{R}\ .$$

This is given by the cup-product in cohomology

$$H^1(G_r,H^1(G,l^p(G)))\times H^1(G_r,H^1(G_l,l^{p'}(G)))\to H^2(G_r,H^2(G_l,l^p(G)\otimes l^{p'}(G)))$$

followed by the duality $l^p(G) \times l^{p'}(G) \to \mathbb{R}$ and evaluating twice on the fundamental cycle in $H_2(G, \mathbb{R})$. We have also an analytic description, namely a pairing

$$W_p^{1/p}(S^1)/const \times W_{p'}^{1/p'}(S^1)/const \to \mathbb{R}$$

is given on smooth function by $f, g \to \int_{S^1} f dg$ and then extended as a bounded bilinear form. This induces a pairing

$$H^1(G, W_p^{1/p}/const) \times H^1(G, W_{p'}^{1/p'}/const) \to \mathbb{R} \; .$$

Lemma 11.6.[Korevaar-Schoen 1]— Let G be a finitely presented group which is realized as a fundamental group of a compact Riemannian manifold M. Let $\rho : G \to O(\mathcal{H})$ be an orthogonal representation, which does not have almost-invariant vectors. Let $[l] \in H^1(G, \mathcal{H})$. Let E be a flat vector bundle with fiber $\mathcal{H}$ over M, corresponding to ρ. Then there is a harmonic 1-form $\omega \in \Omega^1(M, E)$, corresponding to $[l]$.

Proof.—This is a reformulation of [Korevaar-Schoen 1].

Corollary 11.7.— Let M be Kähler. Then if ρ is as in the previous lemma, then
1) There is a natural complex structure in $H^1(G, \mathcal{H})$, making it a complex Hilbert space ;
2) A pairing

$$H^1(G, \mathcal{H}) \times H^1(G, \mathcal{H}) \to \mathbb{R} \; ,$$

given by $[l_1], [l_2] \to ([\omega]^{n-1}([l_1], [l_2]), [M])$ where $[\omega]$ is a Kähler class, $[M]$ is the fundamental class and $([l_1], [l_2]) \in H^2(G, \mathbb{R})$ is a cup-product composed with the scalar product $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$, is a non-degenerate symplectic structure in $H^1(G, \mathcal{H})$.

Proof is the same as for finite-dimensional $\mathcal{H}$, once we have the Hodge theory of the previous lemma.

We now ready to prove that the symmetric pairing

$$\mathcal{H}_{2,g} \times \mathcal{H}_{2,g} \to \mathbb{R}$$

is nondegenerate. Realize G as a lattice in $SO(1,2)$. Then $H^1(G, l^p(G)) = W_p^{1/p}(S^1)/const$. Let H denote the Hilbert transform. It is a bounded operator

$$H : L^p(S^1)/const \to L^p(S^1)/const \quad (p > 1)$$

defined as follows: for $u \in L^p(S^1)$ let f be its harmonic extension and g a conjugate harmonic function, then $Hu = g|S^1$. Since

$$\int_{\mathcal{H}^2} |\nabla f|^p = \int_{\mathcal{H}^2} |\nabla g|^p.$$

H restricts to $W_p^{1/p}(S^1)$ as an isometry.

Now, the symplectic duality $\int f\ dg$ in $W_2^{1/2}(S^1)/const$ is simply equal to (Hf, g). Moreover, H is $SO(1,2)$-invariant. Then the Corollary 11.7 implies that the pairing of Theorem 11.2 is also nondegenerate.

We still have to prove that $\mathcal{H}_{g,p} \neq 0$ and for $p = 2$ is infinite-dimensional. We first describe an element of $\mathcal{H}_{g,p}$ associated to a given realization $G \hookrightarrow SO(1,2)$ as a cocompact lattice,which we will call a principal state.

Recall that if M is a smooth compact oriented manifold, $\mathcal{D}iff^1(M)$ a group of orientation-preserving diffeomorphism of class C^1, then one has a cocycle l in $Z^1(\mathcal{D}iff^1(M), C^0(M))$ defined as [Bott 1]

$$l(\phi) = \log \frac{\phi_* \mu}{\mu},$$

where μ is any smooth density on M, and $\phi_*\mu$ a left action. The class $[l] \in H^1(\mathcal{D}iff^1(M), C^0(M))$ does not depend on μ. For $r \geq 1$ one similarly gets a class in $H^1(\mathcal{D}iff^r(M), C^{r-1}(M))$. Now, let $M = S^{n-1}$ and consider a standard conformal action of $SO^+(1,n)$ on S^{n-1}. We get a class

$$[l]_p \in H^1(SO^+(1,n), W_p^{(n-1)/p}(S^{n-1})/const)$$

for all $p > 1$ simply because $C^\infty(S^{n-1}) \subset W_p^{(n-1)/p}(S^{n-1})$. We claim $[l]_p \neq 0$ for $n = 2$ and $p > n-1$. Since the action is isometric, it follows from the following lemma (we prove and use it only for $n = 2$).

Lemma 11.8.— Fix $z_0 \in B^n$ and let $r(g) = \rho_h(z_0, g^{-1}z_0)$. Then for any fixed μ, $\|l(g)\|_{W_p^{(n-1)/p}(S^{n-1})/const} \to \infty$ as $r(g) \to \infty$.

Proof.(Only for $n = 2$)— We choose for μ the harmonic (Poisson) measure μ_0, associated with z_0. Then $l(g) = \log \frac{g_*\mu_0}{\mu_0}$. For $\beta \in S^{n-1}$, $l(g)(\beta) = B_\beta(z_0, gz_0)$ where $B_\beta(z_0, \cdot)$ is a Busemann function of B^n corresponding to $\beta \in \partial B^n$ and normalized at z_0, that is, $B_\beta(z_0, z_0) = 0$ (see, for example, [Besson-Courtois-Gallot 1]).

We will make the computation only for $n = 2$. Let $z_0 = 0$, $gz_0 = w$, then

$$B_\beta(0, w) = \log \frac{1 - |w|^2}{|w - \beta|^2}.$$

Notice that $\log |\frac{\beta - w}{1 - \bar{w}\beta}| = 0$, since $|\beta| = 1$, so

$$B_\beta(0, w) = \log(1 - |w|^2) - 2\log|1 - \bar{w}\beta| = -2\log|1 - \bar{w}\beta| (mod\ const).$$

Notice that $\log|1 - \bar{w}z|$ is defined and is harmonic in $|z| \leq 1$, so

$$\begin{aligned} \|B_\beta(0, w)\|^p_{W_p^{1/p}(S^1)/const} &= 2^p \int_{B^2} [\nabla(\log|1 - \bar{w}z)|)]_h^p \; d\mu_h = \\ &= 2^p \int_{B^2} \frac{|w|^p}{|1 - \bar{w}z|^p} \frac{1}{(1 - |z|^2)^{2-p}} \; dz d\bar{z} \end{aligned} \qquad (*).$$

Sublemma.— An integral $(*)$ grows as $\log(1 - |w|)$ as $|w| \to 1$.

Proof.— Computing in polar coordinates, we have

$$\int_0^1 dr \frac{1}{(1 - r^2)^{2-p}} \int_0^{2\pi} \frac{d\theta}{|1 - r|w|e^{i\theta}|^p}.$$

It is elementary to see that the inner integral grows as $\frac{1}{(1 - r|w|)^{p-1}}$, so we arrive at

$$\int_0^1 dr \frac{1}{(1 - r)^{2-p}} \frac{1}{(1 - r|w|)^{p-1}} \sim \int_0^a \frac{ds}{s^{2-p}(A + s)^{p-1}}$$

where $a > 0$ is fixed and $A = 1 - |w|$. Further we have $(s = At)$

$$\int_0^{a/A} \frac{dt}{t^{2-p}(1 + t)^{p-1}} \sim \int_0^{a/A} \frac{dt}{t} \sim \log|A|,$$

which proves the Sublemma.

Finally,

$$\|B_\beta(0,w)\|_{W_p^{1/p}(S^1)/const} \sim [\log(1-|w|)]^{1/p},$$

where $\sim$ means that the ratio converges to a constant.

The proof for $n > 2$ will be given elsewhere.

Notice that for $p = 2$ we have (for $n = 2$)

$$\|l(g)\|_{W_2^{1/2}(S^1)/const} \sim \|g\|^{1/2}$$

where $\|g\|$ is a hyperbolic length of a (pointed) geodesic loop, representing g. This exponent in the RHS is the maximal possible. We will later prove a general theorem (Theorem III.3.1) showing that for any orthogonal or unitary representation of $G = \pi_1(\Sigma)$ in a Hilbert space $\mathcal{H}$ and any cocycle $l \in Z^1(G, \mathcal{H})$,

$$\|l(g)\| \leq const \cdot length(g)^{1/2} \log\log length(g)$$

as g converges nontangentially to almost all $\theta \in S^1 = \partial G$.

Coming back to principal states $[l]_p \in H^1(G, W_p^{(n-1)/p}(S^{n-1})/const)$, let E be a flat affine bundle over $M = \mathcal{H}^n/G$ with fiber $W_p^{(n-1)/p}(S^{n-1})$, associated with an affine action

$$g \mapsto R_g + l(g).$$

Notice that

$$s : z \mapsto log\frac{\mu(z)}{\mu(z_0)}$$

is an G-equivariant section of the lift of E on $\widetilde{M} = \mathcal{H}^n$, or, equivalently, defines a section of E. We claim that this section is harmonic. This immediately reduces to a statement that $B_\beta(z_0, z)$ is harmonic mod const as a function of z. In the upper half-plane model it simply means that $(x, y) \mapsto \log y$ is harmonic mod const. The harmonic section just defined does not lift to a harmonic section of the flat affine bundle with fiber $W_p^{(n-1)/p}(S^{n-1})$. For $n = 2$ we can say more. Let

$$H : W_p^{1/p}(S^1)/const \to W_p^{1/p}(S^1)/const$$

be the Hilbert transform. It makes $W_p^{1/p}(S^1)/const$ into a complex Banach space. Then a direct inspection shows that the section of E defined above is (anti)holomorphic (depending on the choice of a sign of H). This will be used later. Equivalently, ds is an (anti)-holomorphic one-form on $\mathcal{H}^2/G$, valued in E. Again, this holomorphic form does not lift to a d and δ-closed form of a flat bundle with fiber $W_p^{1/p}(S^1)$ even for $p = 2$. This latter bundle is a flat bundle with fiber a Hilbert space, but whose monodromy is not orthogonal. The Hodge theory of [Korevaar-Schoen 1] and [Jost 1] does not apply and in fact not every cohomology class is represented by a d and δ-closed form. We will discuss these subtle obstructions to the Hodge theory in a sequel to this paper [

Reznikov 10].

As an application of the computation made above, we will complete the proof of Lemma 5.6 for $p > 1$. Let $u \in W_p^{1/p}(S^1)/const$ and let $f : B^2 \to \mathbb{R}$ be a harmonic extension of u. We claim that

$$|f(w)| \leq c \cdot [\log(1 - |w|)]^{1/p'}.$$

Since the Hilbert transform is invertible in $W_p^{1/p}(S^1)/const$, we can assume that the Fourier coefficients $\hat{u}(n) = 0$ for $n < 0$, so that f is holomorphic:

$$\begin{aligned}
& |f(w)| = |\tfrac{1}{2\pi i}\int_{S^1} \tfrac{u(\xi)d\xi}{\xi - w}| = \tfrac{1}{2\pi}|\int_0^{2\pi} \tfrac{u(e^{i\theta})e^{i\theta}}{e^{i\theta} - w} d\theta| = \\
= \; & \tfrac{1}{2\pi}|\int_0^{2\pi} \tfrac{u(e^{i\theta})d\theta}{1 - w\cdot e^{-i\theta}}| = \tfrac{1}{2\pi}|\int_0^{2\pi} \tfrac{u(e^{-i\theta})d\theta}{1 - w\cdot e^{i\theta}}| = \\
= \; & |\tfrac{1}{2\pi i}\int_0^{2\pi} \tfrac{[u(e^{-i\theta})\cdot e^{-i\theta}]ie^{i\theta}d\theta}{1 - we^{i\theta}}| = | - \tfrac{1}{2\pi i w}\int_0^{2\pi}[u(e^{-i\theta})\cdot e^{-i\theta}][\log(1 - we^{i\theta})]' d\theta| = \\
= \; & | - \tfrac{1}{2\pi i w} < u(e^{-i\theta})\cdot e^{-i\theta}, \log(1 - we^{i\theta}) > | \leq \\
\leq \; & \tfrac{1}{2\pi |w|}\|u(e^{-i\theta})e^{-i\theta}\|_{W_p^{1/p}(S^1)/const} \cdot \|\log(1 - we^{i\theta})\|_{W_{p'}^{1/p'}/const} \leq \\
\leq \; & c\|u\|_{W_p^{1/p}(S^1)/const} |\log(1 - |w|)|^{1/p'}.
\end{aligned}$$

It is very plausible that the result is true, for $u \in W_p^{\frac{n-1}{p}}(S^{n-1})/const$ for $n \geq 3$. Our proof obviously does not work in this case.

We now start to prove that $\mathcal{H}_{2,g}$ is infinite-dimensional. Let M_0, M_0' be factors generated by the left (respectively, right) actions of G in $l^2(G)$

[Murray-von Neumann 1]. Notice that $H^1(G_l, l^2(G))$ can be viewed as a cohomology of a complex

$$l^2(G) \xrightarrow{d_0} \bigoplus_{i=1}^{2g} l^2(G) \xrightarrow{d_1} l^2(G) \qquad (*),$$

computed from the standard CW-decomposition of Σ^g with one zero-dimensional cell, $2g$ one-dimensional cells and one two-dimensional cell. Notice that d_0, d_1 are given by matrices with entries in $\mathbb{Z}[G]$, acting on $l^2(G)$ from the left. Letting $\Delta_l = d_0 d_0^* + d_1^* d_1$ we can view $H^1(G, l^2(G))$ as $Ker\Delta_l$. Notice that $\Delta_l \in M_0$. It follows that $H^1(G, l^2(G))$ is a module over M_0'. Now, since M_0 is type II, there is a decomposition

$$W_2^{1/2}(S^1)/const = H^1(G_l, l^2(G)) = Ker\ \Delta_l = \bigoplus_{j=1}^{m} H_m,$$

for any $m \geq 1$ where H_m are isomorphic right G-modules. Since we know already that $H^1(G_r, W_2^{1/2}(S^1)/const) \neq 0$, and H_j are all isomorphic, it follows that $H(G_r, H_j) \neq 0$ for all j, therefore $\dim H^1(G, W_2^{1/2}(S^1))/const \geq m$. This finally proves Theorem 11.2.

There are natural invariant von Neumann algebras acting in $\mathcal{H}_{2,g}$. Indeed, let M_1' be a double commutant of M_0' in $H^1(G, l^2(G)) = Ker\Delta_l$ and M_1 be a commutant of M_0'. We could define M_1' as a von Neumann algebra , generated by the right action of G in $H^1(G, l^2(G))$ and M_1 as a commutant of M_1'. It follows that M_1, M_1' do not depend on the choice of the complex $(*)$ and therefore $\mathcal{M}ap_{g,1} = Aut(G)$ acts in $H^1(G, l^2(G))$ leaving M_1', M_1 invariant. Now consider $\mathcal{H}_{2,g} = H^1(G_r, H^1(G_l, l^2(G))$. Then $\mathcal{H}_{2,g} = Ker\ \Delta_r : \bigoplus_{i=1}^{2g} H^1(G_l, l^2(G)) \to \bigoplus_{i=1}^{2g} H^1(G_l, l^2(G))$ where a right Laplacian is defined exactly as the left one. It follows that $\mathcal{H}_{2,g}$ is a module over M_1. Let M_2 be a double commutant of M_1 and M_2' be its commutant. We have proved a following theorem, except for the last statement.

Theorem 11.9.— There are infinite-dimensional von Neumann algebras M_2, M_2' acting in $\mathcal{H}_{2,g}$, which are invariant under the action of $\mathcal{M}ap_g$. Moreover, there is an involution τ of $\mathcal{H}_{2,g}$ which commutes with the $\mathcal{M}ap_g$-action and permutes M_2, M_2'.

Proof.— Everything is already proved except for the last statement. Notice that there is an involution $\tau : l^2(G) \to l^2(G)$ defined by $\tau f(g) = f(g^{-1})$. A Lyndon-Serre-Hochschild spectral sequence of the extension $1 \to G \to G \times G \to G \to 1$ shows that $\mathcal{H}_{2,g} = H^2(G \times G, l^2(G))$. Let σ be an involution of $G \times G$ defined by $\sigma(g,h) = (h,g)$. Then one has $\tau[(g,h)v] = (\sigma(g,h))\tau(v)$ where $g, h \in G$ and $v \in l^2(G)$. It follows that τ induces an involution, which we also call τ, in $\mathcal{H}_{2,g}$, which obviously commutes with $\mathcal{M}ap_g$-action and permutes M_2 and M_2'. This completes the proof of Theorem 11.9.

Note that since the unitary representation of G in $H^1(G_l, l^2(G)) = W_2^{1/2}(S^1)/const$ extends to an irreducible representation of $PSL_2(\mathbb{R})$, the commutator M_1 of G in $W_2^{1/2}(S^1)/const$ possesses a faithful trace defined by

$$tr(a) \cdot Id = \int_{PSL_2(\mathbb{R})/G} gag^{-1} dg.$$

Proposition 11.10.— Let $\widetilde{\mathcal{H}}_{2,g}$ be a completion of M_1 under the norm $tr\ xx^*$. Then $\widetilde{\mathcal{H}}_{2,g}$ is a Hilbert space and there is a representation

$$\tilde{\rho} : \mathcal{M}ap_g \to Aut(\widetilde{\mathcal{H}}_{2,g}),$$

leaving invariant a nondegenerate form $x \mapsto tr\ x^2$.

I don't know at the time of writing if $\widetilde{\mathcal{H}}_{2,g}$ is isomorphic to $\mathcal{H}_{2,g}$ as $\mathcal{M}ap_g$-module.

We now turn to the holomorphic realization of $\mathcal{H}_{2,g}$. Fix a realization of G as a cocompact lattice in $SO^+(1,2)$, then $\mathcal{H}_{2,g} = H^1(G, W_2^{1/2}(S^1)/const)$. Recall that G commutes with the Hilbert transform in $W_2^{1/2}(S^1)/const$. Let $S = \mathcal{H}^2/G$, then S is a hyperbolic Riemann surface, homeomorphic to Σ^g. For any element $w \in H^1(G, W_2^{1/2}(S^1)/const)$ we have by Lemma 11.3 and Lemma 11.6 a unique harmonic form in a flat Hilbert bundle E with fiber $W_2^{1/2}(S^1)/const$, associated with the action of G.

Uniqueness should be explained. We have a following general fact.

Lemma 11.11.— Let M be a compact Riemannian manifold. $\rho : \pi_1(M) \to$

$O(H)$ an orthogonal representation in a real Hilbert space, without fixed vectors, $\omega \in H^1(\pi_1(M), H)$. Then there at most one harmonic form, $\omega \in \Omega^1(M, E)$, representing ω.

Proof.— If ω_1, ω_2 are two such forms, then $\omega_1 - \omega_2$ is a derivative of a harmonic section of M. But standard Bochner vanishing theorem shows that such section should be self-parallel, so ρ has a fixed vector, a contradiction.

Notice that H makes $W_2^{1/2}(S^1)/const$ into a complex Hilbert space. Then $\frac{1}{2}(\omega - H(\omega \circ J))$, where J is a complex structure on S, will be a holomorphic 1-form in E, whereas $\frac{1}{2}(\omega + H(\omega \circ J))$ will be an anti-holomorphic 1-form. Let $\mathcal{H}_{2,g}^{\pm}$ be the spaces of holomorphic(respectively, anti-holomorphic) 1-forms in E, then $\mathcal{H}_{2,g} = \mathcal{H}_{2,g}^{+} \bigoplus \mathcal{H}_{2,g}^{-}$. Now, $W_2^{1/2}(S^1)/const$ is identified with exact L^2-harmonic 1-forms in the hyperbolic plane $\mathcal{H}^2$, which is isomorphic as a complex Hilbert space (with a complex structure, defined by the Hodge star operator) to the space of exact L^2-holomorphic 1-form in $\mathcal{H}^2$. So any element in $\mathcal{H}_{2,g}^{+}$ defines a holomorphic 1-form on S valued in a bundle with fibers L^2-holomorphic 1-forms on $\mathcal{H}^2$. In other words, let G act diagonally in $\mathcal{H}^2 \times \mathcal{H}^2$ and

$$Q = \mathcal{H}^2 \times \mathcal{H}^2 / G,$$

then we have an L^2 holomorphic 2-form on Q. The space $\mathcal{H}_{2,g}^{+}$ therefore is identified with the space of L^2 holomorphic 2-forms on Q. Similarly, $\mathcal{H}_{2,g}^{-}$ is identified with the space of L^2 holomorphic 1-form on

$$Q' = \mathcal{H}^2 \times \overline{\mathcal{H}^2} / G,$$

where $\overline{\mathcal{H}^2}$ is obtained from $\mathcal{H}^2$ by reversing the complex structure (i.e. $\bar{J} = -J$). Notice that as complex surfaces, Q and Q' are not biholomorphic: Q contains a compact curve (the quotient of the diagonal) whereas Q' does not. We have proved the following:

Theorem 11.12.(Holomorphic realization of quantum moduli space)— Fix an embedding $G \hookrightarrow SO^+(1,2)$ as a cocompact surface, then $\mathcal{H}_{2,g}$ splits as $\mathcal{H}_{2,g}^{+} \oplus \mathcal{H}_{2,g}^{-}$ where $\mathcal{H}_{2,g}^{+}$ (respectively, $\mathcal{H}_{2,g}^{-}$) is identified with a space of L^2

holomorphic 2-forms on $Q = \mathcal{H}^2 \times \mathcal{H}^2/G$ (respectively, $Q' = \mathcal{H}^2 \times \overline{\mathcal{H}^2}/G$). Moreover, the splitting is orthogonal with respect to the canonical symmetric scalar product in $\mathcal{H}_{2,g}$ and the restriction of this scalar product on $\mathcal{H}^{\pm}_{2,g}$ is positive (respectively, negative).

Example.— The principal state $[l]_2$ lies in $\mathcal{H}^{-}_{2,g}$. We do not know at the time of writing if $\mathcal{H}^{+}_{2,g} = 0$.

1.12 $\mathcal{H}_{p,g}$ as operator spaces and the vacuum vector

In this section we will develop an algebraic and an analytic theory of $\mathcal{H}_{p,g}$ as spaces of operators between $W^{1/q}_q(S^1)/const$, which commute with the action of G. We use rather rough estimates of matrix elements, so the ranges of indices for which the action is established is certainly not the best possible. We start with a lemma.

Lemma 12.1.— Let $u \in W^{1/p}_p(S^1)$ and $a \in l^q(G)$. Then $\sum a(g)R_g u \in W^{1/r}_r(S^1)$ where $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$.

Remark.— R_g means the action of G in $W^{1/p}_p(S^1)$ (a reminiscent of the actions from the right in $l^2(G)$).

Proof.— Let f be a harmonic extension of u, so that

$$\int_{\mathcal{H}^2} |\nabla f|^p \, d\mu_h < \infty.$$

By Young-A.Weil inequality [Hewitt-Ross 1], $l^p * l^q \subset l^r$, so if $h = \sum a_g R_g f$, we have

$$\begin{aligned}
\int_{\mathcal{H}^2} |\nabla h|^r \, d\mu_h &= \int_{\mathcal{H}^2/G} d\mu_h(z) \sum_g |\nabla h(g^{-1}z)|^r \\
&\leq \int_{\mathcal{H}^2/G} d\mu_h(z) \|\nabla h(g^{-1}z)\|_{l^r} \\
&\leq c \cdot \int_{\mathcal{H}^2/G} d\mu_h(z) \|a(g)\|_{l^q(G)} \|\nabla f(g^{-1}z)\|_{l^p} \\
&\leq c \cdot \|a(g)\|_{l^q(G)} \int_{\mathcal{H}^2} |\nabla f|^p d\mu_h.
\end{aligned}$$

The result follows with an estimate

$$\|\sum a(g) R_g u\|_{W_r^{1/r}(S^1)/const} \leq c \cdot \|a(g)\|_{l^q(G)} \|u\|_{W_p^{1/p}(S^1)/const}.$$

Now recall that we have a canonical pairing

$$B : W_r^{1/r}(S^1)/const \times W_{r'}^{1/r'}(S^1)/const \to \mathbb{R},$$

so that a formula

$$(u, v) \mapsto (a \mapsto B(\sum a(g) R_g u, v))$$

defines a map

$$W_p^{1/p}(S^1)/const \times W_{r'}^{1/r'}/const \to l^{q'}(G).$$

Now an element of $W_{r'}^{1/r'}(S^1)/const$ defines an element of

$$Hom_G(W_p^{1/p}(S^1)/const, l^{q'}(G))$$

and an induced map

$$\begin{gathered}
Hom(H^1(G, W_p^{1/p}(S^1)/const, H^1(G, l^{q'}(G)) = \\
= Hom(H^1(G, W_p^{1/p}(S^1)/const, W_{q'}^{1/q'}(S^1)/const).
\end{gathered}$$

In other words, we have a map

$$H^1(G, W_p^{1/p}(S^1)/const) \to Hom_{\mathbb{R}}(W_{r'}^{1/r'}(S^1)/const \to W_{q'}^{1/q'}(S^1)/const),$$

and it is immediate to check that the image lies in Hom_G. So we have a

Proposition 12.2.— The construction above defines a map

$$\mathcal{H}_{p,g} \to Hom_G(W_{r'}^{1/r'}(S^1)/const, W_{q'}^{1/q'}(S^1)/const)$$

for p, q', r' satisfying $\frac{1}{p} \geq 1 + \frac{1}{q'} - \frac{1}{r'}$, which is $\mathcal{M}ap_g$-equivariant.

An induced map in $H^1(G, \cdot)$ produces a bounded $\mathcal{M}ap_g$-equivariant product

$$\mathcal{H}_{p,g} \times \mathcal{H}_{r',g} \to \mathcal{H}_{q',g}.$$

We stress again that the range of indices for which this product is defined should be improved. We will see that viewing $\mathcal{H}_{p,g}$ as an operator space helps to understand $\mathcal{M}ap_g$-action. We turn now to an analytic description of the above. Let $l \in Z^1(G, W_p^{1/p}(S^1)/const)$. A construction of Theorem 4.1 produces a smooth map

$$F : \mathcal{H}^2 \to W_p^{1/p}(S^1)/const,$$

satisfying $F(g^{-1}z) = R_g F(z) + l(g), g \in G$, In particular, $g^*(\nabla F)(g^{-1}z) = R_g(\nabla F)(z)$. Now let $v \in W_{r'}^{1/r'}(S^1)/const$, where $r \geq p$. Then we have a scalar function

$$< F, v >: \mathcal{H}^2 \to \mathbb{R},$$

where $< \cdot, \cdot >$ is a pairing $W_r^{1/r}(S^1)/const \times W_{r'}^{1/r'}(S^1)/const \to \mathbb{R}$ defined in Theorem 7.2. Since $r \geq p$, $W_p^{1/p}(S^1) \subset W_r^{1/r}(S^1)$, so $< F, v >$ is defined. Without futher assumption one can only say that

$$|\nabla < F, v > | \leq const,$$

but if we assume $r > p$, say $\frac{1}{p} = 1 + \frac{1}{q'} - \frac{1}{r'}$, then $< F, v >$ will satisfy

$$\sum_g |\nabla < F, v > (gz)|^{q'} < const$$

for all $z \in \mathcal{H}^2$. Integrating over $\mathcal{H}^2/G$, we get

$$\int_{\mathcal{H}^2} |\nabla < F, v > |^{q'} < \infty,$$

so there exists $< F, v > |S^1 \in W_{q'}^{1/q'}(S^1)$. This defines a desired map

$$H^1(G, W_p^{1/p}(S^1)/const) \to Hom_G(W_{r'}^{1/r'}(S^1)/const, W_{q'}^{1/q'}(S^1)/const).$$

We will use this description now to compute the operator, associated with the principal state

$$[l]_p \in H^1(G, W_p^{1/p}(S^1)/const).$$

Proposition 12.3.— For $p, r', q' > 1$, $\frac{1}{p} \geq 1 + \frac{1}{q'} - \frac{1}{r'}$, an operator in

$$Hom_G(W_{r'}^{1/r'}(S^1)/const, W_{q'}^{1/q'}(S^1)/const),$$

associated to the principal state $[l]_p$ is proportional to the Hilbert transform

$$H : W_{r'}^{1/r'}(S^1)/const \to W_{r'}^{1/r'}(S^1)/const,$$

followed by the embedding $W_{r'}^{1/r'}(S^1) \hookrightarrow W_{q'}^{1/q'}(S^1)$.

Proof.— First, we notice that the Hilbert transform acts as an isometric operator in $W_p^{1/p}(S^1)/const$ for all $p > 1$. This follows at once from the definition of the norm as

$$\|u\| = \int_{\mathcal{H}^2} |\nabla f|^p d\mu_h,$$

where $\Delta f = 0$ and $f|S^1 = u$ (mod const). We will prove the proposition by a direct unimaginative computation. Let

$$g(z) = \frac{z + z_0}{1 + \bar{z}_0 z}, \qquad |z_0| < 1, |z| < 1,$$

so that $g(0) = z_0$. Then the Jacobian of g on the unit circle is

$$\frac{1 - |z_0|^2}{|z - z_0|^2},$$

so

$$l(g) = \log(1 - |z_0|^2) - \log|z - z_0|^2.$$

Let $\varphi : S^1 \to \mathbb{R}$ be smooth. Then

$$< \varphi, l(g) >= \int_{S^1} \varphi'(\theta) \cdot [\log(1 - |z_0|^2) - \log|e^{i\theta} - re^{i\varphi}|^2] d\theta,$$

where $z_0 = re^{i\varphi}$. Obviously,

$$\int_{S^1} \varphi'(\theta) \log(1 - |z_0|^2) = 0,$$

so

$$\begin{array}{rcl} < \varphi, l(g) > & = & -\int_{S^1} \varphi(\theta) \cdot [\log|e^{i\theta} - re^{i\varphi}|^2]' \\ & = & -\int \varphi(\theta) \cdot \frac{2r\sin(\theta-\varphi)}{1+r^2-2r\cos(\theta-\varphi)} \ . \end{array}$$

As $z_0 = re^{i\varphi} \underset{r \to 1}{\to} e^{i\varphi}$, this converges to

$$-v.p. \int \varphi(\theta) \frac{2\sin(\theta - \varphi)}{2 - 2\cos(\theta - \varphi)} = v.p. \int \varphi(\theta) \cdot \frac{1}{tg\ \frac{\theta - \varphi}{2}} = \pi H\varphi(\theta)$$

almost everywhere on S^1. The Proposition is proved, since smooth functions are dense in $W_{r'}^{1/r'}(S^1)$.

Notice that since H commutes with the action of $SO^+(1,2)$, for any cocycle $m \in Z^1(G, W_p^{1/p}(S^1)/const)$, Hm is also an cocycle. In particular, $H[l]_p \in H^1(G, W_p^{1/p}(S^1)/const)$. We wish to compute a corresponding operator in $Hom(W_{r'}^{1/r'}(S^1)/const \to W_{q'}^{1/q'}(S^1)/const)$. Let F, as above, be a smooth map

$$F : \mathcal{H}^2 \to W_p^{1/p}(S^1)/const,$$

satisfying $F(g^{-1}z) = R_g F(z) + l_p(g)$. For $v \in W_{r'}^{1/r'}(S^1)$ we need to find a limit on the boundary of $< HF, v >$. But H repects the pairing $< \cdot, \cdot >$ and $H^2 = -1$, so $< HF, v >= - < F, Hv >$, whose limit on S^1 is $\pi H(-Hv) = \pi v$. We have proved the following lemma.

Lemma 12.4.— For $p, r', q' > 1$, $\frac{1}{p} \geq 1 + \frac{1}{q'} - \frac{1}{r'}$, an operator in

$$Hom_G(W_{r'}^{1/r'}(S^1)/const, W_{q'}^{1/q'}(S^1)/const),$$

associated with $\frac{1}{\pi} H[l]_p$, is the identity.

*Theorem 12.5.*A.— An element $v = H[l]_2 \in \mathcal{H}_{2,g}$ does not depend on the choice of the lattice $G \hookrightarrow SO^+(1,2)$

B.— The action of $\mathcal{M}ap_g$ in $\mathcal{H}_{2,g}$ fixes v.

Remark.— The Theorem is beyond doubt true for all $p > 1$ and not only $p \geq 2$, however I can't prove this at the moment of writing this paper (July,1999). (**Added January, 2000**). This is in fact true. The proof will appear in [Reznikov 10]).

The vector v is called a vacuum vector.

Proof.— Consider two embeddings $i_1, i_2 : G \to SO^+(1,2)$ as cocompact lattices and let v_1, v_2 be corresponding elements. We view v_1, v_2 as elements of $H^1(G_r, H^1(G_l, l^2(G))$. Let A_1, A_2 be associated operators

$$A_1, A_2 : H^1(G_l, l^{r'}(G)) \to H^1(G_l, l^{q'}(G)).$$

We know that $A_1 = A_2 = id$. It follows that an operator, associated with $v_1 - v_2$ is zero. We are going to show that $v_1 - v_2$ is zero. Since

$$v_1 - v_2 \in H^1(G_r, V),$$

where V stands for $H^1(G_l, l^2(G)) \simeq W_2^{1/2}(S^1)/const$, by a result of [Korevaar-Schoen 1] cited above (Lemma 11.6) there exists a harmonic section F of the affine Hilbert bundle over $M = \mathcal{H}^2/G$ with fiber V and nonodromy

$$g \mapsto R_g(\cdot) + m(g),$$

where $m(g)$ is any cocycle, representing $v_1 - v_2$. Let $v \in W_{r'}^{1/r'}(S^1)/const$, then, denoting by F again the lift of this section on $\widetilde{M} = \mathcal{H}^2$, we see that $< F, v >$ is a harmonic function such that

$$\int_{\mathcal{H}^2} |\nabla(< F, v >)|^{q'} \, d\mu_h < \infty,$$

and the trace of $< F, v >$ on S^1 is constant. It follows that $< F, v >$ is constant itself, therefore (v is arbitrary!) $F = w = const$ and

$$m(g) = R_g w - w,$$

so $v_1 - v_2 = 0$. This proves A. Now, if $\phi \in \mathcal{M}ap_{g,1} = Aut(\pi_1(\Sigma^g))$, simply apply A to i_1 and $i_1 \circ \phi$.

We wish to compute v. $[l]_2$ is given by a cocycle

$$g \mapsto -2 \log |\beta - w|,$$

$\beta \in S^1,\ w = g(0)$. This is equal to $2 \log |1 - \bar{w}\beta|$. The latter function is a real part of $2 \log(1 - \bar{w}z)$ which is holomorphic in $|z| \leq 1$, so the Hilbert transform is $2Arg(1 - \bar{w}\beta)$. This means that a cocycle

$$m(g)(\beta) = 2Arg(W - \beta) \ (\ mod\ const)$$

where $W = 1/\bar{w}, w = g(0)$, represents v.

Theorem 12.6.— $H^1(\mathcal{M}ap_{g,1}, H^1(G_l, l^p(G))) \neq 0$ for $p \geq 2$.

Proof.— We embed G as a lattice in $SO^+(1,2)$ and identify $H^1(G_l, l^p(G))$ and $W_p^{1/p}(S^1)/const$. We know that

$$H^0(\mathcal{M}ap_g, H^1(G_r, W_p^{1/p}(S^1)/const) \ni v \neq 0.$$

Notice that $H^0(G_r, W_p^{1/p}(S^1)/const) = 0$ since any G-invariant harmonic 1-form in $\mathcal{H}^2$ has infinte p-energy. So in the spectral sequence

$$E^2_{i,j} : H^i(\mathcal{M}ap_g, H^j(G_r, W_p^{1/p}(S^1)/const)) \Longrightarrow H^{i+j}(\mathcal{M}ap_{g,1}, W_p^{1/p}(S^1)/const)$$

the second differential

$$d_2 : H^0(\mathcal{M}ap_g, H^1(G_r, W_p^{1/p}(S^1)/const) \Longrightarrow H^2(\mathcal{M}ap_g, H^0(G_r, W_p^{1/p}(S^1)/const)$$

must be zero. Therefore the vacuum vector v survives in E^∞.

It is plausible that, in fact,

$$H^1(\mathcal{G}_1, W_p^{1/p}(S^1)/const) \neq 0 \qquad (p > 1)$$

for the group $\mathcal{G}_1$ of quasisymmetric homeomorphisms. (**Added January, 2000**). This is in fact true. A formula

$$\Phi \rightarrow Arg\Phi^{-1}(\beta) - Arg(\beta) \text{ mod const}$$

defines a cocycle of $\mathcal{G}_1$ in $W_p^{1/p}(S^1)/const$ for any $p > 1$. The proof will in [Reznikov 10]).

1.13 Equivariant mapping of the Teichmüller Space, a space of quasifuchsian representations and a space of all discrete representations into $\mathcal{H}_{p,g}$

*Theorem 13.1.*A.— A map which associates to a discrete cocompact representation

$$G \to SO^+(1,2)$$

its principal state

$$[l]_p \in H^1(G_r, H^1(G_l, l^p(G))$$

is an $\mathcal{M}ap_g$-equivariant map of the Teichmüller space $\mathbf{T}_{6g-6}$ to $\mathcal{H}_{p,g}$ for all $p > 1$.
B.— Let $\varphi : G \to SO^+(1,3)$ be a discrete representation. Let $\alpha_\varphi : S^1 \to S^2$ be the limit map of the boundaries

$$S^1 = \partial\widetilde{\Sigma} \to \partial\mathcal{H}^3 = S^2,$$

defined in section 6, associated to φ. For $p > 2$ let

$$[l]_p \in H^1(SO^+(1,3), W_p^{2/p}(S^2)/const)$$

be the principle state. A map

$$\varphi \mapsto A_\varphi\varphi^*[l]_p \in H^1(G_r, H^1(G_l, l^p(G))),$$

defined by first pulling back $[l]_p$ to $\varphi^*[l]_p \in H^1(G, W_p^{2/p}(S^2)/const)$ and then applying the operator

$$A_\varphi : W_p^{2/p}(S^2)/const \to W_p^{1/p}(S^1)/const,$$

induced by α_φ and defined in section 6, is an $\mathcal{M}ap_g$ equivariant map

$$Hom_{discrete}(G, SO^+(1,3))/SO^+(1,3) \to \mathcal{H}_{p,g}$$

for all $p > 2$.
C.— A restriction of the map, defined in B to

$$Hom_{quasifuchsian}(G, SO^+(1,3))/SO^+(1,3)$$

is contained in $\mathcal{H}_{2,g}$.

Proof.— is already contained in section 6–12. We notice that from the operator viewpoint the map of A sends any realization of G as a lattice in $SO^+(1,2)$ to a Hilbert transform of $W_p^{1/p}(S^1)/const$, followed by an identification

$$H^1(G_l, l^p(G)) \simeq W_p^{1/p}(S^1)/const,$$

which depends on the lattice. In other words, fix one lattice embedding

$$\beta_0 : G \to SO^+(1,2).$$

Then any other lattice embedding

$$\beta : G \to SO^+(1,2)$$

can be written as

$$\beta(g) = \Phi_{\beta_0,\beta}\beta_0(g)\Phi^{-1}_{\beta_0,\beta},$$

where $\Phi_{\beta_0,\beta} \in \mathcal{G}_1$ is a quasisymmetric map. Then an operator, associated with β is

$$\Phi_{\beta_0,\beta} H \Phi^{-1}_{\beta_0,\beta} \in Aut(W_p^{1/p}(S^1)/const).$$

This gives an $\mathcal{M}ap_g$-equivariant map

$$\mathbf{T}_{6g-6} \to Aut_G(W_p^{1/p}(S^1)/const)$$

For $p = 2$ one gets a map

$$\mathbf{T}_{6g-6} \to Sp_G(W_2^{1/2}(S^1)/const)$$

because the Hilbert transform and $\mathcal{G}_1$-action are symplectic(section 7), which can be described as follows. First, one embeds $\mathbf{T}_{6g-6}$ in the universal Teichmüller space

$$\mathbf{T} = \mathcal{G}_1/SO^+(1,2).$$

Then using the representation

$$\mathcal{G}_1 \to Sp(W_2^{1/2}(S^1)/const)$$

defined in section 7, one defines an embedding to Sp/U:

$$\mathbf{T} \to Sp(W_2^{1/2}(S^1)/const)/U$$

where U is a group of operator in Sp which commutes with H seen as a complex structure in $W_2^{1/2}(S^1)/const$. Finally, one uses the Cartan embedding

$$Sp/U \to Sp.$$

Theorem 13.2.(Linearization of pseudoAnosov automorphisms)— Let $\phi \in \mathcal{M}ap_{g,1} = Aut(\pi_1(\Sigma^g))$ is a pseudoAnosov automorphism. Then for any $p > 1$ there exists a nontrivial element $S_p \in \mathcal{H}_{p,g}$ with the following properties:
1) for $p_1 < p_2$, S_{p_2} is an image of S_{p_1}, under the natural map $\mathcal{H}_{p_1,g} \to \mathcal{H}_{p_2,g}$;
2) S_p is invariant under $\bar{\phi} \in \mathcal{M}ap_g$;
3) there is a cocycle $\tilde{l}_p \in Z^1(G, W_p^{1/p}(S^1)/const)$, representing S_p, such that for any $g \in G$

$$\sum_{n\in\mathbb{Z}} \|\tilde{l}_p(g) \circ \Phi^n\|_{W_p^{1/p}(S^1)/const} < \infty$$

where $\Phi : S^1 \to S^1$ is a quasisymmetric homeomorphism, associated with ϕ (or, in other words,

$$\sum_{n\in\mathbb{Z}} \|A_\varphi^m \tilde{l}_p(g)\| < \infty$$

where $A_\varphi \in Aut(H^1(G_l, l^p(G))$ is induced by ϕ)

Proof.— is an immediate corollary of [Thurston 2] (see also an exposition in [Otal 1]), which shows that the mapping torus of any homeomorphism $\Psi : \Sigma \to \Sigma$, representing φ is a hyperbolic 3-manifold, Theorem 13.1, Theorem 9.1 and Theorem 9.3.

It is plausible that such S_p is unique up to a multiplier. Knowing S_p is essentially equivalent to knowing the hyperbolic volumes of all ideal simplices with vertices on the limit curve $S^1 \to S^2$.

Chapter 2

A theory of groups acting on the circle

Our first main result in this Chapter is Theorem 1.7 which says, roughly, that Kazhdan group cannot act on the circle. This general theorem draws a line after many years of study and various special results concerning the actions of lattices in Lie groups, see [Witte 1], [Farb-Shalen 1], [Ghys 1]. One can see here a historic parallel with a similar, but easier, general theorem of [Alperin 1] and [Watatani 1] concerning Kazhdan groups acting on trees, which also followed a study of the actions of lattices. Our technique is absolutely different from the cited papers and uses a fundamental cocycle, introduced and studied in section 1. We also use standard facts from Kazhdan groups theory [de la Harpe-Valette 1].

In Sections 2,3 we quantize equivariant maps between boundaries of universal covers, studied in Chapter I, Section 6. Our main tool is a harmonic map theory into infinite-dimensional spaces, as developed in [Korevaar-Schoen 1], see also [Jost 1]. In Section 4 we review some facts about Banach-Lie groups and regulators. In Section 5 we describe a series of higher characteristic classes of subgroups of $\mathcal{D}iff^{1,\alpha}(S^1)$. There are two construction given. One uses an extension to a restricted linear group of a Hilbert space of classes originally defined in [Feigin-Tsygan 1] for infinite Jacobian matrices. Another construction uses the action of a restricted symplectic group $Sp(W_2^{1/2}(S^1)/const)$ on the infinite-dimensional Siegel half-plane. In both construction we use an embedding of $\mathcal{D}iff^{1,\alpha}$ into a restricted linear group, by the unitary and symplectic representation of $\mathcal{D}iff$, respectively. Using the geometry of the Siegel half-plane, we prove that our classes have polynomial growth.

There is a striking similarity between the theory of this Chapter and a theory of symplectomorphism group, see Chapter IV, [Reznikov 2] and [Reznikov 4]. Note that the extended mapping class group action is not $C^{1,\alpha}$ smooth, so the results of this Chapter do not apply to this group. On the other hand, $\mathcal{M}ap_g$ does act symplectically on a smooth compact symplectic manifold.

2.1 Fundamental cocycle

By $\mathcal{D}iff^{1,\alpha}(S^1)$ we denote a group of orientation-preserving diffeomorphisms with derivative in the Hölder space $C^\alpha(S^1)$, which consists of functions f such that

$$|f(x) - f(y)| < c|x-y|^\alpha.$$

There is a series of unitary representations of $\mathcal{D}iff^{1,\alpha}(S^1)$ in $L^2_{\mathbb{C}}(S^1, d\theta)$ given by

$$(\pi(g)(f))(x) = f(g^{-1}x) \cdot [(g^{-1})'(x)]^{\frac{1}{2}+i\beta}, \quad \beta \in \mathbb{R}.$$

We will mostly consider $\beta = 0$, in which case one has an orthogonal representation in $L^2_{\mathbb{R}}(S^1, d\theta)$. An invariant meaning is, of course a representation in half-densities on S^1. Now consider a Hilbert transform H as an operator in $L^2_{\mathbb{R}}(S^1, d\theta)$ given by a usual formula

$$Hf(\varphi) = \frac{1}{\pi} v.p. \int_{S^1} \frac{f(\theta)}{tg\ \frac{\varphi-\theta}{2}} d\theta.$$

We wish to consider $[H, \pi(g^{-1})]$. This is a bounded operator in $L^2(S^1, d\theta)$ given by an integral kernel which we are going to compute. Notice that

$$\frac{1}{tg\ \frac{\varphi-\theta}{2}} = \frac{2}{\varphi - \theta} + smooth\ kernel.$$

A computation of [Pressley-Segal 1] shows that

$$H[\pi(g)f](\varphi) = \frac{2}{\pi} v.p. \int_{S^1} \frac{d\theta}{\varphi - \theta} f(g^{-1}(\theta))[(g^{-1}(\theta))']^{1/2} + smooth\ kernel \circ \pi(g),$$

so

$$(\pi(g^{-1})H\pi(g)f)(\varphi) = [g'(\varphi)]^{1/2} \cdot \tfrac{2}{\pi} v.p. \int_{S^1} \tfrac{d\theta f(g^{-1}(\theta))[(g^{-1}(\theta))']^{1/2}}{g(\varphi)-\theta} \\ + \pi(g^{-1}) \circ smooth\ kernel \circ \pi(g) \quad .$$

Letting $\theta = g(\eta)$ we have

$$(\pi(g^{-1})H\pi(g)f)(\varphi) = [g'(\varphi)]^{1/2} \tfrac{2}{\pi} v.p. \int_{S^1} \tfrac{f(\eta)\cdot[g'(\eta)]^{1/2}}{g(\varphi)-g(\eta)} d\eta + \\ + \pi(g^{-1}) \circ smooth\ kernel \circ \pi(g) = \\ = \tfrac{2}{\pi} v.p. \int_{S^1} \tfrac{[g'(\varphi)g'(\eta)]^{1/2}}{g(\varphi)-g(\eta)} f(\eta) d\eta + \pi(g^{-1}) \circ smooth\ kernel \circ \pi(g)$$

Finally,

$$[(\pi(g^{-1})H\pi(g) - H)](\varphi) = \\ = \tfrac{1}{\pi} \int_{S^1} \tfrac{[g'(\varphi)g'(\eta)]^{1/2}(\varphi-\eta)-(g(\varphi)-g(\eta))}{(g(\varphi)-g(\eta))(\varphi-\eta)} f(\eta) d\eta + \pi(g^{-1}) \circ smooth\ kernel \circ \pi(g) + \\ + smooth\ kernel. \tag{1.1}$$

For a Hilbert space $\mathcal{H}$ and $p \geq 1$ we denote by $J_p(\mathcal{H})$ a Shatten class of operators such that a sum of the p-th powers of their singular numbers converges. By $J_{p+}(\mathcal{H})$ we mean the intersection of all $J_q(\mathcal{H})$ with $q > p$.

Now recall that $g \in \mathcal{D}iff^{1,\alpha}(S^1)$. A following proposition sharpens that of [Pressley-Segal 1] for $\mathcal{D}iff^{\infty}(S^1)$:

Propositin 1.1.
A. For $\alpha > 1/2$, $\quad \pi(g^{-1})H\pi(g) - H \in J_2(L^2(S^1, d\theta))$.
B. For $\alpha > 0$, $\quad \pi(g^{-1})H\pi(g) - H \in J_{1/\alpha+}(L^2(S^1), d\theta)$.

Proof.— As $\varphi - \eta \to 0$,

$$\frac{[g'(\varphi)g'(\eta)]^{1/2}(\varphi-\eta) - (g(\varphi) - g(\eta))}{(g(\varphi) - g(\eta))(\varphi - \eta)} < const \cdot (\varphi - \eta)^{\alpha - 1},$$

so the kernel in (1.1) is in $L^2(S^1 \times S^1, d\theta \otimes d\theta)$ for $\alpha > 1/2$. This proves A.

To prove B we notice that by [Pietsch 1], the estimate on the kernel implies that the operator lies in $\mathcal{J}_{1/\alpha+}$. Strictly speaking, the conditions of [Pietsch 1] require C^∞ smoothness off the diagonal, whereas we have only the Hölder continuity, but the result stays true.

Now notice that $GL(L^2(S^1, d\theta))$ acts in J_p by conjugation. We deduce the following

Proposition 1.2.— A map

$$l : g \mapsto \pi(g)H\pi(g^{-1}) - H$$

is a 1-cocycle of $\mathcal{D}iff^{1,\alpha}(S^1)$ in $J_p(L^2(S^1, d\theta))$ for $p > 1/\alpha$. In particular, l is a 1-cocycle of $\mathcal{D}iff^{1,\alpha}(S^1)$ in J_2 for $\alpha > 1/2$.

We will call l a fundamental cocycle of $\mathcal{D}iff^{1,\alpha}(S^1)$.

Now let G be a subgroup of $\mathcal{D}iff^{1,\alpha}(S^1)$. We obtain a class in $H^1(G, J_p(L^2(S^1, d\theta))$ by restricting l on G. We are going to show that this class is never zero, except for completely pathological actions of G on S^1.

Proposition 1.3.— Let G be a subgroup of $\mathcal{D}iff^{1,\alpha}(S^1)$, $0 < \alpha < 1$. Suppose $p > 1/\alpha$. If $[l] \in H^1(G, J_p)$ zero, then the unitary action of G in $L^2_{\mathbb{C}}(S^1, d\theta)$ is reducible. Moreover, if $H^1(G, J_p) = 0$ then $L^2_{\mathbb{C}}(S^1, d\theta)$ a direct sum of countably many closed invariant subspaces.

Proof.— If $[l] = 0$ then there is $A \in J_p$ such that

$$\pi(g)H\pi(g^{-1}) - H = \pi(g)A\pi(g^{-1}) - A$$

so that $[\pi(g), H - A] = 0$. Since H has two different eigenvalues with infinitely-dimensional eigenspaces, $H - A \neq const \cdot Id$, so the action of G in $L^2_{\mathbb{C}}(S^1, d\theta)$ is reducible.

Next, consider an operator R in $L^2(S^1, d\theta)$ with a kernel

$$K(\varphi, \eta) = \frac{1}{|tg\ \frac{\varphi - \eta}{2}|}.$$

One sees immediately that R is a self-adjoint unbounded operator. Repeating the computation above, we deduce that $\pi(g)R\pi(g^{-1}) - R \in J_p$, so $\tilde{l}(g) = \pi(g)R\pi(g^{-1}) - R$ is another cocycle. If this cocycle is trivial, then we get an unbounded self-adjoint operator $R - A$ which commutes with the action of G. An application of the spectral theorem shows that $L^2(S^1, d\theta)$ is a countable sum of invariant subspaces.

Corollary 1.4.— A restriction of $l, \tilde{l}$ on $SO^+(1,2)$ is not zero, for all $\alpha > 0$.

Proof.— $SO^+(1,2)$ act in $L^2_{\mathbb{C}}(S^1, d\theta)$ as a representation of principal series, which are irreducible.

We now specialize for $\alpha = 1/2$ and $p = 2$. Since $[\tilde{l}] \in H^1(SO^+(1,2), J_2)$ is nonzero, $\|\tilde{l}(g)\|_{J_2}$ is unbounded as a function of g [de la Harpe-Valette 1]. In fact, one has the following

Proposition 1.5.— Let $\pi : SO^+(1,2) \to U(H)$ be a unitary representation and let $l : SO^+(1,2) \to H$ be a continuous cocycle. Suppose $[l] \neq 0$. Then
A. For any cocompact lattice $G \subset SO^+(1,2)$, $[l]|G \neq 0$.
B. $\|l(g^n)\|$ is unbounded as $n \to \infty$ for any hyperbolic g.
C. $\|l(\gamma^n)\|$ is unbounded as $n \to \infty$ for any parabolic $\gamma \neq 1$.

Proof.— Let $V \subset SO^+(1,2)$ be compact and such that $V \cdot G = SO^+(1,2)$. For $v \in V, g \in G$ we have

$$l(vg) = \pi(v)l(g) + l(v),$$

so $\|l(vg)\| \leq \|l(g)\| + \|l(v)\|$. If $l|G$ is bounded, then so is l. This proves A. Next, let P be the image of $SO^+(1,2)/K$ under Cartan embedding, where K is a maximal compact subgroup. By the same reason as above, $l|P$ is unbounded. Let $S^1 \subset P$ be a nontrivial orbit of K in $P \approx \mathcal{H}^2$. Notice that P is closed under raising into an integral power and there is a compact

$V \subset SO^+(1,2)$ such that

$$P \subseteq \bigcup_{n\geq 1} (S^1)^n \cdot V$$

where $(S^1)^n$ is an image of S^1 under raising to n-th power. We deduce that $l| \cup_{n\geq 1} (S^1)^n$ is unbounded. Let $\gamma \in S^1$. Then any element in $(S^1)^n$ is of the form $k\gamma^n k^{-1}$, $k \in K$, so

$$\|l(k\gamma^n k^{-1})\| \leq \|l(k)\| + \|l(k^{-1})\| + \|l(\gamma^n)\|$$

So $\|l(\gamma^n)\|$ is unbounded. Since γ can be any hyperbolic element, B follows. Notice that we proved that $\|l(g_k)\|$ is unbounded for any sequence $g_k \in P$, which escapes all compact sets. Now let $g \in SO^+(1,2)$ be parabolic $\neq 1$, and let τ be the involution fixing K. Then $\tau(g^n) \cdot g^{-n} \in P$ and escapes all compact sets, so $\|l[(\tau g^n) \cdot g^{-n}]\|$ is unbounded. It follows that either $\|l(\tau g^n)\|$ or $\|l(g^{-n})\|$ is unbounded. But all parabolics are conjugate in $SO^+(1,2)$, so C follows.

Proposition 1.6.— Let $G \subset \mathcal{D}iff^{1,\alpha}(S^1)$, $\alpha > 1/2$. Suppose that G contains an element g which is conjugate in $\mathcal{D}iff^{1,\alpha}(S^1)$ to a hyperbolic or a nontrivial parabolic fractional-linear transformation. Then $[l]|G \neq 0$ in $H^1(G, J_2)$.

Proof.— Any such g is conjugate in $\mathcal{D}iff^{1,\alpha}(S^1)$ to an element $g' \in SO^+(1,2)$ for which $\|l(g'^n)\|$ is unbounded, so $\|l(g^n)\|$ is unbounded as well.

We are ready to formulate the main result of this section.

Theorem 1.7.— Let $G \subset \mathcal{D}iff^{1,\alpha}(S^1)$, $\alpha > 1/2$. Suppose that either
1) a natural unitary action ($\beta = 0$) of G in $L^2(S^1, d\theta)$ given by

$$\pi(g)(f)(\varphi) = f(g^{-1}(\varphi)) \cdot [(g^{-1}(\varphi))']^{1/2},$$

is irreducible or is a direct sum of finitely many irreducible factors, or
2) G contains an element, conjugate in $\mathcal{D}iff^{1,\alpha}(S^1)$ to a hyperbolic fractional-linear transformation, or

3) G contains an element, conjugate in $\mathcal{D}iff^{1,\alpha}(S^1)$ to a parabolic ($\neq 1$) fractional-linear transformation, or
4)

$$\sup_{g\in G}\int\int_{S^1}\left[\frac{\sqrt{g'(\varphi)g'(\eta)}(\varphi-\eta)-(g(\varphi)-g(\eta))}{(g(\varphi)-g(\eta))(\varphi-\eta)}\right]^2 d\varphi d\eta=\infty$$

Then G is not Kazhdan.

Proof follows from the formula (II.1.1), Proposition 1.3, Proposition 1.5 and Proposition 1.6.

2.2 Construction of $N = 2$ quantum fields with lattice symmetry

It is possible that the physical time-space is discrete. Correspondingly, in the axiomatic quantum field theory it is possible that the fields must yield invariance not under the whole Poincaré group, but only under a lattice in it. See [Michailov 1], [Belavin 1] in this respect. We are going to construct mathematical objects, which yield such invariance on one hand, and quantize the equivariant measurable maps considered in I.6.3, on the other.

Theorem 2.1.— Let G be a cocompact lattice in $SO^+(1,2)$. Let $\mathcal{H} = L^2_{\mathbb{R}}(S^1, d\theta)$ with the orthogonal action π, corresponding to $\beta\in\mathbb{R}$. Then there exists a measurable map to the space of bounded operators

$$S^1 \xrightarrow{\rho} \mathcal{B}(H)$$

with the following properties.
1) Equivariance: for $s\in S^1$ and $g\in G$

$$\rho(gs)=\pi(g)\rho(s)\pi(g^{-1})$$

almost everywhere on S^1.
2)One has

$$\int_{S^1}(\rho(s)-H)\psi(s)ds\in J_2$$

for $\psi \in C^\infty(S^1)$.
3)There exists $J \in J_2(\mathcal{H})$ such that $\rho(s)$ is a weak nontangential limit

$$\rho(s) = \lim_{g \to s} \pi(g)(H + J)\pi(g^{-1})$$

as $g \in G$ converges nontangentially to $s \in S^1 = \partial G$ a.e. on S^1.

Proof.— As a Hilbert space with orthogonal G-action, $J_2 = L^2(S^1 \times S^1, d\theta \otimes d\theta)$. By the proof of Lemma I.11.3, G does not have almost invariant vectors in J_2. Let $\Sigma = \mathcal{H}^2/G$ and let E be a flat affine vector bundle over Σ with a fiber J_2 and monodromy

$$g \mapsto Ad\pi(g) + l(g).$$

Then by a result of [Korevaar-Schoen 1], and [Jost 1] (lemma I.11.6), there exists a harmonic map

$$\tilde{f} : \mathcal{H}^2 \to J_2$$

satisfying

$$\tilde{f}(gx) = \pi(g)\tilde{f}(x)\pi(g^{-1}) + l(g)$$

Consider $f(x) = \tilde{f}(x) + H$. Then

$$f(gx) = \pi(g)f(x)\pi(g^{-1}),$$

in particular, $\|f(x)\|$ is bounded in operator norm. An operator version of Fatou theorem [Naboko 1 and references therein] shows that f has nontangential limit values a.e. on S^1, say $\rho(s)$. Obviously, ρ is G-invariant. On the other hand, $\tilde{f}$ is a Bloch harmonic J_2-valued function, that is,

$$\sup_{x \in \mathcal{H}^2} \|\nabla \tilde{f}\|_{J_2} < \infty.$$

It follows that $\|\tilde{f}(w)\|_{J_2} < c \cdot \log(1 - |w|)$, $w \in B^2 = \mathcal{H}^2$. This implies by a standard argument (see e.g. [Gorbačuk 1] that $\tilde{f}$ has a limit on S^1 as an element of $\mathcal{D}'(S^1, J_2)$. So for $\psi \in C^\infty(S^1)$,

$$\int_{S^1} (\rho - H)\psi \in J_2,$$

which proves the Theorem.

Remarks.— 1)As was mentioned above, the invariant meaning of the representation π is that $L^2(S^1, d\theta)$ should be regarded as a space of half-densities. Correspondingly, an integral operator is defined by a kernel which is a half-density on $S^1 \times S^1$ of the type $K(\varphi, \eta)(d\varphi d\eta)^{1/2}$. If $K(\varphi, \eta)$ is smooth and has a zero of second order on the diagonal $\Delta \subset S^1 \times S^1$, then one has an invariant definition of its residue or second derivative, which is a quadratic differential. A direct computation which we leave to the reader shows that for $g \in \mathcal{D}iff^\infty(S^1)$
: 1) $l(g) = \pi(g)H\pi(g^{-1}) - H$ is given by a kernel which has a zero of second order on Δ;
2) a corresponding residue $S(g)$ is the Schwartzian of g.

This shows that $l(g)$ is a quantization of the Schwartzian cocycle. The operator field $\rho(s)$ of Theorem 1.8 seems therefore to be related to objects axiomatized, but not constructed, in [Belavin-Polyakov-Zamolodchikov 1].
2)The Theorem and the proof stay valid for any representation

$$\varphi : G \to \mathcal{D}iff^{1,\alpha}(S^1),$$

$\alpha > 1/2$, such that the action on $S^1 \times S^1$ satisfies the very mild conditions of Lemma I.11.3.

2.3 Construction of $N = 3$ quantum fields with lattice symmetry

A theory developed have for $\mathcal{D}iff(S^1)$ does not generalize to $\mathcal{D}iff(S^n), n \geq 2$. The reason is that the action of $\mathcal{D}iff(S^1)$ on S^1 is conformal. There are two ways to generalize various aspects of the theory to higher dimensions, by either considering $SO^+(1,n)$ acting on S^{n-1} or, very surprisingly, a group of symplectomorphisms of a compact symplectic manifold M (see Chapter IV). Here we consider the action of $SO^+(1,3) \simeq PSL_2(\mathbb{C})$ on S^2. We set $d(x,y)$ to be a spherical distance in S^2. Let $d\theta$ denote the spherical measure and let $\mathcal{H} = L^2(S^2, d\theta)$. For $g \in SO^+(1,3)$ let $\mu_g(x)$ denote a conformal factor, that is $\mu_g^2(x)$ is a Jacobian of g with respect to $d\theta$. A formula

$$\pi(g)f(x) = f(g^{-1}(x)) \cdot \mu_{g^{-1}}^{1+i\beta}(x), \quad \beta \in \mathbb{R},$$

defines a unitary representation of $SO^+(1,3)$ in $\mathcal{H}$. Now we introduce an operator H with the kernel

$$K(\varphi,\theta) = \frac{1}{d^2(\varphi,\eta)}.$$

This operator is self-adjoint and unbounded. Our goal is to compute

$$\pi(g)H\pi(g^{-1}) - H = l(g).$$

Proposition 3.1.— $l(g) \in J_2$ for all $g \in SO^+(1,3)$ and $\beta = 0$.

Proof.— A direct computation. One needs to show that as $d(x,y) \to 0$,

$$d^2(g(x),g(y)) - \mu_g(x)\mu_g(y)d^2(x,y)$$

is of order $d^4(x,y)$. In other words, for a fractional-linear trangformation g of $\mathbb{C}$ one needs to show that as $x \to y$, $Im\ x, Im\ y > 0,\ g(x) = x$,

$$\left|\frac{g(x)-g(y)}{g(x)-\overline{g(y)}}\right|^2 - |g'(x)||g'(y)|\frac{Im\ y}{Im\ g(y)}\left|\frac{x-y}{x-\bar{y}}\right|^2$$

is of order $|x-y|^4$. This verifies the result for hyperbolic metric instead of spherical metric, which is of course equivalent. One computes directly using Taylor series for holomorphic function g.

Now arguing as in section 2 we arrive at the following result.

Theorem 3.2.— Let G be a cocompact lattice in $SO^+(1,3)$. Let $\mathcal{H} = L^2_{\mathbb{R}}(S^2,d\theta)$ with orthogonal action of G corresponding to $\beta = 0$. Then there exists a harmonic map

$$\mathcal{H}^3 \xrightarrow{\psi} J_2(\mathcal{H})$$

with the property that $z \mapsto \psi(z) + H$ is equivariant:

$$\psi(gz) + H = \pi(g)(\psi(z) + H)\pi(g^{-1})$$

for all $g \in G$ and $z \in \mathcal{H}^3$.

Since H is unbounded, the boundary value of $\psi(z)+H$ does not exist as a measurable map to the space of bounded operators. It is possible that there is a more clever choice of a conformally natural singular integral operator which is bounded, but I don't know how to do it. Note in this respect that there is a very different realization of an orthogonal representation of $SO^+(1,3)$ in the space of functions on S^2 , discovered in [Reznikov 1]. Namely, look at the natural action of $SO^+(1,3)$ on smooth half co-densities, that is, sections of $\sqrt{\Lambda^2 TS^2}$. Using the spherical metric, we can identify this space with $C^\infty(S^2)$. Then the above-mentioned action leaves invariant a nonnegative quadratic form

$$Q(f) = \int_{S^2} ((\Delta f)^2 - 2|\nabla f|^2) darea$$

whose kernel consists of constants and linear functions. It is possible that there are G-equivariant quantum fields valued in operators acting in the associated Hilbert space.

2.4 Banach-Lie groups and regulators: an overview

A Banach-Lie group is a Banach manifold with a compatible group structure. Usual Lie theory largely extends to this case. In particular, if $\mathcal{G}$ is a Banach-Lie group and $\mathfrak{g}$ its Banach-Lie algebra, then a continuous n-cocycle on $\mathfrak{g}$ defines a left-invariant closed form on $\mathcal{G}$, so that one has a homomorphism

$$H^n_{cont}(\mathfrak{g}, \mathbb{K}) \to H^n_{top}(\mathcal{G}, \mathbb{K})$$

where H^*_{top} is a cohomology of a topological space. In [Reznikov 2] we defined $\mathbb{K}$-homotopy groups of a Lie algebra, so that there is a map

$$\pi_i(\mathcal{G}) \otimes \mathbb{K} \to \pi_i(\mathfrak{g})$$

which in the case $\mathcal{G} = SL_n(C^\infty(M))$, M a compact manifold, $n >> 1$, reduces to the Chern character

$$K_i^{top}(M) \to HC_i(C^\infty(M)) = \Omega^i(M)/d\Omega^{i-1}(M) \oplus H^{i-2}(M, \mathbb{K}) \oplus \cdots$$

($\mathcal{G}$ is not a Banach-Lie group but a Frechét-Lie group in this case). More interesting is a secondary class (=regulator) map. Define an algebraic K-theory of $\mathcal{G}$ as

$$K_i^{alg}(\mathcal{G}) = \pi_i((B\mathcal{G}^\delta)^+)$$

and the augmented K-theory as a kernel of the map $K_i^{alg} \to K_i^{top}$:

$$0 \to \overline{K}_i^{alg}(\mathcal{G}) \to K_i^{alg}(\mathcal{G}) \to \pi_i(B\mathcal{G}) = \pi_{i-1}(\mathcal{G}).$$

Then the regulator map is a homomorphism

$$r : \overline{K}_i^{alg}(\mathcal{G}) \to coker(\pi_i(\mathcal{G}) \otimes \mathbb{K} \to \pi_i(\mathfrak{g})).$$

Lifting this map to cohomology, that is , constructing a map

$$H^*_{cont}(\mathfrak{g}, \mathbb{K}) \to H^*(\mathcal{G}^\delta, \mathbb{K})$$

meets obstructions described in the van Est spectral sequence. If $\mathcal{K} \subset \mathcal{G}$ is a closed subgroup such that $\mathcal{G}/\mathcal{K}$ is contractible, then these obstructions vanish and one gets a map

$$H^*_{cont}(\mathfrak{g}, \mathfrak{k}) \to H^*(\mathcal{G}^\delta)$$

given explicitly by a Dupont-type construction [Dupont 1]. This is essentially the same as geometric construction of secondary classes of flat $\mathcal{G}$-bundles, described in [Reznikov 3]. In case $\mathcal{G} = SL_n(C^\infty(M))$ this gives a usual regulator map in algebraic K-theory. However, for various diffeomorphism groups one construct new interesting classes. For symplectomorphism groups two series of classes, mentioned in the Introduction to Chapter 4, were constructed in [Reznikov 2] and [Reznikov 4], and a new class associated to a Lagrangian submanifold, will be constructed in Chapter 4. The symmetric spaces for $Sympl(M)$, used in [Reznikov 2] are sort of continuous direct products of finite-dimensional Siegel upper half-planes. On the other hand, a symmetric space which we will use in this chapter to construct classes in $H^*(\mathcal{D}iff^{1,\alpha}(S^1))$ is an infinite-dimensional Siegel half-plane. The trouble is, however, that, for a compact manifold Y, (say, S^1) a group of diffeomorphisms of finite smoothness, like $\mathcal{D}iff^k(Y)$, is not a Banach-Lie group: the multiplication from the right is not a diffeomorphism (the multiplication from the left is). This is neatly explained in [Adams-Ratiu-Schmid 1]. Luckily, to construct secondary classes we only use the fact that the multiplication from the left is a diffeomorphism.

2.5 Charateristic classes of foliated circle bundles

As is well known, the continuous cohomology of $\mathcal{D}iff^\infty(S^1)$ is generated by the Euler class and by the integrated Godbillon-Vey class [Geldfand-Fuks 1], [Fuks 1 and references therein]. Moreover, the square of the Euler class is zero. This already shows that the degree of smoothness is crucial. For if one considers the action of the extended mapping class group

$$\mathcal{M}ap_{g,1} \hookrightarrow \mathcal{G}_1 \hookrightarrow Homeo(S^1),$$

then the pull-back of the Euler class has nonzero powers to a degree which goes to infinity with g [Miller 1], [Morita 1], [Mumford 1]. It appears that the scarcity of the cohomology of $\mathcal{D}iff(S^1)$ is a consequence of an (artificial) restriction of excessive degree of smoothness. Notice that the proofs in [Fuks 1] depend hopelessly on C^∞- smoothness. We will give two constructions of a series of new classes in $H^*(\mathcal{D}iff^{1,\alpha}(S^1))$, $0 < \alpha < 1$ using both the unitary representation in $L^2(S^1, d\theta)$ and the symplectic representation in $Sp(W_2^{1/2}(S^1)/const)$. As in the case of the powers of the Euler class, a nonvanishing of these classes is an obstruction to smoothability, i.e. to a conjugation to a subgroup of $\mathcal{D}iff^\infty(S^1)$. We will also prove that our classes are of polynomial growth if $\alpha > 1/2$. A related result (but not the argument) for C^∞ Gelfand-Fuks cohomology in all dimensions is to be found in [Connes-Gromov-Moscovici 1]. Both in spirit and technology, the construction of the classes in $H^*(\mathcal{D}iff^{1,\alpha}(S^1))$ resembles our construction of a series of classes in $H^k_{cont}(Sympl(M), \mathbb{R})$, $k = 2, 6, 10, \cdots$, where M is a compact symplectic manifold and $Sympl(M)$ is its symplectomorphism group [Reznikov 4].

We start with the construction using the unitary representation. By Proposition 1.1, $\pi(g)H\pi(g^{-1}) - H \in J_p$ where $g \in \mathcal{D}iff^{1,\alpha}(S^1), \quad p > 1/\alpha$, π is a unitary action in $L^2_{\mathbb{C}}(S^1, d\theta)$, and H is a complexification of the Hilbert transform. That is $H(e^{in\theta}) = sgn(n) \cdot e^{in\theta}$. The group of $\Phi \in GL(\mathcal{H})$, $\mathcal{H} = L^2_{\mathbb{C}}(S^1, d\theta)$ such that $\Phi H \Phi^{-1} - H \in J_p$ will be denoted $GL_{J_p}(\mathcal{H})$, following [Pressley-Segal 1]. The unitary subgroup of $GL_{J_p}(\mathcal{H})$ is denoted $U_{J_p}(\mathcal{H})$. Let $\mathcal{H}_+, \mathcal{H}_-$ be the eigenspaces of H with eigenvalues $+1$ and -1 respectively. By $Gr_{J_p}(\mathcal{H})$ we denote the restricted Grassmanian $U_{J_p}/U(\mathcal{H}_+) \times U(\mathcal{H}_-)$. Then $Gr_{J_p}(\mathcal{H})$ is a Banach manifold, modelled by the Banach space J_p. The Banach-Lie group $GL_{J_p}(\mathcal{H})$ acts smoothly on $Gr_{J_p}(\mathcal{H})$. On the other hand, though $\mathcal{D}iff^{1,\alpha}(S^1)$ is a group and a Banach manifold, it is not a Banach-Lie group [Adams-Ratiu-Schmid 1]. However, multiplication from the left $L_g(h) = gh$ is a diffeomorphism (but not a multiplication from the right). The embedding

$$\mathcal{D}iff^{1,\alpha}(S^1) \to U_{J_p} \to GL_{J_p}(\mathcal{H})$$

is not continuous. However, an induced action of $\mathcal{D}iff^{1,\alpha}(S^1)$ on $Gr_{J_p}(\mathcal{H})$ is smooth [Pressley-Segal 1].

We will introduce a series of U_{J_p}-invariant differential forms on $Gr_{J_p}(\mathcal{H})$. These forms induce cohomology classes in the Lie algebra cohomology $H^*(Lie(U_{J_p}))$, extending the classes introduced in [Feigin-Tsygan 1] for the Lie algebra of Jacobian matrices. Notice that a tangent space to the origin of $Gr_{J_p}(\mathcal{H})$ can be identified with matrices of the form

$$C = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}$$

where $A \in J_p(\mathcal{H}_+, \mathcal{H}_-)$ and $B \in J_p(\mathcal{H}_-, \mathcal{H}_+)$. Let $C_1, \cdots, C_{2k}, \quad (k \text{ odd})$ be a collection of such matrices. Define

$$\mu_k(C_1, \cdots, C_{2k}) = \sum_{\sigma \in S_{2k}} sgn(\sigma) P_k(\rho(C_{\sigma(1)}, C_{\sigma(2)}), \cdots, \rho(C_{\sigma(2k-1)}, C_{\sigma(2k)})$$

where P_k is the k-th invariant symmetric functions of k matrices, which is a polarization of $tr\ A^k$ (not an elementary symmetric polynomial, as in

[Fuks 1]). Now, $\rho(C_1, C_2)$ is defined as follows: let $\pi(C)$ is the left upper corner of C, i.e. an operator in $\mathcal{B}(\mathcal{H}_+)$. Then $\rho(C_1, C_2) = \pi([C_1, C_2]) - [\pi(C_1), \pi(C_2)]$. The "meaning" of π is that of a connection of a principal bundle on something like the classifying space of the Lie algebra $Lie(GL_{J_p})$, and of ρ is that of the curvature of this connection. Then μ_k becomes a characteristic class, somewhat analogous to the characteristic classes in the standard Chern-Weil theory. Notice that μ_k is defined for all $k \geq [1/\alpha] + 1$. In [Feigin-Tsygan 1], $\rho(C_1, C_2) \in \mathfrak{gl}(\infty, \mathbb{K})$ and μ_k is defined for all k. The form μ_2 defines the famous "Japanese cocycle", [Verdier 1].

Lemma 5.1.— μ_k is U_{J_p}-invariant and closed.

Proof.— The invariance is obvious. The proof of closedness is standard and left to the reader, see the remarks above and [Feigin-Tsygan 1].

Pulling back to $\mathcal{D}iff^{1,\alpha}(S^1)$ (this is possible by the remarks made above) we obtain a left-invariant closed differential form on $\mathcal{D}iff^{1,\alpha}(S^1)$. Pulling back to the universal cover $\widetilde{\mathcal{D}iff}^{1,\alpha}(S^1)$, we obtain a left-invariant closed differential form $\tilde{\mu}_k$ on $\widetilde{\mathcal{D}iff}^{1,\alpha}(S^1)$. A following theorem follows.

Theorem 5.2.— The secondary characteristic class, corresponding to $\tilde{\mu}_k$ is a well-defined class $r(\tilde{\mu}_k)$ in $H^{2k}([\widetilde{\mathcal{D}iff}^{1,\alpha}]^{\delta}, \mathbb{R})$.

Proof.— $\widetilde{\mathcal{D}iff}^{1,\alpha}(S^1)$ is contractible.

Notice that for $\alpha > 1/2$ the class $\mu_1 \in H^2(\mathcal{D}iff^{1,\alpha}(S^1))$ is defined, which is just the integrated Godbillon-Vey class.

Our second construction uses the symplectic action. For simplicity, we only treat the case $\alpha > 1/2$. Recall (Corollary I.7.3) that $\mathcal{G}_1$ acts symplectically in $V = W_2^{1/2}(S^1)/const$. Restricting on $\mathcal{D}iff^{1,\alpha}(S^1)$, we obtain a representation

$$\mathcal{D}iff^{1,\alpha}(S^1) \xrightarrow{\pi} Sp(V).$$

Let H be the Hilbert transform in V, normalized such that $H^2 = -1$. Denote by Sp_{J_p} a subgroup of $A \in Sp(V)$ such that $[A, H] \in J_p$. Denote $U = U(V)$ the unitary group of such A that $[A, H] = 0$. Denote

$$X = Sp_{J_p}/U$$

a restricted Siegel half-plane. This is a Banach contractible manifold [Palais 1]. For $p = 2$ this is a Hilbert manifold with canonical Sp_{J_2}-invariant Riemannian metric of nonpositive curvature. The metric is defined as follows. The tangent space $T_H(X)$ is identified with operators A such that $A \in Lie(Sp_{J_2})$ and $AH = -HA$. It follows that $A \in J_2$, and $A = A^*$. Then the metric is defined as trA^2. This definition is dimension-free and so the proof that the curvature is nonpositive follows from the explicit formulae, as in finitely-dimentional case.

Lemma 5.3.— For $\alpha > 1/2$, $\pi(\mathcal{D}iff^{1,\alpha}(S^1)) \subset Sp_{J_2}(V)$.

Proof.— We will use the computation of [Segal 1]. Let $g \in \mathcal{D}iff^{1,\alpha}(S^1)$. We need to show that

$$S = \sum_{n,m>0} \frac{m}{n} \left| \int_{S^1} e^{i(ng(\theta)+m\theta)}\ d\theta \right|^2 < \infty.$$

As in [Segal 1] we have, using a trick of Kazhdan,

$$S = \sum_{N=1}^{\infty} \sum_{m=1}^{N-1} \frac{m}{n} \left| \int_{S^1} e^{iN\varphi} \cdot [g_\beta^{-1}]'(\varphi) d\varphi \right|^2,$$

where $\beta = \frac{n}{N}$, $n = N - m$, $g_\beta(\theta) = \beta g(\theta) + (1-\beta)\theta$, $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. For $0 \le \beta \le 1$, g_β^{-1} are uniformly in $\mathcal{D}iff^{1,\alpha}(S^1)$ with $\alpha > 1/2$, so

$$\int_{S^1} e^{iN\varphi}[g_\beta^{-1}]'(\varphi)\ d\varphi \le const \cdot N^{-\alpha} \cdot c_N$$

with $\sum_{N=1}^{\infty} c_N^2 < \infty$. Since $\sum_{m=1}^{N-1} \frac{m}{n} \sim \log N$, we have

$$S \le const \cdot \sum_{N=1}^{\infty} N \log N N^{-2\alpha} \cdot c_N^2 < \infty.$$

Now let k be odd, $A_1, \cdots, A_{2k} \in T_H X$ and

$$\nu_k(A_1, A_2, \cdots, A_{2k}) =$$

Lemma 5.4.— ν_k is closed and Sp_{J_2}-invariant.

Proof.— is identical to the finite-dimensional case [Borel 1].

Theorem 5.5.— The secondary characteristic class, corresponding to ν_k defines an element $r(\nu_k)$ in $H^{2k}_{cont}(Sp_{J_2}(V))$ and in $H^{2k}([\mathcal{D}iff^{1,\alpha}(S^1)]^\delta, \mathbb{R})$, $\alpha > 1/2$. All these classes are of polynomial growth.

Proof.— Only the last statement needs a proof. For $x_0, \cdots, x_s \in X$ denote a geodesic span $\sigma(x_0, \cdots, x_s)$ in the following inductive way: $\sigma(x_0, x_1)$ is a geodesic segment joining x_0 and x_1 and $\sigma(x_0, \cdots, x_s)$ is a union of geodesic segments joining x_0 and points of $\sigma(x_1, \cdots, x_s)$. By standard comparison theorems $Vol_s(\sigma(x_0, \cdots, x_s)) \le const \cdot [\max_{0 \le i \le j \le s} \rho(x_i, x_j)]^s$, where $\rho(\cdot, \cdot)$ is the distance function (this is where we use non-positive curvature). By [Dupont 1], $r(\nu_k)$ can be represented by a cocycle

$$g_1, \cdots, g_{2k} \mapsto \int_{\sigma(x_0, g_1 x_0, g_1 g_2 x_0, \cdots, g_1, g_2 \cdots g_{2k} x_0)} \nu_k$$

where $g_i \in Sp_{J_2}$ and $x_0 \in X$ is fixed. Since ν_k is uniformly bounded, the result follows.

We will give an independent proof of polynomial growth of $\mu_2 \in H^2(\mathcal{D}iff^{1,1}(S^1))$. Let $Var(S^1)$ be a space of functions of bounded variation on S^1 mod constants. Then for $f_1, f_2 \in Var(S^1)$,

$$\int_{S^1} f_1 \cdot d\ f_2 \le c \|f_1\|_{Var} \cdot \|f_2\|_{Var}.$$

Now, $Homeo(S^1)$ acts isometrically in $Var(S^1)$ and there is a cocycle $\psi \in H^1(\mathcal{D}iff^{1,1}(S^1), Var)$ given by $g \mapsto \log(g^{-1})'$. By an formula of Thurston, μ_2 can be represented as

$$\int_{S_1} \psi(g_1)\ d\psi(g_2).$$

The result now follows from Lemma I.1.1. For μ_2 as a class in $H^2(\mathcal{D}iff^{\infty}(S^1))$ see also [Connes-Gromov-Moscovici 1].

2.6 Examples

A typical example of a group in $\mathcal{D}iff^{1,\alpha}(S^1)$ is a following one. Let $K \subset \mathbb{R}$ be a subfield (i.e., a number field). By $S^1(K)$ denote K-rational points of $S^1 \subset \mathbb{R}^2$. Define G_K as a group of C^1-diffeomorphism g such that there are points $x_0, \cdots, x_n = x_0 \in S^1(K)$ in this order such that $g_k = g_{[x_k, x_{k+1}]}$ is a restriction of an element of $PSL_2(K)$. The C^1-condition simply means that $g'_k(x_{k+1}) = g'_{k+1}(x_{k+1})$. Then automatically $G_K \subset \mathcal{D}iff^{1,1}(S^1)$. Groups of this type, or rather their obvious analogues which act by piecewise-affine transformations on S^1 viewed as $\mathbb{R}/\mathbb{Z}$ appeared in [Thompson 1], [Greenberg-Sergiesku 1,2], [Brown-Georghegan 1], etc. where various properties were studied. The "proper" Thompson group can be smoothed , that is, embedded in $\mathcal{D}iff(S^1)$ [Ghys-Sergiesku 1] so that the Theorem 1.7 applies. However, it also acts on a tree so it is not Kazhdan already by a result of [Alperin 1], [Watatani 1]. Generally, subgroups of $\mathcal{D}iff^{1,\alpha}(S^1)$ like described above, do not have any obvious action on a tree and one needs our Theorem 1.7 to show that they are not Kazhdan. A parallel theorem for symplectomorphism groups will be given in Chapter IV. Notice also that the proof that our characteristic classes constructed in Section 5 are in polynomial cohomology agrees with a recent result on the growth of the Dehn function of the Thompson group [Guba 1].

Chapter 3

Geometry of unitary cocycles

In this Chapter we return to the asymptotic geometry of finitely generated groups. If G is not Kazhdan, then an orthogonal cocycle $l \in Z^1(G, \mathcal{H})$ should be viewed as a way to linearize the geometry of G. Our first result is a convexity theorem 2.1 which says that the embedding of G into the Hilbert space $\mathcal{H}$ given by l coarsely respects the geometry in a sense that inner points of big "domains" in G are mapped inside the convex hull of the image of boundary points.

We have seen in Chapter I that primitive functions $\mathcal{F} : G \to \mathbb{R}$ of cocycles in $Z^1(G, l^p(G))$ of a surface group satisfy

$$|\mathcal{F}(g)| < c \cdot length(g)^{1/p'}$$

Here, we start a general study of cocycle growth. We show in Theorem 3.1 that for any orthogonal cocycle $l : G \to \mathcal{H}$,

$$\|l(g)\| < c(\theta)[length(g) \log\log\ length(g)]^{1/2}$$

for almost all $\theta \in \partial G \simeq S^1$ and $g \to \theta$ nontangentially. We use in proof an adjusted version of Makarov's law of iterated logarithm. The result extends to all complex hyperbolic cocompact lattices of any dimension.

Using another deep result of Makarov, we show the following in Proposition 3.3. Let G be a surface group, $\beta : G \to \mathbb{Z}$ a surjective homomorphism and $G_0 = Ker\beta$. Then the conical limit set of G_0 has Hausdorff dimension

1, in particular, the exponent $\delta(G_0) = 1$. We do not know if this set has a full Lebesgue measure (it is certainly a doable problem).

Notice that the proof of Lemma I.11.8 shows that the estimate on $\|l(g)\|$ is essentially sharp. It also shows that this estimate does not hold in other Banach spaces. However, imposing various restricitons on a Banach space, one still hopes to get an estimate, reflecting a fine structure of G.

3.1 Smooth and combinatorial harmonic sections

Let G be a finitely generated group. $\pi : G \to O(\mathcal{H})$ an orthogonal representation without almost invariant vectors and $l : G \to \mathcal{H}$ a nontrivial cocycle. If M is a compact Riemannian manifold with $\pi_1(M) = G$ (so that G is finitely presented) then one forms a flat affine bundle E over M with fiber $\mathcal{H}$ and monodromy

$$g \mapsto (v \mapsto \pi(g)v + l(g))$$

A result of [Korevaar-Schoen 1] and [Jost 1] (Lemma I.11.6) states that there is a harmonic section f of E. If M is Kähler then there is another cocycle $m : G \to \mathcal{H}$ so that a complex affine bundle with fiber $E \otimes \mathbb{C}$ and monodromy

$$g \mapsto (v + iw \mapsto \pi(g)v + i\pi(g)w + l(g) + im(g))$$

admits a holomorphic section. Our first result is a combinatorial version of this theorem.

Let $\{\gamma_i\}$ be a finite set of generators for G. Let V be a space of "sections", that is, G-equivariant maps

$$f : G \to \mathcal{H}.$$

This simply means that $f(g^{-1}x) = \pi(g)f(x) + l(g)$. Obviously, such map is determined by $f(1) \in \mathcal{H}$. Therefore, $V \approx \mathcal{H}$. A combinatorial Laplacian is defined as

$$\triangle f(x) = \sum_i f(\gamma_i x) + f(\gamma_i^{-1}x) - 2f(x).$$

Proposition 2.1.— There exists an equivariant $f : G \to \mathcal{H}$ with $\triangle f = 0$.

Proof.— Let $v = f(1)$, then $f(x^{-1}) = xv + l(x)$. Therefore

$$\begin{array}{rcl}\triangle f(x^{-1}) & = & \sum f(\gamma_i x^{-1}) + f(\gamma_i^{-1} x^{-1}) - 2f(x^{-1}) \\ & = & \sum x\gamma_i^{-1}v + l(x\gamma_i^{-1}) + x\gamma_i v + l(x\gamma_i) - 2xv - 2l(x) \\ & = & \sum x(\gamma_i^{-1}v + \gamma_i v - 2v) + \sum xl(\gamma_i^{-1}) + l(x) + xl(\gamma_i) + l(x) - 2l(x) \\ & = & x\sum(\gamma_i^{-1} + \gamma_i - 2)v + x\sum[l(\gamma_i^{-1}) + l(\gamma_i)],\end{array}$$

so that we need only to solve an equation

$$\sum(\gamma_i^{-1} + \gamma_i - 2)v = -\sum[l(\gamma_i^{-1}) + l(\gamma_i)].$$

Notice that $\widetilde{\triangle} : \mathcal{H} \to \mathcal{H}$ defined by $v \mapsto \sum(\gamma_i^{-1} + \gamma_i - 2)v$ is selfadjoint. Moreover, since $\widetilde{\triangle} = -\sum(\pi(\gamma_i) - 1)^*(\pi(\gamma_i) - 1)$, $\widetilde{\triangle}$ is nonpositive and if $0 \in spec(\widetilde{\triangle})$, then $\pi : G \to O(\mathcal{H})$ has almost invariant vectors. Therefore, $\widetilde{\triangle}$ is invertible and the result follows.

3.2 A convexity theorem

We keep the notation of 3.1. Any cocycle $l : G \to \mathcal{H}$ can be seen as an embedding of G in the Hilbert space. If $\|l(g)\| \to \infty$ as $length(g) \to \infty$, then this embedding is uniform in the sense that $\|l(g) - l(h)\| \to \infty$ as $\rho(g, h) \to \infty$ for any word left-invariant metric on G. For instance, Proposition I.2.1 implies that any group G acting discretely (but possibly not cocompactly) on an Hadamard manifold of pinched negative curvature, admits a uniform embedding into $l^p(G)$, $p > 1$. We are, however, interested in a finer geometry of the cocycle embeddings.

For a finite $A \subset G$ and $C > 0$, a C-interior $int_C(A)$ is defined as $\{x | \rho(x, y) < C \Rightarrow y \in A\}$. A C-boundary $\partial_C(A)$ is defined as $A \backslash int_C(A)$.

Theorem 2.2.— Let $\pi : G \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation without almost-invariant vectors. Let $l : G \to \mathcal{H}$ be a cocycle for π. Then there are constants $C_1, C_2(l) > 0$ such that for any finite $A \subset G$ and any $x \in A$,

$$dist_{\mathcal{H}}(l(x) - \overline{conv}(l(\partial_{C_1} A))) \leq C_2. \qquad (*)$$

Proof.— Let $f : G \to \mathcal{H}$ be an equivariant harmonic map of Proposition 1.1. Since $\|f(x^{-1}) - l(x)\| = \|f(1)\| = const$, we can replace $(*)$ by a condition

$$dist_{\mathcal{H}}(f(x) - \overline{conv} f(\partial_{C_1}(A)) \leq C_2',$$

where however, one uses a right-invariant word metric on G in definition of $\partial_C(A)$. This result follows from the maximum principle of harmonic functions. Indeed, let $x \in int_{C_1}(A)$ be such that $dist_{\mathcal{H}}(f(x) - \overline{conv} f(\partial_{C_1}(A)))$ is maximal possible (and $> C_2$)(a choice of C_1, C_2 will be made later). Let v be a unit vector, such that

$$(f(x) - y, v) = dist_{\mathcal{H}}(f(x) - \overline{conv} f(\partial_{C_1}(A))$$

for some $y \in \overline{conv} f(\partial_{C_1}(A))$. Let $h(z) = (f(z) - y, v)$. Then $h(x) > C_2$ and $h(\partial_{C_1}(A)) \subset (-\infty, 0]$. Moreover, $\widetilde{\triangle} h = 0$ and $h(z) \leq h(x)$ for $z \in int_{C_1}(A)$. It follows that $h(\gamma_i x) = h(x)$ for all i. Replacing x by $\gamma_i x$ and continuing until we hit $\partial_{C_1} A$, we arrive to a contradiction with $C_1 = 2$, $C_2 = 2\|f(1)\| + 1$.

3.3 Cocycle growth for a surface group

In this section we continue, for general representations, a subject started in I.5.2. Recall that, for any group G, any primitive function $\mathcal{F} : G \to \mathbb{R}$ of a class in $H^1(G, l^p(G))$ satisties

$$|\mathcal{F}(g)| \leq const \cdot length(g)$$

at least of G is finitely presented. However, if $G = \pi_1(\Sigma)$, a surface group, then one has much finer estimate, established in Theorem I.5.2:

$$|\mathcal{F}(g)| \leq const \cdot length(g)^{1/p'}.$$

Theorem 3.1.— Let $G = \pi_1(\Sigma)$ be a surface group. Let $\pi : G \to O(\mathcal{H})$ be an orthogonal representation without almost-invariant vectors and let $l : G \to \mathcal{H}$ be a cocycle. Then for almost all $\theta \in S^1 \approx \partial G$,

$$\|l(g)\| \leq const(\theta)[length(g) \cdot \log\log\ length(g)]^{1/2} \qquad (*)$$

as g converges nontangentially to θ. Here "almost all" corresponds to a Lebesgue measure on ∂G, identified with S^1 under some lattice embedding $G \hookrightarrow SO^+(1,2)$.

Remark.— Nontangential convergence of points of B^2 to $\theta \in \partial B^2$ is an invariant of quasi-conformal homeomorphism [???]. Therefore $(*)$ is $\mathcal{M}ap_{g,1}$-invariant. Let $A \subset S^1$ be an exceptional set where $(*)$ does not hold. It follows that the Lebesgue measure:

$$meas\ \varphi(A) = 0$$

for all $\varphi \in \mathcal{M}ap_{g,1}$, considered as a quasisymmetric homeomorphism of S^1.

Proof.— Complexifying, we find a holomorphic section of an affine bundle $E_{\mathbb{C}}$ as in section 1. Lifting to $\mathcal{H}^2$, we obtain an equivariant holomorphic map (we replace $\mathcal{H}$ by $\mathcal{H} \otimes \mathbb{C}$)

$$\widetilde{f} : \mathcal{H}^2 \to \mathcal{H}.$$

Notice that $\widetilde{f}$ is a Bloch function, that is, $\|\nabla \widetilde{f}\| \leq const$. The result now follows from a version of the Makarov law [Makarov 1] of iterated logarithm for Hilbert-space-valued Bloch functions.

Proposition.— Let $\psi : B^2 \to \mathcal{H}$ be holomorphic and $\|\nabla \psi\|_h \leq const$. Then for almost all $\theta \in S^1$,

$$\limsup_{z \to \theta} \frac{\|\psi(z)\|}{\sqrt{\log(1-|z|) \log\log\log(1-|z|)}} < \infty.$$

Proof.— We will simply note which changes should be made in a proof for complex-valued functions [Pommerenke 1]. The Hardy identity [Pommerenke 1, page 174] holds in the following form. Let S be a Riemannian surface, $z_0 \in S$, $g : S \to \mathcal{H}$ a holomorphic function, (x,y) normal coordinates in the neighbourhood of z_0. Let n be a positive integer. Then

$$\frac{\partial}{\partial x}(g,g)^{n+1} = (n+1)(g,g)[(g'_x, g) + (g, g'_x)],$$

$$\tfrac{\partial^2}{\partial x^2}(g,g)^{n+1} = n(n+1)(g,g)^{n-1}[g'_x,g)+(g,g'_x)]^2 + (n+1)(g,g)^n[2(g'_x,g'_x)+(g''_x,g)+(g,g''_x)]$$

and the same for $\frac{\partial^2}{\partial y^2}$. Summing up, we have

$$\begin{aligned}\triangle(g,g)^{n+1} &= (\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2})(g,g)^{n+1}\\ &= n(n+1)(g,g)^{n-1}\cdot 4|(g',g)|^2+(n+1)(g,g)^n\cdot 2(g',g'),\end{aligned}$$

because $\triangle g = 0$ and $g'_y = \sqrt{-1}g'_x$. If S is a unit disc then in polar coordinates $z = re^{it}$

$$\triangle = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})+\frac{1}{r^2}\frac{\partial^2}{\partial t^2} = \frac{1}{r^2}[(r\frac{\partial}{\partial r})^2+\frac{\partial^2}{\partial t^2}].$$

So

$$\frac{1}{r^2}((r\frac{\partial}{\partial r})^2+\frac{\partial^2}{\partial t^2})(g,g)^{n+1} = 4n(n+1)(g,g)^{n-1}|(g',g)|^2+2(n+1)(g,g)^n|g'|^2.$$

Integrating over $0 \le t \le 2\pi$ and using Cauchy-Schwartz inequality, we arrive at the inequality of [Pommerenke 1, Theorem 8.9]. The rest of the proof will go unchanged once we know the Hardy-Littlewood maximal theorem for $(g,g)^n$, which is used in [Pommerenke 1, page 187]. Let

$$g^*(s,\xi) = \max_{0\le r\le 1-e^{-s}} |g(r\xi)|, e \le s < \infty, \xi \in S^1.$$

Since $g : B^2 \to \mathcal{H}$ is holomorphic, it is also harmonic and yields the Poisson formula. Then a proof of the Hardy-Littlewood maximal theorem given in [Koosis 1] applies, since it reduces it to the Hardy-Littlewood inequality for the maximal function of $|g|$.

Remark 3.2.— Theorem 3.1 holds for complex hyperbolic cocompact lattices. This is because Makarov's law of iterated logarithm holds for the complex hyperbolic space, as we can see by passing to totally geodesic spaces of complex dimension 1. It is plausible that a version of Theorem 3.1 holds for real hyperbolic lattices (but not quaternionic and Cayley, as these are

Kazhdan, see a new proof in Chapter VI). On the other hand, another deep result of [Makarov 2] saying that Bloch functions are nontangentially bounded for a limit set of Hausdorff dimension one, fails for Hilbert space valued functions. In fact, we have shown in Chapter I that there are unitary cocycles on a surface group such that $\|l(g)\| \to \infty$ as $length(g) \to \infty$.

If G is any finitely generated group, and we are given an orthogonal representation $\pi : G \to O(\mathcal{H})$ and a cocycle $l \in Z^1(G, \mathcal{H})$ with a control on $\|l(g)\|$ from below, then for any embedding of the surface group $\pi_1(\Sigma)$ into G we immediately have a comparison inequality between the word lengthes of elements of $\pi_1(\Sigma)$ in $\pi_1(\Sigma)$ and G. To get a nontrivial result, we need a low bound on $\|l(g)\|$ better then $[length(g) \log\log \ length(g)]^{1/2}$. To find such groups and cocycles seems to be a very attractive problem.

We will now use similar ideas to estimate the Hausdorff dimension of limit sets of some infinite index subgroups of $G = \pi_1(\Sigma)$.

Theorem 3.2.— Let $\beta : G \to \mathbb{Z}$ be a surjective homomorphism and let $G_0 = Ker\beta$. Let A be a conical limit set of G_0. Then $\dim A = 1$.

Proof.— Let $[\beta] \in H^1(G, \mathbb{Z})$ be an induced class. Realize G as a cocompact lattice in $SO^+(1,2)$ so that $S = \mathcal{H}^2/G$ is a hyperbolic surface. Let ω be a holomorphic 1-form on S such that $Re[\omega] = [\beta]$ and let $\widetilde{\omega}$ be a lift of ω on $\mathcal{H}^2$. Let $f : \mathcal{H}^2 \to \mathbb{C}$ be holomorphic with $df = \omega$. Then f is a Bloch function. By a result of [Makarov 2], there is a set $B \subset S^1$ with $\dim B = 1$ such that

$$\limsup_{z \to \theta} |f(z)| < \infty$$

for any $\theta \in B$ and nontangential convergence. Notice that $f(gz) = f(z) + ([\omega], [g])$ where $g \in G$ and $[g]$ is an image of g in $H_1(G, \mathbb{Z})$. Now it is clear that $B \subseteq A$, so $\dim A = 1$.

Remark.— This result does not contradict a theorem of [Sullivan 1] and [Tukia 1] because G_0 is infinitely generated.

In the opposite direction we have the following . Let Σ_1, Σ_2 be two closed surfaces and let $\psi : \Sigma_1 \to \Sigma_2$ be a smooth ramified covering. Let $G_i = \pi_1(\Sigma_i)$ and let $G_0 = Ker\psi_* : G_1 \to G_2$. Let $G_1 \hookrightarrow SO^+(1,2)$ be a realization of G_1 as a lattice. Then for any $z \in B^2$,

$$\sum_{g \in G_0} |1 - gz| < \infty.$$

In other words, either $\delta(G_0) < 1$ or $\delta(G_0) = 1$ and G_0 is of convergence type. In the latter case, the Patterson measure of the conical limit set of G_0 is zero. To see this, notice that we can find hyperbolic structures on $\Sigma_i,\ i = 1, 2$ so that ψ is holomorphic. Let $\widetilde{\psi}$ be a lift of ψ as a map $\widetilde{\psi} : B^2 \to B^2$. Since $\widetilde{\psi}$ is a bounded holomorphic function, $\widetilde{\psi}$ has limit values almost everywhere on S^1. By I.6.5., $|\widetilde{\psi}|S^1| = 1$ almost everywhere. So $\widetilde{\psi}$ is an inner function. Let $C \subset B^2$ be a countable set of zeros of $\widetilde{\psi}$. We claim that C is a finite union of orbits of G_0. First, it is clear that C is G_0-invariant. Let $Q \subset B^2$ be compact which contains a fundamental domain for G_1. Then $\widetilde{\psi}(Q)$ is compact so there is a finite set $R \subset G_2$ such that $g(0) \notin \widetilde{\psi}(Q)$ if $g \notin R$. Let $T \subset G_1$ be finite and such that $\psi_*(T) \supseteq R$. Let $Q_1 = \bigcup_{g \in T^{-1}} gQ$ so that Q_1 is compact and therefore $C \cap Q$ is finite. Let $x \in C$, then $x = gy$ with $y \in Q$. So $0 = \widetilde{\psi}(x) = \psi_*(g)\widetilde{\psi}(y)$, i.e., $\psi_*(g^{-1})(0) = \widetilde{\psi}(y) \in \widetilde{\psi}(Q)$. This means $\psi_*(g^{-1}) \in R$ so $g^{-1} \in TG_0$, and $g \in G_0T^{-1}$, say $g = g_0t^{-1}$, $g_0 \in G_0$, $t \in T$. Then $t^{-1}y \in C$ and $t^{-1}y \in Q_1$, so there are finitely many options for $t^{-1}y$.

We deduce that there are $x_1, \cdots, x_n$ such that $C = \bigcup_{i=1}^n G_0x_i$. The decomposition formula for inner functions implies that

$$\widetilde{\psi}(z) = c \cdot \prod_{\substack{g_0 \in G_0 \\ 1 \le i \le n}} \frac{\overline{g_0x_i}}{g_0x_i} \frac{z - g_0x_i}{1 - \overline{g_0x_i}z}$$

which gives an explicit formula for holomorphic maps between hyperbolic Riemann surfaces (one still needs to find x_i). By a well-known result on zeros of a bouded holomorphic function [Koosis 1, IV: B, Theorem 1],

$$\sum_{g_0 \in G_0} (1 - |g_0x_i|) < \infty.$$

The rest follows from [Nicholls 1].

Chapter 4

A theory of groups of symplectomorphisms

We already have noticed an intriguing similarity between groups acting on the circle and groups acting symplectically on a compact sympletic manifold. The two leading topics studied in Chapter 2, namely, (non-) Kazhdan groups acting on S^1 and characteristic classes, have exact analogues for *Sympl(M)*. In fact, a theory of characteristic classes parallel to II.5 , has already been presented in [Reznikov 2] and [Reznikov 4]. In the second cited paper, we noticed that the Kähler action of *Sympl(M)* on the twistor variety allows us to define a series of classes in $H^{2k}_{cont}(Sympl(M), \mathbb{R})$, k odd , which are highly non-trivial. In the first cited paper,we introduced bi-invariant forms on *Sympl(M)* and the classes in $H^{odd}_{top}(Sympl(M))$ and $H^{odd}(Sympl(M)^{\delta}, \mathbb{R}/A)$ (cohomology of a topological space and a discrete group) where A is a group of periods of the above-mentioned forms. Here we present a fundamental class in $H^1(Sympl(M), L^2(M))$ whose nontriviality on a subgroup $G \subset Sympl(M)$ implies that G is not Kazhdan, similarly to the situation in $\mathcal{D}iff^{1,\alpha}(S^1)$. ¿From the nature of our class it is clear that its vanishing imposes severe restriction on the symplectic action, roughly, the transformations of G should satisfy a certain PDE . We give an explicit formula for our class in the case of a flat torus.

We then introduce a characteristic class in $H^{n+1}(Sympl^{\delta}(M^{2n}), \mathbb{R})$ associated with a compact Lagrangian immersed submanifold. This class is a sympletic counterpart, and a generalization, of the Thurston-Bott class [Bott 1]. We use this class to give a formula for the volume of compact negatively curved manifold through Euclidean volumes of “Busemann bodies” (the images of the manifold under Busemann functions).

4.1 Deformation quantization : an overview

Let F be a field and $A|F$ a (commutative) algebra . A deformation of A is an algebra structure over $F[[\hbar]]$ of $A[[\hbar]]$ extending that A, so that if $x, y \in A$,

$$x * y = x \cdot y + b_1(x,y)\hbar + b_2(x,y)\hbar^2 + \cdots$$

where $x \cdot y$ is a multiplication in A and $x * y$ is a deformed multiplication .

If $F = \mathbb{R}$, $A = C^\infty(M)$, where M is a symplectic manifold , then a deformation quantization is a deformation of A with $b_1(f,g) = \{f,g\}$, a Poisson bracket . A deformation quantization always exists by a result of [Moyal 1], [Vey 1], [Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer 1] [Fedosov 1]. For any algebra $A|F$ one defines a Hochschild collomology $HH^k(A) = Ext^k_{A\otimes A}(A,A)$. There is a natural Lie superalgebra structure in $HH^*(A)$ [Gerstenhaber 1]. There exists a simple explicit complex , computing $HH^k(A)$ with $C^k(A) = Hom_F(\bigotimes_{i=1}^k A, A)$. In particular , b_1 above is a cocycle (for any deformation). If $F = \mathbb{R}$ and A is a topological algebra, one modifies the definitions to obtain topological Hochschild cohomology . If M is a smooth manifold and $A = C^\infty(M)$ with a pointwise multiplication , then

$$HH^k(A) = \Gamma(M, \Lambda^k TM),$$

a space of poly-vector fields . The Lie superalgebra structure coincides with a classical bracket of poly-vector fields.

We will need an explicit form of the cocycle condition for a 2-cocycle $b : A \otimes A \to \mathbb{R}$:

$$xb(y,z) - b(xy,z) + b(x,yz) - b(x,y)z = 0.$$

4.2 A fundamental cocycle in $H^1(Sympl(M), L^2(M))$

Let (M^{2n}, ω) be a compact symplectic manifold . Fix a deformation quantization

$$f * g = f \cdot g + \{f, g\}\hbar + \sum_{i=2}^{\infty} c_i(f, g) \cdot \hbar^i$$

Let $\Phi : M \to M$ be symplectic and let

$$\begin{aligned} f \tilde{*} g &= (f \circ \Phi^{-1} * g \circ \Phi^{-1}) \circ \Phi \\ &= f \cdot g + \{f, g\}\hbar + \textstyle\sum_{i=2}^{\infty} c'_i(f, g) \cdot \hbar^i \end{aligned}$$

Lemma 2.1.— Let $A|F$ be an algebra and let

$$f * g = f \cdot g + c_1(f, g)\hbar + \cdots + c_{k-1}(f, g)\hbar^{k-1} + \sum_{i=k}^{\infty} c_i(f, g) \cdot \hbar^i$$

and

$$f \tilde{*} g = f \cdot g + c_1(f, g)\hbar + \cdots + c_{k-1}(f, g)\hbar^{k-1} + \sum_{i=k}^{\infty} c'_i(f, g) \cdot \hbar^i$$

be two deformations, which coincide up to the order $\hbar^{k-1}$. Then

$$c_i - c'_i : A \otimes A \to A$$

is a Hochschild cocycle .

Proof.—

$$(f*g)*p - (f \tilde{*} g) \tilde{*} p = c_k(f, g) \cdot p + c_k(f \cdot g, p) - c'_k(f, g) \cdot p - c'_k(f \cdot g, p) \qquad (mod\ \hbar^{k+1})$$

Similarly,

$$f*(g * p) - f \tilde{*} (g \tilde{*} p) = f \cdot c_k(g, p) + c_k(f, g \cdot p) - f c'_k(g, p) - c'_k(f, g \cdot p) \qquad (mod\ \hbar^{k+1})$$

So for $c = c_k - c'_k$,

$$f \cdot c(g, p) + c(f, g \cdot p) - c(f, g)p - c(f \cdot g, p) = 0,$$

which means that c is a 2-cocycle.

Lemma 2.2.— A formula

$$\Phi \mapsto \left[(f,g) \mapsto c_2(f\circ\Phi^{-1}, g\circ\Phi^{-1})\circ\Phi - c_2(f,g)\right]$$

defines a smooth cocycle of *Sympl(M)* in the space $Z^2(C^\infty(M), C^\infty(M))$ of Hochschild 2-cocycles for $C^\infty(M)$.

Proof. Follows from Lemma 2.1.

Passing to Hochschild cohomology, we obtain a 1-cocycle of $Sympl(M)$ in

$$HH^2(C^\infty(M)) = \Gamma(M, \Lambda^2 TM).$$

Using the symplectic structure ,we identify $\Gamma(M,\Lambda^2 TM)$ with $\Omega^2(M)$, a space of 2-forms on M. Multilying by ω^{n-1} we obtain a cocycle

$$\mu \in H^1(Sympl(M), C^\infty(M)).$$

4.3 Computation for a flat torus and the main theorem

If M is a coadjoint orbit of a compact Lie group, one can find an explicit formula for the deformation quantization $f*g$. A classical case $M = T^{2n}$, a flat torus, is due to H.Weyl.

Proposition 3.1.— One has a following deformatiom quantization on T^{2n} :

$$f*g = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{i\hbar}{2}\right)^k \sigma^{i_1j_1}\cdots\sigma^{i_kj_k}\frac{\partial^k f}{\partial y_{i_1}\cdots\partial y_{i_k}}\frac{\partial^k g}{\partial y_{j_1}\cdots\partial y_{j_k}}$$

where σ^{ij} are elements of the matrix, inverse to the matrix (σ_{ij}) of a (constant) symplectic form , and the “repeated indices” summation agreement is applied .

Now, since our definition of a fundamental cocycle is completely explicit, one can derive an explicit formula for μ in this case. We give an answer for T^2 (the formula for T^{2n} is completely analogous). The computation is tedious (takes several pages) but straightforward and is left to reader . Here is the formula for T^2 :

$$\Phi \mapsto \frac{\partial^2\Phi_2}{\partial y_1^2}\frac{\partial^2\Phi_1}{\partial y_2^2} - \frac{\partial^2\Phi_1}{\partial y_1^2}\frac{\partial^2\Phi_2}{\partial y_2^2}$$

where $\Phi = (\Phi_1, \Phi_2)$ a symplectomorphism of the form T^2. Summing up, we have

Theorem 3.2.— Let M^{2n} be a compact symplectic manifold , let $Sympl(M)$ its symplectomorphism group , acting orthogonally on a Hilbert space $L^2(M)$. There exists a cocycle

$$\mu \in Z^1(Sympl(M), L^2(M)),$$

defined canonically by a given deformation quantization of $C^\infty(M)$ with the following properties :

A. Let

$$\begin{aligned} f\tilde{*}g &= (f\circ\Phi^{-1} * g\circ\Phi^{-1})\circ\Phi = f\cdot g + \{f,g\}\hbar + c_2'(f,g)\cdot\hbar^2 + \cdots , \\ f * g &= f\cdot g + \{f,g\}\hbar + c_2(f,g)\cdot\hbar^2 + \cdots \end{aligned}$$

and let us indentify the class of the Hochschild cocycle $c_2' - c_2$ with a section ν of $\Lambda^2 TM$. Let $\hat{\nu}$ be a 2-form obtained from ν by lifting the indices using the symplectic form . Then

$$\mu(\Phi)\cdot\omega^n = \hat{\nu}.\omega^{n-1}.$$

B. $\mu(\Phi)$ depends only on the second jet of Φ.

C. For $M = T^2$ and the Weyl deformation quantization , $\Phi = (\Phi_1, \Phi_2)$,

$$\mu(\Phi) = \frac{\partial^2\Phi_2}{\partial y_1^2}\frac{\partial^2\Phi_1}{\partial y_2^2} - \frac{\partial^2\Phi_1}{\partial y_1^2}\frac{\partial^2\Phi_2}{\partial y_2^2}.$$

D. If G is a Kazhdan subgroup of $Sympl(M)$, then

$$\|\mu(\Phi)\|_{L^2} < const \qquad (\Phi \in G).$$

Examples.—
1) $M = T^{2n}$, $G = Sp(2n, Z)$ (Kazhdan for $n \geq 2$). Then μ is identically zero.
2)Let Γ be a surface group, and let M be a component of

$$Hom(\Gamma, SO(3))/SO(3),$$

consisting of representations with nontrivial Stiefel-Whitney class. Then M is a compact symplectic manifold and Map_g acts symplectically on M. We do not know if part D of Theorem 3.2 holds in this case and if Map_g is Kazhdan or not. There is a "Teichmüller structure" on M defined by a holomorphic map of the Teichmüller space into the twistor variety of M, described in [Reznikov 4], see also Chapter 5.

Remark.— The case of two-dimensional M^2 is much easier, simply because $SL_2(\mathbb{R})$ is not Kazhdan . If $Sympl(M, x_0)$ is a subgroup fixing $x_0 \in M$ then one gets a nontrivial unitary cocycle on $Sympl(M, x_0)$ by pulling back from $SL_2(\mathbb{R})$ under the tangent representation. Using the measurable transfer (= Shapiro's lemma) one constructs a cocycle of $Sympl(M)$. See [Zimmer 1] for details .

4.4 Invariant forms on the space of Lagrangian immersions and new regulators for symplectomorphism groups

In this section we will "symplectify" the Thurston-Bott class in the cohomology of diffeomorphism groups. Let M be any (possibly noncompact) symplectic manifold, and let $L_0 \hookrightarrow M$ be a Lagrangian immersion of a compact oriented manifold L_0. Let $Lag(L_0, M)$ be a space of Lagrangian immersions of L into M which can be jointed to L_0 by an exact Lagrangian homotopy. This means the following. If $f_t \to M$ is a smooth family of Lagrangian immersions, than $\frac{d}{dt} f_t|_{t=0}$ is a vector field along L_0. Projecting to the normal bundle NL_0 and accounting that NL_0 is canonically isomorphic to T^*L_0, we get a 1-form on L_0 which is immediately seen to be closed. A Lagrangian homotopy f_t is exact, if this form is exact for all t. There is therefore a well-defined function F (mod const) on L which can be seen as a tangent vector of the deformation .

Definition.— A canonical (n+1)-form ν on $Lag(L, M)$ is defined by

$$\nu(F_0 \cdots F_n) = \int_L F_0 dF_1 \cdots F_n = Vol_{n+1}(\widetilde{Q}) \qquad (*),$$

where $\widetilde{Q}$ is any chain in $\mathbb{R}^{n+1}$ spanning $(F_0, \cdots, F_n)(L)$.

Proposition 4.1—ν is closed.

Proof is an exercise for reader.

Let $Sympl_0(M)$ be a group of Poissonian transformations of M. Then $Lag(L, M)$ is invariant under $Sympl_0(M)$.

Proposition 4.2.— ν is $Sympl_0(M)$-invariant.

Proof is obvious.

A standard theory of regulators [Reznikov 3], [Reznikov 2] implies that, first, one has an induced class in $H^{n+1}(\mathfrak{g}, \mathbb{R})$. where $\mathfrak{g} = Lie(Sympl_0(M))$= $C^\infty(M)/const$ given by $(*)$, where now $F_i \in C^\infty(M)$ and second, a class in

$$Hom(\pi_{n+1}(BSympl_0^\delta(M)^+, \mathbb{R}/A)) \qquad (n+1 \geq 5),$$

where A is a group of periods of ν on maps $\Sigma^{n+1} \to Sympl_0(M)$ of homology spheres to $Sympl_0(M)$. This class often lifts to a class in $H^{n+1}(Sympl_0^\delta(M), \mathbb{R})$ under suitable conditions on topology of $Sympl_0(M)$ (see the discussion in the papers cited above).

As an example, let Q be a compact oriented simply connected manifold, $M = T^*Q$ and $L_0 = Q$, a zero section. Then we obtain a class

$[\nu]$ in $H^{n+1}(Sympl_0(T^*Q),\mathbb{R})$. Notice that the restriction of this class on $\mathcal{D}iff(Q) \hookrightarrow Sympl_0(T^*Q)$ is zero, as $\mathcal{D}iff(Q)$ fixes the zero section. However , our class is an extension of Thurston-Bott class [Bott 1] in $\mathcal{D}iff(Q)$ by means of the following construction. Let $G \subset Sympl_0(T^*Q)$ be a subgroup of symplectomorphisms of the form

$$p_x \mapsto \phi^* p_x + df(x),$$

where $f \in C^\infty(Q)$, $\phi \in \mathcal{D}iff(Q)$, $x \in Q$, $p_x \in T^*_x Q$. Then G is an extension

$$0 \to C^\infty(Q)/const \to G \to \mathcal{D}iff(Q) \to 1.$$

Any 1-cocycle $\psi \in Z^1(\mathcal{D}iff(Q), C^\infty(Q)/const)$ induces a spliting of this exact sequence:

$$S_\psi : \mathcal{D}iff(Q) \to G.$$

Now let μ be a smooth density on Q then $\psi = \frac{\phi_*\mu}{\mu}$ is a 1-cocycle, so it defines such a splitting. A pull-back $S^*_\psi([\nu]|G)$ of our class on $\mathcal{D}iff(Q)$ is precisely the Thurston-Bott class.

We sum up :

Theorem 4.3.—
A. A formula

$$\nu(F_0, \cdots, F_n) = \int_L F_0 dF_1 \cdots dF_n = Vol_{n+1}(\widetilde{Q})$$

defines an $Sympl_0(M)$-invariant closed $(n+1)$-form in $Lag(L,M)$. It induces a class $[\nu] \in H^{n+1}(Lie(Sympl_0(M),\mathbb{R})$ and a regulator

$$[\nu] : \pi_{n+1}(BSympl_0^+(M)) \to \mathbb{R}, \qquad n+1 \geq 5,$$

which lifts to a class

$$[\nu] \in H^{n+1}(Sympl_0^\delta(M),\mathbb{R})$$

if $\widetilde{H}_i(Lag(L,M),\mathbb{R})) = 0, \quad 0 \leq i \leq n+1.$
B. In particular , if Q is a smooth oriented simply-connected closed manifold, then

$$[\nu] \in H^{n+1}(Sympl^\delta(T^*Q),\mathbb{R})$$

pulls back to the Thurston-Bott class under any splittting

$$\mathcal{D}iff(Q) \to C^\infty(Q)/const \rtimes \mathcal{D}iff(Q),$$

coming from a smooth density on Q.

4.5 A volume formula for negatively-curved manifolds

This section is ideologically influenced by [Hamenstädt 1] and discussions with G.Besson (Grenoble, 1996). Let N^n be an Hadamard manifold. Let CN be the space of oriented geodesic of N, which is a symplectic manifold of dimension $2n-2$. Any point $x \in N$ defines a Lagrangian sphere $S_x \subset CN$ of geodesics passing through x.

Lemma 5.1.— A pull-back $S^*\nu$ of the form $\nu \in \Omega^n(CN)$ to N is the Riemannian volume form on N times a constant .

Proof.— An exercise in Jacobi fields.

Now if G acts discretely and cocompactly on N, we have

$$[S^*\nu, \text{fundamental class of } N/G] = c \cdot Vol(N/G).$$

Corollary 5.2.— $[\nu] \neq 0$ in $H^n(Sympl^\delta(N), \mathbb{R})$.

Now we assume that the curvature of N is strictly negative and moreover, the induced action of G on the sphere at infinity S_∞ is of class $C^{1,\frac{n-1}{n}}$. For $n = 2$ this is always the case [Hurder-Katok 1], whereas for $n \geq 3$ seems to require a pinching of the curvature. Notice that the map

$$s_+ : CN \to S_\infty,$$

sending any geodesic $\gamma(t)$ to $\gamma(\infty)$, is a Lagrangian fibration. Therefore if we fix a Lagrangian section of s_+, we will have a symplectomorphism

$CN \simeq T^*(S_\infty)$. Fix $p_0 \in N$, then S_{p_0} is such a section. Notice that an induced homomorphism $G \to Sympl(T^*S_\infty)$ is given by ,

$$g \mapsto (z \mapsto \pi(g)z + dF(p_0, g^{-1}p_0, \theta)),$$

where $g \in G,\ z \in T^*_\theta S_\infty,\ \pi : G \to \mathcal{D}iff^{1,\frac{n-1}{n}}(S_\infty) \to Sympl(T^*S_\infty)$ is induced by the action of G on S_∞ and $B(p_0, g^{-1}p_0, \theta))$ is the Buseman function. Our assumption imply that $B(p_0, p_1, \cdot) \in C^{\frac{n-1}{n}}(S_\infty) \subset W_n{}^{\frac{n-1}{n}}(S_\infty)$. Recall that for $F_1, \cdots, F_n \in W_n{}^{\frac{n-1}{n}}$ we have an n-form

$$\int_{S_\infty} F_1 dF_2 \cdots dF_n = \int_{B^n} du_1 \cdots du_n,$$

where u_i is a harmonic extension of F_i.

We derive a

Corollary 5.3.— Let N^n/G be a compact negatively curved manifold such that the induced action of G on S_∞ is of class $C^{1,\frac{n-1}{n}}$. If the fundamental class of G is

$$\sum_i [g_1{}^{(i)} \cdots g_n{}^{(i)}],$$

then the following volume formula holds :

$$Vol(N/G) = c(n) \cdot \sum_i \int_{S_\infty} F_1{}^{(i)} dF_2{}^{(i)} \cdots dF_n{}^{(i)},$$

where $F_k{}^{(i)}(\theta) = B\left(p_0, \left(g_k{}^{(i)}\right)^{-1} p_0, \theta\right)$.

One can say that a volume of a negatively curved manifold is a sum of Euclidean volumes of Busemann bodies in $\mathbb{R}^n$ bounded by $(F_1, \cdots, F_n)(S_\infty)$.

Replacing the Busemann cocycle by a Jacobian cocycle $\frac{g*\mu}{\mu}$, where μ is a smooth density on S_∞, we arrive to a similar formula for Godbillon-Vey-Thurston-Bott invariant of N/G,under the same regularity assumptions. This seems to have been also accomplished in a preprint [Hurder 1] cited in [Hurder-Katok 1], though I was unable to obtain this paper from its author. The case $n = 2$ is ,however, covered in [Hurder-Katok 1].

Chapter 5

A theory of groups of volume-preserving diffeomorphisms and the nonlinear superrigidity alternative

In this chapter, we shift the focus from linear functional analytic techniques to nonlinear PDE, notably harmonic maps into nonlocally compact spaces, a theory recently developed in [Korevaar-Schoen 1] and [Jost 1]. The main idea is to use twistor varietes, which were in a center of the characteristic classes construction of [Reznikov 4], for a deeper study of volume-preserving actions of groups. We introduce an invariant of a volume-preserving action, which we call Λ, which is a sort of a $\log L^2$-version of a *sup*-displacement studied in [Zimmer 2]. Our first main result, Theorem 2.3, states that if G is a Kazhdan group acting on a compact manifold M preserving volume, then either $\Lambda > 0$ or G fixes a $\log L^2$-metric. A much weaker analogue of this result for the special case of lattices in Lie groups and *sup*-displacement was known before [Zimmer 2, Theorem 4.8].

We then apply our technique to a major open problem in the field, that of the nonlinear superrigidity of volume-preserving actions of lattices in Lie groups. From a nonlinear version of Margulis theorem given in [Zimmer 3] one knows that a volume preserving action of a lattice in a semisimple Lie group of rank ≥ 2 on a low dimensional (with respect to the group) manifold fixes a **measurable** Riemannian metric. Since measurable metrics do not define a geometry on a manifold, one wishes ,of course, to prove a much stronger result: that the action preserves a smooth metric. Zimmer noticed [Zimmer 2 and references therein] that such strong result would follow if one is able to find an invariant metric whose dilations with respect to any smooth metric are in the class L^2_{loc}. The central question of how to find such a "bounded" invariant metric was left completely open. We present a completely new approach to the problem which leads to Theorem 3.1. It states that if a cocompact lattice acts on M preserving volume, then either it **nearly** preserves a $\log L^2$-metric, or a sort of G-structure. This theorem, though constituting a clear progress in solution of the main problem is still less than what one wants in two respects: first, we deal with $\log L^2$-metrics, not L^2-metrics, second, we leave open a very delicate situation when an action nearly preserves a $\log L^2$-metric, but does not exactly preserve such a metric. This situation is purely infinite-dimensional (if an action on a finitely dimensional space of nonpositive curvature nearly preserves a point, it actually preserves a point at infinity). As already said, we use a heavy machinery : harmonic maps into twistor varietes and vanishing results of [Mok-Siu-Yeung 1] and [Corlette 1]. These results will also be applied in the next Chapter to study quaternionic Kähler groups.

As is well-known, an original Kolmogorov's definition of entropy used extremum over all partitions and only became computable after it had been realized by Kolmogorov and Sinai that certain partitions realize entropy. In a way of a pleasant similarity, we show how to compute our invariant Λ for $G = \mathbb{Z}$ in case G leaves a geodesic in the twistor space invariant, like a hyperbolic element of $SL(n, \mathbb{Z})$ acting on T^n. This clearly shows an advantage of $\log L^2$-displacement over *sup*-displacement.

5.1 $\log L^2$-twistor spaces

Twistor varietes (C^∞) were used in [Reznikov 4] to define secondary characteristic classes for volume-preserving and symplectic actions. More specifically, we have defined, for a compact oriented manifold M equipped with a volume form ν, a series of classes in $H^*_{cont}(\mathcal{D}iff_\nu(M))$ of dimension $5, 9, 13, \cdots$ (where $\mathcal{D}iff_\nu(M))$ is a group of volume-preserving diffeomorphisms). Likewise, for a compact symplectic manifold M we have defined classes in $H^*_{cont}(Sympl(M))$ of dimensions of $2, 6, 10, \cdots$. For purposes of the present paper, we will need to work with a $\log L^2$-version of twistor varietes, defined below.

Remark 1.1.— I would like to use an opportunity to note that by some strange reason I have overlooked an integrated Euler class in $H^n_{cont}(\mathcal{D}iff_\nu(M^n))$. The definition is exactly like that in [Reznikov 4] for classes in dimensions 5, 9, $\cdots$ if one realizes that there exists an n-form on the twistor variety for M, which is $\mathcal{D}iff_\nu$-invariant. Alternatively, if $\mathcal{D}iff_\nu(M, p_0)$ is a subgroup fixing a point p_0, then one pulls back the Euler class of $SL_n(\mathbb{R})$ under the tangent representation

$$\mathcal{D}iff_\nu(M, p_0) \to SL_n(\mathbb{R}),$$

and then applies a measurable transfer (see the above cited paper). The just defined class viewed as a class in $H^n(\mathcal{D}iff^s_\nu(M))$ is bounded. This follows from the fact that the Euler class is bounded [Sullivan 2] exactly in the same manner as in [Reznikov 4].

We now define the $\log L^2$-twistor variety X for (M, ν). First, one defines a bundle $\mathcal{P}$ of metrics with volume form ν as an $SL(n)/SO(n)$-bundle, associated with a principal $SL(n)$-bundle, defined by ν. Fix a smooth section (=a Riemannian metric with volume form ν) g_0 of this bundle. For any other measurable section g of $\mathcal{P}$ define

$$\rho^2(g_0, g) = \int_M \rho_x^2(g_0, g) d\nu, \qquad (*)$$

where ρ_x is a distance in $\mathcal{P}_x$ induced by (fixed once forever) $SL(n)$-invariant metric on $SL(n)/SO(n)$. Now the twistor variety X consists of $\log L^2$-metrics, that is,

$$\rho(g_0, g) < \infty.$$

Alternatively, let A_x be a self-adjoint (with respect to $(g_0)_x$) operator such that $g_x = g_0(A_x \cdot, \cdot)$. Then $(*)$ can be written as

$$\int_M \| \log A_x \|^2 d\nu < \infty.$$

A crucial fact about $\mathcal{P}$ is a following

Proposition 1.2.— $\mathcal{P}$ is a complete Hilbert Riemannian manifold with non-positive curvature operator. The action of $\mathcal{D}iff_\nu(M)$ on $\mathcal{P}$ is isometric.

Proof.— We will only define a metric, leaving all routine checks to the reader. A tangent space at g_0 consists of L^2-sections of S^2T^*M, with trace identically zero. If A is such a section (so that A_x is g_0-self-adjoint for all $x \in M$,) then we define a square of the length of A as

$$\int_M tr A^2 d\nu.$$

This metric is invariant under $SO(n)$-valued gauge transformations. Now we define a $\log L^2$ $SL(n)$-gauge group as a group of measurable sections of $Aut(TM)$ such that with respect to g_0,

$$\int_M \| \log(A^* A) \|^2 d\nu < \infty.$$

Then $\mathcal{P}$ is a homogeneous space under the action of this group. We define a metric on $\mathcal{P}$ as a unique invariant metric, which agrees at g_0 with the metric just defined.

Now let (M^{2n}, ω) be a compact symplectic manifold. Let $\mathfrak{T}$ be the twistor bundle, that is, an $Sp(2n)/U(n)$-bundle, associated with the principal $Sp(2n)$-bundle, defined by ω. A smooth section of $\mathfrak{T}$ is exactly a tamed almost-complex structure. One then defines a space Z of $\log L^2$-sections of $\mathfrak{T}$ as above (the C^∞-version was used in [Reznikov 4]).

Prosition 1.3.— The spaces X and Z are Alexandrov and Busemann non-positively curved.

Proofs are standard.

5.2 A new invariant of smooth volume-preserving dinamical systems

Let (M, ν) be a compact oriented manifold with volume form ν. Let G be a finitely generated group which acts on M by smooth transformations, preserving ν. We are going to define a new dinamical invariant which we call Λ. This is a nonnegative real number. Though it depends on the choice of a system of generators of G, the crucial fact of whether $\Lambda > 0$ or $\Lambda = 0$ does not. This relates our Λ to Kolmogorov's entropy [Kolmogorov 1]. The invariant Λ is highly nontrivial already for $G = \mathbb{Z}$, that is, as a new invariant of a volume-preserving diffeomorphism. It is also an invariant under conjugation in $\mathcal{D}iff_\nu(M)$. A central result of this section is Theorem 2 below stating that if G is a Kazhdan group then either $\Lambda > 0$ or G fixes a $\log L^2$-Riemannian metric (again this connects Λ to the Kolmogorov's entropy).

Let $g_1, \cdots, g_n$ be a system of generators for G. Let X be the twistor variety for (M, ν). Let ρ be the distance function for X, introduced in Section 1. We define Λ as the displacement of G-action:

$$\Lambda = \inf_{z \in X} \max_i \rho(g_i z, z).$$

Proposition 2.1.— Λ is invariant under conjugation in $\mathcal{D}iff_\nu(M)$.

Proof.— ρ is $\mathcal{D}iff_\nu$-invariant.

Proposition 2.2.— Let $M = (T^n, can)$ and let $G = \mathbb{Z}$ act by iterations of a hyperbolic element of $SL(n, \mathbb{Z})$. Then $\Lambda > 0$.

Proof.— The proof is based on an observation about Alexandrov non-positively curved spaces and a trick from [Reznikov 4].

Lemma.— Let X be an Alexandrov non-positively curved space and let $\phi : X \to X$ be an isometry which leaves invariant a geodesic γ of X. Then the displacement of ϕ is realized on the points of γ, that is, for $y \in \gamma$,

$$\rho(y, \phi y) = \min_{x \in X} \rho(x, \phi x).$$

Proof.— For $x \in X$ let $y \in \gamma$ be a point which realizes the distance from x to γ. Then $\rho(y, \phi y) \le \rho(x, \phi x)$.

Now let X be the twistor space of T^n and let $Y \subset X$ be the space of metrics, invariant under shifts (we view T^n as a Lie group). Then Y is totally geodesic in X, because it is a manifold of fixed points of a family of isometries. As a Riemannian manifold, $Y \simeq SL(n)/SO(n)$. Any hyperbolic matrix ϕ by definition leaves invariant a geodesic in Y. The result follows.

A main result in the theory of invariant Λ is as follows.

Theorem 2.3.— Let G be a Kazhdan group acting on a compact oriented manifold (M, ν) preserving a volume form ν. Then either $\Lambda > 0$ or G fixes a log L^2-metric on M.

Proof.— Consider an isometric action on X. If the displacement function $\sup_i \rho(g_i z, z)$ is not bounded away from zero, then either there is a fixed point $z_0 \in X$ for G, or G is not Kazhdan, by a result of [Kovevaar-Schoen 1]. The result follows.

5.3 Non-linear superrigidity alternative

Theorem 3.1.—Let G be either a semisimple Lie group of rank$\geq$ 2, or $Sp(n, 1)$ or $Iso(\mathbb{C}a\mathbb{H}^2)$. Let $\Gamma \subset G$ be a cocompact lattice. Let (M^n, ν)

be a compact oriented manifold, on which Γ acts preserving the volume form ν. Then either

a) Γ preserves a $\log L^2$- metric on M, or

b) there exists a sequence $g_0, g_1, \cdots$ of smooth Riemannian metrics on M with volume form ν such that

$$\int_M \|\log A_i\|^2_{g_0} d\nu \to \infty,$$

where $g_i = g_0(A_i \cdot, \cdot)$ and

$$0 < const_1 < \sup_j \int_M \|\log B_{ij}\|^2_{g_i} d\nu < const_2, \qquad (i \to \infty),$$

where $\gamma_j^* g_i = g_i(B_{ij} \cdot, \cdot)$, $\{\gamma_j\}$ is a fixed finite set of generators for Γ, or

c) there is a nonconstant totally geodesic Γ-invariant map

$$\Psi : G/K \to X,$$

where K is a maximal compact subgroup of G.

Remarks.-

1) In case b) we say that Γ nearly fixes a $\log L^2$-metric on M.

2) the case c) implies, for G simple, that $\dim G/K \leq \dim SL(n)/SO(n)$, a so-called Zimmer conjecture.

3) for $G = SL(m, \mathbb{R}), m \geq 3$ and $n = m$, one deduces in case c) an existence of a measurable frame field $\hat{e}(x), \hat{e} = (e_1, \cdots, e_n)$, such that for almost all $x \in M$,

$$\pi(\gamma)_*[\hat{e}(x)] = \gamma \hat{e}(\pi(\gamma)x)$$

where $\gamma \in \Gamma$ and $\pi(\gamma)$ is an action of γ on M.

4) Conversely, a standard action of $SL(n, \mathbb{Z})$ on T^n does not satisfy a) (which is well-known) and b). To see this, we notice that $SL(n, \mathbb{Z})$ leaves invariant a totally geodesic space Y introduced in the proof of Proposition 2.2. The argument of this proof implies that it is enough to show that the

displacement function of the action of Γ on Y diverges to ∞ as one escapes all compact sets of Y. This follows from the fact that Y is a Riemannian symmetric space of non-compact type and Γ does not fix a point at infinity of Y.

5) The statement of Theorem constitutes a definite progress in the non-linear superrigidity problem. There is still a mystery in the option b) where one would prefer a statement that Γ fixes a "point at infinity" of the space of metrics X, perhaps a measurable distribution of k-dimension planes, $k \leq n$. At the time of writing this chapter (August, 1999) I am unable to make such a reduction.

Proof. follows a long-established tradition [Siu 1], [Corlette 1], [Mok-Siu-Yeung 1], see also a treatment of [Jost-Yau 1], in a new infinite-dimensional target context. If neither a) or b) holds then, accounting that Γ is Kazhdan, we deduce that the displacement function of Γ tends to infinity as one escapes all bounded sets in X. Let $F \to \Gamma \setminus G/K$ be a flat fibration with fiber X, corresponding to the action of Γ in X. A theorem of [Kovevaar-Schoen 1], or [Jost 1] implies that there is a harmonic section of F. By Propositon 1.2 and main theorem of [Corlette 1] and [Mok-Siu-Yeung 1], this section must be totally geodesic. The result follows.

In the case of symplectic action of lattice Γ on a compact symplectic manifold (M, ω) we have a comletely similar theorem, as follows.

Theorem 3.2.— Let G be either a semi-simple Lie group of rank≥ 2, or $Sp(n,1)$ or $Iso(\mathbb{C}a\mathbb{H}^2)$, Γ a cocompact lattice in G which acts symplectically on a compact symplectic manifold (M^{2n}, ω). Then either

a) Γ fixes a $\log L^2$ tamed almost-complex structure J, or

b) there exists a sequence of tamed smooth almost-complex structures $J_i \in Z$ with $\rho(J_0, J_i) \to \infty$ and

$$0 < const_2 < \sup_j \rho(\gamma_j J_i, J_i) < const_1, \qquad \text{or}$$

c) there is a Γ-invariant totally geodesic map

$$\Psi : G/K \to Z.$$

Proof. is exactly as above.

In case c) and G simple it follows that $\dim G/K \leq \dim Sp(2n)/U(n)$. If $M = (T^{2n}, can)$, $G = Sp(2n, \mathbb{R})$ and case c) one deduces an existence of a measurable symplectic frame $\hat{e}(x) = (e_1, \cdots, e_{2n}(x))$, such that for $\gamma \in \Gamma$,

$$\pi(\gamma)_*[\hat{e}(x)] = \gamma\hat{e}(\pi(\gamma)(x)).$$

Chapter 6

Kähler and quaternionic Kähler groups

In a letter to the author [Deligne 1] P.Deligne asked if one can extend the author's theorem on rationality of secondary characteristic classes of a flat bundle over a projective variety to quasiprojective varieties. In 1994 the author was able to answer this question positively for the special case of noncompact ball quotients using an analytic technique of [Gromov-Schoen 1] and the scheme of the original proof for projective varietes. Here we present a full answer to Deligne's question, Theorem 1.1, using an analytic technique of [Jost-Zuo 1], who produced harmonic maps of infinite energy but controlled growth.

We then turn to a well-known open problem of finding restriction on topology of compact quaternionic Kähler manifolds. In case of positive scalar curvature the situation is well-understood, but in case of negative scalar curvature the twistor spaces of [Solomon 1] are not Kähler and its technique fails. The only result known was a theorem of [Corlette 1] stating that the fundamental group does not have infinite linear representations unless the manifold is locally symmetric. Our result, Theorem 2.2, states that the fundamental group is Kazhdan. This is of course, a severe restriction (Kazhdan groups are rare). As a by-product of our technique, we obtain a new proof of a classical theorem, stating that the lattices in semisimple

Lie groups of rank ≥ 2, $Sp(n,1)$ and $Iso(\mathbb{C}a\mathbb{H}^2)$ are Kazhdan. We also show using I.1 that the classes in second cohomology space of a Kähler non-Kazhdan group, constructed in [Reznikov 6] and shown there nontrivial, are of polynomial growth. This again is very rare for "just a group", as polynomial growth in cohomology is connected to a polynomial isoperimetric inequality in the Cayley graph, which needs a special reason to hold. This means Kähler groups are rare, too.

6.1 Rationality of secondary classes of flat bundler over quasiprojective varietes

A rationality theorem for secondary classes of flat bundles over compact Kähler manifolds (previously known as Bloch conjecture [Bloch 1]) has been proved in 1993 in [Reznikov 3] and [Reznikov 5]. In a letter to the author [Deligne 1] P.Deligne asked if one can prove such a statement for local system with logarithmic singularities over a quasiprojective variety. The answer happens to be yes.

Theorem 1.1.— Let X be a quasiprojective variety, $\rho : \pi_1(X) \to SL(n,\mathbb{C})$ a representation. Let $b_i(\rho)$ be the imaginary part and $ChS_i(\rho)$ the $\mathbb{R}/\mathbb{Z}$-part of the secondary class $c_i(\rho) \in H^{2i-1}(X,\mathbb{C}/\mathbb{Z})$ of the flat bundle with monodromy ρ. Then

A. $b_i(\rho) = 0 \quad (i \geq 2)$ (the Vanishing Theorem).

B. $ChS_i(\rho) \in H^{2i-1}(X,\mathbb{Q}/\mathbb{Z})$ (the Rationality Theorem).

Proof.— For any smooth manifold, A implies B, as explained in the above cited papers. So we only prove A. Again it is explained in the above cited papers that we may assume ρ to be irreducible. Then by a recent result [Jost-Zuo 1] an associated $SL(n,\mathbb{C})/SU(n)$ flat bundle over X possesses a pluriharmonic section s which satisfies the Sampson degeneration condition. This means the following. The derivative Ds_x, $x \in X$ can be viewed as a $\mathbb{R}$-linear map to the space P of Hermitian matrices. Let $(Ds_x)^{\pm}_{\mathbb{C}}(Y) = (Ds_x(Y) \pm \sqrt{-1}Ds_x(\sqrt{-1}Y))$ be a map of TX to $P \otimes \mathbb{C}$.

Then the image of $(Ds_x)^{\pm}_{\mathbb{C}}$ consists of commuting matrices. Now a first proof of the Main Theorem in [Reznikov 5] applies word-to-word and the result follows.

6.2 Kazhdan property T for Kähler and quaternionic Kähler groups

There are two ways to geometrize group theory. One approach (a time geometry in the terminology of [Reznikov 7]) is to consider finitely generated groups which act on a (usually compact) space with some structure (a volume form, a symplectic form, a tree, a circle, a conformal structure, etc). An amazing phenomenon, amply demonstrated in the previous chapters is that these groups tend to be not Kazhdan. Another approach (a space geometry) is to consider groups which are fundamental groups of a compact (or closed to compact) manifold with some structure (like Kähler). It happens that these groups tend to be Kazhdan. Therefore these two families of "geometric" groups are essentically disjoint. A following result is a main theorem of [Reznikov 6].

Theorem.— Let G be a fundamental group of a compact Kähler manifold. If G is not Kazhdan, then $H^2(G,\mathbb{R}) \neq 0$. Moreover, if H is not Kazhdan and $\psi : G \to H$ is surjective then $0 \neq \psi^* : H^2(H,\mathbb{R}) \to H^2(G,\mathbb{R})$.

I would like to notice an important property, which I overlooked in [Reznikov 6].

Proposition 2.1.—Under the conditions of the Theorem, there is a nontrivial class of polynomial growth in $H^2(G,\mathbb{R})$.

Proof.— There is a unitary representation $\rho : G \to U(\mathcal{H})$ and a class $l \in H^1(G,\mathcal{H})$ such that a class γ in $H^2(G,\mathbb{R})$ given by $\langle l,l \rangle$ is nonzero, where $\langle \cdot,\cdot \rangle$ is an imaginary part of the scalar product in $\mathcal{H}$. This is proved in [Reznikov 6]. Now the result follows from Lemma I 1.1.

It is extremely rare for a finitely generated group to have nonzero polynomial cohomology.

We now turn to quaternionic Kähler manifolds. If a scalar curvature is positive, then the topology is very well understood [Solomon 1]. On the contrary, if the scalar curvature is negative, the only result known is that the fundamental group satisfy the geometric superrigidity [Corlette 1]. This means if $\pi_1(X)$ admits a Zariski-dense representation in an algebraic Lie group, then $\pi_1(X)$ is a lattice, and X a symmetric space of a known type. However, it is a rare occasion for a group to have any finite dimensional linear representation with infinite image. Using a combination of ideas of [Corlette 1] and [Reznikov 6] which is based on [Korevaar-Schoen 1] we now prove a very stong structure theorem.

Theorem 2.2.—Let X be a quaternionic Kähler manifold of negative scalar curvature. Then $\pi_1(X)$ is Kazhdan.

Proof.—Suppose not. Then by [Korevaar-Schoen 1] there exists an affine flat Hilbert bundle E over X with a nonparallel harmonic section. By a vanishing result of [Corlette 1] this section must be totally geodesic. Then X must be covered by a flat torus, a contradiction.

Remark.— The same argument provides a new proof of the classical theorem [Kazhdan 1], [Kostant 1] , that the (cocompact) lattices in semisimple Lie groups of rank≥ 2, $Sp(n,1)$ and $Iso(\mathbb{C}a\mathbb{H}^2)$ are Kazhdan. One uses a vanishing result of [Mok-Siu-Yeung 1] (see also a treatment in [Jost-Yau 1]) for lattices in semisimple Lie groups of rank≥ 2, and the above-mentioned result of [Corlette 1] for $Sp(n,1)$ and $Iso(\mathbb{C}a\mathbb{H}^2)$. Once established for cocompact lattices, the result follows for all lattices because a Lie group and a lattice in it are or are not Kazhdan at the same time.

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