Slow Motion of Charges Interacting Through the Maxwell Field

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Abstract

We study the Abraham model for N charges interacting with the Maxwell field. On the scale of the charge diameter, R_φ, the charges are a distance $\varepsilon^{-1}R_\varphi$ apart and have a velocity $\sqrt{\varepsilon}c$ with ε a small dimensionless parameter. We follow the motion of the charges over times of the order $\varepsilon^{-3/2}R_\varphi/c$ and prove that on this time scale their motion is well approximated by the Darwin Lagrangian. The mass is renormalized. The interaction is dominated by the instantaneous Coulomb forces, which are of the order ε^2. The magnetic fields and first order retardation generate the Darwin correction of the order ε^3. Radiation damping would be of the order $\varepsilon^{7/2}$.

1 Introduction

Classical charges interact through Coulomb forces, as one learns in every course on electromagnetism. Presumably the best realization in nature is a strongly ionized gas, for which the Darwin correction to the Coulomb forces is of importance, since under standard conditions the velocities cannot be considered small as compared to the velocity of light, cf. [7, §65]. Thus, given N charges, with positions r_α, velocities u_α, charges e_α, and masses m_α, $\alpha = 1, \ldots, N$, their motion is governed by the Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{D}} &= \sum_{\alpha=1}^{N} \Big(\frac{1}{2} m_\alpha u_\alpha^2 + \frac{1}{8c^2} m_\alpha^* u_\alpha^4\Big) - \frac{1}{2} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \frac{e_\alpha e_\beta}{4\pi |r_\alpha - r_\beta|} \\ &\quad + \frac{1}{4c^2} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \frac{e_\alpha e_\beta}{4\pi |r_\alpha - r_\beta|} \Big(u_\alpha \cdot u_\beta + |r_\alpha - r_\beta|^{-2} (u_\alpha \cdot [r_\alpha - r_\beta])(u_\beta \cdot [r_\alpha - r_\beta])\Big), \qquad (1.1)\end{aligned}$$

c denoting the velocity of light. The first term is the kinetic energy with a u_α^4-correction of a strength m_α^* depending on the precise model ($m_\alpha^* = m_\alpha$ for a relativistic particle). The second term is the Coulomb potential, whereas the third term is the Darwin potential, which decays as the Coulomb potential and has a velocity dependent strength.

On a more fundamental level, the forces between the charges are mediated through the electromagnetic field. The instantaneous Coulomb-Darwin interaction is a derived concept only. To understand the emergence of such an interaction, in this paper we will investigate the coupled system, charges and Maxwell field, and we will prove that in a certain limit the motion of the charges is well approximated by the Lagrange equations for $\mathcal{L}_{\mathrm{D}}$.

Let us first describe how the charges are coupled to the Maxwell field. To avoid short-distance singularities, we assume that the charge is spread out over a distance R_φ, which physically is of order of the classical electron radius. Thus charge α has a charge distribution ρ_α which for simplicity we take to be of the form

$$\rho_\alpha(x) = e_\alpha \varphi(x), \quad x \in \mathbb{R}^3,$$

where the form factor φ satisfies

$$0 \le \varphi \in C_0^\infty(\mathbb{R}^3)\,, \quad \varphi(x) = \varphi_r(|x|)\,, \quad \varphi(x) = 0 \quad \text{for} \quad |x| \ge R_\varphi\,. \qquad (C)$$

To distinguish the true solution from the approximation (1.1), the position of a charge α in the coupled system is denoted by q_α and its velocity by v_α, $\alpha = 1, \ldots, N$. The charges then generate the charge distribution ρ and the current j given by

$$\rho(x,t) = \sum_{\alpha=1}^{N} \rho_\alpha(x - q_\alpha(t)) \quad \text{and} \quad j(x,t) = \sum_{\alpha=1}^{N} \rho_\alpha(x - q_\alpha(t)) v_\alpha(t), \qquad (1.2)$$

which satisfy charge conservation by fiat. The Maxwell field, consisting of the electric field E and the magnetic field B, evolves according to

$$c^{-1} \frac{\partial}{\partial t} B(x,t) = -\nabla \wedge E(x,t), \quad c^{-1} \frac{\partial}{\partial t} E(x,t) = \nabla \wedge B(x,t) - c^{-1} j(x,t) \qquad (1.3)$$

with the constraints

$$\nabla \cdot E(x,t) = \rho(x,t), \quad \nabla \cdot B(x,t) = 0. \qquad (1.4)$$

The charges generate the electromagnetic field which in turn determines the forces on the charges through the Lorentz force equation

$$\frac{d}{dt}\Big(m_{\mathrm{b}\alpha}\gamma_\alpha v_\alpha(t)\Big) = \int d^3x\, \rho_\alpha(x - q_\alpha(t))\Big[E(x,t) + v_\alpha(t) \wedge B(x,t)\Big], \quad t \in \mathbb{R}, \tag{1.5}$$

for $\alpha = 1, \ldots, N$. Here $m_{\mathrm{b}\alpha}$ is the bare mass of charge α and γ_α the relativistic factor $\gamma_\alpha = (1 - v_\alpha^2/c^2)^{-1/2}$, which ensures $|v_\alpha| < c$. Note that there are no direct forces acting between the particles. Eqns. (1.2)–(1.5) are known as Abraham model for N charges.

We define the energy function by

$$\mathcal{H}(E, B, \vec{q}, \vec{v}) = \sum_{\alpha=1}^{N} m_{\mathrm{b}\alpha}\gamma_\alpha + \frac{1}{2}\int d^3x\, [E^2(x) + B^2(x)], \tag{1.6}$$

with $\vec{q} = (q_1, \ldots, q_N)$ and $\vec{v} = (v_1, \ldots, v_N)$. It then may be seen that the initial value problem corresponding to (1.2)–(1.5) has a unique weak solution of finite energy and that $\mathcal{H}$ is conserved by this solution, compare with [4] for the case of a single particle.

We assume that initially the particles are very far apart on the scale set by R_φ. Thus we require, for $\alpha \neq \beta$, that

$$|q_\alpha(0) - q_\beta(0)| \cong \varepsilon^{-1} R_\varphi \tag{1.7}$$

with $\varepsilon > 0$ small. If particles would come together as close as R_φ, our equations of motion are not trustworthy anyhow. In addition, we require that the initial velocities be small compared to the speed of light,

$$|v_\alpha(0)| \cong \sqrt{\varepsilon} c. \tag{1.8}$$

Subject to these restrictions, in essence, the initial electromagnetic field is chosen such as to minimize the energy function $\mathcal{H}$ from (1.6), cf. Section 5.1 for precise statements and estimates. With these initial conditions, for the particles to travel a distance of order $\varepsilon^{-1}R_\varphi$ it will take a time of order $\varepsilon^{-3/2}R_\varphi/c$, which will be the time scale of interest. Thus physically we consider slow particles that are far apart, and we want to follow their motion over long times.

Next note that it takes a time of order $\varepsilon^{-1}R_\varphi/c$ for a signal to travel between the particles. This means that on the time scale of interest, retardation effects are small. If particles interact through Coulomb forces, as will have to be proved, the strength of the forces is of order ε^2 since the distance is of order $\varepsilon^{-1}R_\varphi$. Followed over a time span $\varepsilon^{-3/2}R_\varphi/c$, this yields a change in velocity of order $\sqrt{\varepsilon}c$. On this basis we expect the orders of magnitude (1.7) and (1.8) to remain valid over times of order $\varepsilon^{-3/2}R_\varphi/c$. There is one subtle point here, however. The self-interaction of a charge with the fields renormalizes its mass. Thus in (1.1) the quantity m_α cannot be the bare mass of the charge, the electromagnetic mass has to be added.

In theoretical physics it is common practice to count the post-Coulombian corrections in orders of v/c relative to the motion through pure Coulomb forces. Thus the Darwin term is the first correction and of order $(v/c)^2$. The next correction is of order $(v/c)^3$ and accounts for damping through radiation. If we push the Taylor expansion in Section 3 one term further, one obtains

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial u_\alpha}\Big) = \frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial r_\alpha} + (e_\alpha/\, 6\pi c^3)\sum_{\beta=1}^{N} e_\beta \dddot{v}_\beta, \tag{1.9}$$

$\alpha = 1, \ldots, N$. The physical solution has to be on the center manifold for (1.9). At the present level of precision it suffices to substitute the Hamiltonian dynamics to lowest order, which yields

$$
\frac{d}{dt}\Big(\frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial u_\alpha}\Big) = \frac{\partial \mathcal{L}_{\mathrm{D}}}{\partial r_\alpha}
$$
$$
+\frac{e_\alpha}{6\pi c^3}\frac{1}{2}\sum_{\substack{\beta,\beta'=1\\ \beta\neq\beta'}}^{N}(\frac{e_\beta}{m_\beta}-\frac{e_{\beta'}}{m_{\beta'}})\frac{e_\beta e_{\beta'}}{4\pi|r_\beta - r_{\beta'}|^3}\Big((u_\beta-u_{\beta'})-3\frac{(r_\beta-r_{\beta'})\cdot(u_\beta-u_{\beta'})}{|r_\beta-r_{\beta'}|^2}(r_\beta-r_{\beta'})\Big).
$$

Note that if the ratio e_α/m_α does not depend on α, then the radiation reaction vanishes and the system does not emit dipole radiation. The next order correction is $(v/c)^4$ and of Lagrangian form. It is discussed in [7] and [1].

In general relativity, there is a huge effort to obtain corrections to the Newtonian orbits, which as a problem is similar to the one discussed here. The most famous example is the Hulse-Taylor binary pulsar, where two highly compact neutron stars of roughly solar mass revolve around each other with a period of 7.8 h [9]. In this case $(v/c) = 10^{-3}$. For gravitational systems there is only quadrupole radiation which is of order $(v/c)^5$. To this order the theory agrees with the observed radio signals within 0,3%. In newly designed experiments one expects highly improved precision which will require corrections up to order $(v/c)^{11}$.

2 Main results

We recall the initial conditions for the Abraham model (1.2)–(1.5), where we set $c = 1$ throughout for simplicity. For the initial positions $q^0_\alpha = q_\alpha(0)$ we require

$$
C_1\varepsilon^{-1} \le |q^0_\alpha - q^0_\beta| \le C_2\varepsilon^{-1}, \quad \alpha\neq\beta, \tag{2.1}
$$

for some constants $C_1, C_2 > 0$. For the initial velocities $v^0_\alpha = v_\alpha(0)$ we assume

$$
|v^0_\alpha| \le C_3\sqrt{\varepsilon} \tag{2.2}
$$

with $C_3 > 0$. The initial fields are a sum over charge solitons,

$$
E(x,0) = E^0(x) = \sum_{\alpha=1}^{N} E_{v^0_\alpha}(x-q^0_\alpha) \quad \text{and} \quad B(x,0) = B^0(x) = \sum_{\alpha=1}^{N} B_{v^0_\alpha}(x-q^0_\alpha). \tag{2.3}
$$

Here

$$
E_v(x) = -\nabla\phi_v(x) + (v\cdot\nabla\phi_v(x))v \quad \text{and} \quad B_v(x) = -v\wedge\nabla\phi_v(x) \tag{2.4}
$$

and the Fourier transform of ϕ_v is given by

$$
\hat\phi_v(k) = e\hat\varphi(k)/[k^2-(k\cdot v)^2], \tag{2.5}
$$

where it is understood that in $\phi_{v^0_\alpha}$ we have to set $e = e_\alpha$. For this choice of data, the constraints (1.4) are satisfied for $t = 0$ and therefore for all t. In case $N = 1$, the particle would travel freely, $q_1(t) = q^0_1 + v^0_1 t$, $t \ge 0$, and the co-moving electromagnetic fields would maintain their form (2.3).

In spirit, the bounds (2.1) and (2.2) should propagate in time and the form (2.3) of the electromagnetic fields, at least in approximation. On the other hand, for two particles with opposite charge one particular solution is the head on collision which violates the lower bound in (2.1).

Considerably more delicate are solutions where some particles reach infinity in finite time, [8, 10]. Thus we simply require that for given constants $C_*, C^* > 0$ the bound

$$C_* \varepsilon^{-1} \leq \sup_{t \in [0,\, T\varepsilon^{-3/2}]} |q_\alpha(t) - q_\beta(t)| \leq C^* \varepsilon^{-1}, \quad \alpha \neq \beta, \tag{2.6}$$

holds, which implicitly defines the first time, T, at which (2.6) is violated. In fact (2.6) looks like an uncheckable assumption. But, as to be shown, the optimal T can be computed on the basis of the approximation dynamics generated by the Lagrangian (1.1).

Under the assumption (2.6) the velocity bound propagates through the conservation of energy. We define the electrostatic energy of the charge distributions as

$$\mathcal{E}_{\text{stat}} = \sum_{\alpha=1}^{N} e_\alpha^2 \Big(\frac{1}{2} \int d^3k \, |\hat{\varphi}(k)|^2 k^{-2} \Big). \tag{2.7}$$

and compute the energy (1.6) for the given initial data. Then

$$\mathcal{H}(0) := \mathcal{H}(t=0) = \sum_{\alpha=1}^{N} m_{\mathrm{b}\alpha} \gamma(v_\alpha^0) + \mathcal{E}_{\text{stat}} + \mathcal{O}(\varepsilon)$$

with $\gamma(v) = (1 - v^2)^{-1/2}$. We minimize the electromagnetic field energy $\mathcal{H}_{\mathrm{f}}(t) = \frac{1}{2} \int d^3x \, [E^2(x,t) + B^2(x,t)]$ at time t for given ρ and j, i.e., for given positions $\vec{q}(t)$ and velocities $\vec{v}(t)$. Using (2.6) it may be shown that

$$\mathcal{H}(t) \geq \sum_{\alpha=1}^{N} m_{\mathrm{b}\alpha} \gamma(v_\alpha(t)) + \mathcal{E}_{\text{stat}} + \mathcal{O}(\varepsilon).$$

Since by energy conservation $\mathcal{H}(0) = \mathcal{H}(t)$ and since the dominant contributions $\mathcal{E}_{\text{stat}}$ cancel exactly, we thus will continue to have the bound $|v_\alpha(t)| \cong C\sqrt{\varepsilon}$. (We refer to Section 5.1 in Appendix A for the complete argument). Therefore

$$\sup_{t \in [0,\, T\varepsilon^{-3/2}]} |v_\alpha(t)| \leq C_v \sqrt{\varepsilon} \tag{2.8}$$

with some constant $C_v > 0$.

As a next step we solve the inhomogeneous Maxwell equations for the fields and insert them into the Lorentz force equations. According to the retarded part of the fields, retarded positions $q_\alpha(s)$, $s \in [0,t]$, will show up. To control the Taylor expansion of $q_\alpha(t) - q_\alpha(s)$ and thus of the retarded force, including the Darwin term, we will need bounds not only on positions and velocities, but also on $\dot{v}_\alpha$ and $\ddot{v}_\alpha$. Implicitly they use that the true fields remain close to the fields of the form (2.3) evaluated at current positions and velocities.

Lemma 2.1 *Let the initial data for the Abraham model satisfy (2.1), (2.2), and (2.3). Moreover, assume*

$$C_* \varepsilon^{-1} \leq \sup_{t \in [0, T\varepsilon^{-3/2}]} |q_\alpha(t) - q_\beta(t)|, \quad \alpha \neq \beta, \tag{2.9}$$

for some $T > 0$. Then there exist constants $C^*, C_v > 0$ such that (2.6) and (2.8) hold. In particular, $\sup_{t \in [0, T\varepsilon^{-3/2}]} |v_\alpha(t)| \leq \bar{v} < 1$ for some $\bar{v}$. In addition, we find $C > 0$ and $\bar{e} > 0$ such that

$$\sup_{t \in [0, T\varepsilon^{-3/2}]} |\dot{v}_\alpha(t)| \leq C\varepsilon^2 \quad \textit{and} \quad \sup_{t \in [0, T\varepsilon^{-3/2}]} |\ddot{v}_\alpha(t)| \leq C\varepsilon^{7/2} \tag{2.10}$$

in case that $|e_\alpha| \leq \bar{e}$, $\alpha = 1, \ldots, N$. In the estimates (2.6), (2.8), and (2.10), C and $\bar{e}$ do depend only on T and the bounds for the initial data, but not on ε.

The proof of this lemma is rather technical and will be given in Appendix A. Using the bounds of Lemma 2.1, we expand the Lorentz force up to an error of order $\varepsilon^{7/2}$, cf. Lemma 3.5, which is the order of radiation damping (the Coulomb force is order ε^2 and radiation damping a relative order $\varepsilon^{3/2}$ smaller). The terms up to order ε^3 then can be collected in the form of the Darwin Lagrangian (1.1). We set

$$m_\alpha = m_{\mathrm{b}\alpha} + \frac{4}{3}e_\alpha^2 m_e \quad \text{and} \quad m_\alpha^* = m_{\mathrm{b}\alpha} + \frac{16}{15}e_\alpha^2 m_e$$

with the electromagnetic mass $m_e = \frac{1}{2}\int d^3k \, |\hat{f}(k)|^2 k^{-2}$ and the Darwin Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{D}}(\vec{r},\vec{u}) \;\; = \;\; & \sum_{\alpha=1}^{N}\Big(\frac{1}{2}m_\alpha u_\alpha^2 + \frac{\varepsilon}{8}m_\alpha^* u_\alpha^4\Big) - \frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\frac{e_\alpha e_\beta}{4\pi|r_\alpha - r_\beta|} \\ & + \frac{\varepsilon}{4}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\frac{e_\alpha e_\beta}{4\pi|r_\alpha - r_\beta|}\Big(u_\alpha\cdot u_\beta + |r_\alpha - r_\beta|^{-2}(u_\alpha\cdot[r_\alpha - r_\beta])(u_\beta\cdot[r_\alpha - r_\beta])\Big)\end{aligned}$$

for $\vec{r} = (r_1,\dots,r_N)$ and $\vec{u} = (u_1,\dots,u_N)$. The comparison dynamics is then

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}_{\mathrm{D}}}{\partial u_\alpha}\Big) = \frac{\partial\mathcal{L}_{\mathrm{D}}}{\partial r_\alpha}, \quad \alpha = 1,\dots,N. \tag{2.11}$$

It conserves the energy

$$\mathcal{H}_{\mathrm{D}}(\vec{r},\vec{u}) = \sum_{\alpha=1}^{N}\Big(\frac{1}{2}m_\alpha u_\alpha^2 + \varepsilon\frac{3}{8}m_\alpha^* u_\alpha^4\Big) + \frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\frac{e_\alpha e_\beta}{4\pi|r_\alpha - r_\beta|}\,. \tag{2.12}$$

Because of the Coulomb singularity, in general the solutions to (2.11) will exist only locally in time, the only exception being when all charges have the same sign, in which case energy conservation yields global existence. In the corresponding gravitational problem, for a set of positive phase space measure, mass can be transported to infinity in a finite time, [10]. We do not know whether this can happen also for the Coulomb problem.

We set

$$q_\alpha^0 = \varepsilon^{-1}r_\alpha^0 \quad \text{and} \quad v_\alpha^0 = \sqrt{\varepsilon}u_\alpha^0, \quad \alpha = 1,\dots,N, \tag{2.13}$$

with $r_\alpha^0 \neq r_\beta^0$ for $\alpha\neq\beta$. Then (2.1) and (2.2) are satisfied. During the initial time slip of order ε^{-1} the fields build up the forces between particles and adjust to their motion. Thus during that period the dynamics of the particles is not well approximated by the Darwin Lagrangian and we correct the initial data of the comparison dynamics to the true positions and velocities only at the end of the initial time slip. To take into account that the comparison dynamics will have no global solutions in time, in general, we define $\tau \in]0,\infty]$ to be the first time when either $\lim_{t\to\tau^-}|r_\alpha(t) - r_\beta(t)| = 0$ for some $\alpha\neq\beta$ or $\lim_{t\to\tau^-}|r_\alpha(t)| = \infty$ for some α holds for the comparison dynamics (2.11).

As our main approximation result we state

Theorem 2.2 *Let $T > 0$ be fixed. Define $\tau \in]0,\infty]$ as above and fix some $\delta_0 \in]0,\tau[$. For the Abraham model let the initial data be given by (2.13) and (2.3). Furthermore we require $|e_\alpha| \leq \bar{e}$, with $\bar{e} = \bar{e}(T,\mathrm{data}) > 0$ from Lemma 2.1. Let $t_0 = 4(R_\varphi + C^*\varepsilon^{-1})$. We adjust the initial data of the comparison dynamics such that $q_\alpha(t_0) = \varepsilon^{-1}r_\alpha(\varepsilon^{3/2}t_0)$ and $v_\alpha(t_0) = \sqrt{\varepsilon}u_\alpha(\varepsilon^{3/2}t_0)$, $\alpha = 1,\dots,N$.*

Then there exists a constant $C > 0$ such that for all $t \in [t_0, \min\{\tau - \delta_0, T\}\,\varepsilon^{-3/2}]$ we have

$$|q_\alpha(t) - \varepsilon^{-1}r_\alpha(\varepsilon^{3/2}t)| \leq C\sqrt{\varepsilon}, \quad |v_\alpha(t) - \sqrt{\varepsilon}u_\alpha(\varepsilon^{3/2}t)| \leq C\varepsilon^2, \quad \alpha = 1,\dots,N. \tag{2.14}$$

Remarks (i) If we are satisfied with the precision from the pure Coulomb dynamics, then in (2.14) we loose one power in ε. In this case, we can adjust the initial data of the comparison dynamics at time $t = 0$, and then (2.14) holds for all $t \in [0, \min\{\tau - \delta_0, T\}\, \varepsilon^{-3/2}]$.
(ii) In fact the initial data need not to be adjusted exactly at $t = t_0$, a bound

$$|q_\alpha(t_0) - \varepsilon^{-1} r_\alpha(\varepsilon^{3/2} t_0)| \sim \sqrt{\varepsilon} \quad \text{and} \quad |v_\alpha(t_0) - \sqrt{\varepsilon} u_\alpha(\varepsilon^{3/2} t_0)| \sim \varepsilon^2$$

would be sufficient.

3 Self–action and mutual interaction

In this section we expand the Lorentz force term

$$F_\alpha(t) = \int d^3x\, \rho_\alpha(x - q_\alpha(t))[E(x,t) + v_\alpha(t) \wedge B(x,t)]. \tag{3.1}$$

Since the fields (E, B) are a solution to the inhomogeneous Maxwell's equations, we may decompose them in the initial and the retarded fields,

$$E(x,t) = E^{(0)}(x,t) + E^{(r)}(x,t) \quad \text{and} \quad B(x,t) = B^{(0)}(x,t) + B^{(r)}(x,t),$$

where

$$\begin{aligned}
\hat{E}^{(0)}(k,t) &= \cos|k|t\, \hat{E}(k,0) - i\, \frac{\sin|k|t}{|k|}\, k \wedge \hat{B}(k,0), \\
\hat{B}^{(0)}(k,t) &= \cos|k|t\, \hat{B}(k,0) + i\, \frac{\sin|k|t}{|k|}\, k \wedge \hat{E}(k,0), \\
\hat{E}^{(r)}(k,t) &= -\int_0^t ds\, \cos|k|(t-s)\, \hat{\jmath}(k,s) + i \int_0^t ds\, \frac{\sin|k|(t-s)}{|k|}\, \hat{\rho}(k,s)k, \\
\hat{B}^{(r)}(k,t) &= -i \int_0^t ds\, \frac{\sin|k|(t-s)}{|k|}\, k \wedge \hat{\jmath}(k,s),
\end{aligned}$$

cf. [6, Section 4], with $j(x,t)$ and $\rho(x,t)$ from (1.2). Accordingly we can rewrite $F_\alpha(t)$ in (3.1) as

$$\begin{aligned}
F_\alpha(t) &= \int d^3x\, \rho_\alpha(x - q_\alpha(t))[E^{(0)}(x,t) + v_\alpha(t) \wedge B^{(0)}(x,t)] \\
&\quad + \int d^3x\, \rho_\alpha(x - q_\alpha(t))[E^{(r)}(x,t) + v_\alpha(t) \wedge B^{(r)}(x,t)] \\
&= F_\alpha^{(0)}(t) + F_\alpha^{(r)}(t).
\end{aligned} \tag{3.2}$$

First we consider $F_\alpha^{(0)}(t)$.

Lemma 3.1 *For $t \in [t_0, T\varepsilon^{-3/2}]$, with $t_0 = 4(R_\varphi + C^*\varepsilon^{-1})$, we have $F_\alpha^{(0)}(t) = 0$.*

Proof: If $S(t)$ denotes the solution group generated by the free wave equation in $D^{1,2}(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, it follows from (2.3) through Fourier transform that

$$\begin{pmatrix} E^{(0)}(x,t) \\ \dot{E}^{(0)}(x,t) \end{pmatrix} = \left[S(t) \begin{pmatrix} E^{(0)}(\cdot,0) \\ \dot{E}^{(0)}(\cdot,0) \end{pmatrix} \right](x) = -\sum_{\beta=1}^N e_\beta \int_{-\infty}^0 ds\, [S(t-s)\Phi_E^\beta(\cdot - q_\beta^0 - v_\beta^0 s)](x),$$

where $\Phi_E^\beta(x) = (\varphi(x)v_\beta^0, \nabla\varphi(x))$. The analogous formula is valid for $B^{(0)}(x,t)$, with Φ_E^β to be replaced with $\Phi_B^\beta(x) = (0, v_\beta^0 \wedge \nabla\varphi(x))$. For fixed $1 \le \beta \le N$ and $x \in \mathbb{R}^3$ with $|x - q_\beta^0| \le t - R_\varphi$ assumption (C) yields $\left[S(t-s)\Phi_E^\beta(\cdot - q_\beta^0 - v_\beta^0 s)\right]_1(x) = 0$ for all $s \le 0$ by means of Kirchhoff's formula and Lemma 2.1, $[\ldots]_1$ denoting the first component. As for $t \in [t_0, T\varepsilon^{-3/2}]$ and $|x - q_\beta^0| > t - R_\varphi$ we obtain

$$\begin{aligned} |x - q_\alpha(t)| &\ge |x - q_\beta^0| - |q_\alpha(t) - q_\beta(t)| - |q_\beta(t) - q_\beta^0| \ge t - R_\varphi - C^*\varepsilon^{-1} - C\sqrt{\varepsilon}\, t \\ &\ge t_0/2 - R_\varphi - C^*\varepsilon^{-1} \ge R_\varphi \end{aligned}$$

for ε small by Lemma 2.1, the claim follows. $\square$

Turning then to $F_\alpha^{(r)}(t)$ in (3.2), we write this term in Fourier transformed form and use (1.2) to obtain

$$F_\alpha^{(r)}(t) = e_\alpha^2 F_{\alpha\alpha}^{(r)}(t) + \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} e_\alpha e_\beta F_{\alpha\beta}^{(r)}(t), \tag{3.3}$$

with

$$\begin{aligned} F_{\alpha\beta}^{(r)}(t) = \int_0^t ds \int dk\, |\hat\varphi(k)|^2 e^{-ik\cdot[q_\alpha(t) - q_\beta(s)]} \Big\{ &- \cos|k|(t-s)\, v_\beta(s) + i\,\frac{\sin|k|(t-s)}{|k|}\, k \\ &- i\,\frac{\sin|k|(t-s)}{|k|}\, v_\alpha(t) \wedge (k \wedge v_\beta(s)) \Big\}, \end{aligned} \tag{3.4}$$

$\alpha, \beta = 1, \ldots, N$. The term $F_{\alpha\alpha}^{(r)}(t)$ accounts for the self-force, whereas $F_{\alpha\beta}^{(r)}(t)$ for $\beta \neq \alpha$ represents the mutual interaction force between particle α and particle β. These both contributions are dealt with separately in the following two subsections.

Before going on to this, we state an auxiliary result.

Lemma 3.2 *Let $1 \le \alpha, \beta \le N$, $\alpha \neq \beta$. For $t \in [t_0, T\varepsilon^{-3/2}]$ we have*

(a) $$\begin{aligned} &-\int_0^t ds \int dk\, |\hat\varphi(k)|^2 e^{-ik\cdot[q_\alpha(t) - q_\beta(s)]} \cos|k|(t-s)\, v_\beta(s) \\ &= -\int_0^\infty d\tau \int dk\, |\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \cos|k|\tau \left\{ v_\beta - i\tau(k\cdot v_\beta)v_\beta - \tau\dot v_\beta \right\} + \mathcal{O}(\varepsilon^{7/2}), \end{aligned}$$

(b) $$\begin{aligned} &i\int_0^t ds \int dk\, |\hat\varphi(k)|^2 e^{-ik\cdot[q_\alpha(t) - q_\beta(s)]} \frac{\sin|k|(t-s)}{|k|} k \\ &= i\int_0^\infty d\tau \int dk\, |\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \frac{\sin|k|\tau}{|k|} k \left\{ 1 - ik\cdot\left[\tau v_\beta - \frac{1}{2}\tau^2 \dot v_\beta\right] - \frac{1}{2}\tau^2 (k\cdot v_\beta)^2 \right\} + \mathcal{O}(\varepsilon^{7/2}), \end{aligned}$$

(c) $$\begin{aligned} &(-i)\int_0^t ds \int dk\, |\hat\varphi(k)|^2 e^{-ik\cdot[q_\alpha(t) - q_\beta(s)]} \frac{\sin|k|(t-s)}{|k|}\, v_\alpha(t) \wedge (k \wedge v_\beta(s)) \\ &= (-i)\int_0^\infty d\tau \int dk\, |\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \frac{\sin|k|\tau}{|k|}\, v_\alpha \wedge (k \wedge v_\beta) + \mathcal{O}(\varepsilon^{7/2}). \end{aligned}$$

Here $v_\alpha = v_\alpha(t)$, etc., and $\xi_{\alpha\beta} = q_\alpha(t) - q_\beta(t)$.

The proof is somewhat tedious and given in Appendix B.

3.1 Self–action

For $t \in [t_0, T\varepsilon^{-3/2}]$ we have

$$F_{\alpha\alpha}^{(r)}(t) = \int_0^\infty d\tau \int dk\, |\hat\varphi(k)|^2 e^{-i(k\cdot v_\alpha)\tau}\Big(1+\frac{i}{2}(k\cdot\dot v_\alpha)\tau^2\Big)\Big\{-\cos|k|\tau\,[v_\alpha-\dot v_\alpha\tau] + i\,\frac{\sin|k|\tau}{|k|}\,k - i\,\frac{\sin|k|\tau}{|k|}\,v_\alpha\wedge(k\wedge[v_\alpha-\dot v_\alpha\tau])\Big\} + \mathcal{O}(\varepsilon^{7/2}). \tag{3.5}$$

The rigorous proof of this relation is omitted since it is very similar to the proof of Lemma 3.2 given in Appendix B. It once more relies on the fact that we may Taylor expand

$$q_\alpha(s) \cong q_\alpha - v_\alpha\tau + \frac{1}{2}\dot v_\alpha\tau^2 + \mathcal{O}(\varepsilon^{7/2}), \quad v_\alpha(s) \cong v_\alpha - \dot v_\alpha\tau + \mathcal{O}(\varepsilon^{7/2})$$

by Lemma 2.1, with $q_\alpha = q_\alpha(t)$ etc. and $\tau = t - s$, whence

$$e^{-ik\cdot[q_\alpha(t)-q_\alpha(s)]} \cong e^{-i(k\cdot v_\alpha)\tau}\Big(1+\frac{i}{2}(k\cdot\dot v_\alpha)\tau^2\Big) + \mathcal{O}(\varepsilon^{7/2}).$$

Introducing

$$I_p = \int_0^{\bar t} d\tau \frac{\sin(|k|\tau)}{|k|} e^{-i(k\cdot v_\alpha)\tau}\,\tau^p, \quad J_p = \int_0^{\bar t} d\tau\cos(|k|\tau)e^{-i(k\cdot v_\alpha)\tau}\,\tau^p, \quad p\in\mathbb{N}_0,$$

Equ. (3.5) may be rewritten as

$$\begin{aligned} F_{\alpha\alpha}^{(r)}(t) = \lim_{\bar t\to\infty}\Big(&-\int dk\,|\hat\varphi(k)|^2\Big\{v_\alpha J_0 - \dot v_\alpha J_1 + \frac{i}{2}(k\cdot\dot v_\alpha)v_\alpha J_2\Big\} \\ &+\int dk\,|\hat\varphi(k)|^2\Big\{i\,[(1-v_\alpha^2)k + (k\cdot v_\alpha)v_\alpha]I_0 + i\,[(v_\alpha\cdot\dot v_\alpha)k - (k\cdot v_\alpha)\dot v_\alpha]I_1 \\ &-\frac{1}{2}(k\cdot\dot v_\alpha)[(1-v_\alpha^2)k + (k\cdot v_\alpha)v_\alpha]I_2\Big\}\Big) + \mathcal{O}(\varepsilon^{7/2}), \end{aligned} \tag{3.6}$$

since $\dot v_\alpha^2 = \mathcal{O}(\varepsilon^4)$. Denote the term containing the J_p by $\mathcal{J}$ and the one containing the I_p by $\mathcal{I}$. To evaluate the limits $\bar t\to\infty$, we can rely on the results from [6, Section 4]. We first recall that

$$\int dk\,|\hat\varphi(k)|^2\,J_0 \to 0, \quad \int dk\,|\hat\varphi(k)|^2\,J_1 \to -2m_e\gamma_\alpha^2 \quad \text{as} \quad \bar t\to\infty,$$

with $\gamma_\alpha = (1-v_\alpha^2)^{-1/2}$ and $m_e = \frac{1}{2}\int dk\,|\hat\varphi(k)|^2 k^{-2}$. Moreover, $\nabla_v J_1 = -ikJ_2$, and therefore

$$\begin{aligned} \mathcal{J} &\to (-2m_e\gamma_\alpha^2)\dot v_\alpha + \frac{1}{2}\,\dot v_\alpha\cdot\nabla_v(-2m_e\gamma_\alpha^2)\,v_\alpha = -2m_e\gamma_\alpha^2\Big(\dot v_\alpha + \gamma_\alpha^2(v_\alpha\cdot\dot v_\alpha)v_\alpha\Big) \\ &= -2m_e\Big((1+v_\alpha^2)\dot v_\alpha + (v_\alpha\cdot\dot v_\alpha)v_\alpha\Big) + \mathcal{O}(\varepsilon^4) \end{aligned} \tag{3.7}$$

as $\bar t\to\infty$, the latter equality according to the expansion $\gamma_\alpha^2 = 1 + v_\alpha^2 + \mathcal{O}(v_\alpha^4) = 1 + v_\alpha^2 + \mathcal{O}(\varepsilon^2)$ and $\gamma_\alpha^4 = 1 + \mathcal{O}(\varepsilon)$.

What concerns $\mathcal{I}$, we know from [3, 6] that $\int dk\,|\hat\varphi(k)|^2\,kI_0 \to 0$,

$$\int dk\,|\hat\varphi(k)|^2\,I_0 \to 2m_e|v_\alpha|^{-1}\mathrm{arth}|v_\alpha|, \quad \int dk\,|\hat\varphi(k)|^2\,(k\cdot\dot v_\alpha)kI_2 \to -2m_e\mu(v_\alpha)\dot v_\alpha$$

as $\bar{t} \to \infty$, where

$$\mu(v)z = \Big(\frac{\gamma^2}{v^2} - |v|^{-3}\mathrm{arth}|v|\Big)\, z + \Big(\frac{\gamma^4}{v^4}(5v^2-3) + 3|v|^{-5}\,\mathrm{arth}|v|\Big)\,(v\cdot z)v$$

for $|v|<1$ and $z \in \mathbb{R}^3$. Consequently, since $s^{-1}\mathrm{arth}(s) = 1 + s^2/3 + s^4/5 + \mathcal{O}(s^6)$ for s close to zero, it thus follows after some calculation that

$$\begin{aligned}
\mathcal{I} \ \ \to \ \ & -(v_\alpha\cdot\dot v_\alpha)\nabla_v\Big(2m_{\mathrm{e}}|v_\alpha|^{-1}\mathrm{arth}|v_\alpha|\Big) + \dot v_\alpha\, v_\alpha\cdot\nabla_v\Big(2m_{\mathrm{e}}|v_\alpha|^{-1}\mathrm{arth}|v_\alpha|\Big) \\
& -\frac{1}{2}(1-v_\alpha^2)\Big(-2m_{\mathrm{e}}\mu(v_\alpha)\dot v_\alpha\Big) - \frac{1}{2}v_\alpha\, v_\alpha\cdot\Big(-2m_{\mathrm{e}}\mu(v_\alpha)\dot v_\alpha\Big) \\
= \ \ & \Big(\frac{2}{3}+\frac{22}{15}v_\alpha^2\Big)m_{\mathrm{e}}\dot v_\alpha + \frac{14}{15}m_{\mathrm{e}}(v_\alpha\cdot\dot v_\alpha)v_\alpha + \mathcal{O}(\varepsilon^4). \qquad (3.8)
\end{aligned}$$

Summarizing (3.6), (3.7), and (3.8), we arrive at

Lemma 3.3 *For $t\in[t_0, T\varepsilon^{-3/2}]$ we have*

$$F_{\alpha\alpha}^{(r)}(t) = -\Big(\frac{4}{3}+\frac{8}{15}v_\alpha^2\Big)m_{\mathrm{e}}\dot v_\alpha - \frac{16}{15}m_{\mathrm{e}}(v_\alpha\cdot\dot v_\alpha)v_\alpha + \mathcal{O}(\varepsilon^{7/2}).$$

3.2 Mutual interaction

In this section we expand $F_{\alpha\beta}^{(r)}(t)$ from (3.4) with $\beta\neq\alpha$. For $p\in\mathbb{N}_0$ we have that

$$A_p := \int_0^\infty d\tau \int dk\,|\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}}\,\frac{\sin|k|\tau}{|k|}\,\tau^p = (4\pi)^{-1}\int\!\!\int dxdy\,\varphi(x)\varphi(y)|\xi_{\alpha\beta}+x-y|^{p-1}$$

and

$$\begin{aligned}
B_p \ &:= \int_0^\infty d\tau\int dk\,|\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}}\,\cos(|k|\tau)\,\tau^p \\
&= (-p)(4\pi)^{-1}\int\!\!\int dxdy\,\varphi(x)\varphi(y)|\xi_{\alpha\beta}+x-y|^{p-2} = (-p)A_{p-1},
\end{aligned}$$

as may be seen through Fourier transform. We hence obtain from Lemma 3.2 that for $\beta\neq\alpha$ and $t\in[t_0,T\varepsilon^{-3/2}]$

$$\begin{aligned}
F_{\alpha\beta}^{(r)}(t) \ &= \ -v_\beta(v_\beta\cdot\nabla_\xi)B_1 + \dot v_\beta B_1 - \nabla_\xi A_0 + \frac{1}{2}(\dot v_\beta\cdot\nabla_\xi)\nabla_\xi A_2 - \frac{1}{2}(v_\beta\cdot\nabla_\xi)^2\nabla_\xi A_2 \\
&\quad +(v_\alpha\cdot v_\beta)\nabla_\xi A_0 - v_\beta(v_\alpha\cdot\nabla_\xi)A_0 + \mathcal{O}(\varepsilon^{7/2}), \qquad (3.9)
\end{aligned}$$

taking also into account that $A_1 = (4\pi)^{-1}$, thus $\nabla_\xi A_1 = 0$. As a consequence of $|\xi_{\alpha\beta}| = \mathcal{O}(\varepsilon^{-1})$, cf. Lemma 2.1, of assumption (C), and of Lemma 2.1, it follows that in (3.9) we have $-\nabla_\xi A_0 = \mathcal{O}(\varepsilon^2)$, while all other terms are $\mathcal{O}(\varepsilon^3)$. Since e.g.

$$\Big|(v_\alpha\cdot v_\beta)\nabla_\xi A_0 - (v_\alpha\cdot v_\beta)\Big(-\frac{\xi_{\alpha\beta}}{4\pi|\xi_{\alpha\beta}|^3}\Big)\Big| \le C\varepsilon^4,$$

with an obvious similar estimate for the other terms besides $-\nabla_\xi A_0$, we find from (3.9) and after some calculation that for $\beta \neq \alpha$ and $t \in [t_0, T\varepsilon^{-3/2}]$

$$\begin{aligned}
F^{(r)}_{\alpha\beta}(t) &= v_\beta (v_\beta \cdot \nabla_\xi)\Big(\frac{1}{4\pi|\xi_{\alpha\beta}|}\Big) - \dot{v}_\beta\Big(\frac{1}{4\pi|\xi_{\alpha\beta}|}\Big) - \nabla_\xi A_0 + \frac{1}{2}(\dot{v}_\beta \cdot \nabla_\xi)\nabla_\xi\Big(\frac{|\xi_{\alpha\beta}|}{4\pi}\Big) \\
&\quad -\frac{1}{2}(v_\beta \cdot \nabla_\xi)^2 \nabla_\xi\Big(\frac{|\xi_{\alpha\beta}|}{4\pi}\Big) + (v_\alpha \cdot v_\beta)\nabla_\xi\Big(\frac{1}{4\pi|\xi_{\alpha\beta}|}\Big) - v_\beta (v_\alpha \cdot \nabla_\xi)\Big(\frac{1}{4\pi|\xi_{\alpha\beta}|}\Big) + \mathcal{O}(\varepsilon^{7/2}) \\
&= -\nabla_\xi A_0 - \frac{1}{8\pi|\xi_{\alpha\beta}|}\,\dot{v}_\beta - \frac{(\dot{v}_\beta \cdot \xi_{\alpha\beta})}{8\pi|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} + \frac{v_\beta^2}{8\pi|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} - \frac{3(v_\beta \cdot \xi_{\alpha\beta})^2}{8\pi|\xi_{\alpha\beta}|^5}\,\xi_{\alpha\beta} \\
&\quad -\frac{(v_\alpha \cdot v_\beta)}{4\pi|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} + \frac{(v_\alpha \cdot \xi_{\alpha\beta})}{4\pi|\xi_{\alpha\beta}|^3}\,v_\beta + \mathcal{O}(\varepsilon^{7/2}).
\end{aligned}$$

Finally, to deal with the lowest-order term we observe that with $\vec{n} = \xi_{\alpha\beta}/|\xi_{\alpha\beta}|$

$$\Big|\nabla_\xi A_0 + \frac{\xi_{\alpha\beta}}{4\pi|\xi_{\alpha\beta}|^3}\Big| = \frac{1}{4|\xi_{\alpha\beta}|^2}\Big|\int\!\!\int dxdy\,\varphi(x)\varphi(y)\Big(\frac{\vec{n} + \frac{x-y}{|\xi_{\alpha\beta}|}}{|\vec{n} + \frac{x-y}{|\xi_{\alpha\beta}|}|^3} - \vec{n}\Big)\Big|. \tag{3.10}$$

Defining $R = (x-y)/|\xi_{\alpha\beta}| = \mathcal{O}(\varepsilon)$ for $|x|, |y| \le R_\varphi$, we can expand $\psi(R) = (\vec{n} + R)/|\vec{n} + R|$ to obtain that $\psi(R) = \vec{n} + R - 3(\vec{n}\cdot R)\vec{n} + \mathcal{O}(\varepsilon^2)$. As $\int\int dxdy\,\varphi(x)\varphi(y)(x-y) = 0$, we hence conclude that the right-hand side of (3.10) is $\mathcal{O}(\varepsilon^4)$. Thus we can summarize our estimates on the mutual interaction force as follows.

Lemma 3.4 *For $\beta \neq \alpha$ and $t \in [t_0, T\varepsilon^{-3/2}]$ we have*

$$\begin{aligned}
F^{(r)}_{\alpha\beta}(t) &= \frac{\xi_{\alpha\beta}}{4\pi|\xi_{\alpha\beta}|^3} - \frac{1}{8\pi|\xi_{\alpha\beta}|}\,\dot{v}_\beta - \frac{(\dot{v}_\beta \cdot \xi_{\alpha\beta})}{8\pi|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} + \frac{v_\beta^2}{8\pi|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} - \frac{3(v_\beta \cdot \xi_{\alpha\beta})^2}{8\pi|\xi_{\alpha\beta}|^5}\,\xi_{\alpha\beta} \\
&\quad -\frac{(v_\alpha \cdot v_\beta)}{4\pi|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} + \frac{(v_\alpha \cdot \xi_{\alpha\beta})}{4\pi|\xi_{\alpha\beta}|^3}\,v_\beta + \mathcal{O}(\varepsilon^{7/2}).
\end{aligned}$$

3.3 Summary of the estimates

By (3.1), (3.2), and Lemma 3.1 we find $F_\alpha(t) = F^{(r)}_\alpha(t)$ for $t \in [t_0, T\varepsilon^{-3/2}]$. According to (3.3) and Lemmas 3.3 and 3.4 we hence have obtained the following expansion of the Lorentz force in (3.1). For $t \in [t_0, T\varepsilon^{-3/2}]$ we have

$$\begin{aligned}
F_\alpha(t) &= -\Big(\frac{4}{3} + \frac{8}{15}v_\alpha^2\Big)m_{\mathrm{e}}\dot{v}_\alpha - \frac{16}{15}m_{\mathrm{e}}(v_\alpha \cdot \dot{v}_\alpha)v_\alpha + G_\alpha(\vec{q}, \vec{v}, \dot{\vec{v}}) + \mathcal{O}(\varepsilon^{7/2}), \\
G_\alpha(\vec{q}, \vec{v}, \dot{\vec{v}}) &= \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{e_\alpha e_\beta}{4\pi}\Big(\frac{\xi_{\alpha\beta}}{|\xi_{\alpha\beta}|^3} - \frac{1}{2|\xi_{\alpha\beta}|}\,\dot{v}_\beta - \frac{(\dot{v}_\beta \cdot \xi_{\alpha\beta})}{2|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} + \frac{v_\beta^2}{2|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} - \frac{3(v_\beta \cdot \xi_{\alpha\beta})^2}{2|\xi_{\alpha\beta}|^5}\,\xi_{\alpha\beta} \\
&\qquad -\frac{(v_\alpha \cdot v_\beta)}{|\xi_{\alpha\beta}|^3}\,\xi_{\alpha\beta} + \frac{(v_\alpha \cdot \xi_{\alpha\beta})}{|\xi_{\alpha\beta}|^3}\,v_\beta\Big),
\end{aligned} \tag{3.11}$$

where $t_0 = 4(R_\varphi + C^*\varepsilon^{-1})$, $\xi_{\alpha\beta} = q_\alpha(t) - q_\beta(t)$, $v_\alpha = v_\alpha(t)$, and $v_\beta = v_\beta(t)$. Due to the Lorentz equation $\frac{d}{dt}(m_{\mathrm{b}\alpha}\gamma_\alpha v_\alpha) = F_\alpha(t)$, cf. (1.5), we finally obtain the following lemma by calculating the right-hand side and expanding γ_α.

Lemma 3.5 *For $t \in [t_0, T\varepsilon^{-3/2}]$ we have*

$$M_\alpha(v_\alpha)\dot{v}_\alpha = G_\alpha(\vec{q}, \vec{v}, \dot{\vec{v}}) + \mathcal{O}(\varepsilon^{7/2}), \quad 1 \le \alpha \le N,$$

with G_α from (3.11) and $M_\alpha(v)$ the (3×3)-matrix $M_\alpha(v)(z) = (m_\alpha + \frac{1}{2}m_\alpha^* v^2)z + m_\alpha^*(v \cdot z)v$ for $v, z \in \mathbb{R}^3$.

4 Proof of Theorem 2.2

We need to compare a solution $(q_\alpha(t), v_\alpha(t))$ of (1.2)–(1.5) with data (2.13) to $(\tilde{r}_\alpha(t), \tilde{u}_\alpha(t))$, where we let

$$\tilde{r}_\alpha(t) = \varepsilon^{-1} r_\alpha(\varepsilon^{3/2} t), \quad \tilde{u}_\alpha(t) = \sqrt{\varepsilon}\, u_\alpha(\varepsilon^{3/2} t), \tag{4.1}$$

and where the $(r_\alpha(t), u_\alpha(t))$ are the solution to the system induced by (2.11) with data (r_α^0, u_α^0).

A somewhat lengthy but elementary calculation shows that $(\tilde{r}_\alpha(t), \tilde{u}_\alpha(t))$ satisfy

$$M_\alpha(\tilde{u}_\alpha)\dot{\tilde{u}}_\alpha = G_\alpha(\vec{\tilde{r}}, \vec{\tilde{u}}, \dot{\vec{\tilde{u}}}), \quad 1 \le \alpha \le N, \tag{4.2}$$

cf. Lemma 3.5 for the notation. Recalling that $\tau \in]0, \infty]$ was defined to be the first time when either $\lim_{t \to \tau^-} |r_\alpha(t) - r_\beta(t)| = 0$ for some $\alpha \ne \beta$ or $\lim_{t \to \tau^-} |r_\alpha(t)| = \infty$ for some α holds, we find that (4.2) is valid for $t \in [0, (\tau - \delta_0)\varepsilon^{-3/2}]$, for any $\delta_0 \in]0, \tau[$ which we consider to be fixed throughout. This leads to some useful estimates on the effective dynamics.

Lemma 4.1 *For suitable constants $C_0, C^0, C > 0$ (depending on τ, δ_0, and the data) we have*

$$C_0 \varepsilon^{-1} \le \sup_{t \in [0,\, (\tau - \delta_0)\varepsilon^{-3/2}]} |\tilde{r}_\alpha(t) - \tilde{r}_\beta(t)| \le C^0 \varepsilon^{-1}, \quad \alpha \ne \beta, \tag{4.3}$$

and

$$\sup_{t \in [0,\, (\tau - \delta_0)\varepsilon^{-3/2}]} |\tilde{u}_\alpha(t)| \le C\sqrt{\varepsilon}. \tag{4.4}$$

Proof: The bounds in (4.3) follow from (4.1) and the fact that $|r_\alpha(t) - r_\beta(t)| \ge \delta_1$ and $|r_\alpha(t)| \le C$ on $[0, \tau - \delta_0]$ for some $\delta_1 > 0$, $C > 0$, by definition of τ. Concerning (4.4), by conservation of the energy $\mathcal{H}_{\mathrm{D}}$ from (2.12) we obtain $C \ge \mathcal{H}_{\mathrm{D}}(\vec{r}(0), \vec{u}(0)) = \mathcal{H}_{\mathrm{D}}(\vec{r}(t), \vec{u}(t)) \ge \frac{1}{2} m_\alpha u_\alpha^2(t)$ as long as the solution exists, in particular for $t \in [0, \tau - \delta_0]$. □

To simplify the presentation, we henceforth omit the tilde and write (r, u) instead of $(\tilde{r}, \tilde{u})$ to denote the rescaled solution. Utilizing the bounds from Lemma 2.1 and from (4.3), (4.4), it may be seen after some calculation that

$$\Big|G_\alpha(\vec{q}, \vec{v}, \dot{\vec{v}})(t) - G_\alpha(\vec{r}, \vec{u}, \dot{\vec{u}})(t)\Big| \le C \sum_{\beta=1}^{N} \Big(\varepsilon^3 |q_\beta(t) - r_\beta(t)| + \varepsilon^{5/2} |v_\beta(t) - u_\beta(t)| + \varepsilon |\dot{v}_\beta(t) - \dot{u}_\beta(t)|\Big) \tag{4.5}$$

for $1 \le \alpha \le N$ and $t \in [0, T\varepsilon^{-3/2}] \cap [0, (\tau - \delta_0)\varepsilon^{-3/2}] = [0, \min\{\tau - \delta_0, T\}\varepsilon^{-3/2}]$. Note that the term $\varepsilon^3 |q_\beta - r_\beta|$ appears through comparison of $\xi_{\alpha\beta}/|\xi_{\alpha\beta}|^3$ to $r_{\alpha\beta}/|r_{\alpha\beta}|^3$, cf. the form of G_α in (3.11).

Next, a general (3×3)-matrix $M(v) = a(v)\mathrm{id} + b(v \otimes v)$ has the inverse

$$M(v)^{-1} = a(v)^{-1}\mathrm{id} + \frac{b}{a(v)[a(v) + bv^2]}(v \otimes v).$$

This remark shows $|M_\alpha(v_\alpha)^{-1}| = \mathcal{O}(1)$ and $|M_\alpha(v_\alpha)^{-1} - M_\alpha(u_\alpha)^{-1}| \le C\sqrt{\varepsilon}|v_\alpha - u_\alpha|$ for $t \in [0, \min\{\tau - \delta_0, T\}\varepsilon^{-3/2}]$. Since $|G_\alpha(\vec{q}, \vec{v}, \dot{\vec{v}})| = \mathcal{O}(\varepsilon^2)$ it follows from Lemma 3.5, (4.2), and (4.5) that

$$|\dot{v}_\alpha(t) - \dot{u}_\alpha(t)| \le C \sum_{\beta=1}^{N} \Big(\varepsilon^3 |q_\beta(t) - r_\beta(t)| + \varepsilon^{5/2} |v_\beta(t) - u_\beta(t)| + \varepsilon |\dot{v}_\beta(t) - \dot{u}_\beta(t)| \Big) + \mathcal{O}(\varepsilon^{7/2})$$

for $1 \le \alpha \le N$ and $t \in [t_0, \min\{\tau - \delta_0, T\}\varepsilon^{-3/2}]$. Summation over α and choosing $\varepsilon > 0$ sufficiently small this results in

$$\sum_{\alpha=1}^{N} |\dot{v}_\alpha(t) - \dot{u}_\alpha(t)| \le C \sum_{\alpha=1}^{N} \Big(\varepsilon^3 |q_\alpha(t) - r_\alpha(t)| + \varepsilon^{5/2} |v_\alpha(t) - u_\alpha(t)| \Big) + \mathcal{O}(\varepsilon^{7/2}) \tag{4.6}$$

for $t \in [t_0, \min\{\tau - \delta_0, T\}\varepsilon^{-3/2}]$. To use this basic estimate, we write $d_\alpha(t) = q_\alpha(t) - r_\alpha(t)$ as

$$d_\alpha(t) = d_\alpha(t_0) + (t - t_0)\dot{d}_\alpha(t_0) + \int_{t_0}^{t} (t - s)\ddot{d}_\alpha(s)\, ds, \quad \dot{d}_\alpha(t) = \dot{d}_\alpha(t_0) + \int_{t_0}^{t} \ddot{d}_\alpha(s)\, ds.$$

We then obtain for $t \in [t_0, \min\{\tau - \delta_0, T\}\varepsilon^{-3/2}]$ from (4.6) that

$$\begin{aligned} D(t) &\le D(t_0) + (t - t_0)\bar{D}(t_0) + C\varepsilon^3 \int_{t_0}^{t} (t - s) D(s)\, ds \\ &\quad + C\varepsilon^{5/2} \int_{t_0}^{t} (t - s)\bar{D}(s)\, ds + C\sqrt{\varepsilon}, & (4.7) \\ \bar{D}(t) &\le \bar{D}(t_0) + C\varepsilon^3 \int_{t_0}^{t} D(s)\, ds + C\varepsilon^{5/2} \int_{t_0}^{t} \bar{D}(s)\, ds + C\varepsilon^2, & (4.8) \end{aligned}$$

where

$$D(t) = \max_{1 \le \alpha \le N} \max_{s \in [t_0, t]} |d_\alpha(s)| \quad \text{and} \quad \bar{D}(t) = \max_{1 \le \alpha \le N} \max_{s \in [t_0, t]} |\dot{d}_\alpha(s)|.$$

Application of Gronwall's lemma to (4.8) yields

$$\bar{D}(t) \le C\Big(\bar{D}(t_0) + \varepsilon^2 + \varepsilon^3 \int_{t_0}^{t} D(s)\, ds \Big), \tag{4.9}$$

and utilizing this in (4.7) implies

$$D(t) \le D(t_0) + (t - t_0)\bar{D}(t_0) + C\sqrt{\varepsilon} + C\varepsilon^{-1/2}(\bar{D}(t_0) + \varepsilon^2) + C\varepsilon^3 \int_{t_0}^{t} (t - s) D(s)\, ds.$$

Finally, $(t - s) \le C\varepsilon^{-3/2}$ yields upon a further application of Gronwall's lemma that

$$D(t) \le C\Big(D(t_0) + \varepsilon^{-3/2}\bar{D}(t_0) + \sqrt{\varepsilon}\, \Big), \quad t \in [t_0, \min\{\tau - \delta_0, T\}\varepsilon^{-3/2}]. \tag{4.10}$$

By assumption $D(t_0) = 0 = \bar{D}(t_0)$. Therefore (4.10) and (4.9) imply (2.14). This completes the proof of Theorem 2.2. □

5 Appendix A: Proof of Lemma 2.1

This appendix concerns the proof of Lemma 2.1. We split the proof into three subsections.

5.1 Bounding the particle distances and the velocities

We intend to use energy conservation to show (2.8), and for that reason we calculate with (2.3) the field energy

$$\begin{aligned}\mathcal{H}_{\mathrm{F}}(0) &= \frac{1}{2}\int d^3x\,[E^2(x,0)+B^2(x,0)]\\ &= \frac{1}{2}\sum_{\alpha=1}^{N}\int d^3x\,[E^2_{v^0_\alpha}(x-q^0_\alpha)+B^2_{v^0_\alpha}(x-q^0_\alpha)]+\frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\int d^3x\,\Big[E_{v^0_\alpha}(x-q^0_\alpha)\cdot E_{v^0_\beta}(x-q^0_\beta)\\ &\qquad\qquad +B_{v^0_\alpha}(x-q^0_\alpha)\cdot B_{v^0_\beta}(x-q^0_\beta)\Big].\end{aligned}$$

According to (2.4) and [6, Section 2] the first term equals

$$\mathcal{H}^{(1)}_{\mathrm{F}}(0)=\sum_{\alpha=1}^{N}e_\alpha^2\Big(\frac{1}{2}\int d^3k\,|\hat{\varphi}(k)|^2k^{-2}\Big)\Big[\frac{1}{|v^0_\alpha|}\,\log\frac{1+|v^0_\alpha|}{1-|v^0_\alpha|}-1\Big].$$

Denoting the term in $[\ldots]$ as $\psi(|v^0_\alpha|)$, $\psi(r)$ is odd, and hence Taylor expansion implies $\psi(r)=1+\mathcal{O}(r^2)$ for r small. Therefore (2.2) yields

$$\mathcal{H}^{(1)}_{\mathrm{F}}(0)=\mathcal{E}_{\mathrm{Coul}}+\mathcal{O}(\varepsilon),$$

with $\mathcal{E}_{\mathrm{Coul}}$ from (2.7). To deal with the contributions for $\alpha\neq\beta$ in the second term, we obtain by passing to Fourier transformed form and observing (2.2) that e.g.

$$\int d^3x\,E_{v^0_\alpha}(x-q^0_\alpha)\cdot E_{v^0_\beta}(x-q^0_\beta)=e_\alpha e_\beta\int d^3k\,|\hat{\varphi}(k)|^2k^{-2}e^{ik\cdot(q^0_\alpha-q^0_\beta)}+\mathcal{O}(\varepsilon)=\mathcal{O}(\varepsilon),$$

the latter with (2.1) and by passing to polar coordinates. Thus we have shown

$$\mathcal{H}_{\mathrm{F}}(0)=\mathcal{E}_{\mathrm{Coul}}+\mathcal{O}(\varepsilon). \tag{5.1}$$

Next we will investigate the field energy at time $t>0$. We claim that

$$\mathcal{H}_{\mathrm{F}}(t)=\frac{1}{2}\int d^3x\,[E^2(x,t)+B^2(x,t)]\geq\frac{1}{2}\int d^3x\,E^2(x,t)\geq-\frac{1}{2}\Big(\rho(\cdot,t),\Delta^{-1}\rho(\cdot,t)\Big)_{L^2(\mathbb{R}^3)}. \tag{5.2}$$

The easiest way to see this is to introduce potentials A and ϕ,

$$B(x,t)=\nabla\wedge A(x,t),\quad E(x,t)=-\nabla\phi(x,t)-F(x,t),\quad\text{with}\quad F(x,t)=\frac{\partial A}{\partial t}(x,t),$$

for the electromagnetic field. Then $\rho=\nabla\cdot E=-\Delta\phi-\nabla\cdot F$, and the estimate in (5.2) follows by passing to Fourier transformed form. On the other hand, substituting ρ from (1.2) into

$$-\frac{1}{2}\Big(\rho(\cdot,t),\Delta^{-1}\rho(\cdot,t)\Big)_{L^2(\mathbb{R}^3)}=\frac{1}{2}\int d^3k\,|\hat{\rho}(k,t)|^2k^{-2},$$

by assumption (2.9) we can argue exactly as before to show that the terms with $\alpha\neq\beta$ are $\mathcal{O}(\varepsilon)$, and thus

$$\mathcal{H}_{\mathrm{F}}(t)\geq\frac{1}{2}\sum_{\alpha=1}^{N}\int d^3k\,|\hat{\rho}_\alpha(k)|^2k^{-2}+\mathcal{O}(\varepsilon)=\mathcal{E}_{\mathrm{Coul}}+\mathcal{O}(\varepsilon) \tag{5.3}$$

for $t \in [0, T\varepsilon^{-3/2}]$. Consequently for $t \in [0, T\varepsilon^{-3/2}]$ by energy conservation, cf. (1.6), by (5.1) and (5.3)

$$\begin{aligned} \sum_{\alpha=1}^{N} m_{\mathrm{b}\alpha}\gamma(v_\alpha^0) + \mathcal{E}_{\mathrm{Coul}} + \mathcal{O}(\varepsilon) &= \sum_{\alpha=1}^{N} m_{\mathrm{b}\alpha}\gamma(v_\alpha^0) + \mathcal{H}_F(0) = \sum_{\alpha=1}^{N} m_{\mathrm{b}\alpha}\gamma(v_\alpha(t)) + \mathcal{H}_F(t) \\ &\geq \sum_{\alpha=1}^{N} m_{\mathrm{b}\alpha}\gamma(v_\alpha(t)) + \mathcal{E}_{\mathrm{Coul}} + \mathcal{O}(\varepsilon), \end{aligned}$$

with $\gamma(v) = (1-v^2)^{-1/2}$. Thus

$$\sum_{\alpha=1}^{N} m_{\mathrm{b}\alpha}\gamma(v_\alpha^0) + C\varepsilon \geq \sum_{\alpha=1}^{N} m_{\mathrm{b}\alpha}\gamma(v_\alpha(t)), \quad t \in [0, T\varepsilon^{-3/2}] \tag{5.4}$$

with some constant C depending on C_1, C_3, C_*, T. This estimate now allows to prove (2.8). Define

$$I_+ = \{\alpha \in \{1,\dots,N\} : \gamma(v_\alpha(t)) \leq \gamma(v_\alpha^0)\} \quad \text{and} \quad I_- = \{\alpha \in \{1,\dots,N\} : \gamma(v_\alpha(t)) > \gamma(v_\alpha^0)\}.$$

For $\alpha \in I_+$ we have $|v_\alpha(t)| \leq |v_\alpha^0| \leq C_3\sqrt{\varepsilon}$ by (2.2). Thus for ε so small that $C_3^2\varepsilon \leq 1/2$, $\gamma(v_\alpha^0) - \gamma(v_\alpha(t)) \leq \sqrt{2}|(v_\alpha^0)^2 - (v_\alpha(t))^2| \leq C\varepsilon$. Therefore by (5.4)

$$C\varepsilon \geq \sum_{\alpha\in I_-} m_{\mathrm{b}\alpha}\Big(\gamma(v_\alpha(t)) - \gamma(v_\alpha^0)\Big).$$

Since $m_{\mathrm{b}\alpha} > 0$ we deduce that

$$\gamma(v_\alpha(t)) \leq \gamma(v_\alpha^0) + C\varepsilon, \quad \alpha \in I_-,$$

and according to $|v_\alpha^0| \leq C_3\sqrt{\varepsilon}$ it then follows that $|v_\alpha(t)| \leq C\sqrt{\varepsilon}$ also for $\alpha \in I_-$. This concludes the proof of (2.8).

Using (2.1) and (2.8) it is finally easy to derive the upper bound in (2.6), since for $t \in [0, T\varepsilon^{-3/2}]$ we have

$$|q_\alpha(t) - q_\beta(t)| \leq |q_\alpha^0 - q_\beta^0| + |q_\alpha(t) - q_\alpha^0| + |q_\beta(t) - q_\beta^0| \leq C_2\varepsilon^{-1} + 2C_vT\sqrt{\varepsilon}\varepsilon^{-3/2} = C^*\varepsilon^{-1},$$

with $C^* = C_2 + 2C_vT$. We remark that for the estimates in this section the smallness of the e_α was not needed.

5.2 Bounding $|\dot{v}_\alpha(t)|$

Since

$$\frac{d}{dt}\Big(m_{\mathrm{b}\alpha}\gamma_\alpha v_\alpha(t)\Big) = m_{0\alpha}(v_\alpha(t))\dot{v}_\alpha(t),$$

with the (3×3)-matrices $m_{0\alpha}(v_\alpha)$ given through $m_{0\alpha}(v_\alpha)(z) = m_{\mathrm{b}\alpha}(\gamma_\alpha z + \gamma_\alpha^3(v_\alpha\cdot z)v_\alpha)$, $z \in \mathbb{R}^3$, we obtain from (1.5) that for $\alpha = 1,\dots,N$

$$\begin{aligned} \dot{v}_\alpha &= m_{0\alpha}(v_\alpha)^{-1}\int d^3x\,\rho_\alpha(x-q_\alpha)\Big([E(x) - E_{v_\alpha}(x-q_\alpha)] + v_\alpha \wedge [B(x) - B_{v_\alpha}(x-q_\alpha)]\Big) \\ &= m_{0\alpha}(v_\alpha)^{-1}\int d^3x\,\rho_\alpha(x)\Big(Z_1(x+q_\alpha,t) + v_\alpha \wedge Z_2(x+q_\alpha,t)\Big) + R_\alpha(t), \end{aligned} \tag{5.5}$$

where $m_{0\alpha}(v_\alpha)^{-1}z = m_{b\alpha}{}^{-1}\gamma_\alpha^{-1}(z-(v_\alpha\cdot z)v_\alpha)$, $z\in \mathbb{R}^3$, is the matrix inverse of $m_{0\alpha}(v_\alpha)$. For (5.5) it is important to note that adding the $E_{v_\alpha}(x-q_\alpha)$-term and the $v_\alpha\wedge B_{v_\alpha}(x-q_\alpha)$-term does not change the integral, as may be seen through Fourier transform using (2.4) and (2.5). Moreover, in (5.5) we have set

$$R_\alpha(t) = m_{0\alpha}(v_\alpha)^{-1}\Big(\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^N \int d^3x\,\rho_\alpha(x-q_\alpha)\big[E_{v_\beta}(x-q_\beta)+v_\alpha\wedge B_{v_\beta}(x-q_\beta)\big]\Big) \tag{5.6}$$

and

$$Z(x,t) = \begin{pmatrix} Z_1(x,t)\\ Z_2(x,t)\end{pmatrix} = \begin{pmatrix} E(x,t)-\sum_{\beta=1}^N E_{v_\beta(t)}(x-q_\beta(t))\\ B(x,t)-\sum_{\beta=1}^N B_{v_\beta(t)}(x-q_\beta(t))\end{pmatrix}.$$

Maxwell's equations and the relations $(v\cdot\nabla)E_v(x) = -\nabla\wedge B_v(x)+e\varphi(x)v$, $(v\cdot\nabla)B_v(x)=\nabla\wedge E_v(x)$, $e=e_\alpha$ for index α, yield

$$\dot Z(t) = \mathcal{A}Z(t) - f(t)\,,\quad \text{with}\quad \mathcal{A} = \begin{pmatrix} 0 & \nabla\wedge\\ -\nabla\wedge & 0\end{pmatrix} \tag{5.7}$$

and

$$f(x,t) = \sum_{\beta=1}^N \begin{pmatrix} (\dot v_\beta(t)\cdot\nabla_v)E_{v_\beta}(x-q_\beta(t))\\ (\dot v_\beta(t)\cdot\nabla_v)B_{v_\beta}(x-q_\beta(t))\end{pmatrix}. \tag{5.8}$$

The Maxwell operator $\mathcal{A}$ generates a C^0-group $U(t)$, $t\in\mathbb{R}$, of isometries in $L^2(\mathbb{R}^3)^3\oplus L^2(\mathbb{R}^3)^3$; see [2, p. 435; (H2)]. Therefore we have the mild solution representation

$$Z(x,t) = [U(t)Z(\cdot,0)](x) - \int_0^t ds\,[U(t-s)f(\cdot,s)](x). \tag{5.9}$$

According to (2.3), $Z(0)=0$, so the first term drops out. To estimate the remaining term, we first state and prove some auxiliary lemmas that will be used frequently.

Lemma 5.1 *For given $f=(f_1,f_2)$ with $\nabla\cdot f_1=0$ and $\nabla\cdot f_2=0$ we have for $W(t,s,x) = (W_1(t,s,x),W_2(t,s,x)) = [U(t-s)f(\cdot,s)](x)$*

$$\begin{aligned} W_1(t,s,x) &= \frac{1}{4\pi(t-s)^2}\int_{|y-x|=(t-s)} d^2y\,\big[(t-s)\nabla\wedge f_2(y,s)+f_1(y,s)+((y-x)\cdot\nabla)f_1(y,s)\big],\\ W_2(t,s,x) &= \frac{1}{4\pi(t-s)^2}\int_{|y-x|=(t-s)} d^2y\,\big[-(t-s)\nabla\wedge f_1(y,s)+f_2(y,s)+((y-x)\cdot\nabla)f_2(y,s)\big].\end{aligned}$$

Proof: See [6, Lemma 8.1]. □

Lemma 5.2 *(a) Let $\xi(s)\geq 0$ be some function. Assume that for $y\in\mathbb{R}^3$, $s\in[0,t]$, and some $f(y,s)=(f_1(y,s),f_2(y,s))$ with $\nabla\cdot f_1 = 0 = \nabla\cdot f_2$*

$$|f_1(y,s)|+|f_2(y,s)| \;\leq\; C\xi(s)\sum_{\beta=1}^N \frac{1}{1+|y-q_\beta(s)|^2}, \tag{5.10}$$

$$|\nabla f_1(y,s)|+|\nabla f_2(y,s)| \;\leq\; C\xi(s)\sum_{\beta=1}^N \frac{1}{1+|y-q_\beta(s)|^3}, \tag{5.11}$$

Then for each $\alpha = 1, \ldots, N$, $t \in [0, T\varepsilon^{-3/2}]$, and $|x| \le R_\varphi$

$$\Big| \int_0^t ds\, [U(t-s) f(\cdot, s)](x + q_\alpha(t)) \Big| \le C\Big(\sup_{s \in [0,t]} \xi(s) \Big).$$

(b) Under the hypotheses of (a), if instead of (5.10) and (5.11) it holds for fixed $1 \le \alpha \le N$ that

$$|f_1(y,s)| + |f_2(y,s)| \;\le\; C\xi(s) \sum_{\substack{\beta=1 \\ \beta \ne \alpha}}^N \frac{1}{1 + |y - q_\beta(s)|^3}, \tag{5.12}$$

$$|\nabla f_1(y,s)| + |\nabla f_2(y,s)| \;\le\; C\xi(s) \sum_{\substack{\beta=1 \\ \beta \ne \alpha}}^N \frac{1}{1 + |y - q_\beta(s)|^4}, \tag{5.13}$$

then for $t \in [0, T\varepsilon^{-3/2}]$ and $|x| \le R_\varphi$ we have even that

$$\Big| \int_0^t ds\, [U(t-s) f(\cdot, s)](x + q_\alpha(t)) \Big| \le C\Big(\sup_{s \in [0,t]} \xi(s) \Big)\varepsilon.$$

(c) Let $\xi(\tau, s) \ge 0$ be some function. Assume that for $y \in \mathbb{R}^3$, $\tau \in [0,t]$, $s \in [0,\tau]$, and some $g(y,\tau,s) = (g_1(y,\tau,s), g_2(y,\tau,s))$ with $\nabla \cdot g_1 = 0 = \nabla \cdot g_2$ that

$$|g_1(y,\tau,s)| + |g_2(y,\tau,s)| \;\le\; C\xi(\tau,s) \sum_{\alpha=1}^N \frac{1}{1 + |y - q_\alpha(s)|^3}, \tag{5.14}$$

$$|\nabla g_1(y,\tau,s)| + |\nabla g_2(y,\tau,s)| \;\le\; C\xi(\tau,s) \sum_{\alpha=1}^N \frac{1}{1 + |y - q_\alpha(s)|^4}. \tag{5.15}$$

Then for each $\alpha = 1, \ldots, N$, $t \in [0, T\varepsilon^{-3/2}]$, and $|x| \le R_\varphi$

$$\Big| \int_0^t d\tau \int_0^\tau ds\, [U(t-s) g(\cdot, \tau, s)](x + q_\alpha(t)) \Big| \le C\Big(\sup_{(\tau,s) \in \Delta_t} \xi(\tau, s) \Big),$$

where $\Delta_t = \{(\tau, s) : \tau \in [0,t], s \in [0,\tau]\}$.

In (a)–(c), all constants C on the right-hand sides are independent of α, t, and x.

Proof: (a) Define W as in Lemma 5.1. We derive the estimates with W_1. Fix $1 \le \alpha \le N$, $t \in [0, T\varepsilon^{-3/2}]$, $s \in [0,t]$, and $|x| \le R_\varphi$. According to Lemma 5.1, (5.10), and (5.11)

$$|W_1(t,s,x + q_\alpha(t))| \le C \frac{\xi(s)}{(t-s)^2} \sum_{\beta=1}^N I_{\alpha\beta}^{(2)}(t,s,x),$$

with

$$I_{\alpha\beta}^{(n)}(t,s,x) = \int_{|y - x - q_\alpha(t)| = (t-s)} d^2y \left[\frac{(t-s)}{1 + |y - q_\beta(s)|^{n+1}} + \frac{1}{1 + |y - q_\beta(s)|^n} \right]. \tag{5.16}$$

In the sum in (5.16), with general $n \ge 2$, we first consider the term $I_{\alpha\alpha}^{(n)}(t,s,x)$, i.e., the one with $\beta = \alpha$. In this case according to (2.8), $|y - q_\beta(s)| \ge |y - x - q_\alpha(t)| - |x| - |q_\alpha(t) - q_\alpha(s)| \ge$

$(t-s) - R_\varphi - C\sqrt{\varepsilon}(t-s) \geq (t-s)/2 - R_\varphi$ for ε small. Therefore $|y - q_\beta(s)| \geq (t-s)/4$ for $s \leq t - 4R_\varphi$. We hence obtain for $\beta = \alpha$ and $s \leq t - 4R_\varphi$

$$I^{(n)}_{\alpha\alpha}(t,s,x) \leq C\,\frac{(t-s)^2}{1+(t-s)^n}. \tag{5.17}$$

On the other hand, for $s \in [t - 4R_\varphi, t]$

$$\begin{aligned} I^{(n)}_{\alpha\alpha}(t,s,x) &\leq C(t-s)^2[(t-s)+1] \leq C(t-s)^2[4R_\varphi+1] \leq C(t-s)^2\frac{1}{1+(4R_\varphi)^n} \\ &\leq C\,\frac{(t-s)^2}{1+(t-s)^n}. \end{aligned}$$

Hence (5.17) shows that the latter estimate holds for any $s \in [0,t]$. Since

$$\int_0^t \frac{ds}{1+(t-s)^2} \leq C, \quad \int_0^t d\tau \int_0^\tau \frac{ds}{1+(t-s)^3} \leq C,$$

the term with $\beta = \alpha$ will satisfy the claimed estimates not only in (a), but also in (c).

Next we turn to deriving a bound for $I^{(2)}_{\alpha\beta}(t,s,x)$ with $\beta \neq \alpha$. First note that for some portion of the interval $[0,t]$ the preceding argument applies again. For this, define $t_0 = 4(R_\varphi + C^*\varepsilon^{-1})$. Then for $s \leq t - t_0$ we find by (2.8) for ε small that on the y-sphere

$$\begin{aligned} |y - q_\beta(s)| &\geq |y - x - q_\alpha(t)| - |x| - |q_\alpha(t) - q_\beta(s)| \\ &\geq (t-s) - R_\varphi - |q_\alpha(t) - q_\beta(t)| - |q_\beta(t) - q_\beta(s)| \\ &\geq (t-s) - R_\varphi - C^*\varepsilon^{-1} - C\sqrt{\varepsilon}(t-s) \geq (t-s)/2 - R_\varphi - C^*\varepsilon^{-1} \geq (t-s)/4. \end{aligned}$$

Therefore as in (5.17) for general $n \geq 2$

$$I^{(n)}_{\alpha\beta}(t,s,x) \leq C\,\frac{(t-s)^2}{1+(t-s)^n}, \quad s \in [0, t-t_0], \tag{5.18}$$

and it remains to estimate $I^{(2)}_{\alpha\beta}(t,s,x)$ for $\beta \neq \alpha$ and $s \in [t - t_0, t]$. To do so, we note that an explicit computation shows for $z_1, z_2 \in \mathbb{R}^3$ and $\gamma \geq 0$

$$\begin{aligned} \int_{|y-z_1|=\gamma} d^2y\,\frac{1}{(1+|y-z_2|^2)} &= \frac{\pi\gamma}{|z_1 - z_2|}\,\log\left(\frac{1+(\gamma+|z_1-z_2|)^2}{1+(\gamma-|z_1-z_2|)^2}\right) \\ &= \frac{\pi\gamma}{|z_1 - z_2|}\,\log\left(1+\frac{4\gamma|z_1-z_2|}{1+(\gamma-|z_1-z_2|)^2}\right) \\ &\leq \frac{4\pi\gamma^2}{1+(\gamma-|z_1-z_2|)^2}, \end{aligned} \tag{5.19}$$

as $\log(1+A) \leq A$ for $A \geq 0$. Similarly, for $n \geq 2$

$$\begin{aligned} &\int_{|y-z_1|=\gamma} d^2y\,\frac{1}{(1+|y-z_2|^{n+1})} \\ &= 2\pi\gamma^2 \int_{-1}^1 \frac{dr}{1+\left(|z_1-z_2|^2 + 2\gamma|z_1-z_2|r + \gamma^2\right)^{(n+1)/2}} \end{aligned}$$

$$\leq \; C\gamma^2 \int_{-1}^{1} \frac{dr}{\left(1+|z_1-z_2|^2+2\gamma|z_1-z_2|r+\gamma^2\right)^{(n+1)/2}}$$

$$= \; C_n \frac{\gamma}{|z_1-z_2|}\left(\frac{1}{\left[1+(|z_1-z_2|-\gamma)^2\right]^{(n-1)/2}} - \frac{1}{\left[1+(|z_1-z_2|+\gamma)^2\right]^{(n-1)/2}}\right). \tag{5.20}$$

So in particular

$$\int_{|y-z_1|=\gamma} d^2y \, \frac{1}{(1+|y-z_2|^{n+1})} \leq C \frac{\gamma}{|z_1-z_2|}, \quad n \geq 2. \tag{5.21}$$

Below we will also need some more refined estimates, and for this purpose we note that according to (5.20) also

$$\int_{|y-z_1|=\gamma} d^2y \, \frac{1}{(1+|y-z_2|^3)} \leq C \frac{\gamma^2}{1+(|z_1-z_2|+\gamma)^2} \leq C \frac{\gamma^2}{|z_1-z_2|^2}. \tag{5.22}$$

Analogously we obtain

$$\int_{|y-z_1|=\gamma} d^2y \, \frac{1}{(1+|y-z_2|^4)} \leq C \, \frac{1}{1+(|z_1-z_2|-\gamma)^2} \, \min\Big\{1, \, \frac{\gamma^2}{|z_1-z_2|^2}\Big\}. \tag{5.23}$$

As to bound $I^{(2)}_{\alpha\beta}(t,s,x)$ for $\beta \neq \alpha$ and $s \in [t-t_0, t]$ we then use (5.21) and (5.19) with $z_1 = x+q_\alpha(t)$, $z_2 = q_\beta(s)$, and $\gamma = t-s$ to obtain for $s \in [t-t_0, t]$

$$I^{(2)}_{\alpha\beta}(t,s,x) \leq C\left(\frac{(t-s)^2}{|x+q_\alpha(t)-q_\beta(s)|} + \frac{(t-s)^2}{1+[(t-s)-|x+q_\alpha(t)-q_\beta(s)|]^2}\right). \tag{5.24}$$

Therefore by (5.18) and (5.24)

$$\begin{aligned}
&\int_0^t ds \, \frac{\xi(s)}{(t-s)^2} \, I^{(2)}_{\alpha\beta}(t,s,x) \\
&\leq \; \Big(\sup_{s\in[0,t]} \xi(s)\Big)\left(\int_0^{t-t_0} \frac{ds}{(t-s)^2} \, I^{(2)}_{\alpha\beta}(t,s,x) + \int_{t-t_0}^{t} \frac{ds}{(t-s)^2} \, I^{(2)}_{\alpha\beta}(t,s,x)\right) \\
&\leq \; C\Big(\sup_{s\in[0,t]} \xi(s)\Big)\left(\int_0^{t-t_0} \frac{ds}{1+(t-s)^2} + \int_{t-t_0}^{t} \frac{ds}{|x+q_\alpha(t)-q_\beta(s)|}\right. \\
&\qquad\qquad \left. + \int_{t-t_0}^{t} \frac{ds}{1+[(t-s)-|x+q_\alpha(t)-q_\beta(s)|]^2}\right).
\end{aligned} \tag{5.25}$$

The first of the three integrals is bounded by a constant. Concerning the second, we have

$$|x+q_\alpha(t)-q_\beta(s)| \geq |q_\alpha(t)-q_\beta(t)| - |x| - |q_\beta(t)-q_\beta(s)| \geq C_*\varepsilon^{-1} - R_\varphi - C\sqrt{\varepsilon}\,(t-s)$$

by (2.6) and (2.8). In the domain of integration $[t-t_0, t]$ it holds that $t-s \leq t_0 \leq C\varepsilon^{-1}$, whence

$$|x+q_\alpha(t)-q_\beta(s)| \geq C_*\varepsilon^{-1} - R_\varphi - C\varepsilon^{-1/2} \geq (C_*/2)\varepsilon^{-1}, \quad s \in [t-t_0, t], \quad \beta \neq \alpha, \quad |x| \leq R_\varphi, \tag{5.26}$$

for ε small. Therefore the second integral can be bound by $C\varepsilon \int_{t-t_0}^{t} ds \leq C\varepsilon t_0 \leq C$. To estimate the last integral $=: J$ on the right-hand side of (5.25), we substitute $\theta = t-s$ to obtain

$$J = \int_0^{t_0} \frac{d\theta}{1+[\theta - r(\theta)]^2} \tag{5.27}$$

with $r(\theta) = |x + q_\alpha(t) - q_\beta(t-\theta)|$. Observe that $|\dot r(\theta)| \le |\dot q_\beta(t-\theta)| \le C\sqrt{\varepsilon}$ by (2.8). Thus $\theta \mapsto \chi(\theta) = \theta - r(\theta)$ is strictly increasing, and we can substitute $\theta = \theta(\chi)$ to get

$$J = \int_{\chi(0)}^{\chi(t_0)} \frac{d\chi}{1-\dot r(\theta)} \left(\frac{1}{1+\chi^2}\right) \le C \int_{\mathbb{R}} \frac{d\chi}{1+\chi^2} \le C.$$

Summarizing these estimates we obtain the bound claimed in part (a) of the lemma.

(b) Defining $I^{(n)}_{\alpha\beta}$ as in (5.16), we need to show

$$\int_0^t \frac{ds}{(t-s)^2}\, I^{(3)}_{\alpha\beta}(t,s,x) \le C\varepsilon, \quad \beta \ne \alpha. \tag{5.28}$$

By (5.18),

$$\int_0^{t-t_0} \frac{ds}{(t-s)^2}\, I^{(3)}_{\alpha\beta}(t,s,x) \le C \int_0^{t-t_0} \frac{ds}{(t-s)^2} \frac{(t-s)}{1+(t-s)^2}.$$

In the domain of integration, $(t-s) \ge t_0 \le C\varepsilon^{-1}$, and hence

$$\int_0^{t-t_0} \frac{ds}{(t-s)^2}\, I^{(3)}_{\alpha\beta}(t,s,x) \le C\varepsilon \int_0^t \frac{ds}{1+(t-s)^2} \le C\varepsilon. \tag{5.29}$$

Thus it remains to estimate the part of the integral in (5.28) for $s \in [t-t_0, t]$. Firstly, by (5.22),

$$\begin{aligned} &\int_{t-t_0}^t \frac{ds}{(t-s)^2} \int_{|y-x-q_\alpha(t)|=(t-s)} d^2y \, \frac{1}{(1+|y-q_\beta(s)|^3)} \\ &\quad \le C \int_{t-t_0}^t \frac{ds}{|x+q_\alpha(t)-q_\beta(s)|^2} \le C\varepsilon^2 \int_{t-t_0}^t ds = C\varepsilon^2 t_0 \le C\varepsilon. \end{aligned} \tag{5.30}$$

Here we have used $|x+q_\alpha(t)-q_\beta(s)| \ge (C_*/2)\varepsilon^{-1}$ for ε small, cf. (5.26). Reference to this is possible, since we again have that $\beta \ne \alpha$. Analogously we infer from (5.23) that

$$\begin{aligned} &\int_{t-t_0}^t \frac{ds}{(t-s)^2} \int_{|y-x-q_\alpha(t)|=(t-s)} d^2y \, \frac{(t-s)}{(1+|y-q_\beta(s)|^4)} \\ \le\ & \int_{t-t_0}^t ds \, \frac{(t-s)}{|x+q_\alpha(t)-q_\beta(s)|^2} \left(\frac{1}{1+[(t-s)-|x+q_\alpha(t)-q_\beta(s)|]^2}\right) \\ \le\ & C\varepsilon^2 t_0 \int_{t-t_0}^t \frac{ds}{1+[(t-s)-|x+q_\alpha(t)-q_\beta(s)|]^2} = C\varepsilon J \le C\varepsilon, \end{aligned}$$

with the bounded J from (5.27). This together with (5.30) and (5.29) shows that (5.28) is satisfied.

(c) Due to the remarks in (a), (5.14), and (5.15) we only have to prove

$$\int_0^t d\tau \int_0^\tau \frac{ds}{(t-s)^2}\, I^{(3)}_{\alpha\beta}(t,s,x) \le C, \quad \beta \ne \alpha, \quad t \in [0, T\varepsilon^{-3/2}], \quad |x| \le R_\varphi. \tag{5.31}$$

We decompose the domain of integration $\Delta_t = \{(\tau,s) : \tau \in [0,t], s \in [0,\tau]\}$ in $\Delta_{t,1} = \Delta_t \cap \{(\tau,s) : s \in [0, t-t_0]\}$ and $\Delta_{t,2} = \{(\tau,s) : \tau \in [t-t_0, t], s \in [t-t_0, \tau]\}$. On $\Delta_{t,1}$ we can utilize (5.18) to get

$$\int\int_{\Delta_{t,1}} d\tau ds \, \frac{1}{(t-s)^2}\, I^{(3)}_{\alpha\beta}(t,s,x) \le C \int_0^t d\tau \int_0^\tau ds \, \frac{1}{1+(t-s)^3} \le C. \tag{5.32}$$

Since again $t-s \le t_0 \le C\varepsilon^{-1}$ for $(\tau,s)\in\Delta_{t,2}$, by (5.26) and (5.21)

$$\begin{aligned}
&\int\!\!\int_{\Delta_{t,2}} d\tau ds\,\frac{1}{(t-s)^2}\int_{|y-x-q_\alpha(t)|=(t-s)}\frac{d^2y}{1+|y-q_\beta(s)|^3}\\
&\le\; C\int\!\!\int_{\Delta_{t,2}} d\tau ds\,\frac{1}{(t-s)}\,\frac{1}{|x+q_\alpha(t)-q_\beta(s)|}\\
&\le\; C\varepsilon\int_{t-t_0}^{t} d\tau\int_{t-t_0}^{\tau}\frac{ds}{t-s} = C\varepsilon\int_{t-}^{t} ds = C\varepsilon t_0 \le C. \qquad (5.33)
\end{aligned}$$

In addition, by (5.23)

$$\begin{aligned}
&\int\!\!\int_{\Delta_{t,2}} d\tau ds\,\frac{1}{(t-s)}\int_{|y-x-q_\alpha(t)|=(t-s)}\frac{d^2y}{1+|y-q_\beta(s)|^4}\\
&\le\; \int\!\!\int_{\Delta_{t,2}} d\tau ds\,\frac{1}{(t-s)}\,\frac{1}{1+[(t-s)-|x+q_\alpha(t)-q_\beta(s)|]^2}\\
&=\; \int_{t-t_0}^{t}\frac{ds}{1+[(t-s)-|x+q_\alpha(t)-q_\beta(s)|]^2}\le C, \qquad (5.34)
\end{aligned}$$

since the last integral is just J from (5.27) and hence bounded. By (5.32), (5.33), and (5.34) we thus have proved (5.31). □

Lemma 5.3 *Define $\phi_v(x)$ through $\hat\phi_v(k) = e\hat\varphi(k)/[k^2-(k\cdot v)^2]$. Then for $x\in\mathbb{R}^3$ and $|v|\le\bar v<1$, with $\nabla=\nabla_x$,*

$$\begin{aligned}
|\nabla\phi_v(x)|+|\nabla_v\nabla\phi_v(x)|+|\nabla_v^2\nabla\phi_v(x)|+|\nabla_v^3\nabla\phi_v(x)| &\le C|e|(1+|x|)^{-2},\\
|\nabla^2\phi_v(x)|+|\nabla_v\nabla^2\phi_v(x)|+|\nabla_v^2\nabla^2\phi_v(x)|+|\nabla_v^3\nabla^2\phi_v(x)| &\le C|e|(1+|x|)^{-3},\\
|\nabla^3\phi_v(x)|+|\nabla_v\nabla^3\phi_v(x)|+|\nabla_v^2\nabla^3\phi_v(x)|+|\nabla_v^3\nabla^3\phi_v(x)| &\le C|e|(1+|x|)^{-4},\\
|\nabla^4\phi_v(x)|+|\nabla_v\nabla^4\phi_v(x)|+|\nabla_v^2\nabla^4\phi_v(x)|+|\nabla_v^3\nabla^4\phi_v(x)| &\le C|e|(1+|x|)^{-5}.
\end{aligned}$$

Proof: Tedious calculations; see also the appendices of [5, 6]. □

Now we can estimate $\int_0^t ds\,[U(t-s)f(\cdot,s)](x+q_\alpha(t))$, cf. (5.9), for $t\in[0,T\varepsilon^{-3/2}]$ and $|x|\le R_\varphi$, using Lemma 5.1 and Lemma 5.2(a), with $f=(f_1,f_2)$ defined by (5.8). Since $\nabla\cdot B_v=0$, and $\nabla\cdot E_v=e\varphi$ is independent of v, we have $\nabla\cdot f_1 = 0 = \nabla\cdot f_2$. Concerning (5.10) and (5.11), note $|\nabla_v E_v(x)|+|\nabla_v B_v(x)|\le C(|\nabla\phi_v(x)|+|\nabla_v\nabla\phi_v(x)|)\le C|e|(1+|x|)^{-2}$ and $|\nabla_v\nabla E_v(x)|+|\nabla_v\nabla B_v(x)|\le C(|\nabla^2\phi_v(x)|+|\nabla_v\nabla^2\phi_v(x)|)\le C|e|(1+|x|)^{-3}$ by Lemma 5.3. Thus (5.10) and (5.11) are satisfied with $\xi(s)=\Big(\max_{1\le\beta\le N}|\dot v_\beta(s)|\Big)\Big(\max_{1\le\beta\le N}|e_\beta|\Big)$. As $Z(x,0)=0$, hence (5.9) in conjunction with Lemma 5.2(a) yields for $\alpha=1,\dots,N$

$$|Z(x+q_\alpha(t),t)|\le C\Big(\sup_{s\in[0,t]}\max_{1\le\beta\le N}|\dot v_\beta(s)|\Big)\Big(\max_{1\le\beta\le N}|e_\beta|\Big),\quad t\in[0,T\varepsilon^{-3/2}],\quad |x|\le R_\varphi. \qquad (5.35)$$

We will utilize this further in (5.5), and to this end we also need to bound $R_\alpha(t)$ from (5.6). For fixed $\beta\ne\alpha$ one calculates for the interaction terms

$$\begin{aligned}
\Psi_{\alpha\beta}(t) &= \int d^3x\,\rho_\alpha(x-q_\alpha(t))\nabla\phi_{v_\beta(t)}(x-q_\beta(t))\\
&= (-i)\,e_\alpha e_\beta\int d^3k\,k\frac{|\hat\varphi(k)|^2}{k^2-(k\cdot v_\beta(t))^2}\,e^{ik\cdot[q_\beta(t)-q_\alpha(t)]}\\
&= \frac{e_\alpha e_\beta}{4\pi}\int\!\!\int d^3x d^3y\,\varphi(x-q_\alpha(t))\varphi(y-q_\beta(t))\nabla\zeta_{v_\beta(t)}(x-y), \qquad (5.36)
\end{aligned}$$

with

$$\zeta_v(x) = \frac{1}{\left[(1-v^2)x^2 + (x\cdot v)^2\right]^{1/2}}, \quad \hat{\zeta}_v(k) = \sqrt{\frac{2}{\pi}}\,\frac{1}{k^2-(k\cdot v)^2}, \quad |v|<1. \tag{5.37}$$

Then $\sup_{t\in[0,T\varepsilon^{-3/2}]}|\nabla\zeta_{v_\beta(t)}(x)| \le C(1+|x|)^{-2}$ due to (2.8). By (C), in (5.36) we only need to integrate over (x,y) that have $|x-q_\alpha(t)|\le R_\varphi$ and $|y-q_\beta(t)|\le R_\varphi$. Then by (2.6), $|x-y|\ge |q_\alpha(t)-q_\beta(t)|-2R_\varphi \ge C_*\varepsilon^{-1}-2R_\varphi \ge (C_*/2)\varepsilon^{-1}$ for ε small. Therefore (5.36) shows

$$|\Psi_{\alpha\beta}(t)| \le C\varepsilon^2, \quad t\in[0,T\varepsilon^{-3/2}], \quad \alpha\ne\beta. \tag{5.38}$$

By definition of $B_v(x)$ and $E_v(x)$ we have

$$R_\alpha(t) = m_{0\alpha}(v_\alpha(t))^{-1}\sum_{\beta\ne\alpha}\Big(-\Psi_{\alpha\beta}(t)+[v_\beta(t)\cdot\Psi_{\alpha\beta}(t)]v_\beta(t)+v_\alpha(t)\wedge[-v_\beta(t)\wedge\Psi_{\alpha\beta}(t)]\Big) \tag{5.39}$$

cf. (5.6), and therefore (5.38) together with (2.8) implies

$$|R_\alpha(t)|\le C\varepsilon^2, \quad t\in[0,T\varepsilon^{-3/2}]. \tag{5.40}$$

Hence (5.5), (5.35), and (5.40) finally yield

$$|\dot v_\alpha(t)| \le C\Big(\sup_{s\in[0,t]}\max_{1\le\beta\le N}|\dot v_\beta(s)|\Big)\Big(\max_{1\le\beta\le N}|e_\beta|\Big)+C\varepsilon^2,$$

for every $\alpha=1,\ldots,N$ and $t\in[0,T\varepsilon^{-3/2}]$. Choosing $\max_{1\le\beta\le N}|e_\beta|\le\bar e$ with $\bar e>0$ sufficiently small, we find that for $\alpha=1,\ldots,N$

$$\sup_{t\in[0,\,T\varepsilon^{-3/2}]}|\dot v_\alpha(t)|\le C\varepsilon^2. \tag{5.41}$$

For later reference we also note that then according to (5.35)

$$|Z(x+q_\alpha(t),t)|\le C\varepsilon^2, \quad \alpha=1,\ldots,N, \quad t\in[0,T\varepsilon^{-3/2}], \quad |x|\le R_\varphi. \tag{5.42}$$

5.3 Bounding $|\ddot v_\alpha(t)|$

By (2.8) we have in particular that

$$|v_\alpha(t)-v_\beta(t)|\le C\sqrt{\varepsilon}, \quad t\in[0,T\varepsilon^{-3/2}]. \tag{5.43}$$

In order to estimate the derivative of Equ. (5.5), first note that using the explicit form of $m_{0\alpha}(v_\alpha)^{-1}$ we obtain from (5.41) that

$$\left|\frac{d}{dt}\,m_{0\alpha}(v_\alpha(t))^{-1}\right|\le C|\dot v_\alpha(t)|\le C\varepsilon^2. \tag{5.44}$$

Hence by (5.5), (5.42) and (5.40)

$$|\ddot v_\alpha(t)|\le C\Big(\varepsilon^4+|M_\alpha(t)|+|\dot R_\alpha(t)|\Big), \tag{5.45}$$

with R_α defined in (5.6), and

$$M_\alpha(t)=\int d^3x\,\rho_\alpha(x)\Big[(L_\alpha(t)Z_1)(x+q_\alpha(t),t)+v_\alpha(t)\wedge(L_\alpha(t)Z_2)(x+q_\alpha(t),t)\Big], \tag{5.46}$$

where $L_\alpha(t)\phi = (v_\alpha(t)\cdot\nabla)\phi + \partial_t\phi$ for a function $\phi = \phi(x,t)$. We first estimate $M_\alpha(t)$. Let $\Sigma_\alpha(x,t) = (L_\alpha(t)Z)(x,t)$. Since generally $\frac{d}{dt}[L_\alpha(t)\phi] = L_\alpha(t)\dot{\phi} + (\dot{v}_\alpha\cdot\nabla)\phi$ and, see (5.7), $\dot{Z} = \mathcal{A}Z - f$ with f from (5.8), we obtain

$$\dot{\Sigma}_\alpha = \mathcal{A}\Sigma_\alpha + (\dot{v}_\alpha\cdot\nabla)Z - L_\alpha(t)f.$$

According to (2.3) it may be shown that $\Sigma_\alpha(x,0) = 0$. We hence get

$$\Sigma_\alpha(x+q_\alpha(t),t) = \int_0^t d\tau\,\Big[U(t-\tau)\Big((\dot{v}_\alpha(\tau)\cdot\nabla)Z(\cdot,\tau) - L_\alpha(\tau)f(\cdot,\tau)\Big)\Big](x+q_\alpha(t)).$$

As a consequence of $\frac{d}{dt}(\nabla Z) = \nabla(\mathcal{A}Z - f) = \mathcal{A}(\nabla Z) - \nabla f$ and $Z(x,0) = 0$, we obtain from the group property of $U(\cdot)$ that

$$\begin{aligned}
\Sigma_{\alpha,1}(x+q_\alpha(t),t) &= \int_0^t d\tau\,\Big[U(t-\tau)\Big((\dot{v}_\alpha(\tau)\cdot\nabla)Z(\cdot,\tau)\Big)\Big](x+q_\alpha(t))\\
&= -\int_0^t d\tau\int_0^\tau ds\,\Big[U(t-s)\Big((\dot{v}_\alpha(\tau)\cdot\nabla)f(\cdot,s)\Big)\Big](x+q_\alpha(t)).
\end{aligned}$$

With $g(y,\tau,s) = \dot{v}_\alpha(\tau)\cdot\nabla f(y,s)$ it follows from the definitions of f, $E_v(x)$, and $B_v(x)$ that $\nabla\cdot g = 0$. Moreover, by (5.41) and Lemma 5.3 we find that (5.14) and (5.15) are satisfied with $\xi(\tau,s) = \varepsilon^4$. Therefore Lemma 5.2(c) applies to yield for $\alpha = 1,\ldots,N$

$$|\Sigma_{\alpha,1}(x+q_\alpha(t),t)| \le C\varepsilon^4, \quad t\in[0,T\varepsilon^{-3/2}], \quad |x|\le R_\varphi. \tag{5.47}$$

To estimate

$$\Sigma_{\alpha,2}(x+q_\alpha(t),t) = -\int_0^t d\tau\,\Big[U(t-\tau)\Big(L_\alpha(\tau)f(\cdot,\tau)\Big)\Big](x+q_\alpha(t)),$$

observe that

$$\begin{aligned}
[L_\alpha(\tau)f(\cdot,\tau)](x) &= v_\alpha(\tau)\cdot\nabla f(x,\tau) + \partial_t f(x,\tau)\\
&= \sum_{\beta=1}^N\Big\{(\ddot{v}_\beta\cdot\nabla_v)\Phi_{v_\beta}(x-q_\beta) + (\dot{v}_\beta\cdot\nabla_v)^2\Phi_{v_\beta}(x-q_\beta)\Big\}\\
&\quad + \sum_{\substack{\beta=1\\ \beta\ne\alpha}}^N \nabla^2_{xv}\Phi_{v_\beta}(x-q_\beta)(v_\alpha - v_\beta, \dot{v}_\beta)\\
&=: f^\natural(\tau,y) + f^\flat(\tau,y),
\end{aligned}$$

with all time arguments taken at time τ, and $\Phi_v = (E_v, B_v)$. Since $\nabla\cdot B_v = 0$ and $\nabla\cdot E_v = e\varphi$ is independent of v, we have that $\nabla\cdot f^\natural = 0 = \nabla\cdot f^\flat$. In addition, $f^\natural$ satisfies (5.10) and (5.11) with

$$\xi^\natural(\tau) = \Big(\max_{1\le\beta\le N}|\ddot{v}_\beta(\tau)|\Big)\Big(\max_{1\le\beta\le N}|e_\beta|\Big) + \varepsilon^4.$$

Because $f^\flat$ has an additional x-derivative, moreover (5.12) and (5.13) hold for $f^\flat$, with

$$\xi^\flat(\tau) = \Big(\max_{1\le\beta\le N}|v_\alpha(\tau) - v_\beta(\tau)|\Big)\varepsilon^2,$$

as again follows from Lemma 5.3 and (5.41). Thus Lemma 5.2(a) and (b) imply that for all $\alpha = 1, \ldots, N$, $t \in [0, T\varepsilon^{-3/2}]$, and $|x| \le R_\varphi$

$$\begin{aligned}
&|\Sigma_{\alpha,2}(x + q_\alpha(t), t)| \\
&\le \Big|\int_0^t d\tau\, [U(t-\tau) f^\natural(\cdot,\tau)](x + q_\alpha(t))\Big| + \Big|\int_0^t d\tau\, [U(t-\tau) f^\flat(\cdot,\tau)](x + q_\alpha(t))\Big| \\
&\le C\Big(\sup_{\tau\in[0,t]} \xi^\natural(\tau) + \sup_{\tau\in[0,t]} \xi^\flat(\tau)\varepsilon\Big) \\
&\le C\Big[\varepsilon^4 + \Big(\sup_{\tau\in[0,t]} \max_{1\le\beta\le N} |\ddot v_\beta(\tau)|\Big)\Big(\max_{1\le\beta\le N} |e_\beta|\Big) + \Big(\sup_{\tau\in[0,t]} \max_{1\le\beta\le N} |v_\alpha(\tau) - v_\beta(\tau)|\Big)\varepsilon^3\Big].
\end{aligned}$$

Hence by (5.47) and (5.43) for $\alpha = 1, \ldots, N$, $t \in [0, T\varepsilon^{-3/2}]$, and $|x| \le R_\varphi$,

$$|\Sigma_\alpha(x + q_\alpha(t), t)| \le C\Big[\varepsilon^{7/2} + \Big(\sup_{\tau\in[0,t]} \max_{1\le\beta\le N} |\ddot v_\beta(\tau)|\Big)\Big(\max_{1\le\beta\le N} |e_\beta|\Big)\Big].$$

According to the definition of $M_\alpha(t)$ in (5.46) we therefore have

$$\begin{aligned}
|M_\alpha(t)| &= \Big|\int_{|x|\le R_\varphi} d^3x\, \rho_\alpha(x)\Big[\Sigma_{\alpha,1}(x + q_\alpha(t), t) + v_\alpha(t) \wedge \Sigma_{\alpha,2}(x + q_\alpha(t), t)\Big]\Big| \\
&\le C\Big[\varepsilon^{7/2} + \Big(\sup_{\tau\in[0,t]} \max_{1\le\beta\le N} |\ddot v_\beta(\tau)|\Big)\Big(\max_{1\le\beta\le N} |e_\beta|\Big)\Big]. \qquad (5.48)
\end{aligned}$$

To further estimate the right-hand side of (5.45), we have to bound $\dot R_\alpha(t)$, with $R_\alpha(t)$ from (5.6). Calculating $\dot R_\alpha(t)$ explicitly we obtain

$$\begin{aligned}
&\dot R_\alpha(t) \\
&= \Big(\frac{d}{dt} m_{0\alpha}(v_\alpha)^{-1}\Big) m_{0\alpha}(v_\alpha) R_\alpha(t) \\
&\quad + m_{0\alpha}(v_\alpha)^{-1}\Big(\sum_{\substack{\beta=1\\ \beta\ne\alpha}} \int d^3x\, \rho_\alpha(x - q_\alpha)\Big[(\dot v_\beta \cdot \nabla_v) E_{v_\beta}(x - q_\beta) + v_\alpha \wedge (\dot v_\beta \cdot \nabla_v) B_{v_\beta}(x - q_\beta)\Big]\Big) \\
&\quad + m_{0\alpha}(v_\alpha)^{-1}\Big(\sum_{\substack{\beta=1\\ \beta\ne\alpha}} \int d^3x\, \rho_\alpha(x - q_\alpha)\Big[((v_\alpha - v_\beta) \cdot \nabla) E_{v_\beta}(x - q_\beta) \\
&\qquad\qquad + v_\alpha \wedge ((v_\alpha - v_\beta) \cdot \nabla) B_{v_\beta}(x - q_\beta)\Big]\Big) \\
&\quad + m_{0\alpha}(v_\alpha)^{-1}\Big(\sum_{\substack{\beta=1\\ \beta\ne\alpha}} \int d^3x\, \rho_\alpha(x - q_\alpha)\, \dot v_\alpha \wedge B_{v_\beta}(x - q_\beta)\Big) \\
&=: \dot R_{\alpha,1}(t) + \dot R_{\alpha,2}(t) + \dot R_{\alpha,3}(t) + \dot R_{\alpha,4}(t)
\end{aligned}$$

with all time arguments at time t. Firstly,

$$|\dot R_{\alpha,1}(t)| = \Big|\Big(\frac{d}{dt} m_{0\alpha}(v_\alpha)^{-1}\Big) m_{0\alpha}(v_\alpha) R_\alpha(t)\Big| \le C\varepsilon^4 \qquad (5.49)$$

for $\alpha = 1, \ldots, N$ and $t \in [0, T\varepsilon^{-3/2}]$ by (5.44) and (5.40). Since $B_v(x) = -v \wedge \nabla\phi_v(x)$, by (5.41), (2.8), and (5.38) also

$$|\dot R_{\alpha,4}(t)| \le C\varepsilon^{9/2}. \qquad (5.50)$$

What concerns $\dot{R}_{\alpha,2}(t)$, we may repeat the calculation in (5.36) to obtain

$$\begin{aligned}\nabla_v \Psi_{\alpha\beta}(t) &:= \int d^3x\, \rho(x - q_\alpha(t)) \nabla^2_{xv} \phi_{v_\beta(t)}(x - q_\beta(t)) \\ &= \frac{1}{4\pi} \int\int d^3x d^3y\, \rho(x - q_\alpha(t)) \rho(y - q_\beta(t)) \nabla^2_{xv} \zeta_{v_\beta(t)}(x - y),\end{aligned}$$

with $\zeta_v(x)$ from (5.37). Since $\sup_{t\in[0,T\varepsilon^{-3/2}]} |\nabla^2_{xv}\zeta_{v_\beta(t)}(x)| \le C(1+|x|)^{-2}$, we get as before that

$$|\nabla_v \Psi_{\alpha\beta}(t)| \le C\varepsilon^2, \quad t \in [0, T\varepsilon^{-3/2}], \quad \alpha \neq \beta,$$

and hence by (5.41)

$$|\dot{R}_{\alpha,2}(t)| \le C\varepsilon^4. \tag{5.51}$$

So finally we have to bound $\dot{R}_{\alpha,3}(t)$, and this relies on a similar argument. Here we have

$$\begin{aligned}\nabla \Psi_{\alpha\beta}(t) &:= \int d^3x\, \rho(x - q_\alpha(t)) \nabla^2 \phi_{v_\beta(t)}(x - q_\beta(t)) \\ &= \frac{1}{4\pi} \int\int d^3x d^3y\, \rho(x - q_\alpha(t)) \rho(y - q_\beta(t)) \nabla^2 \zeta_{v_\beta(t)}(x - y),\end{aligned}$$

and $\sup_{t\in[0,T\varepsilon^{-3/2}]} |\nabla^2\zeta_{v_\beta(t)}(x)| \le C(1+|x|)^{-3}$. This in turn yields

$$|\nabla \Psi_{\alpha\beta}(t)| \le C\varepsilon^3, \quad t \in [0, T\varepsilon^{-3/2}], \quad \alpha \neq \beta.$$

Using the explicit form of $E_v(x)$ and $B_v(x)$, as in (5.39), we then get for $t \in [0, T\varepsilon^{-3/2}]$

$$|\dot{R}_{\alpha,3}(t)| \le C\varepsilon^3\, |v_\alpha(t) - v_\beta(t)| \le C\varepsilon^{7/2}, \tag{5.52}$$

by (5.43). Summarizing (5.49), (5.50), (5.51), and (5.52) it follows that

$$|\dot{R}_\alpha(t)| \le C\varepsilon^{7/2}, \quad \alpha = 1, \ldots, N, \quad t \in [0, T\varepsilon^{-3/2}]. \tag{5.53}$$

Consequently, by (5.45), (5.48), and (5.53) for $\alpha = 1, \ldots, N$ and $t \in [0, T\varepsilon^{-3/2}]$

$$|\ddot{v}_\alpha(t)| \le C\Big(\varepsilon^4 + |M_\alpha(t)| + |\dot{R}_\alpha(t)|\Big) \le C\Big[\varepsilon^{7/2} + \Big(\sup_{\tau\in[0,t]} \max_{1\le\beta\le N} |\ddot{v}_\beta(\tau)|\Big)\Big(\max_{1\le\beta\le N} |e_\beta|\Big)\Big].$$

Choosing $\max_{1\le\beta\le N} |e_\beta| \le \bar{e}$ with $\bar{e}$ sufficiently small we hence obtain

$$\sup_{t\in[0,T\varepsilon^{-3/2}]} |\ddot{v}_\alpha(t)| \le C\varepsilon^{7/2}, \quad \alpha = 1, \ldots, N.$$

This completes the proof of Lemma 2.1. □

6 Appendix B: Proof of Lemma 3.2

Here we give the proof of Lemma 3.2. We verify e.g. (b). To compare the left-hand side to the right-hand side of the assertion, we will insert some additional terms and estimate the corresponding differences $D_j(t)$, $j=1,2,3$, for $t\in[t_0,T\varepsilon^{-3/2}]$, where $t_0=4(R_\varphi+C^*\varepsilon^{-1})$. First we introduce

$$\begin{aligned} D_1(t) &= i\int_0^t d\tau \int d^3k\,|\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}}\Big\{e^{-ik\cdot[q_\beta(t)-q_\beta(t-\tau)]}-e^{-ik\cdot[\tau v_\beta-\frac12\tau^2\dot v_\beta]}\Big\}\frac{\sin|k|\tau}{|k|}k \\ &= -\nabla_\xi\int_0^t d\tau \int d^3k\,|\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}}\Big\{e^{-ik\cdot[q_\beta(t)-q_\beta(t-\tau)]}-e^{-ik\cdot[\tau v_\beta-\frac12\tau^2\dot v_\beta]}\Big\}\frac{\sin|k|\tau}{|k|} \\ &= -\nabla_\xi\int\int d^3xd^3y\,\varphi(x)\varphi(y) \\ &\qquad\times\int_0^t d\tau\,\Big\{\psi_\tau\Big([\xi_{\alpha\beta}+x-q_\beta(t-\tau)]-[y-q_\beta(t)]\Big)-\psi_\tau\Big([x-\frac12\tau^2\dot v_\beta]-[y-\tau v_\beta]\Big)\Big\}, \end{aligned} \tag{6.1}$$

as follows through application of the Fourier transform, with $\xi_{\alpha\beta}=q_\alpha(t)-q_\beta(t)$, and $\psi_\tau(x)=(4\pi|x|)^{-1}$ for $|x|=\tau$ whereas $\psi_\tau(x)=0$ otherwise. We claim that for $x,y\in\mathbb{R}^3$ with $|x|,|y|\le R_\varphi$ and $t\in[t_0,T\varepsilon^{-3/2}]$ there exists a unique $\tau_0=\tau_0(x,y,t,\xi_{\alpha\beta})\in[0,t_0]\subset[0,t]$ such that

$$\tau_0=\Big|[\xi_{\alpha\beta}+x-q_\beta(t-\tau_0)]-[y-q_\beta(t)]\Big|. \tag{6.2}$$

To see this, observe with $\theta(\tau)=\tau-|[\xi_{\alpha\beta}+x-q_\beta(t-\tau)]-[y-q_\beta(t)]|$ that $0\ge\theta(0)\ge-(2R_\varphi+C^*\varepsilon^{-1})$ and $\theta'(\tau)\ge 1-C_v\sqrt{\varepsilon}$ by (2.6) and (2.8). For ε so small that $1-C_v\sqrt{\varepsilon}\ge 1/2$ we hence obtain $\theta(t_0)\ge-(2R_\varphi+C^*\varepsilon^{-1})+t_0/2=2C^*\varepsilon^{-1}$. This shows θ has a unique zero $\tau_0\in[0,t_0]$. Moreover (6.2) together with (2.6) implies

$$\tau_0\ge|\xi_{\alpha\beta}|-|x-q_\beta(t-\tau_0)]-[y-q_\beta(t)]|\ge C_*\varepsilon^{-1}-2R_\varphi-C_v\sqrt{\varepsilon}\tau_0,$$

whence also $\tau_0\ge C\varepsilon^{-1}$ for ε small. Similarly, we find a unique $\tau_1=\tau_1(x,y,t,\xi_{\alpha\beta})$ satisfying

$$\tau_1=\Big|[\xi_{\alpha\beta}+x-\frac12\tau_1^2\dot v_\beta]-[y-\tau_1 v_\beta]\Big|, \tag{6.3}$$

with τ_1 having the same properties as τ_0. By definition of ψ_τ we therefore may simply write

$$D_1(t)=-\int\int d^3xd^3y\,\varphi(x)\varphi(y)\,\nabla_\xi\Big(\tau_0^{-1}-\tau_1^{-1}\Big). \tag{6.4}$$

To estimate this, we calculate from (6.2) that

$$\begin{aligned} \nabla_\xi\tau_0^{-1} &= -\tau_0^{-3}\Big\{\Big([\xi_{\alpha\beta}+x-q_\beta(t-\tau_0)]-[y-q_\beta(t)]\Big) \\ &\qquad+\Big([\xi_{\alpha\beta}+x-q_\beta(t-\tau_0)]-[y-q_\beta(t)]\Big)\cdot v_\beta(t-\tau_0)\nabla_\xi\tau_0\Big\}, \end{aligned}$$

with an analogous expression for $\nabla_\xi\tau_1^{-1}$. Therefore

$$\begin{aligned} \Big|\nabla_\xi\Big(\tau_0^{-1}-\tau_1^{-1}\Big)\Big| &\le C\Big(\tau_0^{-3}|q_\beta(t-\tau_0)-q_\beta(t-\tau_1)|\Big[1+|v_\beta(t-\tau_1)||\nabla_\xi\tau_1|\Big] \\ &\qquad+|\tau_0^{-3}-\tau_1^{-3}|\Big|[\xi_{\alpha\beta}+x-q_\beta(t-\tau_1)]-[y-q_\beta(t)]\Big|\Big[1+|v_\beta(t-\tau_1)||\nabla_\xi\tau_1|\Big] \\ &\qquad+\tau_0^{-2}|v_\beta(t-\tau_0)-v_\beta(t-\tau_1)||\nabla_\xi\tau_1| \\ &\qquad+\tau_0^{-2}|v_\beta(t-\tau_0)||\nabla_\xi(\tau_0-\tau_1)|\Big). \end{aligned} \tag{6.5}$$

¿From (6.2), (6.3), and according to the Taylor expansion

$$q_\beta(t-\tau) = q_\beta(t) - \tau v_\beta + \frac{1}{2}\tau^2 \dot v_\beta + \mathcal{O}(\varepsilon^{7/2}\tau^3),$$

cf. Lemma 2.1, it follows that

$$\begin{aligned} |\tau_0 - \tau_1| &\le \Big|\tau_0 v_\beta - \frac{1}{2}\tau_0^2 \dot v_\beta - \tau_1 v_\beta + \frac{1}{2}\tau_1^2 \dot v_\beta\Big| + \mathcal{O}(\varepsilon^{7/2}\tau_0^3) \\ &\le C\sqrt{\varepsilon}\,|\tau_0 - \tau_1| + C\varepsilon^2(\tau_0+\tau_1)|\tau_0-\tau_1| + \mathcal{O}(\varepsilon^{7/2}\tau_0^3), \end{aligned}$$

whence

$$|\tau_0 - \tau_1| = \mathcal{O}(\sqrt{\varepsilon}\,), \quad |\tau_0^{-3} - \tau_1^{-3}| = \mathcal{O}(\varepsilon^{9/2}),$$

recall $C\varepsilon^{-1} \le \tau_0, \tau_1 \le t_0 = \mathcal{O}(\varepsilon^{-1})$. Differentiating (6.2) and (6.3) w.r. to $\xi = \xi_{\alpha\beta}$ we moreover get $|\nabla_\xi \tau_0| + |\nabla_\xi \tau_1| = \mathcal{O}(1)$, and after a longer calculation which we omit also $|\nabla_\xi(\tau_0 - \tau_1)| \le C\Big(\varepsilon^{3/2} + \sqrt{\varepsilon}\,|\nabla_\xi(\tau_0-\tau_1)|\Big)$, thus

$$|\nabla_\xi(\tau_0 - \tau_1)| \le C\varepsilon^{3/2}.$$

Utilizing these estimates and Lemma 2.1 in (6.5), we consequently obtain $|\nabla_\xi(\tau_0^{-1} - \tau_1^{-1})| \le C\varepsilon^{7/2}$. Hence (6.4) yields

$$\sup_{t\in[t_0,\, T\varepsilon^{-3/2}]} |D_1(t)| \le C\varepsilon^{7/2} \tag{6.6}$$

as desired. Next, with

$$\begin{aligned} D_2(t) &= i\int_0^t d\tau \int d^3k\, |\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \\ &\quad \times \Big\{ e^{-ik\cdot[\tau v_\beta - \frac{1}{2}\tau^2 \dot v_\beta]} - \Big(1 - ik\cdot\Big[\tau v_\beta - \frac{1}{2}\tau^2\dot v_\beta\Big] - \frac{1}{2}\tau^2 (k\cdot v_\beta)^2\Big)\Big\} \frac{\sin|k|\tau}{|k|} k \end{aligned}$$

it may be shown in a a similar way that

$$\sup_{t\in[t_0,\, T\varepsilon^{-3/2}]} |D_2(t)| \le C\varepsilon^{7/2}. \tag{6.7}$$

Finally we need to compare $\int_0^t d\tau(\ldots)$ to the infinite $d\tau$-integral and thus let

$$D_3(t) = i\int_t^\infty d\tau \int d^3k\, |\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \Big(1 - ik\cdot\Big[\tau v_\beta - \frac{1}{2}\tau^2 \dot v_\beta\Big] - \frac{1}{2}\tau^2 (k\cdot v_\beta)^2\Big) \frac{\sin|k|\tau}{|k|} k.$$

With the notation

$$K_p = e^{-ik\cdot\xi_{\alpha\beta}} \int_t^\infty d\tau\, \frac{\sin|k|\tau}{|k|}\, \tau^p, \quad p = 0,\ldots,2,$$

this may be rewritten as

$$D_3(t) = \int d^3k\, |\hat\varphi(k)|^2 \Big(-\nabla_\xi K_0 - (v_\beta\cdot\nabla_\xi)\nabla_\xi K_1 + \frac{1}{2}(\dot v_\beta \cdot \nabla_\xi)\nabla_\xi K_2 - \frac{1}{2}(v_\beta\cdot\nabla_\xi)^2 \nabla_\xi K_2\Big).$$

Thus we only need to estimate

$$\begin{aligned} \int d^3k\, |\hat\varphi(k)|^2 K_p &= \int d^3k\, |\hat\varphi(k)|^2 e^{-ik\cdot\xi_{\alpha\beta}} \int_t^\infty d\tau\, \frac{\sin|k|\tau}{|k|}\, \tau^p \\ &= \int\!\!\int d^3x d^3y\, \varphi(x)\varphi(y) \int_t^\infty d\tau\, \psi_\tau(\xi_{\alpha\beta} + x - y)\, \tau^p, \end{aligned} \tag{6.8}$$

the latter equality follows analogously to (6.1). However, for $|x|, |y| \le R_\varphi$ and $t \in [t_0, T\varepsilon^{-3/2}]$ we obtain in case $\tau = |\xi_{\alpha\beta} + x - y|$ from (2.6) the contradiction

$$4(R_\varphi + C^*\varepsilon^{-1}) = t_0 \le t \le \tau \le 2R_\varphi + |\xi_{\alpha\beta}| \le 2R_\varphi + C^*\varepsilon^{-1}.$$

This shows the term in (6.8) is identically zero for $t \in [t_0, T\varepsilon^{-3/2}]$, and thus $D_3(t) = 0$ for $t \in [t_0, T\varepsilon^{-3/2}]$. Together with (6.6) and (6.7) this completes the proof of Lemma 3.2(b). □

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