Frobenius$_\infty$ invariants of homotopy Gerstenhaber algebras, I

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Abstract

We construct a functor from the derived category of homotopy Gerstenhaber algebras, $\mathfrak{g}$, with finite-dimensional cohomology to the purely geometric category of so-called F_∞-manifolds. The latter contains Frobenius manifolds as a subcategory (so that a pointed Frobenius manifold is itself a homotopy Gerstenhaber algebra). If $\mathfrak{g}$ happens to be formal as a L_∞-algebra, then its F_∞-manifold comes equipped with the Gauss-Manin connection. Mirror Symmetry implications are discussed.

1 Introduction

Frobenius manifolds play a central role in the usual formulation of Mirror Symmetry, as may be seen in the following diagram,

$$\begin{tikzcd} \boxed{\begin{array}{c}\textsf{A category of compact}\\ \textsf{symplectic manifolds}\end{array}} \arrow[r, "\mathsf{GW}"] & \boxed{\begin{array}{c}\textsf{A category of}\\ \textsf{Frobenius manifolds}\end{array}} & \boxed{\begin{array}{c}\textsf{A category of compact}\\ \textsf{Calabi} - \textsf{Yau manifolds}\end{array}} \arrow[l, "\mathsf{BK}"'] \end{tikzcd}$$

where morphisms in all categories are just diffeomorphisms preserving relevant structures, and GW and BK stand, respectively, for the Gromov-Witten (see, e.g., [Ma2]) and Barannikov-Kontsevich [BK, Ba] functors. A pair $(\widetilde{M}, M)$ consisting of a symplectic manifold $\widetilde{M}$ and a Calabi-Yau manifold M is said to be *Mirror* if $\mathsf{GW}(\widetilde{M}) = \mathsf{BK}(M)$. According to Kontsevich [Ko1], this equivalence is a shadow of a more fundamental equivalence of natural A_∞-categories attached to M and $\widetilde{M}$.

This paper is much motivated by the Barannikov-Kontsevich construction [BK, Ba] of the functor from the right in the above diagram, and by Manin's comments [Ma1] on their construction. The roots of the BK functor lie in the extended deformation theory of complex structures on M, more precisely in very special properties of the (differential) Gerstenhaber algebra $\mathfrak{g}$ "controlling" such deformations. One of the miracle features of Calabi-Yau manifolds, the one which played a key role in the BK construction, is that deformations of their complex structures are non-obstructed, always producing a *smooth* versal moduli space[1]. In the language of Gerstenhaber algebras, the exceptional algebraic properties necessary to produce a Frobenius manifold out of $\mathfrak{g}$, have been axiomatized in

[1] A similar phenomenon occurs in the extended deformation theory of Lefschetz symplectic structures which also produces, via the same BK functor, Frobenius manifolds [Me1]. These should not be confused with GW.

[Ma1, Ma2]. As a result, the functor

$$\boxed{\begin{array}{c}\text{“Exceptional”}\\ \text{Gerstenhaber algebras}\end{array}} \quad \stackrel{\mathsf{BK}}{\longrightarrow} \quad \boxed{\begin{array}{c}\text{Frobenius}\\ \text{manifolds}\end{array}}$$

is now well understood.

One of our purposes in this paper is to extend the BK functor from the category of Calabi-Yau manifolds to the category of arbitrary compact complex manifolds. Which means the study of a diagram

$$\boxed{\begin{array}{c}\text{Arbitrary}\\ \text{Gerstenhaber algebras}\end{array}} \quad \stackrel{?}{\longrightarrow} \quad \boxed{\begin{array}{c}?\\ \end{array}}$$

Generically, the extended deformation theory of complex structures is obstructed, and it would be naive to expect that the question mark above stands for the category of Frobenius manifolds. In fact, it is not, and the answer is captured in the following notion.

1.1. Definition. An F_∞*-manifold* is the data $(\mathcal{H}, E, \partial, [\mu_*], e)$, where

(i) $\mathcal{H}$ is a formal pointed $\mathbb{Z}$-graded manifold,

(ii) E is the Euler vector field on $\mathcal{H}$, $Ef := \frac{1}{2}|f|f$, for all homogeneous functions on $\mathcal{H}$ of degree $|f|$,

(iii) ∂ is an odd homological (i.e. $\partial^2 = 0$) vector field on $\mathcal{H}$ such that $[E, \partial] = \frac{1}{2}\partial$ and $\partial I \subset I^2$, I being the ideal of the distinguished point in $\mathcal{H}$,

(iv) $[\mu_n : \otimes^n \mathcal{T}_\mathcal{H} \to \mathcal{T}_\mathcal{H}]$, $n \in \mathbb{N}$, is a homotopy class of smooth unital strong homotopy commutative (C_∞) algebras defined on the tangent sheaf, $\mathcal{T}_\mathcal{H}$, to $\mathcal{H}$, such that $Lie_E \mu_n = \frac{1}{2} n \mu_n$, for all $n \in \mathbb{N}$, and μ_1 is given by

$$\begin{array}{rccc}\mu_1: & \mathcal{T}_\mathcal{H} & \longrightarrow & \mathcal{T}_\mathcal{H} \\ & X & \longrightarrow & \mu_1(X) := [\partial, X].\end{array}$$

(v) e is the unit, i.e. an even vector field on $\mathcal{H}$ such that $[\partial, e] = 0$, $\mu_2(e, X) = X$, $\forall X \in \mathcal{T}_\mathcal{H}$, and $\mu_n(\ldots, e, \ldots) = 0$ for all $n \geq 3$.

Clearly, the category of F_∞-manifolds containes Frobenius manifolds as a subcategory.

On any F_∞-manifold the vector field ∂ defines an integrable distribition, $\mathsf{Im}\,\mu_1$, which is tangent to its subspace of zeros, “$\mathsf{zeros}(\partial)$”. The structures (i)-(v) make the tangent sheaf to the smooth part of the associated quotient, “$\mathsf{zeros}(\partial)$”$/\mathsf{Im}\,\mu_1$, into a sheaf of graded unital associative algebras.

Theorem A. *For any differential unital (graded commutative) Gerstenhaber algebra* $\mathfrak{g}$*, its cohomology,* $\mathbf{H}(\mathfrak{g})$*, if finite-dimensinal, is canonically an* F_∞*-manifold.*

The resulting diagram

$$\boxed{\begin{array}{c}\text{Differential unital}\\ \text{Gerstenhaber algebras}\end{array}} \xrightarrow{F_\infty} \boxed{\begin{array}{c}F_\infty\\ \text{manifolds}\end{array}}$$

implies, in turn, a diagram,

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{A category of compact}\\ \text{symplectic manifolds}\end{array}} \arrow[r, "F_\infty"] & \boxed{\begin{array}{c}\text{A category of}\\ F_\infty-\text{manifolds}\end{array}} & \boxed{\begin{array}{c}\text{A category of compact}\\ \text{complex manifolds}\end{array}} \arrow[l, "F_\infty"'] \\
& \boxed{\begin{array}{c}\text{A category of}\\ \text{holomorphic vector bundles}\\ \text{on compact complex manifolds}\end{array}} \arrow[u, "F_\infty"] &
\end{tikzcd}$$

through the Gerstenhaber algebras controlling extended deformations of symplectic, complex and holomorphic vector bundle structures. Moreover, the F_∞-functor enjoys the correct "classical limit": when restricted to exceptional Gerstenhaber algebras (i.e. the ones satisfying Manin's axioms [Ma1]), the F_∞-functor coincides precisely with the Barannikov-Kontsevich construction [BK], and hence takes values in the subcategory of Frobenius manifolds.

Let us emphasize once again that, in the above diagram, $F_\infty(\mathsf{symplectic\ manifolds})$ has nothing to do with Gromov-Witten invariants[2]. Nevertheless, to rather different mathematical objects we can canonically attach invariants lying in one and the same geometric category. Hence we can use these F_∞-invariants for a classification, and even speak about *dull mirror symmetry* when

$$F_\infty(\mathsf{Object}) = F_\infty(\widetilde{\mathsf{Object}}).$$

Such a relation may be a shadow of something conceptually more interesting (cf. [Ko1]).

Theorem A is explained and generalized by the following

Theorem B. *There is a canonical functor, F_∞, from the derived category of unital homotopy Gerstenhaber (G_∞) algebras with finite-dimensional cohomology to the category of F_∞-manifolds.*

This result implies that the cohomology space of any homotopy Gerstenhaber algebra is, if finite-dimensional, canonically an F_∞-manifold,

$$\boxed{\begin{array}{c}\text{Homotopy}\\ \text{Gerstenhaber algebras}\end{array}} \xrightarrow{F_\infty} \boxed{\begin{array}{c}F_\infty\\ \text{manifolds}\end{array}}$$

The recent proof of Deligne's conjecture [Ta1, Ko3, V, MS] gives the following diagrammatic corollary of Theorem B,

$$\boxed{\begin{array}{c}A_\infty\\ \text{algebra } \mathcal{A}\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{Hochschild complex}\\ C^\bullet(\mathcal{A},\mathcal{A})\end{array}} \xrightarrow{F_\infty} \boxed{\begin{array}{c}F_\infty\\ \text{manifold}\end{array}}$$

[2] At best, this is a very weak shadow of the Mirror Symmetry, see below.

which, probably, has a direct relevance to the Mirror symmetry through the following specializations, C and D, of statement B.

Any G_∞-algebra $\mathfrak{g}$ is, in particular, a L_∞-algebra so that its cohomology, $\mathbf{H}(\mathfrak{g})$, has the induced structure, $[\ ,\ ]_{\text{ind}}$, of Lie algebra. If there exists a quasi-isomorphism of L_∞-algebras,

$$(\mathfrak{g}, L_\infty\text{–component of the } G_\infty\text{–structure}) \xrightarrow{F} (\mathbf{H}(\mathfrak{g}), [\ ,\ ]_{\text{ind}})\,,$$

then $\mathfrak{g}$ is said to be L_∞-*formal*, and F is called a *formality map*. In terms of the associated F_∞-invariant, the L_∞-formality of a G_∞-algebra $\mathfrak{g}$ gets translated into a canonical flat structure, the Gauss-Manin connection ∇, such that $\nabla_X e = 0$ and $\nabla_X\nabla_Y\nabla_Z\partial = 0$ for any flat vector fields X, Y and Z. This specialization of F_∞-structure is called *pre-Frobenius*$_\infty$ structure.

Theorem C. *There is a canonical functor from the category of pairs $(\mathfrak{g}, F)$, where $\mathfrak{g}$ is a L_∞-formal homotopy Gerstenhaber algebra and F a formality map, to the category of pre-Frobenius$_\infty$ manifolds.*

In fact, a pre-Frobenius$_\infty$ manifold (a Frobenius manifold, in particular) is itself a homotopy Gerstenhaber algebra.

According to Kontsevich's celebrated Formality Theorem [Ko2], for any compact complex manifold M, the Hochschild differential Lie algebra, $C^\bullet(\mathcal{A}, \mathcal{A})$, associated to the algebra of Dolbeault forms, $\mathcal{A} = (\Gamma(M, \Omega_M^{0,\bullet}), \bar\partial)$, is L_∞-formal so that Theorem C has a wide area of applications.

Theorem D. *If a homotopy Gerstenhaber algebra $\mathfrak{g}$ is quasi-isomorphic, as a L_∞-algebra, to an Abelian differential Lie algebra, then the tangent sheaf, $\mathcal{T}_{\mathcal{H}}$, to its cohomology* $\mathbf{H}(\mathfrak{g})$ *viewed as a linear supermanifold is canonically a sheaf of unital graded commutative associative algebras.*

The point is that the Hochschild complex built out of the Dolbeault algebra, $\mathcal{A} = (\Gamma(\mathcal{M}, \Omega_M^{0,\bullet}), \bar\partial)$, of a Calabi-Yau manifold $\mathcal{M}$, satisfies the conditions of Theorem D. In fact, the canonically induced associative product on the tangent sheaf to the associated linear supermanifold, $\mathbf{H}^\bullet(\mathcal{M}, \wedge^\bullet T_{\mathcal{M}})$, is, for an appropriate formality map, potential and satisfies the WDVV equations. The resulting composition,

$$\boxed{\begin{array}{c}\mathsf{Calabi-Yau}\\ \mathsf{manifold\ M}\end{array}} \longrightarrow \boxed{\begin{array}{c}(C^\bullet_{\text{Hoch}}(\mathcal{A},\mathcal{A}), \mathsf{formality\ map, trace})\\ \mathsf{where}\ \mathcal{A} = (\Gamma(\mathsf{M}, \Omega^{0,*}_{\mathsf{M}}), \bar\partial)\end{array}} \xrightarrow{F_\infty} \boxed{\begin{array}{c}\mathsf{Frobenius}\\ \mathsf{manifold}\end{array}}$$

together with its analogue for the de Rham algebra of a compact Lefschetz symplectic manifold, will be discussed in the second part of this paper.

The paper is organized as follows:

Section 2: the origin of the data (i)-(iii) in Definition 1.1 of an F_∞-manifold is explained via the deformation theory. Here we use only the L_∞-component of a G_∞-structure. The main technical tool is a modified version of the classical deformation functor which is proved to be non-obstructed. This part of the story is, probably, of independent interest.

Section 3: we use a homotopy techique to explain the origin of the data (iv)-(v) in Definition 1.1, and to prove Theorems A-D.

Section 4: we give second proofs of the main claims of this paper using perturbative solutions of algebro-differential equations.

2 Deformation functors

2.1. Odd Lie superalgebras. Let k be a field with characteristic $\neq 2$. An *odd Lie superalgebra over* k is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\tilde{0}} \oplus \mathfrak{g}_{\tilde{1}}$ equipped with an odd k-linear map [Ma2]

$$\begin{array}{rccc} [\bullet]: & \mathfrak{g} \otimes \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & a \otimes b & \longrightarrow & [a \bullet b], \end{array}$$

which satisfies the following conditions

(a) odd skew-symmetry: $[a \bullet b] = -(-1)^{(\tilde{a}+1)(\tilde{b}+1)}[b \bullet a]$,

(b) odd Jacobi identity:

$$[a \bullet [b \bullet c]] = [[a \bullet b] \bullet c] + (-1)^{(\tilde{a}+1)(\tilde{b}+1)}[b \bullet [a \bullet c]],$$

for all $a, b, c \in \mathfrak{g}_{\tilde{0}} \cup \mathfrak{g}_{\tilde{1}}$.

The parity change functor transforms this structure into the usual Lie superalgebra brackets, $[\ ,\ ]$, on $\Pi\mathfrak{g}$. Thus odd Lie superlgebras are nothing but Lie superalgebras in the "awkwardly" chosen $\mathbb{Z}_2$-grading. For this reason we sometimes omit the prefix *odd*, and treat $(\mathfrak{g}, d, [\ \bullet\ ])$ and $(\Pi\mathfrak{g}, d, [\ ,\ ])$ as different representations of one and the same object. One advantage of of working with $[\bullet]$ rather than with the usual Lie brackets $[\ ,\ ]$ is that this awkward $\mathbb{Z}_2$ grading induces the correct, for our purposes, $\mathbb{Z}_2$-grading on the associated cohomology supermanifold (see below). Another advantage will become clear below, when we introduce on $\mathfrak{g}$ one more algebraic structure (an *even*, in this awkward $\mathbb{Z}_2$-grading, associative product) making $\mathfrak{g}$ into a Gerstenhaber algebra (cf. [Ma1, Ma2]).

2.2. Cohomology as a formal supermanifold. A data $(\mathfrak{g}, [\ \bullet\ ], d)$ with $(\mathfrak{g}, [\ \bullet\ ])$ being a Lie superalgebra and

$$d : \mathfrak{g} \longrightarrow \mathfrak{g}$$

an odd k-linear map satisfying

$$d[a \bullet b] \quad = \quad [da \bullet b] - (-1)^{\tilde{a}}[a \bullet db]$$

is called a *differential* Lie superalgebra, or shortly *dLie-algebra*. This triple $(\mathfrak{g}, [\bullet], d)$ is often abbreviated to $\mathfrak{g}$.

The cohomology of $\mathfrak{g}$,

$$\mathbf{H}(\mathfrak{g}) := \mathsf{Ker}\, d/\mathsf{Im}\, d\,,$$

inherits the structure of Lie superalgebra. We always assume in this paper that $\mathbf{H}(\mathfrak{g})$, which we often abbreviate to $\mathbf{H}$, is a finite dimensional superspace, say $\dim \mathbf{H} = p|q$. Let

$\{[e_i], i = 1, \ldots, p+q\}$ be a basis consisting of homogeneous elements with parity denoted by $\tilde{i}$, and $\{t^i, i = 1, \ldots, p+q\}$ the associated dual basis in $\mathbf{H}^*$. The supercommutative ring, $k[[t^1, \ldots, t^{p+q}]]$, of formal power series will be abbreviated to $k[[t]]$. The (purely) notational advantage of working with $k[[t]]$ rather than with the invariantly defined object $\odot^\bullet \mathbf{H}^*$ is that we shall want viewing

(i) $\mathbf{H}$ as a smooth formal pointed $(p|q)$-dimensional supermanifold denoted (to emphasize this change of thought) by $\mathcal{H}$ or $\mathcal{H}_\mathfrak{g}$,

(ii) $\{t^i\}$ as linear coordinates on $\mathcal{H}$,

(iii) $k[[t]]$ as the space of global sections of the structure sheaf, $\mathcal{O}_\mathcal{H}$, on $\mathcal{H}$.

The ideal sheaf of the origin, $0 \in \mathcal{H}$, will be denote by I.

There is a canonical map

$$\begin{array}{rrcl} s: & k[[t]] \otimes \mathbf{H} & \longrightarrow & H^0(\mathcal{H}, \mathcal{T}_\mathcal{H}) \\ & \sum a^i(t)[e_i] & \longrightarrow & \sum a^i(t) \frac{\partial}{\partial t^i}, \end{array}$$

where $H^0(\mathcal{H}, \mathcal{T}_\mathcal{H})$ stands for the space of global sections of the sheaf, $\mathcal{T}_\mathcal{H}$, of formal vector fields on $\mathcal{H}$. There is a well defined action of $H^0(\mathcal{H}, \mathcal{T}_\mathcal{H})$ on both $k[[t]] \otimes \mathbf{H}$ and $k[[t]] \otimes \mathfrak{g}$ through the first factor. If X is a formal vector field on $\mathcal{H}$ and Γ is an element of $k[[t]] \otimes \mathfrak{g}$ (or of $k[[t]] \otimes \mathbf{H}$), then the result of this action is denoted by $\overrightarrow{X}\,\Gamma$.

Any element Γ in $k[[t]] \otimes \mathfrak{g}$ (or in $k[[t]] \otimes \mathbf{H}$) can be uniquely decomposed,

$$\Gamma = \Gamma_{[0]} + \Gamma_{[1]} + \ldots + \Gamma_{[n]} + \ldots$$

into homogeneous polynomials, $\Gamma_{[n]}$, of degree n in the variables t^i. The sum of the first n terms in the above decomposition is denoted by $\Gamma_{(n)}$, i.e. $\Gamma_{(n)} = \Gamma \mod I^{n+1}$.

We shall call an element $\Gamma \in k[[t]] \otimes \mathfrak{g}$ *versal* if it is even, $\Gamma \mod I = 0$, $\Gamma_{[1]} = \Gamma \mod I^2 \in I \otimes \mathsf{Ker}\ d$ and $\Gamma_{[1]} \mod \mathsf{Im}\ d = \sum_{i=1}^{p+q} t^i [e_i]$.

The sheaf $\mathcal{T}_\mathcal{H}$ comes canonically equipped with a flat torsion-free affine connection ∇ whose horizontal sections are, by definition, the linear span of $s([e_i])$, $i = 1, \ldots, p+q$, i.e.

$$\nabla X = 0$$

if and only of $s^{-1}(X)$ is a "constant" (independent of t^i) element in $k[[t]] \otimes \mathbf{H}$. This connection memorizes the origin of $\mathcal{H}$ as a vector superspace.

It will be important, in this paper, to ignore sometimes the flat structure and view $\mathcal{H}$ only as a smooth formal supermanifold with a distinguished point 0 but no preferred coordinate system. To avoid possible confusion, we adopt from now on this latter viewpoint unless the flat connection ∇ is explicitly mentioned.

2.2.1. $\mathbb{Z}$-grading. Differential Lie superalgebras $(\mathfrak{g}, d, [\bullet])$ which we often encounter in geometry have their $\mathbb{Z}_2$-grading induced from a finer structure, $\mathbb{Z}$-*grading*, which is, by definition, a decomposition of $\mathfrak{g}$ into a direct sum,

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^i,$$

with the following consistency conditions

(a) $d\mathfrak{g}^i \subset \mathfrak{g}^{i+1}$, and

(b) $[\mathfrak{g}^i \bullet \mathfrak{g}^j] \subset \mathfrak{g}^{i+j-1}$.

The $\mathbb{Z}_2$-grading associated to this structure is then simply $\mathfrak{g}_{\tilde{0}} := \bigoplus_{i\in 2\mathbb{Z}} \mathfrak{g}^i$ and $\mathfrak{g}_{\tilde{1}} := \bigoplus_{i\in 2\mathbb{Z}+1} \mathfrak{g}^i$.

Clearly, there is an induced $\mathbb{Z}$-grading on the cohomology Lie superalgebra, $\mathbf{H} = \oplus_{i\in\mathbb{Z}}\mathbf{H}^i(\mathfrak{g})$, as well as on the structure sheaf, $\mathcal{O}_{\mathcal{H}}$, of the associated cohomology supermanifold.

2.3. Classical deformation functor. One of the approaches to constructing a (versal) deformation space of a given mathematical structure $\mathcal{A}$ consists of the following steps (see, e.g. [GM, Ko2, Ba], and references therein):

1) Associate to $\mathcal{A}$ a "controlling" differential $\mathbb{Z}$-graded Lie algebra $(\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}^k, d, [\,\bullet\,])$ over a field k (which is usually $\mathbb{R}$ or $\mathbb{C}$).

2) Define the deformation functor

$$\mathsf{Def}^{\,0}_{\mathfrak{g}} : \left\{ \begin{array}{l} \text{the category of Artin} \\ k\text{-local algebras} \end{array} \right\} \longrightarrow \{\text{the category of sets}\}$$

as follows

$$\mathsf{Def}^0_{\mathfrak{g}}(\mathcal{B}) = \left\{ \Gamma \in (\mathfrak{g}\otimes m_{\mathcal{B}})^2 \mid d\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] = 0 \right\} / \exp{(\mathfrak{g}\otimes m_{\mathcal{B}})^1},$$

where $m_{\mathcal{B}}$ is the maximal ideal of the Artin algebra $\mathcal{B}$, the latter is viewed as a $\mathbb{Z}$-graded algebra concentrated in degree zero (so that $(\mathfrak{g}\otimes m_{\mathcal{B}})^i = \mathfrak{g}^i \otimes m_{\mathcal{B}}$), and the quotient is taken with respect to the following representation of the gauge group $\exp{(\mathfrak{g}\otimes m_{\mathcal{B}})^1}$,

$$\Gamma \to \Gamma^g = e^{\mathrm{ad}_g}\Gamma - \frac{e^{ad_g} - 1}{\mathrm{ad}_g} dg, \qquad g \in (\mathfrak{g}\otimes m_{\mathcal{B}})^1,$$

where ad is just the usual internal automorphism of $\mathfrak{g}$, $\mathrm{ad}_g\Gamma := [g \bullet \Gamma]$.

3) Try to represent the deformation functor by a topological (pro-Artin) algebra $\mathcal{O}_{\mathcal{M}}$ so that
$$\mathsf{Def}^{\,0}_{\mathfrak{g}}(\mathcal{B}) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}_{\mathcal{M}}, \mathcal{B}).$$
This associates to the mathematical structure $\mathcal{A}$ the formal moduli space $\mathcal{M}$ whose "ring of functions" is $\mathcal{O}_{\mathcal{M}}$.

In geometry, one often continues with a fourth step by constructing a cohomological splitting of $\mathfrak{g}$ and applying the Kuranishi method [Ku, GM] to represent versally the deformation functor by the ring of analytic (rather than formal) functions on the Kuranishi space.

The tangent space, $\mathsf{Def}^0_{\mathfrak{g}}(k[\varepsilon]/\varepsilon^2)$, to the functor $\mathsf{Def}^0_{\mathfrak{g}}$ is isomorphic to the cohomology group $\mathbf{H}^2(\mathfrak{g})$ of the complex $(\mathfrak{g}, d)$. If one extends in the obvious way the above deformation functor to the category of arbitrary $\mathbb{Z}$-graded k-local Artin algebras (which may not be concentrated in degree 0), one gets the functor $\mathsf{Def}^*_{\mathfrak{g}}$ with the tangent space isomorphic to the full cohomology group $\oplus_{i\in\mathbb{Z}}\mathbf{H}^i(\mathfrak{g})$.

When working with the extended deformation functor $\mathsf{Def}^*_{\mathfrak{g}}$ it is often no loss of essential information to forget the $\mathbb{Z}$-grading on $\mathfrak{g}$ and keep only the associated $\mathbb{Z}_2$-grading. One gets then the following equivalent definition of $\mathsf{Def}^*_{\mathfrak{g}}$:

$$\mathsf{Def}^*_{\mathfrak{g}}: \left\{\begin{array}{c}\text{the category of Artin}\\ k\text{-local superalgebras}\end{array}\right\} \longrightarrow \{\text{the category of sets}\}$$

$$\mathsf{Def}^*_{\mathfrak{g}}(\mathcal{B}) := \left\{\Gamma \in (\mathfrak{g}\otimes m_{\mathcal{B}})_{\tilde{0}} \mid d\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] = 0\right\} / \exp{(\mathfrak{g}\otimes m_{\mathcal{B}})_{\tilde{1}}}.$$

This functor is representable by a *smooth* formal moduli space $\mathcal{M}$ if there exists a versal (in the sense of Sect. 2.2) solution,

$$\Gamma = \sum_a e_a t^a + \sum_{a_1,a_2} \Gamma_{a_1 a_2} t^{a_1} t^{a_2} + \ldots \in (\mathfrak{g}\otimes k[[t]])_{\tilde{0}}, \tag{1}$$

to the so-called *Maurer-Cartan equation*,

$$d\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] = 0.$$

Due to versality of Γ, any other solution over an arbitrary Artin algebra $\mathcal{B}$ is equivalent to this one by a base change $k[[t]] \to \mathcal{B}$.

2.3.1. L_∞-morphisms, part I. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two dLie-algebras. To formulate the basic theorem of the classical deformation theory, we shall need the following notion: a sequence of linear maps

$$F_n : \odot^n \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2, \quad n = 1, 2, \ldots, \qquad \tilde{F}_n = \tilde{n} + 1,$$

defines a L_∞-*morphism* from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ if

$$\begin{aligned} dF_n(\gamma_1, \gamma_2, \ldots, \gamma_n) + \sum_{i=1}^n \pm F_n(\gamma_1, \ldots, d\gamma_i, \ldots, \gamma_n) = \\ = \frac{1}{2}\sum_{\substack{k+l=n\\ k,l\geq 1}} \frac{1}{k!l!} \sum_{\sigma\in\Sigma_n} \pm \left[F_k(\gamma_{\sigma_1}, \ldots, \gamma_{\sigma_k}) \bullet F_l(\gamma_{\sigma_{k+1}}, \ldots, \gamma_{\sigma_n})\right] + \\ + \sum_{i<j} \pm F_{n-1}([\gamma_i \bullet \gamma_j], \gamma_1, \ldots, \gamma_n), \end{aligned}$$

for arbitrary $\gamma_1, \ldots, \gamma_n \in \mathfrak{g}_1$.

In particular, the first map F_1 is a morphism of complexes which respects the Lie brackets up to homotopy defined by the second map F_2.

A L_∞-morphism $F = \{F_n\} : \mathfrak{g}_1 \to \mathfrak{g}_2$ is called a *quasi-isomorphism* if its linear part F_1 induces an isomorphism, $\mathbf{H}(\mathfrak{g}_1) \to \mathbf{H}(\mathfrak{g}_2)$, of associated cohomology groups.

2.3.2. Basic Theorem of Deformation Theory [Ko2] . *An L_∞-morphism $F = \{F_n\} : \mathfrak{g}_1 \to \mathfrak{g}_2$ defines a natural transformation of the functors*

$$\begin{array}{rccc} F_* : & \mathsf{Def}^*_{\mathfrak{g}_1} & \longrightarrow & \mathsf{Def}^*_{\mathfrak{g}_2} \\ & \Gamma & \longrightarrow & F_*(\Gamma) := \sum_{n=1}^{\infty} \frac{1}{n!} F_n(\Gamma, \dots, \Gamma), \end{array}$$

i.e. if Γ is a solution to Maurer-Cartan equations in $(\mathfrak{g}_1 \otimes m_{\mathcal{B}})_{\tilde{0}}$, then $F_*(\Gamma)$ is a solution to Maurer-Cartan equations in $(\mathfrak{g}_2 \otimes m_{\mathcal{B}})_{\tilde{0}}$.

Moreover, if F is a quasi-isomorphism, then F_* is an isomorphism.

2.3.3. Corollary. *If a dLie-algebra $\mathfrak{g}$ is quasi-isomorphic to an Abelian dLie-algebra, then $\mathsf{Def}^*_{\mathfrak{g}}$ is versally representable by a smooth formal pointed supermanifold $\mathcal{H}_{\mathfrak{g}}$ (the cohomology supermanifold of $\mathfrak{g}$).*

Proof. If $\mathfrak{h}$ is an Abelian dLie-algebra, then, in the notations of Sect. 2.2,

$$\Gamma = \sum_{i=1}^{\dim \mathbf{H}(\mathfrak{h})} t^i [e_i]$$

is a versal solution of Maurer-Cartan equations. Hence $\mathsf{Def}^*_{\mathfrak{h}}$ is representable by $\mathcal{H}_{\mathfrak{h}}$.

If $\mathfrak{g}$ is quasi-isomorphic to $\mathfrak{h}$, the required statement follows from Theorem 2.3.2 and the isomorphism $\mathcal{H}_{\mathfrak{g}} = \mathcal{H}_{\mathfrak{h}}$. $\square$

There are two remarkable examples, one dealing with extended deformations of complex structures on a Calabi-Yau manifold [BK] and another with extended deformations of the symplectic structure on a Lefschetz manifold [Me1], when the rather strong condition of Corollary 2.3.3 holds true[3]. In general, however, there will be obstructions to constructing a versal solution to the Maurer-Cartan equations, and we will have to resort to other technical means such as modifying the deformation functor as explained below in Sect. 2.4 below or further extending the deformation problem to the category of L_∞-algebras.

2.3.4. Example (deformations of complex manifolds). It is well known that the total space of the cotangent bundle, $\Omega^1_{\mathbb{R}}$, to a real n-dimensional manifold M carries a natural Poisson structure $\{\ ,\ \}$ making the structure sheaf $\mathcal{O}_{\Omega^1_{\mathbb{R}}}$ into a sheaf of Lie algebras. In a natural local coordinate system $(x^a, p_a := \partial/\partial x^a)$,

$$\{f, g\} = \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial x^a} - \frac{\partial f}{\partial x^a}\frac{\partial g}{\partial p_a}.$$

If we now change the parity of the fibers of the natural projection $\Omega^1_{\mathbb{R}} \to M$ (which is allowed since they are vector spaces), we will get an $(n|n)$-dimensional supermanifold $\Pi\Omega^1_{\mathbb{R}}$ equipped with a natural odd Poisson structure $\{\ \bullet\ \}$ making the structure sheaf $\mathcal{O}_{\Pi\Omega^1_{\mathbb{R}}}$ into

[3]One more example (of a different technical origin though of the same mirror symmetry flavor) when a naturally extended deformation problem gives rise to a *smooth* extended moduli space is discussed in [Me2].

a sheaf of odd Lie superalgebras. In a natural local coordinate system $(x^a, \psi_a := \Pi\partial/\partial x^a)$ on $\Pi\Omega^1_{\mathbb{R}}$,

$$\{f \bullet g\} = \frac{\partial f}{\partial \psi_a}\frac{\partial g}{\partial x^a} + (-1)^{\tilde{g}(\tilde{f}+1)}\frac{\partial g}{\partial \psi_a}\frac{\partial f}{\partial x^a}.$$

The smooth functions on $\Pi\Omega^1_{\mathbb{R}}$ have a simple geometric interpretation in term of the underlying manifold M — they are just smooth polyvector fields. Indeed, a standard power series decomposition in odd variables gives

$$f = \sum_{k=0}^{n}\sum_{a_1,\dots,a_k} f^{a_1\dots a_k}(x)\psi_{a_1}\dots\psi_{a_k}$$

implying the isomorphism of sheaves $\mathcal{O}_{\Pi\Omega^1_{\mathbb{R}}} = \Lambda^\bullet T_{\mathbb{R}}$, where $T_{\mathbb{R}}$ is the real tangent bundle to M.

Therefore, the sheaf $\Lambda^\bullet T_{\mathbb{R}}$ with the $\mathbb{Z}_2$-grading,

$$(\Lambda^* T_{\mathbb{R}})_{\tilde{0}} := \Lambda^{\mathrm{even}} T_{\mathbb{R}}, \qquad\qquad (\Lambda^* T_{\mathbb{R}})_{\tilde{1}} := \Lambda^{\mathrm{odd}} T_{\mathbb{R}},$$

induced from that on $\mathcal{O}_{\Pi\Omega^1_{\mathbb{R}}}$, is naturally a sheaf of odd Lie superalgebras. The odd Poisson bracket $\{\bullet\}$ is called, in this incarnation, the *Schouten* bracket and is often denoted by $[\bullet]_{\mathrm{Sch}}$.

If M is a complex manifold, then the canonical odd Poison structure on the parity changed holomorphic cotangent bundle, $\Pi\Omega^1_M$, is itself holomorphic giving rise thereby to the structure of odd Lie superalgebra on the sheaf, $\Lambda^\bullet T_M$, of holomorphic polyvector fields. This can be used to make the vector space

$$\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}^k, \qquad \mathfrak{g}^k := \bigoplus_{i+j=k} \Gamma(M, \Lambda^i T_M \otimes \Lambda^j \overline{T}^*_M),$$

into a $\mathbb{Z}$-graded differential algebra by taking $\bar{\partial}$, the $(0,1)$ part of the de Rham operator, as a differential, and the map,

$$\begin{array}{rccc} [\bullet]: & \Gamma(M, \Lambda^{i_1}T_M \otimes \Lambda^{j_1}\overline{T}^*_M) \times \Gamma(M, \Lambda^{i_2}T_M \otimes \Lambda^{j_2}\overline{T}^*_M) & \to & \Gamma(M, \Lambda^{i_1+i_2-1}T_M \otimes \Lambda^{j_1+j_2}\overline{T}^*_M) \\ & X_1 \otimes \overline{w}_1 \times X_2 \otimes \overline{w}_2 & \to & [X_1 \otimes \overline{w}_1 \bullet X_2 \otimes \overline{w}_2], \end{array}$$

given by

$$[X_1 \otimes \overline{w}_1 \bullet X_2 \otimes \overline{w}_2] := (-1)^{\tilde{j}_1\tilde{i}_2}[X_1 \bullet X_2]_{\mathrm{Sch}} \otimes (\overline{w}_1 \wedge \overline{w}_2).$$

as the (odd) Lie brackets.

The importance of this dLie-algebra stems from the fact that the associated deformation functors $\mathsf{Def}^0_{\mathfrak{g}}$ and $\mathsf{Def}^*_{\mathfrak{g}}$ describe, respectively, ordinary and extended deformations of the given complex structure on a smooth manifold M. Indeed, a complex structure on a real $2n$-dimensional manifold M is a decomposition, $\mathbb{C}\otimes T_{\mathbb{R}} = T_M \oplus \overline{T}_M$, of the complexified real tangent bundle into a direct sum of complex integrable distributions, T_M and its complex conjugate $\overline{T}_M$. Another decomposition like that, $\mathbb{C}\otimes T_{\mathbb{R}} = T'_M \oplus \overline{T}'_M$, can be described in terms of the original complex structure by the graph, $\overline{T}'_M$, of a linear map $\Gamma : \overline{T}_M \to T_M$, i.e. by an element $\Gamma \in \mathfrak{g}^2$. The integrability of T'_M amounts then to the Maurer-Cartan equation in $\mathfrak{g}^2$,

$$\overline{\partial}\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma]_{\mathrm{Sch}} = 0.$$

By solving (if possible) the above equation in the full Lie algebra $\mathfrak{g}$ rather than in its subalgebra $\mathfrak{g}^2$ (and taking the quotient by the gauge group describing equivalent deformations), one gets a so-called extended complex structure on M whose geometric meaning is not yet fully understood. It is understood [Ko1], however, that this structure does have an important Mirror Symmetry aspect (at least for Calabi-Yau manifolds): Kontsevich noticed that his Formality Theorem [Ko2] identifies the moduli space of extended complex structures on a given complex manifold M with the moduli space of A_∞-deformations of the derived category of coherent sheaves on M, mirror counterpart of the conjectured Fukaya category built out of a dual complex manifold $\widetilde{M}$.

If M is a Calabi-Yau manifold, then, as was shown by Barannikov and Kontsevich [BK], the Maurer-Cartan equations in $\mathfrak{g}$ admit a versal solution of the form (1) implying that the moduli space of extended deformations of complex structures on M is smooth, and is isomorphic[4] to $\mathcal{H}$. In fact they have shown much more [BK, Ba]: in this case $\mathcal{H}$ has an induced structure of Frobenius manifold which conjecturally coincides with the Frobenius manifold structure on $H_*(\widetilde{M}, \mathbb{C})$ constructed via the Gromov-Witten invariants of the dual Calabi-Yau manifold $\widetilde{M}$. Barannikov [Ba] checked this conjecture for projective complete intersections Calabi-Yau manifolds.

For a general complex manifold M, the (extended) deformations are obstructed and the Maurer-Cartan equations associated to $\mathfrak{g}$ have no versal solutions of the form (1). It is one of the main tasks of this paper to understand what happens to the Barannikov-Kontsevich's Frobenius structure on $\mathcal{T}_{\mathcal{H}}$ in the presence of obstructions.

2.3.5. Example (deformations of Poisson and symplectic manifolds). It is well known that a 2-vector field, $\nu_0 \in \Gamma(M, \Lambda^2 T_{\mathbb{R}})$, defines a Poisson structure,

$$\{f, g\} = \nu_0(df \otimes dg), \qquad f, g \in \mathcal{O}_M,$$

on a smooth real n-dimensional manifold M if and only if

$$[\nu_0 \bullet \nu_0]_{\mathrm{Sch}} = 0.$$

Then a deformed 2-vector field, $\nu_0 + \nu \in \Gamma(M, \Lambda^2 T_{\mathbb{R}})$, is again a Poisson structure if and only if ν satisfies the Maurer-Cartan equation,

$$d\nu + \frac{1}{2}[\nu \bullet \nu]_{\mathrm{Sch}} = 0,$$

in the differential $\mathbb{Z}$-graded algebra

$$\left(\mathfrak{g} = \bigoplus_{i=0}^{n} \Gamma(M, \Lambda^i T_{\mathbb{R}}),\ [\ \bullet\ ]_{\mathrm{Sch}},\ d := [\nu_0 \bullet \ldots]_{\mathrm{Sch}} \right).$$

Hence the associated deformation functors $\mathsf{Def}^0_{\mathfrak{g}}/\mathsf{Def}^*_{\mathfrak{g}}$ describe (extended) deformations of the given Poisson structure ν_0 on M.

[4]Strictly speaking, this isomorphism holds true in the category of formal manifolds, in which we work in this paper. It is no problem to choose the power series (1) convergent thereby inducing on the extended moduli space a smooth analytic structure. The latter is then analytically isomorphic to an open neighbourhood of zero in $\mathbf{H}$ which we denote by the same symbol $\mathcal{H}$ (and continue doing this every time the analyticity aspect emerges).

For generic ν_0, the associated cohomology group $\oplus_i \mathbf{H}^i(\mathfrak{g})$ may not be finite-dimensional even for compact manifolds. If, however, ν_0 comes from a symplectic 2-form ω on M, the situation is very different. In this case one may use the "lowering indices map" $\omega : T_{\mathbb{R}} \to \Omega^1_{\mathbb{R}}$ to identify $(\mathfrak{g}, d)$ with the de Rham complex on M — it is not hard to check that this map sends the differential $[\nu_0 \bullet \ldots]_{\mathsf{Sch}}$ on $\Lambda^\bullet T_{\mathbb{R}}$ into the usual de Rham differential on $\Omega^\bullet_{\mathbb{R}}$. The (odd) Lie brackets induced on $\Omega^*_{\mathbb{R}}$ from the Schouten brackets on $\Lambda^\bullet T_{\mathbb{R}}$ we denote by $[\ \bullet\ ]_\omega$ to emphasize its dependence on the symplectic structure. Hence the deformation functors $\mathsf{Def}^{\,0}_{\mathfrak{g}}/\mathsf{Def}^{\,*}_{\mathfrak{g}}$ associated with the dLie-algebra

$$\left(\mathfrak{g} = \bigoplus_{i=0}^{n} \Gamma(M, \Omega^i_{\mathbb{R}}),\ [\ \bullet\ ]_\omega,\ d = \text{de Rham differential}\right)$$

describe (extended) deformations of a symplectic structure ω on M. Its cohomology $\mathbf{H}$ is nothing but the de Rham cohomology of M.

A compact symplectic manifold (M, ω) is called *Lefschetz* if the the natural cup product on the de Rham cohomology,

$$[\omega^k] : H^{m-k}(M, \mathbb{R}) \longrightarrow H^{m+k}(M, \mathbb{R})$$

is an isomorphism for any $k \leq m := \frac{1}{2} \dim M$. This class of manifolds (which includes the class of Kähler manifolds by the Hard Lefschetz Theorem) is of interest to us for the extended deformation functor $\mathsf{Def}^{\,*}_{\mathfrak{g}}$ associated with an arbitrary Lefschetz symplectic manifold is non-obstructed and is representable by a smooth moduli space isomorphic to $\mathcal{H}$; moreover, this moduli space of "extended symplectic structures" is always a Frobenius manifold [Me1]. This result is more than parallel to the Barannikov-Kontsevich's construction of extended moduli spaces/Frobenius structures for Calabi-Yau manifolds — it is just another example when their beautiful machinery works (see a nice exposition of Manin [Ma2]).

The extended deformation functor associated with a generic compact symplectic manifold seems to be obstructed, and one should employ different techniques (see below) to study geometric structures induced on moduli spaces of extended symplectic structures.

2.3.6. Example (deformations of holomorphic vector bundles). Let $E \to M$ be a holomorphic vector bundle on a complex n-dimensional manifold M. There is an associated differential $\mathbb{Z}$-graded Lie algebra

$$\left(\mathfrak{g} = \Gamma(M, \mathrm{End}E \otimes \overline{\Omega}^{\bullet}_M[-1]),\ [\ \bullet\ ],\ \bar{\partial}\right)$$

with the Lie brackets,

$$\begin{array}{rccc} [\bullet]: & \Gamma(M, \mathrm{End}E \otimes \overline{\Omega}^{i_1-1}_M) \times \Gamma(M, \mathrm{End}E \otimes \overline{\Omega}^{i_2-1}_M) & \to & \Gamma(M, \mathrm{End}E \otimes \overline{\Omega}^{(i_1+i_2-1)-1}_M) \\ & A_1 \otimes \overline{w}_1 \times A_2 \otimes \overline{w}_2 & \to & [A_1 \otimes \overline{w}_1 \bullet A_2 \otimes \overline{w}_2], \end{array}$$

given by

$$[A_1 \otimes \overline{w}_1 \bullet A_2 \otimes \overline{w}_2] := (A_1 A_2 - A_2 A_1) \otimes (\overline{w}_1 \wedge \overline{w}_2)\,.$$

The deformation functor $\mathsf{Def}^{\,0}_{\mathfrak{g}}$ associated with this algebra describes deformations of the holomorphic structure in the vector bundle E. It is tempting to view its extension $\mathsf{Def}^{\,*}_{\mathfrak{g}}$

as a tool for studying *extended* deformations, but we reserve this role for the functor $\mathsf{Def}^*_{\dots}$ associated with a larger differential algebra constructed in 3.1.5 below.

In general, all these functors are obstructed.

One may combine this differential Lie algebra (or its extension 3.1.5) together with the one of Example 2.3.4 into their natural semi-direct product to study *joint* (extended) deformations of the pair $E \to M$.

2.4. L_∞-algebras. These algebras will play only an auxiliary, purely technical, role in this paper.

By definition, a *strong homotopy Lie algebra*, or shortly L_∞-*algebra*, is a vector superspace $\mathfrak{h}$ equipped with linear maps,

$$\begin{array}{cccc} \mu_k : & \Lambda^k\mathfrak{h} & \longrightarrow & \mathfrak{h} \\ & v_1 \wedge \ldots \wedge v_k & \longrightarrow & \mu_k(v_1, \ldots, v_k), \qquad k \geq 1, \quad \tilde{\mu}_k = \tilde{k}, \end{array}$$

satisfying, for any $n \geq 1$ and arbitrary $v_1, \ldots, v_n \in \mathfrak{h}_{\tilde{0}} \cup \mathfrak{h}_{\tilde{1}}$, the following *higher order Jacobi identities*,

$$\sum_{k+l=n+1} \sum_{\sigma \in Sh(k,n)} (-1)^{\tilde{\sigma}+k(l-1)} e(\sigma; v_1, \ldots, v_n) \mu_l \left(\mu_k(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), v_{\sigma(k+l)}, \ldots, v_{\sigma(n)}\right) = 0,$$

where $Sh(k,n)$ is the set of all permutations $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ which satisfy $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots < \sigma(n)$. The symbol $e(\sigma; v_1, \ldots, v_n)$ (which we abbreviate from now on to $e(\sigma)$) stands for the *Koszul sign* defined by the equality

$$v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(n)} = (-1)^{\tilde{\sigma}} e(\sigma) v_1 \wedge \ldots \wedge v_n,$$

$\tilde{\sigma}$ being the parity of the permutation σ.

The $\mathbb{Z}$-graded version of this definition would require μ_k to be homogeneous of degree $2 - k$.

This notion as well as the associated notion of A_∞-algebra (reminded below) are due to Stasheff [S].

The first three higher order Jacobi identities have the form

$n = 1$**:** $\quad d^2 = 0,$

$n = 2$**:** $\quad d[v_1, v_2] = [dv_1, v_2] + (-1)^{\tilde{v}_1}[v_1, dv_2],$

$n = 3$**:** $\quad [[v_1, v_2], v_3] + (-1)^{(\tilde{v}_1+\tilde{v}_2)\tilde{v}_3}[[v_3, v_1], v_2] + (-1)^{\tilde{v}_1(\tilde{v}_2+\tilde{v}_3)}[[v_2, v_3], v_1] = -d\mu_3(v_1, v_2, v_3) - \mu_3(dv_1, v_2, v_3) - (-1)^{\tilde{v}_1}\mu_3(v_1, dv_2, v_3) - (-1)^{\tilde{v}_1+\tilde{v}_2}\mu_3(v_1, v_2, dv_3),$

where we denoted $dv_1 := \mu_1(v_1)$ and $[v_1, v_2] := \mu_2(v_1, v_2)$.

Therefore, L_∞-algebras with $\mu_k = 0$ for $k \geq 3$ are nothing but the usual differential Lie superalgebras with the differential μ_1 and the Lie bracket μ_2. If, furthermore, $\mu_1 = 0$, one gets the class of usual Lie superalgebras.

2.4.1. Odd L_∞-algebras. To make the above picture consistent with the choices made in Sect. 2.1, we should change the parity of $\mathfrak{h}$. Hence we shall work from now on with $\mathfrak{g} := \Pi\mathfrak{h}$ and denote

$$\mu_n(v_1 \bullet v_2 \bullet \ldots \bullet v_n) := \Pi\mu_n(\Pi v_1, \Pi v_2, \ldots, \Pi v_n), \quad \forall v_1, \ldots, v_n \in \mathfrak{g},$$

for all $n \geq 1$. This change of grading also unveils, through the following three observations, a rather compact image of the L_∞-structure itself:

1) The vector superspace $\odot^\bullet \mathfrak{g} = \bigoplus_{n=1}^\infty \odot^n \mathfrak{g}$ has a natural structure of cosymmetric *coalgebra*,

$$\Delta(w_1 \odot \ldots \odot w_n) = \sum_{i=1}^{n} \sum_{\sigma \in Sh(i,n)} e(\sigma) \left(w_{\sigma(1)} \odot \ldots \odot w_{\sigma(i)}\right) \otimes \left(w_{\sigma(i+1)} \odot \ldots \odot w_{\sigma(n)}\right).$$

2) Every coderivation of this coalgebra, i.e. an odd map $Q : \odot^\bullet \mathfrak{g} \to \odot^\bullet \mathfrak{g}$ satisfying $\Delta \circ Q = (Q \otimes \mathrm{Id} + \mathrm{Id} \otimes Q) \circ \Delta$, is equivalent to an arbitrary series of odd linear maps, $\mu_n : \odot^n \mathfrak{g} \to \mathfrak{g}$.

3) A codifferential $Q = \{\mu_*\}$ is a differential, i.e. $Q^2 = 0$, if and only if μ_n satisfy the higher order Jacobi identities.

In conclusion, an (odd) L_∞-structure on $\mathfrak{g}$ is equivalent to a codifferential on $(\odot^\bullet \mathfrak{g}, \Delta)$.

2.4.2. L_∞-morphisms, part II. Given two L_∞-algebras, $(\mathfrak{g}, \mu_*)$ and $(\mathfrak{g}', \mu'_*)$. A L_∞-morphism F from the first one to the second is, by definition, a differential coalgebra homomorphism

$$F : (\odot^\bullet \mathfrak{g}, \Delta, Q) \longrightarrow (\odot^\bullet \mathfrak{g}', \Delta, Q'),$$

i.e. a linear map $F : \odot^\bullet \mathfrak{g} \to \odot^\bullet \mathfrak{g}'$ satisfying $(F \otimes F) \circ \Delta = \delta' \circ F$ and $F \circ Q = Q' \circ F$. The first of these equations says that F is completely determined by a set of linear maps $F_n : \odot^n \mathfrak{g} \to \mathfrak{g}'$ of parity $\tilde{n} + \tilde{1}$, while the second one imposes on these F'_n a sequence of linear equations. If both input and output of F are usual differential Lie superalgebras, these equations are presicely the ones written down in Sect. 2.3.1.

A L_∞-morphism $F : (\mathfrak{g}, \mu_*) \to (\mathfrak{g}', \mu'_*)$ is called a *quasi-isomorphism* if its first component $F_1 : \mathfrak{g} \to \mathfrak{g}'$ induces an isomorphism between cohomology groups of complexes $(\mathfrak{g}, \mu_1)$ and $(\mathfrak{g}', \mu'_1)$. It is called a *homotopy of the L_∞-algebras*, if $F_1 : \mathfrak{g} \to \mathfrak{g}'$ is an isomorphism of underlying vector graded superspaces.

If the L_∞-morphism $F : (\mathfrak{g}, \mu_*) \to (\mathfrak{g}', \mu'_*)$ is a quasi-isomorphism, then, as was proved in [Ko2], there exists a L_∞-morphism $F' : (\mathfrak{g}', \mu'_*) \to (\mathfrak{g}, \mu_*)$ which induces the inverse isomorphism between cohomology groups of complexes $(\mathfrak{g}, \mu_1)$ and $(\mathfrak{g}', \mu'_1)$.

Two L_∞-morphisms, $F, G : (\mathfrak{g}, \mu_*) \to (\mathfrak{g}', \mu'_*)$, are said to be *homotopy equivalent* if there is an odd linear map $h : \odot^\bullet \mathfrak{g} \to \mathfrak{g}'$ such that

$$\Delta' \circ h = F \otimes h + h \otimes G, \quad \text{and} \quad F = G + Q' \circ h + h \circ Q.$$

This map is completely determined by a set of linear maps, $\{h_n : \odot^n \mathfrak{g} \to \mathfrak{g}', \tilde{h_n} = \tilde{n}, n = 1, 2, \ldots\}$, and is called a *homotopy of morphisms*. The resulting homotopy relation on the set of all L_∞-morphisms from $(\mathfrak{g}, \mu_*)$ to $(\mathfrak{g}', \mu'_*)$ is an equivalence relation.

2.4.3. A geometric interpretation of a L_∞-algebra $\mathfrak{g}$. The dual of the free cocommutative coalgebra $\odot^\bullet \mathfrak{g}$ can be identified with the algebra of formal power series on the vector superspace $\mathfrak{g}$ viewed as a formal pointed supermanifold (to emphasize this change of thought we denote the supermanifold structure on $\mathfrak{g}$ by $M_\mathfrak{g}$). With this identification, the L_∞-structure μ_* on $\mathfrak{g}$, that is the codifferential Q on $\odot^\bullet \mathfrak{g}$, goes into an odd vector field Q on $M_\mathfrak{g}$ satisfying [Ko2]

a) $Q^2 = 0$,

b) $QI \subset I$,

where I is the ideal of the distinguished point, $0 \in M_\mathfrak{g}$. (An odd vector field on a formal pointed superspace satisfying the above two conditions is usually called *homological*.)

A L_∞-morphism F between two L_∞-algebras $(\mathfrak{g}, \mu_*)$ and $(\mathfrak{g}', \mu'_*)$ is nothing but a Q-equivariant map between the associated formal pointed homological supermanifolds, $(M_\mathfrak{g}, Q)$ and $(M_{\mathfrak{g}'}, Q')$.

2.5. A modified deformation functor. For a general dLie-algebra $\mathfrak{g}$, the classical deformation functor $\mathsf{Def}^*_\mathfrak{g}$ is not representable by a *smooth* versal moduli space. At best one can use Kuranishi technique to represent $\mathsf{Def}^*_\mathfrak{g}$ by a singular analytic space. There is, however, a simple geometric way to keep track of versality and smoothness. The idea is as follows.

First, we extend the input category used in the construction of $\mathsf{Def}^*_\mathfrak{g}$ in Sect. 2.3 to a category of *differential* Artin superalgebras whose Obs are pairs, $(\mathcal{B}, \partial)$, consisting of an Artin superalgebra $\mathcal{B}$ together with a differential $\partial : \mathcal{B} \to \mathcal{B}$ satisfying $\partial m_\mathcal{B} \subset m_\mathcal{B}^2$, and whose Mors are morphisms of Artin superalgebras commuting with the differentials. A $\mathbb{Z}$-graded version of this definition would require ∂ to have degree $+1$.

Second, to the "controlling" differential Lie algebra $\mathfrak{g}$ we associate a new deformation functor,

$$
\begin{array}{rccc}
\text{Деф}^*_\mathfrak{g} : & \left\{ \begin{array}{c} \text{the category of differential} \\ \text{Artin superalgebras} \end{array} \right\} & \longrightarrow & \{\text{the category of sets}\} \\
& (\mathcal{B}, \partial) & \longrightarrow & \text{Деф}^*_\mathfrak{g}(\mathcal{B}, \partial)
\end{array}
$$

by setting

$$
\text{Деф}^*_\mathfrak{g}(\mathcal{B}, \partial) = \left\{ \Gamma \in (\mathfrak{g} \otimes m_\mathcal{B})_{\tilde{0}} \mid d\Gamma + \vec{\partial}\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] = 0 \right\} / \exp{(\mathfrak{g} \otimes m_\mathcal{B})_{\tilde{1}}}.
$$

Here the quotient is taken with respect to the following representation of the gauge group,

$$
\Gamma \to \Gamma^g = e^{\mathrm{ad}_g}\Gamma - \frac{e^{ad_g} - 1}{\mathrm{ad}_g}(d + \vec{\partial})g, \quad \forall g \in (\mathfrak{g} \otimes m_\mathcal{B})_{\tilde{1}}.
$$

On the subcategory $(\mathcal{B}, 0)$ the deformation functor $\text{Деф}^*_\mathfrak{g}$ coincides precisely with the classical one $\mathsf{Def}^*_\mathfrak{g}$.

2.5.1. Remark. If, for a derivation $\partial : \mathcal{B} \to \mathcal{B}$, an element $\Gamma \in (\mathfrak{g} \otimes m_{\mathcal{B}})_{\tilde{0}}$ satisfies the equation (which we sometimes call the *Master equation*),

$$d\Gamma + \vec{\partial}\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] = 0,$$

then

$$\begin{aligned} 0 &= d\left(d\Gamma + \vec{\partial}\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] \right) \\ &= -\vec{\partial} d\Gamma + [d\Gamma \bullet \Gamma] \\ &= -\vec{\partial} d\Gamma - [\vec{\partial}\Gamma \bullet \Gamma] - \frac{1}{2}\left[[\Gamma \bullet \Gamma] \bullet \Gamma\right] \\ &= -\vec{\partial}\left(d\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] \right) \\ &= -\vec{\partial}^2 \Gamma, \end{aligned}$$

motivating our assumption above that ∂ is a differential in $\mathcal{B}$ rather than merely a derivation.

2.5.2. L_∞-extension of Деф*. This extension will be used later only as a technical tool in the study of $\text{Деф}^*_{\mathfrak{g}}$ for usual differential Lie superalgebras $\mathfrak{g}$.

Given a L_∞-algebra $(\mathfrak{g}, \mu_*)$, we define,

$$\begin{array}{rccc} \text{Деф}^*_{\mathfrak{g}} : & \left\{ \begin{array}{c} \text{the category of differential} \\ \text{Artin superalgebras} \end{array} \right\} & \longrightarrow & \{\text{the category of sets}\} \\ & (\mathcal{B}, \partial) & \longrightarrow & \text{Деф}^*_{\mathfrak{g}}(\mathcal{B}, \partial) \end{array}$$

by setting

$$\text{Деф}^*_{\mathfrak{g}}(\mathcal{B}, \partial) = \left\{ \Gamma \in (\mathfrak{g} \otimes m_{\mathcal{B}})_{\tilde{0}} \mid \vec{\partial}\Gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{n!} \mu_n(\Gamma \bullet \ldots \bullet \Gamma) \right\} / \sim$$

Here the quotient is taken with respect to the gauge equivalence, $\sim$, which is best described using the following geometric model of the Деформation functor.

In $\mathsf{Category}^{op}$, both the differential Artin superalgebra, $(\mathcal{B}, \partial)$, and the L_∞-algebra, $(\mathfrak{g}, \mu_*)$, are represented by formal pointed analytic homological superspaces, $(M_{\mathcal{B}}, 0, \partial)$ and, respectively, $(M_{\mathfrak{g}}, 0, Q)$. Then the set

$$S = \left\{ \Gamma \in (\mathfrak{g} \otimes m_{\mathcal{B}})_{\tilde{0}} \mid \vec{\partial}\Gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{n!} \mu_n(\Gamma \bullet \ldots \bullet \Gamma) \right\}$$

is just the set of all formal maps of pointed supermanifolds,

$$\Gamma : (M_{\mathcal{B}}, 0) \longrightarrow (M_{\mathfrak{g}}, 0),$$

satisfying the equivariency condition

$$d\Gamma(\partial) = \Gamma^*(Q).$$

The latter is precisely the (L_∞-generalization of) the Master equation.

Both superpaces, $(M_{\mathcal{B}}, 0, \partial)$ and $(M_{\mathfrak{g}}, 0, Q)$, are foliated by integrable distributions,

$$\begin{aligned} \mathcal{D}_\partial &:= \{X \in TM_{\mathcal{B}} \mid X = [\partial, Y] \text{ for some } Y \in TM_{\mathcal{B}}\}, \\ \mathcal{D}_Q &:= \{X' \in TM_{\mathfrak{g}} \mid X' = [Q, Y'] \text{ for some } Y' \in TM_{\mathfrak{g}}\}, \end{aligned}$$

and $d\Gamma(\mathcal{D}_\partial) \subset \Gamma^*(\mathcal{D}_Q)$ for any $\Gamma \in S$. Hence any such Γ defines a map, $\hat{\Gamma}$, through the following commutative diagram,

$$\begin{array}{ccc} M_{\mathcal{B}} & \xrightarrow{\Gamma} & M_{\mathfrak{g}} \\ \downarrow & & \downarrow \\ M_{\mathcal{B}}/\mathcal{D}_\partial & \xrightarrow{\hat{\Gamma}} & M_{\mathfrak{g}}/\mathcal{D}_Q \end{array}$$

We say that two elements in S are *gauge equivalent*, $\Gamma_1 \sim \Gamma_2$, if $\hat{\Gamma}_1 = \hat{\Gamma}_2$. Infinitesimally, the gauge equivalence is given by

$$\Gamma \sim \Gamma + (d + \vec{\partial})g - \sum_{n=2}^{\infty} \frac{(-1)^{n(n+1)/2}}{(n-1)!} \mu_n(g \bullet \Gamma \bullet \ldots \bullet \Gamma), \quad \forall g \in (\mathfrak{g} \otimes m_{\mathcal{B}})_{\tilde{1}}.$$

2.5.4. Remark. If, for a derivation $\partial : \mathcal{B} \to \mathcal{B}$, an element $\Gamma \in (\mathfrak{g} \otimes m_{\mathcal{B}})_{\tilde{0}}$ satisfies the Master equation,

$$\vec{\partial}\Gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{n!} \mu_n(\Gamma \bullet \ldots \bullet \Gamma),$$

then, using the higher Jacobi identities (as in 2.5.1), one gets an implication,

$$\vec{\partial}^2\Gamma = 0.$$

2.5.5. Basic Theorem of Деформation Theory. *Let* $(\mathfrak{g}_1, Q_1)$ *and* $(\mathfrak{g}_2, Q_2)$ *be two* L_∞*-algebras (in partucilar, dLie-algebras). An* L_∞*-morphism* $F = \{F_n\} : \mathfrak{g}_1 \to \mathfrak{g}_2$ *defines a natural transformation of the functors,*

$$\begin{array}{rccc} F_* : & \text{Деф}^*_{\mathfrak{g}_1} & \longrightarrow & \text{Деф}^*_{\mathfrak{g}_2} \\ & \Gamma & \longrightarrow & F_*(\Gamma) := \sum_{n=1}^{\infty} \frac{1}{n!} F_n(\Gamma, \ldots, \Gamma). \end{array}$$

Moreover, if F *is a quasi-isomorphism, then* F_* *is an isomorphism.*

Proof. Assume $\Gamma \in (\mathfrak{g}_1 \otimes m_{\mathcal{B}})_{\tilde{0}}$ satisfies the Master equation,

$$d\Gamma(\partial) = \Gamma^*(Q_1).$$

The L_∞-morphism F, when viewed as map, $M_{\mathfrak{g}_1} \to M_{\mathfrak{g}_2}$, of pointed formal manifold, satisfies,

$$dF(Q_1) = F^*(Q_2).$$

Hence $F_*(\Gamma)$, which the same as $F \circ \Gamma$, obviously satisfies the Master equation in $(\mathfrak{g}_2, Q_2)$. $\square$

2.5.6. Smoothness Theorem. *The deformation functor* Деф^* *is unobstructed, i.e. for any dLie-algebra* $\mathfrak{g}$ *with finite-dimensional cohomology* $\mathbf{H}(\mathfrak{g})$*, the functor* $\text{Деф}^*_{\mathfrak{g}}$ *is versally representable by a smooth pointed formal* $\dim \mathbf{H}(\mathfrak{g})$*-dimensional homological supermanifold* $(\mathcal{H}_{\mathfrak{g}}, \partial)$*.*

Moreover, the diffeomorphism class of $(\mathcal{H}_{\mathfrak{g}}, \partial)$ *is an invariant of* $\mathfrak{g}$*.*

Proof. It is enough to show that

(i) there exists a versal element $\Gamma \in k[[t]] \otimes \mathfrak{g}$ and a differential $\partial : k[[t]] \to k[[t]]$ satisfying the Master equation

$$d\Gamma + \vec{\partial}\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] = 0, \tag{2}$$

(ii) the differential ∂, when viewed as a vector field on the cohomological supermanifold $\mathcal{H}_{\mathfrak{g}}$, is an invariant of $\mathfrak{g}$.

Unless $\mathfrak{g}$ is formal, there is no quasi-isomorphism from $(\mathfrak{g}, [\ \bullet\ ], d)$ to its cohomology $(\mathbf{H}(\mathfrak{g}), [\ \bullet\ ]_{\text{ind}}, 0)$. However, there *always* exists a L_∞-structure, $\{\mu_*, \text{with } \mu_1 = 0\}$, on $\mathbf{H}(\mathfrak{g})$ which is quasi-isomorphic, via some L_∞-morphism F, to $(\mathfrak{g}, [\ \bullet\ ], d)$. Moreover, this structure is defined uniquely up to a homotopy.

Setting $\Gamma_{[1]} = \sum_i t^i [e_i]$, in the notations of Sect. 2.2, we define a derivation, $\partial : k[[t]] \to k[[t]]$, by the formula

$$\vec{\partial}\Gamma_{[1]} = \sum_{n=2}^{\infty} \frac{(-1)^{n(n+1)/2}}{n!} \mu_n(\Gamma_{[1]} \bullet \ldots \bullet \Gamma_{[1]}).$$

By Remark 2.5.4, this derivation satisfies $\partial^2 = 0$. Hence $(\Gamma_{[1]}, \partial)$ is a versal solution of the Master equation in $(\mathbf{H}(\mathfrak{g}), \mu_*)$, while $(F_*(\Gamma_{[1]}), \partial)$ is, by Theorem 2.5.5, a versal solution of the Master equation in $(\mathfrak{g}, [\ \bullet\ ], d)$. This proves claim (i).

The L_∞-structure $\{\mu_*\}$ on $\mathbf{H} \simeq \mathbb{R}^{p|q}$ is well-defined only up to a homotopy, $\{\eta_{(n)} : \odot^n \mathbf{H} \to \mathbf{H}, n \geq 2, \widetilde{\eta_n} = \tilde{0}\}$. Is is easy to check that a homotopy change of the induced L_∞-structure

$$\mu_* \xrightarrow{\eta_{(*)}} \mu'_*,$$

does change the differential,

$$\partial \longrightarrow \partial',$$

but in a remarkably geometric way,

$$\partial' = d\eta(\partial),$$

where $\eta : \mathbb{R}^{p|q} \to \mathbb{R}^{p|q}$ is just a formal change of coordinates

$$t^i \to t'^i = t^i + \sum_{j,k} \pm \eta^i_{(2)\,jk} t^j t^k + \sum_{j,k,l} \pm \eta^i_{(3)\,jkl} t^j t^k t^l + \ldots \ .$$

Put another way, a homotopy change of μ_* affects only the coordinate representation of the vector field ∂ on $\mathcal{H}_{\mathfrak{g}}$. As a geometric entity, this is an invariant of $\mathfrak{g}$. $\square$

2.5.7. Corollary. *The derived category of L_∞-algebras with finite-dimensional cohomology is canonically equivalent to the (purely geometric) category of pointed formal homological supermanifolds, $(\mathcal{H}, \partial, 0)$, with ∂ satisfying $\partial I \subset I^2$, I being the ideal of the distinguished point $0 \in \mathcal{H}$.*

Proof. It is well-known that each quasi-isomorphism of L_∞-algebras is a homotopy equivalence. Thus the derived category of L_∞-algebras is equivalent to the their homotopy category. Then the required statement follows immediately from an observation made in the proof of Theorem 2.5.6 that, for any homotopy class of L_∞-algebras $[\mathfrak{g}]$, the associated homotopy class of L_∞-structures induced on the cohomology $\mathbf{H}(\mathfrak{g})$ is isomorphically mapped into one and the same homological manifold $(\mathcal{H}_{\mathfrak{g}}, \partial, 0)$. $\square$

2.5.8. Remarks. (i) The origin of the vector field ∂ in Theorem 2.5.6 can be traced back to Chen's power series connection [C]. This will be made apparent in Section 4 where we give another, perturbative, proof of the above Theorem. From now on we call ∂ the *Chen's differential* or *Chen's vector field.*

(ii) The higher order tensors μ_* induced on the cohomology $\mathbf{H}(\mathfrak{g})$ by a L_∞-quasi-isomorphism from a dLie-algebra $\mathfrak{g}$ coincide precisely with the Massey products when they are well-defined and univalued. Thus the Chen's differential gives a compact (and invariant) representation of the homotopy class of Massey products.

2.5.9. Extended Kuranishi moduli space. Since the Chen's vector field ∂ on $\mathcal{H}$ is homological, the distribution

$$\mathcal{D}_\partial = \{X \in \mathcal{T}_\mathcal{H} \mid X = [\partial, Y\,] \text{ for some } Y \in \mathcal{T}_\mathcal{H}\}$$

is integrable (cf. Sect. 2.5.3). Indeed, the Jacobi identities imply

$$[[\partial, X], [\partial, Y]] = [\partial, [X, [\partial, Y]]]\,.$$

Consider an affine subscheme,

$$\text{“}\mathsf{Zeros}(\partial)\text{”} := \mathsf{Spec}\ k[[t]]/ < \partial t^1, \ldots, \partial t^{p+q} >,$$

of zeros of the vector field ∂. The distribution $\mathcal{D}_\partial$ is tangent to “$\mathsf{Zeros}(\partial)$” since $[\partial, [\partial, X]] = 0$. We define the *extended Kuranishi space*, $\mathcal{K}_{\mathfrak{g}}$, as the so called non-linear homology [BK, Ma2, Ba] of the Chen differential, i.e. as the quotient “$\mathsf{Zeros}(\partial)$”$/\mathcal{D}_\partial$. (For our purposes it is enough to understand the latter as $\mathsf{Spec}\, k[[t]] \cap \mathsf{Ker}\, \partial / < \partial t^1, \ldots, \partial t^{p+q} >$.) This passage from $(\mathcal{H}, \partial)$ to $\mathcal{K}_{\mathfrak{g}}$ establishes a clear link between the unobstructed deformation functor Деф$^*_{\mathfrak{g}}$ and the classical one $\mathsf{Def}^*_{\mathfrak{g}}$.

Kuranishi spaces, $\mathsf{K}_{\mathfrak{g}}$, originally emerged [Ku, GM] in the context of the deformation functor $\mathrm{Def}^0_{\mathfrak{g}}$ associated to a cohomologically split $\mathbb{Z}$-graded dLie algebra $\mathfrak{g}$. It is not hard to see that $\mathsf{K}_{\mathfrak{g}}$ is a proper subspace of the extended Kuranishi space $\mathcal{K}_{\mathfrak{g}}$. We will see below that for a rich class of dLie algebras $\mathfrak{g}$ — the so-called (homotopy) Gerstenhaber algebras — the tangent sheaves to the smooth parts, $\mathcal{K}_{\text{smooth}}$, of the associated extended Kuranishi spaces are canonically sheaves of associative algebras. This structure is not visible if working in the category of original (non-extended) Kuranishi spaces K only.

2.6. Cohomological splitting. It is very easy to compute Chen's differential once a cohomological decomposition of a dLie-algebra $\mathfrak{g}$ under investigation is chosen. The latter

means the data (i, p, Q), where $i : \mathbf{H}(\mathfrak{g}) \longrightarrow \mathfrak{g}$ is a linear injection, $p : \mathfrak{g} \longrightarrow \mathbf{H}(\mathfrak{g})$ a linear surjection, and $Q : \mathfrak{g} \longrightarrow \mathfrak{g}$ an odd linear operator, all satisfying the conditions,

$$p \circ i = \mathrm{Id} = i \circ p \oplus dQ \oplus Qd,$$

in $\mathrm{End}_k(\mathfrak{g})$.

Such a decomposition of $\mathfrak{g}$ often occurs in (complex) differential geometry [K, Ku], where typical dLie-algebras come equipped with a norm $|| \ ||$ and their cohomologies $\mathbf{H}$ get identified with harmonic subspaces, $\mathrm{Harm} := \mathsf{Ker}\, d \cap \mathsf{Ker}\, d^* \subset \mathfrak{g}$, d^* being the adjoint of d with respect to $|| \ ||$. The operator Q is then Gd^*, where G is the Green function of the Laplacian $\Box = dd^* + d^*d$. In this situation the formal power series solution, Γ, of the Master equation as well as the associated Chen's vector field can be chosen to be convergent inducing, thereby, the structure of analytic (rather than formal) homological supermanifold on $\mathcal{H}$.

It is not hard to check that, given a cohomological splitting of $\mathfrak{g}$, the pair, (Γ, ∂), given recursively by

$$\begin{aligned}
\Gamma_{[1]} &= \sum_i t^i e_i \in \mathsf{Ker}\, Q \cap \mathsf{Ker}\, d \\
\Gamma_{[2]} &= -\frac{1}{2} Q[\Gamma_{[1]}(t) \bullet \Gamma_{[1]}(t)], \\
\Gamma_{[3]} &= -\frac{1}{2} Q \left([\Gamma_{[1]}(t) \bullet \Gamma_{[2]}(t)] + [\Gamma_{[2]}(t) \bullet \Gamma_{[1]}(t)] \right), \\
&\cdots \\
\Gamma_{[n]} &= -\frac{1}{2} Q \left(\sum_{k=1}^{n-1} [\Gamma_{[k]}(t) \bullet \Gamma_{[n-k]}(t)] \right) \qquad (3) \\
&\cdots
\end{aligned}$$

and

$$\vec{\partial} p(\Gamma_{[1]}) := -\frac{1}{2} p\left([\Gamma \bullet \Gamma] \right),$$

give an explicit versal solution of the Master equation in $\mathfrak{g}$.

The above power series for Γ is well known in the classical Deformation Theory [K, Ku] where it plays a key role in constructing Kuranishi analytic moduli spaces. This series is essentially an inversion of the Kuranishi map [Ku] in the category of L_∞-algebras.

2.7. Formality and flat structures. If an algebra $(\mathfrak{g}, d, [\ \bullet\ ])$ is formal, then, as follows from the proof of Theorem 2.5.6, the associated homological supermanifold $(\mathcal{H}_{\mathfrak{g}}, \partial, 0)$ has a canonical flat structure ∇. In the associated flat coordinates t^i, the Chen's vector field ∂ has coefficients which are polynomials in t^i of order ≤ 2. (This observation can, in fact, be made into a geometric criterion of formality.) More precisely, the following is true.

2.7.1. Theorem. *For any formal dLie-algebra $\mathfrak{g}$ there is a canonical isomorphism of sets,*

$$\left\{ \begin{array}{c} \text{Flat structures on } \mathcal{H}_{\mathfrak{g}} \text{ such that } \nabla_X \nabla_Y \nabla_Z \partial = 0 \\ \text{for any horizontal vector fields } X, Y \text{ and } Z \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Homotopy classes} \\ \text{of formality maps} \end{array} \right\}$$

Let us choose a basis, $\{s_i, i = 1, \ldots, p+q\}$, in the $(p|q)$-dimensional vector superspace $\mathbf{H}(\mathfrak{g})$, and let $\{t^i\}$ be the associated linear coordinates. We shall need, for a short time, a category $\mathsf{Artin}_{k[[t]]}$ consisting of Artin superalgebras of the form

$$\mathcal{A}_N := k[[t^1, \ldots, t^{p+q}]] / < (t^1)^{N_1} \cdots (t^{p+q})^{N_{p+q}} > .$$

Denoting the maximal ideal of such a superalgebra by m_N, we set $(\mathfrak{g} \otimes m_N)_{\mathsf{versal}}$ to be a linear subspace in $\mathfrak{g} \otimes m_N$ consisting of even elements, Γ, satisfying $\Gamma \bmod m_N = 0$, $\Gamma \bmod m_N^2 \in \mathsf{Ker}\, d$, and

$$\left(\Gamma \bmod m_N^2\right) \bmod \mathsf{Im}\, d = \sum_{i=1}^{p+q} t^i s_i.$$

This set is invariant under the action of the gauge group $\exp(\mathfrak{g} \otimes m_N)_{\tilde{1}}$ (see Sect. 2.5).

2.7.2. Lemma. *For any formal dLie-algebra $\mathfrak{g}$ there is a canonical isomorphism of sets,*

$$\lim_{\longleftarrow} \frac{\left\{\Gamma \in (\mathfrak{g} \otimes m_N)_{\mathsf{versal}} \mid d\Gamma + \vec{\partial}\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] = 0\right\}}{\textsf{gauge group}} = \frac{\textsf{Formality maps}}{\textsf{homotopy equivalence}},$$

where the projective limit is taken over the category $\mathsf{Artin}_{k[[t]]}$.

Proof. For any $\mathcal{A}_N \in \mathsf{Artin}_{k[[t]]}$, the Master equation in the Lie algebra $(\mathbf{H}(\mathfrak{g}) \otimes \mathcal{A}_N, [\ \bullet\ ]_{\mathrm{ind}})$ has a canonical versal solution,

$$\Gamma_0 = \sum_{i=1}^{p+q} t^i s_i, \quad \partial = \sum_{i,j,k=1}^{p+q} (-1)^{\tilde{j}(\tilde{i}+1)} t^i t^j C_{ij}^k \frac{\partial}{\partial t^k},$$

where C_{ij}^k are the structure constants of $[\ \bullet\ ]_{\mathrm{ind}}$.

If $\mathfrak{g}$ is formal, and $F = \{F_n : \odot^n \mathbf{H}(\mathfrak{g}) \to \mathfrak{g}, n = 1, 2, \ldots\}$ is a formality map, then, by Theorem 2.5.5,

$$\Gamma := \sum_{n=1}^{\infty} \frac{1}{n!} F_n(\Gamma_0, \ldots, \Gamma_0), \quad \textsf{the same } \partial,$$

is a versal solution of the Master equation in $\mathfrak{g} \otimes \mathcal{A}_N$.

It is easy to check that an arbitrary homotopy change,

$$F \to F^h,$$

of the formality map, say the one induced by a set of linear maps $h = \{h_n : \odot^n \mathbf{H}(\mathfrak{g}) \to \mathfrak{g}, \tilde{h}_n = \tilde{n}, n = 1, 2, \ldots\}$, change the versal solution Γ into a *gauge equivalent* one,

$$\Gamma \to \Gamma^g,$$

where

$$g = \sum_i h_1(e_i) t^i + \sum_{i,j} \pm h_2(e_i, e_j) t^i t^j + \sum_{i,j,k} \pm h_3(e_i, e_j, e_k) t^i t^j t^k + \ldots\ .$$

Hence there is a canonical map,

$$\begin{array}{ccc} \frac{\textsf{Formality maps}}{\textsf{homotopy equivalence}}, & \longrightarrow & \frac{\left\{\Gamma\in(\mathfrak{g}\otimes m_N)_{\textsf{versal}} | d\Gamma+\vec{\partial}\Gamma+\frac{1}{2}[\Gamma\bullet\Gamma]=0\right\}}{\textsf{gauge group}} \\ F=\{F_n : \odot^n \mathbf{H}(\mathfrak{g})\rightarrow \mathfrak{g}, n=1,2,\ldots\}/\sim & \longrightarrow & \sum_{n=1}^{\infty}\frac{1}{n!}F_n(\Gamma_0,\ldots,\Gamma_0)/\sim \end{array}$$

which implies (almost immediately) the desired result. □

2.7.3. Proof of Theorem 2.7.1. Let Diff_0 be the group of all formal diffeomorphisms of $\mathcal{H}_{\mathfrak{g}} = \mathbf{H}(\mathfrak{g})$ into itself preserving the origin, and set

$$\mathsf{Diff}_{0,\partial} := \left\{\phi \in \mathsf{Diff}_0 \mid \phi_*(\partial) \text{ is quadratic in } t^i\right\}.$$

Note that $\mathsf{Diff}_{0,\partial} = \mathsf{Diff}_0$ if the Chen's vector field ∂ vanishes. In general,

$$GL(p+q) \subseteq \mathsf{Diff}_{0,\partial} \subseteq \mathsf{Diff}_0.$$

There is an obvious isomorphism,

$$\left\{\begin{array}{c} \textsf{Flat sructures on } \mathcal{H}_{\mathfrak{g}} \textsf{ such that } \nabla_X\nabla_Y\nabla_Z\partial = 0 \\ \textsf{for any horizontal vector fields } X, Y \textsf{ and } Z \end{array}\right\} = \frac{\mathsf{Diff}_{0,\partial}}{GL(p+q)}.$$

On the other hand, by Theorem 4.2.2 (see below),

$$\frac{\mathsf{Diff}_{0,\partial}}{GL(p+q)} = \lim_{\longleftarrow} \frac{\left\{\Gamma \in (\mathfrak{g}\otimes m_N)_{\textsf{versal}} \mid d\Gamma + \vec{\partial}\Gamma + \frac{1}{2}[\Gamma\bullet\Gamma] = 0\right\}}{\textsf{gauge group}}.$$

The final link in the chain of canonical isomorphisms is provided by Lemma 2.7.2. □

2.7.4. Corollary. *For any compact Calabi-Yau manifold, there is a canonical isomorphism of sets,*

$$\left\{\begin{array}{c} \textsf{Flat connections on Barannikov} - \textsf{Kontsevich's} \\ \textsf{moduli space of extended complex structures} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{c} \textsf{Homotopy classes} \\ \textsf{of formality maps} \end{array}\right\}$$

3 Homotopy Gerstenhaber algebras

3.1. Differential Gerstenhaber algebras. A *differential Gerstenhaber algebra*, or shortly, a *dG-algebra*, is the data $(\mathfrak{g}, d, [\ \bullet\ ],\ \cdot\ )$ where

(i) $(\mathfrak{g}, d, [\ \bullet\ ])$ is a $\mathbb{Z}$-graded dLie-algebra as defined in Sect. 2.2.1;

(ii) $(\mathfrak{g}, d,\ \cdot\ )$ is a differential $\mathbb{Z}$-graded associative algebra with the product

$$\begin{array}{rccc} \cdot : & \mathfrak{g}\otimes\mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & a\otimes b & \longrightarrow & a\cdot b, \end{array}$$

having degree 0;

(iii) the binary operations $[\,\bullet\,]$ and $\cdot$ satisfy the odd Poisson identity,

$$[a\bullet(b\cdot c)]=[a\bullet b]\cdot c+(-1)^{(\tilde{a}+1)\tilde{b}}b\cdot[a\bullet c],$$

for all homogeneous $a,b,c\in\mathfrak{g}$.

A dG-algebra is called *graded commutative* if such is the dot product.

The *identity* in $\mathfrak{g}$ is an even element e_0 such that $de_0=0$, $e_0\cdot a=a\cdot e_0=a$ and $[e_0\bullet a]=0$ for any $a\in\mathfrak{g}$. It defines a cohomology class $[e_0]$ in $\mathbf{H}$, and a constant vector field on $\mathcal{H}$ which we denote by e.

3.1.1. Remark. If $\mathfrak{g}$ is a unital dG-algebra, then a versal solution, Γ, of the Master equation (2) in $\mathfrak{g}$ can (and will) be always normalized in such a way that

$$\vec{e}\,\Gamma=e_0.$$

3.1.2. Differential Gerstenhaber-Batalin-Vilkovisky algebras. Let $(\mathfrak{g},\ \cdot\ )$ be a $\mathbb{Z}$-graded commutative associative algebra over a field k. Let us say that the zero operator, $0:\mathfrak{g}\rightarrow\mathfrak{g}$, is of order -1, and let us denote the linear operator, $x\rightarrow a\cdot x$, of left multiplication by an element $a\in\mathfrak{g}$ by l_a. A homogeneous linear operator, $D:\mathfrak{g}\rightarrow\mathfrak{g}$, is said to be an *operator of order* k if the operator $[D,l_a]$ is of order $k-1$ for any homogeneous a in $\mathfrak{g}$.

Assume now that $(\mathfrak{g},\ \cdot\ )$ comes equipped with

(i) a degree $+1$ linear operator, $d:\mathfrak{g}\rightarrow\mathfrak{g}$, of order 1, and

(ii) a degree -1 linear operator, $\Delta:\mathfrak{g}\rightarrow\mathfrak{g}$, of order 2,

satisfying the conditions,

$$d^2=0,\qquad \Delta^2=0,\qquad d\Delta+\Delta d=0.$$

In this case the data

$$(\mathfrak{g},d,\ \cdot\ ,[\,\bullet\,])$$

with

$$[a\bullet b]:=(-1)^{\tilde{a}}\Delta(a\cdot b)-(-1)^{\tilde{a}}(\Delta a)\cdot b-a\cdot(\Delta b),\qquad\forall a,b\in\mathfrak{g},$$

defines a dG-algebra [Ma2]. The dG-algebras arising in this way are often called *exact* or dGBV-algebras.

3.1.3. Example (complex manifolds). For any n-dimensional comlex manifold M the differential Lie algebra of Example 2.3.1,

$$\mathfrak{g}=\left(\Gamma(M,\Lambda^{\bullet}T_M\otimes\Lambda^{\bullet}\overline{T}^*_M),\ \bar{\partial}\ ,\ [\,\bullet\,]\right),$$

equipped with a supercommutative product,

$$\begin{array}{rccl}\wedge: & \Gamma(M,\Lambda^{i_1}T_M\otimes\Lambda^{j_1}\overline{T}^*_M)\times\Gamma(M,\Lambda^{i_2}T_M\otimes\Lambda^{j_2}\overline{T}^*_M) & \rightarrow & \Gamma(M,\Lambda^{i_1+i_2}T_M\otimes\Lambda^{j_1+j_2}\overline{T}^*_M)\\ & X_1\otimes\overline{w}_1\times X_2\otimes\overline{w}_2 & \rightarrow & (-1)^{\tilde{j}_1\tilde{i}_2}(X_1\wedge X_2)\otimes\overline{w}_1\wedge\overline{w}_2,\end{array}$$

is a unital $\mathbb{Z}$-graded commutative dG-algebra.

If M admits a nowhere vanishing global holomorphic volume form, $\Omega \in \Gamma(M, \Omega^n_M)$, then the above dG-algebra is actually exact [Ti, To, BK] with Δ being the composition,

$$\Delta: \ \Lambda^i T_M \xrightarrow{i_\Omega} \Omega^{n-i}_M \xrightarrow{\partial} \Omega^{n-i+1}_M \xrightarrow{i_\Omega^{-1}} \Lambda^{i-1} T_M.$$

Here $i_\Omega : \Lambda^\bullet T_M \to \Omega^\bullet_M$ is the natural isomorphism given by contraction with the holomorphic volume form, and ∂ is the $(1,0)$-part of the de Rham operator.

3.1.4. Example (symplectic manifolds). For any symplectic manifold (M, ω) the dLie algebra of Example 2.3.5,

$$\mathfrak{g} = (\Gamma(M, \Omega^\bullet_{\mathbb{R}}), \ d \ , \ [\ \bullet\ ]_\omega\,) \,,$$

together with a graded commutative product, $a \cdot b := a \wedge b$, is a unital $\mathbb{Z}$-graded dG-algebra. Moreover, it is a dGBV-algebra with the 2-nd order differential given by

$$\Delta|_{\Omega^k_{\mathbb{R}}} = (-1)^{k+1} * d * .$$

Here $* : \Omega^k_{\mathbb{R}} \to \Omega^{2m-k}_{\mathbb{R}}$ is the symplectic analogue of the Hodge duality operator defined by the condition, $\beta \wedge (*\alpha) = \langle \beta, \alpha \rangle \omega^m / m!$, with $\langle\, ,\, \rangle$ being the pairing between k-forms induced by the symplectic form.

3.1.5. Example (vector bundles). Let M be a complex manifold, and $\pi : E \to M$ a holomorphic vector bundle. There is a complex of $\mathbb{Z}$-graded sheaves $(\odot^\bullet E \otimes \Lambda^\bullet E^*, \Delta)$,

$$\ldots \xrightarrow{\Delta} \odot^{k+1} E \otimes \Lambda^{l+1} E^* \xrightarrow{\Delta} \odot^k E \otimes \Lambda^l E^* \xrightarrow{\Delta} \odot^{k-1} E \otimes \Lambda^{l-1} E^* \xrightarrow{\Delta} \ldots,$$

where the differential Δ is just the contraction, the $\mathbb{Z}$-grading is induced from the one on $\Lambda^\bullet E$, and we set $\odot^k E = \Lambda^k E^* = 0$ for $k < 0$. It is easy to see that Δ is a linear operator of oder 2 with respect to the natural supercommutative product,

$$\begin{array}{ccc} \odot^\bullet E \otimes \Lambda^\bullet E^* \, \times \, \odot^\bullet E \otimes \Lambda^\bullet E^* & \xrightarrow{\cdot} & \odot^\bullet E \otimes \Lambda^\bullet E^* \\ (a_1 \otimes b_1^*) \, \times (a_2 \otimes b_2^*) & \longrightarrow & (a_1 \odot a_2) \otimes (b_1^* \wedge b_2^*). \end{array}$$

Hence the data

$$\mathfrak{g} = \left(\bigoplus_k \mathfrak{g}^k, \mathfrak{g}^k := \bigoplus_{i+j=k} \Gamma(M, \odot^\bullet E \otimes \Lambda^i E^* \otimes \overline{\Omega}^j_M), \ \Delta \ , \ \bar{\partial} \right)$$

is a unital dGBV-algebra. It extends the dLie algebra of Example 2.3.6, as the following calculation shows.

Proposition. *The bracket* $[\ \bullet\ ]$ *on* $\odot^\bullet E \otimes \Lambda^\bullet E^*$*, when restricted to* $E \otimes E^*$*, coincides, up to a sign, with the usual commutator of morphisms.*

Proof. Let us consider a pair of germs, $C_1 = a_1 \otimes b_1^*$ and $C_2 = a_2 \otimes b_2^*$, in the same stalk of $E \otimes E^*$. Then

$$\begin{aligned}
[C_1 \bullet C_2] &= (-1)^{\tilde{C}_1} \Delta(C_1 \cdot C_2) - (-1)^{\tilde{C}_1} \Delta(C_1) \cdot C_2 - C_1 \cdot \Delta(C_2) \\
&= -\Delta\left((a_1 \odot a_2) \otimes (b_1^* \wedge b_2^*)\right) + \Delta(a_1 \otimes b_1^*) \cdot (a_2 \otimes b_2) - (a_1 \otimes b_1) \cdot \Delta(a_2 \otimes b_2^*) \\
&= -\langle a_1, b_1^* \rangle\, a_2 \otimes b_2^* + \langle a_1, b_2^* \rangle\, a_2 \otimes b_1^* - \langle a_2, b_1^* \rangle\, a_1 \otimes b_2^* \\
&\quad + \langle a_2, b_2^* \rangle\, a_1 \otimes b_1^* + \langle a_1, b_1^* \rangle\, a_2 \otimes b_2^* - \langle a_2, b_2^* \rangle\, a_1 \otimes b_1^* \\
&= \langle a_1, b_2^* \rangle\, a_2 \otimes b_1^* - \langle a_2, b_1^* \rangle\, a_1 \otimes b_2^* \\
&= -\left(C_1 C_2 - C_2 C_1\right),
\end{aligned}$$

where angular brackets stand for the usual pairing between a vector and a 1-form. □

There is a problem with the constructed dGBV-algebra — its cohomology may not be finite dimensional even for compact manifolds. It is can be resolved by passing to its dGBV-subalgebra,

$$\mathfrak{g}_E = \left(\bigoplus_k \mathfrak{g}^k, \mathfrak{g}^k := \bigoplus_{i+j=k} \Gamma(M, \odot^i E \otimes \Lambda^i E^* \otimes \overline{\Omega}^j_M),\ \Delta\ ,\ \bar{\partial} \right).$$

If necessary, the asymmetry of E and E^* can be eliminated by taking the tensor product $\mathfrak{g}_E \otimes \mathfrak{g}_{E^*}$.

3.1.6. Example (Hochschild cohomology). Let A be an associative algebra over a field k. The $\mathbb{Z}$-graded vector space of *Hochschild cochains*,

$$C^\bullet(A, A) := \bigoplus_{n=0}^{\infty} \mathrm{Hom}_k(A^{\otimes n}, A),$$

can be made into a *Hochschild complex* with the differential, $d : C^n(A, A) \to C^{n+1}(A, A)$, given by

$$\begin{aligned}
(df)(a_1 \otimes \ldots \otimes a_{n+1}) \;:=\;& a_1 f(a_2 \otimes \ldots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^I f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) \\
& + (-1)^{n+1} f(a_1 \otimes \ldots \otimes a_n) a_{n+1}
\end{aligned}$$

for any $f \in C^n(A, A)$.

One can define two binary operations, $C^\bullet(A, A) \otimes C^\bullet(A, A) \to C^\bullet(A, A)$, the degree 0 *dot product*,

$$(f \cdot g)(a_1 \otimes \ldots \otimes a_{k+l}) := (-1)^{kl} f(a_1 \otimes \ldots \otimes a_k) g(a_1 \otimes \ldots \otimes a_l), \ \ \forall f \in C^k(A, A), g \in C^l(A, A),$$

and the degree -1 *bracket*,

$$[f \bullet g] := f \circ g - (-1)^{(k+1)(l+1)} g \circ f,$$

where

$$(f \circ g)(a_1 \otimes \ldots \otimes a_{k+l-1}) := \sum_{i=1}^{k-1} (-1)^{(i+1)(l+1)} f(a_1 \otimes \ldots \otimes a_i \otimes g(a_{i+1} \otimes \ldots \otimes a_{i+l}) \otimes \ldots \otimes a_{k+l-1}).$$

These two make the Hochschild complex into a $\mathbb{Z}$-graded differential associative algebra and a differential (odd) Lie algebra respectively. Though $(C^\bullet(A,A), d, [\ \bullet\ ],\ \cdot\ )$ is not a dG-algebra, it is a remarkabale fact that the associated Hochschild cohomology,

$$\mathrm{Hoch}^\bullet(A,A) = \frac{\mathsf{Ker}\, d}{\mathsf{Im}\, d},$$

carries the structure of graded commutative dG-algebra with respect to the naturally indiced dot product, Lie bracket and the zero differential.

3.2. A_∞-algebras. A *strong homotopy algebra*, or shortly A_∞-*algebra*, is by definition a vector superspace V equipped with linear maps,

$$\begin{array}{rccc} \mu_k : & \otimes^k V & \longrightarrow & V \\ & v_1 \otimes \ldots \otimes v_k & \longrightarrow & \mu_k(v_1, \ldots, v_k), \qquad k \geq 1, \end{array}$$

of parity $\tilde{k}$ satisfying, for any $n \geq 1$ and any $v_1, \ldots, v_n \in V$, the following *higher order associativity conditions*,

$$\sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^r \mu_k\left(v_1, \ldots, v_j, \mu_l(v_{j+1}, \ldots, v_{j+l}), v_{j+l+1}, \ldots, v_n\right) = 0, \tag{4}$$

where $r = \tilde{l}(\tilde{v}_1 + \ldots + \tilde{v}_j) + \tilde{j}(\tilde{l}-1) + (\tilde{k}-1)\tilde{l}$ and $\tilde{v}$ denotes the parity of $v \in V$.

Denoting $dv_1 := \mu_1(v_1)$ and $v_1 \cdot v_2 := \mu_2(v_1, v_2)$, we can spell the first three conditions from the above infinite series as follows,

$n = 1$**:** $\quad d^2 = 0,$

$n = 2$**:** $\quad d(v_1 \cdot v_2) = (dv_1) \cdot v_2 + (-1)^{\tilde{v}_1} v_1 \cdot (dv_2),$

$n = 3$**:** $\quad v_1 \cdot (v_2 \cdot v_3) - (v_1 \cdot v_2) \cdot v_3 = d\mu_3(v_1, v_2, v_3) + \mu_3(dv_1, v_2, v_3) + (-1)^{\tilde{v}_1} \mu_3(v_1, dv_2, v_3) + (-1)^{\tilde{v}_1 + \tilde{v}_2} \mu_3(v_1, v_2, dv_3),$

Therefore A_∞-algebras with $\mu_k = 0$ for $k \geq 3$ are nothing but the differential associative superalgebras with the differential μ_1 and the associative multiplication μ_2. If, furthermore, $\mu_1 = 0$, one recovers the usual associative superalgebras.

There is a (finer) $\mathbb{Z}$-graded version of the above definition in which the maps μ_n are required to be homogeneous (usually of degree $n-2$) with respect to the given $\mathbb{Z}$-grading on V.

3.2.1. Identity. An element e in the A_∞-algebra is called the *identity* if $\mu_1(e) = 0$, $\mu_2(e, v) = \mu_2(v, e) = v$ and $\mu_n(v_1, \ldots, e, \ldots, v_{n-1}) = 0$ for all $n \geq 3$ and arbitrary $v, v_1, \ldots, v_{n-1} \in V$.

3.2.2. Homotopy classes of A_∞-algebras. For a pair of A_∞-algebras, (V, μ_*) and $(\tilde{V}, \tilde{\mu}_*)$, there is a natural notion of a A_∞-morphism from V to $\tilde{V}$ which is, by definition, a set of linear maps

$$F = \{f_n : V^{\otimes n} \longrightarrow \tilde{V},\ n \geq 1\},$$

of parity $\tilde{n}+1$ (or of degree $1-n$ in the $\mathbb{Z}$-graded case) which satisfy

$$\sum_{1\leq k_1<k_2<\ldots<k_i=n} (-1)^{i+r}\tilde{\mu}_i(f_{k_1}(v_1,\ldots,v_{k_1}), f_{k_2-k_1}(v_{k_1+1},\ldots,v_{k_2}),\ldots, f_{n-k_{i-1}}(v_{k_{i-1}+1},\ldots,v_n))$$

$$= \sum_{k+l=n+1}\sum_{j=0}^{k-1}(-1)^{l(\tilde{v}_1+\ldots+\tilde{v}_j+n)+j(l-1)} f_k(v_1,\ldots,v_j,\mu_l(v_{j+1},\ldots,v_{j+l}),v_{j+l+1},\ldots,v_n).$$

The first three floors in the above infinite tower are

$n=1$: $\tilde{\mu}_1 = \mu_2 =: d$,

$n=2$: $\tilde{\mu}_2(v_1,v_2) = \mu_2(v_1,v_2) + (df_2)(v_1,v_2)$,

$n=3$: $\tilde{\mu}_3(v_1,v_2,v_3) + \tilde{\mu}_2(f_2(v_1,v_2),v_3) - (-1)^{\tilde{v}_1}\tilde{\mu}_2(a_1, f_2(v_3,v_4)) =$
$\mu_3(v_1,v_2,v_3) - f_2(\mu_2(v_1,v_2),v_3) + f_2(v_1,\mu_2(v_3,v_4)) + (df_3)(v_1,v_2,v_3)$,

where we naturally extended the differential $d: V\rightarrow V$ to $d:\otimes^k V^*\otimes V\rightarrow \otimes^k V^*\otimes V$ (so that, for example, $(df_2)(v_1,v_2) = df_2(v_1,v_2) + f_2(dv_1,v_2) + (-1)^{\tilde{v}_1} f_2(v_1,dv_2)$)

A morphism $F=\{f_n\}$ of the A_∞-algebra (V,μ_*) to itself is called a *homotopy* if f_1 is an isomorphism. If (V,μ_*) has the identity e, then by a homotopy of (V,μ_*,e) we understand a homotopy of (V,μ_*) satisfying the additional conditions, $f_n(v_1,\ldots,e,\ldots,v_{n-1}) = 0$ for all $n\geq 2$ and arbitrary $v_1,\ldots,v_{n-1}\in V$.

It is not hard to see that homotopy defines an equivalence relation in the set of all possible (unital) A_∞-structures on a given vector superspace V.

3.2.3. Remark. For future reference we rewrite the n-th order associativity condition (4) as

$$\Lambda_n(v_1,\ldots,v_n) = (d\mu_n)(v_1,\ldots,v_n)$$

where

$$\Lambda_n(v_1,\ldots,v_n) := \sum_{\substack{k+l=n-1\\ k,l\geq 1}} (-1)^{r'}\mu_{k+1}(v_1,\ldots,v_j,\mu_{l+1}(v_{j+1},\ldots,v_{j+l+1}),v_{j+l+2},\ldots,v_n)$$

and $r' = (l+1)(\tilde{v}_1+\ldots+\tilde{v}_j) + jl + k(l+1) + 1$.

3.2.4. Remark. It follows from (4) for $n=3$ that the cohomology,

$$H(V) := \frac{\mathsf{Ker}\ \mu_1}{\mathsf{Im}\ \mu_1}$$

of a (unital) A_∞-algebra (V, μ_*) is canonically a (unital) *associative* algebra. Moreover, a homotopy class of (unital) A_∞-structures on V induces one and the same structure of (unital) associative algebra on $H(V)$.

3.2.5. The bar construction. There is a conceptually better interpretation [S] of an A_∞-structure on the vector superspace V as a co-differential on the bar-construction of V. Here are the details:

(i) The vector space
$$\mathsf{B}(V) := \bigoplus_{n=1}^{\infty} (V[1])^{\otimes n}$$
is naturally a co-algebra with the co-product given by
$$\Delta(w_1 \otimes \ldots \otimes w_n) = \sum_{i=1}^{n} (w_1 \otimes \ldots \otimes w_i) \otimes (w_i \otimes \ldots \otimes w_n)\,.$$

(ii) A linear map $Q : \mathsf{B}(V) \to \mathsf{B}(V)$ is said to be a *co-derivation* if $\Delta Q = Q \otimes \mathrm{Id} + \mathrm{Id} \otimes Q$. There is a one-to-one correspondence between such co-derivations and Hochschild cochains understood as elements of $\mathrm{Hom}(\mathsf{B}(V), V)$.

(iii) A homogeneous (of degree -2) Hochschild cochain $\mu_* : \mathsf{B}(V) \to V$ defines an A_∞-structure on V if and only if the associated co-derivation Q is a *co-differential*, i.e. satisfies $Q^2 = 0$.

In this setup, a morphism $(V, \mu_*) \to (\tilde{V}, \tilde{\mu}_*)$ as in 3.8.1 is precisely a morphism of the associated bar-constructions respecting co-differentials.

3.3. C_∞-algebras. This notion is a supercommutative analogue of the notion of A_∞-algebra.

Let V be a $\mathbb{Z}$-graded vector space and $\mathsf{B}(V)$ its bar construction. One can make the latter into an associative and graded commutative algebra by defining the *shuffle tensor product*, $\circledast : \mathsf{B}(V) \otimes \mathsf{B}(V) \to \mathsf{B}(V)$, as follows
$$(w_1 \otimes \ldots \otimes w_k) \circledast (w_{k+1} \otimes \ldots \otimes w_n) := \sum_{\sigma \in Sh(k,n)} e(\sigma; w_1, \ldots, w_n) w_{\sigma(1)} \otimes \ldots \otimes w_{\sigma(n)}.$$

Here we used the notations explained in Sect. 2.4.

By definition [GJ], a *strong homotopy commutative algebra*, or shortly, C_∞-algebra is an A_∞-algebra (V, μ_*) such that the associated Hochschild cochain $\mu_* : \mathsf{B}(V) \to V$ factors through the composition[5]
$$\mu_* :\ \mathsf{B}(V) \xrightarrow{\substack{\text{natural}\\ \text{projection}}} \mathsf{B}(V)/\mathsf{B}(V) \circledast \mathsf{B}(V) \longrightarrow V.$$

This implies, in particular, that
$$\mu_2(v_1, v_2) = (-1)^{\tilde{v}_1 \tilde{v}_2} \mu_2(v_2, v_1)$$

[5] Such cochains are often called *Harrison* cochains.

for any $v_1, v_2 \in V$.

One defines notions of unital C_∞-algebras, of a morphism of C_∞-algebras, of their homotopy etc. in the same way as in the A_∞-case.

3.4. G_∞-algebras. Let V be a $\mathbb{Z}$-graded vector space and let

$$\mathsf{Lie}(V[1]^*) = \sum_{k=1}^{\infty} \mathsf{Lie}^k(V[1]^*)$$

the free graded Lie algebra generated by the shifted dual vector space $V[1]^*$, i.e.

$$\mathsf{Lie}^1(V[1]^*) := V[1]^*, \quad \mathsf{Lie}^k(V[1]^*) := \left[V[1]^*, \mathsf{Lie}^{k-1}(V[1]^*)\right].$$

The Lie bracket on $\mathsf{Lie}(V[1]^*)$ extends in a usual way to the skew-symmetric associative algebra

$$\wedge^\bullet \mathsf{Lie}(V[1]^*) = \sum_{k=0}^{\infty} \wedge^k \mathsf{Lie}(V[1]^*),$$

making the latter into a Gerstenhaber algebra.

3.4.1. Definition [Ta1, TT]. *A homotopy Gerstenhaber algebra*, or shortly G_∞-*algebra* is a graded vector space V together with a degree one linear operator

$$Q : \wedge^\bullet \mathsf{Lie}(V[1]^*) \longrightarrow \wedge^\bullet \mathsf{Lie}(V[1]^*)$$

such that $Q^2 = 0$ and Q is a derivation with respect to both the product and the bracket.

A G_∞-*morphism*, $V \to V'$, of G_∞-algebras is by definition a morphism, $(\wedge^\bullet \mathsf{Lie}(V[1]^*), Q) \to (\wedge^\bullet \mathsf{Lie}(V'[1]^*), Q')$, of associated differential Gerstenhaber algebras.

The definition 3.4.1 makes sense only in the case when V is finite-dimensional. However, an obvious dualization fixes the problem [TT]:

(i) The dual of $\mathsf{Lie}(V[1]^*)$ can be identified with the quotient $\mathsf{B}(V)/\mathsf{B}(V) \circledast \mathsf{B}(V)$, $\circledast$ being the shuffle tensor product.

(ii) Derivations of $\wedge^* \mathsf{Lie}(V[1]^*)$ can be identified with arbitrary collections of linear maps,

$$m^*_{k_1,\ldots,k_n} : V[1]^* \longrightarrow \mathsf{Lie}^{k_1}(V[1]^*) \wedge \ldots \wedge \mathsf{Lie}^{k_n}(V[1]^*),$$

which upon dualization go into linear homogeneous maps,

$$m_{k_1,\ldots,k_n} : \frac{V^{\otimes k_1}}{\text{shuffle products}} \odot \ldots \odot \frac{V^{\otimes k_n}}{\text{shuffle products}} \longrightarrow V,$$

of degree $3 - n - k_1 - \ldots - k_n$.

(iii) the condition $Q^2 = 0$ translates into a well-defined set of quadratic equations for $m_{k_1,\ldots,k_n}$ which say, in particular, that m_1 is a differential on V and that the product, $v_1 \cdot v_2 := (-1)^{\tilde{v}_1} m_2(v_1, v_2)$, together with the Lie bracket, $[v_1 \bullet v_2] := -(-1)^{\tilde{v}_1} m_{1,1}(v_1, v_2)$, satisfy the Poisson identity up to a homotopy given by $m_{2,1}$. Hence the associated cohomology space $\mathbf{H}$ is a graded commutative Gerstenhaber algebra with respect to the binary operations induced by m_2 and $m_{1,1}$.

The *identity* in a G_∞-algebra V is an even element e such that all $m_{k_1,\dots,k_n}(\dots,e,\dots)$ vanish except $m_2(e,v)=v$.

3.4.2. Theorem-construction. *There is a canonical functor from the derived category of unital G_∞-algebras with finite-dimensional cohomology to the category of F_∞-manifolds.*

Proof. Since each quasi-isomorphism of G_∞-algebras is an equivalence relation, the derived category of G_∞-algebras coincides with their homotopy category.

We construct the desired functor,

$$\boxed{\begin{array}{c}\text{Derived category}\\ \text{of } G_\infty \text{ algebras}\end{array}} \quad \xrightarrow{F_\infty} \quad \boxed{\begin{array}{c}F_\infty\\ \text{manifolds}\end{array}}$$

in two steps.

Step 1. Suppose we are given a homotopy class, [ ], of G_∞-structures on a graded vector space V. By Kontsevich's Lemma 1 in [Ko3], a cohomological splitting of the complex (V,m_1) transfers [ ] into a homotopy class, $[\wedge^\bullet \mathsf{Lie}(\mathbf{H}[1]^*),Q]$, of minimal G_∞-algebras on the finite-dimensional cohomology space of the above complex. Moreover, this class does not depend on the choice of a particular cohomological splitting, and it is homotopy equivalent to the original one.

Step 2. Let $\mathcal{I}$ be the multiplicative ideal in $\wedge^\bullet \mathsf{Lie}(\mathbf{H}[1]^*)$ generated by the commutant of $\mathsf{Lie}(\mathbf{H}[1]^*)$. Any differential Q from the induced homotopy class preserves this ideal and induces, through the quotient $\wedge^\bullet \mathsf{Lie}(\mathbf{H}[1]^*)/\mathcal{I}$, a homotopy class of L_∞-structures on V which, by Corollary 2.5.7, can be identified with an odd vector field ∂ on the associated cohomological supermanifold $\mathcal{H}$ satisfying $\partial I\subset I^2$ and $[E,\partial]=\partial$, I being the ideal of the distinguished point $0\in\mathcal{H}$ and E the Euler vector field. We claim that the rest of the data listed in Definition 1.1 gets induced on $\mathcal{H}$ through the quotient $\wedge^\bullet \mathsf{Lie}(\mathbf{H}[1]^*)/\mathcal{I}^2$. Indeed, what is left of a differential Q on this quotient can be described as a collection of tensors, $m_{k,1,\dots,1}$, which, in a basis $\{e_a\}$ of $\mathbf{H}$, are represented by their components, $\mu^a_{b_1\dots b_k,c_1,\dots,c_l}$, $k\geq 1, l\geq 0$. The Chen's vector field ∂ and the tensors μ_k defining the structure of a C_∞-algebra on the tangent sheaf $\mathcal{T}_{\mathcal{H}}$ are then given by formal power series,

$$\partial=\sum_{l\geq 0}\pm\mu^a_{b_1,c_1,\dots,c_l}t^{b_1}t^{c_1}\dots t^{c_l}\frac{\partial}{\partial t^a}$$

and

$$\mu^a_{b_1\dots b_k}=\sum_{l\geq 0}\pm\mu^a_{b_1\dots b_k,c_1,\dots,c_l}t^{c_1}\dots t^{c_l}.$$

where t^a are the associated linear coordinates on $\mathcal{H}$ to which we assign degree $2-|e_a|$. It is easy to see that the G_∞-identities for $m_{k,1,\dots,1}$ get transformed into the right identities for the tensor fields ∂ and μ_k on $\mathcal{H}$. This completes the construction. $\square$

3.4.3. Corollary. *For any unital G_∞-algebra with finite dimensional cohomology, the tangent sheaf to the smooth part of the extended Kuranishi space, $\mathcal{M}=$ “$\mathsf{zeros}(\partial)$”$/\mathsf{Im}\,\mu_1$, is canonically a sheaf of induced (unital) associative algebras.*

It will be interesting to find out when $\mathcal{M}_{\text{smooth}}$ with its canonically induced structure 3.4.3 is an F-manifold in the sense of Hertling and Manin [HM].

3.5. Remark. Different "resolutions" of the chain operad in the little disk operad give different notions of homotopy Gerstenhaber algebra [V]. The definition 3.4.1 is the most canonical one. However, the functor F_∞ is not an equivalence in this case.

The proof of Theorem 3.4.2 suggests one more version: *a reduced homotopy Gerstenhaber algebra* is a graded vector space V together with the structure of G_∞-algebra such that all composition maps $m_{k_1,\ldots,k_n}$ vanish except $m_{k_1,1,\ldots,1}$. The derived category of such algebras is equivalent to the category of F_∞-manifolds (cf. Theorem 2.5.7).

3.6. Formality and Gauss-Manin connections. A *pre-Frobenius$_\infty$ manifold* is the data $(\mathcal{H}, E, \nabla, \partial, [\mu_*], e)$, where

(i) $\mathcal{H}$ is a formal pointed $\mathbb{Z}$-graded manifold,

(ii) E is the Euler vector field on $\mathcal{H}$, $Ef := \frac{1}{2}|f|f$, for all homogeneous functions on $\mathcal{H}$ of degree $|f|$,

(iii) ∇ is a flat torsion-free affine connection, called the Gauss-Manin connection, on $\mathcal{H}$,

(iv) ∂ is an odd homological (i.e. $\partial^2 = 0$) vector field on $\mathcal{H}$ such that $[E, \partial] = \partial$, $\nabla_X \nabla_Y \nabla_Z \partial = 0$ for any horizontal vector fields, X, Y and Z on $\mathcal{H}$, and $\partial I \subset I^2$, I being the ideal of the distinguished point in $\mathcal{H}$,

(v) $[\mu_n : \otimes^n \mathcal{T}_\mathcal{H} \to \mathcal{T}_\mathcal{H}]$, $n \in \mathbb{N}$, is a homotopy class of smooth unital strong homotopy commutative (C_∞) algebras defined on the tangent sheaf, $\mathcal{T}_\mathcal{H}$, to $\mathcal{H}$, such that $Lie_E \mu_n = \frac{1}{2} n \mu_n$, for all $n \in \mathbb{N}$, and μ_1 is given by

$$\begin{array}{rccc} \mu_1 : & \mathcal{T}_\mathcal{H} & \longrightarrow & \mathcal{T}_\mathcal{H} \\ & X & \longrightarrow & \mu_1(X) := [\partial, X]. \end{array}$$

(v) e is the flat unit, i.e. an even vector field on $\mathcal{H}$ such that $[\partial, e] = 0$, $\nabla e = 0$, $\mu_2(e, X) = X$, $\forall X \in \mathcal{T}_\mathcal{H}$, and $\mu_n(\ldots, e, \ldots) = 0$ for all $n \geq 3$.

3.6.1. Theorem. *There is a canonical functor from the category of pairs $(\mathfrak{g}, F)$, where $\mathfrak{g}$ is L_∞-formal unital homotopy Gerstenhaber algebra and F a formality map, to the category of pre-Frobenius$_\infty$ manifolds.*

Proof. The desired statement follows immediately from Theorem 2.7.1 and a version of Theorem-Construction 3.4.2 where the formality map F is used to transfer the G_∞-structure from the algebra to its cohomology. □

3.6.2. Theorem. *If a homotopy Gerstenhaber algebra $\mathfrak{g}$ is quasi-isomorphic, as a L_∞-algebra, to an Abelian dLie algebra, then the tangent sheaf, $\mathcal{T}_\mathcal{H}$, to its cohomology viewed*

as a linear supermanifold is canonically a sheaf of unital graded commutative associative algebras.

Proof. In this case $\partial = 0$ and μ_2, which is now defined uniquely, makes $\mathcal{T}_{\mathcal{H}}$ into a sheaf of unital graded commutative associative algebras. $\square$

4 Perturbative construction of F_∞-invariants

The purpose of this section is to give second "down-to-earth" proofs of some of the main claims of this paper. Our approach here is based on perturbative solutions of algebro-differential equations rather than on the homotopy technique used in the two previous Sections.

First comes a perturbative proof of the Smoothness Theorem 2.5.6.

4.1. Theorem (Chen's construction). *For any differential Lie superalgebra $\mathfrak{g}$, there exists a versal element, $\Gamma \in k[[t]] \otimes \mathfrak{g}$, and an odd derivation, $\partial : k[[t]] \longrightarrow k[[t]]$, such that $\partial^2 = 0$ and the equation,*

$$d\Gamma + \vec{\partial}\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] = 0$$

holds. Moreover, for any quasi-isomorphism of complexes of vector spaces, $\phi : (\mathfrak{g}, d) \longrightarrow (\mathbf{H}, 0)$, Γ may be normalized so that $\phi(\Gamma_{[n]}) = 0$ for all $n \geq 2$.

We shall prove this Theorem by induction using (twice) the following Lemma which is merely a trancated version of Remark 2.5.1.

4.1.1. Lemma. *Assume the elements $\Gamma_{(n)} = \sum_{k=0}^{n} \Gamma_{[k]} \in k[[t]] \otimes \mathfrak{g}$ and $\partial_{(n)} = \sum_{k=0}^{n} \partial_{[n]} \in \mathsf{Der}\, k[[t]]$ satisfy*

$$d\Gamma_{(n)} + \vec{\partial}_{(n)}\Gamma_{(n)} + \frac{1}{2}\left[\Gamma_{(n)} \bullet \Gamma_{(n)}\right] = 0 \mod I^{n+1}.$$

Then

$$\psi_{[n+1]} := d\Gamma_{(n)} + \vec{\partial}_{(n)}\Gamma_{(n)} + \frac{1}{2}\left[\Gamma_{(n)} \bullet \Gamma_{(n)}\right] \mod I^{n+2}$$

satisfies

$$d\psi_{[n+1]} = -\vec{\partial}^{2}_{(n)}\left(\Gamma_{(n)}\right) \mod I^{n+2}.$$

4.1.2. Proof of the Theorem. Let

$$\phi : (\mathfrak{g}, d) \longrightarrow (\mathbf{H}, 0),$$

be a quasi-isomorphism, i.e. a morphism of complexes inducing an isomorphism on cohomology. Since $\mathfrak{g}$ is defined over a field, such a quasi-isomorphism always exists (note that we do not ask for any sort of a relationship between ϕ and the Lie brackets).

Let e_i be any representatives of the cohomology classes $[e_i]$ in $\mathsf{Ker}\ d \subset \mathfrak{g}$. We may assume without loss of generality that $\phi(e_i) = [e_i]$. Then choosing $\Gamma_{[0]} = 0$, $\Gamma_{[1]} := \sum_{i=1}^{p+q} t^i e_i$, and $\partial_{[0]} = \partial_{[1]} = 0$ we get the data $(\Gamma_{(1)}, \partial_{(1)})$ satisfying the Master equation modulo terms in I^2 and the nilpotency condition $\partial^2 = 0$ modulo terms in I^3.

Assume we have constructed a versal element $\Gamma_{(n)} = \sum_{k=1}^{n} \Gamma_{[k]} \in k[[t]] \otimes \mathfrak{g}$ and an odd vector field $\partial_{(n)} = \sum_{k=2}^{n} \partial_{[n]}$ on $\mathcal{H}$ such that the equations

$$P_n : \begin{cases} d\Gamma_{(n)} + \vec{\partial}_{(n)}\Gamma_{(n)} + \frac{1}{2}\left[\Gamma_{(n)} \bullet \Gamma_{(n)}\right] = 0 \mod I^{n+1} \\ \partial^2_{(n)} = 0 \mod I^{n+2}. \end{cases}$$

are satisfied.

Let us show that there exists $\Gamma_{[n+1]} \in k[[t]] \otimes \mathfrak{g}$ and $\partial_{[n+1]} \in H^0(\mathcal{T} M_{\mathbf{H}})$ such that

$$\Gamma_{(n+1)} = \Gamma_{(n)} + \Gamma_{[n+1]}, \qquad \partial_{(n+1)} = \partial_{(n)} + \partial_{[n+1]},$$

satisfy the equations P_{n+1}.

Note that, in the notations of Lemma 4.4.1, one has

$$d\Gamma_{(n+1)} + \vec{\partial}_{(n+1)}\Gamma_{(n+1)} + \frac{1}{2}\left[\Gamma_{(n+1)} \bullet \Gamma_{(n+1)}\right] \mod I^{n+2} = d\Gamma_{[n+1]} + \psi_{[n+1]} + \vec{\partial}_{[n+1]}\Gamma_{[1]}.$$

Let us now define $\vec{\partial}_{[n+1]}$ by setting

$$\vec{\partial}_{[n+1]}\Gamma_{[1]} := -\phi(\psi_{[n+1]}).$$

As $d\psi_{[n+1]} = 0$ by Lemma 4.4.1 and the second equation of P_n, we conclude that

$$\psi_{[n+1]} + \vec{\partial}_{[n+1]}\Gamma_{[1]} \in (\mathsf{Ker}\,\phi \cap \mathsf{Ker}\,d) \otimes k[[t]]_{[n+1]}.$$

Since ϕ is a quasi-isomorphism, $\mathsf{Ker}\ \phi \cap \mathsf{Ker}\ d = \mathsf{Im}\ d$. Hence, there exists $\Gamma_{[n+1]} \in k[[t]] \otimes \mathfrak{g}$ such that

$$d\Gamma_{[n+1]} = -\psi_{[n+1]} - \vec{\partial}_{[n+1]}\Gamma_{[1]}.$$

Thus the first equation of the system P_{n+1} holds. This implies, by Lemma 2.4.2,

$$\begin{aligned} d\psi_{[n+2]} &= -\vec{\partial}^2_{(n+1)}\Gamma_{(n+1)} \mod I^{n+3} \\ &= -\vec{\partial}^2_{(n+1)}\Gamma_{[1]} \mod I^{n+3}. \end{aligned}$$

Applying ϕ to both sides of this equation, we get

$$\vec{\partial}^2_{(n+1)}\phi(\Gamma_{[1]}) = 0$$

implying the second equation of the system P_{n+1},

$$\vec{\partial}^2_{(n+1)} = 0 \mod I^{n+3},$$

and completing thus the inductive procedure.

Finally, we note that $\mathsf{Ker}\,d + \mathsf{Ker}\,\phi = \mathfrak{g}$ for ϕ is a quasi-isomorphism. Hence we can always adjust $\Gamma_{[n+1]}$, $n \geq 1$, so that it lies in $\mathsf{Ker}\,\phi$. $\square$.

4.1.3. Remarks. (i) The role of ∂ in the Chen's construction is to absorb all the obstructions so that constructing a versal solution to the Master equation poses no problem (cf. Smoothness Theorem 2.5.6).

(ii) The Chen's differential ∂ is completely determined by Γ. Indeed, the Master equations imply,

$$\vec{\partial}\phi(\Gamma) = -\frac{1}{2}\phi\left([\Gamma \bullet \Gamma]\right).$$

Decomposing,

$$\phi(\Gamma) = \sum_i f^i(t)[e_i],$$

we note that $f^i(t) = t^i \bmod I^2$. Hence the functions $f^i(t)$ define a coordinate system on $\mathcal{H}$ and the values, $\partial f^i(t)$, completely determine the differential ∂.

In particular, if Γ is ϕ-*normalized*, i.e. $\phi(\Gamma_{[n\geq 2]}) = 0$, then ∂ can be computed by the formula

$$\vec{\partial}\left(\sum_{i=1}^{p+q} t^i[e_i]\right) = -\frac{1}{2}\phi\left([\Gamma \bullet \Gamma]\right).$$

(iii) We shall understand from now on a versal solution, Γ, of the Master equations and the associated Chen differential ∂ as, respectively, global sections of the sheaves $\mathfrak{g}\otimes\mathcal{O}_{\mathcal{H}}$ and $\mathcal{T}_{\mathcal{H}}$ on $\mathcal{H}$ (in practical terms, this essentially fixes their transformation properties under arbitrary changes of coordinates on the cohomology supermanifold).

We call sometimes Γ a *Master function*.

(iv) The argument in (ii) also downplays the role of the quasi-isomorphism ϕ used in the Chen construction. If Γ is normalised with respect to a quasi-isomorphism $\phi : (\mathfrak{g}, d) \to (\mathbf{H}, 0)$, then, for any other quasi-isomorphism ϕ', the *same* Γ can be viewed as ϕ'-normalized, but in a new coordinate system $t'^i = f^i(t^j)$ given by $\phi'(\Gamma) = f^i(t^j)[e_i]$. Thus varying quasi-isomorphism ϕ used in the construction of Γ amounts to varying flat structure on the pointed supermanifold $\mathcal{H}$.

(v) Chen has actually invented his differential ∂ in the context of differential associative algebras [C]. Its Lie algebra analogue, Theorem 4.1, is due to Hain [H].

4.2. Gauge equivalence. Let us consider the following action, called a *gauge transformation*, of $\mathfrak{g}_{\tilde{1}} \otimes I$ on $\mathfrak{g}\otimes k[[t]]$:

$$\begin{array}{ccc} \mathfrak{g}\otimes I \times \mathfrak{g}\otimes k[[t]] & \longrightarrow & \mathfrak{g}\otimes k[[t]] \\ g\otimes\Gamma & \longrightarrow & \Gamma^g := e^{\mathrm{ad}_g}\Gamma - \frac{e^{\mathrm{ad}_g}-1}{\mathrm{ad}_g}(d+\vec{\partial})g. \end{array}$$

4.2.1. Lemma. *If $\Gamma \in \mathfrak{g}\otimes k[[t]]$ is a Master function, then, for any $g \in \mathfrak{g}_{\tilde{1}}\otimes I$, the function Γ^g is also a Master function, and both these share the same Chen differential.*

Proof. We have to show that the equation $d\Gamma + \overrightarrow{\partial}\Gamma + \frac{1}{2}[\Gamma\bullet\Gamma] = 0$ implies

$$d\Gamma^g + \overrightarrow{\partial}\Gamma^g + \frac{1}{2}[\Gamma^g \bullet \Gamma^g] = 0.$$

This follows immediately from the well-known formulae [GM],

$$e^{\mathrm{ad}_g} d e^{-\mathrm{ad}_g} = d - \mathrm{ad}_{\frac{e^{\mathrm{ad}_g}-1}{\mathrm{ad}_g} dg}, \qquad e^{\mathrm{ad}_g} \partial e^{-\mathrm{ad}_g} = \partial - \mathrm{ad}_{\frac{e^{\mathrm{ad}_g}-1}{\mathrm{ad}_g} \vec{\partial} g}, \qquad e^{\mathrm{ad}_g} \mathrm{ad}_\Gamma e^{-\mathrm{ad}_g} = \mathrm{ad}_{e^{\mathrm{ad}_g}\Gamma},$$

and

$$e^{\mathrm{ad}_g}\left[(\ldots) \bullet (\ldots)\right] = \left[e^{\mathrm{ad}_d}(\ldots) \bullet e^{\mathrm{ad}_g}(\ldots)\right].$$

□

4.2.2. Theorem. *Let $\mathfrak{g}$ be a differential Lie algebra. For any two Master functions on $\mathcal{H}$, Γ and Γ', there is a gauge function $g \in \Gamma(\mathcal{H}, \mathfrak{g} \otimes I)$ and a diffeomorphism $f : (\mathcal{H}, 0) \to (\mathcal{H}, 0)$ such that $\Gamma' = f^*(\Gamma^g)$ and $\partial = f_*(\partial')$.*

A sketch of the proof. Let us fix a quasi-isomorphism $\phi : (\mathfrak{g}, d) \to (\mathbf{H}, 0)$ of complexes of Abelian groups, and a coordinate system on $\mathcal{H}$ in which Γ' is ϕ-normalized.

We have $\Gamma'_{[1]} = \Gamma_{[1]} - dg_{[1]}$, for some $g_{[1]} \in \Gamma(\mathcal{H}, \mathfrak{g} \otimes I)$, and $\partial'_{[1]} = \partial_{[1]} = 0$. Hence $\Gamma' = \Gamma^{g_{[1]}} \mod I^2$ and there is a unique diffeomorphism $f_1 : \mathcal{H} \to \mathcal{H}$ such that the Master function $\Gamma'' := f_1^*(\Gamma^{g_{[1]}})$ is ϕ-normalized and $\Gamma''_{[1]} = \Gamma'_{[1]}$. Hence,

$$d(\Gamma'_{[2]} - \Gamma''_{[2]}) + \overrightarrow{(\partial'_{[2]} - \partial''_{[2]})}\Gamma_{[1]} = 0$$

implying $\partial'_{[2]} = \partial''_{[2]}$ and $d(\Gamma'_{[2]} - \Gamma''_{[2]}) = 0$. Since $\phi(\Gamma'_{[2]} - \Gamma''_{[2]}) = 0$ and ϕ is a quasi-isomorphism, there exists $g_{[2]} \in \mathfrak{g} \otimes I^2$ such that $\Gamma'_{[2]} - \Gamma_{[2]} = dg_{[2]}$. Hence

$$\Gamma' = (\Gamma'')^{g_{[2]}} = f_1^*(\Gamma^{g_{(2)}}) \mod I^3.$$

Continuing by induction and using Lemma 4.2.1 one easily obtains the desired result.
□

4.2.3. Corollary. *The Chen's vector field ∂ on $\mathcal{H}$ is an invariant of $\mathfrak{g}$.*

4.3. Differential on $\mathcal{T}_{\mathcal{H}}$. We fix from now on a dG-algebra $\mathfrak{g}$ and a Master function Γ on $\mathcal{H}$. The latter is not defined canonically, though the associated Chen differential ∂ *is*.

We also fix a quasi-isomorphism, $\phi : (\mathfrak{g}, d) \to (\mathbf{H}, 0)$, of complexes of Abelian groups. This puts *no* restriction whatsoever on the dG-algebra under consideration. Moreover, our main results will not depend on the particular choices of Γ and ϕ we have made — these two are no more than working tools.

The global vector field ∂ on $\mathcal{H}$ makes $\mathcal{T}_{\mathcal{H}}$ into a sheaf of complexes with the differential

$$\begin{array}{rccc} \delta : & \mathcal{T}_{\mathcal{H}} & \longrightarrow & \mathcal{T}_{\mathcal{H}} \\ & X & \longrightarrow & \delta X := [\partial, X], \end{array}$$

where [,] stands for the usual commutator of (germs) of vector fields. Indeed,

$$(\delta)^2 X = [\partial, [\partial, X]] = \frac{1}{2}[[\partial, \partial], X] = [\partial^2, X] = 0,$$

where we have used the Jacobi identity and the fact that $\partial^2 = 0$.

This, of course, induces a differential on the sheaf of tensor products, $\mathcal{T}_{\mathcal{H}}^{\otimes m} \otimes (\mathcal{T}_{\mathcal{H}}^*)^{\otimes n}$ (and on the associated vector space of global sections), which we denote by the same symbol δ.

4.4. Deformed dG-algebra. It is easy to check that the map

$$\begin{array}{rccc} d^\Gamma: & k[[t]] \otimes \mathfrak{g} & \longrightarrow & k[[t]] \otimes \mathfrak{g} \\ & a & \longrightarrow & d^\Gamma a := da + \vec{\partial} a + [\Gamma \bullet a]. \end{array}$$

satisfies

(i) $(d^\Gamma)^2 = 0$,

(ii) $d^\Gamma(a \cdot b) = (d^\Gamma a) \cdot b + (-1)^{\tilde{a}} a \cdot d^\Gamma b$

(iii) $d^\Gamma [a \bullet b] = \left[d^\Gamma a \bullet b\right] - (-1)^{\tilde{a}} \left[a \bullet d^\Gamma b\right]$

implying that the data $\left(k[[t]] \otimes \mathfrak{g}, [\,\bullet\,], \cdot, d^\Gamma\right)$ is a dG-algebra.

The differentials d^Γ and δ make the sheaf $\mathfrak{g} \otimes (\mathcal{T}_{\mathcal{H}}^*)^{\otimes k}$ on $\mathcal{H}$ into a sheaf of complexes with the differential which we denote by D^Γ. For example, for any germ $\Phi \in \mathfrak{g} \otimes \mathcal{T}_{\mathcal{H}}^*$ and any germ $X \in \mathcal{T}_{\mathcal{H}}$ over the same point in $\mathcal{H}$,

$$(D^\Gamma \Phi)(X) := d^\Gamma \Phi(X) - (-1)^{\tilde{\Phi}} \Phi(\delta X).$$

The vector space $\mathrm{Hom}(\mathcal{T}_{\mathcal{H}}^{\otimes k}, \mathfrak{g} \otimes \mathcal{O}_{\mathcal{H}})$ is also a complex with the differential denoted by the same symbol D^Γ.

4.5. Morphism of sheaves of complexes. The versal solution Γ gives rise to a morphism of $\mathcal{O}_{\mathcal{H}}$-modules,

$$\begin{array}{rccc} \Upsilon: & \mathcal{T}_{\mathbf{H}} & \longrightarrow & \mathcal{O}_{\mathcal{H}} \otimes \mathfrak{g} \\ & X & \longrightarrow & \Upsilon(X) := \overrightarrow{X}\Gamma. \end{array}$$

It is not hard to check that Υ is a monomorphism.

4.5.1. Lemma. *The element $\Upsilon \in \mathrm{Hom}(\mathcal{T}_{\mathcal{H}}, \mathfrak{g} \otimes \mathcal{O}_{\mathcal{H}})$ is cyclic, i.e.*

$$D^\Gamma \Upsilon = 0.$$

Proof. Applying $X \in \mathcal{T}_{\mathcal{H}}$ to both sides of the equation

$$d\Gamma + \vec{\partial}\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] = 0$$

we get

$$(-1)^{\tilde{X}} d^\Gamma(\overrightarrow{X}\Gamma) + \overrightarrow{[X, \partial]}\,\Gamma = 0,$$

implying $(D^\Gamma \Upsilon)(X) = 0$. $\square$

4.5.2. Corollary. *For any* $X \in \mathcal{T}_{\mathcal{H}}$, $d^{\Gamma}(\overrightarrow{X}\Gamma) = \overrightarrow{\delta X}\,\Gamma$.

4.5.3 Corollary. *For any* $\chi \in \mathrm{Hom}(\otimes^k \mathcal{T}_{\mathcal{H}}, \mathcal{T}_{\mathcal{H}})$ *one has*

$$D^{\Gamma}(\Upsilon \circ \chi) = \Upsilon \circ (\delta\chi).$$

Proof. We have, using Corollary 4.5.2,

$$\begin{aligned} D^{\Gamma}(\Upsilon \circ \chi)(X_1, \dots, X_k) &= d^{\Gamma}\left(\overrightarrow{\chi(X_1,\dots,X_k)}\Gamma\right) - (-1)^{\tilde{\chi}}\overrightarrow{\chi(\delta X_1,\dots,X_k)}\Gamma \\ &\quad - \dots - (-1)^{\tilde{\chi}+\tilde{X}_1+\dots+\tilde{X}_{k-1}}\overrightarrow{\chi(X_1,\dots,\delta X_k)}\Gamma \\ &= \left(\overrightarrow{\delta\chi(X_1,\dots,X_k)}\Gamma\right) - (-1)^{\tilde{\chi}}\overrightarrow{\chi(\delta X_1,\dots,X_k)}\Gamma \\ &\quad - \dots - (-1)^{\tilde{\chi}+\tilde{X}_1+\dots+\tilde{X}_{k-1}}\overrightarrow{\chi(X_1,\dots,\delta X_k)}\Gamma \\ &= (\Upsilon \circ \delta\chi)(X_1,\dots,X_k) \end{aligned}$$

for arbitrary $X_1, \dots, X_k \in \mathcal{T}_{\mathcal{H}}$. $\square$

Therefore, the map

$$\Upsilon : (\mathcal{T}_{\mathcal{H}}, \delta) \to \left(\mathfrak{g} \otimes \mathcal{O}_{\mathcal{H}}, d^{\Gamma}\right)$$

is a morphism of sheaves of complexes. Note that the "projection" map $s \circ \phi : \mathfrak{g} \otimes \mathcal{O}_{\mathcal{H}} \to \mathcal{T}_{\mathcal{H}}$ satisfies $s \circ \phi \circ \Upsilon = \mathrm{Id}$ but does *not*, in general, respect the differentials.

Analogously one shows that the morphism $\Upsilon \cdot \Upsilon$ defined by the commutative diagram

$$\begin{tikzcd} \mathcal{T}_{\mathcal{H}} \otimes \mathcal{T}_{\mathcal{H}} \arrow[r, "\Upsilon\otimes\Upsilon"] \arrow[dr, "\Upsilon\cdot\Upsilon"'] & \mathfrak{g} \otimes \mathfrak{g} \otimes \mathcal{O}_{\mathcal{H}} \arrow[d, "\cdot\otimes Id"] \\ & \mathfrak{g} \otimes \mathcal{O}_{\mathcal{H}} \end{tikzcd}$$

defines a cyclic element in $(\mathrm{Hom}(\mathcal{T}_{\mathcal{H}}^{\otimes 2}, \mathfrak{g} \otimes \mathcal{O}_{\mathcal{H}}), D^{\Gamma})$. In a similar way one uses muliplicative structure in $\mathfrak{g}$ to construct cyclic elements $\Upsilon \cdot \Upsilon \cdot \Upsilon$ etc.[6]

For future reference we define a morphism $\Upsilon_{(n)} \in \mathrm{Hom}(\mathcal{T}, \mathfrak{g} \otimes \mathcal{O}_{\mathcal{H}})$ by setting $\Upsilon_{(n+1)}(X) := \overrightarrow{X}\Gamma_{(n+1)}$. Similarly one defines $\Upsilon_{(n)} \cdot \Upsilon_{(n)}$, etc.

4.6. Multiplicative structure in $\mathcal{T}_{\mathcal{H}}$. We will show in this subsection that, for any dG-algebra $\mathfrak{g}$, the associated tangent sheaf $\mathcal{T}_{\mathcal{H}}$ is always a sheaf of differential associative algebras (defined uniquely up to a homotopy).

4.6.1. Theorem. *There exists an even morphism of sheaves,* $\mu \in \mathrm{Hom}(\mathcal{T}_{\mathcal{H}}^{\otimes 2}, \mathcal{T}_{\mathcal{H}})$, *such that* $\delta\mu = 0$ *and the diagram*

$$\begin{tikzcd} \mathcal{T}_{\mathcal{H}} \otimes \mathcal{T}_{\mathcal{H}} \arrow[r, "\mu"] \arrow[dr, "\Upsilon\cdot\Upsilon"'] & \mathcal{T}_{\mathcal{H}} \arrow[d, "\Upsilon"] \\ & \mathfrak{g} \otimes \mathcal{O}_{\mathcal{H}} \end{tikzcd}$$

[6] The cyclicity of $\Upsilon \cdot \Upsilon$, etc., relies on the Poisson identity holding in $(\mathfrak{g}, [\bullet], \cdot, d)$.

is commutative at the cohomology level, i.e.

$$[\Upsilon \cdot \Upsilon] = [\Upsilon \circ \mu]$$

in the cohomology sheaf $\mathsf{Ker}\, D^\Gamma / \mathsf{Im}\, D^\Gamma$ *associated with the sheaf of complexes* $(Hom(\mathcal{T}_\mathcal{H}^{\otimes 2}, \mathfrak{g}\otimes \mathcal{O}_\mathcal{H}), D^\Gamma)$.

Proof. We have to show that there exists $\mu \in \mathrm{Hom}(\mathcal{T}_\mathcal{H}^{\otimes 2}, \mathcal{T}_\mathcal{H})$ such that

$$\delta\mu(X, Y) = \mu(\delta X, Y) + (-1)^{\tilde{X}}\mu(X, \delta Y) \tag{5}$$

and

$$\overrightarrow{X}\Gamma \cdot \overrightarrow{Y}\Gamma = \overrightarrow{\mu(X, Y)}\,\Gamma + (D^\Gamma A)(X, Y) \tag{6}$$

for some $A \in \mathrm{Hom}(\mathcal{T}_\mathcal{H}^{\otimes 2}, \mathfrak{g} \otimes \mathcal{O}_\mathcal{H})$ and any $X, Y \in \mathcal{T}_\mathcal{H}$. We shall proceed by induction and assume, without loss of generality, that the vector fields X and Y are constant, i.e. $\nabla X = \nabla Y = 0$.

The above equations can obviously be satisfied $\mathrm{mod} I$: just set

$$\mu_{[0]}(X, Y) := \phi(\overrightarrow{X}\Gamma_{[1]} \cdot \overrightarrow{Y}\Gamma_{[1]}).$$

Indeed,

$$\overrightarrow{X}\Gamma_{[1]} \cdot \overrightarrow{Y}\Gamma_{[1]} - \overrightarrow{\mu_{[0]}(X, Y)}\,\Gamma_{[1]} \in \mathsf{Ker}\ \phi \cap \mathsf{Ker}\ d$$

and hence this expression is d-exact. Denote it by $dA_{[0]}(X, Y)$. (We can always normalise $A_{[0]}$ so that it lies in $\mathrm{Hom}(\mathcal{T}_\mathcal{H}^{\otimes 2}, \ker\phi \otimes \mathcal{O}_\mathcal{H})$.) This solves (6) $\mathrm{mod}\, I$. The equation (5) mod I is trivial (recall that $\partial_{[<2]} = 0$).

Assume now that we have constructed $\mu_{(n)}$ and $A_{(n)}$ so that the equations

$$\delta_{(n)}\mu_{(n-1)}(X, Y) = \mu_{(n-1)}(\delta_{(n)}X, Y) + (-1)^{\tilde{X}}\,\mu_{(n-1)}(X, \delta_{(n)}Y) \mod I^{n+1} \tag{7}$$

$$\overrightarrow{X}\Gamma_{(n+1)} \cdot \overrightarrow{Y}\Gamma_{(n+1)} = \overrightarrow{\mu_{(n)}(X, Y)}\,\Gamma_{(n+1)} + ({D^\Gamma}_{(n)}A_{(n)})(X, Y) \mod I^{n+1} \tag{8}$$

hold.

The Theorem will be proved if we find $\mu_{[n+1]}$ and $A_{[n+1]}$ satisfying

$$\delta_{(n+1)}\left(\mu_{(n)}(X, Y) + \mu_{[n+1])}(X, Y)\right) = \mu_{(n)}(\delta_{(n+1)}X, Y) + (-1)^{\tilde{X}}\mu_{(n)}(X, \delta_{(n+1)}Y) \mod I^{n+2}$$

and

$$\overrightarrow{X}\Gamma_{(n+2)} \cdot \overrightarrow{Y}\Gamma_{(n+2)} - \overrightarrow{\mu_{(n)}(X, Y)}\,\Gamma_{(n+2)} - \overrightarrow{\mu_{[n+1]}(X, Y)}\,\Gamma_{[1]} - ({D^\Gamma}_{(n+1)}A_{(n)})(X, Y) =$$
$$= dA_{[n+1]}(X, Y) \mod I^{n+2}$$

Defining

$$\mu_{[n+1]}(X, Y) := \phi\left(\overrightarrow{X}\Gamma_{(n+2)} \cdot \overrightarrow{Y}\Gamma_{(n+2)} - \overrightarrow{\mu_{(n)}(X, Y)}\,\Gamma_{(n+2)} - ({D^\Gamma}_{(n+1)}A_{(n)})(X, Y)\right)$$

we ensure that the morphism

$$\lambda_{[n+1]}(X, Y) := \left(\Upsilon_{(n+1)} \cdot \Upsilon_{(n+1)} - \Upsilon_{(n+1)} \circ \mu_{(n)} - {D^\Gamma}_{(n+1)}A_{(n)}) - \Upsilon_{(0)} \circ \mu_{[n+1]}\right)(X, Y) \mod I^{n+2}$$

take values in the sheaf in $\mathsf{Ker}\,\phi \otimes \mathcal{O}_{\mathcal{H}}$. Since it vanishes modulo I^{n+1}, we have, modulo I^{n+2},

$$
\begin{aligned}
d\lambda_{[n+1]}(X, Y) &= (D^{\Gamma}_{(n+1)}\lambda_{[n+1]})(X, Y) \\
&= -\left(D^{\Gamma}_{(n+1)}(\Upsilon_{(n+1)} \circ \mu_{(n)})\right)(X, Y) \\
&= -d^{\Gamma}_{n+1}(\overrightarrow{\mu_{(n)}(X, Y)}\Gamma) + \overrightarrow{\mu_{(n)}(\delta_{(n+1)}X, Y)}\Gamma + (-1)^{\tilde{X}}\,\overrightarrow{\mu_{(n)}(X, \delta_{(n+1)}Y)}\Gamma \\
&= -\overrightarrow{(\delta_{(n+1)}\mu_{(n)}(X, Y) - \mu_{(n)}(\delta_{(n+1)}X, Y) - (-1)^{\tilde{X}}\mu_{(n)}(X, \delta_{(n+1)}Y))}\Gamma
\end{aligned}
$$

where we have used Corollary 4.5.2 and the fact that $D^{\Gamma}(\Upsilon \cdot \Upsilon) = 0$. Applying ϕ to the last equation, we get

$$
\delta_{(n+1)}\mu_{(n)}(X, Y) = \mu_{(n)}(\delta_{(n+1)}X, Y) + (-1)^{\tilde{X}}\mu_{(n)}(X, \delta_{(n+1)}Y) \mod I^{n+2}
$$

and hence

$$
d\lambda_{[n+1]}(X, Y) = 0.
$$

Since $\mathsf{Ker}\,\phi \cap \mathsf{Ker}\,d \subset \mathsf{Im}\,d$, there exists $A_{[n+1]}(X, Y)$ (which can be chosen to lie in $\mathsf{Ker}\,\phi \otimes \mathcal{O}_{\mathcal{H}}$) such that

$$
\lambda_{[n+1]}(X, Y) = dA_{[n+1]}(X, Y).
$$

This completes the inductive procedure and hence the proof of the Theorem. □

4.6.2. Definition. An even morphism of sheaves, $\mu \in \mathrm{Hom}(\otimes^2\mathcal{T}_{\mathcal{H}}, \mathcal{T}_{\mathcal{H}})$, satisfying the conditions of Theorem 4.6.1 is called *induced*. The associated data $(\mathcal{T}_{\mathcal{H}}, \delta, \mu)$ is called a sheaf of *induced differential algebras*.

Clearly, an induced product on $\mathcal{T}_{\mathcal{H}}$ is supercommutative if the product $\cdot$ in $\mathfrak{g}$ is supercommutative.

4.7. (Non)Uniqueness. How unique is the product μ induced on the tangent sheaf $\mathcal{T}_{\mathcal{H}}$ by Theorem 4.6.1? When is it associative? To address these questions we shall need the following technical result.

4.7.1. Lemma. *If* $\tau \in \mathrm{Hom}(\otimes^k\mathcal{T}_{\mathcal{H}}, \mathcal{T}_{\mathcal{H}})$ and $B \in \mathrm{Hom}(\otimes^k\mathcal{T}_{\mathcal{H}}, \mathfrak{g} \otimes \mathcal{O}_{\mathcal{H}})$ *satisfy the equation*

$$
\Upsilon \circ \tau = D^{\Gamma}B
$$

then there exists $\chi \in \mathrm{Hom}(\otimes^k\mathcal{T}_{\mathcal{H}}, \mathcal{T}_{\mathcal{H}})$ and $C \in \mathrm{Hom}(\otimes^k\mathcal{T}_{\mathcal{H}}, \mathsf{Ker}\,\phi \otimes \mathcal{O}_{\mathcal{H}})$ *such that*

(1) $B = \Upsilon \circ \chi + D^{\Gamma}C$,

(2) $\tau = \delta\chi$, *i.e.*

$$
\begin{aligned}
\tau(X_1, \ldots, X_k) &= \delta\chi(X_1, \ldots, X_k) - (-1)^{\tilde{\chi}}\chi(\delta X_1, \ldots, X_k) \\
&\quad - \ldots - (-1)^{\tilde{\chi}+\tilde{X}_1+\ldots+\tilde{X}_{k-1}}\chi(X_1, \ldots, \delta X_k)
\end{aligned}
$$

for any $X_1, \ldots, X_k \in \mathcal{T}_{\mathcal{H}}$.

Proof. Without loss of generality we may assume that (germs of) vectors fields $X_1, \dots, X_k$ are constant. In view of Corollary 4.5.3 and injectivity of the map Υ, it is enough to show that the equation

$$\overrightarrow{\tau(X_1, \dots, X_k)}\Gamma = (D^\Gamma B)(X_1, \dots, X_k) \tag{9}$$

implies

$$B(X_1, \dots, X_k) = \overrightarrow{\chi(X_1, \dots, X_k)}\Gamma + (D^\Gamma C)(X_1, \dots, X_k)$$

for some $\chi \in \mathrm{Hom}(\otimes^k \mathcal{T}_{\mathcal{H}}, \mathcal{T}_{\mathcal{H}})$ and $C \in \mathrm{Hom}(\otimes^k \mathcal{T}_{\mathcal{H}}, \mathsf{Ker}\, \phi \otimes \mathcal{O}_{\mathcal{H}})$. We shall proceed by induction.

The equation (9) mod$I$ is

$$\overrightarrow{\tau_{[0]}(X_1, \dots, X_k)}\Gamma_{[1]} = dB_{[0]}(X_1, \dots, X_k).$$

Hence $\tau_{[0]} = 0$ and $dB_{[0]} = 0$. Set

$$\chi_{[0]}(X_1, \dots, X_k) := \phi\left(B_{[0]}(X_1, \dots, X_k)\right).$$

Then $B_{[0]}(X_1, \dots, X_k) - \overrightarrow{\chi_{[0]}(X_1, \dots, X_k)}\Gamma_{[1]}$ lies in $(\mathsf{Ker}\, \phi \cap \mathsf{Ker}\, d) \otimes \mathcal{O}_{\mathcal{H}}$ and hence equals $dC_{[0]}(X_1, \dots, X_k)$ for some $C_{[0]} \in \mathrm{Hom}(\otimes^k \mathcal{T}_{\mathcal{H}}, \mathsf{Ker}\, \phi \otimes \mathcal{O}_{\mathcal{H}})$.

Assume that $\chi_{(n)}$ and $C_{(n)}$ are constructed so that the equations

$$B_{(n)}(X_1, \dots, X_k) = \overrightarrow{\chi_{(n)}(X_1, \dots, X_k)}\Gamma_{(n+1)} + (D^\Gamma_{(n)} C_{(n)})(X_1, \dots, X_k) \mod I^{n+1}$$

holds.

Let us show that there exist $\chi_{[n+1]}$ and $C_{[n+1]}$ satisfying

$$\begin{aligned} B_{(n+1)}(X_1, \dots, X_k) &= \overrightarrow{\chi_{(n)}(X_1, \dots, X_k)}\Gamma_{(n+2)} + \overrightarrow{\chi_{[n+1]}(X_1, \dots, X_k)}\Gamma_{[1]} \\ &\quad + (D^\Gamma_{(n+1)} C_{(n)})(X_1, \dots, X_k) + dC_{[n+1]}(X_1, \dots, X_k) \mod I^{n+2}, \end{aligned}$$

or, equivalently, satisfying

$$dC_{[n+1]}(X_1, \dots, X_k) = \psi_{[n+1]}(X_1, \dots, X_k) - \overrightarrow{\chi_{[n+1]}(X_1, \dots, X_k)}\Gamma_{[1]} \mod I^{n+2},$$

where we have set

$$\psi_{[n+1]}(X_1, \dots, X_k) := B_{(n+1)}(X_1, \dots, X_k) - \overrightarrow{\chi_{(n)}(X_1, \dots, X_k)}\Gamma_{(n+2)} - (D^\Gamma_{(n+1)} C_{(n)})(X_1, \dots, X_k).$$

Since $\psi_{[n+1]}(X_1, \dots, X_k)$ vanishes modI^{n+1}, this is a monom of degree $n+1$ and t^i, and hence, modulo I^{n+2},

$$\begin{aligned} d\psi_{[n+1]}(X_1, \dots, X_k) &= (D^\Gamma_{(n+1)} \psi_{[n+1]})(X_1, \dots, X_k) \\ &= (D^\Gamma_{(n+1)} B_{(n+1)})(X_1, \dots, X_k) - D^\Gamma_{(n+1)}(\Upsilon_{(n+1)} \circ \chi_{(n)})(X_1, \dots, X_k) \\ &= \overrightarrow{\left(\tau_{(n+1)}(X_1, \dots, X_k) - (\delta_{(n+1)} \chi_{(n)})(X_1, \dots, X_k)\right)}\,\Gamma. \end{aligned}$$

Applying ϕ to both sides of these equations we get

$$\tau_{(n+1)}(X_1, \dots, X_k) = (\delta_{(n+1)} \chi_{(n)})(X_1, \dots, X_k) \mod I^{n+1}$$

and hence

$$d\psi_{[n+1]}(X_1,\dots,X_k) = 0 \mod I^{(n+1)},$$

We define $\chi_{[n+1]}$ by

$$\chi_{[n+1]}(X_1,\dots,X_k) := \phi(\psi_{[n+1]}(X_1,\dots,X_k)).$$

Then $\psi_{[n+1]}(X_1,\dots,X_k) - \overrightarrow{\chi_{[n+1]}(X_1,\dots,X_k)}\Gamma_{[1]} \in \mathsf{Ker}\, d \cap \mathsf{Ker}\, \phi \subset \mathsf{Im}\, d$. This proves the existence of $C_{[n+1]}$ and hence completes the proof of the theorem. □

If $\mu', \mu'' \in \mathrm{Hom}(\otimes^2 \mathcal{T}_{\mathcal{H}}, \mathcal{T}_{\mathcal{H}})$ are two products as in Theorem 4.6.1, then

$$\varUpsilon \circ (\mu' - \mu'') = D^\Gamma B$$

for some $B \in \mathrm{Hom}(\otimes^2 \mathcal{T}_{\mathcal{H}}, \mathfrak{g} \otimes \mathcal{O}_{\mathcal{H}})$, and hence, by Lemma 4.7.1,

$$\mu' - \mu'' = \delta\chi$$

for some odd $\chi \in \mathrm{Hom}(\otimes^2 \mathcal{T}_{\mathcal{H}}, \mathcal{T}_{\mathcal{H}})$, i.e. μ' and μ'' are what is called *homotopy equivalent*.

On the other hand, if μ'' is a product with the properties stated by Theorem 4.6.1, then, for any odd $\chi \in \mathrm{Hom}(\otimes^2 \mathcal{T}_{\mathcal{H}}, \mathcal{T}_{\mathcal{H}})$, the product

$$\mu' := \mu'' + \delta\chi$$

also enjoyes the properties of Theorem 4.6.1. Indeed, by Corollary 4.6.3,

$$\begin{aligned} \varUpsilon \circ \mu'' &= \varUpsilon \circ \mu' + \varUpsilon \circ (\delta\chi) \\ &= \varUpsilon \circ \mu' + D^\Gamma(\varUpsilon \circ \chi) \end{aligned}$$

and hence

$$[\varUpsilon \cdot \varUpsilon] = [\varUpsilon \circ \mu'] = [\varUpsilon \circ \mu']$$

in the cohomology sheaf $\mathsf{Ker}\, D^\Gamma / \mathsf{Im}\, D^\Gamma$.

Thus the set of products μ induced on $\mathcal{T}_{\mathcal{H}}$ by Theorem 4.6.1 is a principal homogeneous space over the Abelian group $\delta \mathrm{Hom}_{\tilde{1}}(\otimes^2 \mathcal{T}_{\mathcal{H}}, \mathcal{T}_{\mathcal{H}})$. Hence all induced products on each stalk of $\mathcal{T}_{\mathcal{H}}$ combine into a single homotopy class which we call *induced*.

4.7.2. Theorem. *The sheaf $\mathcal{T}_{\mathcal{H}}$ is canonically a sheaf of induced homotopy classes of differential algebras.*

Proof. We have to show that the homotopy class of products induced on $\mathcal{H}$ is an invariant of the dG-algebra under consideration, i.e. that it is independent of the choice of a quasi-isomorphism ϕ and on the choice of a Master function Γ used in its construction. In view of Remark 4.1.3(iv), it is enough to check the invariance of the product under the gauge transformations,

$$\Gamma \longrightarrow \Gamma^g := e^{\mathrm{ad}_g}\Gamma - \frac{e^{\mathrm{ad}_g} - 1}{\mathrm{ad}_g}(d + \vec{\partial})g, \qquad g \in \Gamma(\mathcal{H}, \mathfrak{g}_{\tilde{1}} \otimes \mathcal{H}).$$

A straightforward analysis of the basic equation (5) shows that gauge transformation changes the tensor A,

$$A^g = e^{\mathrm{ad}_g}\left(A(X,Y) - G\cdot\Upsilon - \Upsilon\cdot G + G\cdot D^\Gamma G + G\circ\mu\right),$$

where $G \in \mathrm{Hom}(\mathcal{T}_\mathcal{H}, \mathfrak{g}\otimes\mathcal{O}_\mathcal{H})$ is given by

$$G(X) := (-1)^{\tilde{X}}\frac{e^{\mathrm{ad}_g}-1}{\mathrm{ad}_g}(d+\vec{\partial})\overrightarrow{X}g,$$

but leaves the product invariant, $\mu^g = \mu$. $\square$

4.8. Identity in $\mathfrak{g} \Rightarrow$ identity in $\mathcal{T}_\mathcal{H}$. If the dG-algebra under consideration, $\mathfrak{g}$, has the identity e_0, and the versal solution Γ is approprietly normalized (see Remark 3.1.1), then

$$\begin{aligned}\overrightarrow{\delta(e)}\,\Gamma &= \overrightarrow{[\partial, e]}\,\Gamma\\ &= \vec{\partial}e_0 + \vec{e}(d\Gamma + \frac{1}{2}[\Gamma\bullet\Gamma])\\ &= 0 + de_0 + [e_0\bullet\Gamma]\\ &= 0,\end{aligned}$$

so that $\delta(e) = 0$. We shall show next that the induced homotopy class of differential algebras on each stalk of $\mathcal{T}_\mathcal{H}$ containes a canonical subclass of *unital* differential algebras.

4.8.1. Theorem. *If $\mathfrak{g}$ has the identity e_o, then $\mathcal{T}_\mathcal{H}$ is canonically a sheaf of induced homotopy classes of differential algebras with the identity e.*

A sketch of the proof. It is enough to show that there exists a δ-closed element, $\mu \in \mathrm{Hom}_{\tilde{0}}(\otimes^2\mathcal{T}_\mathcal{H}, \mathcal{T}_\mathcal{H})$, such that, for arbitrary (constant) X, $Y \in \mathcal{T}_\mathcal{H}$, the equation

$$\overrightarrow{X}\Gamma\cdot\overrightarrow{Y}\Gamma = \overrightarrow{\mu(X,Y)}\,\Gamma + (D^\Gamma A)(X,Y)$$

holds for some $A \in \mathrm{Hom}(\mathcal{T}_\mathcal{H}^{\otimes 2}, \mathfrak{g}\otimes\mathcal{O}_\mathcal{H})$ satisfying $A(X,e) = X$ and $A(e,Y) = Y$ (cf. (6)).

Recall (see the proof of Theorem 4.6.1) that at the lowest order we have,

$$\begin{aligned}\mu_{[0]}(X,Y) &= \phi(\overrightarrow{X}\Gamma_{[1]}\cdot\overrightarrow{Y}\Gamma_{[1]})\\ dA_{[0]}(X,Y) &= \overrightarrow{X}\Gamma_{[1]}\cdot\overrightarrow{Y}\Gamma_{[1]} - \overrightarrow{\mu_{[0]}(X,Y)}\,\Gamma_{[1]}.\end{aligned}$$

and hence $\mu_{[0]}(X,e) = \mu_{[0]}(e,X) = X$ and $dA_{[0]}(X,e) = dA_{[0]}(e,X) = 0$. We claim that $A_{[0]}$ can be chosen to satisfy $A_{[0]}(X,e) = A_{[0]}(e,X) = 0$. This can be achieved by a replacement,

$$\begin{aligned}A_{[0]}(X,Y) \;\to\; A'_{[0]}(X,Y) \;:=&\; A_{[0]}(X,Y) - A_{[0]}(X,e)\cdot\overrightarrow{Y}\Gamma - (-1)^{\tilde{X}}\overrightarrow{X}\Gamma\cdot A_{[0]}(e,Y)\\ &+ (-1)^{\tilde{X}}\overrightarrow{X}\Gamma\cdot A_{[0]}(e,e)\overrightarrow{Y}\Gamma,\end{aligned}$$

which satisfies,

$$dA'_{[0]}(X,Y) = dA_{[0]}(X,Y), \qquad A'_{[0]}(X,e) = A'_{[0]}(e,X) = 0.$$

This observation allows us to include into the inductive procedure of the proof of Theorem 4.6.1 the additional assumptions

$$\mu_{(n)}(X,e)=\mu_{(n)}(e,X)=X, \qquad A_{(n)}(X,e)=A_{(n)}(e,X)=0,$$

and show, by exactly the same argument as in the case $n=0$ above, that they hold true for $n+1$.

Thus there does exist a product μ from the induced homotopy class satisfying $\mu(X,e)=\mu(e,X)=X$. It is defined uniquely up to a transformation

$$\mu \longrightarrow \mu+\delta\chi$$

with χ satisfying $\chi(X,e)=\chi(e,X)=0$ for arbitrary $X\in\mathcal{T}_{\mathcal{H}}$. Thus what is well-defined is the induced homotopy class of *unital* differential algebras. □

4.9. Theorem. *For any (unital) dG-algebra $\mathfrak{g}$, the tangent sheaf $\mathcal{T}_{\mathcal{H}}$ to its cohomology supermanifold is canonically a sheaf of homotopy classes of (unital) A_∞-algebras with*

(i) $\mu_1=[\partial,\ldots]$, *∂ being the Chen's vector field, and*

(ii) the homotopy class of μ_2 being the induced homotopy class as in Theorem 4.6.1.

A sketch of the proof. By Theorem 4.6.1, there exists a product $\mu_2\in\mathrm{Hom}(\otimes^2\mathcal{T}_{\mathcal{H}},\mathcal{T}_{\mathcal{H}})$ satisfying the equation

$$\overrightarrow{\mu_2(X_1,X_2)}\Gamma=\overrightarrow{X_1}\Gamma\cdot\overrightarrow{X_2}\Gamma+(D^\Gamma A_2)(X_1,X_2)$$

for some odd $A_2\in\mathrm{Hom}(\otimes^2\mathcal{T}_{\mathcal{H}},\mathfrak{g}\otimes\mathcal{O}_{\mathcal{H}})$ and arbitrary $X_1,X_2\in\mathcal{T}_{\mathcal{H}}$. We have, in the notations of subsection 3.2.3,

$$\begin{aligned}\overrightarrow{\Lambda_3(X_1,X_2,X_3)}\Gamma &= \overrightarrow{\mu_2(X_1,\mu_2(X_2,X_3))-\mu_2(\mu_2(X_1,X_2),X_3)}\Gamma\\ &= \overrightarrow{X_1}\Gamma\cdot\left(\overrightarrow{X_2}\Gamma\cdot\overrightarrow{X_3}\Gamma\right)+(D^\Gamma A_2)(X_1,\mu_2(X_2,X_4))+\Upsilon(X_1)\cdot(D^\Gamma A_2)(X_2,X_3)\\ &\quad-\left(\overrightarrow{X_1}\Gamma\cdot\overrightarrow{X_2}\Gamma\right)\cdot\overrightarrow{X_3}\Gamma-(D^\Gamma A_2)(\mu_2(X_1,X_2),X_3)-(D^\Gamma A_2)(X_1,X_2)\cdot\Upsilon(X_3)\\ &= (D^\Gamma B_3)(X_1,X_2,X_3),\end{aligned}$$

where

$$\begin{aligned}B_3(X_1,X_2,X_3) &:= (-1)^{\tilde{X}_1}\Upsilon(X_1)\cdot A_2(X_2,X_3)-A_2(X_1,X_2)\cdot\Upsilon(X_3)\\ &\quad+A_2(X_1,\mu_2(X_2,X_3))-A_2(\mu_2(X_1,X_2),X_3).\end{aligned}$$

Here we used associativity of the dot product in $\mathfrak{g}$, D^Γ-closedness of Υ and δ-closedness of μ_2.

By Lemma 4.7.1, there exists $\mu_3\in\mathrm{Hom}_{\tilde{0}}(\otimes^3\mathcal{T}_{\mathcal{H}},\mathcal{T}_{\mathcal{H}})$ such that the 3rd order associativity condition, $\Lambda_3=\delta\mu_3$, is satisfied, and

$$\overrightarrow{\mu_3(X_1,X_2,X_3)}\Gamma=B_3(X_1,X_2,X_3)+(D^\Gamma A_3)(X_1,X_2,X_3)$$

for some $A_3\in\mathrm{Hom}_{\tilde{0}}(\otimes^3\mathcal{T}_{\mathcal{H}},\mathfrak{g}\otimes\mathcal{O}_{\mathcal{H}})$.

Exactly the same procedure constructs inductively all the higher order products $\mu_n\in\mathrm{Hom}_{\tilde{n}}(\otimes^n\mathcal{T}_{\mathcal{H}},\mathcal{T}_{\mathcal{H}})$ satisfying the higher order associativity conditions:

Step 1. Assume that we have constructed $\mu_k \in \mathrm{Hom}_{\tilde{k}}(\otimes^k \mathcal{T}_{\mathcal{H}}, \mathcal{T}_{\mathcal{H}})$ and $A_k \in \mathrm{Hom}_{\tilde{k}+\tilde{1}}(\otimes^k \mathcal{T}_{\mathcal{H}}, \mathfrak{g}\otimes \mathcal{O}_{\mathcal{H}})$, $k = 2, \ldots, n-1$, such that $\Lambda_k = \delta\mu_k$ (k-th order associativity condition) and

$$\begin{aligned}
\overrightarrow{\mu_k(X_1, \ldots, X_k)}\Gamma \quad &= \quad \sum_{i+j=k} (-1)^{(j+1)(\tilde{X}_1+\ldots+\tilde{X}_i)+i+1} A_i(X_1, \ldots, X_i) \cdot A_i(X_{i+1}, \ldots, X_k) \\
&\quad + \sum_{\substack{i+j=k+1\\ i\geq 2\\ j\geq 2}} \sum_{l=0}^{i-1} (-1)^r A_i(X_1, \ldots, X_l, \mu_j(X_{l+1}, \ldots, X_{l+j}), X_{l+j+1}, \ldots, X_k) \\
&\quad + (D^\Gamma A_k)(X_1, \ldots, X_k) \\
&=: \; B_k(X_1, \ldots, X_k) + (D^\Gamma A_k)(X_1, \ldots, X_k),
\end{aligned}$$

where we have set $A_1 := \Upsilon$ and $r = \tilde{j}(\tilde{X}_1 + \ldots \tilde{X}_1) + \tilde{l}(\tilde{j} - 1) + (\tilde{i} - \tilde{1})\tilde{j} + 1$.

Step 2. Use the above expressions for $\Upsilon \circ \mu_k$, $k = 2, \ldots, n-1$, to show that

$$\overrightarrow{\Lambda_n(X_1, \ldots, X_n)}\Gamma = (D^\Gamma B_n)(X_1, \ldots, X_n).$$

Step 3. Apply Lemma 3.6.1 to conclude that there exists μ_n such that $\Lambda_n = \delta\mu_n$ (n-th order associativity condition) and $\Upsilon \circ \mu_n = B_n + D^\Gamma A_n$ for some $A_n \in \mathrm{Hom}_{\tilde{n}+\tilde{1}}(\otimes^k \mathcal{T}_{\mathcal{H}}, \mathfrak{g} \otimes \mathcal{O}_{\mathcal{H}})$

Finally, we note that at each stage of the above construction the nth product μ_n is defined only up to a δ-exact term, δf_n. These arbitrary terms combine all together into a homotopy of the A_∞-structure $(\mathcal{T}_{\mathcal{H}}, \mu_*)$. $\square$

4.9.1. Corollary. *The cohomology sheaf on $\mathcal{H}$,*

$$\mathcal{H}_{\mathbf{H}} := \frac{\mathsf{Ker}\ \delta}{\mathsf{Im}\ \delta},$$

is canonically a sheaf of induced (unital) associative algebras.

4.9.2. Corollary. *The tangent sheaf, $\mathcal{T}\mathcal{M}_{\text{smooth}}$, to the smooth part of the extended Kuranishi space is canonically a sheaf of induced (unital) associative algebras.*

4.9.3. The Euler field. If the dG-algebra $\mathfrak{g}$ under consideration is $\mathbb{Z}$-graded, then the cohomology $\mathbf{H}$ and hence its dual $\mathbf{H}^*$ are also $\mathbb{Z}$-graded. We make $k[[t]]$ into a $\mathbb{Z}$-graded ring my setting

$$k[[t]] = \odot^\bullet \mathbf{H}^*[2].$$

This also induces $\mathbb{Z}$-grading in the sheaves $\mathcal{O}_H$ and $\mathcal{T}_{\mathcal{H}}$ on the supermanifold $\mathcal{H}$.

If $\{[e_i]\}$ is a basis in $\mathbf{H}$ and $\{t^i\}$ are the associated linear coordinates on $\mathcal{H}$ as in Sect. 2.2, then

$$|t^i| = 2 - |[e_i]|.$$

With this choice of $\mathbb{Z}$-grading on $\mathcal{O}_{\mathcal{H}}$ we ensure that $|\Gamma| = 2$ and hence $|\partial| = 1$, $|\delta| = 1$, and $|\mu_n| = n$ for all the induced higher order products on $\mathcal{T}_{\mathcal{H}}$.

The *Euler field* on M is, by definition, the derivation E of $k[[t]]$ given by

$$Ef = \frac{1}{2}|f|f, \qquad \forall f \in k[[t]].$$

In coordinates,

$$E = \frac{1}{2}\sum_i |t^i| t^i \frac{\partial}{\partial t^i}.$$

This vector field generates the scaling symmetry on $(\mathcal{H}, \mu_*)$ (cf. [BK, Ma2]). If we decompose

$$\mu_n\left(\frac{\partial}{\partial t^{i_1}}, \ldots, \frac{\partial}{\partial t^{i_n}}\right) = \sum_k \mu^k_{i_1\ldots i_n}(t)\frac{\partial}{\partial t^k},$$

then, as follows from the explicit construction of μ_n given in the proof of Theorem 4.9,

$$E\,\mu^k_{i_1\ldots i_n} = \frac{1}{2}\left(|t^k| - |t^{i_1}| - \ldots - |t^{i_n}| + n\right)\mu^k_{i_1\ldots i_n}.$$

Note also that in the presence of identity, $[e, E] = e$.

4.9.4. The perturbative proof of Theorem A. The required statement follows immediately from the graded commutative version of Theorem 4.9 and Sect. 4.9.3. □

4.9.5. A generalization to G_∞-algebras. In the perturbative construction of the F_∞-functor for dG-algebras the odd Poisson identity was used in a few places. For example, in the construction of μ_2 the only place where we relied on it was the cyclicity of $\Upsilon \cdot \Upsilon$,

$$D^\Gamma(\Upsilon \cdot \Upsilon) = 0.$$

However, a glance at the basic equation (6) (and its higher order analogues in Sect. 4.9) shows that the perturbative argument stands if the cyclicity (and its analogues) holds only up to a homotopy. Therefore, the generalization from dG-algebras to G_∞-algebras is straightforward, affecting only auxiliary tensors A_n.

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