Topology Change and Vector Modules on Noncommutative Surfaces of Rotation

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Abstract

A non associative, noncommutative algebra is defined that may be interpreted as a set of vector modules over a noncommutative surface of rotation. Two of these vector modules are identified with the analogues of the tangent and cotangent spaces in noncommutative geometry, via the definition of an exterior derivative. This derivative reduces to the standard exterior derivative in the commutative limit. The representation of this algebra is used to investigate a simple topology change where two connected compact surfaces coalesce to form one such surface.

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I Introduction

Noncommutative geometry has been proposed by many people [1][2] as a candidate for the formulation of quantum gravity, principally because it combines the noncommutative structure of quantum mechanics with the geometrical structure of general relativity. Noncommutative geometry has also been proposed [3] in the membrane picture of particle mechanics.

There are two main tasks in noncommutative geometry: The first is to find a one-parameter set of algebras $\mathcal{A}(\varepsilon)$ which are noncommutative for $\varepsilon \neq 0$ and commutative when $\varepsilon = 0$. We require that we can embed $\mathcal{A}(\varepsilon = 0)$ into $C(\mathcal{M}, \mathbb{C})$ the commutative algebra of complex valued function on a manifold $\mathcal{M}$. Here ε plays the rôle of $\hbar$ in quantum mechanics. This is similar to the quantisation of Poisson manifolds and investigations of $\star$-products. We call $\mathcal{A}(\varepsilon)$ the noncommutative version of $\mathcal{M}$, although it is the algebra not the manifold which fails to commute. Thus if $\mathcal{M}$ is the sphere $\mathcal{A}(\varepsilon)$ is called the noncommutative sphere [4]. In this article $\mathcal{M}$ will be a surface of rotation. For a function $\rho : \mathbb{R} \mapsto \mathbb{R}$ we generate a surface of rotation by rotating the section of the graph $y = \rho(x)$ which is below the x-axis, about the x-axis. Thus the corresponding algebra $\mathcal{A}(\varepsilon)$ is called the noncommutative surfaces of rotation (from now on this phrase is abbreviated to **NCSR**). These are defined in section II and were first described in [5], together with some simple facts such as their Poisson structure and representations.

The second task is to write down the objects studied in differential geometry, such as tangent bundles, cotangent bundles, exterior algebras, metric tensors, connections and curvature, in terms of elements of the algebra $C(\mathcal{M}, \mathbb{C})$ and then extend these definitions for $\mathcal{A}(\varepsilon)$.

There are two key properties required of tangent vector fields. Firstly that they should be derivatives, i.e. follow Leibniz rule, and secondly that they should form a module over the algebra of functions. (That is one can multiply a vector with a scalar to give another vector.) It turns out that for noncommutative geometry these two properties are incompatible, and one must choose either to have vectors which are derivatives, or vectors which form a module.

The standard method is to choose vectors which form derivatives [1]. If the underlying algebra $\mathcal{A}$ is a matrix algebra then it is easy to show that all such vectors are inner. That is if $\xi : \mathcal{A} \mapsto \mathcal{A}$ such that $\xi(fg) = \xi(f)g + f\xi(g)$ for all $f, g \in \mathcal{A}$ then there exists $h \in \mathcal{A}$ such that $\xi = \mathrm{ad}_h$ where $\mathrm{ad}_h f = [h, f] = hf - fh$. Clearly if ξ is inner then $f\xi$ is not inner. In section IV.3 we show that the same is true for the algebra of functions on a NCSR.

In [6] the author gives an alternative method of defining tangent vectors on the noncommutative sphere. These vectors do form a (one sided) module over the noncommutative sphere but are derivatives only in the commutative limit. That is $\xi(fg) = \xi(f)g + f\xi(g) + O(\varepsilon)$. This article may be seen as the result of giving a NCSR a vector bundle structure similar to that defined for the sphere in [6].

One consequence of giving a NCSR a vector bundle structure is topology change. The idea that quantum gravity should lead to topology change is not new, but up to now has been quite vague. The topology change described here is quite simple. It consists of two or more disjoint surfaces each topologically equivalent to the sphere coalescing to form one such surface. Looking at figure 1 we see that as a curve is raise and lowered the corresponding surface of rotation has a different number of disjoint connected components. This is an example from Morse theory.

I.1 Structure of article

We start in section II with a review of NCSR. It is necessary to define three separate but related algebra $\mathcal{A}^C$, $\mathcal{A}^R$ and $\mathcal{A}^N$. It is $\mathcal{A}^R$ which is closest to the algebra defined in [5]. All three algebras are defined with respect to a C^1 real function ρ and a parameter $\varepsilon > 0$. For $\mathcal{A}^R$ a third parameter $R \in \mathbb{R}$ is given. In section II.1 we give the finite dimensional unitary representations of these three algebras.

When $\varepsilon = 0$ the algebras $\mathcal{A}^C$, $\mathcal{A}^R$ and $\mathcal{A}^N$ become commutative algebras. In section II.2 we give a topological meaning to the algebra $\mathcal{A}^R$. This algebra forms a dense subalgebra of $C(\mathcal{M}, \mathbb{C})$ the algebra of complex valued continuous functions from the surface of rotation $\mathcal{M}$. The surface $\mathcal{M}$ which depends on ρ and R is a collection of disjoint surface, each surface topologically equivalent to the sphere. If the curve ρ has more than one local minima then the number of disjoint surface and hence topology of $\mathcal{M}$ will depend on R. This can be seen in figure 1.

The limit as $\varepsilon \to 0$ of the commutator in $\mathcal{A}^R(\rho, \varepsilon)$ gives $\mathcal{M}$ a Poisson structure. In fact this structure is symplectic. This is calculated in section II.3, and this is used to write the exterior derivative, metric, hodge dual and Laplace equation in a form easiest to convert to the noncommutative case.

Throughout sections III and IV we assume that ρ is trivial, that is, amongst other requirements, that it has just one local minima. Thus all corresponding surfaces $\mathcal{M}(\rho, R)$ will be connected and there is no change in topology to different R. However since ρ is trivial we can define the algebra $\mathcal{B}$, which depends on ρ and $\varepsilon > 0$. An important subalgebra of $\mathcal{B}$ is $\mathcal{A}^N$.

The representation of $\mathcal{B}$, given in section III.1, is a Hilbert space $\mathcal{G}$ called a trivial multi-topology lattice, since it may be thought of as a two dimensional lattice. This lattice is decomposed into the direct some of vector spaces V_n. Each V_n is an n dimensional unitary representation of the algebras $\mathcal{A}^N$ and $\mathcal{A}^C$. Also V_n is a unitary representation of $\mathcal{A}^R$ if R has a certain value.

In section III.3 we show how, if $\rho(z) = z^2$ then the algebra $\mathcal{A}^R$ is isomorphic to the algebra $su(2)$, whilst the algebra $\mathcal{B}$ is equivalent to the the product of two Heisenberg-Weyl algebras. Thus the embedding of $\mathcal{A}^R \subset \mathcal{B}$ corresponds to the Jordan-Schwinger representation of $su(2)$. In [6] we have used this Jordan-Schwinger representation to construct a space of vector modules over $\mathcal{A}^R(\rho, \varepsilon)$ for $\rho = z^2$. In section IV this process is repeated for a general trivial ρ. We first define the nonassociative algebra (Ψ, μ) which because it is nonassociative the product μ is given explicitly. This is decomposed into a set of right modules Ψ_r over $\mathcal{A}^R$, where $\Psi_0 = \mathcal{A}^R$, and $r \in \mathbb{Z}$. Using the results of section II.3, in section IV.2 we show how to interpret Ψ_{-2} as the space of 1-forms over $\mathcal{M}$ and construct an exterior derivative d, which satisfies Leibniz only in the commutative limit. We also interpret Ψ_2 as the analogous space (module) of tangent vector fields $T\mathcal{M}$. This defines a vector as an object which can be multiplied by a scalar to give a vector, but which is a derivative only in the commutative limit. In section IV.3 we show that any operator that obeys Leibniz must be inner, and thus the fact the our operator does not obey Leibniz must be accepted.

In section V we return our attention to the more general ρ. We can no longer form the algebra $\mathcal{B}$, however we can still investigate the multi-topology lattices $\mathcal{G}$. Here, once again, $\mathcal{G}$ is a direct sum $\oplus_s V_s$ where each V_s is a representation of $\mathcal{A}^N$. Since $\mathcal{G}$ encodes the different topologies of $\mathcal{M}$, this justifies the name multi-topology lattice.

Unfortunately problems may occur near the topology change, and this requires one of four compromises to be made. The detail of the construction of $\mathcal{G}$ and the compromises is given in sections V.1 to V.4, where we demonstrate that one can always construct a multi-topology lattice. In section V.5 we describe some operators which can be interpreted as operators for topology change. We indicate that a topology change can be thought of as a block diagonal matrix. Finally in section V.6 we propose an operator algebra which may model the dynamics of surfaces which, although they remain axially symmetric, change shape and interact.

Finally in section VI we discuss some of the areas of research that follow from this article.

II Review of Noncommutative Surfaces of Rotation (NCSR)

For the purposes of this article we will review three closely related but different algebra $\mathcal{A}^C, \mathcal{A}^R, \mathcal{A}^N$. All three can be referred to as the NCSR so we will use the correct symbol if we need to be precise. The algebra $\mathcal{A}^R$ is equivalent the algebra given in [5] where NCSR were first introduced.

Let us define the domains $\mathcal{C}_{\mathbb{R}} = C^1(\mathbb{R} \mapsto \mathbb{C})$ and $\mathcal{C}_{\mathbb{R}^2} = C^1(\mathbb{R} \times \mathbb{R} \mapsto \mathbb{C})$. We have chosen C^1 but similar results exist for C^k or C^ω. We note that both these domains are commutative algebras where the product fg is the pointwise multiplication.

The algebras $\mathcal{A}^C, \mathcal{A}^R, \mathcal{A}^N$ are all defined with respect to a function ρ. Throughout this article we will assume that

- $\rho \in C^1(\mathbb{R} \mapsto \mathbb{R})$,
- $\rho \to \infty$ as $x \to \pm\infty$,
- ρ has a finite number of turning points
- there is no interval in which ρ is a constant.

(1)

All three algebras also require that we specify $\varepsilon \in \mathbb{R}$ with $\varepsilon \geq 0$. There are some results which can be reformulated for negative or even complex ε but for this article we will assume that $\varepsilon \geq 0$. For the algebra $\mathcal{A}^R$ there is a third parameter $R \in \mathbb{R}$ for which we make no further assumptions.

The three algebras are given in table I together with a fourth algebra $\mathcal{A}^{NC}$ which is used to relate the other three algebras to each other. We shall call the elements X_+ and X_- the ladder operators, and the elements X_0 and N_0 the diagonal operators. These names come from representations. The expression $X_\pm^r$ means

$$X_\pm^r = \begin{cases} (X_+)^r & \text{if } r \geq 0 \\ (X_-)^{-r} & \text{if } r < 0 \end{cases} \tag{2}$$

We see from the list of generators for each algebra that the ladder operators X_+, X_- are handle differently from the diagonal operators X_0, N_0. In general any $\mathcal{C}_{\mathbb{R}}$ function of X_0 or $\mathcal{C}_{\mathbb{R}^2}$ function of (X_0, N_0) are allowed, but only polynomials of X_+ and X_-. We have to handle $\mathcal{C}_{\mathbb{R}}$ function of X_0 because $\rho \in \mathcal{C}_{\mathbb{R}}$ and ρ appears in the defining equations of the

algebra. If we were to allow $\mathcal{C}_{\mathbb{R}}$ functions of X_+ for example this would cause problems with the existence or otherwise of limits.

Lemma 1. *For all four algebras*

$$[X_0, X_\pm] = \pm\varepsilon X_\pm \tag{3}$$

whilst for $\mathcal{A}^N$ and $\mathcal{A}^{NC}$

$$[N_0, X_\pm] = 0 \tag{4}$$

for $\mathcal{A}^R$ we have

$$X_- X_+ = \rho(R) - \rho(X_0 + \varepsilon) \tag{5}$$

and for $\mathcal{A}^N$ we have

$$X_- X_+ = \rho(N_0) - \rho(X_0 + \varepsilon) \tag{6}$$

Every element can be written uniquely as (83), (86), (89), or (93). This form is known as normal ordering.

Proof. (3) and (4) follow from the respective quotient equations. (5) and (6) follow by considering $X_+X_-X_+$. □

The algebra $\mathcal{A}^{NC}$ may be thought of as the extensions of $\mathcal{A}^C$ with the central element N_0. This algebra is defined so we have the follow lemma.

Lemma 2. *The relationship between the three algebras for a NCSR is given by the following diagram:*

$$\begin{tikzcd} \mathcal{A}^C \arrow[r, hook] \arrow[rrd, "q_2"] & \mathcal{A}^{NC} \arrow[r, "q_1"] & \mathcal{A}^N \arrow[d, "q_3"] \\ & & \mathcal{A}^R \end{tikzcd} \tag{7}$$

where the hooked arrows refer to the natural embedding, q_1 is the quotient $X_+X_- - \rho(N_0) - \rho(X_0) \sim 0$, q_2 is the quotient $X_+X_- - \rho(R) - \rho(X_0) \sim 0$, and q_3 is the quotient $N_0 - R \sim 0$.

Proof. Trivial. □

II.1 "Standard" Representations of ρ

A **standard unitary representation** of $\mathcal{A}^C$, $\mathcal{A}^R$ is $\mathcal{A}^N$ defined with respect to the pair (J, V) where $J = \{z \mid z^\downarrow \le z \le z^\uparrow\} \subset \mathbb{R}$ is an interval, called the **representation interval** such that

$$\begin{aligned} &|J|/\varepsilon = (z^\uparrow - z^\downarrow)/\varepsilon \in \mathbb{N} \\ &\rho(z^\downarrow) = \rho(z^\uparrow) \\ &\rho(z) \le \rho(z^\uparrow)\ \forall z \in J \end{aligned} \tag{8}$$

and V is a finite dimensional vector space with dimension $\dim(V) = |J|/\varepsilon$. The basis of V is $|m\rangle$ where $m \in \mathbb{Z}$ and $m^\downarrow \le m \le m^\uparrow$. Here $m^\downarrow$ is an arbitrary integer and $m^\uparrow = m^\downarrow + \dim V - 1$.

The standard unitary representation of $\mathcal{A}^C$ with respect to the pair (J, V) is given by

$$\begin{aligned} f(X_0)|m\rangle &= f(z^\downarrow - \varepsilon m^\downarrow + \varepsilon m)|m\rangle \\ X_+|m\rangle &= (\rho(z^\downarrow) - \rho(z^\downarrow - \varepsilon m^\downarrow + \varepsilon m + \varepsilon))^{1/2}|m+1\rangle \\ X_-|m\rangle &= (\rho(z^\downarrow) - \rho(z^\downarrow - \varepsilon m^\downarrow + \varepsilon m))^{1/2}|m-1\rangle \end{aligned} \tag{9}$$

The standard representation of $\mathcal{A}^N$ and $\mathcal{A}^{NC}$ with respect to (J, V) is given by

$$f(X_0, N_0)|m\rangle = f(z^\downarrow - \varepsilon m^\downarrow + \varepsilon m, z^\downarrow)|m\rangle \tag{10}$$

and the last two equations of (9).

There exists a standard representation of $\mathcal{A}^R(\rho, \varepsilon, R)$ with respect to (J, V) if and only if $R = z^\downarrow$. In this case the representation is given again by (9).

Therefore unlike $\mathcal{A}^C$ and $\mathcal{A}^N$ where there exists an infinite set of standard representation, for $\mathcal{A}^R$ there exists either one or zero standard representation. The infinite set of standard representation of $\mathcal{A}^N$ is used in the constructions of the multi-topology lattice representation of the NCSR. See section III.1 for the representation of trivial NCSR, and section V for the representation of non-trivial NCSR.

II.2 Topology of ρ

For a given $R \in \mathbb{R}$, let $\mathcal{M} = \mathcal{M}(\rho, R)$ be the surface in $\mathbb{R}^3$

$$\mathcal{M} = \mathcal{M}(\rho, R) = \{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 = \rho(R) - \rho(x^3)\} \tag{11}$$

If $\rho(R)$ is the value of a maxima of ρ then the set of points $\mathcal{M}$ obeying (11) do not form a manifold, since two of the surfaces are glued at a point. If $\rho(R)$ is the value of a minima of ρ then $\mathcal{M}$ describes a manifold but one of the disjoint pieces is simply a point. The set of R where $\rho(R)$ is the value of a maxima or minima of ρ are called singular. Since we are interested in describing surfaces we must assume that R is nonsingular.

We give a coordinate system for $\mathcal{M}$ as (ϕ, z) where

$$z \in \{z \mid \rho(R) \geq \rho(z)\} \tag{12}$$

and $0 \leq \phi < 2\pi$ and give the coordinate chart : $(\phi, z) \mapsto \mathcal{M}$ as

$$x^1 = (\rho(R) - \rho(z))^{1/2} \cos\phi \qquad x^2 = (\rho(R) - \rho(z))^{1/2} \sin\phi \qquad x^3 = z \tag{13}$$

For a given nonsingular R, the surface $\mathcal{M}(\rho, R)$ is the disjoint union of connected surfaces, each one topologically equivalent to the sphere. The surfaces are in one to one correspondence with the intervals in the set (12).

We have followed the standard procedure of noncommutative geometry by saying the algebra $\mathcal{A}^R(\rho, 0, R)$ is "equivalent" to the manifold $\mathcal{M}(\rho, R)$, and that $\mathcal{A}^R(\rho, \varepsilon, R)$ for $\varepsilon \neq 0$ is the noncommutative analogue of $\mathcal{M}(\rho, R)$ or, alternatively, that $\mathcal{A}^R(\rho, \varepsilon, R)$ is an ε perturbation of $\mathcal{M}(\rho, R)$. Of course an algebra is not equivalent to a manifold. What we mean is that there exists the map

$$\begin{gathered} : \mathcal{A}^R \mapsto C^0(\mathcal{M} \mapsto \mathbb{C}) \\ \sum_r X_\pm^r f_r(X_0) \mapsto \sum_r e^{ir\phi} f_r(z) \end{gathered} \tag{14}$$

The set $C^0(\mathcal{M} \mapsto \mathbb{C})$ is given a commutative algebraic structure via pointwise multiplication, so that (14) is a homomorphism. (14) is also and injection, so we can view $\mathcal{A}^R(\rho, 0, R) \subset C^0(\mathcal{M} \mapsto \mathbb{C})$. We give $C^0(\mathcal{M} \mapsto \mathbb{C})$ the L^∞ topology so $\mathcal{A}^R$ forms a dense subset of $C^0(\mathcal{M} \mapsto \mathbb{C})$. Thus we can approximate any continuous function on $\mathcal{M}$ with a series of functions in $\mathcal{A}^R$ and then deform these functions to give elements in $\mathcal{A}^R(\rho, \varepsilon, R)$.

We define the **Topology intervals** $\mathbf{I}_t$ which is a set of intervals corresponding to a single connected surface as $\rho(R)$ moves up and down. The topology intervals are labelled $\mathbf{I}_t$ where $t \in \mathcal{T}$ and $\mathcal{T}$ is a finite index set. These are defined as:

- If $I = \{z | z^\downarrow \le z \le z^\uparrow\}$ then $I \in \mathbf{I}_t$ for some $t \in \mathcal{T}$ if and only if
$$\rho(z^\downarrow) = \rho(z^\uparrow) \qquad \text{and} \qquad \rho(z) < \rho(z^\downarrow) \qquad \forall\, z \in I \backslash \{z^\downarrow, z^\uparrow\} \tag{15}$$
- If $I \in \mathbf{I}_t$ and $I' \in \mathbf{I}_t$ then either $I \subset I'$ or $I' \subset I$ and there are no maxima in the sets $I \backslash I'$ or $I' \backslash I$.

In figure 1 the curve ρ has 5 topology intervals, with I_1, I_2, I_3 and I_4 belonging to different intervals. I_3 and I_5 belong to the same topology interval.

Lemma 3. *The topology interval $\mathbf{I}_t$ may be described uniquely with respect to four parameters $z_1^{(t)}$, $z_2^{(t)}$, $z_3^{(t)}$, $z_4^{(t)} \in \mathbb{R} \cup \pm\infty$ which satisfy the following*

- $z_1^{(t)} \le z_2^{(t)} \le z_3^{(t)} \le z_4^{(t)}$,
- $\rho(z_1^{(t)}) = \rho(z_4^{(t)})$, *and* $\rho(z_2^{(t)}) = \rho(z_3^{(t)})$,
- ρ *is decreasing between* $z_1^{(t)}$ *and* $z_2^{(t)}$
- ρ *is increasing between* $z_3^{(t)}$ *and* $z_4^{(t)}$
- $\rho(z) \le \rho(z_2^{(t)})$ *for all* $z_2^{(t)} \le z \le z_3^{(t)}$
- *Either* $-z_1^{(t)} = z_4^{(t)} = \infty$ *or* $z_1^{(t)}$ *is a local maxima or* $z_4^{(t)}$ *is a local maxima*
- *Either* $z_2^{(t)} = z_3^{(t)}$ *or there exists a point* $z_3^{(t)} < z_5^{(t)} < z_4^{(t)}$ *such that* $\rho(z_5^{(t)}) = \rho(z_3^{(t)})$

$\mathbf{I}_t$ *is now defined as the set of intervals*

$$\mathbf{I}_t = \left\{ I = \{z | z^\downarrow \le z \le z^\uparrow\} \;\middle|\; \rho(z^\downarrow) = \rho(z^\uparrow)\,,\ z_1^{(t)} \le z^\downarrow < z_2^{(t)}\,,\ z_3^{(t)} < z^\uparrow \le z_4^{(t)} \right\} \tag{16}$$

Proof. Trivial. (See Figure 2.) □

Lemma 4. *The number of topology intervals is bounded by*

$$|\mathcal{T}| \le \textit{number of maxima} + \textit{number of minima} \tag{17}$$

with equality when all the maxima have different values.

Proof. If the number of maxima is finite then there is one topology interval which extend to infinity. Each maxima now creates two new topology intervals, hence result. □

As R increases and crosses a singular point, the topology of the manifold undergoes a transition, as two or more adjacent intervals coalesce or a single interval splits. This corresponds to two or more surfaces coalescing or a single surface bifurcating. These changes can be encoded into a function $\pi_{\mathcal{T}} : \mathcal{T} \mapsto \mathcal{T} \cup \{\infty\}$ where $\mathbf{I}_{\pi_{\mathcal{T}}(t)}$ is directly above $\mathbf{I}_t$ or $\mathbf{I}_t = \infty$ if it is the highest topology interval. Thus

$$\rho(z_2^{(t_2)}) = \rho(z_1^{(t_1)}) \qquad \text{when} \quad t_2 = \pi_{\mathcal{T}}(t_1) \tag{18}$$

If $\pi_{\mathcal{T}}^{-1}\{s\} = \{t_1, t_2, \ldots, t_N\}$ then there is a transition where the topology intervals $\mathbf{I}_{t_1}, \mathbf{I}_{t_2}, \ldots, \mathbf{I}_{t_N}$ coalescing into the interval $\mathbf{I}_s$. This implies there are $N-1$ maxima all of the same value, one maxima between successive $\mathbf{I}_{t_i}$.

For example in figure 1, if we let $I_1 \in \mathbf{I}_1$, $I_2 \in \mathbf{I}_2$, $I_3, I_5 \in \mathbf{I}_3$, $I_4 \in \mathbf{I}_4$ and let $\mathbf{I}_5$ be the unbounded topology interval then $\pi_{\mathcal{T}}(1) = \pi_{\mathcal{T}}(2) = 4$, $\pi_{\mathcal{T}}(3) = \pi_{\mathcal{T}}(4) = 5$ and $\pi_{\mathcal{T}}(5) = \infty$.

II.3 The metric and differential structure of $\mathcal{M}$ in terms of the Poisson structure

The limit of the noncommutative structure gives $\mathcal{M}$ a Poisson structure [5]:

$$\{f, h\} = \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon} [f, h] \right) = -i \left(\frac{\partial f}{\partial z} \frac{\partial h}{\partial \phi} - \frac{\partial h}{\partial z} \frac{\partial f}{\partial \phi} \right) \tag{19}$$

We can write some of the standard objects of differential geometry simply in terms of the Poisson structure, the elements of $C(\mathcal{M}, \mathbb{C})$ and the elements dX_+, dX_- and dX_0. This is useful since these expression are the easiest to extend to the noncommutative case. Although in this article we will attempt only a definition of df by extending (20).

$$df = \frac{1}{\rho(R) - \rho(X_0)} X_- \Big(dX_+ \{X_0, f\} - dX_0 \{X_+, f\} \Big) \tag{20}$$

$$df = \frac{-1}{\rho(R) - \rho(X_0)} X_+ \Big(dX_- \{X_0, f\} - dX_0 \{X_-, f\} \Big) \tag{21}$$

The metric g on $\mathcal{M}$ is given by the pull back of the metric $\mathbb{R}^3$ using the mapping given by (13).

$$\begin{aligned} g &= dz \otimes dz + \tfrac{1}{2} dX_+ \otimes dX_- + \tfrac{1}{2} dX_- \otimes dX_+ \\ &= \frac{C(z)}{4(\rho(R) - \rho(z))} dz \otimes dz + (\rho(R) - \rho(z)) d\phi \otimes d\phi \end{aligned} \tag{22}$$

where

$$C(z) = \rho'(z)^2 + 4\rho(R) - 4\rho(z) \tag{23}$$

Using g we construct the map $\tilde{}: T^*\mathcal{M} \mapsto T\mathcal{M}$. Thus we can express the metric g solely in terms of functions on $\mathcal{M}$, i.e. elements of $C(\mathcal{M}, \mathbb{C})$.

$$g(\widetilde{df}, \widetilde{dh}) = \star^{-1}(df \wedge \star dh) = \frac{-2}{C(X_0)} \Big(\{X_+, f\}\{X_-, h\} + \{X_-, f\}\{X_+, h\} + 2\{X_0, f\}\{X_0, h\} \Big) \tag{24}$$

The metric g also defines a Hodge dual $\star : \Lambda(\mathcal{M}) \mapsto \Lambda(\mathcal{M})$. This gives

$$\star df = \frac{2i}{C(X_0)^{1/2}}\Big(dX_-\{f,X_+\} + dX_+\{f,X_-\} + 2dX_0\{f,X_0\}\Big) \tag{25}$$

and the Laplace operator

$$\begin{aligned}\star^{-1}(d\star df) = \frac{-2}{C(X_0)}\Big(\{X_-,\{X_+,f\}\} + \{X_+,\{X_-,f\}\} + 2\{X_0,\{X_0,f\}\}\Big)\\ + \frac{2}{C(X_0)^2}\rho'(X_0)(\rho''(X_0)-2)\Big(X_-\{X_+,f\} - X_+\{X_-,f\}\Big)\end{aligned} \tag{26}$$

As stated in the introduction we wish to find corresponding definitions for these geometric objects, but for $f \in \mathcal{A}^R$ instead of $f \in C^0(\mathcal{M},\mathbb{C})$. The idea is that these new definitions reduce to the above expressions when $\varepsilon \to 0$. In this article, we only suggest a new definitions to the exterior derivative, which reduces to (20) and (21) in the limit $\varepsilon \to 0$.

III Trivial NCSR: The algebra $\mathcal{B}$

Throughout this section we will assume that ρ is trivial. This means that as well as the assumptions given by (1) we also assume

- There exists a single minima at z_0.
- $\rho''(z_0) \neq 0$ (27)
- $\rho'(z) \neq 0$ for all $z \neq z_0$

Such a ρ will have just one topology interval and in the classical limit $\mathcal{M}(\rho,R)$, for all $R \neq 0$, is connected surface topologically equivalent to the sphere. Thus there is no topology change associated with a ρ obeying (27).

We set $D = \rho(z_0)$ and define the functions

$$\tau : \mathbb{R} \mapsto \mathbb{R} \qquad \tau(x) = (\rho(x) - D)^{1/2} \qquad \text{and } \tau \text{ is decreasing} \tag{28}$$

$$\omega : \mathbb{R} \mapsto \mathbb{R} \qquad \text{implicitly by} \qquad \tau(\omega(x)) + \tau(\omega(x)+x) = 0 \tag{29}$$

For $k \in \mathbb{Z}$ we define the operators N_k and X_k (where X_1 and X_{-1} are not to be confused with X_+ and X_-) via

$$N_k = \omega(\omega^{-1}(N_0) + \varepsilon k) \qquad \text{and} \qquad X_k = N_k - N_0 + X_0 \tag{30}$$

Lemma 5. *The functions τ and ω and their inverses τ^{-1} and ω^{-1} are all well defined, strictly decreasing and belong to $\mathcal{C}_\mathbb{R}$. The operator N_k is an $\mathcal{C}_\mathbb{R}$ function of N_0 whilst the operator X_k is a $\mathcal{C}_{\mathbb{R}^2}$ function of (X_0, N_0).*

Proof. Follows from showing of $\tau'(z) < 0$ and $\omega'(z) < 0$ for all $z \in \mathbb{R}$. □

Let us define the algebra $\mathcal{B} = \mathcal{B}(\rho,\varepsilon)$ as that generated by a_+, a_-, b_+, b_- and $\mathcal{C}_{\mathbb{R}^2}$ functions of (X_0, N_0) quotiented by the following relationships:

$$[X_0, N_0] = 0 \quad f(X_j,N_k)a_\pm = a_\pm f(X_{j\pm1} \pm \varepsilon, N_{k\pm1}) \quad f(X_j,N_k)b_\pm = b_\pm f(X_{j\pm1}, N_{k\pm1}) \tag{31}$$

$$\begin{array}{ll} a_-a_+ = \tau(N_0) - \tau(X_0+\varepsilon) & b_-b_+ = \tau(N_0) + \tau(X_0) \\ a_+a_- = \tau(N_{-1}) - \tau(X_{-1}) & b_+b_- = \tau(N_{-1}) + \tau(X_{-1}) \end{array} \tag{32}$$

$$\begin{aligned} b_+a_+ &= a_+b_+\left(\frac{(\tau(N_0)-\tau(X_0+\varepsilon))(\tau(N_1)+\tau(X_1+\varepsilon))}{(\tau(N_1)-\tau(X_1+\varepsilon))(\tau(N_0)+\tau(X_0))}\right)^{1/2} \\ b_-a_- &= a_-b_-\left(\frac{(\tau(N_{-1})-\tau(X_{-1}))(\tau(N_{-2})+\tau(X_{-2}-\varepsilon))}{(\tau(N_{-2})-\tau(X_{-2}))(\tau(N_{-1})+\tau(X_{-1}))}\right)^{1/2} \\ b_-a_+ &= a_+b_-\left(\frac{\rho(N_0)-\rho(X_0+\varepsilon)}{(\tau(N_{-1})-\tau(X_{-1}+\varepsilon))(\tau(N_{-1})+\tau(X_{-1}))}\right)^{1/2} \\ b_+a_- &= a_-b_+\left(\frac{(\tau(N_{-1})-\tau(X_{-1}))(\tau(N_{-1})+\tau(X_{-1}-\varepsilon))}{\rho(N_0)-\rho(X_0)}\right)^{1/2} \end{aligned} \tag{33}$$

The elements a_+, a_-, b_+, b_- are known as hopping operators. This name comes from the two dimensional lattice representation that we discuss in the following section. There is a Hermitian conjugate on $\mathcal{B}$ given by

$$\begin{array}{l} \dagger : \mathcal{B} \mapsto \mathcal{B} \qquad (a_\pm)^\dagger = a_\mp \qquad (b_\pm)^\dagger = b_\mp \qquad (f(X_0,N_0))^\dagger = \overline{f}(X_0,N_0) \quad \text{where} \quad \overline{f}(x,y) = \overline{f(\overline{x},\overline{y})} \\ \quad\text{and} \qquad (\xi\zeta\lambda)^\dagger = \overline{\lambda}\zeta^\dagger\xi^\dagger\,, \qquad \text{for} \qquad \lambda\in\mathbb{C}\,,\ \xi,\zeta\in\mathcal{B} \end{array} \tag{34}$$

The algebra $\mathcal{B}$ satisfies the following theorem:

Theorem 6. *The definition of $\mathcal{B}$ is consistent with (34). All the subexpression in (33) which don't contain $a_\pm, b_\pm$ are $\mathcal{C}_{\mathbb{R}^2}$ functions of (X_0,N_0) and are real and strictly positive. Thus the commutation relations are well defined. The general element of $\mathcal{B}$ can be written*

$$\xi = \sum_{rs} a_\pm^r b_\pm^s \xi_{rs}(X_0,N_0) \tag{35}$$

where the sum is finite, $\xi_{rs}\in\mathcal{C}_{\mathbb{R}^2}$ and $a_\pm^r$ and $b_\pm^r$ are defined as in (2).

The algebra $\mathcal{A}^N$ is a subalgebra of $\mathcal{B}$ where we set

$$X_+ = b_-a_+ \qquad \textit{and} \qquad X_- = a_-b_+ \tag{36}$$

Given $\xi\in\mathcal{B}$, the following are equivalent

- $\xi\in\mathcal{A}^N$ (37)
- $[N_0,\xi]=0$ (38)
- *we can write ξ as* $\sum_r a_\pm^r b_\pm^{-r}\xi_r(X_0,N_0)$ (39)

Proof. We show that (34) is constant with (31-33) by direct substitution. We see $(N_k)^\dagger = N_k$, so for example

$$(N_k a_\pm)^\dagger = a_\mp N_k = N_{k\pm1}a_\mp = (a_\pm N_{k\pm1})^\dagger$$

To show that the expressions in (X_0,N_0) on the right hand side of (33) are in $\mathcal{C}_{\mathbb{R}^2}$ and strictly positive consider the subexpression

$$f(X_0,N_0) = \frac{\tau(N_1)+\tau(X_1+\varepsilon)}{\tau(N_0)+\tau(X_0)}$$

which occurs in the first equation of (33), this may be expressed

$$f(z,y) = \frac{\tau(y^+) + \tau(y^+ - y + z + \varepsilon)}{\tau(y) + \tau(z)}$$

where $z = X_0$, $y = N_0$ and $y^+ = \omega(\omega^{-1}(y) + \varepsilon)$. Using (29) we see

$$f(z,y) = \frac{\tau(y^+ - y + z + \varepsilon) - \tau(y^+ + \omega^{-1}(y) + \varepsilon)}{\tau(z) - \tau(y + \omega^{-1}(y))}$$

Since τ is single valued then the numerator and denominator of the above equation are zero when $z = y + \omega^{-1}(y)$. The value of f at this point is

$$f(y + \omega^{-1}(y), y) = \frac{\tau'(y^+ + \omega^{-1}(y) + \varepsilon)}{\tau'(y + \omega^{-1}(y))} > 0$$

This result is the similar for all the fractions, hence result.

Equations (32) to (33) that define $\mathcal{B}$ are sufficient to reduce any expression in the hopping operators and $\mathcal{C}_{\mathbb{R}^2}$ functions of (X_0, N_0) into (35). To see this take word constructed from generators. Use the commutation relations (31) and (33) to push all the a_+ and a_- to the left. Now use (32) to remove all a_+a_- pairs. Keep the resulting term in $a_\pm^r$, and push the $\mathcal{C}_{\mathbb{R}^2}$ function in (X_0, N_0) to the right. Now do the same with the b_+ and b_- terms.

To show that $\mathcal{A}^N$ is a subalgebra of $\mathcal{B}$ we reproduce the defining equations of $\mathcal{A}^N$. We already have $[N_0, X_0] = 0$. For $f \in \mathcal{C}_{\mathbb{R}^2}$

$$f(X_0,N_0)X_+ = f(X_0,N_0)b_-a_+ = b_-f(X_{-1},N_{-1})a_+ = b_-a_+f(X_0+\varepsilon,N_0) = X_+f(X_0+\varepsilon,N_0)$$

and likewise for X_-, hence (90).

$$\begin{aligned}
X_+X_- &= b_-a_+a_-b_+ \\
&= b_-(\tau(N_{-1}) - \tau(X_{-1}))b_+ \\
&= b_-b_+(\tau(N_0) - \tau(X_0)) \\
&= (\tau(N_0) + \tau(X_0))(\tau(N_0) - \tau(X_0)) \\
&= \tau(N_0)^2 - \tau(X_0)^2 \\
&= (\rho(N_0) - D) - (\rho(X_0) - D) = \rho(N_0) - \rho(X_0)
\end{aligned}$$

Hence (91) Thus the subalgebra generated X_+, X_- given by (36) and $\mathcal{C}_{\mathbb{R}^2}$ function of (X_0, N_0) is indeed $\mathcal{A}^N$.

Clearly if $f \in \mathcal{A}^N$ then $[N_0, f] = 0$. If $[N_0, f] = 0$ then write f is in (35). For each term in the sum, (31) implies that the number a_+ must equal the number of b_-, and the number of b_+ must equal the number of a_-, thus (39).

If f is written in (39) then use (33) to permute the $a_\pm$ and $b_\pm$ to give a sequence $a_-b_+a_-b_+\cdots$ or $b_-a_+b_-a_+\cdots$ which is replaced with X_+^r or X_-^r. This gives (89). □

If we were to admit $\mathcal{C}_{\mathbb{R}}$ functions of the hopping operators then we could construct the elements X_0 and N_0 as follows:

$$\begin{aligned}
N_0 &= \omega(\omega^{-1}(\tau^{-1}(\tfrac{1}{2}b_+b_- + \tfrac{1}{2}a_+a_-)) + \varepsilon) \\
X_0 &= \tau^{-1}(\tfrac{1}{2}b_+b_- - \tfrac{1}{2}a_+a_-) - \tau^{-1}(\tfrac{1}{2}b_+b_- + \tfrac{1}{2}a_+a_-) + \omega(\omega^{-1}(\tau^{-1}(\tfrac{1}{2}b_+b_- + \tfrac{1}{2}a_+a_-)) + \varepsilon)
\end{aligned} \tag{40}$$

Clearly ε is a parameter for the noncommutativity since when $\varepsilon = 0$ then $\mathcal{B}(\rho, \varepsilon = 0)$ reduces to a commutative algebra. This algebra is isomorphic to a dense subalgebra of continuous functions on $\mathbb{R}^4$. To see this set $a_\pm = x^1 \pm ix^2$ and $b_\pm = y^1 \pm iy^2$. This give the map $\mathcal{B}(\rho, 0) \hookrightarrow C^0(\mathbb{R}^4 \mapsto \mathbb{C})$. If we give $C^0(\mathbb{R}^4 \mapsto \mathbb{C})$ the standard continuity norm then $\mathcal{B}(\rho, 0)$ forms a dense set. The diagonal operators are given by $N_0 = \tau^{-1}(\|x\| + \|y\|)$ and $X_0 = \tau^{-1}(\|x\| - \|y\|)$.

The noncommutative structure of $\mathcal{B}$ gives rise to a Poisson structure on $\mathbb{R}^4$. This structure is complicated to write down in terms of the coordinates x^1, x^2, y^1, y^2.

III.1 Lattice representations of a trivial NCSR

The reason for the complicated commutation relations given by (33) is that there is a natural representation of $\mathcal{B}$ as a two dimensional lattice.

Let $\mathcal{G} = \mathcal{G}(\rho, \varepsilon)$ be a Hilbert space with orthonormal basis $\left|{n \atop m}\right\rangle$ with $n, m \in \mathbb{Z}$, and $n \geq 0$, $0 \leq m \leq n-1$. Let the dual of $\left|{n \atop m}\right\rangle$ be written $\left\langle{n \atop m}\right|$. There is a representation of $\mathcal{B}$ given by

$$\begin{aligned} f(X_0, N_0)\left|{n \atop m}\right\rangle &= f(\omega(\varepsilon n) + \varepsilon m, \omega(\varepsilon n))\left|{n \atop m}\right\rangle \\ a_+\left|{n \atop m}\right\rangle &= \big(\tau(\omega(\varepsilon n)) - \tau(\omega(\varepsilon n) + \varepsilon m + \varepsilon)\big)^{1/2}\left|{n+1 \atop m+1}\right\rangle \\ a_-\left|{n \atop m}\right\rangle &= \big(\tau(\omega(\varepsilon n - \varepsilon)) - \tau(\omega(\varepsilon n - \varepsilon) + \varepsilon m)\big)^{1/2}\left|{n-1 \atop m-1}\right\rangle \\ b_+\left|{n \atop m}\right\rangle &= \big(\tau(\omega(\varepsilon n)) + \tau(\omega(\varepsilon n) + \varepsilon m)\big)^{1/2}\left|{n+1 \atop m}\right\rangle \\ b_-\left|{n \atop m}\right\rangle &= \big(\tau(\omega(\varepsilon n - \varepsilon)) + \tau(\omega(\varepsilon n - \varepsilon) + \varepsilon m)\big)^{1/2}\left|{n-1 \atop m}\right\rangle \end{aligned} \tag{41}$$

Lemma 7. *The representation given above by (41) is indeed a representation and it is unitary.*

Proof. This simply involves showing all the relations (31-33) are consistent with (41). This should not surprise us since the relations where constructed so that (41) was a representation. This representation of $\mathcal{B}$ is unitary since the hermitian conjugate of f is the adjoint:

$$\begin{aligned} \left\langle{n+1 \atop m+1}\right|a_+\left|{n \atop m}\right\rangle &= \big(\tau(\omega(\varepsilon n)) - \tau(\omega(\varepsilon n) + \varepsilon m + \varepsilon)\big)^{1/2} = \left\langle{n \atop m}\right|a_-\left|{n+1 \atop m+1}\right\rangle \\ \left\langle{n+1 \atop m}\right|b_+\left|{n \atop m}\right\rangle &= \big(\tau(\omega(\varepsilon n)) + \tau(\omega(\varepsilon n) + \varepsilon m)\big)^{1/2} = \left\langle{n \atop m}\right|b_-\left|{n+1 \atop m}\right\rangle \end{aligned}$$

□

For each $n \in \mathbb{N}$ let $V_n \subset \mathcal{G}$ be the subspace $V_n = \mathrm{span}\{\left|{n \atop m}\right\rangle \mid m = 0, \ldots, n-1\}$ and let J_n be the interval $J_n = \{z \mid z_n^\downarrow \leq z \leq z_n^\uparrow\}$ where $z_n^\downarrow = \omega(\varepsilon n)$ and $z_n^\uparrow = \omega(\varepsilon n) + \varepsilon n$.

Lemma 8. *The Hilbert space $\mathcal{G}$ is also a representation of $\mathcal{A}^N \subset \mathcal{B}$ given by (41) and (36). This is given by*

$$\begin{aligned} f(X_0, N_0)\left|{n \atop m}\right\rangle &= f(z_n^\downarrow + \varepsilon m, z_n^\downarrow)\left|{n \atop m}\right\rangle \\ X_+\left|{n \atop m}\right\rangle &= (\rho(z_n^\downarrow) - \rho(z_n^\downarrow + \varepsilon m + \varepsilon))^{1/2}\left|{n \atop m+1}\right\rangle \\ X_-\left|{n \atop m}\right\rangle &= (\rho(z_n^\downarrow) - \rho(z_n^\downarrow + \varepsilon m))^{1/2}\left|{n \atop m-1}\right\rangle \end{aligned} \tag{42}$$

For each $n \in \mathbb{N}$, the interval J_n is a representation interval, and the pair (J_n, V_n) define a standard unitary representation of $\mathcal{A}^C$, $\mathcal{A}^N$ and $\mathcal{A}^R(\rho, \varepsilon, R = z_n^\downarrow)$ given by (42).

Proof. By substituting (41) into (39) we get (42).

Clearly $|J_n|/\varepsilon = n \in \mathbb{N}$ and $\rho(z_n^{\downarrow}) = \rho(z_n^{\uparrow})$. Since ρ has one minima which must lie between $z_n^{\downarrow}$ and $z_n^{\uparrow}$ then J_n obeys (8) and J_n is a representation interval. Equations (42) coincide with (9) and (10) when $z^{\downarrow} = z_n^{\downarrow}$ and $m^{\downarrow} = 0$.

For the algebra $\mathcal{A}^R(\rho, \varepsilon, R = z_n^{\downarrow})$ use the last four equations in (9). □

In figure 3 we view each basis vector $\left|{n \atop m}\right\rangle$ as a point in $\mathbb{R}^2$ with coordinates

$$x - \text{coord} = \left\langle{n \atop m}\middle|X_0\middle|{n \atop m}\right\rangle \qquad\qquad y - \text{coord} = \left\langle{n \atop m}\middle|\rho(N_0)\middle|{n \atop m}\right\rangle \tag{43}$$

In figure 3 these points are represented by crosses. The last point on the right of each J_n does not represent a basis vector and so is drown with a circle. Since the y-coordinate is independent of m the n points in each V_n lie on the same horizontal line. This line is labelled J_n although the representation interval J_n refers only to the x-coordinate.

Clearly all the crosses lie on or above the curve ρ since $\rho(z_n^{\downarrow}) - \rho(z) \geq 0$ for $z \in J_n$. The y-coordinate of the lines J_n increases with n since

$$\left\langle{n+1 \atop m}\middle|\rho(N_0)\middle|{n+1 \atop m}\right\rangle \geq \left\langle{n \atop m}\middle|\rho(N_0)\middle|{n \atop m}\right\rangle \tag{44}$$

The arrow representing a_+ always point to the top right whilst the arrow representing b_+ always point to the top left. This is because

$$\left\langle{n+1 \atop m}\middle|X_0\middle|{n+1 \atop m}\right\rangle \leq \left\langle{n \atop m}\middle|X_0\middle|{n \atop m}\right\rangle \leq \left\langle{n+1 \atop m+1}\middle|X_0\middle|{n+1 \atop m+1}\right\rangle \tag{45}$$

Although the representation is not faithful we do have the following lemma.

Lemma 9. *If $f(x, y, \varepsilon)$ is $\mathcal{C}_{\mathbb{R}^2}$ in (X_0, N_0) and analytic in ε in a domain about $\varepsilon = 0$ and if*

$$f(X_0, N_0, \varepsilon)\left|{n \atop m}\right\rangle = 0 \tag{46}$$

for all $\left|{n \atop m}\right\rangle \in \mathcal{G}(\rho, \varepsilon)$ and for all $\varepsilon > 0$ then $f \equiv 0$.

If ρ is analytic and ξ is a word constructed from the generators of $\mathcal{B}$ without explicit ε and

$$\xi\left|{n \atop m}\right\rangle = 0 \tag{47}$$

for all $\left|{n \atop m}\right\rangle \in \mathcal{G}(\rho, \varepsilon)$ and for all $\varepsilon > 0$ then $\xi \equiv 0$.

Proof. Fix y. Let $y_1 > y$ satisfy $\rho(y) = \rho(y_1)$. Now fix x so that $(x-y)/(y_1-x) \in \mathbb{Q}$ and let ε_0 be the highest common factor of $(x-y)$ and (y_1-x). So $N = (y_1-y)/\varepsilon_0 \in \mathbb{N}$ and $M = (x-y)/\varepsilon_0 \in \mathbb{N}$.

For each $t \in \mathbb{N}$ choose $\varepsilon = \varepsilon_0/t$, $n = tN$ and $m = tM$

$$0 = \left\langle{n \atop m}\middle|f(X_0, N_0, \varepsilon)\middle|{n \atop m}\right\rangle = f(x, y, \varepsilon)$$

and since $f(x, y, \varepsilon)$ is analytic in ε about $\varepsilon = 0$, this implies $f(x, y, \varepsilon) = 0$ for all ε.

The conditions on x implies that $f(x, y, \varepsilon) = 0$ for a dense set of x, and since f is $\mathcal{C}_{\mathbb{R}}$ in x it is true for all x. Finally since we were free to choose y we have $f \equiv 0$.

If ρ is analytic and ξ is generated as stated then ξ can be rewritten as (35) with the $f_{rs} \in \mathcal{C}_{\mathbb{R}}$ with an implicit analytic dependence on ε. Thus we can apply the first part of this lemma. □

III.2 Under-hopping operators

Let us consider an alternative set of hopping operators given by

$$
\begin{aligned}
a'_+ &= a_+ \left(\frac{\tau(N_1)-\tau(X_1+\varepsilon)}{\tau(N_0)-\tau(X_0+\varepsilon)}\right)^{1/2} & \qquad b'_+ &= b_+ \left(\frac{\tau(N_1)+\tau(X_1+\varepsilon)}{\tau(N_0)+\tau(X_0)}\right)^{1/2} \\
a'_- &= a_- \left(\frac{\tau(N_0)-\tau(X_0)}{\tau(N_{-1})-\tau(X_{-1})}\right)^{1/2} & \qquad b'_- &= b_- \left(\frac{\tau(N_0)+\tau(X_0+\varepsilon)}{\tau(N_{-1})+\tau(X_{-1})}\right)^{1/2}
\end{aligned}
\tag{48}
$$

The operators a'_+, a'_-, b'_+, b'_- are clearly in $\mathcal{B}$ since the right hand side of (48) are $\mathcal{C}_{\mathbb{R}^2}$ functions of (X_0, N_0). This is similar to the proof of theorem 6.

It is easy to show that

$$
X_+ = a'_+ b'_- \qquad \text{and} \qquad X_- = b'_+ a'_- \tag{49}
$$

Thus we could have constructed $\mathcal{B}$ using these alternative hopping operators instead of $a_\pm, b_\pm$. All the results would have been similar and the algebra $\mathcal{B}$ constructed using $a'_\pm, b'_\pm$ is isomorphic to the algebra constructed using $a_\pm, b_\pm$. This is not the case for the non-trivial NCSR as we will see in the next section.

To distinguish between the two sets of hopping operators we will call the set $\{a_\pm, b_\pm\}$ the over-hopping operators, whilst the set $\{a'_\pm, b'_\pm\}$ the under-hopping operators. This is due to the following diagram.

$$
\begin{tikzcd}
& \left|\begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix}\right\rangle \arrow[dr, "b_-"] & \\
\left|\begin{smallmatrix} n \\ m \end{smallmatrix}\right\rangle \arrow[ur, "a_+"] \arrow[rr, "X_+"] \arrow[dr, "a'_+"'] & & \left|\begin{smallmatrix} n \\ m+1 \end{smallmatrix}\right\rangle \\
& \left|\begin{smallmatrix} n-1 \\ m \end{smallmatrix}\right\rangle \arrow[ur, "b'_-"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \left|\begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix}\right\rangle \arrow[dl, "a_-"'] & \\
\left|\begin{smallmatrix} n \\ m \end{smallmatrix}\right\rangle & & \left|\begin{smallmatrix} n \\ m+1 \end{smallmatrix}\right\rangle \arrow[ul, "b_+"'] \arrow[ll, "X_-"'] \arrow[dl, "b'_+"] \\
& \left|\begin{smallmatrix} n-1 \\ m \end{smallmatrix}\right\rangle \arrow[ul, "a'_-"] &
\end{tikzcd}
\tag{50}
$$

The representation of the under-hopping operators are given by

$$
\begin{aligned}
a'_+ \left|\begin{smallmatrix} n \\ m \end{smallmatrix}\right\rangle &= \big(\tau(\omega(\varepsilon n+\varepsilon)) - \tau(\omega(\varepsilon n+\varepsilon)+\varepsilon m+\varepsilon)\big)^{1/2} \left|\begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix}\right\rangle \\
a'_- \left|\begin{smallmatrix} n \\ m \end{smallmatrix}\right\rangle &= \big(\tau(\omega(\varepsilon n)) - \tau(\omega(\varepsilon n)+\varepsilon m)\big)^{1/2} \left|\begin{smallmatrix} n-1 \\ m-1 \end{smallmatrix}\right\rangle \\
b'_+ \left|\begin{smallmatrix} n \\ m \end{smallmatrix}\right\rangle &= \big(\tau(\omega(\varepsilon n+\varepsilon)) + \tau(\omega(\varepsilon n+\varepsilon)+\varepsilon m+\varepsilon)\big)^{1/2} \left|\begin{smallmatrix} n+1 \\ m \end{smallmatrix}\right\rangle \\
b'_- \left|\begin{smallmatrix} n \\ m \end{smallmatrix}\right\rangle &= \big(\tau(\omega(\varepsilon n)) + \tau(\omega(\varepsilon n)+\varepsilon m)\big)^{1/2} \left|\begin{smallmatrix} n-1 \\ m \end{smallmatrix}\right\rangle
\end{aligned}
\tag{51}
$$

III.3 Symmetric trivial NCSR and the noncommutative sphere

The situation is even simpler if ρ is an even function with one (quadratic) minima at $\rho(0) = 0$. In this case

$$
\rho(-x) = \rho(x) \qquad \tau(-x) = -\tau(x) \qquad \omega(x) = -\tfrac{1}{2}x \tag{52}
$$

Thus we have the much simpler results:

$$
\begin{aligned}
N_k &= N_0 - \tfrac{1}{2}\varepsilon k & \qquad X_k &= X_0 - \tfrac{1}{2}\varepsilon k & \qquad [N_0, a_\pm] &= \mp\tfrac{\varepsilon}{2} a_\pm \\
[N_0, b_\pm] &= \mp\tfrac{\varepsilon}{2} b_\pm & \qquad [X_0, a_\pm] &= \pm\tfrac{\varepsilon}{2} a_\pm & \qquad [X_0, b_\pm] &= \mp\tfrac{\varepsilon}{2} b_\pm
\end{aligned}
\tag{53}
$$

Finally we give the connection between this and [6]. If we let $\rho(z) = z^2$ then $\tau(x) = -z$ and from (31),(32) and (33) we deduce that $[a_\pm, b_\pm] = 0$ and $[a_-, a_+] = [b_-, b_+] = \varepsilon$. So $\mathcal{B}$ is simply the the product of two Heisenberg-Weyl algebras.

If we let $J_0 := X_0 + \frac{1}{2}\varepsilon$, $J_\pm = X_\pm$ and $K_0 := -N_0$ we obtain the algebra given by the Jordan-Schwinger representation of $su(2)$. This is the starting point for the analysis of vectors and spinors on the noncommutative sphere. To obtain the representation in that article we relabel the vectors $\left|{n' \atop m'}\right\rangle$ where $n' = -\frac{1}{2}(n-1)$ and $m' = m - \frac{1}{2}(n-1)$.

IV Vector module over NCSR

In this section we still assume that ρ is trivial that is it obeys the constraints (1) and (27). At the end of the last section we saw that the algebra $\mathcal{B}$ was, for the special $\rho(z) = z^2$, simply the product of two Heisenberg-Weil algebras. In [6] we used this to produce analogues of Vector and spinor fields for the noncommutative sphere. In this section we repeat the process for NCSR.

IV.1 The algebra (Ψ, μ)

Given $R, \varepsilon \in \mathbb{R}$ and C^1 function $\rho : \mathbb{R} \mapsto \mathbb{R}$ we define the non-associative algebra (Ψ, μ). Since this is a non-associative algebra we write the product μ explicitly. Ψ is the set

$$\Psi = \left\{ \xi = \sum_{r,s} {a_\pm}^r {b_\pm}^s \xi_{rs}(X_0) \;\middle|\; \xi_{rs} \in \mathcal{C}_\mathbb{R} \text{ finite sum} \right\} \tag{54}$$

We define the non-associative product $\mu : \Psi \times \Psi \mapsto \Psi$ as follows: Given two elements $\xi, \zeta \in \Psi$, these may also be considered elements of $\mathcal{B}$. We write the element $\xi\zeta \in \mathcal{B}$ in the form (35). Now make the identity $N_0 \to R$ to produce an element in Ψ called $\mu(\xi, \zeta)$. In other words $(\Psi, \mu) = \mathcal{B}/\mathcal{I}(N_0 - R)$ where we quotient $\mathcal{B}$ on the right by the ideal generated by $N_0 - R$.

We decompose Ψ as:

$$\Psi = \bigoplus_{r \in \mathbb{Z}} \Psi_r \qquad \text{where} \qquad \xi \in \Psi_r \iff N_0 \xi = \xi N_r \tag{55}$$

or equivalently

$$\xi \in \Psi_r \iff \xi = \sum_m a_\pm^{r+m} b_\pm^{r-m} f_m^\xi(X_0) \tag{56}$$

The set Ψ_0 is a subalgebra of Ψ, and it is equivalent to $\mathcal{A}^R$. All other sets Ψ_r are right modules over $\mathcal{A}^R$. We wish to identify these modules as vector fields, covectors fields, spinor fields, etc. In this article we only interpret Ψ_2 and Ψ_{-2} as the space of vector and covector fields respectively. This is done in the following section.

The relationship between our five algebras is given by:

$$\begin{tikzcd}
\mathcal{A}^C \arrow[r, hook] \arrow[drr, "q_2"] & \mathcal{A}^{NC} \arrow[r, "q_1"] & \mathcal{A}^N \arrow[r, hook] \arrow[d, "q_3"] & \mathcal{B} \arrow[d, "q_4"] \\
 & & \mathcal{A}^R \arrow[r, hook] & \Psi
\end{tikzcd} \tag{57}$$

where the hooked arrows refer to the natural embedding q_1, q_2, q_3, are given in lemma 2 and q_4 is the quotient $N_0 - R \sim 0$ one the right.

IV.2 One forms over $\mathcal{A}^R$

We wish to define the space of one forms $\Omega^1(\mathcal{A}^R)$ and the exterior derivative $d : \mathcal{A}^R \mapsto \Omega^1(\mathcal{A}^R)$. To do this we say that $\Omega^1(\mathcal{A}^R)$ is a right module over $\mathcal{A}^R$ which is spanned by $\{\xi_0, \xi_+, \xi_-\}$ (which are not independent), such that

$$\xi_0 = dX_0 \qquad\qquad \xi_+ = dX_+ \qquad\qquad \xi_- = dX_- \tag{58}$$

and there is a formula for d which is consistent with (58) and reduces to (20) in the limit $\varepsilon \to 0$.

Unfortunately, like most problems with quantisation, this procedure is not unique. We are free to choose (21) instead of (20), we can choose the ordering of the elements, and we can always add a random term which vanishes when $\varepsilon = 0$.

We shall choose

$$\begin{aligned} df = {} & \xi_+ X_- \varepsilon^{-1}(\rho(R) - \rho(X_0))^{-1}[X_0, f] - \xi_0 X_- \varepsilon^{-1}(\rho(R) - \rho(X_0))^{-1}[X_+, f] \\ & - \tfrac{1}{2}\xi_0 X_- \varepsilon^{-1}(\rho(R) - \rho(X_0+\varepsilon))^{-1}\big([\rho(X_0+\varepsilon), f] - \varepsilon^{-1}(\rho(X_0+\varepsilon) - \rho(X_0))\big) \end{aligned} \tag{59}$$

where ξ_+ and ξ_0 are independent elements of Ω^1. We can see that d is no longer a derivative for $\varepsilon \neq 0$ but that

$$d(fh) = d(f)h + fd(h) + O(\varepsilon) \tag{60}$$

Clearly $\xi_0 = dX_0$ and $\xi_+ = dX_+$. To be consistent with (58) we let $\xi_- = dX_-$ giving

$$\xi_- = -\xi_+(\rho(R) - \rho(X_0+\varepsilon))^{-1}X_-^2 - \tfrac{1}{2}\xi_0\varepsilon^{-1}\frac{(\rho(X_0+2\varepsilon) - \rho(X_0))}{(\rho(R) - \rho(X_0+\varepsilon))}X_- \tag{61}$$

and rearranging

$$\xi_+ = -\xi_-(\rho(R) - \rho(X_0))^{-1}X_+^2 - \tfrac{1}{2}\xi_0\varepsilon^{-1}\frac{(\rho(X_0+2\varepsilon) - \rho(X_0))}{(\rho(R) - \rho(X_0+\varepsilon))}X_+ \tag{62}$$

We can now write df in terms of ξ_0 and ξ_- giving

$$\begin{aligned} df = {} & -\xi_-\varepsilon^{-1}(\rho(R) - \rho(X_0))^{-1}X_+[X_0, f] - \xi_0\varepsilon^{-1}(\rho(R) - \rho(X_0+\varepsilon))^{-1}[X_-, f]X_+ \\ & + \tfrac{1}{2}\xi_0\varepsilon^{-1}(\rho(R) - \rho(X_0+\varepsilon))^{-1}\big([\rho(X_0+\varepsilon), f] - \varepsilon^{-1}(\rho(X_0+2\varepsilon) - \rho(X_0+\varepsilon))[X_0, f]\big) \end{aligned} \tag{63}$$

We have chosen the final term in (59) in order to have the maximum similarity between (59) and (63)

Both $\Omega^1(\mathcal{A}^R)$ and Ψ_{-2} are right modules over $\mathcal{A}^R$. We now make the identification $\Omega^1(\mathcal{A}^R) = \Psi_{-2}$ via the definitions

$$\xi_+ = a_-^2 \text{ and } \xi_- = b_-^2 \tag{64}$$

and ξ_0 is given by (62).

We can now construct the noncommutative analogue of the tangent bundle $T\mathcal{M}$, a subset of which, we identify as Ψ_2. Given an element $\xi \in \Psi_2$ we define the noncommutative analogue of the vector field as the function $V_\xi : \mathcal{A}^R \mapsto \mathcal{A}^R$ given by $V_\xi(f) = \mu(\xi, df)$. We can see that for $\varepsilon \neq 0$ then X does not obey Leibniz rule. As sated in the introduction this must be the case since $T\mathcal{M}$ defined here is a right module.

By extending the results of [6], we can also interpret the modules Ψ_1 and Ψ_{-1} as spinors over $\mathcal{A}^R$. This would mean the module $\Psi_r \oplus \Psi_{-r}$ is the module of spin $r/2$ fields. This approach for spinors differs from the standard approach for noncommutative geometry, using the supersymmetric group $SU(2|1)$ (for example [7]).

IV.3 Derivatives of $\mathcal{A}^R$ are inner

As stated in the introduction, if we identify $T\mathcal{M}$ with the space of derivations (obeying Leibniz) then it will not form a module over $\mathcal{A}^R$. This is because, as we shall show here, all derivations are inner and it is easy to show that these do not form a module.

Theorem 10. *All Leibniz derivations on $\mathcal{A}^R$ are inner.*

Proof. Let $\xi : \mathcal{A}^R \mapsto \mathcal{A}^R$ be a derivations. That is for any two function $f, g \in \mathcal{A}^R$, $\xi(fg) = \xi(f)g + f\xi(g)$. We are required to show that there exists an $f \in \mathcal{A}^R$ such that $\xi = \mathrm{ad}_f$. Since ξ is a derivation it is only necessary to show that $\xi(X_0) = \mathrm{ad}_f(X_0)$ and $\xi(X_\pm) = \mathrm{ad}_f(X_\pm)$. All other functions can be derived from these.

Expanding $\xi(X_0)$ in terms of the eigenstates of ad_{X_0}, written in normal form, we have

$$\xi(X_0) = \sum_{r=-\infty}^{\infty} X_\pm^r p_r(X_0)$$

For some set of functions p_r. Let

$$\widehat{f} = \sum_{r=-\infty, r\neq 0}^{\infty} \tfrac{1}{r} X_\pm^r p_r(X_0)$$

Then

$$\xi(X_0) = [\widehat{f}, X_0] + p_0(X_0)$$

Since ξ and $\mathrm{ad}_{\widehat{f}}$ are both derivative then this formula extends to any function of X_0 as

$$\xi(h(X_0)) = [\widehat{f}, h(X_0)] + h'(X_0)p_0(X_0)$$

Let us define

$$g_\pm = \xi(X_\pm) - [\widehat{f}, X_\pm]$$

Taking the ξ derivative of (3) gives

$$[\xi(X_0), X_\pm] + [X_0, X_\pm] = \pm\varepsilon\xi(X_\pm)$$

Expanding and substituting the above expressions give

$$[X_0, g_\pm] = \pm\varepsilon g_\pm + [X_\pm, p_0(X_0)]$$

By expanding in normal form g_+ we see that

$$[p_0(X_0), X_+] = \varepsilon \sum_{r=-\infty}^{\infty} (1-r) X_\pm^r g_r(X_0)$$

Which is only consistent if $p_0 = 0$. This means that $[X_0, g_+] = \varepsilon g_+$. This means we can write $g_+ = X_+(g(X_0+\varepsilon) - g(X_0))$ for some function g. Let $f = \widehat{f} + g(X_0)$. Then

$$\xi(X_0) = [f, X_0] \qquad\qquad \xi(X_+) = [f, X_+]$$

Now taking the derivative of (88) and expanding gives

$$\begin{aligned}
\xi(X_+)X_- + X_+\xi(X_-) &= \xi(\rho(R) - \rho(X_0)) \\
[\widehat{f}, X_+]X_- + g_+X_- + X_+[\widehat{f}, X_-] + X_+g_- &= -[\widehat{f}, \rho(X_0)] \\
g_+X_- + X_+g_- &= 0 \\
X_+(g(X_0+\varepsilon) - g(X_0))X_- + X_+g_- &= 0
\end{aligned}$$

which gives $g_- = [g(X_0), X_-]$. Which implies $\xi(X_-) = [f, X_-]$. $\square$

V Representations of non trivial NCSR

Having established how to represent trivial NCSR we know turn our attention to how to represent non-trivial NCSR. Thus we assume that ρ obeys the conditions given by (1), and that it has at least two local minima.

There are problems associated with the construction of the multi-topology representations of ρ. Within each topology interval the representation of ρ is similar to the representation of a trivial ρ. It is at the boundaries of the topology intervals that the problems occur. This is not surprising since it is here that the topology changes occurs.

In this section we do not try to construct the algebra $\mathcal{B}$ but simply try to construct a Hilbert space $\mathcal{G}$ and the hopping operators. An example $\mathcal{G}$ together with over hopping operators is given in figure 4.

In section V.1 we construct the Hilbert space $\mathcal{G}_0$, and in section V.2 we construct the hopping operators. As indicated in section V.3, these do not form a true representation of $\mathcal{A}^N$ and it is necessary to compromise. There are at least four possible compromises. Two of them use over-hopping operators and two use under-hopping operators. Unlike the case of a trivial ρ, in general there is no isomorphism between the over-hopping operators and the under-hopping operators.

V.1 The Hilbert space $\mathcal{G}_0$

We construct first the Hilbert space $\mathcal{G}_0$. For two of the four choices for representations we use $\mathcal{G}_0$ directly for the others we have to modify $\mathcal{G}_0$. Given a ρ which obeys (1) and an $\varepsilon > 0$, the Hilbert space $\mathcal{G}_0 = \mathcal{G}_0(\rho, \varepsilon)$ is defined as follows:

Let $\{J_s \subset \mathbb{R},\ s \in \mathbb{N}\}$ be a set of intervals obeying (8). The end points of the interval J_s are given by points $z_s^{\downarrow}$ and $z_s^{\uparrow}$ so $J_s = \{z \in \mathbb{R} \mid z_s^{\downarrow} \le z \le z_s^{\uparrow}\}$. Define the function $\pi : \mathbb{N} \mapsto \mathbb{N}$ as: Given J_s then $J_{\pi(s)}$ is the smallest $J_t \neq J_s$ such that $J_s \subset J_t$.

Recall that since $\mathcal{T}$ is a finite set there exists an element t_∞ such that $\pi_{\mathcal{T}}(t_\infty) = \infty$. This is the unbounded topology interval. For each $s \in \mathcal{S}$ we define integers n_s, $m_s^{\downarrow}$ and $m_s^{\uparrow}$ as: Choose an s_0 such that $J_{s_0} \in \mathbf{I}_{t_\infty}$. Let $n_{s_0} = 0$ and $m_{s_0}^{\downarrow} = 0$. Now for all $J_s \in \mathbf{I}_{t_\infty}$ let $n_{\pi(s)} = n_s + 1$ and $m_s^{\downarrow} = 0$. For the other s let $n_s = n_{\pi(s)} - 1$ and

$$m_s^{\downarrow} = \left\lfloor \frac{z_s^{\downarrow} - z_{\pi(s)}^{\downarrow}}{\varepsilon} \right\rfloor + m_{\pi(s)}^{\downarrow} \qquad\qquad m_s^{\uparrow} = m_s^{\downarrow} + \frac{|J|}{\varepsilon} - 1 \tag{65}$$

For each $s \in \mathcal{S}$ let V_s be the vector space spanned by the basis vectors $\{\left|{n_s \atop m}\right\rangle\}$ where $m \in \mathbb{Z}$, $m_s^{\downarrow} \le m \le m_s^{\uparrow}$.

Finally we define $\mathcal{G}_0$ as $\mathcal{G}_0 = \oplus_{s\in\mathbb{N}} V_s$. An example of a $\mathcal{G}_0$ is given in figure 4.

We have required that ρ has only a finite number of maxima since otherwise it is possible to construct ρ such that no, or only a finite number of J_s exist. See figure 5.

The choice of s for each J_s is arbitrary in the construction of $\mathcal{G}_0$ as is the initial s_0, but once these are set, that fixes n and m for each basis vector $\left|{n \atop m}\right\rangle$.

V.2 Hopping operators

For each $\left|{n \atop m}\right\rangle \in \mathcal{G}_0$ there is a unique $s \in \mathbb{N}$ such that $\left|{n \atop m}\right\rangle \in V_s$. The effect of the diagonal operators are given by

$$f(X_0, N_0) = f(z_s^{\downarrow} + \varepsilon m - \varepsilon m_s^{\downarrow}, z_s^{\downarrow})\left|{n \atop m}\right\rangle \tag{66}$$

For each $s \in \mathbb{N}$ and $m_s^\downarrow \le m \le m_s^\uparrow$ let

$$D_s = \min\{\rho(z), z \in J_s\} \tag{67}$$

$$(\sigma_m^s)^2 = \rho(x_s^\downarrow - \varepsilon m_s^\downarrow + \varepsilon m) - D_s \tag{68}$$

$$C_s = \sigma^s_{m_s^\downarrow} = -\sigma^s_{m_s^\uparrow+1} = \big(\rho(x_s^\downarrow) - D_s\big)^{1/2} > 0 \tag{69}$$

The sign of σ_m^s is arbitrary except for $m = m_s^\downarrow$ and $m = m_s^\uparrow + 1$. The construction of $\mathcal{G}_0$ guarantees that if $\left|{n \atop m}\right\rangle \in V_s$ then $\left|{n+1 \atop m}\right\rangle \in \mathcal{G}_0$ and $\left|{n+1 \atop m+1}\right\rangle \in \mathcal{G}_0$. In contrast to the trivial case, the over-hopping operators and under-hopping operators are no longer equivalent. The over-hopping operators are given by

$$\begin{aligned}
a_+\left|{n \atop m}\right\rangle &= (C_s - \sigma^s_{m+1})^{1/2}\left|{n+1 \atop m+1}\right\rangle \\
a_-\left|{n \atop m}\right\rangle &= \begin{cases} (C_t - \sigma^t_m)^{1/2}\left|{n-1 \atop m-1}\right\rangle & \text{if } \left|{n-1 \atop m-1}\right\rangle \in V_t \text{ for some } V_t \subset \mathcal{G}_0 \\ 0 & \text{if } \left|{n-1 \atop m-1}\right\rangle \notin \mathcal{G}_0 \end{cases} \\
b_+\left|{n \atop m}\right\rangle &= (C_s + \sigma^s_m)^{1/2}\left|{n+1 \atop m}\right\rangle \\
b_-\left|{n \atop m}\right\rangle &= \begin{cases} (C_t + \sigma^t_m)^{1/2}\left|{n-1 \atop m}\right\rangle & \text{if } \left|{n-1 \atop m}\right\rangle \in V_t \text{ for some } V_t \subset \mathcal{G}_0 \\ 0 & \text{if } \left|{n-1 \atop m}\right\rangle \notin \mathcal{G}_0 \end{cases}
\end{aligned} \tag{70}$$

The under-hopping operators are given (with $t = \pi(s)$) by

$$\begin{aligned}
a'_+\left|{n \atop m}\right\rangle &= (C_t - \sigma^t_{m+1})^{1/2}\left|{n+1 \atop m+1}\right\rangle \\
a'_-\left|{n \atop m}\right\rangle &= \begin{cases} (C_s - \sigma^s_m)^{1/2}\left|{n-1 \atop m-1}\right\rangle & \text{if } \left|{n-1 \atop m-1}\right\rangle \in \mathcal{G}_0 \\ 0 & \text{if } \left|{n-1 \atop m-1}\right\rangle \notin \mathcal{G}_0 \end{cases} \\
b'_+\left|{n \atop m}\right\rangle &= (C_t + \sigma^t_m)^{1/2}\left|{n+1 \atop m}\right\rangle \\
b'_-\left|{n \atop m}\right\rangle &= \begin{cases} (C_s + \sigma^s_m)^{1/2}\left|{n-1 \atop m}\right\rangle & \text{if } \left|{n-1 \atop m-1}\right\rangle \in \mathcal{G}_0 \\ 0 & \text{if } \left|{n-1 \atop m}\right\rangle \notin \mathcal{G}_0 \end{cases}
\end{aligned} \tag{71}$$

In order to compare the hopping operators with the ladder operators, we define the action of the ladder operators on $\mathcal{G}_0$ as:

$$\begin{aligned}
X_+\left|{n \atop m}\right\rangle &= (\rho(z_s^\downarrow) - \rho(z_s^\downarrow - \varepsilon m_s^\downarrow + \varepsilon m + \varepsilon))^{1/2}\left|{n \atop m+1}\right\rangle \\
X_-\left|{n \atop m}\right\rangle &= (\rho(z_s^\downarrow) - \rho(z_s^\downarrow - \varepsilon m_s^\downarrow + \varepsilon m))^{1/2}\left|{n \atop m-1}\right\rangle
\end{aligned} \tag{72}$$

These are consistent with the (J_s, V_s) representation of $\mathcal{A}^N$ given by (9).

Lemma 11. *The operators N_0 and X_0 are self-adjoint and the operators a_+, b_+, a'_+, b'_+ are the adjoints of a_-, b_-, a'_-, b'_- respectively. The points given by (43) are well placed since (66) satisfies (44) and (45). The hopping operators are related to the ladder operators in*

the following circumstances:

$$\begin{aligned}
&\textit{if } \left|{}^{n}_{m}\right\rangle \in V_s\,,\ \left|{}^{n}_{m+1}\right\rangle \in V_s \textit{ then } b_-a_+\left|{}^{n}_{m}\right\rangle = (C_s^2-(\sigma^s_{m+1})^2)^{1/2}\left|{}^{n}_{m+1}\right\rangle = X_+\left|{}^{n}_{m}\right\rangle\\
&\textit{if } \left|{}^{n}_{m}\right\rangle \in V_s\,,\ \left|{}^{n}_{m-1}\right\rangle \in V_s \textit{ then } a_-b_+\left|{}^{n}_{m}\right\rangle = (C_s^2-(\sigma^s_{m+1})^2)^{1/2}\left|{}^{n}_{m-1}\right\rangle = X_-\left|{}^{n}_{m}\right\rangle\\
&\textit{if } \left|{}^{n}_{m}\right\rangle \in V_s\,,\ \left|{}^{n}_{m+1}\right\rangle \notin \mathcal{G}_0 \textit{ then } b_-a_+\left|{}^{n}_{m}\right\rangle = 0\\
&\textit{if } \left|{}^{n}_{m}\right\rangle \in V_s\,,\ \left|{}^{n}_{m-1}\right\rangle \notin \mathcal{G}_0 \textit{ then } a_-b_+\left|{}^{n}_{m}\right\rangle = 0
\end{aligned}$$

$$\begin{aligned}
&\textit{if } \left|{}^{n}_{m}\right\rangle \in V_s\,,\ \left|{}^{n}_{m+1}\right\rangle \in V_s\,,\ \left|{}^{n-1}_{m}\right\rangle \in \mathcal{G}_0 \textit{ then } a'_+b'_-\left|{}^{n}_{m}\right\rangle = (C_s^2-(\sigma^s_{m+1})^2)^{1/2}\left|{}^{n}_{m+1}\right\rangle = X_+\left|{}^{n}_{m}\right\rangle\\
&\textit{if } \left|{}^{n}_{m}\right\rangle \in V_s\,,\ \left|{}^{n}_{m-1}\right\rangle \in V_s\,,\ \left|{}^{n-1}_{m-1}\right\rangle \in \mathcal{G}_0 \textit{ then } b'_+a'_-\left|{}^{n}_{m}\right\rangle = (C_s^2-(\sigma^s_{m})^2)^{1/2}\left|{}^{n}_{m-1}\right\rangle = X_-\left|{}^{n}_{m}\right\rangle\\
&\textit{if } \left|{}^{n}_{m}\right\rangle \in V_s\,,\ \left|{}^{n}_{m+1}\right\rangle \notin \mathcal{G}_0 \textit{ or } \left|{}^{n}_{m}\right\rangle \in V_s\,,\ \left|{}^{n-1}_{m}\right\rangle \notin \mathcal{G}_0 \textit{ then } a'_+b'_-\left|{}^{n}_{m}\right\rangle = 0\\
&\textit{if } \left|{}^{n}_{m}\right\rangle \in V_s\,,\ \left|{}^{n}_{m+1}\right\rangle \notin \mathcal{G}_0 \textit{ or } \left|{}^{n}_{m}\right\rangle \in V_s\,,\ \left|{}^{n-1}_{m-1}\right\rangle \notin \mathcal{G}_0 \textit{ then } b'_+a'_-\left|{}^{n}_{m}\right\rangle = 0
\end{aligned}$$

Thus (J_s, V_s) is a representation of $\mathcal{A}^N$ using over-hopping operators where $X_+ = b_-a_+$ and $X_- = a_-b_+$ if

$$\dim V_s \geq \sum_{t\in\pi^{-1}\{s\}} (\dim V_t + 1) \tag{73}$$

On the other hand (J_s, V_s) is a representation of $\mathcal{A}^N$ using under-hopping operators where $X_+ = a'_+b'_-$ and $X_- = b'_+a'_-$ if

$$\dim V_s = \left(\sum_{t\in\pi^{-1}\{s\}} \dim V_t\right) + 1 \tag{74}$$

Proof. The first and second parts follow from direct substitution. The conditions for $\mathcal{G}_0$ to be a representation of $\mathcal{A}^N$ follows from the requirement that $X_+\left|{}^{n}_{m}\right\rangle = 0$ if and only if $n = n_s$ and $m = m_s^{\uparrow}$ for some $s \in \mathbb{N}$. An example which obeys (73) is given in figure 4. □

Clearly for a given ρ and ε both (73) and (74) cannot be satisfied. To see which, if either, is satisfied consider figure (6). Here J_1, J_2, J_3 are intervals satisfying (8). Thus

$$|J_1| = \varepsilon\lfloor \Delta x_1/\varepsilon\rfloor, \qquad |J_2| = \varepsilon\lfloor \Delta x_2/\varepsilon\rfloor, \qquad |J_3| = \varepsilon\lceil (\Delta x_1 + \Delta x_2)/\varepsilon\rceil \tag{75}$$

where $\lfloor x\rfloor$ and $\lceil x\rceil$ are the nearest integer below and above x respectively. As a result either $|J_3| = |J_1|+|J_2|+\varepsilon$ obeying (74) or $|J_3| = |J_1|+|J_2|+2\varepsilon$ obeying (73). If ρ has just one maxima then we can choose over-hopping or under-hopping operators appropriately. However if ρ has more than one maxima it may be impossible to satisfy either (74) for all $s \in \mathbb{N}$ or (73) for all $s \in \mathbb{N}$.

V.3 Four compromises

For each $s \in \mathbb{N}$ we have a classical surface $\mathcal{M}_s$ given by

$$\mathcal{M}_s = \{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 = \rho(R) - \rho(x^3), x^3 \in J_s\} \tag{76}$$

We would like to have $\mathcal{G}_0$ a unitary representation of $\mathcal{A}^N$ and each V_s as the standard unitary representation of $\mathcal{A}^R(\rho, \varepsilon, R)$, where $R = z_s^{\downarrow}$ and thus the noncommutative analogue

of $\mathcal{M}_s$. As noted, in general, either (74) or (73) is not satisfied. It is therefore necessary either to accept that there exists some intervals J_s where we do not have a true standard unitary of $\mathcal{A}^N$, or to alter $\mathcal{G}_0$ in some way. There are at least four possible compromises, two of which use over-hopping operators and two of which use under-hopping operators. The disadvantages of each compromise are summarised in the following table:

Comprise	Hop op	Uses $\mathcal{G}_0$	Rep of $\mathcal{A}^N$	Obeys (44)	Obeys (45)
1. Merging V_s	over	yes	no	yes	yes
2. Splitting V_s	under	yes	no	yes	yes
3. Removing V_s	over	no	yes	yes	no
4. Add a vector	under	no	yes	yes	yes

For the following we assume that ρ has only one maxima and two minima. However we can see how this generalises for any finite number of maxima.

V.3.1 Compromise 1: Merging of V_ss

We use the Hilbert space $\mathcal{G} = \mathcal{G}_0$ and the over-hopping operators. Consider figure 7 where $\dim V_8 = \dim V_3 + \dim V_7 + 1$ and hence violates (73). If we let $V_3 = \operatorname{span}\{\left|{}^4_0\right\rangle, \left|{}^4_1\right\rangle, \left|{}^4_2\right\rangle\}$ and $V_7 = \operatorname{span}\{\left|{}^4_3\right\rangle, \left|{}^4_4\right\rangle, \left|{}^4_5\right\rangle, \left|{}^4_6\right\rangle\}$ then

$$X_+\left|{}^4_2\right\rangle = (4C_3C_7)^{1/2}\left|{}^4_3\right\rangle \qquad\qquad X_-\left|{}^4_3\right\rangle = (4C_3C_7)^{1/2}\left|{}^4_2\right\rangle$$

If V_3 and V_7 were both representations of $\mathcal{A}^N$ then $X_+\left|{}^4_2\right\rangle = X_-\left|{}^4_3\right\rangle = 0$. We say that V_3 and V_7 have merged. We also see that

$$\left\langle{}^4_3\right|[N_0, X_+]\left|{}^4_2\right\rangle = (4C_3C_7)^{1/2}(z_7^{\downarrow} - z_3^{\downarrow}) \neq 0 \tag{77}$$

thus violating (4). For all other V_s except V_3 and V_7 there is a valid representation of $\mathcal{A}^N$.

An interpretation of this is that the two surfaces $\mathcal{M}_3$ and $\mathcal{M}_7$ are less apart than the distance of the parameter ε. Since we interpret the noncommutative versions of $\mathcal{M}_3$ and $\mathcal{M}_7$ as somehow smearing out the classical surfaces then this causes $\mathcal{M}_3$ and $\mathcal{M}_7$ to interact.

V.3.2 Compromise 2: Splitting V_ss

We use the Hilbert space $\mathcal{G} = \mathcal{G}_0$ and the under-hopping operators. Consider figure 8 where $\dim V_6 = \dim V_2 + \dim V_5 + 2$ and hence violates (74). If we let $V_6 = \operatorname{span}\{\left|{}^3_0\right\rangle, \left|{}^3_1\right\rangle, \ldots, \left|{}^3_6\right\rangle\}$ then

$$X_+\left|{}^3_2\right\rangle = X_-\left|{}^3_3\right\rangle = 0 \tag{78}$$

thus violating (90).

This has the opposite interpretation to the compromise above. Because the classical surface $\mathcal{M}_6$ has a small waist its noncommutative analogue is equivalent to two surfaces.

V.3.3 Compromise 3: Removing V_s

Here we use the over-hopping operators, but modify $\mathcal{G}_0$ if there exist a V_s which violates (73).

Let $\mathcal{G} = \oplus_{s\in\mathcal{S}}$ where $\mathcal{S} \in \mathbb{N}$ is the set of all V_s which obey (73), i.e. we remove from $\mathcal{G}_0$ all V_s which violate (73). We then have to redefine the hopping operators.

Consider figure 9 where $\dim V_8 = \dim V_3 + \dim V_7 + 1$ and hence violates (73). We do not include V_8 from $\mathcal{G}$. We have to relabel $\left|{n \atop m}\right\rangle$ for all V_s where $J_s \subset J_8$. If we let $V_9 = \mathrm{span}\{\left|{5 \atop 0}\right\rangle, \left|{5 \atop 1}\right\rangle, \dots, \left|{5 \atop 8}\right\rangle\}$ then we must make $V_3 = \mathrm{span}\{\left|{4 \atop 0}\right\rangle, \left|{4 \atop 1}\right\rangle, \left|{4 \atop 2}\right\rangle\}$ and $V_7 = \mathrm{span}\{\left|{4 \atop 4}\right\rangle, \left|{4 \atop 5}\right\rangle, \left|{4 \atop 6}\right\rangle, \left|{4 \atop 7}\right\rangle\}$ so that there is a vector missing between V_3 and V_7. Otherwise these spaces would merge. This requirement may mean that (45) is violated as in figure 9.

It is difficult to see how to interpret this compromise but at least $\mathcal{G}$ is a unitary representation of $\mathcal{A}^N$ and all the V_s that remain in $\mathcal{G}$ are unitary representation of $\mathcal{A}^R$.

V.3.4 Compromise 4: Adding an extra vector

If we use the under-hopping operators so $X_+ = a'_+ b'_-$ and $X_- = b'_+ a'_-$ and modify $\mathcal{G}_0$ if there exist a V_s that violates (74).

Consider figure 10 where $\dim V_6 = \dim V_2 + \dim V_5 + 2$ and hence violates (74). Let $V_6 = \mathrm{span}\{\left|{3 \atop 0}\right\rangle, \left|{3 \atop 1}\right\rangle, \dots, \left|{3 \atop 6}\right\rangle\}$, $V_2 = \mathrm{span}\{\left|{2 \atop 0}\right\rangle, \left|{2 \atop 1}\right\rangle\}$ and $V_5 = \mathrm{span}\{\left|{2 \atop 3}\right\rangle, \left|{2 \atop 4}\right\rangle, \left|{2 \atop 5}\right\rangle\}$. Now let $\mathcal{G} = \mathcal{G}_0 \oplus \mathrm{span}\{\left|{2 \atop 2}\right\rangle\}$. By enlarging $\mathcal{G}$ we have avoided the problem in compromise 2. We define a_+ and b_+ on $\left|{2 \atop 2}\right\rangle$ by considering $a_-\left|{3 \atop 3}\right\rangle$ and $b_-\left|{3 \atop 2}\right\rangle$.

Again $\mathcal{G}$ is a unitary representation of $\mathcal{A}^N$ and all the V_s that remain in $\mathcal{G}$ are unitary representation of $\mathcal{A}^R$.

V.4 Representation of the topology change

We say that a lattice representation $\mathcal{G}(\rho, \varepsilon)$ reflects the topology of ρ if

- for all $t \in \mathcal{T}$ there exists $s \in \mathbb{N}$ such that $J_s \in \mathbf{I}_t$.
- for all $s \in \mathbb{N}$ there exists $t \in \mathcal{T}$ such that $J_s \in \mathbf{I}_t$. Either $J_{\pi(s)} \in \mathbf{I}_t$ or $J_{\pi(s)} \in \mathbf{I}_{\pi_{\mathcal{T}}(t)}$.
- for all $t \in \mathcal{T}$ such that $\pi_{\mathcal{T}}(t) \neq \infty$ then there exists a unique $s \in \mathbb{N}$ such that $J_s \in \mathbf{I}_t$ and $J_{\pi(s)} \in \mathbf{I}_{\pi_{\mathcal{T}}(t)}$.

This definition gives the relationship between the maps $\pi : \mathbb{N} \mapsto \mathbb{N}$ and $\pi_{\mathcal{T}} : \mathcal{T} \mapsto \mathcal{T} \cup \{\infty\}$.
Theorem 12. *For all four compromise there exist an $\varepsilon_0 > 0$ such that for al $0 < \varepsilon < \varepsilon_0$ there exists a lattice representation of the NCSR which respects the topology of ρ.*

Proof. Clearly for all ε we can construct the space $\mathcal{G}_0$ and hence the space $\mathcal{G}$. If we let

$$\varepsilon_0 = \min_{t \in \mathcal{T}} \left(\sup_{I \in \mathbf{I}_t}(|I|) - \inf_{I \in \mathbf{I}_t}(|I|) \right)$$

This guarantees that $\mathcal{G}_0$ respects the topology of ρ. Thus the result for compromises 1, 2, and 4. For compromise 4 we note that the added vectors do not correspond to any interval J_s and thus are not in a topology interval $\mathbf{I}_t$.

For compromise 3 it is necessary to replace ε_0 with $\varepsilon_0/|\mathcal{T}|$ where $|\mathcal{T}|$ are the number of elements in $\mathcal{T}$. This guarantees that even if we remove the maximum number of spaces V_s there is still a J_s in each $\mathbf{I}_t$. $\square$

V.5 Operators for topology change

For each V_s let $L(V_s)$ be the set of operators on V_s and let $\pi^{-1}(V_s) = \oplus_{t\in\pi^{-1}(s)}V_t$. Let us define the linear operators

$$\begin{array}{ll} A_+, B_+ : L(\pi^{-1}(V_s)) \mapsto L(V_s) & A_-, B_- : L(V_s) \mapsto L(\pi^{-1}(V_s)) \\ A_+(f) = a_+(a_-a_+)^{-1} f a_- & A_-(f) = a_- f a_+(a_-a_+)^{-1} \\ B_+(f) = b_+(b_-b_+)^{-1} f b_- & B_-(f) = b_- f b_+(b_-b_+)^{-1} \end{array} \tag{79}$$

Lemma 13. *The maps $A_\pm, B_\pm$ are well defined. A_+ and B_+ are homomorphism i.e. $A_+(fg) = A_+(f)A_+(g)$. Given $f \in L(\pi^{-1}(V_s)$ then*

$$A_-(A_+(f)) = B_-(B_+(f)) = f \tag{80}$$

We can write the elements in $L(V_s)$ as matrices using the basis $\left|{n \atop m}\right\rangle$. As a matrix, $A_+(f)$ has zeros in the first row and column, and $B_+(f)$ has zeros in the last row and column. Furthermore $A_+(f)$ and $B_+(f)$ are block diagonal matrices with each block corresponding to a $V_t \subset \pi^{-1}(V_s)$.

A_- and B_- are not homomorphism, and it is impossible to define such a homomorphism.

Proof. For $\left|{n \atop m}\right\rangle \in V_s$ then, $a_-a_+\left|{n \atop m}\right\rangle = (C_s - \sigma^s_{m+1})\left|{n \atop m}\right\rangle \neq 0$ so $(a_-a_+)^{-1}$ is well defined via $(a_-a_+)^{-1}\left|{n \atop m}\right\rangle = (C_s - \sigma^s_{m+1})^{-1}\left|{n \atop m}\right\rangle$. The same is true for $(b_-b_+)^{-1}$ with $(b_-b_+)^{-1}\left|{n \atop m}\right\rangle = (C_s + \sigma^s_m)^{-1}\left|{n \atop m}\right\rangle$. So $A_\pm, B_\pm$ are well defined.

The homomorphism of A_+ and B_+ follow from simple substitution, as does (80). That A_- and B_- are not homomorphism follows from the non existence of linear homomorphism from the matrix algebra $M_n(\mathbb{C})$ to $M_{n-1}(\mathbb{C})$

□

These maps may be considered operators for topology change. For example in figure 9, J_9 represents the noncommutative analogue of a single connected surface as do J_3 and J_7. We have the maps $A_+, B_+ : L(V_9) \mapsto L(V_3 \oplus V_7)$ this the noncommutative analogue of a map for one surface to two. The maps $A_-, B_- : L(V_3 \oplus V_7) \mapsto L(V_9)$ are the reverse.

V.6 Generalised Multi-topology Lattice

Finally in this section we give an example of a further generalisation of the multi-topology lattice $\mathcal{G}$. This lattice no longer refers to a specific ρ but each space V_s corresponds to a different ρ_s. This may have applications to the motion of closed surfaces which, whilst they remain axially symmetric, change shape.

There is clearly an asymmetry between the operators A_+, B_+ and A_-, B_- defined by (79), since A_+, B_+, which reflect the coalescing of surfaces, are homomorphisms, whilst A_-, B_-, which reflect the bifurcating of a surface, are not homomorphisms. Thus the study of the generalised multi-topology lattice will enables one to study surfaces which place coalescence and bifurcation on equal terms.

We define a **generalised multi-topology lattice** as a Hilbert space, $\mathcal{G}$ which has an orthonormal basis $\{\left|{n \atop m}\right\rangle\}$ where the set $\{(n, m)\}$ is any subset of $\mathbb{Z} \times \mathbb{Z}$. The dual of $\left|{n \atop m}\right\rangle$ is written $\left\langle{n \atop m}\right|$. We decompose $\mathcal{G}$ into orthogonal subspaces

$$\mathcal{G} = \bigoplus_{s\in\mathbb{N}} V_s\,, \qquad \text{where } V_s = \mathrm{span}\{\left|{n_s \atop m}\right\rangle, m^\downarrow_s \leq m \leq m^\uparrow_s\} \tag{81}$$

The hopping operators are given by the operators $a_\pm, b_\pm : \mathcal{G} \mapsto \mathcal{G}$

$$
\begin{aligned}
a_+\left|{}^n_m\right\rangle &= \begin{cases} (C_s - \sigma^s_{m+1})^{1/2}\left|{}^{n+1}_{m+1}\right\rangle & \text{if } \left|{}^n_m\right\rangle \in V_s \text{ and } \left|{}^{n+1}_{m+1}\right\rangle \in \mathcal{G} \\ 0 & \text{if } \left|{}^{n+1}_{m+1}\right\rangle \notin \mathcal{G} \end{cases} \\
a_-\left|{}^n_m\right\rangle &= \begin{cases} (C_t - \sigma^t_{m})^{1/2}\left|{}^{n-1}_{m-1}\right\rangle & \text{if } \left|{}^{n-1}_{m-1}\right\rangle \in V_t \text{ for some } V_t \subset \mathcal{G} \\ 0 & \text{if } \left|{}^{n-1}_{m-1}\right\rangle \notin \mathcal{G} \end{cases} \\
b_+\left|{}^n_m\right\rangle &= \begin{cases} (C_s + \sigma^s_{m})^{1/2}\left|{}^{n+1}_{m}\right\rangle & \text{if } \left|{}^n_m\right\rangle \in V_s \text{ and } \left|{}^{n+1}_{m}\right\rangle \in \mathcal{G} \\ 0 & \text{if } \left|{}^{n+1}_{m}\right\rangle \notin \mathcal{G} \end{cases} \\
b_-\left|{}^n_m\right\rangle &= \begin{cases} (C_t + \sigma^t_{m})^{1/2}\left|{}^{n-1}_{m}\right\rangle & \text{if } \left|{}^{n-1}_{m}\right\rangle \in V_t \text{ for some } V_t \subset \mathcal{G} \\ 0 & \text{if } \left|{}^{n-1}_{m}\right\rangle \notin \mathcal{G} \end{cases}
\end{aligned}
\tag{82}
$$

where $\sigma^s_m \in \mathbb{R}$ and $C_s \in \mathbb{R}$ are constrained so that all the square roots are real and nonnegative. σ^s_m depends on s and m whilst C_s depends only on s and is given by $C_s = |\sigma^s_{m^\downarrow_s}| = |\sigma^s_{m^\uparrow_s+1}|$

Clearly with the appropriate choice of σ^s_m we can make $\mathcal{G}$ correspond to one of the compromise representation of a NCSR with over-hopping operators. A similar definition for under-hopping operators can also be given.

The full implications of such a lattice is being researched.

VI Discussion

This article is the result of extending the definition of tangent and cotangent vector fields given in [6] for the noncommutative sphere, to the noncommutative surfaces of rotation given in [5]. For trivial ρ we can define the algebras $\mathcal{B}$, and Ψ and use them to define such vector fields. The results, although more complicated, are similar to those for the case of the noncommutative sphere. For non-trivial ρ it is not possible to define $\mathcal{B}$ or Ψ. However we can still define the multi-topology lattice and use it to examine the changes in topology. This involved one of four possible compromises, which were given in detail.

In order to interpret our system as a toy model for a quantised spacetime, we need to define and interpret curvature and hence gravity. We indicated how one defines tangent and co-tangent vectors. We can use the results in section II.3 to provide noncommutative versions of the the metric (24), hodge dual (25), and Laplace operator (26). These will all generate problems with the choice of ordering. There is a great deal of interest in connections and curvature [8] [9]. One can extend this philosophy to write the connection $\nabla_{\widetilde{df}}(\widetilde{dh})$ and curvature $R(\widetilde{df}_1, \widetilde{df}_2, \widetilde{df}_3, df_4)$ in terms of the Poisson structure on $\mathcal{M}$. However we note the rôles of f and h in $\nabla_{\widetilde{df}}(\widetilde{dh})$ are different. Perhaps one should look at $\nabla_X(dh)$ where $X = \mathrm{ad}_h$. That is to consider the two different types of vectors!

An alternative application of this theory is as a model of a particle. The idea is that $\mathcal{M}$ is the surfaces of a particle in space. For this we have to choose ρ, an element $\mathcal{H} \in \mathcal{A}^{\mathcal{R}}$ such that $\mathcal{H} = \mathcal{H}^\dagger$ and a value of ε. Once these are fixed we can calculate the spectrum of $\mathcal{H}$. If we interpret $\mathcal{H}$ as a Hamiltonian, then we could call the spectrum of $\mathcal{H}$ a mass spectrum. The main problem with this is choosing ρ and $\mathcal{H}$ since there is not much physics we can use to guide us. If ρ is not trivial and we choose $\mathcal{H}$ to contain the hopping operators then this could be used as a model of interacting particles.

Acknowledgement

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Name	$\mathcal{A}^C = \mathcal{A}^C(\rho, \varepsilon)$	$\mathcal{A}^R = \mathcal{A}^R(\rho, \varepsilon, R)$
Parameters Generators	$\rho \in C^1(\mathbb{R} \mapsto \mathbb{R})$, $\varepsilon \in \mathbb{R}$ X_+, X_-, and $f(X_0)$ where $f \in \mathcal{C}_{\mathbb{R}}$	$\rho \in C^1(\mathbb{R} \mapsto \mathbb{R})$, $R, \varepsilon \in \mathbb{R}$ X_+, X_-, and $f(X_0)$ where $f \in \mathcal{C}_{\mathbb{R}}$
General normal ordered element	$$f = \sum_{r=0}^{r_{\max}} \sum_{s=0}^{s_{\max}} X_+^r X_-^s f_{rs}(X_0) \qquad (83)$$ with $f_{rs} \in \mathcal{C}_{\mathbb{R}}$	$$f = \sum_{r=0}^{r_{\max}} X_\pm^r f_r(X_0) \qquad (86)$$ with $f_r \in \mathcal{C}_{\mathbb{R}}$
Quotient equations	$$f(X_0)X_\pm = X_\pm f(X_0 \pm \varepsilon) \qquad (84)$$ $$[X_+, X_-] = \rho(X_0 + \varepsilon) - \rho(X_0) \qquad (85)$$	$$f(X_0)X_\pm = X_\pm f(X_0 \pm \varepsilon) \qquad (87)$$ $$X_+X_- = \rho(R) - \rho(X_0) \qquad (88)$$

Name	$\mathcal{A}^N = \mathcal{A}^N(\rho, \varepsilon)$	$\mathcal{A}^{NC} = \mathcal{A}^{NC}(\rho, \varepsilon, R)$
Parameters Generators	$\rho \in C^1(\mathbb{R} \mapsto \mathbb{R})$, $\varepsilon \in \mathbb{R}$ X_+, X_-, and $f(X_0, N_0)$ where $f \in \mathcal{C}_{\mathbb{R}^2}$	$\rho \in C^1(\mathbb{R} \mapsto \mathbb{R})$, $\varepsilon \in \mathbb{R}$ X_+, X_-, and $f(X_0, N_0)$ where $f \in \mathcal{C}_{\mathbb{R}^2}$
General normal ordered element	$$f = \sum_{r=-r_{\max}}^{r_{\max}} X_\pm^r f_r(X_0, N_0) \qquad (89)$$ with $f_r \in \mathcal{C}_{\mathbb{R}^2}$	$$f = \sum_{r=0}^{r_{\max}} \sum_{s=0}^{s_{\max}} X_+^r X_-^s f_{rs}(X_0, N_0) \quad (93)$$ with $f_{rs} \in \mathcal{C}_{\mathbb{R}^2}$
Quotient equations	$$f(X_0, N_0)X_\pm = X_\pm f(X_0 \pm \varepsilon, N_0) \qquad (90)$$ $$X_+X_- = \rho(N_0) - \rho(X_0) \qquad (91)$$ $$[X_0, N_0] = 0 \qquad (92)$$	$$f(X_0, N_0)X_\pm = X_\pm f(X_0 \pm \varepsilon, N_0) \qquad (94)$$ $$[X_+, X_-] = \rho(X_0 + \varepsilon) - \rho(X_0) \qquad (95)$$ $$[X_0, N_0] = 0 \qquad (96)$$

Table I: Properties of the algebras $\mathcal{A}^C, \mathcal{A}^R, \mathcal{A}^N, \mathcal{A}^{NC}$

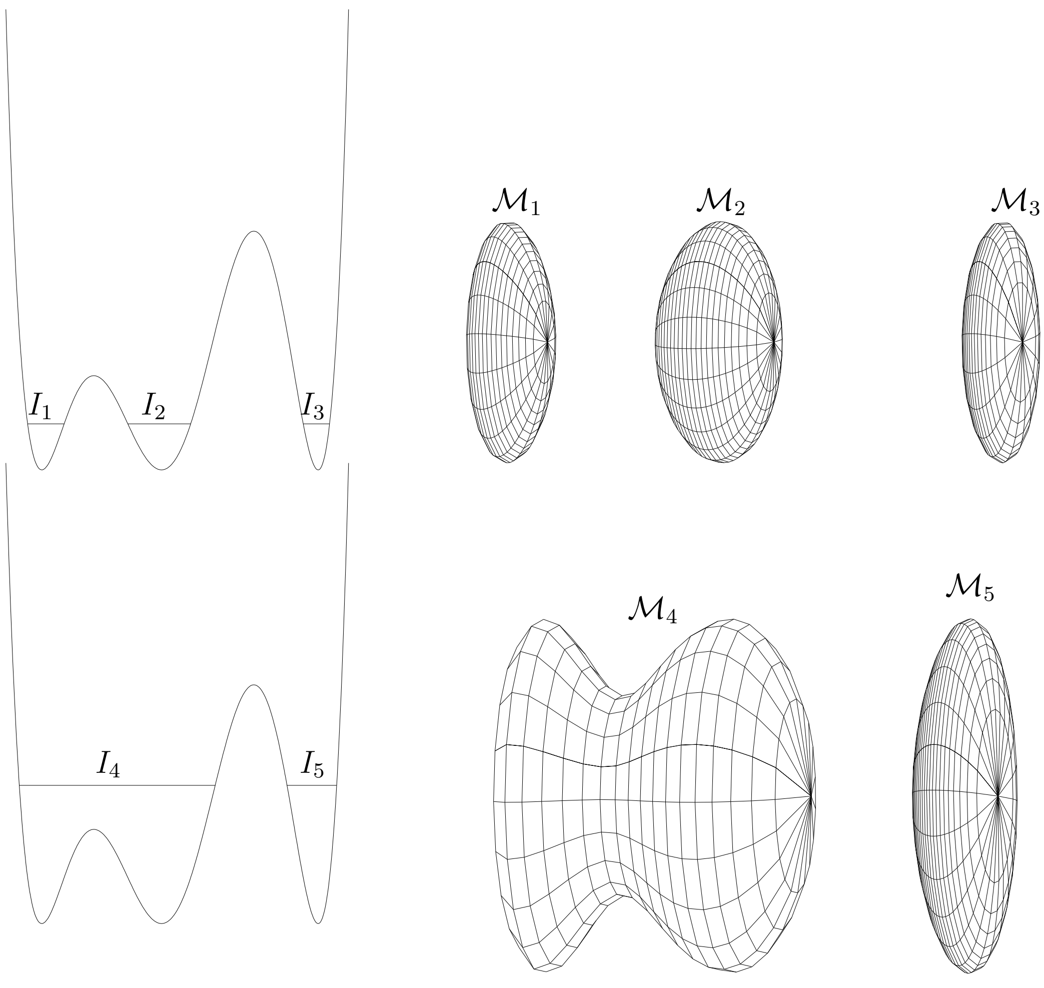


Figure 1: Surface of rotation corresponding to the same ρ for different values of R

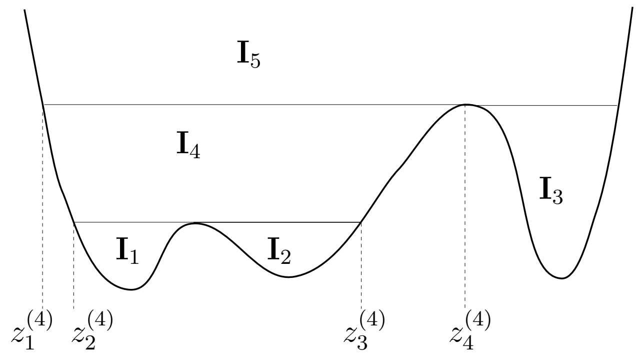


Figure 2: example of topology intervals

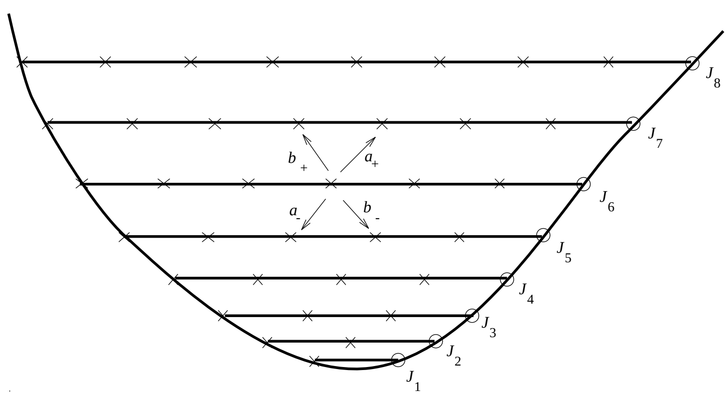
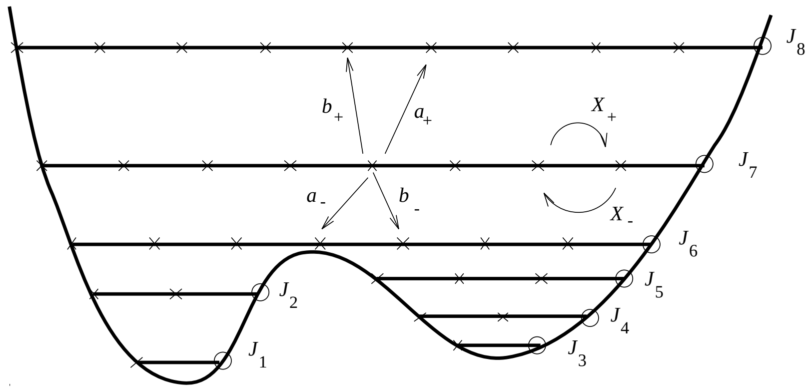


Figure 3: First few V_n for a trivial ρ



$$V_8 \to \left|{6 \atop 0}\right\rangle \quad \left|{6 \atop 1}\right\rangle \quad \left|{6 \atop 2}\right\rangle \quad \left|{6 \atop 3}\right\rangle \quad \left|{6 \atop 4}\right\rangle \quad \left|{6 \atop 5}\right\rangle \quad \left|{6 \atop 6}\right\rangle \quad \left|{6 \atop 7}\right\rangle \quad \left|{6 \atop 8}\right\rangle$$

$$V_7 \to \left|{5 \atop 0}\right\rangle \quad \left|{5 \atop 1}\right\rangle \quad \left|{5 \atop 2}\right\rangle \quad \left|{5 \atop 3}\right\rangle \quad \overset{b_+ \nwarrow \ \nearrow a_+}{\underset{a_- \swarrow \ \searrow b_-}{\left|{5 \atop 4}\right\rangle}} \quad \left|{5 \atop 5}\right\rangle \underset{X_-}{\overset{X_+}{\rightleftarrows}} \left|{5 \atop 6}\right\rangle \quad \left|{5 \atop 7}\right\rangle$$

$$V_6 \to \left|{4 \atop 0}\right\rangle \quad \left|{4 \atop 1}\right\rangle \quad \left|{4 \atop 2}\right\rangle \quad \left|{4 \atop 3}\right\rangle \quad \left|{4 \atop 4}\right\rangle \quad \left|{4 \atop 5}\right\rangle \quad \left|{4 \atop 6}\right\rangle$$

$$V_2 \to \left|{3 \atop 0}\right\rangle \quad \left|{3 \atop 1}\right\rangle \qquad V_5 \to \left|{3 \atop 3}\right\rangle \quad \left|{3 \atop 4}\right\rangle \quad \left|{3 \atop 5}\right\rangle$$

$$V_1 \to \left|{2 \atop 0}\right\rangle \qquad V_4 \to \left|{2 \atop 3}\right\rangle \quad \left|{2 \atop 4}\right\rangle$$

$$V_3 \to \left|{1 \atop 3}\right\rangle$$

[t]

space	dim	basis
V_1	1	$\{\left\|{2 \atop 0}\right\rangle\}$
V_2	2	$\{\left\|{3 \atop 0}\right\rangle, \left\|{3 \atop 1}\right\rangle\}$
V_3	1	$\{\left\|{1 \atop 3}\right\rangle\}$
V_4	2	$\{\left\|{2 \atop 3}\right\rangle, \left\|{2 \atop 4}\right\rangle\}$

space	dim	basis
V_5	3	$\{\left\|{3 \atop 3}\right\rangle, \left\|{3 \atop 4}\right\rangle, \left\|{3 \atop 5}\right\rangle\}$
V_6	7	$\{\left\|{4 \atop 0}\right\rangle, \dots, \left\|{4 \atop 6}\right\rangle\}$
V_7	8	$\{\left\|{5 \atop 0}\right\rangle, \dots, \left\|{5 \atop 7}\right\rangle\}$
V_8	9	$\{\left\|{6 \atop 0}\right\rangle, \dots, \left\|{6 \atop 8}\right\rangle\}$

Figure 4: The First Few Representations and the multi-topology lattice for a non-trivial ρ

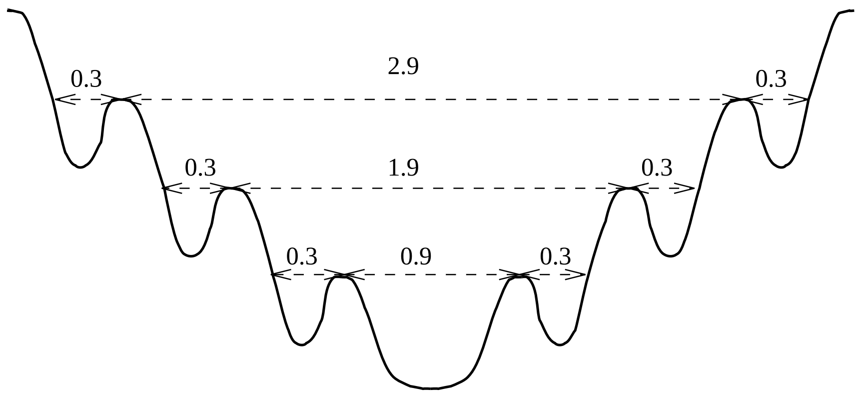


Figure 5: Example of a ρ for which no G-representation exists ($\varepsilon = 1$)

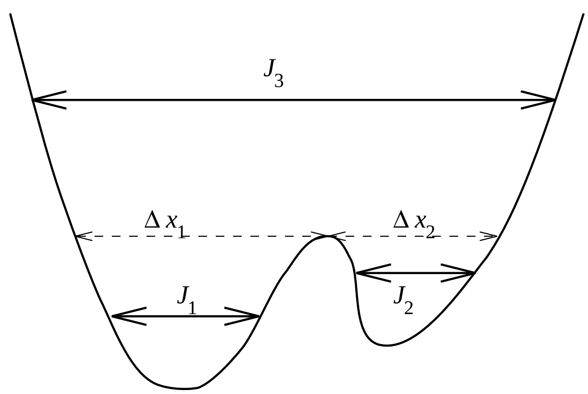


Figure 6: An example of a ρ containing a maxima

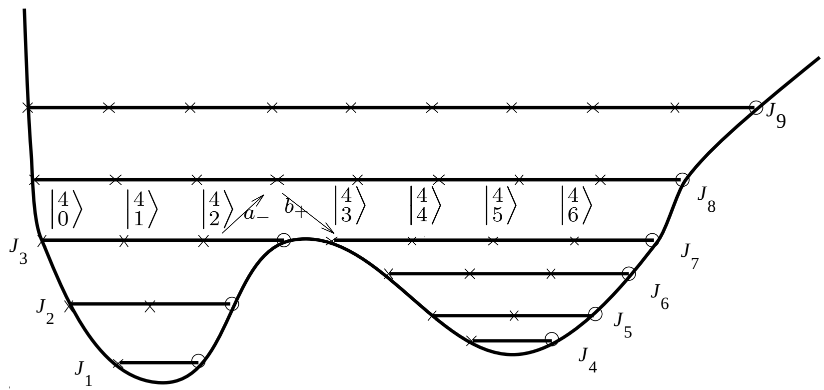


Figure 7: Merging of V_3 and V_7

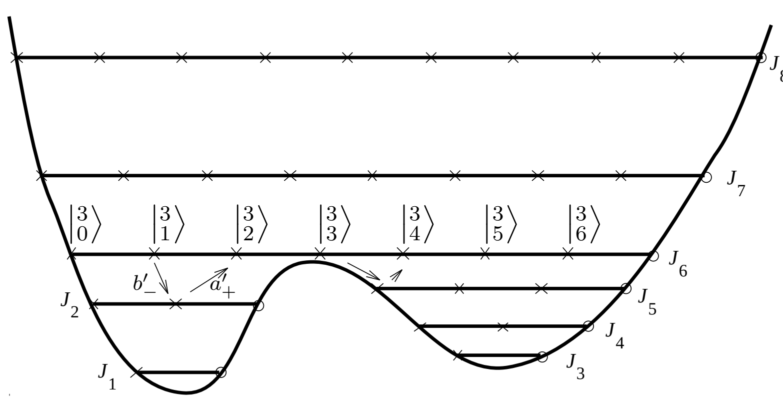


Figure 8: Splitting of V_8

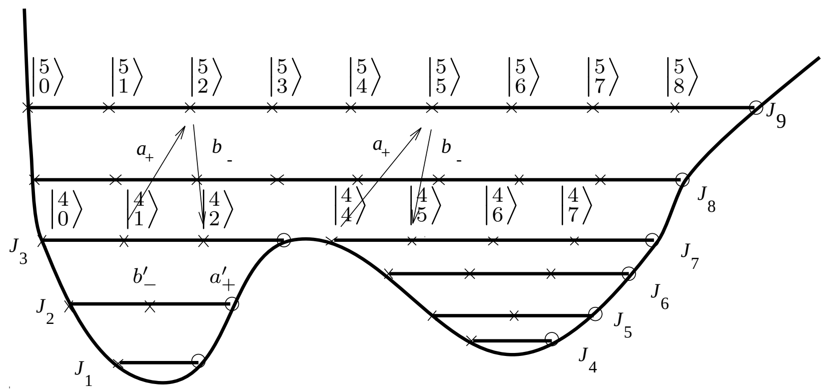


Figure 9: Removal of V_7

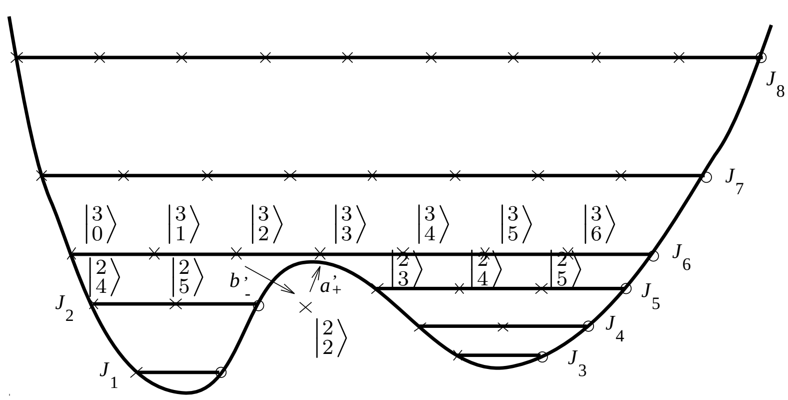


Figure 10: Adding a point V_6