On a Planarity Criterion Coming from Knot Theory

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Abstract

A graph G is called *minimalizable* if a diagram with minimal crossing number can be obtained from an arbitrary diagram of G by crossing changes. If, furthermore, the minimal diagram is unique up to crossing changes then G is called *strongly minimalizable*. In this article, it is explained how minimalizability of a graph is related to its automorphism group and it is shown that a graph is strongly minimalizable if the automorphism group is trivial or isomorphic to a product of symmetric groups. Then, the treatment of crossing number problems in graph theory by knot theoretical means is discussed and, as an example, a planarity criterion for minimalizable graphs is given.

Keywords: Minimalizable Graphs, Automorphism Group, Crossing Number

AMS classification: 05C10; 57M15

1 Introduction

One of the hardest tasks in topological graph theory is the problem to determine the minimal crossing number of graph embeddings. Only very few results are known and there are several outstanding conjectures concerning the crossing numbers of, e.g., the complete graph K_n, the complete bipartite graph $K_{n,m}$, and the product of cycles $C_n \times C_m$ for arbitrary values of $m, n \in \mathbb{N}$.

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A main obstacle for finding lower bounds of crossing numbers by means of knot theory is the fact that, in contrast to knot diagrams, the diagram of a planar graph in general cannot be unknotted by crossing changes. Likewise, it is in general not possible to obtain a diagram with minimal crossing number by crossing changes in an arbitrary diagram. In Section 2 of this paper, the class of graphs having this property is investigated. A graph belonging to this class is called *minimalizable* and, in the case that a minimalizable graph possesses a unique minimal diagram up to crossing changes, it is called *strongly minimalizable*. The rôle that the graph's automorphism group is playing with respect to being minimalizable is investigated and examples of strongly minimalizable graphs are given, namely, those graphs whose automorphism group is either trivial or an appropriate product of symmetric groups.

In Section 3, a general method is described how crossing number problems in graph theory can be reduced to determine crossing numbers of finitely many graph diagrams by knot theoretical means. As an example, a planarity criterion for minimalizable graphs that arises from knot theory is stated and it is proved in Section 4.

2 Minimalizable Graphs

The graphs considered in the following are allowed to have multiple edges and loops. A *topological graph* is a 1-dimensional cell complex which is related to an abstract graph in the obvious way. If G is a topological graph, then a *graph $\mathcal{G}$ in* $\mathbb{R}^3$ is the image of an embedding of G into $\mathbb{R}^3$. Two graphs $\mathcal{G}_1$, $\mathcal{G}_2$ in $\mathbb{R}^3$ are called *equivalent* or *ambient isotopic* if there exists an orientation preserving autohomeomorphism of $\mathbb{R}^3$ which maps $\mathcal{G}_1$ onto $\mathcal{G}_2$. Embeddings of topological graphs in $\mathbb{R}^3$ can be examined via *graph diagrams*, i.e., images under regular projections to an appropriate plane equipped with over-under information at double points. Two graph diagrams D and D' are called *equivalent* or *ambient isotopic* if the corresponding graphs in $\mathbb{R}^3$ are equivalent. Equivalent graph diagrams can be transformed into each other by a finite sequence of so-called *Reidemeister moves* combined with orientation preserving homeomorphisms of the plane to itself, see [7] or [19].

The *crossing number $cr(G)$* of a graph G is the minimal number of double points in a regular projection of any graph embedding in 3-space. In graph theory, the equivalent definition of $cr(G)$ as minimal number of crossings in a *good drawing* of G is more common, see [4] for an introduction.

Definition A graph G is called *minimalizable* if a diagram with minimal crossing number can be obtained from an arbitrary diagram of G by a choice of crossing changes followed by an ambient isotopy. G is called *strongly minimalizable* if it is minimalizable and possesses a minimal diagram that is unique up to crossing changes followed by an ambient isotopy.

Remark It is well-known that the (planar) graph with one vertex and one edge corresponding to a classical knot is strongly minimalizable. There exist graphs, even planar ones, that are not minimalizable, see [15].

Theorem 1 *Any two diagrams of a strongly minimalizable graph are equivalent up to crossing changes, i.e., they can be transformed into each other by a finite sequence of crossing changes and ambient isotopies.*

$\square$

Remark Observe that every finite sequence of crossing changes and ambient isotopies applied to a graph diagram can be reduced such that all crossing changes are realized in a single diagram. This can be seen in the same way as for the equivalent definitions of the unknotting number of a knot via crossing changes and ambient isotopies, see [1].

Theorem 2 *A planar minimalizable graph is strongly minimalizable.*

Proof: This follows immediately from a result by Lipson [10], Corollary 6.
$\square$

Remark In [15], a planar minimalizable graph is called *trivializable* and it is shown that a subgraph of a trivializable graph again is trivializable. In general, a subgraph of a strongly minimalizable graph is not strongly minimalizable, nor even minimalizable. This follows from the fact that every graph is isomorphic to a subgraph of the complete graph K_n for some $n \in \mathbb{N}$ and this is strongly minimalizable (see Corollary 6 below). On the other hand, there are infinitely many examples of graphs that are not minimalizable, see [15], [16], [18].

As mentioned above, a knot represented by a knot diagram always can be unknotted by appropriate crossing changes. Likewise, a knotted arc that connects two points in the plane can be unknotted by crossing changes. This can be achieved by traveling along the arc from one end to the other and choosing self-crossings such that each crossing is passed as an overcrossing when reached for the first time.

Furthermore, up to crossing changes followed by an ambient isotopy, there are only finitely many ways of drawing a graph in the plane. The number of essentially different drawings depends on the graph's automorphism group.

Lemma 3 *Let G be a graph with vertices $v_1, \ldots, v_n$ and let $w_1, \ldots, w_n$ be distinct points in the plane. Then, up to crossing changes followed by an ambient isotopy, G has a unique diagram such that the vertex v_i corresponds to the point w_i for $i = 1, \ldots, n$.*

Proof: The proof is carried out by induction on the number $k \geq 0$ of graph edges. If G has no edge then there is nothing to show since the n points in the plane can be arranged arbitrarily by ambient isotopy.

Now let G have $k \geq 1$ edges and let D and D' be diagrams of G such that $w_1, \ldots, w_n$ correspond to the graph vertices. Choose an arbitrary edge e of G. By induction hypothesis, the diagrams arising from D and D' by deleting the arcs corresponding to e allow a choice of crossing changes such that the resulting diagrams are equivalent. Apply these crossing changes to the diagrams D and D', respectively, and change those crossings in which the arcs a in D and a' in D' related to e are involved as follows. Choose the crossings of a with an arc different from a such that a is always above the other arc and change the self-crossings of a such that a is transformed into an unknotted arc. Apply the same procedure to the arc a' in D'. The two resulting diagrams of G are easily seen to be equivalent.

□

Theorem 4 *A graph G with trivial automorphism group is strongly minimalizable.*

Proof: Let G have n vertices. Since n distinct points in the plane can be moved arbitrarily by ambient isotopy, an embedding of G into the plane is determined by connecting n fixed points by arcs corresponding to the incidence relation given by G. By Lemma 3, the only ambiguity, up to crossing changes followed by an ambient isotopy, to connect the points in the plane by arcs arises from the automorphisms of G. $\square$

Remark If a graph G possesses no vertices of degree two then there is an appropriate subdivsion of G that has trivial automorphism group. The additional vertices are topologically uninteresting but, of course, important for the property of being minimalizable. Adding vertices of degree two in the described way has the same effect as colouring the edges of the graph and considering only graph automorphisms and modifications of graph diagrams that respect colourings.

Theorem 5 *Let G be a graph with n vertices. If the automorphism group of G is isomorphic to $S_{n_1} \times \ldots \times S_{n_k}$ with $n_1 + \ldots + n_k = n$ then G is strongly minimalizable.*

Proof: As explained in the proof of Theorem 4, only the effect of connecting a fixed set of points by arcs in two different ways arising from an automorphism of G has to be investigated. Because of the structure of the automorphism group given, the n points can be partitioned into subsets corresponding to the partition $n = n_1 + \ldots + n_k$ and such that whenever two points belonging to different subsets have to be connected by an arc then every point of the first subset has to be connected with every point of the second subset and vice versa. Likewise, whenever two different points belonging to the same subset have to be connected then any two different points of this subset have to be connected by an arc and if there is a vertex incident with a loop then every vertex of the subset is incident with a loop. Corresponding statements hold in the case of multiple edges. Thus, the arising diagram of G is unique up to crossing changes followed by an ambient isotopy. $\square$

Remark The construction that is given in the proof of Theorem 5 shows that the graph G has K_{n_i,n_j} as a subgraph if two points of the subsets corresponding to n_i and n_j are connected by an arc, and it has K_{n_i} as a subgraph if two points of the subset corresponding to n_i are connected by an arc.

Corollary 6 *For $n, n_1, \ldots, n_k \in \mathbb{N}$, the complete graph K_n and the complete k-partite graph $K_{n_1,\ldots,n_k}$ are strongly minimalizable.*

□

3 Crossing Number Problems

Applying the results of the previous section, an algorithm can be given to reduce the problem of finding the crossing number of a given graph to the problem of determining the crossing numbers of finitely many graph diagrams up to ambient isotopy. Start with an arbitrary diagram of the graph and calculate the crossing numbers of the graph embeddings corresponding to all possible choices of crossing information for the diagram's double points. Then change the diagram by connecting vertices in the plane by arcs in a different manner corresponding to an automorphism of the graph. As before, calculate the crossing numbers of all related graphs in $\mathbb{R}^3$, and carry on until all graph automorphisms have been applied to the starting diagram. The desired crossing number of the graph is the minimum taken over all crossing numbers calculated.

By this procedure, a complicated problem of topological graph theory is transformed into finitely many knot theoretical problems. Of course, the determination of crossing numbers for graphs in $\mathbb{R}^3$ is, in general, no easy task either. Some results concerning the crossing numbers of particular classes of graphs can be found in [13] and [14].

The planarity criterion that is stated in the following theorem is an example how results on crossing numbers for graphs can be deduced from the calculation of crossing numbers of graph embeddings in $\mathbb{R}^3$. A proof of the theorem is given in Section 4 where the knot theoretical ingredients are described, too.

Theorem 7 *Let G be a minimalizable graph and let D be an arbitrary diagram of G. Furthermore, let there exist a graph vertex v in D of degree four such that the following conditions are fulfilled.*

i) For each of the four pairs of neighbouring edges of v, there exists a cycle that contains the two edges such that the two cycles in D belonging to opposite pairs only meet in v.

ii) If D' arises from D by arbitrary crossing changes then replacing v with an appropriate crossing gives a diagram of an embedded graph with crossing number at least two.

Then G is non-planar.

Remark Because of the fact that a graph that possesses a non-planar subgraph is non-planar itself, Theorem 7 may be applied to graphs that have no (suitable) vertex of degree four by considering an appropriate subgraph.

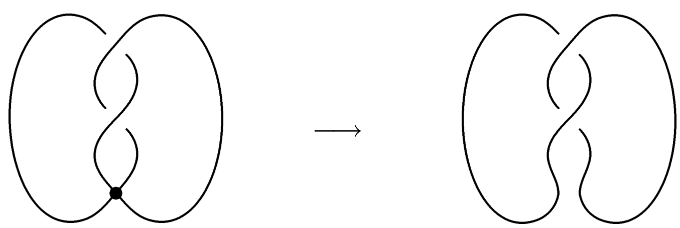

Figure 1: Forbidden situation yields Hopf link

Observe that the second part of condition i) in Theorem 7 excludes the situation depicted in Fig. 1, i.e., cutting through the vertex gives two curves that are linked.

Example K_5 is non-planar as can easily be seen by applying Theorem

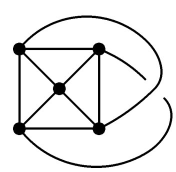

Figure 2: A minimal diagram of K_5

7 to the diagram depicted in Fig. 2 for an arbitrary vertex v. Condition i) obviously is fulfilled. For both choices of over-under information for the diagram's single crossing (indeed, the two arising diagrams are equivalent), v can be replaced by an appropriate crossing such that the resulting diagram contains a Hopf link. Thus condition ii) is fulfilled, too.

Of course, it is well-known that K_5 is non-planar and, by Kuratowski's theorem, that every non-planar graph contains a K_5- or a $K_{3,3}$-minor. The graph considered in the next example contains a $K_{3,3}$-minor.

Example With the same argumentation as in the previous example, the

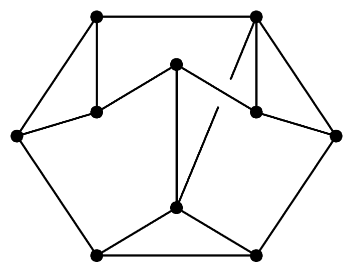

Figure 3: Non-planar graph

graph depicted in Fig. 3 is non-planar. Again, both of the two vertices of degree four are appropriate for applying Theorem 7.

In general, very little is known about the behaviour of the crossing number when a new graph is constructed from one or more given graphs. For example, it is an open problem wether the crossing number is additive with respect to a *connected sum* $G_1 \# G_2$ of two graphs G_1 and G_2, i.e., two edges $e_1 = \{v_1, v_2\} \in G_1$ and $e_2 = \{w_1, w_2\} \in G_2$ that are not loops are replaced by edges $e'_1 = \{v_1, w_1\}$ and $e'_2 = \{v_2, w_2\}$ (the definition can be extended to subdivisions of G_1 and G_2). The corresponding problem for graph embeddings is well-known in knot theory, and the additivity of crossing numbers for connected sums of knots and links is an old outstanding conjecture. In the case of abstract graphs, it is not even clear if a connected sum of two (strongly) minimalizable graphs is (strongly) minimalizable.

Theorem 8 *Let G and G' be (strongly) minimalizable graphs. Then the following hold.*

a) *The disjoint union $G \sqcup G'$ of G and G' is (strongly) minimalizable and*

$$cr(G \sqcup G') \;=\; cr(G) + cr(G')\,.$$

b) *Any one-point union $G \bullet G'$ of G and G' is (strongly) minimalizable and*

$$cr(G \bullet G') \;=\; cr(G) + cr(G')\,.$$

Proof: For arbitrary diagrams of $G \sqcup G'$ and $G \bullet G'$, respectively, a sequence of crossing changes can be chosen such that arcs corresponding to G always overcross arcs corresponding to G' at the diagram's double points. Denote the arising diagrams by $D^{\sqcup}$ and $D^{\bullet}$, respectively.

For the graph in $\mathbb{R}^3$ that belongs to $D^{\sqcup}$, it may be assumed that the part corresponding to G lies in the half space above the projection plane and the part corresponding to G' lies beneath the projection plane. Clearly, there is an equivalent diagram $D \sqcup D'$ which is the disjoint union of a diagram D of G and D' of G'. Since both graphs are minimalizable, there are sequences of crossing changes that realize $cr(G)$ and $cr(G')$ in D and D', respectively, showing that $cr(G \sqcup G') \geq cr(G) + cr(G')$. The opposite inequality holds trivially and it follows that the crossing number is additive with respect to connected sums. Furthermore, crossing changes in $D \sqcup D'$ realize $cr(G \sqcup G')$ and therefore $G \sqcup G'$ is (strongly) minimalizable.

Similarly, the graph in $\mathbb{R}^3$ that belongs to $D^{\bullet}$ may be thought to consist of a part corresponding to G lying in the upper half space and a part corresponding to G' lying in the lower half space except one point v in which the graphs intersect. Deform the graph corresponding to G' such that its projection is contained completely in one region of the subdiagram of $D^{\bullet}$ belonging to G (except for the point v). This gives an equivalent diagram $D \bullet D'$ that is a one-point union of a diagram D of G and D' of G'. The rest of the proof is completely the same as in the case of $G \sqcup G'$.

□

4 Proof of Theorem 7

For the proof of Theorem 7, some knot theoretical definitions and results have to be given. The objects under consideration, as they are needed here, are described in more detail in [13]. For general knot theoretical terminology see, e.g., [1], [2], [8], [9], [12]. In the following, a *link* is an embedding of a graph in $\mathbb{R}^3$ that consists of one or more disjoint loops.

A *tangle* is a part of a link diagram in the form of a disk with four arcs emerging from it, see Fig. 4, where the tangle's position is indicated by labeling the emerging arcs with letters a, b, c, d in a clockwise ordering (or simply one of them with "a"). An equivalence relation for tangles is given via ambient isotopy that fixes the ends of the tangle's arcs. For a tangle t,

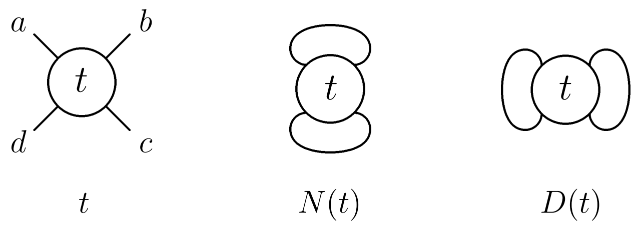

Figure 4: A tangle and its closures

there are two possible ways to connect the four ends by two arcs in the plane that do not intersect, see Fig. 4. The arising link diagrams $N(t)$ and $D(t)$ are called *closures* of t.

Denote the two different tangles with crossing number zero by 0 and ∞, and the two diffent tangles with crossing number one by 1 and $\overline{1}$. They belong to the class $\mathcal{R}$ of *rational tangles*. There is an important connection between rational tangles and *Reidemeister moves of type V* at a graph vertex as depicted in Fig. 5. Indeed, a rational tangle can be defined as the result

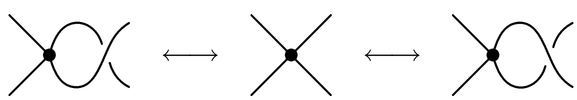

Figure 5: Reidemeister moves of type V

of applying a sequence of moves as depicted in Fig. 5 to one of the tangles 0, ∞, 1, $\overline{1}$ instead of a graph vertex.

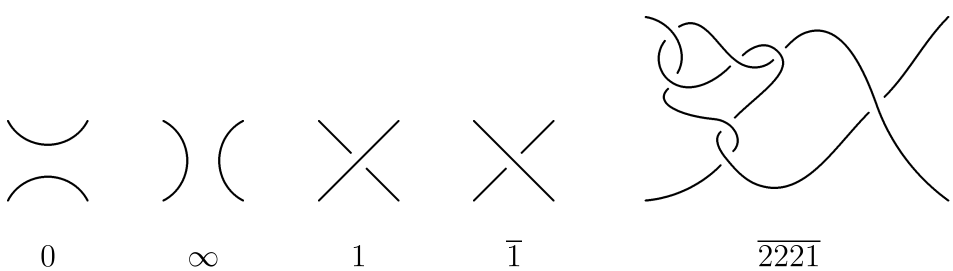

Figure 6: Rational tangles in normal form

Rational tangles can be uniquely classified by rational numbers, see [3] and [5], and there is a normal form for rational tangles from which the crossing number and an alternating diagram that realizes this number can easily be read off, see Fig. 6. For a rational tangle r, let $|r|$ denote its crossing number.

From a given graph diagram D that contains a vertex of degree four, there can be obtained new graph diagrams by substituting rational tangles for the graph vertex, see [13]. To do this in a well-defined way, it is necessary to give an *orientation* to the vertex, i.e., labeling an edge incident with the vertex with the letter a. Then a rational tangle can be substituted for the graph vertex in the obvious way such that the edge labeled with "a" fits together with the corresponding arc emerging from r. An example is given in Fig. 7.

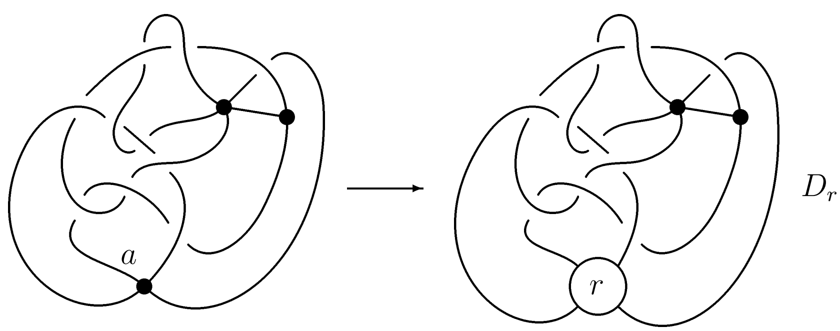

Figure 7: Replacing a vertex by a rational tangle

In the same manner as in [13], where only diagrams of 4-regular graphs are considered, the following theorem can be shown.

Theorem 9 *Let D be a graph diagram that has a vertex of degree four for which an orientation is chosen. Then, the set $\{D_r \mid r \in \mathcal{R}\}$ is invariant with respect to ambient isotopy.*

□

A link diagram D is called *reduced* if it does not contain a crossing point p such that $D \setminus \{p\}$ has more components than D as depicted in Fig. 8. Since a reduced alternating link diagram has minimal crossing number, see

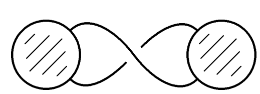

Figure 8: Non-reduced diagram

[6], [11], [17] for proofs of this famous *Tait Conjecture*, the following lemma can readily be deduced from a rational tangle's normal form.

Lemma 10 *For every rational tangle r with $|r| \geq 2$, either $N(r)$ or $D(r)$ has crossing number $|r|$.*

$\square$

Using the same technique as for the proof of Lemma 10, namely, the span of the Jones polynomial which is additive with respect to connected sums $D_1 \# D_2$ of knot diagrams D_1, D_2, the following easily can be shown.

Lemma 11 *For a rational tangle r with $|r| \geq 2$ and arbitrary knot diagrams D_1, D_2, either $D_1 \# N(r) \# D_2$ or $D_1 \# D(r) \# D_2$ has crossing number $\geq |r|$.*

$\square$

Remark Indeed, Lemma 11 holds for any reduced alternating diagram instead of $N(r)$ or $D(r)$, respectively, and for connected sums with finitely many diagrams $D_1, \ldots, D_k$. The typical situation that occurs in the proof of Theorem 7 is depicted in Fig. 9.

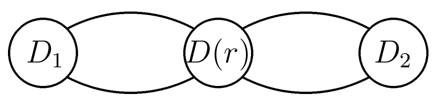

Figure 9: Connected sum $D_1 \# D(r) \# D_2$

Proof of Theorem 7: Assume that $\widetilde{D}$ is a crossing-free diagram of G. Following Theorem 2, there exists a diagram D' equivalent with $\widetilde{D}$ which arises from D by crossing changes. Choose a vertex-orientation for v in D' and consider the diagrams D'_r with rational tangles r. If r has crossing number at least two then D'_r likewise has crossing number at least two because of condition i), which obviously is fulfilled for the diagram D' as well as for D, since a subdiagram of D'_r has crossing number at least two by Lemma 11. Furthermore, either D'_1 or $D'_{\overline{1}}$ has crossing number at least two because of condition ii). Thus, there are at most three diagrams D'_r, namely, the diagrams D'_0, D'_∞, D'_1 or the diagrams D'_0, D'_∞, $D'_{\overline{1}}$, that have crossing number strictly less than two.

But considering diagrams $\widetilde{D}_r$, corresponding to an arbitrary vertex of degree four in $\widetilde{D}$ with chosen vertex-orientation, immediately yields a contradiction since each of the four diagrams $\widetilde{D}_0$, $\widetilde{D}_\infty$, $\widetilde{D}_1$, $\widetilde{D}_{\overline{1}}$ obviously has crossing number zero or one.

$\square$

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