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On the Structure of the Bäcklund Transformations for the Relativistic Lattices

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Abstract

The Bäcklund transformations for the relativistic lattices of the Toda type and their discrete analogues can be obtained as the composition of two duality transformations. The condition of invariance under this composition allows to distinguish effectively the integrable cases. Iterations of the Bäcklund transformations can be described in the terms of nonrelativistic lattices of the Toda type. Several multifield generalizations are presented.

1 Introduction

In this paper we consider four classes of difference and differential-difference equations, which contain, in particular, the Toda lattice introduced by Toda [1], the relativistic Toda lattice introduced by Ruijsenaars [2, 3], their generalizations introduced by Yamilov [4] and Suris [5, 6, 7] and their discretizations introduced by Hirota [8] and Suris [9, 10, 11]. The systematic account is given of the method proposed in the papers [12, 13] for studying of these classes of equations, and some classification results are presented. The main idea of the method can be described in a few words.

For the given lattice equation $e[q] = 0$ we define, in a certain way, the pair of transformations $T_+ : q \to Q$ and $T_- : q \to \tilde{Q}$. In general, variables Q and $\tilde{Q}$ satisfy the distinct equations. We say that the lattice $e[q] = 0$ admits the duality transformations $T_\pm$ if both variables Q, $\tilde{Q}$ satisfy the same equation $E[Q] = 0$ which is called the dual equation. Obviously, the composition of the duality transformations can be used for reproducing of solutions. Denoting iterations of the transformations $T_l = T_-^{-1}T_+$ and $T_l^+ = T_+T_-^{-1}$ by superscript l one obtains the commutative diagram displayed on the Figure 1.

In other words, transformations T_l and T_l^+ define the Bäcklund transformations for the given lattice and its dual. This explains the connection between duality transformations and integrability since existence of the Bäcklund transformation is an indispensable feature of any integrable system. Quite analogously, the Bäcklund transformations for the KdV and mKdV equations are obtained by the composition of two slightly different Miura

$$
\begin{tikzcd}
q^{l-1} \arrow[rr, "T_l"] & & q^l \arrow[rr, "T_l"] \arrow[dl, "T_-"'] \arrow[dr, "T_+"] & & q^{l+1} \arrow[dl, "T_-"'] & \cdots \\
\cdots & Q^{l-1} \arrow[rr, "T_l^+"'] & & Q^l \arrow[rr, "T_l^+"'] & & Q^{l+1}
\end{tikzcd}
$$

Figure 1. Bäcklund transformation as the composition of duality transformations.

maps and the Schlesinger transformation for PII is constructed from two substitutions into PXXXIV.

The requirement that the variables Q and $\tilde{Q}$ satisfy the same equation is stringent enough and allows to distinguish effectively the integrable relativistic lattices. However, it does not work for the subclass of nonrelativistic lattices, which are characterized by the property $T_+ = T_-$. In contrast to the relativistic case, the duality transformation is now irrelevant to integrability and cannot be used for classification. Nevertheless, this subclass does not require special treatment since it arises as a by-product of already obtained results. Namely, it turns out that the iterations of the transformation T_l for the relativistic lattice admitting duality transformation are described by some lattice of the Toda type and the same is true for their discrete analogues.

The above scheme is applied to the discrete relativistic lattices in the Section 2. In the Section 2.1 we introduce the notions of the duality transformation and the dual equation, and prove that the dual equation belongs to the same class. In the Section 2.3 we study the Bäcklund transformation T_l in more details and prove that it is equivalent to some discrete lattice of the Toda type. The classification of integrable equations is performed in the Section 2.2, several multifield generalizations are presented in the Section 4. The Section 3 devoted to the lattices of the relativistic Toda type is, in fact, exact continuous double of the Section 2. Joint of both discrete and continuous theories is performed in the Section 3.4.

2 The lattices of the discrete relativistic Toda type

This section deals with the difference equations on the real or complex variable q defined on the two-dimensional integer lattice. As a rule, we use abridged notation $q = q_{mn}$, $q_{ij} = q_{m+i,n+j}$. The shift operators on the first and second subscripts will be denoted T_m and T_n respectively. Next, it is convenient to introduce notation for the differences (see Figure 2)

$$x = q - q_{-1,0}, \quad y = q - q_{0,-1}, \quad z = q - q_{-1,-1}.$$

Obviously, one of these differences can be expressed through the other two, e.g. $z = x + y_{-1,0} = y + x_{0,-1}$, and the following identity is valid

$$(T_m - 1)y_{0,1} = (T_n - 1)x_{1,0}. \tag{2.1}$$

The uppercase letters Q, X, Y, Z (reserved for the dual variable and its differences) are used in quite similar manner.

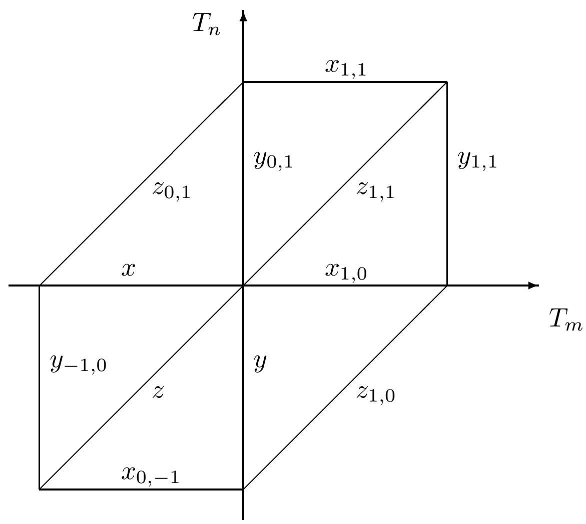

Figure 2. Pattern for the discrete relativistic lattice.

2.1 Duality transformations

The lattices of the discrete relativistic Toda type

$$(T_m - 1)f(x) + (T_n - 1)g(y) + (T_mT_n - 1)h(z) = 0 \tag{2.2}$$

are the Euler equations for the Lagrangians of the form

$$\mathcal{L} = \sum_{m,n}(a(x) + b(y) + c(z)) \tag{2.3}$$

where $f = a'$, $g = b'$, $h = c'$. In this section we assume that $f'g'h' \neq 0$ in order to eliminate the case of the nonrelativistic lattices, so that equation (2.2) involves 7 nodes of the lattice as shown on the Figure 2. It is clear that the roles of the shifts T_m, T_n and $T_m^{-1}T_n^{-1}$ in (2.2) are equal (actually, one can assume them as the generators of the regular hexagonal lattice).

Equation (2.2) can be rewritten in two equivalent forms of the momentum conservation law:

$$\begin{aligned} &(T_m - 1)(f(x) + h(z_{0,1})) + (T_n - 1)(g(y) + h(z)) = 0 \\ &\ \Leftrightarrow\ (T_m - 1)(f(x) + h(z)) + (T_n - 1)(g(y) + h(z_{1,0})) = 0. \end{aligned}$$

This allows to introduce the pair of transformations acting on the differences:

$$T_+ :\ (X, Y_{-1,0}) = T(x, y_{0,1}), \qquad T_- :\ (\tilde{X}_{0,-1}, \tilde{Y}) = T(x_{1,0}, y) \tag{2.4}$$

where the mapping $T : (x, y) \to (X, Y)$ is given by the formulae

$$X = g(y) + h(x + y), \quad Y = -f(x) - h(x + y). \tag{2.5}$$

Obviously, the variables Q and $\tilde{Q}$ corresponding to X, Y and $\tilde{X}, \tilde{Y}$ are defined only up to the addition of an arbitrary constant. However, the equations for these variables contain only differences and can be derived by solving (2.4), (2.5) with respect to x, y and using the identity (2.1). Generally, these equations are distinct from each other.

Definition 1. *The lattice (2.2) admits the duality transformations (2.4), (2.5) if the mapping T is invertible and both variables Q, $\tilde{Q}$ satisfy the same lattice which is called dual to (2.2).*

The following Theorem characterizes equations admitting duality transformations in terms of the mapping T. It also demonstrates that the duality transformations do not lead out off the class (2.2). The immediate corollary is that the equation which is dual to the dual equation coincides with the original one.

Theorem 1. *Equation (2.2) admits duality transformations if and only if the inverse of (2.5) is of the form*

$$x = G(Y) + H(X+Y), \quad y = -F(X) - H(X+Y). \tag{2.6}$$

In this case the dual equation is of the form

$$(T_m - 1)F(X) + (T_n - 1)G(Y) + (T_m T_n - 1)H(Z) = 0. \tag{2.7}$$

Proof. Let T^{-1} be of the stated form, then one can easily check that the elimination of x, y by means of the identity (2.1) brings to equation (2.7) for both transformations (2.4).

Conversely, assume that equations for Q and $\tilde{Q}$ coincide. Let T^{-1} be of the form

$$x = \Phi(X, Y), \quad y = \Psi(X, Y)$$

then inverses of the transformations (2.4) are given by the formulae (the tilde in the second one is omitted)

$$\begin{aligned} T_+^{-1}: &\quad x = \Phi(X, Y_{-1,0}), \quad y_{0,1} = \Psi(X, Y_{-1,0}), \\ T_-^{-1}: &\quad x_{1,0} = \Phi(X_{0,-1}, Y), \quad y = \Psi(X_{0,-1}, Y). \end{aligned}$$

The identity (2.1) yields the equations

$$\Phi(X_{1,1}, Y_{0,1}) - \Phi(X_{1,0}, Y) - \Psi(X_{1,0}, Y) + \Psi(X, Y_{-1,0}) = 0,$$

$$\Phi(X, Y_{0,1}) - \Phi(X_{0,-1}, Y) - \Psi(X_{1,0}, Y_{1,1}) + \Psi(X, Y_{0,1}) = 0$$

which must be equivalent to each other. In order to compare these equations, rewrite the last one in the form

$$\Phi(X, Y_{0,1}) - \Phi(X + Y_{-1,0} - Y, Y) - \Psi(X_{1,0}, X_{1,1} + Y_{0,1} - X_{1,0}) + \Psi(X, Y_{0,1}) = 0,$$

so that both equations contain the variables $X_{1,1}$, $X_{1,0}$, X, $Y_{0,1}$, Y, $Y_{-1,0}$. Now let us consider $X_{1,1}$ as function on the rest variables from this set, then

$$\begin{aligned} \frac{\partial X_{1,1}}{\partial Y_{0,1}} &= -\frac{\partial_{Y_{0,1}} \Phi(X_{1,1}, Y_{0,1})}{\partial_{X_{1,1}} \Phi(X_{1,1}, Y_{0,1})} \\ &= \frac{\partial_{Y_{0,1}} [\Phi(X, Y_{0,1}) - \Psi(X_{1,0}, X_{1,1} + Y_{0,1} - X_{1,0}) + \Psi(X, Y_{0,1})]}{\partial_{X_{1,1}} \Psi(X_{1,0}, X_{1,1} + Y_{0,1} - X_{1,0})}. \end{aligned}$$

The last equality must be satisfied identically. Differentiating it with respect to X and $X_{1,0}$ yields

$$\partial_X\partial_{Y_{0,1}}(\Phi(X,Y_{0,1})+\Psi(X,Y_{0,1}))=0,\quad \partial_{X_{1,0}}\partial_{Y_{0,1}}\Psi(X_{1,0},X_{1,1}+Y_{0,1}-X_{1,0})=0$$

and hence

$$\Phi(X,Y)+\Psi(X,Y)=G(Y)-F(X),\quad \Psi(X,Y)=K(X)-H(X+Y). \tag{2.8}$$

Analogously, differentiating the relation

$$\begin{aligned}\frac{\partial Y_{-1,0}}{\partial X}&=-\frac{\partial_X\Psi(X,Y_{-1,0})}{\partial_{Y_{-1,0}}\Psi(X,Y_{-1,0})}\\ &=\frac{\partial_X[\Phi(X,Y_{0,1})-\Phi(X+Y_{-1,0}-Y,Y)+\Psi(X,Y_{0,1})]}{\partial_{Y_{-1,0}}\Phi(X+Y_{-1,0}-Y,Y)}\end{aligned}$$

with respect to Y yields

$$\partial_X\partial_Y\Phi(X+Y_{-1,0}-Y,Y)=0\ \Rightarrow\ \Phi(X,Y)=L(Y)+M(X+Y).$$

Comparing with (2.8) completes the proof. ∎

2.2 Classification theorem

The Definition 1 turns out to be severe enough and allows to obtain the finite list of integrable equations (2.2). In virtue of the Theorem 1 it is sufficient to find all functions f,g,h such that inverse of the transformation (2.5) is given by (2.6). This means that the Jacobian $\Delta=f'g'+g'h'+h'f'$ of the map (2.5) must be nonzero and the following identities must hold

$$x_{XY}+y_{XY}=0,\quad x_{XY}=x_{XX},\quad y_{XY}=y_{YY}.$$

These three relations are equivalent. Indeed, the Jacobi matrix is

$$\begin{pmatrix} x_X & y_X\\ x_Y & y_Y\end{pmatrix}=\frac{1}{\Delta}\begin{pmatrix} -h' & f'+h'\\ -g'-h' & h'\end{pmatrix}$$

that is $x_X=-y_Y$. Straightforward computation proves that functions $f(x)$, $g(y)$, $h(x+y)$ must satisfy the equation

$$(g'+h')\frac{f''}{f'}+(f'+h')\frac{g''}{g'}=(f'+g')\frac{h''}{h'}.$$

The designations $f'=1/u$, $g'=1/v$, $h'=1/w$ rewrite it in more convenient form

$$[v(y)+w(x+y)]u'(x)+[u(x)+w(x+y)]v'(y)=[u(x)+v(y)]w'(x+y). \tag{2.9}$$

The classification problem is reduced to solving of this functional equation. Of course, functions u,v,w can be multiplied by an arbitrary constant and the linear transformation

$$\tilde{x}=c(x-x_0),\quad \tilde{y}=c(y-y_0)\ \Leftrightarrow\ \tilde{q}_{m,n}=c(q_{m,n}-mx_0-ny_0-\text{const})$$

can be applied, as well as the permutation of the x, y, z axes.

At first let us consider some degenerate cases. Assume that two of three functions are constant, say u and v. Then (2.9) yields that either w is constant as well, or $u = -v$ and w is arbitrary. In the first case the equation (2.2) is linear and in the second one it is of the form

$$\alpha(q_{1,0} + q_{-1,0} - q_{0,1} - q_{0,-1}) + h(q_{1,1} - q) - h(q - q_{-1,-1}) = 0$$

and admits "integration":

$$\alpha(q_{m+1,n} - q_{m,n+1}) + h(q_{m+1,n+1} - q_{m,n}) = c_{m-n}.$$

Other degenerate case corresponds to the vanishing of the Jacobian, what is equivalent to $u + v + w = 0$ and implies that all three functions are linear. In this case one can prove that equation (2.2) can be reduced to the equation on the variable $p = y_{0,1}/x$. Further on we will not consider these degenerate cases.

Lemma 2. *The functions u, v, w satisfy the equations*

$$(u')^2 = \delta u^2 + 2\alpha u + \varepsilon, \quad (v')^2 = \delta v^2 + 2\beta v + \varepsilon, \quad (w')^2 = \delta w^2 + 2\gamma w + \varepsilon. \tag{2.10}$$

Proof. At first prove that functions $u(x)$ and $v(y)$ satisfy the equation

$$(u'' - v'')(u + v) - (u')^2 + (v')^2 = k(u - v), \quad k = \mathrm{const}\,. \tag{2.11}$$

Let us eliminate w from (2.9). Applying the operator $\partial_x - \partial_y$ one obtains the linear system on w, w' :

$$\begin{pmatrix} u' + v' & -u - v \\ u'' - v'' & v' - u' \end{pmatrix} \begin{pmatrix} w + u \\ w' \end{pmatrix} = (u - v) \begin{pmatrix} u' \\ u'' \end{pmatrix}.$$

Its determinant Δ is exactly the left hand side of the equation (2.11). If it is identically zero then (2.11) is proved, otherwise one finds

$$w + u = \frac{u - v}{\Delta}(uu'' - (u')^2 + u''v + u'v'), \quad w' = \frac{u - v}{\Delta}(u''v' + u'v'')$$

and consequently $((u - v)/\Delta)_y(uu'' - (u')^2 + u''v + u'v') = 0$. Assume that the expression in the second bracket vanishes. If $u' \not\equiv 0$ then $v' = -u''v/u' + u' - uu''/u'$, $v'' = -u''v'/u'$, but then, as one easily checks, $\Delta = 0$. If $u' = 0$ then $w + u = 0$, that is we come to the degenerate solution excluded above. Therefore $((u - v)/\Delta)_y = 0$. Due to the symmetry between u and v one can prove analogously $((u - v)/\Delta)_x = 0$ and obtain (2.11).

Further on, rewriting (2.11) in the form

$$\left(\frac{u'}{u + v}\right)_x = \frac{v'' + k}{u + v} - \frac{(v')^2 + 2kv}{(u + v)^2},$$

multiplying by $u'/(u + v)$ and integrating with respect to x yield

$$(u')^2 = \delta(y)(u + v)^2 - 2(v'' + k)(u + v) + (v')^2 + 2kv.$$

Replacing u and v one obtains

$$(v')^2 = \tilde{\delta}(x)(u+v)^2 - 2(u''+k)(u+v) + (u')^2 + 2ku.$$

Subtracting one equation from another and using (2.11) one obtains $\tilde{\delta} = \delta = \text{const}$. Summing and dividing by $u+v$ give $u''+v'' = \delta(u+v)-k$. The separation of the variables yields $(u')^2 = \delta u^2 + 2\alpha u + \varepsilon$, $(v')^2 = \delta v^2 + 2\beta v + \tilde{\varepsilon}$, where $\alpha+\beta=-k$, and substitution into (2.11) proves $\varepsilon = \tilde{\varepsilon}$. The last of the equations (2.10) is obtained in virtue of the symmetry of x, y, z axes. ■

It is clear that the solutions of equations (2.10) must satisfy also some additional relations. However their analysis is not in principle difficult, and the direct examination of all solutions brings to the following list.

Theorem 3. *The equations (2.2) admitting duality transformations are exhausted, up to the changes $\tilde{q}_{m,n} = c(q_{m,n} - mx_0 - ny_0)$ and permutations of x, y, z axes, by the following sets of the functions f, g, h. In formulae (A), (B), (C) the parameters are constrained by relation $\lambda+\mu+\nu=0$, and in (I) by relation $\lambda\mu\nu = -1$.*

(A)	$f = \frac{\mu}{x}$,	$g = \frac{\nu}{y}$,	$h = \frac{\lambda}{z}$,
(B)	$f = \mu \coth x$,	$g = \nu \coth y$,	$h = \lambda \coth z$,
(C)	$f = \frac{1}{2}\log\frac{x+\mu}{x-\mu}$,	$g = \frac{1}{2}\log\frac{y+\nu}{y-\nu}$,	$h = \frac{1}{2}\log\frac{z+\lambda}{z-\lambda}$,
(D)	$f = \log x$,	$g = \log y$,	$h = \log(1-1/z)$,
(E)	$f = -e^x - 1$,	$g = e^{-y}$,	$h = \frac{1}{1+e^z}$,
(F)	$f = \log(e^x - 1)$,	$g = \log(e^y - 1)$,	$h = -\log(e^z - 1)$,
(G)	$f = -\log(e^{-x} - 1)$,	$g = \log(e^y - 1)$,	$h = -z$,
(H)	$f = \log(\lambda^{-1}(e^x + 1))$,	$g = \log(e^{-y} - 1)$,	$h = \log\frac{e^z+\lambda}{e^z+1}$,
(I)	$f = \log\frac{\mu e^x + 1}{e^x + \mu}$,	$g = \log\frac{\nu e^y+1}{e^y+\nu}$,	$h = \log\frac{\lambda e^z+1}{e^z+\lambda}$.

The duality transformations (2.4), (2.5) link together the equations corresponding to solutions (B) and (C), (D) and (E), (F) and (G), while the equations corresponding to solutions (A), (H) and (I) are self-dual. ■

Notice, that the cases (A) and (B) are connected by the point transformation $q = \exp(2\tilde{q})$. It is explained by the fact that the Lagrangian $\sum(\mu\log x + \nu\log y + \lambda\log z)$ of the equation (2.2), (A) is invariant under the dilations $q \to Cq$ as well as under the shifts $q \to q + C$. On the other hand, the inversions $q \to q/(1-Cq)$ preserve the Lagrangian as well but the change $q = 1/\tilde{q}$ which maps this group into the shift group does not bring to a new equation.

2.3 The lattices of the discrete Toda type

The method proposed in the Section 2.1 does not work for the lattices of the discrete Toda type which correspond to the case $h = 0$ (this is equivalent to $f'g'h' = 0$, without loss of generality). Indeed, in this case transformations T_+ and T_- coincide for arbitrary f, g and classification becomes impossible. However, we can dispense with it since these lattices arise as a by-product of already obtained results for the discrete relativistic lattices. Namely, we will demonstrate, using only few basic formulae from the Section 2.1

and without any complicated calculations, that the iterations of the Bäcklund transformation $T_l = T_-^{-1}T_+$ are described by some discrete lattice of the Toda type. Hence we automatically obtain some list of integrable lattices, see Theorem 4 below. Probably, this list is exhaustive (cf. [5, 10, 11, 14]), but unfortunately I do not know any classification results, like Yamilov's Theorem 8, which can be compared with this list.

Let us consider some lattice (2.2) admitting duality transformations and denote iterations of T_l by superscript l, in such a way that tilde in the formula (2.4) corresponds to the value $l-1$. It is possible to rewrite equations (2.2), (2.7) in terms of the mixed variables x, X. Indeed, one obtains directly from (2.4), (2.5) the relations

$$X_{0,-1} = g(y) + h(x_{0,-1} + y), \quad X^{-1} = g(y_{0,1}) + h(x_{1,1} + y_{0,1})$$

and therefore equation (2.2) is equivalent to

$$(T_m - 1)f(x) + X^{-1} - X_{0,-1} = 0.$$

Analogously, in virtue of (2.6) equation (2.7) is equivalent to

$$(T_m - 1)F(X) + x_{1,1} - x^1_{1,0} = 0.$$

"Integrating" this equation with respect to m (recall that $x_{1,0} = (T_m - 1)q$) one obtains $X = \varphi((T_l - T_n)q + c)$ where function φ is inverse of F. The constant c does not depend on m, but may depend on l, n, and it can be set to zero without lost of generality by means of appropriate shift of the variables q. Then eliminating X from the previous equation brings to the lattice of the discrete Toda type

$$(T_m - 1)f(x) - (T_lT_n^{-1} - 1)\varphi((1 - T_l^{-1}T_n)q) = 0. \tag{2.12}$$

So, we have already proved that the Bäcklund transformation for the discrete relativistic lattice is governed by some discrete nonrelativistic lattice. However, the complete picture is even more rich: it turns out that the 3-dimensional lattice generated by the shifts T_l, T_m, T_n contains 3 instances of the discrete nonrelativistic lattices and 4 instances of the discrete relativistic lattices. In order to see this let us remind that the roles of all shifts in equation (2.2) are equal and therefore we can repeat the above calculation starting from the other set of mixed variables. More precisely, rewriting equations (2.2) and (2.7) in terms of y, Y or z, Z one obtains the equations

$$(T_n - 1)g(y) + Y^{-1}_{-1,0} - Y = 0, \quad (T_n - 1)G(Y) + y_{0,1} - y^1_{1,1} = 0,$$
$$(T_mT_n - 1)h(z) + Z - Z^{-1} = 0, \quad (T_mT_n - 1)H(Z) + z^1_{1,1} - z_{1,1} = 0$$

and further elimination of Y and Z yields equations

$$(T_n - 1)g(y) - (T_lT_m - 1)\psi((1 - T_l^{-1}T_m^{-1})q) = 0, \tag{2.13}$$
$$(T_mT_n - 1)h(z) + (T_l - 1)\eta((T_l^{-1} - 1)q) = 0 \tag{2.14}$$

where functions ψ and η are inverses of G and H respectively.

The placement of the variables involved in equations (2.12), (2.13) and (2.14) is shown at the Figure 3. At this picture the variables q involved in the original equation (2.2)

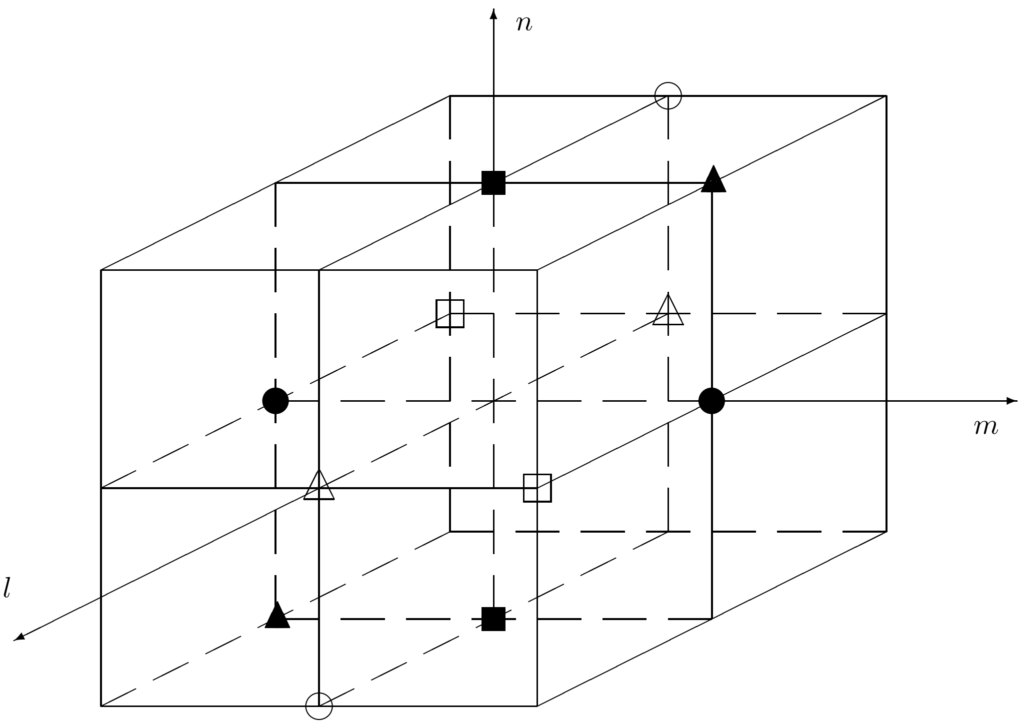

Figure 3. Each of three discrete nonrelativistic lattices involves 2 pairs of the same shape (black and white). Each of four discrete relativistic lattices involves 3 distinct pairs, number of white pairs is even (0 or 2).

lie in the plane (mn) and are marked by black symbols, while the variables obtained by means of the Bäcklund transformations are marked by the white ones. Equation (2.12) constraints two black and two white circles (and, of course, the centre of the cube), squares and triangles correspond to (2.13) and (2.14).

Next, subtracting from the equation (2.2) two of three equations (2.12), (2.13) and (2.14) we obtain some discrete relativistic lattice again, for example subtracting of (2.13), (2.14) yields equation

$$(T_m-1)f(x)-(T_l-1)\eta((T_l^{-1}-1)q)+(T_lT_m-1)\psi((1-T_l^{-1}T_m^{-1})q)=0$$

which involves variables in the plane (lm). Two other choices bring to the discrete relativistic lattices in the plane (ln) and in the plane spanned over the vectors $(0,1,1)$ and $(1,1,0)$.

In conclusion of this section we present the list of the discrete lattices of the Toda type which is obtained by direct examination of the equations listed in the Theorem 3.

Theorem 4. *The equations (2.12), (2.13), (2.14) are equivalent, up to the renaming of the shifts and the linear changes* $\tilde q_{m,n}=c(q_{m,n}-mx_0-ny_0)$*, to the lattices*

$$(T_m-1)\frac{1}{x}=(T_n-1)\frac{1}{y},$$

$$
\begin{aligned}
&(T_m-1)e^x=(T_n-1)e^y,\\
&(T_m-1)\frac{1}{e^x-1}=(T_n-1)\frac{1}{e^y-1},\\
&(T_m-1)\log x=(T_n-1)\log y,\\
&(T_m-1)\log(1-1/x)=(T_n-1)\log(1-1/y),\\
&(T_m-1)\log(e^x-1)=(T_n-1)\log(e^y-1),\\
&(T_m-1)x=(T_n-1)\log(e^y-1),\\
&(T_m-1)\log\left(\frac{e^x-\lambda}{e^x-1}\right)=(T_n-1)\log\left(\frac{e^y-\lambda}{e^y-1}\right).
\end{aligned}
$$

3 The lattices of the relativistic Toda type

Now let us consider the differential-difference equations on the variable $q_n(t)$. As before we will omit the subscript $n : q = q_n,\ q_i = q_{n+i}$ and use uppercase letters for the dual variables. Instead of differences x, y, z we consider the quantities

$$p=\dot q,\quad y=q-q_{-1}$$

which satisfy identity

$$D_t y_1=(T_n-1)p. \tag{3.1}$$

3.1 Duality transformations

The lattices of the relativistic Toda type

$$\dot p=r(p)(h(y_1)p_1-h(y)p_{-1}+g(y_1)-g(y)),\quad \dot y=p-p_{-1} \tag{3.2}$$

are the Euler equations for the Lagrangians of the form

$$\mathcal{L}=\int dt\sum_n(a(p)-b(y)-c(y)p), \tag{3.3}$$

where $r=1/a''$, $g=b'$, $h=c'$. In this section we assume that the nondegeneracy condition $h\neq 0$, $|g'|+|h'|\neq 0$ is fulfilled.

The term $c(y)p$ of the Lagrangian is equivalent, up to the total divergence, to $c(y_1)p$ and this results in two equivalent forms of the momentum conservation law:

$$
\begin{aligned}
&D_t(a'(p)-c(y_1))=(T_n-1)(h(y)p_{-1}+g(y))\\
&\Leftrightarrow\ D_t(a'(p)-c(y))=(T_n-1)(h(y)p+g(y)).
\end{aligned}
$$

This allows to define the pair of transformations

$$T_+ :\ (P,Y)=T(p,y_1),\quad T_- :\ (\tilde P_{-1},\tilde Y)=T(p,y) \tag{3.4}$$

where mapping $T:(p,y)\to(P,Y)$ is defined by the formula

$$P=h(y)p+g(y),\quad Y=a'(p)-c(y). \tag{3.5}$$

Notice, that the lattices (3.2) are the continuous limit of the discrete relativistic lattices (2.2). Indeed, let us consider the family of the functionals of the form (2.3)

$$\mathcal{L}_\varepsilon = \sum_{m,n}(\varepsilon a(x_{m,n}/\varepsilon) - \varepsilon b(y_{m,n}) + k(y_{m,n}) - k(z_{m,n})),$$

and let $k' = c$, $q_{m,n} = q_n(t)$, $t = m\varepsilon$. It is easy to see that passage to the limit $\varepsilon \to 0$ brings exactly to the Lagrangian (3.3). Transformations (3.4) also are obtained from (2.4) by this limit.

Definition 2. *The lattice (3.2) admits the duality transformations (3.4), (3.5) if the mapping T is invertible and both variables Q, $\tilde{Q}$ satisfy the same lattice which is called dual to (3.2).*

Notice that functions a, b, c are defined up to some linear transformations and therefore mapping (3.5) is not uniquely defined. However, it is easy to see that this arbitrariness corresponds to the linear changes of the variables P, Y and all lattices dual to the given equation (3.2) are equivalent under the changes $Q_n \to \alpha Q_n + \beta n + \gamma t$.

Theorem 5. *The lattice (3.2) admits duality transformations if and only if the inverse of the mapping (3.5) is of the form*

$$p = H(Y)P + G(Y), \quad y = A'(P) - C(Y), \quad H = C'. \tag{3.6}$$

In this case the dual equation is of the form ($R = 1/A''$)

$$\dot{P} = R(P)(H(Y_1)P_1 - H(Y)P_{-1} + G(Y_1) - G(Y)), \quad \dot{Y} = P - P_{-1}. \tag{3.7}$$

Proof. Let T^{-1} be of the form (3.6). Then elimination of p, y from (3.4) by means of identity (3.1) brings to equation (3.7) in both cases.

In order to prove inverse statement we need to compare equations on the variables Q and $\tilde{Q}$. Straightforward calculation proves that the variables P, Y satisfy the equations of the form

$$\dot{P} = r(p)(h(y_1)(P - P_{-1}) + \Delta(p, y_1)(p_1 - p)), \quad \dot{Y} = P - P_{-1}, \tag{3.8}$$

where $\Delta(p, y) = h^2(y) + \frac{1}{r(p)}(h'(y)p + g'(y))$ and p_1, p, y_1 have to be expressed in terms of P_1, P, Y_1, Y. Therefore, the dual equation must be linear in P_{-1}. Analogously, considering equation on $\tilde{Q}$ one proves that the dual equation must be linear in P_1. Since in (3.8) only p_1 depends on P_1, hence mapping T must satisfy the condition $\partial^2 p/\partial P^2 = 0$. This proves the first formula in (3.6). The second one follows from the property $\partial y/\partial Y = -\partial p/\partial P$ which is evident from the structure of the Jacobi matrix

$$\begin{pmatrix} \partial p/\partial P & \partial y/\partial P \\ \partial p/\partial Y & \partial y/\partial Y \end{pmatrix} = \frac{1}{\Delta(p, y)} \begin{pmatrix} h(y) & 1/r(p) \\ h'(y)p + g'(y) & -h(y) \end{pmatrix} \tag{3.9}$$

where $\Delta \neq 0$ by Definition 2. ■

As in the discrete case we see that the original lattice is dual for its dual and the composition $T_-^{-1}T_+$ defines the Bäcklund transformation for (3.2) (see Figure 2). We will study this Bäcklund transformation in the Section 3.3.

3.2 Classification theorem

Theorem 6. *The lattices (3.2) admitting the duality transformations are characterized by the following equations for the coefficients:*

$$\begin{aligned} &r = r_2p^2 + r_1p + r_0, \\ &g' = r_2g^2 + R_1g + R_0 - r_0h^2, \quad h' = 2r_2gh - r_1h^2 + R_1h. \end{aligned} \tag{3.10}$$

The coefficients of the dual lattice satisfy equations

$$\begin{aligned} &R = r_2P^2 + R_1P + R_0, \\ &G' = r_2G^2 + r_1G + r_0 - R_0H^2, \quad H' = 2r_2GH - R_1H^2 + r_1H. \end{aligned} \tag{3.11}$$

Proof. Straightforward calculation proves that necessary and sufficient condition $\partial^2 p/\partial P^2 = 0$ is equivalent to the following relation involving the functions $r(p), g(y), h(y)$:

$$h(h''p + g'') - h'(h'p + g') + 2rh^2h' - r'h^2(h'p + g') = 0. \tag{3.12}$$

The second derivative of (3.12) with respect to p is $r'''(h'p + g') = 0$. By the condition of nondegeneracy, this implies that r is a polynomial and its degree is less than 3. Dividing (3.12) by h^2 and integrating the result with respect to y one obtains

$$h'p + g' + 2rh^2 - r'h(hp + g) = h(\alpha + R_1p)$$

where α and R_1 are some constants. Let $r = r_2p^2+r_1p+r_0$, then collecting the coefficients of p in this relation gives the system

$$g' = \alpha h + r_1gh - 2r_0h^2, \quad h' = 2r_2gh - r_1h^2 + R_1h$$

which is equivalent to (3.10) modulo common first integral

$$r_2g^2 + R_1g + R_0 - r_1gh + r_0h^2 = \alpha h. \tag{3.13}$$

The formula for R can be easily obtained from the relation $R = \Delta(p, y)r$. Thus, equations for g and h are uniquely defined by the coefficients of the polynomials r and R. Since due to the Theorem 5 the duality relation is symmetric, hence equations for G and H can be written automatically. ■

Remark. The question about value of the first integral for the system (3.11) is a bit more complicated. However, this value can be calculated by sequential elimination of g, h and p from the formula (3.13) using relations $g = P - hp$, $h = RH/r$ and $p = HP + G$. This brings to relation

$$r_2G^2 + r_1G + r_0 - R_1GH + R_0H^2 = \alpha H,$$

that is, the value of the integration constant α for the dual lattice is the same. Notice, that actually role of this constant is very important since it may depend on n. This brings to the integrable lattices containing an arbitrary parameter in each node [15] (it plays role of the discrete spectrum when constructing solitons). For sake of simplicity we will not consider this generalization.

By use of the first integral (3.13) the system (3.10) is reduced to the equation

$$(h')^2=(r_1^2-4r_2r_0)h^4+(4\alpha r_2-2R_1r_1)h^3+(R_1^2-4r_2R_0)h^2 \tag{3.14}$$

which is solved in elementary functions. Consideration of all the possible choices of parameters and the branches of solutions brings to the following result.

Theorem 7. *Up to the transformations $q_n\to\alpha q_n+\beta t+\gamma n$, $t\to\delta t$, the nondegenerate lattices (3.2) admitting the duality transformations are exhausted by the list* ($\dot y=p-p_{-1}$) :

$$
\begin{array}{ll}
(a) & \dot p=p_1e^{y_1}-p_{-1}e^{y}-e^{2y_1}+e^{2y},\\
(b) & \dot p=p\left(\dfrac{p_1}{y_1}-\dfrac{p_{-1}}{y}+y_1-y\right),\\
(c_{\mu,\nu}) & \dot p=p\left(\dfrac{p_1}{1+\mu e^{-y_1}}-\dfrac{p_{-1}}{1+\mu e^{-y}}+\nu(e^{y_1}-e^{y})\right),\\
(d) & \dot p=p(p+1)\left(\dfrac{p_1}{y_1}-\dfrac{p_{-1}}{y}\right),\\
(e_\mu) & \dot p=p(p-\mu)\left(\dfrac{p_1}{\mu+e^{y_1}}-\dfrac{p_{-1}}{\mu+e^{y}}\right),\\
(f_\mu) & \dot p=(p^2+\mu)\left(\dfrac{p_1-y_1}{\mu+y_1^2}-\dfrac{p_{-1}-y}{\mu+y^2}\right),\\
(g_\mu) & \dot p=\dfrac12(p^2+1-\mu^2)\left(\dfrac{p_1-\sinh y_1}{\mu+\cosh y_1}-\dfrac{p_{-1}-\sinh y}{\mu+\cosh y}\right).
\end{array}
$$

The duality transformations (3.4), (3.5) link together equations (a) and (b), (d) and (e_0), (f_μ) for $\mu\neq0$ and ($g_{\pm1}$), while the rest equations are self-dual. ■

3.3 The lattices of the Toda type

As in the discrete case, the subclass of the Toda type lattices

$$\dot p=r(p)(f(y_1)-f(y)),\quad \dot y=p-p_{-1} \tag{3.15}$$

must be considered separately, since transformations T_+ and T_- coincide. It should be mentioned that in this case the duality transformation was introduced by Toda [1]. It is given by the formula

$$P=f(y_1),\quad Y=a'(p),\quad a''=1/r$$

and the coefficients of the dual lattice are defined by the formulae $f'=R(f)$, $F(a'(p))=p$. For example, the Toda lattice $\ddot q=\exp(q_1-q)-\exp(q-q_{-1})$ is dual to the lattice $\ddot Q=\dot Q(Q_1-2Q-Q_{-1})$.

In contrast to the relativistic case, the duality transformation is irrelevant to integrability and cannot be used for the classification of the lattices (3.15). For the first time this problem was solved by Yamilov in the framework of the symmetry approach.

Theorem 8 (Yamilov, [4]). *The Toda type lattice (3.15) admits the higher symmetries iff*

$$r(p)=r_2p^2+r_1p+r_0,\quad f'=r_2f^2+R_1f+R_0. \tag{3.16}$$

This result demonstrates that integrable lattices (3.15) can be obtained from the integrable lattices of the relativistic Toda type by passing to the limit $h = 0$ in equations (3.10). More interesting link between this two classes of equations can be established along the same arguments as in the Section 2.3.

Let us denote iterations of transformation $T_l = T_-^{-1}T_+$ by the superscript l and let tilde in the formula (3.4) corresponds to the value $l-1$. Then formulae (3.4), (3.5), (3.6) take form

$$P_{-1} = h(y)p_{-1} + g(y), \quad P^{-1} = h(y_1)p_1 + g(y_1) \tag{3.17}$$

$$p^1 = H(Y)P_{-1} + G(Y), \quad p = H(Y)P + G(Y) \tag{3.18}$$

and therefore equations (3.2), (3.7) can be rewritten as a coupled lattice of the Volterra type

$$\dot p = r(p)(P^{-1} - P_{-1}), \quad \dot P = R(P)(p_1 - p^1).$$

Assuming

$$P = f(q_1 - q^1), \quad f' = R(f) \tag{3.19}$$

we obtain the Toda type lattice

$$\ddot q = r(\dot q)(f(q_1^{-1} - q) - f(q - q_{-1}^1)) \tag{3.20}$$

for the variables situated along the line $l + n = \text{const}$. Since functions r, R were described in the Section 3, we immediately repeat the Yamilov's result (3.16).

3.4 Nonlinear superposition principle

Recall, that in the discrete case the 3-dimensional lattice generated by the shifts T_l, T_m, T_n contains 4 instances of the discrete relativistic lattices and 3 instances of nonrelativistic ones (see Figure 3). In the continuous case the picture is more bare: the variables q_{ln} form the 2-dimensional lattice containing 2 instances of the relativistics lattices, 1 instance of the Toda type lattices and 1 instance of the discrete Toda type lattices, as shown on the Figure 4.

The shift T_n corresponds to the lattice (3.2) and the diagonal shift $T_l^{-1}T_n$ corresponds to the lattice (3.20), as was proved in the previous section. The following Theorem demonstrates that the condition of the commutativity of these shifts is equivalent to some discrete lattice of the Toda type which can be interpreted as the nonlinear superposition principle of the equations (3.2) and (3.20).

Theorem 9. *The variables $q, q^{\pm1}, q_{\pm1}$ are related by equation of the form*

$$(T_l - 1)c(q^{-1} - q + \delta) + (T_n - 1)c(q - q_{-1}) = 0. \tag{3.21}$$

Proof. The function φ inverse to A' satisfies equation $\varphi' = R(\varphi)$, that is $f(y) = \varphi(y+\varepsilon)$. Therefore

$$C(Y) = A'(P) - y_1 = A'(f(q_1 - q^1)) - q_1 + q = q - q^1 + \varepsilon,$$

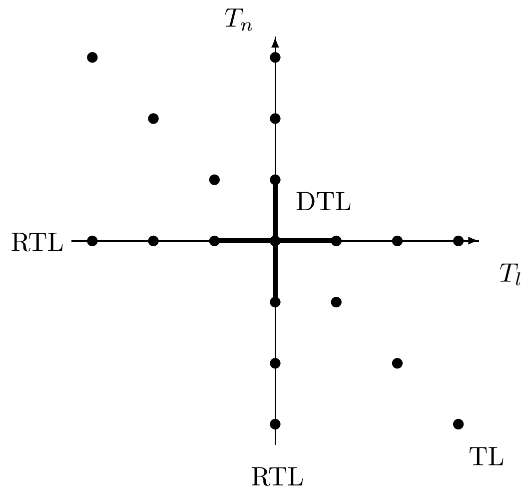

Figure 4. Equations associated with the relativistic lattice.

that is $Y = s(q - q^1 + \varepsilon)$ where s is the inverse function to C. It is easy to prove that s' satisfies the same equation (3.14) as $h = c'$, that is $s(y) = c(y + \tilde{\varepsilon}) + \text{const}$.

Next, eliminating p from the formulae

$$Y = a'(p) - c(y_1), \quad Y^{-1} = a'(p) - c(y)$$

brings to $Y^{-1} - Y = (T_n - 1)c(y)$ and this completes the proof. ■

It is clear that parameter δ can be set to zero by the proper choice of the function f in formula (3.19) or, equivalently, by the shift $q^l_n \to q^l_n + \delta l$. Therefore, the terms in (3.21) are symmetric and the shifts T_l and T_n plays the equal roles, although their origin was different. This suggests that T_l corresponds to some relativistic lattice as well. Indeed, eliminating P from the equation $\dot{p} = r(p)(P^{-1} - P_{-1})$ by means of the relations (3.18) brings to equation

$$\dot{p} = r(p)\left(-\frac{p^1}{H(Y)} + \frac{p^{-1}}{H(Y^{-1})} + \frac{G(Y)}{H(Y)} - \frac{G(Y^{-1})}{H(Y^{-1})}\right)$$

and since $Y = c(q - q^1 + \delta)$ we obtain a relativistic lattice again. The coefficients of this equation coincide with h, g up to the shift of argument.

3.5 Example: Heisenberg chain

In the previous sections the discrete symmetries of integrable lattices were studied. Now we will briefly discuss the continuous symmetries. I restrict myself by example of the equation (f_0)[1]. This choice is motivated by the link between (f_0) and the Heisenberg chain which became very popular due to its applications in discrete geometry [16, 17, 18]. Remind that this model reads

$$s_t = as \times \left(\frac{s_1}{1 + \langle s, s_1\rangle} + \frac{s_{-1}}{1 + \langle s, s_{-1}\rangle}\right) + b\left(\frac{s_1 + s}{1 + \langle s, s_1\rangle} - \frac{s + s_{-1}}{1 + \langle s, s_{-1}\rangle}\right) \quad (3.22)$$

[1]These results were obtained in collaboration with Professors A.B. Shabat and A.P. Veselov.

where $s \in \mathbb{R}^3$, $\langle s, s\rangle = 1$, $\langle , \rangle$ and $\times$ denote standard scalar and vector products respectively, a and b are arbitrary constants. Up to the author knowledge this equation was introduced by Sklyanin [19] in the case $b = 0$ and by Ragnisco and Santini [20] in the general case. The continuous limit of the Sklyanin lattice is the Heisenberg model

$$s_t = s \times s_{xx}, \quad \langle s, s\rangle = 1.$$

It was noticed that r-matrices for these models coincide and, more generally, this property can be accepted as a definition of correct discretization for a given continuous equation [21].

In order to obtain (f_0) let us consider the complexification $s \in \mathbb{R}^3 \to s \in \mathbb{C}^3$ and the stereographic projection

$$s = S(u, v) = \frac{1}{u - v}(1 - uv, i + iuv, u + v). \tag{3.23}$$

It is convenient to represent the flow (3.22) for arbitrary set of parameters a, b as a linear combination of the flows corresponding to the sets $a = i, b = \pm 1$. These flows are given by the following formulae in terms of the variables u, v :

$$u_{t_+} = \frac{(u_1 - u)(u - v)}{u_1 - v}, \quad v_{t_+} = \frac{(u - v)(v - v_{-1})}{u - v_{-1}}, \tag{3.24}$$

$$u_{t_-} = \frac{(u_{-1} - u)(u - v)}{u_{-1} - v}, \quad v_{t_-} = \frac{(u - v)(v - v_1)}{u - v_1}. \tag{3.25}$$

The lattice (3.24) appeared in [22, 23] for the first time.

Next, notice that elimination of P in virtue of (3.19) brings the formulae (3.17) to the form

$$\dot q = \frac{f(q_1 - q^1) - g(q_1 - q)}{h(q_1 - q)}, \quad \dot q^1 = \frac{f(q - q^1_{-1}) - g(q^1 - q^1_{-1})}{h(q^1 - q^1_{-1})}.$$

It is easy to see that this system corresponding to the case (f_0) ($f(y) = g(y) = -1/y$, $h(y) = 1/y^2$) coincide with (3.24) if we assume $q = u$, $q^1 = v$ and $t = t_+$. Since the lattice (3.25) is obtained by reflection $n \to -n$ we immediately come to the following statement.

Proposition 10. *Both variables u and v satisfy the lattices*

$$q_{t_\pm t_\pm} = q^2_{t_\pm}\left(\frac{q_{\pm 1, t_\pm}}{(q_{\pm 1} - q)^2} - \frac{q_{\mp 1, t_\pm}}{(q - q_{\mp 1})^2} - \frac{1}{q_{\pm 1} - q} + \frac{1}{q - q_{\mp 1}}\right)$$

in virtue of the equations (3.24), (3.25).

The lattices (3.24), (3.25) are Hamiltonian. Their Hamiltonians are

$$H_+ = \sum_n \log\frac{u_1 - v}{u - v}, \quad H_- = \sum_n \log\frac{u - v_1}{u - v}$$

respectively and the Poisson brackets are of the form

$$\{u_m, v_n\} = (u_n - v_n)^2\delta_{mn}, \quad \{u_m, u_n\} = \{v_m, v_n\} = 0. \tag{3.26}$$

The next proposition can be proved by straightforward calculations. It demonstrates that the lattices (3.24), (3.25) are the symmetries of each other and the shift $(u_n, v_n) \to (u_{n+1}, v_{n+1})$ defines the Bäcklund transformation for some hyperbolic system.

Proposition 11. *The Hamiltonians H_+ and H_- are in involution, the vector fields ∂_{t_+} and ∂_{t_-} commute and the variables u_n, v_n satisfy the system*

$$u_{t_+t_-}=\frac{2u_{t_+}u_{t_-}}{u-v}-u_{t_+}-u_{t_-},\quad v_{t_+t_-}=\frac{2v_{t_+}v_{t_-}}{v-u}+v_{t_+}+v_{t_-}. \tag{3.27}$$

In terms of the vector (3.23) this system reads

$$s_{t_+t_-}+\langle s_{t_+},s_{t_-}\rangle s+is\times(s_{t_+}+s_{t_-})=0,\quad \langle s,s\rangle=1.$$

Commutativity of the flows ∂_{t_+}, ∂_{t_-} allows to construct compatible zero curvature representations

$$W_{t_+}=U_1^+W-WU^+,\quad W_{t_-}=U_1^-W-WU^-,$$

so that $\operatorname{tr}W_N\dots W_1$ generates the common first integrals of the both lattices under the periodic boundary conditions $u_N=u_0$, $v_N=v_0$. The matrices $W,U^\pm$ are given by the formulae

$$W=\lambda I+P(u,v),\quad U^+=\frac{1}{2\lambda}(P(u,v_{-1})-P(v_{-1},u)),$$

$$U^-=\frac{1}{2(\lambda+1)}(P(v,u_{-1})-P(u_{-1},v))$$

where P denotes the projector

$$P(u,v)=\frac{1}{u-v}\begin{pmatrix}-v & uv\\ -1 & u\end{pmatrix}.$$

The Poisson structure (3.26) can be written in the r-matrix form

$$\{W_m(\lambda)\underset{,}{\otimes}W_n(\mu)\}=\delta_{mn}[r,W_m(\lambda)\otimes W_n(\mu)]$$

with the same r-matrix as for the Heisenberg model:

$$r=\frac{1}{\lambda-\mu}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}.$$

Equation (3.27) gives an example of hyperbolic symmetry. Any lattice (3.2) of the relativistic Toda type admitting duality transformations possesses also evolution symmetries, the simplest representative is given in the following Theorem.

Theorem 12 ([12]). *The Lagrangian (3.3) admits the variational symmetry of the form*

$$q_\tau=r(p)(h(y_1)p_1+h(y)p_{-1}+g(y_1)+g(y))+R_1p^2 \tag{3.28}$$

if and only if the coefficients r,g,h satisfy the system (3.10).

Obviously, one can rewrite this symmetry as the partial differential equation if use the lattice (3.2) itself and assume $q = u$, $q_1 = v$:

$$u_\tau + u_{tt} = 2r(u_t)(h(v-u)v_t + g(v-u)) + R_1 u_t^2,$$
$$v_\tau - v_{tt} = 2r(v_t)(h(v-u)u_t + g(v-u)) + R_1 v_t^2.$$

Returning to the system (3.24) one obtains the symmetry ($t = t_+$)

$$\begin{aligned} iu_\tau - u_{tt} &= 2u_t^2\left(\frac{v_t}{(u-v)^2} - \frac{1}{u-v}\right), \\ iv_\tau + v_{tt} &= 2v_t^2\left(\frac{u_t}{(u-v)^2} - \frac{1}{u-v}\right) \end{aligned} \tag{3.29}$$

which is equivalent, in the geometrical terms (3.23), to the modified Heisenberg model [24]

$$s_\tau = s \times s_{tt} - \frac{i}{2} s_t \langle s_t, s_t\rangle, \quad \langle s, s\rangle = 1. \tag{3.30}$$

So we have proved that the flows defined by the lattice (3.22) at $a = i, b = 1$ and equation (3.30) commute. One can reformulate this result as follows:

Proposition 13. *The shift $s \to s_1$ in the Heisenberg chain (3.22) for $a = i, b = 1$ defines the t-part of the Bäcklund transformation for equation (3.30).*

Notice, that according to (3.24) this shift is equivalent to solving of Riccati equation on the variable v_1 :

$$u_t = \frac{(u_1 - u)(u - v)}{u_1 - v}, \quad v_{1,t} = \frac{(u_1 - v_1)(v_1 - v)}{u_1 - v}.$$

Another choice of dependent variables allows to rewrite (3.29) in the form

$$iu_{1,\tau} = u_{1,tt} - \frac{2u_{1,t}^2}{u_1 - v}, \quad iv_\tau = -v_{tt} - \frac{2v_t^2}{u_1 - v}.$$

This system is equivalent to the Heisenberg model $\sigma_\tau = \sigma \times \sigma_{tt}$, $\langle\sigma, \sigma\rangle = 1$ in terms of the vector $\sigma = S(u_1, v)$, which is related with the vectors (3.23) by the formula

$$\sigma = \frac{s_1 + s - is_1 \times s}{1 + \langle s_1, s\rangle}.$$

4 Multifield examples

In the previous sections we assumed that the variable q was scalar ($q \in \mathbb{F}$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$). The brief analysis proves that the Theorems 1 and 5 remain valid for the vector case $q \in \mathbb{F}^N$ as well. Thus, the multifield generalizations can be obtained by finding of the mappings $T : \mathbb{F}^{2N} \to \mathbb{F}^{2N}$ with the special structure described in these Theorems. This problem is much more difficult than in the scalar case and the classification of such mappings is far from completeness.

However it is not difficult to find some particular examples. The most simple are the generalizations of the discrete Heisenberg equation (A) and the Heisenberg lattice (f_0) related to the Jordan triple systems. It was Svinolupov who recognized the role of the Jordan algebraic structures in the theory of integrable systems for the first time. I provide only necessary information about the Jordan triple systems, the interested reader can find more in [25, 26, 27, 28, 29] and references therein.

4.1 Jordan triple systems

The ternary algebra J with multiplication $\{\}: J^3 \to J$ is called Jordan triple system if the following identities hold

$$\{abc\} = \{cba\}, \tag{4.1}$$
$$\{ab\{cde\}\} - \{cd\{abe\}\} = \{\{cba\}de\} - \{c\{bad\}e\}. \tag{4.2}$$

Some consequences are:

$$\begin{gathered}\{ab\{aca\}\} = \{a\{bac\}a\} = \{\{aba\}ca\}, \quad \{a\{bab\}c\} = \{\{aba\}bc\},\\ 2\{\{abc\}bd\} = \{a\{bcb\}d\} + \{c\{bab\}d\}.\end{gathered} \tag{4.3}$$

Calculations are simplified by use of the linear operators $L_{ab}, M_{ab}, M_a : J \to J$ defined for arbitrary elements $a, b \in J$ as follows:

$$L_{ab}(c) = \{abc\}, \quad M_{ab}(c) = \{acb\}, \quad M_a(c) = M_{aa}(c) = \{aca\}.$$

Notice, that identity (4.2) is equivalent to

$$[L_{ab}, L_{cd}] = L_{\{cba\}d} - L_{c\{bad\}}. \tag{4.4}$$

We are especially interested in rational expressions. They are build from the inverse elements which are defined as $a^{-1} = M_a^{-1}(a)$. In some Jordan triple systems the operator M_a is degenerate. In such cases the notion of inverse element can be partly substituted by the notion of a deformation vector which is the solution of the system $\partial b/\partial a = -M_b$ [30]. However, for sake of simplicity, we shall consider only the cases when $\det M_a \neq 0$ almost everywhere in J. The Jordan triple system with this property are called Jordan triple system with invertible elements.

The following Lemma demonstrates that some properties of a^{-1} are the same as in a ring.

Lemma 14. *Let the operator M_a be invertible then*

$$M_{a^{-1}} = M_a^{-1}, \quad (a^{-1})^{-1} = a, \quad (a^{-1})_x = -M_a^{-1}(a_x) \tag{4.5}$$

and the following rules of cancellation of a and a^{-1} are valid

$$\begin{gathered}\{aa^{-1}b\} = b, \quad \{a^{-1}\{acb\}a^{-1}\} = \{cba^{-1}\},\\ \{a\{a^{-1}ca^{-1}\}b\} = \{ca^{-1}b\}, \quad \{ba\{a^{-1}ca^{-1}\}\} = \{bca^{-1}\}.\end{gathered} \tag{4.6}$$

Proof. Let $b \in J$, $c = M_a(b)$, $d = M_{a^{-1}}(c)$. One obtains using (4.3)

$$\begin{aligned}M_a(d) &= \{a\{a^{-1}ca^{-1}\}a\} = 2\{aa^{-1}\{aa^{-1}c\}\} - \{a\{a^{-1}aa^{-1}\}c\}\\ &= 2\{aa^{-1}\{aa^{-1}\{aba\}\}\} - \{\{aa^{-1}a\}a^{-1}c\} = \{aa^{-1}\{aba\}\}\\ &= \{aba\} = M_a(b),\end{aligned}$$

and therefore $b = d$. Arbitrariness of b implies $M_{a^{-1}}M_a = I$.

Further on, $(a^{-1})^{-1} = M_{a^{-1}}^{-1}(a^{-1}) = M_a(a^{-1}) = a$.

For arbitrary b

$$\{aa^{-1}b\} = \{aa^{-1}\{aM_a^{-1}(b)a\}\} = \{\{aa^{-1}a\}M_a^{-1}(b)a\} = \{aM_a^{-1}(b)a\} = b.$$

Taking this into account when differentiating relation $a = \{aa^{-1}a\}$ one proves the last formula in (4.5).

In virtue of (4.4) and relation $L_{aa^{-1}} = I$ which is already proved one obtains

$$L_{M_{a^{-1}}(c)a} = L_{a^{-1}\{ca^{-1}a\}} + [L_{a^{-1}c}, L_{a^{-1}a}] = L_{a^{-1}c} + [L_{a^{-1}c}, I] = L_{a^{-1}c},$$

and the last formula in (4.6) is proved. The next to the last is equivalent (use operator notation). Finally, (4.2) implies

$$\{a^{-1}\{acb\}a^{-1}\} = 2\{a^{-1}bc\} - L_{M_{a^{-1}}(b)a}(c) = \{a^{-1}bc\}. \qquad \blacksquare$$

Now we can prove the formula ("harmonic mean")

$$(a^{-1}+b^{-1})^{-1} = \{a(a+b)^{-1}b\}. \tag{4.7}$$

Let us denote $a + b = c^{-1}$, then

$$\begin{aligned} M_{(a^{-1}+b^{-1})}(\{acb\}) &= \{a^{-1}\{acb\}a^{-1}\} + \{b^{-1}\{acb\}b^{-1}\} \\ &\quad + 2\{a^{-1}\{ac(c^{-1}-a)\}b^{-1}\} \\ &= \{cba^{-1}\} - \{b^{-1}ac\} + 2b^{-1}. \end{aligned}$$

Symmetrization on a and b gives (4.7).

Analogue of the Killing form in the Jordan triple system is the scalar product $\langle a, b\rangle = \operatorname{tr} L_{ab}$. Relation (4.4) implies the invariance property

$$\langle\{abc\}, d\rangle = \langle a, \{bcd\}\rangle \tag{4.8}$$

of this product. Further on we assume that it is also symmetric and nondegenerate. Notice that if the element a is invertible then the equalities $L_{ab} = M_a M_{a^{-1}b}$, $L_{ba} = M_{a^{-1}b} M_a$ imply $\langle a, b\rangle = \langle b, a\rangle$. Hence, in virtue of continuity, in the Jordan triple systems with invertible elements the symmetry property always holds.

Examples below together with the reductions $a = \pm a^\tau$ of the Examples 1 and 2 (with $M = N$) exhaust all simple Jordan triple systems aside from two exceptional ones.

Example 1. The linear space J of $N \times N$ matrices becomes the Jordan triple system with respect to the triple product defined by means of the standard matrix multiplication as follows

$$\{abc\} = \frac{1}{2}(abc + cba).$$

The operator M_a is invertible iff $\det a \neq 0$, that is almost everywhere. The element a^{-1} coincides with inverse matrix. The subspaces of symmetric and skewsymmetric matrices are Jordan triple systems as well. However, in the Jordan triple system of the skewsymmetric matrices of odd order the operator M_a is degenerate for all a. The scalar product is $\langle a, b\rangle = \operatorname{tr} ab$.

Example 2. Previous example admits generalisation for $N \times M$ matrices:

$$\{abc\} = \frac{1}{2}(ab^\tau c + cb^\tau a),$$

where $^\tau$ denotes transposition. In particular, if $M = 1$ then J turns into the N-dimensional vector space with multiplication

$$\{abc\} = \frac{1}{2}(\langle a, b\rangle c + \langle c, b\rangle a)$$

where $\langle , \rangle$ denotes the standard scalar product. However, the operator M_a is not invertible for $M \neq N$.

Example 3. More interesting triple product in the N-dimensional vector space is given by

$$\{abc\} = \langle a, b\rangle c + \langle c, b\rangle a - \langle a, c\rangle b.$$

The scalar product in J coincides with the standard one. Operator M_a, its inverse and vector a^{-1} are defined by formulae

$$M_a(b) = 2\langle a, b\rangle a - \langle a, a\rangle b, \quad M_a^{-1} = \langle a, a\rangle^{-2} M_a, \quad a^{-1} = \langle a, a\rangle^{-1} a.$$

4.2 Jordan analogues of Heisenberg equations

The lattice (2.2), (A) of discrete relativistic Toda type admits literal generalization

$$\mu(T_m - 1)x^{-1} + \nu(T_n - 1)y^{-1} + \lambda(T_m T_n - 1)z^{-1} = 0, \quad \lambda + \mu + \nu = 0 \tag{4.9}$$

for arbitrary Jordan triple system with invertible elements. Indeed, using relation (4.7) one can prove that the mapping $T : J^2 \to J^2$ given by the formulae

$$X = \nu y^{-1} + \lambda(x + y)^{-1}, \quad Y = -\mu x^{-1} - \lambda(x + y)^{-1}$$

coincide with its inverse and therefore the duality transformations (2.4) map equation (4.9) into itself. The nonrelativistic analogue of this equation corresponding to the set of parameters $\mu = 1$, $\nu = -1$, $\lambda = 0$ can be obtained along the same scheme as in the Section 2.3.

Equation (4.9) is the Euler equation for the Lagrangian

$$\mathcal{L} = \sum_{m,n}(\mu \log\det M_x + \nu \log\det M_y + \lambda \log\det M_z).$$

In order to prove this, note that if $f(u) = \frac{1}{2}\log\det M_u$ then, in virtue of Lemma 14,

$$\langle \frac{\partial f}{\partial u}, v\rangle = \frac{d}{d\varepsilon} f(u + \varepsilon v)|_{\varepsilon=0} = \operatorname{tr}(M_u^{-1} M_{uv}) = \operatorname{tr} L_{u^{-1}v} = \langle u^{-1}, v\rangle$$

and therefore $\partial f/\partial u = u^{-1}$.

Analogously, the lattice (f_0) of the relativistic Toda type admits generalization

$$\dot{y} = p - p_{-1}, \quad \dot{p} = M_p(M_{y_1}^{-1}(p_1) - M_y^{-1}(p_{-1}) - y_1^{-1} + y^{-1})$$

corresponding to the Lagrangian

$$\mathcal{L} = \int dt \sum_n (\frac{1}{2}\log\det M_p - \langle p, y^{-1}\rangle - \frac{1}{2}\log\det M_y).$$

Indeed, if $f(u) = \langle u^{-1}, a\rangle$ then

$$\frac{d}{d\varepsilon} f(u + \varepsilon v)|_{\varepsilon=0} = -\langle M_u^{-1}(v), a\rangle = -\langle M_u^{-1}(a), v\rangle$$

and therefore $\partial f/\partial u = -M_u^{-1}(a)$.

The duality transformations (3.4) are defined by the mapping T of the form

$$Y = p^{-1} - y^{-1}, \quad P = y^{-1} - M_y^{-1}(p)$$

which is involutive. Applying the scheme of the Section 3.3 one obtains the Toda type lattice

$$\ddot{q} = -M_{\dot{q}}((q_1 - q)^{-1} - (q - q_{-1})^{-1})$$

which is equivalent to the Jordan Volterra lattice [29]

$$\dot{p} = M_p(P - P_1), \quad \dot{P} = M_P(p_{-1} - p).$$

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