Noether Theorem and the quantum mechanical operators

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Abstract

We show that the quantum mechanical momentum and angular momentum operators are fixed by the Noether theorem for the classical Hamiltonian field theory we proposed.

Recently, we have proposed a classical Hamiltonian field theory [1],which in the limit of very large Planck frequency,mimics many aspects of a quantum mechanical system. In particular, the Schrodinger Equation will follow from the Hamilton's Field Equation, and the Hamiltonian of the classical field theory will become the energy expectation value of the corresponding quantum mechanical system.

The Hamiltonian density for the classical field theory we proposed contains (1+2n) pairs of canonical conjugate variables (p, q) (P_j, Q_j), $(\pi_j, \eta_j$), j=1,...,n.All of these canonical variables are functions of $x = (x_1, .., x_n)$ and t.And it reads as

$$H = (1/2h)(V(x))(p^2 + q^2) - (1/2h)(mc^2)(P_j^2 + Q_j^2 + \pi_j^2 + \eta_j^2) - (c/2)p\partial_j(Q_j + \eta_j) - (c/2)(P_j + \pi_j)\partial_j q \quad (1)$$

Independent variations of the field variables generate the Hamiltonian Field Equation. If we are interested in the case in which the field variables oscillate with frequencies far smaller than the Planck frequency h/mc, then the variables P_j, Q_j and π_j, η_j will be related to p,q through [1]

$$\begin{aligned} Q_i &= h/2mc\partial_i p \\ P_i &= -h/2mc\partial_i q \\ \eta_i &= h/2mc\partial_i p \\ \pi_i &= -h/2mc\partial_i q, j = 1, ..n \end{aligned} \quad (2)$$

In this paper, we will explore the translational and rotational symmetries of this classical Hamiltonian field theory. It is well known in the literature that these continuous symmetries will lead to conserved physical quantities ; a result called the Noether theorem [2] .And the aim of this paper is to understand how these conserved physical quantities coming from a classical field theory are related to the measurable quantities of the corresponding quantum mechanical system.

Let us first consider the $V(x) = 0$ case.When there is no external potential present, there will be both translational and rotational symmetries for the classical field system. And by Noether theorem, there exist some corresponding conserved quantities. For the translational invariance, the resulting conserved quantities are the components of the second rank tress-energy tensor T^μ_ν,given by Noether as

$$T^\mu_\nu = \Sigma(\partial L/\partial(\partial_\mu u))(\partial_\nu u) - L\delta^\mu_\nu \quad (3)$$

where L is the underlying Lagrangian density for the classical field theory. u stands collectively for all the field variables.Noether theorem requires the conservation law

$$\partial_\mu T^\mu_\nu = 0 \quad (4)$$

We are particularly interested in the vector m_j defined as

$$m_j = \int d^n x T_{0j} \tag{5}$$

These m_j,other than a multiplicative constant that we shall fix later,are always taken as the mementum components carried by the classical fields because they are generated by the translational symmetries. A close look at T_j^0 will show that they are independent of the detailed structures of the Lagrangian density L and has the simple form of

$$T_j^0 = \Sigma(\partial L/\partial(\partial_t u))(\partial_j u) = p\partial_j q + P_i \partial_j Q_i + \pi_i \partial_j \eta_i \tag{6}$$

Using the result given in Eq(2),T_j^0 can be written in terms of p and q as

$$T_j^0 = p\partial_j q - 2(h/2mc)^2 \partial_i p \partial_j \partial_i q \tag{7}$$

For a very large Planck frequency , the second term of Eq(7) drops out, and hence

$$m_j = \int d^n x(-p\partial_j q) \tag{8}$$

If we define the corresponding quantum mechanical wave function by $\psi(x,t)$=$(q(x,t) + ip(x,t))/\sqrt{2}$[1]. It can be seen immediately that

$$m_j = \int d^n x \psi * (-i\partial_j)\psi, \tag{9}$$

after integration by parts.

The physical meaning of the above result is the following: If we use $\star p_j$ to denote the quantum mechanical operator for the j the component of the momentum, and if we use the above ψ to compute the expectation value of the momentum components, then $\star p_j$ must be of the form

$$\star p_j = -i\beta\partial_j \tag{10}$$

where β is a proportional constant that will be shown to be h later.This result can be regarded as a derivation of the most fundamental quantum mechanical prescription

$$\star p_j = -ih\partial_j \tag{11}$$

For the rotational invariance, we assume that p,q,P_j,Q_j,π_j and η_j all transform as scalars under the rotation group.The resulting conserved quantities will then be the components of the third rank angular momentum tensor$M_{\lambda\mu}^{\beta}$, given by Noether as

$$M_{\lambda\mu}^{\beta} = x_\mu T_\lambda^\beta - x_\lambda T_\mu^\beta \tag{12}$$

The components of this third rank tensor that are related to the angular momentum components of the classical fields are M_{lk}^0.Using the result given in Eq(7), it can be shown easily that the integrated components

$$L_{lk} = \int d^n x M_{lk}^0 \tag{13}$$

can be written as

$$L_{lk} = \int d^n x \psi * (-ix_l \partial_k + ix_k \partial_l)\psi \tag{14}$$

And hence the orbital angular momentum $\star L$ in quantum mechanics will have the familiar form

$$\star L = rX \star p \tag{15}$$

In the presence of the potential V(x), we will no longer have translational invariance, and so no more conservation law. Instaed we shall have [3]

$$d/dx_\mu(T^\mu_\nu) = -\partial_\nu L = \partial_\nu H \tag{16}$$

The integrated spatial parts for the above equation read as

$$-\partial_t \int d^n x T_{0j} + \int d^n x d/dx_l T_{lj} = \int d^n x \partial_j H \tag{17}$$

Throwing away the surface term,and using the results given in Eq(1),Eq(8) and Eq(9), Eq(17) will become

$$-\partial_t \int d^n x \psi * (-i\partial_j)\psi = 1/h \int d^n x \psi * (\partial_j V)\psi \tag{18}$$

or

$$\partial_t \int d^n x \psi * (-ih\partial_j)\psi = \int d^n x \psi * (-\partial_j V)\psi \tag{19}$$

This is the Ehrenfest theorem [4] that we always encounter in quantum mechanics. And as we have promised before, we have fixed the proportional constant β that appeared in Eq(10) to be the Planck constant h.

An equation similar to Eq(18) can also be derived for the angular momentum which relates the rate of change of angular momentum with the external applied torque.

So we may conclude our paper by saying that the quantum mechanical operators for the momentum and angular momentum variables will be fixed by the Noether theorem for our classical Hamiltonian field theory.

References

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