Temporal correlation function in $3 - D$ Turbulence.

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We observe oscillatory decay in the two-point, non-equal time, velocity correlation function of homogeneous, isotropic turbulence. We found this through a direct numerical simulation (DNS) of the three dimensional Navier-Stokes ($3 - D$ NS) equation. We give an approximate analytic theory which explains this oscillatory behaviour. The wave-number and frequency dependent effective viscosity turns out to be complex; the imaginary part gives rise to the temporal oscillation. We find that, at least for the decay at short times, data collapse occur among the inertial range velocity wave-vector modes with the long time dynamic exponent $z = 2/3$, but the time period of the temporal oscillation is not universal.

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In homogeneous, isotropic turbulence the main interest is to understand the long-range spatio-temporal correlations exhibited by the velocity field. Towards this end one studies the scaling behaviour of the velocity structure functions $S_p(l,t) \equiv \langle [v_i(0,0) - v_i(l,t)]^p \rangle$ with respect to l and t. Here v_i is the velocity field, l and t are the spatial and temporal separations. Kolmogorov had predicted [1] through his dimensional analysis argument that the equal time structure function $S_p(l,0) = (\epsilon l)^{p/3}$. He had assumed that in the inertial range i.e., $\eta_d \ll l \ll L$ (η_d and L are, respectively, dissipation and forcing scales) $S_p(l,0)$ is a function of ϵ (the mean energy dissipation rate) and l only. But after anomalous scaling properties of $S_p(l)$ were discovered through experiments and simulations [2], the importance of L (which is also called the integral scale) has been recognised. Now it is known that $S_p(l) \sim l^{\zeta_p}$, when ζ_p is a monotonically increasing, convex, nonlinear function of p. The negative correction to the exponent $\delta\zeta_p = \zeta_p - p/3$ has to appear as the exponent of a dimensionless quantity (l/L) in order to keep the dimension of $S_p(l)$ unchanged. In all experiments and simulations of the $3 - D$ NS equation L is finite. Infact a grand challenge for analytic theories of turbulence is to show that finite limit for ζ_p exist for $l \ll L \to \infty$. Such a scheme has been successfully carried out [3] for the passive scalar field advected by a random velocity field in three dimensions (the Kraichnan model [4]). It has been shown that the anomalous scaling exponents of the passive scalar structure functions are independent of L (as $L \to \infty$), but the amplitudes do depend on L. Given that so much effort have been made to understand $S_p(l,0)$, not much is known about $S_p(l,t)$ even for small integer values of p (of course $S_p(l,t)$ is more complicated than $S_p(l,0)$). Literature on $S_p(l,t)$ in NS turbulence or related Burgers turbulence is rather sparse [5]. Dynamic renormalisation group (DRG) calculations [6], one loop self-consistent calculations [7] with the randomly forced $3 - D$ NS equation suggest a long time dynamic exponent $z = 2/3$. In Ref [7] large wavenumber limit of the velocity correlation function was also explored assuming dynamic scaling hypothesis to be valid. In experiments with a mean flow, because of large scale background velocity, one expects to measure $z = 1$. But even with zero mean velocity it has been shown [8], in the context of an $1 - D$ Burgers equation, how $z = 1$ could arise.

In this work we focus on the simplest two point non-equal-time velocity correlation function in the wave-number (k) space. We show that in $3 - D$ fluid turbulence, in the large L limit, the real part of the non-equal time velocity correlation function $C_{ij}(\mathbf{k},t) \equiv R\langle v_i(\mathbf{k},t) v_j(-\mathbf{k},0)\rangle$, for the inertial scales ($l = k^{-1}$), has oscillatory behaviour within a decaying envelope. From the incompressibility ($\nabla.\mathbf{v} = 0$) and isotropy assumptions it follows $C_{ij}(\mathbf{k},t) = c(k,t)P_{ij}(\mathbf{k})$. Here $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$ is the tranverse projector.

An attempt to calculate the $c(k,t)$ has been carried out by L'vov et.al. [10] in the context of a turbulent flow with a mean velocity field V_0. But they treated Navier-Stokes equation at a linear level. They had compensated for the non-linear term to some extent by using a wave number dependent renormalised viscosity instead of the bare viscosity. They predict an oscillatory behaviour for $c(k,t)$, but in the absence of the mean velocity (i.e., $V_0 = 0$) the oscillation vanish (i.e., it is a purely kinematic effect). But our data from a numerical simulation of the $3 - D$ NS equation (with large but finite L and zero mean velocity) clearly reveals presence of oscillations. The data (see Fig.4,5) looks like the displacement of an under-damped harmonic oscillator. Also simulation of the REWA (reduced wave vector set approximation) model by Eggers [11] shows a non exponential decay at short times and a negative minima.

Incompressible NS equation, forced randomly with a scale dependent variance, has been shown [12,13] to be a good model for fluid turbulence as far as multiscaling properties are concerned. But there exist many unresolved theoretical problems with the analytic calculations with this model. Our calculation is based on a variant of this model, where instead of a singular forcing spectrum (which goes as k^{-3}) we use a spectrum which peaks at a small but finite L^{-1}, goes to zero at $k = 0$ and behaves as k^{-3} for $k \gg L^{-1}$. The equation of motion for the velocity field fourier component $\mathbf{v_i}(\mathbf{k})$ is

$$\dot{v}_i(\mathbf{k}) + \nu_0 k^2 v_i(\mathbf{k}) = -i\lambda M_{ijl}(\mathbf{k}) \sum_{\mathbf{q}} v_j(\mathbf{q}) v_l(\mathbf{k}-\mathbf{q}) + f_i(\mathbf{k}, t) \; . \tag{1}$$

The random force $f_i(k,t)$ is a gaussian, white noise with the variancne

$$\langle f_i(\mathbf{k},t) f_j(\mathbf{k}',t')\rangle = \frac{(2\pi)^3 2D_0 k}{(k^2+L^{-2})^2} P_{ij}(\mathbf{k}) \delta(\mathbf{k}+\mathbf{k}') \delta(t-t') \tag{2}$$

Here $M_{ijl} = [k_j P_{il}(\mathbf{k}) + k_l P_{ij}(\mathbf{k})]/2$ and λ is an artificial coupling constant which will be set to 1 later. Henceforth we will denote the variance of the force $2D_0 k/(k^2+L^{-2})^2$ by $D(k)$.

In the theories with singular forcing spectrum (or equivalently infinite integral scale), infrared divergences appear if one tries to calculate effective viscosity perturbatively. Also the scheme cannot handle the so called sweeping effect i.e., the interaction of the bigger eddies (of size q^{-1}) with the eddy of size k^{-1} (when $(q < k)$). Physically the bigger eddies just advect the smaller eddies without distorting them much, so such divergences are basically defect of such a perturbative scheme. But One should remember that this eddy picture is quite heuristic in nature because velocity fourier modes $v(k)$ are global features of the velocity field where as the eddies are spatially correlated patches in the velocity field and hence local in nature. One systematic way to get rid of sweeping divergences in the equal time velocity structure functions, is to go to the lagrangian frame. Another way is to do an RG calculation which excludes the effect of the $(q < k)$ modes on the k mode, so both the infrared divergence and sweeping effect are eliminated. In these calculations because of universality reasons one is mainly interested in the zero frequency limit of the effective viscosity i.e., $\delta\nu(k, \omega \to 0)$ and assumes that for all frequencies ω, the effective propagator $G(k,\omega) \sim (-i\omega + k^2 \delta\nu(k, \omega \to 0))^{-1}$ i.e., remains a Lorentzian in ω. This approximation works well for long time properties. But here since we are interested in $c(k,t)$ for all t (including the short time behaviour) we need the correct behaviour of $G(k,\omega)$ for all ωs'.

Our procedure to calculate $c(k,\omega)$ is a mixture of a self-consistent and a perturbative scheme. We show that if we assume a large but finite integral scale L, due to nonlinear interaction among modes, the effective viscosity is complex. It has the regular renormalized real part and a k,ω dependent imaginary part as well. The oscillation in $c(k,t)$ arises because of this imaginary part. We use an one loop perturbation theory to determine the complex viscosity. The calculation is similar to the standard RG procedure for evaluating zero frequency viscosity. But unlike in the RG procedure, where only modes greater than the external wave-vector k are integrated out, we integrate over all q modes, including the range $[0, k]$. In the zero frequency limit ($\omega = 0$) our calculation is self-consistent (at one loop level), but for finite ω it is a perturbative calculation. Since our forcing spectrum is not singular, there is no infrared divergence in our integrals.

Treating the nonlinear term perturbatively [9] the effective response function $G(k,\omega)$ can be calculated as $G^{-1} = -i\omega + \nu_0 k^2 + \delta\nu(k,\omega)$. In the small k and $\nu_0 \to 0$ limit, $\nu_0 k^2$ is negligible compared to $\delta\nu(k,\omega)$. Hence $G^{-1} = -i[\omega - k^2 I(S)] + k^2 R(S)$, where we have denoted $\delta\nu(k,\omega)$ by $k^2 S$, and the real, imaginary parts by R,I. Using $c(k,\omega) = D(k)|G(k,\omega)|^2$ we get

$$c(k,\omega) = \frac{D(k)}{[\omega - k^2 I(S)]^2 + k^4 R(S)^2} \tag{3}$$

$c(k,t)$ is the inverse fourier transform of $c(k,\omega)$. So our task is to calculate S. Following Ref. [9]

$$-k^2 S P_{lj}(\mathbf{k}) = (i\lambda)^2 M_{lmn}(\mathbf{k}) \int \frac{d^3q\, d\omega'}{(2\pi)^4} M_{nij}(\mathbf{k}-\mathbf{q}) \times P_{im}(\mathbf{q}) D(q) |G(q,\omega')|^2 G(|\mathbf{k}-\mathbf{q}|, \omega-\omega') \tag{4}$$

Multiplying both sides by $P_{jl}(\mathbf{k})$ and contracting over l, j we get

$$S = \frac{M_{jmn}(\mathbf{k})}{2k^2} \int \frac{d^3q\, d\omega'}{(2\pi)^4} M_{nij}(\mathbf{k}-\mathbf{q}) P_{im}(\mathbf{q}) D(q) \times |G(q,\omega')|^2 G(|\mathbf{k}-\mathbf{q}|, \omega-\omega')$$

Integrating the r.h.s. over ω' gives

$$S = \int \frac{d^3q}{(2\pi)^3} b(\mathbf{k}, \mathbf{k}-\mathbf{q}, \mathbf{q}) \frac{D(q)}{2\nu_1 q^z} \frac{1}{-i\omega + \nu_1(q^z + |\mathbf{k}-\mathbf{q}|^z)} \tag{5}$$

$k^2 b(\mathbf{k}, \mathbf{k}-\mathbf{q}, \mathbf{q})$ is obtained by contracting M_{jmn}, M_{nij} and P_{im}. The expression for b is $b(\mathbf{k}, \mathbf{k}-\mathbf{q}, \mathbf{q}) = \frac{|\mathbf{k}-\mathbf{q}|}{k}(xy + z^3)$ [14]. The trio $(\mathbf{k}, \mathbf{k}-\mathbf{q}, \mathbf{q})$ form a triangle and x, y, z are the direction cosines of the angles opposite to $\mathbf{k}, \mathbf{k}-\mathbf{q}$, and $\mathbf{q}$ respectively. We have calculated this integral numerically. We have used $G(k,\omega)^{-1} = -i\omega + \nu_1 k^z$ anticipating a renormalisation of the viscosity in the $\omega \to 0$ limit. Here ν_1 could be a function of L, D_0 which we determine later. But as far as the k,ω dependence of the integral is concerned, for fixed L and D_0, we can treat ν_1 as a constant. From Eq.5 note that $I(S) \to 0$, as $\omega \to 0$. We check numerically that at $\omega = 0$, the integral in Eq.5 scales as k^{-2z}. Also by expanding

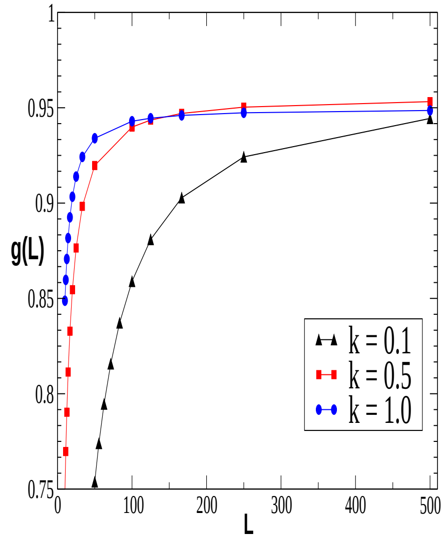

FIG. 1. $\nu_1(L)$ versus L. It shows convergence of $\nu_1(L)$ as $L \to \infty$. In our numerical evaluation of the integral in Eq.5 we choose $D_0 = 0.01$ and first evaluate the integral for $\omega = 0$ in order to get Fig.1. From this and Eq.6 we determine $\nu_1 \sim 0.06$. Then we choose a large enough value for $L(= 100)$ and evaluate the integral for nonzero values of ω to get S.

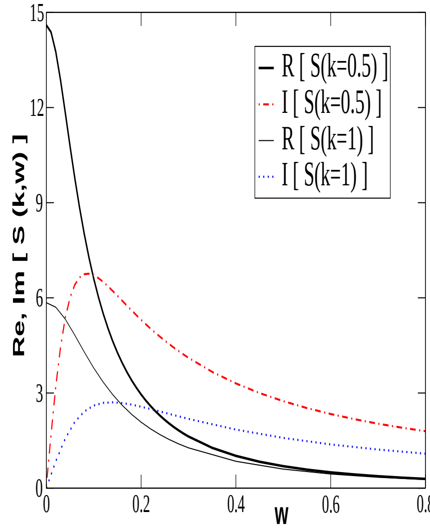

FIG. 2. S versus ω for two different k values. Real and imaginary parts of S are indicated by $R[S]$ and $I[S]$.

the integral in small k and retaining only leading powers in k (as is done in a DRG procedure [6]) one can infer this. If self-consistency has to be achieved at $\omega = 0$, then on the l.h.s. of Eq.5, $R[S(k, 0)] = \nu_1 k^{z-2}$. This fixes $z = 2/3$. Numerical evaluation of the integral with a finite L corroborates the approximate analytic prediction for small k because $L^{-1} << k$. Now we determine ν_1 in a self consistent way [15].

At $\omega = 0$, self consistency of Eq.5 requires

$$\nu_1 k^{-4/3} = \frac{D_0}{(2\pi)^2 \nu_1^2} k^{-4/3} g(L) \;\; , \;\; \text{so} \;\; \nu_1^3 = \frac{D_0}{(2\pi)^2} g(L) \tag{6}$$

Where $g(L)$ is the L dependent integral on the r.h.s. of Eq.5. In Fig.1 we show that $g(L)$ converges to a finite value as L grows large. Using $g(L) \simeq .95$ from Fig.1, we get $\nu_1(L) \simeq 0.28 D_0^{1/3}$. While evaluating the integral for inertial range k modes we used $G^{-1} = -i\omega + \nu_1 k^{2/3}$ (neglecting the $\nu_0 k^2$ term) for all the q and $k - q$ modes in the integrand, though the integral runs over both inertial and dissipation modes. But this is approximately correct because modes very far from the k mode do not contribute (locality in k space).

Returning to Eq.5, the dominant pole of $c(k, \omega)$ will decide the oscillation and the decay of $c(k, t)$. If the pole lies at $\omega = \omega_1 + i\omega_2$ then $\omega_1 = k^2 I(S[\omega_1])$ and $\omega_2 = -k^2 R(S[\omega_2])$. From the shapes of $R, I(S)$ in Fig.2 and the fact that $R, I(S)$ are even,odd functions of ω, we can infer that there will be two solutions to these transcendental equations. They are of the form $\pm\omega_1 + i\omega_2$, when ω_2 is negative. This ensures the causality of the effective response function G. The approximate data collapse for the inertial range k modes ($\eta_D^{-1} > k > L^{-1}$) in our Fig.3 implies that the oscillation period scales with $k^{-2/3}$.

We performed a DNS of the $3 - D$ NS equation and calculated $c(k, t)$ versus t for various $\mathbf{v}(\mathbf{k})$ modes. The data (see Fig.4,5) clearly shows oscillation in time. In our simulation forcing was present only at large length scales (i.e., in the k-space the $\mathbf{v}(\mathbf{k})$ modes in the smallest two shells were forced). Our pseudo-spectral scheme for the DNS is same as in [16]. We used a 32^3 grid with periodic boundary condition. We obtained a short inertial range (shown in Fig.6) with $Re_\lambda \sim 22$. We have obtained very long time series ($\geq 68T(L)$) for averaging $c(k, t)$. A preliminary test run [17] on a 64^3 grid also confirms existence of such oscillations.

Our simulation data in Fig.4 indicates that the oscillation time period approximately scales as $k^{-2/3}$ in the inertial range. But Fig.5 shows that modes closed to the forced modes and the dissipative modes differ widely from the inertial ones in periodicity and decay. We have averaged the data for $\geq 68T(L)$ (when $T(L)$ is the large eddy turnover time). This is sufficienty long averaging

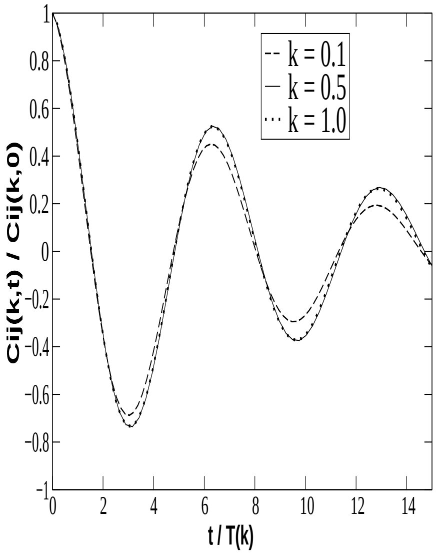

FIG. 3. $c(k,t)$ versus the scaled time $t/T(k)$ (with $T(k) = k^{-2/3}$), for two different k values.

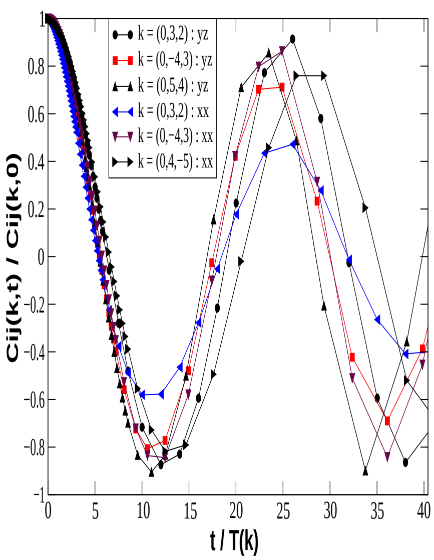

FIG. 4. $C_{ij}(\mathbf{k},t)/C_{ij}(\mathbf{k},0)$ plotted against scaled time $t/T(k)$ (with eddy turnover time $T(k) = A\epsilon^{-1/3}k^{-2/3}$). Plots for different, independent $\mathbf{k}$ vectors (in the inertial range) and different i,j are indicated by $\mathbf{k} = (k_x,k_y,k_z) : i,j$. Approximate data collapse can be seen. Data point shown at the longest time separation and zero time separation are averaged over ~ 68 and ~ 84 Large-eddy turnover times ($T(L)$) respectively.

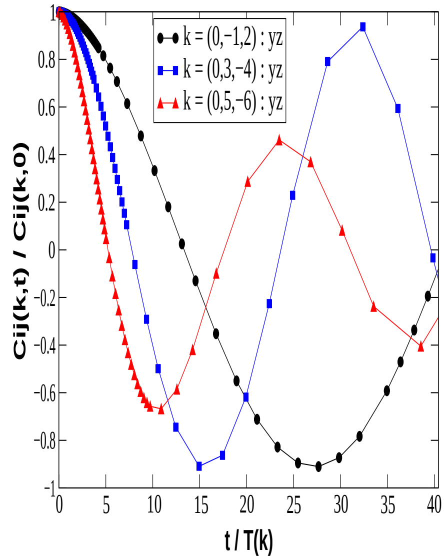

FIG. 5. $C_{ij}(\mathbf{k},t)/C_{ij}(\mathbf{k},0)$ versus $t/T(k)$ plotted for three different $\mathbf{k}$ vectors which lie close to the forced range (circle), in the inertial range (square) and close to the dissipation range (triangle). They show wide separation in periodicity and decay.

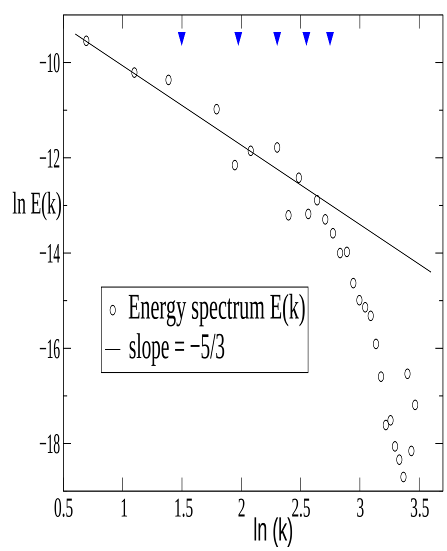

FIG. 6. The energy spectrum $lnE(k)$ versus lnk. The position of the different chosen k values, for which $c(k,t)$ has been plotted in Fig.4 and 5, are indicated at the top of the figure.

time for an equal time, shell averaged, correlation functions to converge; but here for $c(k,t)$ since (a) single $\mathbf{k}$ mode is involved, (b) long time history (few $T(L)$'s) of the mode is important, it requires a longer averaging time. In a simulation with small but finite ν_0 the $\nu_1 k^{2/3}$ term will loose its dominance over $\nu_0 k^2$ as k increases towards the dissipation range. That explains the strong damping seen in the $\mathbf{k} = (0, 5, -6)$ mode in Fig.5.

In the simulation we calculate $T(L)$ using the formula $T(L) = L_{box}/v_{rms}$, where L_{box} is the simulation box size. From dimensional analysis arguments [18] the scale dependent eddy-turnover time $T(k) = A\epsilon^{-1/3}k^{-2/3}$, when A is a constant of $\mathcal{O}(1)$. Equating $T(L) = A\epsilon^{-1/3}(2\pi/L)^{-2/3}$ we get the prefactor $A\epsilon^{-1/3}$ and hence can calculate $T(k)$. Fig.4 shows that the time period of oscillation $\lambda_k \sim 25T(k)$. This ratio may be non-universal. To get a clue let us look at our randomly forced model, where $I(S)$ is an explicit function of D_0, ν_1 and $\nu_1 \propto D_0^{1/3}$ (see Eq.5 and 6). Hence λ_k depends on D_0 in a complicated way. We explore the dependence of $T(k)$ on D_0 below. In this model $T(k)$ cannot be determined in a simple way because here all the k shells are being forced and hence the energy flux is not a constant but increses logarithmically with k [13]. The scale dependent energy flux $\Pi(k) = \int_0^k D(q)d^3q/(2\pi)^3 = \frac{D_0}{2\pi^2}[\ln(a^2+k^2) - k^2(a^2+k^2)^{-1} - 2\ln(a)]$, when $a = L^{-1}$. Again using dimensional analysis we get $T(k) = A.\Pi(k)^{-1/3}k^{-2/3}$ (hence $T(k) \propto D_0^{-1/3}$). Evaluating $\Pi(k)^{-1/3}$ for $D_0 = 0.01$ and $L = 100$ (which we used for our theoretical graphs in Fig.2,3) gives $\Pi(0.1)^{-1/3} = 6.17$, $\Pi(1.0)^{-1/3} = 8.1$. Allthough this k dependence is weak, but the perfect data collapse with respect to $tk^{2/3}$ in Fig.3 implies that the above dimensional analysis estimate is not accurate. In Fig.3 the oscillation period $\lambda_k \sim 6k^{-2/3}$ and hence $\lambda_k \sim T(k)$ (neglecting the constant A of $\mathcal{O}(1)$).

The imaginary part of the viscosity, generated by an one loop perturbation theory, cannot be interpreted as a background velocity which is slowly varying in time. This mis-interpretation may be provoked by the fact that if we had a mean background flow V_0 in the problem, then the nonlinear term would generate an extra term $i(\mathbf{V_0}.\mathbf{k})\mathbf{v}(\mathbf{k},t)$ in the NS equation. With this extra linear term the bare propagator will be $G(k,\omega) \sim (-i\omega - i\mathbf{V_0}.\mathbf{k} + \nu_0 k^2)^{-1}$ [10]. But the k,ω dependent complex viscosity, which we find in our theory, when transformed to $(\mathbf{x},t)$ space, gives an additional memory term $\int d\mathbf{x}'dt'\kappa(|\mathbf{x}-\mathbf{x}'|, t-t')\mathbf{v}(\mathbf{x}',t')$ in the equation of motion (e.o.m.). But it is true that the $\kappa(k,t)$ field oscillates at a slower time scale than that of the $\mathbf{v}(\mathbf{k},t)$ mode itself (as $I\delta\nu(k,\omega)$ has a peak at a lower ω than that of $c(k,\omega)$). Also a significant contribution to the integral for $\delta\nu(k,\omega)$ comes from the $q < k$ modes. So it does resemble sweeping by larger (and hence slower) eddies to some extent. But rigorously convection by a background velocity $\mathbf{V_0}(\mathbf{x},t)$ should look like $\mathbf{V_0}(\mathbf{x},t).\nabla\mathbf{v}(\mathbf{x},t)$ (which is local in $(\mathbf{x},t)$). We note that in the turbulence context, complex effective viscosity has been proposed before by J.K. Bhattacharjee in [7]. The author had assumed dynamic scaling hypothesis (as in dynamic critical phenomena) to be valid for fluid turbulence and had predicted a form for the effective viscosity in the high frequency limit. Then an interpolation scheme had been used to connect the two limits (small ω and large ω) of the effective viscosity.

In conclusion, we have given numerical evidence and an approximate theory for the novel oscillatory decay of the two-point, temporal correlation function in $3-D$ fluid turbulence. This behaviour is similar to viscoelastic effect seen in complex fluids.

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