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Matrix Factorization for an $SO(2)$ Spinning Top and Related Problems

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Abstract

We study the matrix factorization problem associated with an $SO(2)$ spinning top by using the algebro-geometric approach. We derive the explicit expressions in terms of Riemann theta functions and discus some related problems including a non-compact extension and the case when the Lax matrix contains higher-order powers of the spectral parameter.

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1 Introduction

Analytic factorization for matrix valued functions, or the matrix Riemann-Hilbert problem, appears in problems of exactly integrable dynamical systems as well as in problems of elasticity and diffraction.

In the context of exactly integrable models, the factorization problem for the R-matrix associated to an affine Lie algebra is the same as the matrix Riemann-Hilbert problem [1]. In this case exists a general formula for the solution in terms of Riemann theta functions [1], derived from algebro-geometric methods of finite-band integration theory [2]. This construction is based on the fact that the time evolution linearizes on the Jacobian of the spectral curve, and consequently one can express the solution in terms of the Baker-Akheizer functions, which are eigen-vectors of the Lax matrix. Various integrable dynamical systems can be formulated via such R-matrix [1], and most typical are the spinning tops.

In this paper we derive the explicit expressions for the case of an $SO(2)$ spinning top, which is a simple but a non-trivial example of the two-by-two matrix factorization problem. This is done in order to illustrate the general theta-function formula from [1], and also in order to be able to compare to alternative methods of matrix factorization. The neccesity for alternative methods of factorization comes from the fact that the theta-function formula becomes difficult to use in the case when the spectral curve is non-hyperelliptic, which is mostly the case. Also there are cases when only a non-canonical factorization is possible, for which the theta-function formula is not valid. We illustrate this on the example of a non-compact extension of the $SO(2)$ spinning top.

2 Lax formulation and matrix Rieman-Hilbert problem

Consider a rotator with a unit radius and angle ϕ in a potential $V(\phi) = -a^2 \cos 2\phi$, where a is a real constant. The Hamiltonian is given by

$$H = \frac{1}{2} l^2 - a^2 \cos 2\phi \quad , \tag{1}$$

where $l = \dot{\phi} = d\phi/dt$. As a one-dimensional Hamiltonian system, it is integrable, and it is the $n = 2$ case of an n-dimensional integrable spinning top system given by the following n-by-n Lax matrix [1]

$$L = \mathbf{a}\lambda + \mathbf{l} + \mathbf{s}\lambda^{-1} = \mathbf{a}\lambda + \mathbf{l} + R_\phi^{-1} \mathbf{f} R_\phi \lambda^{-1} \quad , \tag{2}$$

where λ is a complex number called the spectral parameter, $\mathbf{a}$ and $\mathbf{f}$ are fixed symmetric matrices, $\mathbf{l}$ is antisymmetric and traceless matrix, while R_ϕ is an $SO(n)$ matrix.

Formula (2) gives a mapping from the phase space of the n-dimensional spinning top $T^*SO(n) = (R_\phi, \mathbf{l})$ to a Hamiltonian orbit in the affine $gl(n)$ algebra $\hat{g}$ defined as

$$\hat{g} = \sum_{j\in \mathbf{Z}} g_j \lambda^j \quad , \quad g_j \in gl(n) \quad .$$

Without loss of generality one can take that the $\mathbf{a}$ and $\mathbf{f}$ matrices are diagonal, and for simplicity we take $\mathbf{a} = \mathbf{f}$ and $tr\mathbf{a} = 0$. The last condition implies $tr\, L = 0$, and this means that we are restricting $\hat{g}$ to the affine $sl(n)$ sub-algebra.

The Hamiltonian (1) is the $n = 2$ case of the $SO(n)$ Hamiltonian

$$H = -\frac{1}{2} Res\, tr(\lambda^{-1} L^2) = -\frac{1}{2} tr\mathbf{l}^2 - tr(\mathbf{a} R_\phi^{-1} \mathbf{f} R_\phi) \quad . \tag{3}$$

The corresponding Lax equations follow from the R-matrix formulation [1], and they are given by

$$\dot{L} = [L, M_\pm] \quad , \quad M_\pm = \frac{1}{2}(R \pm 1) d\varphi(L) = \pm P_\pm L \quad , \tag{4}$$

where $\varphi(L) = \frac{1}{2} tr(L^2)$ is an invariant function on $\hat{g}$ and $R = P_+ - P_-$ is the R-matrix. $P_\pm$ are the projectors onto $\hat{g}_\pm$ subalgebras, which are defined as

$$\hat{g}_+ = \sum_{j\geq 0} g_j \lambda^j \quad , \quad \hat{g}_- = \sum_{j<0} g_j \lambda^j \quad , \tag{5}$$

so that $\hat{g} = \hat{g}_+ \oplus \hat{g}_-$, and therefore

$$M_+ = P_+ L = \mathbf{l} + \mathbf{a}\lambda \quad , \quad M_- = -P_- L = -\mathbf{s}\lambda^{-1} \quad . \tag{6}$$

The equations (4) are solved by

$$L(\lambda, t) = g_\pm(\lambda, t) L(\lambda) g_\pm^{-1}(\lambda, t) \tag{7}$$

where $L(\lambda) = L(\lambda, t = 0)$ and $g_\pm(\lambda, t)$ are solutions of the following matrix factorization problem

$$e^{tL(\lambda)} = g_+^{-1}(\lambda, t) g_-(\lambda, t) \quad , \tag{8}$$

such that g_+ is analytic for $\lambda \neq \infty$ and g_- is analytic for $\lambda \neq 0$. This is a Riemann-Hilbert problem for $e^{tL(\lambda)}$ and a curve which encircles the $\lambda = 0$ point in the complex plain.

Solution of the factorization problem (8) exists for t that satisfies the Gohberg-Feldman bound, which is satisfied for sufficiently small times [1]. Otherwise one has a non-canonical factorization

$$e^{tL(\lambda)} = g_+^{-1}(\lambda, t) D(\lambda) g_-(\lambda, t) \quad , \tag{9}$$

where $D(\lambda) = diag(\lambda^{k_1}, ..., \lambda^{k_n})$.

3 Algebro-geometric factorization

One way to solve the problem (8) is to use the connection with Riemann surfaces. The spectral curve is given by

$$\det(L(\lambda) - \mu) = 0 \quad , \tag{10}$$

and it is time-independent. Its compact model defines the corresponding Riemann surface Γ. Consider the line bundle $E_L(p) \in \mathbf{C}^n$ associated to the eigen-vectors of L

$$L(p)v(p) = \mu(p)v(p) \quad , \quad p \in \Gamma\,. \tag{11}$$

Its time-evolution is linear on $Jac\,\Gamma$, and it is given by $E_t = E \otimes F_t$, where F_t is the line bundle on Γ determined by the transition function $\exp t\mu(p)$ with respect to the covering $\{U_+, U_-\}$, where $U_\pm = \{p\,|\,\lambda^{\pm 1}(p) \neq \infty\}$. In order to solve (8), one needs to consider the holomorphic sections of the dual bundle E_L^*, since the bundle E_L does not have holomorphic sections. Denote these sections as $\psi(p)$, then they satisfy

$$L(p)\psi(p) = \mu(p)\psi(p) \quad , \tag{12}$$

so that $\psi_\pm(p,t) = g_\pm(\lambda(p),t)\psi(p)$ are eigenvectors of $L(\lambda,t)$ which are regular in $U_\pm$. Hence

$$g_\pm(\lambda,t) = \Psi_\pm(\lambda,t)\Psi_\pm^{-1}(\lambda,t=0) \tag{13}$$

where $\Psi_\pm(\lambda,t)_{jk} = \psi^j(p_k,t)$. Hence if one knows the sections $\psi_\pm$, one can solve the factorization problem (8) as well as the dynamics. These sections can be represented via meromorphic functions $\varphi_\pm$ on $U_\pm$ as

$$\varphi_\pm(p,t) = \frac{\psi_\pm(p,t)}{s_\pm(p)} \quad , \tag{14}$$

where $s_\pm(p)$ are sections of E_L^*. These functions are called Baker-Akheizer (BA) functions and satisfy

$$(\varphi_\pm) \geq -D^* \tag{15}$$

where D^* is a divisor of degree $g+n-1$, where g is the genus of Γ.

Given (14) and (15) and the following properties of $\psi_\pm$

$$\frac{d\psi_\pm(p,t)}{dt} = -M_\pm(\lambda,t)\psi_\pm(p,t) \tag{16}$$

and

$$\psi_+(p,t) = e^{-\mu(p)t}\psi_-(p,t) \quad , \tag{17}$$

one can construct the BA functions in terms of the theta functions [3, 4, 1]. Let $P_\infty = \sum_{j=1}^n P_j$ be the divisor of poles of λ, let p_0 be a fixed point in Γ and let D be a divisor of degree g such that the line bundle E_L^* is associated to the divisor

$$D^* = P_\infty + D - p_0 \quad . \tag{18}$$

Let ω_j be a set of normalized holomorphic Abelian differentials on Γ

$$\int_{a_j} \omega_k = 2\pi i \delta_{jk} \quad , \quad \int_{b_j} \omega_k = B_{jk} \quad , \tag{19}$$

where a_j and b_j, $j = 1, ..., g$ is a basis of homology cycles and B is the period matrix. Note that the Riemann theta function is defined on $\mathbf{C}^g$ by quasi-periodicity conditions

$$\theta(z + 2\pi i n) = \theta(z) \quad , \quad \theta(z + Bn) = e^{-\frac{1}{2}(Bn,n)-(z,n)}\theta(z) \quad , \quad n \in \mathbf{Z}^g \tag{20}$$

and can be written as

$$\theta(z) = \sum_{n \in \mathbf{Z}^g} \exp\{\frac{1}{2}(Bn, n) + (z, n)\} \quad , \tag{21}$$

where $(X, Y) = \sum_{j=1}^{g} X_j Y_j$. The theta function can be considered as a section on $Jac\,\Gamma = \mathbf{C}^g/(2\pi i n + Bm)$.

Let $A(p)$ be the Abel transform

$$A(p) = \int_{p_0}^{p} \omega \quad , \tag{22}$$

and let $d\Omega$ be an Abelian differential of the second kind, normalized by

$$\int_{a_j} d\Omega = 0 \quad , \tag{23}$$

which is regular in U_- such that $d\Omega - d\mu$ is regular in U_+. Then

$$\varphi_-^j(p, t) = \gamma_j(t) e^{t\Omega(p)} \frac{\theta(A(p) - c - Vt)\theta(A(p) - c_0)}{\theta(A(p) - c_1)\theta(A(p) - c_j)} \quad , \tag{24}$$

where $\Omega(p) = \int_{p_0}^{p} d\Omega$, and the constants c_k and the function $\gamma_j(t)$ are determined from the following requirements

(a) φ_-^j has no discontinuity across the cuts

(b) φ_-^j is subordinate to the divisor $p_0 - P_j - D$

(c) $Res_{P_j}(\lambda^{-1}\varphi_-^j) = d_j$

where d_j are time-independent constants.

Condition (a) is satisfied if $c + c_0 = c_1 + c_j$ and $V_j = \int_{b_j} d\Omega$. Condition (b) is satisfied if the constants c_k are chosen such that the divisor of zeros of $\theta(A(p) - c_0)$ is $p_0 + \tilde{D}$, the divisor of zeros of $\theta(A(p) - c_1)$ is $D + \tilde{D}$ and the divisor of zeros of $\theta(A(p) - c_j)$ is $P_j + \tilde{D}$, for some divisor $\tilde{D}$. Condition (c) serves to determine the functions $\gamma_j(t)$ and it follows from (16) when $\lambda \to \infty$. The following identity will be useful for our purposes

$$V_j = \int_{b_j} d\Omega = -\sum_{k=1}^{n} Res_{P_k}(\mu\omega_k) \quad . \tag{25}$$

4 The SO(2) Case

In the case of the rotator, $\mathbf{a} = \mathbf{f} = diag(a, -a)$,

$$\mathbf{l} = \begin{pmatrix} 0 & l \\ -l & 0 \end{pmatrix} \quad , \tag{26}$$

and

$$\mathbf{s} = a \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} \quad , \tag{27}$$

so that

$$L = \begin{pmatrix} a\lambda + a\lambda^{-1}\cos 2\phi & l + a\lambda^{-1}\sin 2\phi \\ -l + a\lambda^{-1}\sin 2\phi & -a\lambda - a\lambda^{-1}\cos 2\phi \end{pmatrix} \quad . \tag{28}$$

The spectral curve (10) is then given by

$$\mu^2 = a^2\lambda^2 - 2E + a^2\lambda^{-2} \quad , \tag{29}$$

where $E = H$ is the energy. The equation (29) can be transformed into elliptic form

$$w^2 = \lambda^4 - \frac{2E}{a^2}\lambda^2 + 1 = (\lambda^2 - l_+^2)(\lambda^2 - l_-^2) \quad , \tag{30}$$

so that the spectral surface is a torus. We take the a_1 cycle to encircle the $[-l_+, -l_-]$ cut, while the b_1 cycle goes from the $[-l_+, -l_-]$ cut to the $[l_-, l_+]$ cut, were it passes through to the second sheet and it goes back to the $[-l_+, -l_-]$ cut. The normalized holomorphic Abelian differential is given by

$$\omega = \frac{2\pi i}{\mathcal{A}}\frac{d\lambda}{w} \quad , \quad \mathcal{A} = \int_{a_1} \frac{d\lambda}{w} \quad , \tag{31}$$

so that the period matrix is given by

$$B = \int_{b_1} \omega = 2\pi i \frac{\mathcal{B}}{\mathcal{A}} = 2\pi i \tau \quad . \tag{32}$$

The Abelian differential of the second kind $d\Omega$ must satisfy for $\lambda \to 0$

$$d\Omega(p) = d\mu + O(1)d\lambda = \left[\mp\frac{a}{\lambda^2} + O(1)\right] d\lambda \quad , \tag{33}$$

and it is normalized by

$$\int_{a_1} d\Omega = 0 \quad . \tag{34}$$

Hence the second-order poles are at $p = 0^+$ and $p = 0^-$ and one can construct $d\Omega$ in terms of the prime forms [4]. Alternatively, from (33) it follows that the corresponding Abelian integral $\Omega(p)$ behaves as

$$\Omega(p) = \pm\frac{a}{\lambda} + O(1) \quad , \tag{35}$$

for $p \to 0^{\pm}$, and hence it is a meromorphic function with simple poles at $0^{\pm}$, and with residues $\pm a$. A meromorphic function having only simple poles can be specified by the position of the poles and with the values of the residues, and it can be constructed in terms of the theta functions [5]. This implies that

$$\Omega(p) = a\alpha \left[\frac{\theta_1'(A(p) - A(0^+))}{\theta_1(A(p) - A(0^+))} - \frac{\theta_1'(A(p) - A(0^-))}{\theta_1(A(p) - A(0^-))} \right] \quad , \tag{36}$$

where $\alpha = 2\pi i/\mathcal{A}$ and $\theta_1(z)$ is the odd Jacobi theta function (40) and $\theta'(z) = \frac{d\theta(z)}{dz}$. Also note that the identity (25) gives

$$V = 2a \frac{2\pi i}{\mathcal{A}} = 2a\alpha \quad . \tag{37}$$

We take $D = p_1 \neq e_i$, where e_i are the end-points of the brunch-cuts, and $P_\infty = P_1 + P_2 = \infty^+ + \infty^-$ so that

$$\begin{aligned} c_0 &= \pi i + \frac{1}{2}B \\ c_1 &= A(p_1) + \pi i + \frac{1}{2}B \\ c_j &= A(P_j) + \pi i + \frac{1}{2}B \quad , \end{aligned} \tag{38}$$

and hence

$$c = c_1 + c_j - c_0 = A(p_1) + A(P_j) + \pi i + \frac{1}{2}B \quad . \tag{39}$$

By using

$$\theta(z + \pi i + \frac{1}{2}B) = e^{-B/8 - z/2}\theta_1(z) \quad , \tag{40}$$

where $\theta_1(z) = -i\theta[\frac{1}{2}, \frac{1}{2}](z)$ is the odd Jacobi theta function, we obtain

$$\varphi_-^j(p,t) = \gamma_j(t) e^{t\tilde{\Omega}(p)} \frac{\theta_1(-A(p) + A(p_1) + A(P_j) + Vt)\theta_1(-A(p))}{\theta_1(-A(p) + A(p_1))\theta_1(-A(p) + A(P_j))} \quad , \tag{41}$$

and $\tilde{\Omega}(p) = \Omega(p) - V/2$.

The constants $\gamma_j(t)$ can be determined from the condition (c), and for this one needs the expansion of (41) in $\zeta = \lambda^{-1}$ for $\lambda \to \infty$. The following formulas will be useful

$$A(p) = \int_{p_0}^{\infty^\pm} \omega + \int_{\infty^\pm}^{p} \omega = A(P_j) - z_j \tag{42}$$

and

$$z_j = \alpha_j \zeta + O(\zeta^2) = \pm\alpha\zeta + O(\zeta^2) \quad . \tag{43}$$

One also has for $p \to P_j$

$$\Omega(p) = \Omega_j^0 + \Omega_j^1 \zeta + O(\zeta^2) \quad , \tag{44}$$

so that when $p \to P_j$ we obtain

$$
\begin{aligned}
\varphi_-^j(p,t) &= \gamma_j(t) e^{t\tilde{\Omega}_j^0} \frac{\theta_1(A_1+Vt)\theta_1(-A_j)}{\theta_1(-A_j+A_1)\alpha_j\theta_1'(0)} \left[\zeta^{-1} + \alpha_j \left(\frac{\Omega_j^1}{\alpha_j} t + \frac{\theta_1'(A_1+Vt)}{\theta_1(A_1+Vt)} \right.\right. \\
&+ \left.\left. \frac{\theta_1'(-A_j)}{\theta_1(-A_j)} - \frac{\theta_1'(-A_j+A_1)}{\theta_1(-A_j+A_1)} - \frac{\theta_1''(0)}{2\theta_1'(0)} \right) + O(\zeta) \right] \quad , \qquad (45)
\end{aligned}
$$

where $A(p_k) = A_k$. Also when $p \to P_j^*$, where $p^* = (-w, \lambda)$, we have

$$
\varphi_-^j(p,t) = \gamma_j(t) e^{t\tilde{\Omega}_{j^*}^0} \frac{\theta_1(A_1 + A_j - A_j^* + Vt)\theta_1(-A_j^*)}{\theta_1(-A_j^* + A_1)\theta_1(-A_j^* + A_j)} \left[1 + O(\zeta)\right] \quad . \qquad (46)
$$

From (45) it follows that

$$
d_j = \gamma_j(t) e^{t\tilde{\Omega}_j^0} \frac{\theta_1(A_1+Vt)\theta_1(-A_j)}{\theta_1(-A_j+A_1)\alpha_j\theta_1'(0)} \quad . \qquad (47)
$$

As far as the solutions of the equations of motion are concerned, these can be obtained by inserting the $\lambda \to \infty$ expansions (45) and (46) of the BA functions into (16). Namely, for $\lambda \to \infty$ we have

$$
\varphi_-^j = \lambda \psi_-^j = \lambda \psi_0^j + \psi_1^j + O(1/\lambda) \qquad (48)
$$

so that (16) gives

$$
\frac{d\psi_0^j}{dt} = 0 \quad , \quad \frac{d\psi_1^j}{dt} = \mathrm{s}\psi_0^j \quad . \qquad (49)
$$

By using (49) together with (45) and (46) we get $\psi_0^1 = (d_1, 0)^T$, $\psi_0^2 = (0, d_2)^T$ and

$$
\begin{aligned}
\pm\, \alpha \frac{d}{dt} \left[\frac{\theta_1'(A_1+Vt)}{\theta_1(A_1+Vt)} \right] + \Omega_1^j &= \pm a \cos 2\phi \qquad (50) \\
\frac{d}{dt} \left[\gamma_j(t) e^{t\tilde{\Omega}_{j^*}^0} \frac{\theta_1(A_j - A_j^* + A_1 + Vt)\theta_1(-A_j^*)}{\theta_1(A_1 - A_j^*)\theta_1(A_j - A_j^*)} \right] &= a d_j \sin 2\phi \quad . \qquad (51)
\end{aligned}
$$

From (50) we obtain

$$
\sin^2 \phi = -\frac{1}{2}(\Omega_1/a - 1) - \alpha \frac{V}{2a} \frac{d^2}{dz^2} \log \theta_1(z) \quad , \qquad (52)
$$

where $\Omega_1 = \Omega_1^1 = -\Omega_1^2$ and $z = Vt + A_1$. Due to (25), $V/2a = \alpha$ so that (52) can be written as

$$
\sin^2 \phi = \text{const.} + \alpha^2 \wp(2a\alpha t + A_1) \quad , \qquad (53)
$$

where $\wp$ is the Weierstrass function. Since

$$
\wp(\alpha^{-1}u | \alpha^{-1}\omega_1, \alpha^{-1}\omega_2) = \alpha^2 \wp(u|\omega_1, \omega_2) \quad , \qquad (54)
$$

where $\omega_{1,2}$ are the periods of the Weierstrass function, we get

$$\sin^2 \phi = \wp\left(2at + \frac{A_1}{\alpha}\right) + \text{const.} \quad . \tag{55}$$

The expression (55) can be shown to coincide with the solution of the equations of motion obtained from the energy expression

$$2E = \dot{\phi}^2 - 2a^2 \cos 2\phi \quad . \tag{56}$$

It follows from (56) that

$$\tilde{\omega} t = \pm \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \quad , \tag{57}$$

where $\tilde{\omega} = \sqrt{2(E + a^2)}$, $k^2 = 2a^2(E + a^2)^{-1}$ and $E \geq -a^2$. The elliptic integral (57) implies

$$\sin \phi = \mathrm{sn}(\tilde{\omega} t, k) \quad . \tag{58}$$

However, in order to compare to (55), we rewrite the integral (57) as

$$2at = \pm \int_0^{\sin^2 \phi} \frac{dx}{\sqrt{4x(1-x)(k^{-2} - x)}} \quad . \tag{59}$$

Since the Weierstrass function can be defined as

$$u = \int_{\wp(u)}^{\infty} \frac{dy}{\sqrt{4y^3 - g_2 y - g_3}} \quad , \tag{60}$$

from (59) it follows that

$$\sin^2 \varphi = \wp(2at + u_0) + \text{const.} \quad , \quad Im\, u_0 \neq 0 \quad , \tag{61}$$

where

$$u_0 = \int_0^{\infty} \frac{dx}{\sqrt{4x(1-x)(k^{-2} - x)}} \quad . \tag{62}$$

Clearly the expression (61) coincides with (55).

Now define the matrices $\Phi_{\pm}(\lambda, t)_{jk} = \varphi^j_{\pm}(p_k, t)$, where

$$\varphi_+(p, t) = e^{-\mu(p)t} \varphi_-(p, t) \quad ,$$

then

$$g_{\pm}(\lambda, t) = \Phi_{\pm}(\lambda, t) \Phi_{\pm}^{-1}(\lambda, t = 0) \quad . \tag{63}$$

This formula follows from (13), due to relation (14).

The relation (55) can be satisfied for all times if the choice of p_1 is such that $Im(A_1/\alpha) \neq 0$, in accordance with (61). This follows from $\wp(u) = u^{-2} + O(u^2)$ so that the r.h.s. of (55) diverges for

$$2at + \frac{A_1}{\alpha} = 0 \quad . \tag{64}$$

Therefore if $Im(A_1/\alpha) \neq 0$, one obtains a physical solution.

If p_1 is chosen such that $Im(A_1/\alpha) = 0$, then the factorization (8) will not be valid for t close to $A_1/(2a\alpha)$. In this case one would have a non-canonical factorization given by (9). We will show in the next section that this case corresponds to a dynamical system based on a non-compact extension of the $SO(2)$ group.

5 Related problems

Let us replace ϕ with iq in (1), and let $l = \dot{q}$. In this way we obtain a new dynamical system describing particle on a line moving in a potential $V(q) = -a^2 \cosh 2q$. The solution of the equations of motion can be obtained from the energy

$$2E = \dot{q}^2 - 2a^2 \cosh 2q \quad ,$$

and it is given by

$$2at = \pm \int_{\epsilon_k}^{\sinh^2 q} \frac{dx}{\sqrt{4x(1+x)(k^{-2}+x)}} \quad , \tag{65}$$

where $\epsilon_k = 0$ for $k^2 = 2a^2/(E+a^2) \geq 0$, while $\epsilon_k = -k^{-2}$ for $k^2 < 0$. Note that now there is no restriction $E + a^2 \geq 0$ as in the compact case, since the potential is not bounded from bellow. Then by using the same relations as in the compact case we get

$$\sinh^2 q = \wp(2at + u_0) + \mathrm{const.} \quad , \quad Im\, u_0 = 0 \quad , \tag{66}$$

since

$$u_0 = \int_{\epsilon_k}^{\infty} \frac{dx}{\sqrt{4x(1+x)(k^{-2}+x)}} \quad . \tag{67}$$

This is an example of a dynamical system where q diverges for finite times, and hence one can expect that the corresponding factorization will be non-canonical. The replacement $\phi \to iq$ corresponds to replacing the $SO(2)$ group with the non-compact subgroup of $SO(2, \mathbf{C})$. The corresponding Lax matrix can be obtained from (2) by taking $\mathbf{f} = -\mathbf{a}$ and $\phi = iq$, so that

$$L = \begin{pmatrix} a\lambda - a\lambda^{-1}\cosh 2q & l - ia\lambda^{-1}\sinh 2q \\ -l - ia\lambda^{-1}\sinh 2q & -a\lambda + a\lambda^{-1}\cosh 2q \end{pmatrix} \quad , \tag{68}$$

and $l = i\dot{q}$. The spectral curve is (29) with $E \to -E$, since

$$-l^2 - 2a^2 \cos 2\phi = \dot{q}^2 - 2a^2 \cosh 2q = 2E \quad .$$

Consequently, the derivation of the BA functions is almost the same as in the section 4, and the only difference is in the equation (50), where $a \to -a$ due to the fact that now $\mathbf{f} = -\mathbf{a}$. This gives

$$\sinh^2 q = \wp(2at + A_1/\alpha) + \mathrm{const.} \quad . \tag{69}$$

The physical solutions are obtained for $Im\, A_1/\alpha = 0$, in agreement with (66). Hence the canonical factorization of e^{tL} will not be possible for $2at \to A_1/\alpha$.

Note that for $n \geq 3$ one obtains the spectral curves which are non-hyperelliptic, and in that case finding the BA functions gets difficult, because not much is known about the non-hyperelliptic curves, except in some special cases, like Kowalewski top [3] (see [4] for other examples). For example, for $n = 3$ case the Lax matrix (2) gives a cubic spectral curve

$$\mu^3 + L^{(2)}\mu - \det L = 0 \quad , \tag{70}$$

which is difficult to analyze. It would be interesting to see whether some alternative method of solving the corresponding factorization problem can be used, for example see [6]. Preliminary study indicates that the techniques developed in [6] can be used to solve the $n = 2$ case [7], so that one can expect that the $n = 3$ case could be also solved.

Note that the Lax matrix (2) can be generalized to

$$L = \mathbf{a}\lambda^m + \sum_{j=m-1}^{-k} L_j \lambda^j \quad , \quad m, k \in \mathbf{N} \quad , \tag{71}$$

where $\mathbf{a}$ is a constant matrix while L_j are the dynamical matrices [1]. The dynamics is given by (4) and the corresponding factorization problem is given by (8). In the $n = 2$ case ($L \in gl(2)$ or $sl(2)$) the BA functions can be easily calculated since in that case one obtains a hyper-elliptic spectral curve

$$w^2 = P_{2m+2k}(\lambda) \quad , \tag{72}$$

where $P_{2m+2k}(\lambda)$ is a polynomial of order $2m + 2k$. However, this system is not physically interesting, i.e. the corresponding Hamiltonian

$$H = -\frac{1}{2} Res\, tr(\lambda^{-1} L^2) \quad , \tag{73}$$

has no physical applications. Still, the two-by-two matrices (71) provide non-trivial solvable examples of the Riemann-Hilbert problem for e^{tL}, and one can compare the results to alternative methods of matrix factorization.

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