A GENERALIZED TORELLI THEOREM

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Abstract. Given a smooth projective curve C of positive genus g, Torelli's theorem asserts that the pair $(J(C), W^{g-1})$ determines C. We show that the theorem is true with W^{g-1} replaced by W^d for each d, in the range $1 \leq d \leq g-1$.

§1 Introduction

All curves in subsequent sections will be assumed to be smooth projective curves over $\mathbb{C}$. The genus of C will always be denoted by g. If C is such a curve (with $g > 0$) we will let $J(C)$ denote its Jacobian and

$$u\colon C \to J(C)$$

will be the Abel-Jacobi map. We will let $C^{(d)}$ denote the dth symmetric power of C and for $1 \leq d \leq g-1$, W^d will be the image of $C^{(d)}$ inside the Jacobian under the Abel-Jacobi map. Since by a theorem of Riemann, the theta divisor is a translate of W^{g-1}, Torelli's theorem asserts that the pair $(J(C), W^{g-1})$ determines the curve, meaning that if C' is another curve such that there is an isomorphism $J(C) \cong J(C')$ carrying theta divisors to theta divisors then the curves must be isomorphic. Our aim is to show that an analogous statement holds for each $1 \leq d < g-1$. With this in mind we will assume in all following sections that $g \geq 4$, as smaller genera are covered by existing theorems. Our strategy is largely based on the strategy in [1].

As a corollary we have that two curves are isomorphic if and only if their dth symmeteric powers are isomorphic, where d is an integer smaller than the genus of one (and hence both) of the curves.

This problem and the above mentioned Corollary was originally proposed by Prof. Donu Arapura. Thanks also to the particpants of the Working Algebraic Geometry Seminar at Purdue, in particular to Prof. Kenji Matsuki who pointed out a mistake in an earlier version.

§2 Preliminaries

The Jacobian of a curve C is defined to be

$$J(C) = \mathrm{H}^0(C, \Omega^1_C)^* / \mathrm{H}_1(C, \mathbb{Z}).$$

The Abel-Jacobi map is defined by

$$\begin{aligned} u: \quad C &\longrightarrow J(C) \\ p &\longmapsto \int_{p_0}^{p} \end{aligned}$$

where p_0 is a fixed basepoint. Let $C^{(d)} = C^d/S^d$ be the dth symmetric power of C. We identify the points of $C^{(d)}$ with effective divisors of degree d on C. The Abel-Jacobi map can be extended to a morphism

$$u : C^{(d)} \longrightarrow J(C).$$

We have

(2.1). Theorem (Abel's). *Let $D, D' \in C^{(d)}$. Then*

$$D \sim D' \quad if\ and\ only\ if \quad u(D) = u(D')$$

where the relation $\sim$ is linear equivalence.

Proof. See [4]. □

We let $W^d = u(C^{(d)})$. By Abel's Theorem W^d parameterises complete linear systems of degree d on C. Our aim is to reconstruct C from the pair $(J(C), W^d)$ where $0 < d \leq g-1$. The main tool in doing this will be the Gauss map, defined as follows. Take $p \in W^d_{\mathrm{smooth}}$ and let $\mathrm{T}_p(W^d)$ be its holomorphic tangent space. There is an automorphism, translation by $-p$,

$$\begin{aligned} \tau_p : \quad J(C) &\longrightarrow J(C) \\ x &\longmapsto x - p \end{aligned}$$

This allows us to canonically identify $\mathrm{T}_p(W^d)$ with a d-dimensional subspace of $\mathrm{T}_0(J(C)) \simeq \mathrm{H}^0(C, \Omega^1_C)^*$. This defines the Gauss map

$$\mathcal{G} : W^d_{\mathrm{smooth}} \longrightarrow \mathbb{G}(d-1, g-1),$$

where $\mathbb{G}(d-1, g-1)$ is the Grassmanian parameterizing $d-1$ dimensional linear subvarieties of $\mathbb{P}^{g-1}$ (or equivalently d-dimensional subspaces of $\mathbb{C}^g$. The result we need is:

(2.2). Theorem. *Let $\phi_K \colon C \to (\mathbb{P}^{g-1})^*$ be the canonical morphism and let $D \in C^{(d)}$. Then $u(D) \in W^d_{\mathrm{smooth}}$ if and only if* $\dim|D| = 0$. *If we denote by $\overline{\phi_K(D)}$ the linear span of D on the canonical curve then*

$$\mathcal{G}(u(D)) = \overline{\phi_K(D)}.$$

Proof. This result can be found in §2.7 of [4]. □

Note that the linear span of a multiple of a point is the appropriate osculating plane to C inside $\mathbb{P}^{g-1}$. The condition that $\dim|D| = 0$ forces $\overline{\phi_K(D)}$ to be a $d-1$ dimensional linear subvariety of $\mathbb{P}^{g-1}$. This is by

(2.3). Theorem (Geometric Riemann-Roch)**.** *For D as in the above discussion we have* $\dim|D| = d - 1 - \dim\overline{\phi_K(D)}$.

Proof. Again this can be found in [4]. □

§3 Our Strategy

We first describe the idea behind the proof of the Torelli theorem for curves, due to A. Andreotti, see [1]. The Gauss map

$$\mathcal{G}\colon W^{g-1}_{\text{smooth}} \to (\mathbb{P}^{g-1})^*$$

is a quasi-finite morphism of degree

$$\binom{2g-2}{g-1}.$$

To see this, a hyperplane H intersects the image of a curve C under its canonical morphism in $2g-2$ points $p_1, p_2, \dots, p_{2g-2}$, which are in general posiion for a generic H. By Theorem (2.2) the fibre over H consists of all images of divisors of the form $u(p_{i_1} + p_{i_2} + \dots + p_{i_{g-1}})$ where i_j range over $\{1, 2, \dots, 2g-2\}$. If C is non-hyperelliptic then let C^* be the dual variety to C, that is the locus of all tangent hyperplanes to $\phi_K(C)$ inside $(\mathbb{P}^{g-1})^*$. Now one would expect that the (closure of the) branch locus of $\mathcal{G}$ to be C^* since the fibre over a tangent hyperplane H should have cardinality smaller than

$$\binom{2g-2}{g-1}.$$

(Since $H.C = 2p_1 + \dots p_{2g-3}$, the first point is repeated and there are fewer choices for points in the fibre.) It is known how to recover C from C^*, for example see [5]. In the case that C is hyperelliptic the canonical morphism $\phi_K\colon C \to \mathbb{P}^{g-1}$ is branched at $2g+2$ points labelled $b_1, \dots, b_{2g+2}$. We denote by C^* the dual variety to the rational normal curve $\phi_K(C)$ and b_i^* denotes the locus of all hyperplanes passing through b_i. In the hyperelliptic case, by the same reasoning as in the non-hyperelliptic case, one would expect that the branch locus of $\mathcal{G}$ to be $C^* \cup b_1^* \cup \dots b_{2g+2}^*$. It is known how to recover C from this information.

We would like to try to apply this technique to our situation. Firstly, we may reduce to the case where $(g-1)/2 < d < g-1$. To do this choose an integer n so that $(g-1)/2 < nd \leq g-1$. Then

$$W^{nd} = \underbrace{W^d + W^d + \ldots + W^d}_{n \text{ times}}.$$

The above addition is addition inside the Jacobian.

Fix $\mathbb{P}^{g-1} = \mathbb{P}(\mathrm{H}^0(C, \Omega^1_C)^*)$. Now consider the locus

$$\begin{aligned} \mathrm{F}(d,g) = \{(V,W) &\in \mathbb{G}(d-1, \mathbb{P}^{g-1}) \times \mathbb{G}(d-1, \mathbb{P}^{g-1}) \mid \\ &\overline{V+W} \neq \mathbb{P}^{g-1}\}. \end{aligned}$$

The notation $\overline{V+W}$ means linear span of V and W. So $\mathrm{F}(d,g)$ is the locus of all pairs of $(d-1)$-dimensional linear subvarieties that are contained inside some hyperplane. There is a rational morphism

$$\alpha\colon \mathrm{F}(d,g) \dashrightarrow (\mathbb{P}^{g-1})^*$$

defined by $(V,W) \mapsto \overline{V+W}$. We take $\mathrm{E}(d,g)$ to be the pullback of $\mathrm{F}(d,g)$ under

$$\mathcal{G} \times \mathcal{G}\colon W^d_{\text{smooth}} \times W^d_{\text{smooth}} \to \mathbb{G}(d-1, g-1) \times \mathbb{G}(d-1, g-1).$$

Now let β be the composed rational morphism

$$\beta\colon \mathrm{E}(d,g) \dashrightarrow (\mathbb{P}^{g-1})^*.$$

Arguing as in the case $d = g-1$ we see that the branch locus of β contains enough information to recover C. Note that the hypothesis $(g-1)/2 < d < g-1$ is required to insure that $\mathrm{E}(d,g)$ is not empty.

§4 Generic Determinental Varieties

Two identities that will be useful later are presented in this section.

In this section d and g will be non-negative integers with $(g-1)/2 < d < g-1$. We will need the case $g \geq 4$ later. Let M be the generic $g \times 2d$ matrix,

$$M = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,2d} \\ x_{21} & x_{22} & \cdots & x_{1,2d} \\ \vdots & \vdots & & \vdots \\ x_{g1} & x_{g2} & \cdots & x_{g,2d} \end{pmatrix}$$

over the polynomial ring $\mathbb{C}[x_{ij}]$. We will let $M_{(i_1,i_2,\ldots,i_g)}$, where $i_1 < i_2 < \ldots < i_g$, be the following submatrix of M.

$$M_{(i_1,i_2,\ldots,i_g)} = \begin{pmatrix} x_{1,i_1} & x_{1,i_2} & \cdots & x_{1,i_g} \\ x_{2,i_1} & x_{2,i_2} & \cdots & x_{1,i_g} \\ \vdots & \vdots & & \vdots \\ x_{g,i_1} & x_{g,i_2} & \cdots & x_{g,i_g} \end{pmatrix}.$$

Also let

$$N = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,g-1} \\ x_{21} & x_{22} & \cdots & x_{1,g-1} \\ \vdots & \vdots & & \vdots \\ x_{g1} & x_{g2} & \cdots & x_{g,g-1} \end{pmatrix}.$$

Let f be the product of the $(g-1) \times (g-1)$ minors of N. Let $R = \mathbb{C}[x_{ij}]_f$. Let I be the ideal generated by the $g \times g$ minors of M in $\mathbb{C}[x_{ij}]$. Finally let J be the ideal of $\mathbb{C}[x_{ij}]$ generated by the minors of the form $\det(M_{(1,2,\dots,g-1,i)})$, as i ranges over, $g \leq i \leq 2d$. We wish to prove

(4.1). Proposition. *Consider the ideals I_f, J_f obtained by extending I and J to the ring R. We have $I_f = J_f$.*

Proof. It is clear that $J_f \subseteq I_f$. We proceed by showing that $\sqrt{J_f} = \sqrt{I_f}$ and then showing that J_f is equal to its radical.

We begin by showing $\sqrt{J\langle f\rangle} = \sqrt{I\langle f\rangle}$. Here $\langle f\rangle$ is the ideal generated by f. To show the above it suffices to show that the two ideals have the same zero locus inside $\mathbb{A}^{g\times 2d}$. It is clear that

$$Z(J\langle f\rangle) = Z(J) \cup Z(f) \supseteq Z(I) \cup Z(f) = Z(I\langle f\rangle).$$

Now take $p = (p_{ij})$ in the zero locus of $J\langle f\rangle$. We may assume p is not in the zero locus of f, for otherwise we are done. Consider the matrix

$$M_p = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1,2d} \\ p_{21} & p_{22} & \cdots & p_{2,2d} \\ \vdots & \vdots & & \vdots \\ p_{g1} & p_{g2} & \cdots & p_{g,2d} \end{pmatrix}.$$

Showing that $p \in Z(I)$ is equivalent to showing that $\operatorname{rank}(M_p) \leq g-1$. Since $(p_{ij}) \notin Z(f)$ the first $g-1$ columns of M_p are linearly independent. As $(p_{ij}) \in Z(J)$,

$$\det((M_p))_{(1,2,\dots,g-1,i)} = 0,$$

for $g \leq i \leq 2d$. So the ith column is in the linear span of the first $g-1$ columns and we are done. We have shown $\sqrt{J\langle f\rangle} = \sqrt{I\langle f\rangle}$. An elementary argument now shows that $\sqrt{I_f} = \sqrt{J_f}$.

Finally we need to show that J_f is radical. Notice that J_f is generated by polynomials of the form

$$\begin{aligned} \det(M_{(1,2,\dots,g-1,i)}) = \det(N_1)x_{1i} - \det(N_2)x_{2i} + \dots \\ (-1)^g \det(N_{g-1})x_{g-1,i}. \end{aligned}$$

Here N_j is the submatrix of N obtained by deleting the jth row. Each of the $\det(N_j)$ are units in our ring R. The result follows from the following lemma. $\square$

(4.2). Lemma. *Let A be a reduced ring and consider the polynomial ring $B = A[x_{ij}]$, where $1 \leq i \leq n$ and $1 \leq j \leq m$. Consider elements*

$$f_i = u_{i1}x_{i1} + u_{i2}x_{i2} + \ldots + u_{im}x_{im}.$$

Form the ideal $I = (f_1, f_2, \ldots, f_n)$. If the u_{ij} are units in A then B/I is reduced.

Proof. Observe that $B/I \cong A[x_{ij}]$ but with new index ranges $2 \leq i \leq n$ and $2 \leq j \leq m$. $\square$

Now let M be the matrix

$$M = \begin{pmatrix} x_{11} & \cdots & x_{1,2d} \\ \vdots & & \vdots \\ x_{g1} & \cdots & x_{g,2d} \end{pmatrix}$$

over the polynomial ring $\mathbb{C}[x_{ij}]$. Consider the submatrices

$$A = \begin{pmatrix} x_{11} & \cdots & x_{1,d} \\ \vdots & & \vdots \\ x_{d,1} & \cdots & x_{d,d} \end{pmatrix} \quad B = \begin{pmatrix} x_{1,d+1} & \cdots & x_{1,2d} \\ \vdots & & \vdots \\ x_{d,d+1} & \cdots & x_{d,2d} \end{pmatrix}.$$

Set $f = \det(A)$ and $g = \det(B)$. We will be interested in the following ideals of the ring $\mathbb{C}[x_{ij}]_{fg}$. Let I be ideal of the $g \times g$ minors of M and let J be the ideal of the $g \times g$ minors of

$$N = M \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}.$$

(4.3). Lemma. *The ideals I and J of $\mathbb{C}[x_{ij}]_{fg}$ are equal.*

Proof. The subschemes of $\operatorname{spec}(\mathbb{C}[x_{ij}]_{fg})$ defined by I and J are supported on the same closed subset. So it suffices to show that both I and J are reduced. The fact that I is reduced is the fundamental theorem of invariant theory, see [2]. To show that J is reduced consider the $\mathbb{C}$ algebra automorphism of $\mathbb{C}[x_{ij}]_{fg}$ defined by

$$x_{ij} \mapsto y_{ij}$$

where

$$M \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} y_{11} & \cdots & y_{1,2d} \\ \vdots & & \vdots \\ y_{g1} & \cdots & y_{g,2d} \end{pmatrix}.$$

This automorphism carries J to I so we are done. $\square$

§5 A Subvariety of $\mathbb{G}(d-1,g-1)\times\mathbb{G}(d-1,g-1)$

We let $\mathbb{G}(d-1,g-1)$ denote the Grassmanian paramaterizing $(d-1)$ dimensional linear subspaces of $\mathbb{P}^{g-1}$. Let

$$\begin{aligned}\mathrm{F}(d,g)=\{(V,W)\in\mathbb{G}(d-1,g-1)\times\mathbb{G}(d-1,g-1)\mid\\ V\subseteq H, W\subseteq H \text{ for some hyperplane } H\subseteq\mathbb{P}^{g-1}\}.\end{aligned}$$

In the above V and W are closed points of the Grassmanian. We wish to describe the reduced scheme structure on $\mathrm{F}(d,g)$. First we recall how to cover Grassmanian with open affines isomorphic to $\mathbb{C}^{d(g-d)}$.

Let $V\in\mathbb{G}(d-1,g-1)$ be a closed point. So V can be thought of as the column space of a $g\times d$ matrix A. Write

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d}\\ a_{21} & a_{22} & \cdots & a_{2d}\\ \vdots & \vdots & \vdots & \vdots\\ a_{g1} & a_{g2} & \cdots & a_{gd}\end{pmatrix}.$$

This representation is unique upto the action of $\mathrm{GL}(d,\mathbb{C})$.

Let $I=(i_1,i_2,\dots i_d)$, where $i_j\in\{1,2,\dots,g\}$ and $i_1<i_2<\dots<i_d$. We will denote by A^I the following $d\times d$ submatrix of A:[1]

$$A^I=\begin{pmatrix} a_{i_1 1} & a_{i_1 2} & \cdots & a_{i_1 d}\\ a_{i_2 1} & a_{i_2 2} & \cdots & a_{i_2 d}\\ \vdots & \vdots & \vdots & \vdots\\ a_{i_g 1} & a_{i_g 2} & \cdots & a_{i_g d}\end{pmatrix}.$$

Now since the rank of A is d, the matrix A has a non vanishing $d\times d$ minor. Let this minor be $\det(A^I)$. The matrix $A'=A(A^I)^{-1}$ also has column space equal to V, furthermore it is the unique representative with $(A')^I=\mathrm{Id}_d$. For each $I=(i_1,i_2,\dots i_d)$ as above, set

$$\begin{aligned}U_I=\{V\in\mathrm{G}(d,g)\mid \text{ the } I \text{ minor of a matrix}\\ \text{representative of } V \text{ is invertible}\}.\end{aligned}$$

There is a bijection $U_I\cong\mathbb{C}^{d.(g-d)}$, which is in fact an isomorphism. For further details see [4] or [5].

It follows from the above that $\mathbb{G}(d-1,g-1)\times\mathbb{G}(d-1,g-1)$ has an open affine cover constisting of opens of the form $U_I\times U_J\cong\mathbb{C}^{2d(g-d)}$. Now take $(V,W)\in U_I\times U_J$, with V the column space of a matrix A and W the column space of a matrix B. The locus we are trying to describe, $\mathrm{F}(d,g)$, consists of those pairs (V,W) such that $\mathrm{rank}(A|B)<g$. Here $(A\mid B)$ is the matrix obtained by augmenting the matrix A with the matrix B. Now the rank of $(A|B)<g$ if and only if the $g\times g$ minors

[1] In the preceeding section we defined A_I. In that section the submatrix A_I of A was obtained by choosing columns of A, while here we are choosing rows.

of $(A \mid B)$ vanish. The latter condition holds if and only if the $g \times g$ minors of the matrix $(A \mid B)C$ vanish where

$$C = \begin{pmatrix} A_I^{-1} & 0 \\ 0 & B_J^{-1} \end{pmatrix}.$$

The entries of the matrix $(AB)C$ determine the image of (V, W) under the isomorphism $U_I \times U_J \cong \mathbb{C}^{d(g-d)+d(g-d)}$. So the ideal generated by the $g \times g$ minors of $(A \mid B)C$ determines a scheme structure on $\mathrm{F}(d, g) \cap U_I \times U_J$.It follows from [2] pg. 71 that this scheme structure is reduced, being a specialization of the ideal I_k defined there. Hence these ideal sheaves on $U_I \times U_J$ glue together to give an ideal sheaf for the reduced structure on $\mathrm{F}(d, g)$.

We let

$$U_{\mathrm{F}} = \{(V, W) \in \mathrm{F}(d, g) \mid \mathrm{rank}(A \mid B) = g - 1\}.$$

There is a morphism

$$\alpha\colon U_F \dashrightarrow (\mathbb{P}^{g-1})^*,$$

It takes a closed point (V, W) to the linear span of V and W. We will denote $\overline{U}_F$ by $\mathrm{F}(d, g)_{\mathrm{main}}$.

§6 The construction of $\mathrm{E}(d, g)$

In this section let C be a curve of genus $g \geq 4$. Let $(g-1)/2 < d < g - 1$. We have a morphism

$$\mathcal{G} \times \mathcal{G}\colon W^d_{\mathrm{smooth}} \times W^d_{\mathrm{smooth}} \to \mathbb{G}(d-1, g-1) \times \mathbb{G}(d-1, g-1).$$

Define $\mathrm{E}(d, g) \hookrightarrow W^d_{\mathrm{smooth}} \times W^d_{\mathrm{smooth}}$ to be the fibre over $\mathrm{F}(d, g)$. We take U_{E} to be the preimage of U_{F} and $\mathrm{E}(d, g)_{\mathrm{main}}$ to be the closure of U_{E}.There is morphism

$$\beta\colon U_{\mathrm{E}} \longrightarrow (\mathbb{P}^{g-1})^*.$$

We have, by theorem (2.2),

$$\beta(u(D), u(D')) = \overline{\phi_K(D) \cup \phi_K(D')},$$

where $(D, D') \in C^{(d)} \times C^{(d)}$ are divisors whose image under the Abel-Jacobi map is in W^d_{smooth}. Recall that $\overline{A}$ means linear span of some subset A of $\mathbb{P}^{g-1}$ in $\mathbb{P}^{g-1}$. Notice that $\overline{\phi_K(D) \cup \phi_K(D')}$ is a hyperplane in $\mathbb{P}^{g-1}$, for the condition $(u(D), u(D')) \in \mathrm{E}(d, g)$ forces $\overline{\phi_K(D) \cup \phi_K(D')}$ to be contained in a hyperplane and the condition $(u(D), u(D')) \in U_{\mathrm{E}}$ forces $\overline{\phi_K(D) \cup \phi_K(D')}$ to be exactly a hyperplane.

A generic hyperplane $H \in (\mathbb{P}^{g-1})^*$ intersects C in $2g-2$ points that are in general position , see [2]. So suppose that $H.C = p_1 + p_2 + \ldots + p_{2g-2}$. Then by (2.3), the pair

$$(u(p_1 + p_2 + \ldots + p_d), u(p_{d+1} + p_{d+2} + \ldots + p_{2d})),$$

(notice $2d < 2g-2$) is a closed point of $W^d_{\text{smooth}} \times W^d_{\text{smooth}}$. Furthermore the above pair, gives a point in U_E mapping to H under β. Hence β is dominant. Since a hyperplane can only intersect C in a finite number of points, the map β is quasi-finite. It follows that U_E has dimension $g-1$.

We let C^* denote the dual variety to $\phi_K(C)$.

(6.1). Lemma. *(a) Suppose that C is a non-hyperelliptic curve. Let $H \in (\mathbb{P}^{g-1})^* - C^*$. If $\beta((u(D), u(D'))) = H$ then $(u(D), u(D'))$ lies on a component of* $\mathrm{E}(d, g)$ *of dimension $g-1$ and is in the smooth locus of* $\mathrm{E}(d, g)_{\text{main}}$.
(b) Suppose that C is hyperelliptic. Let $H \in (\mathbb{P}^{g-1})^* - C^*$ and assume also that H does not pass through any of the branch points of the canonical map $\phi_K \colon C \to \mathbb{P}^{g-1}$. If $\beta((u(D), u(D'))) = H$ then $(u(D), u(D'))$ lies on a component of $\mathrm{E}(d, g)$ *of dimension $g-1$ and is in the smooth locus of* $\mathrm{E}(d, g)_{\text{main}}$.

Proof. The following proof is for (a).

Write $D = p_1 + p_2 + \ldots + p_d$ and $D' = p'_1 + p'_2 + \ldots + p'_d$. We choose local coordinates z_i and z'_i on C centred at p_i and p'_i respectively. Now as H is not a tangent hyperplane C, we have $p_i \neq p_j$ and $p'_i \neq p'_j$ for $i \neq j$. It follows that $z_1, z_2 \ldots, z_d$ and $z'_1, z'_2 \ldots, z'_d$ descend to local co-ordinates on $C^{(d)} \times C^{(d)}$ centred at (D, D'). Furthermore, by (2.1), the Abel-Jacobi map is an isomorphism around (D, D'), since $u(D), u(D') \in W^d_{\text{smooth}}$. So we have some local co-ordinates on $W^d \times W^d$ centred at $(u(D), u(D'))$. Let $\omega_1, \ldots \omega_g$ be a basis for $\mathrm{H}^0(\Omega^1_C)$. We write ω_j as $\Omega_{ji}(z_i)dz_i$ in a neighbourhood of p_i and as $\Omega'_{ji}(z'_j)dz'_j$ in a neighbourhood of p'_j. Let

$$M(z) = \begin{pmatrix} \Omega_{11}(z_1) & \ldots & \Omega_{1d}(z_d) & \Omega'_{11}(z'_1) & \ldots & \Omega'_{1d}(z'_d) \\ \Omega_{21}(z_1) & \ldots & \Omega_{2d}(z_d) & \Omega'_{21}(z'_2) & \ldots & \Omega'_{2d}(z'_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ \Omega_{g1}(z_1) & \ldots & \Omega_{gd}(z_d) & \Omega'_{g1}(z'_1) & \ldots & \Omega'_{gd}(z'_d) \end{pmatrix}.$$

In a neighbourhood of $(u(D), u(D'))$, $\mathrm{E}(d, g)$ is defined by the vanishing of the $g \times g$ minors of $M(z)$, by (4.3). Now by (2.3), $\dim \overline{\phi_K(D)} = d-1$, so in a neighbourhood of $(u(D), u(D'))$ the first d columns of $M(z)$ are linearly independent. Since $M(Z)$ has rank $g-1$ at the point $(u(D), u(D'))$ we may reindex the points of D' so that the first

$g-1$ columns of $M(z)$ are linearly independent in a neighbourhood of $(u(D), u(D'))$. Set

$$f_i = \det M(z)_{(1,2,\dots,g-1,i)},$$

where $g-1 < i \leq 2d$. By (4.1), $\mathrm{E}(d,g)$ is defined by f_i in a neighbourhood of $(u(D), u(D'))$. The assertion that $(u(D), u(D'))$ lies on a component of dimension $g-1$ of $\mathrm{E}(d,g)$ follows.

By definition, f_j is independent of the co-ordinates z'_i for $g-d \leq i \leq d$ and $i \neq j-d$. So the Jacobian matrix is of the form

$$\left.\begin{pmatrix}
\frac{\partial f_g}{\partial z_1} & \frac{\partial f_{g+1}}{\partial z_1} & \cdots & \frac{\partial f_{2d}}{\partial z_1} \\
\frac{\partial f_g}{\partial z_2} & \frac{\partial f_{g+1}}{\partial z_2} & \cdots & \frac{\partial f_{2d}}{\partial z_2} \\
\vdots & \vdots & & \vdots \\
\frac{\partial f_g}{\partial z_d} & \frac{\partial f_{g+1}}{\partial z_d} & \cdots & \frac{\partial f_{2d}}{\partial z_d} \\
\frac{\partial f_g}{\partial z'_1} & \frac{\partial f_{g+1}}{\partial z'_1} & \cdots & \frac{\partial f_{2d}}{\partial z'_1} \\
\frac{\partial f_g}{\partial z'_2} & \frac{\partial f_{g+1}}{\partial z'_2} & \cdots & \frac{\partial f_{2d}}{\partial z'_2} \\
\vdots & \vdots & & \vdots \\
\frac{\partial f_g}{\partial z'_{g-d-1}} & \frac{\partial f_{g+1}}{\partial z'_{g-d-1}} & \cdots & \frac{\partial f_{2d}}{\partial z'_{g-d-1}} \\
\frac{\partial f_g}{\partial z'_{g-d}} & 0 & \cdots & 0 \\
0 & \frac{\partial f_{g+1}}{\partial z'_{g-d+1}} & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & \frac{\partial f_{2d}}{\partial z'_d}
\end{pmatrix}\right|_{(u(D),u(D'))}.$$

Suppose that $(u(D), u(D'))$ is a singular point of $\mathrm{E}(d,g)$. This is true if and only if the above matrix has rank smaller than $2d-g+1$. It has rank smaller than $2d-g+1$ if and only if

$$\left.\frac{\partial f_j}{\partial z'_j}\right|_{(u(D),u(D'))} = 0$$

for some j. Now

$$
\begin{aligned}
0 &= \left.\frac{\partial f_j}{\partial z'_j}\right|_{(u(D),u(D'))} \\
&= \begin{vmatrix}
\Omega_{11}(p_1) & \cdots & \Omega_{1d}(p_d) & \Omega'_{11}(p'_1) & \cdots & \Omega'_{1,g-1-d}(p'_{g-1-d}) & \left.\frac{\partial \Omega'_{1j}}{\partial z'_j}\right|_{p_j} \\
\Omega_{21}(p_1) & \cdots & \Omega_{2d}(p_d) & \Omega'_{21}(p'_1) & \cdots & \Omega'_{2,g-1-d}(p'_{g-1-d}) & \left.\frac{\partial \Omega'_{2j}}{\partial z'_j}\right|_{p_j} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
\Omega_{g1}(p_1) & \cdots & \Omega_{gd}(p_d) & \Omega'_{g1}(p'_1) & \cdots & \Omega'_{g,g-1-d}(p'_{g-1-d}) & \left.\frac{\partial \Omega'_{gj}}{\partial z'_j}\right|_{p_j}
\end{vmatrix}
\end{aligned}
$$

The first $g-1$ columns lie inside H. So it follows that the last column is contained in H. This implies the tangent line to p'_j is in H, which in turn contradicts $H \notin C^*$.

A similar argument proves (b). $\square$

§7 Generic Tangent Hyperplanes

Let C be a curve with a fixed non-degenerate embedding $\phi\colon C \hookrightarrow \mathbb{P}^n$, with $n \geq 3$. Recall that all curves are assumed to be smooth and projective. The genus of our curve will also be assume to be ≥ 4. We will denote by C^* the dual variety to C inside $(\mathbb{P}^n)^*$. By forming the incidence correspondence

$$\Sigma = \{(p, H) \mid p \in C,\ H \in (\mathbb{P}^n)^*,\ \mathrm{T}_p(C) \subseteq H\}$$

and using standard arguments we see that C^* is an irreducible hypersurface in $(\mathbb{P}^n)^*$. We use the notation $\mathrm{T}_p(C)$ to denote the tangent line to C at p inside $\mathbb{P}^n$.

Let $\phi_2\colon C \to \mathbb{G}(2, n)$ be the second associated curve to ϕ. So $\phi_2(p)$ is the unique plane having intersection order at least 3 with C at p. (See [4], pg. 263). Let $\Gamma_2 \subseteq C \times \mathbb{G}(2, n)$ be the graph of ϕ_2. We form the incidence correspondence

$$\Sigma'' = \{(p, P, H) \in \Gamma_2 \times (\mathbb{P}^n)^* \mid (p, P) \in \Gamma_2,\ \text{and } P \subseteq H\}.$$

Let $\pi_C\colon$ be the projection $\pi_C\colon \Sigma'' \to C$. The fibre over $p \in C$ is irreducible of dimension $n-3$. It follows that Σ'' is irreducible of dimension $n-2$. The projection from Σ'' to C^* is a finite morphism, hence the locus of hyperplanes having intersection at least 3 at some point of C is an irreducible closed subvariety of codimension 1 inside C^*.

(7.1). Lemma. *Let $\phi_K : C \to \mathbb{P}^{g-1}$ be the canonical morphism.*
(a) Suppose that C is a non-hyperelliptic curve so that ϕ_K is an immersion. Then for a generic $H \in C^* \subseteq (\mathbb{P}^{g-1})^*$,

$$H.C = 2p_1 + p_2 + p_3 + \ldots + p_{2g-3}$$

where the p_i are distinct.
(b) Suppose that C is a hyperelliptic curve so that $\phi_K(C)$ is a rational normal curve . Let C^* be the dual variety to $\phi_K(C)$. Let $b_1, \ldots, b_{2g+2}$ be the branch points of ϕ_K. We denote by $b_i^* \subseteq (\mathbb{P}^{g-1})^*$ the dual variety to b_i, consisting of all hyperplanes through b_i. So b_i^* is a hyperplane in $(\mathbb{P}^{g-1})^*$. Then for a generic

$$H \in C^* \cup b_1^* \cup \ldots \cup b_{2g+2}^*$$

we have that

$$H.C = 2p_1 + p_2 + p_3 + \ldots + p_{2g-3}$$

where the p_i are distinct.

Proof. (a) We have seen, in the discussion preceeding the lemma, that for a generic $H \in C^*$, $H.C$ has no points of multiplicity 3. So we need to show that a generic tangent hyperplane has only one point of multiplicity 2. Form the incidence correspondence

$$\Sigma' = \{(p, q, P, H) \in C \times C \times \mathbb{G}(3, g-1) \times (\mathbb{P}^{g-1})^* \mid p \neq q \\ \mathrm{T}_p(C), \mathrm{T}_q(C) \subseteq P \subseteq H\}.$$

Note that Σ' is only locally closed in $C \times C \times \mathbb{G}(3, g-1) \times (\mathbb{P}^{g-1})^*$. Let

$$\Sigma' = \Sigma'_1 \cup \Sigma'_2 \cup \ldots \cup \Sigma'_l$$

be an irreducible decomposition for Σ'. There is a projection $\Sigma' \to C \times C$. From [6] IV Theorem (3.10) there is a closed subset $X \subseteq C \times C$ such that for each $(p, q) \notin D$, the tangent lines $\mathrm{T}_p(C)$ and $\mathrm{T}_q(C)$ do not meet and X has codimension 1 in $C \times C$. Consider the restricted projection

$$\pi_i : \Sigma'_i \to C \times C.$$

Now if there is a point $(p, q) \notin X$, and in the image of Σ'_i, the fibre over (p, q) has dimension $g - 5$ as $\mathrm{T}_p(C)$ and $\mathrm{T}_q(C)$ span a 3-plane in $\mathbb{P}^{g-1}$. Hence

$$\begin{aligned} \dim \Sigma'_i &\leq \dim C \times C + \dim(\text{fibre}) \\ &= g - 3. \end{aligned}$$

(Note that if $g = 4$, then there is no such (p, q).) If there is no such (p, q) then the projection can be factored as

$$\pi_i : \Sigma'_i \to X.$$

Now the fibre over a point has dimension $g-4$. So as above $\dim\Sigma_i' \leq g-3$. Hence, for the closure $\overline{\Sigma'}$, we have

$$\dim\overline{\Sigma'} \leq g-3.$$

So the image of the projection $\overline{\Sigma'} \to C^*$ has smaller than dimension $g-2$. Since C^* is a hypersurface, the result follows.
(b) First consider $H \in C^*$.

By the remark proceeding the lemma, it suffices to show that for a generic $H \in C^*$, $H.C$ has only one point of multiplicity two and H does not pass through one of the b_i. The first assertion follows as in (a). For the second assertion notice that $C^*, b_1^*, \ldots, b_{2g+2}^*$ are distinct hypersurfaces in $(\mathbb{P}^{g-1})^*$. The result follows.

This last remark also deals with the case $H \in {b_i}^*$. □

§8 Proof of the Generalized Torelli Theorem

We wish to prove

(8.1). Theorem. *Let C be a smooth projective curve over $\mathbb{C}$ of genus $g \geq 1$. If $1 \leq d \leq g-1$ is an integer then the pair $(J(C), W^d)$ determine the curve, that is if $(J(C), W^d(C)) \cong (J(C'), W^d(C'))$ for some other smooth projective curve C' then $C' \cong C$.*

Proof. We may assume $g \geq 4$ as the cases $g = 1, 2, 3$ are covered by the regular Torelli theorem. Furthermore we may reduce to the case $(g-1)/2 < d < g-1$ as follows. If $d = g-1$ we are done by Torelli's theorem. If $d < (g-1)/2$ then choose n so that $(g-1)/2 < nd \leq g-1$. Now we may replace W^d by

$$W^{nd} = \underbrace{W^d + W^d + \ldots + W^d}_{n \text{ times}}.$$

We will study the branch locus of the map

$$\beta \colon \mathrm{E}(d,g)_{\mathrm{main}} \dashrightarrow (\mathbb{P}^{g-1})^*.$$

Note that we can recover the rational map β from the information $(J(C), W^d)$. Now let $U_{\mathrm{E}} \subseteq \mathrm{E}(d,g)$ be the open subset defined at the start of §5. We have a morphism $\beta|_{U_{\mathrm{E}}} \colon U_{\mathrm{E}} \to (\mathbb{P}^{g-1})^*$. Let B be the branch locus of β. This is the image of the ramification locus inside $(\mathbb{P}^{g-1})^*$. A closed point p is in the ramification locus if and only if β fails to be a local analytic isomorphism at p. At this point we break the proof into two cases, the case where C is non-hyperelliptic and the case where C is hyperelliptic.

First we study the case where C is non-hyperelliptic. We will show that $\bar{B} = C^*$. Then C can be recovered from this information, see [5].

First we show that $\bar{B} \subseteq C^*$. Let $H \notin C^*$. Then $H.C = p_1 + p_2 + \ldots + p_{2g-2}$ with the p_i distinct. Let $T \subseteq (\mathbb{P}^{g-1})^*$ be all the hyperplanes having transverse intersection with C, that is $T = (\mathbb{P}^{g-1})^* - C^*$. The incidence correspondece

$$I = \{(p, H) \in C \times T \mid p \in \text{Supp } H.C\} \to T$$

is a $(2g-2)$-sheeted covering space of T,[2] pg.110. Given $(u(D), u(D')) \in U_E$ with $\beta((u(D), u(D'))) = H \in T$. It is claimed that there exists an open neighbourhood V in the usual topology such that

$$\beta|_V : V \to \beta(V)$$

is an injection. To see this, first take $H \in W \subseteq T$, with sheets $W_1, W_2, \ldots, W_{2g-2}$. Let μ_i be the compostion $W \to W_i \to C$, which is holomorphic. Write $D = p_1 + \ldots + p_d$. The p_i are distinct by choice of H, so we may find opens $p_i \in U_i$ such that

(1) $U_i \cap U_j =$ for $i \neq j$

(2) $U_i \subseteq \mu_j(W)$ for some j.

Writing $D' = p'_1 + \ldots + p'_d$ we may find similar opens U'_i. Set $U = U_1 \times \ldots \times U_d$, $U' = U'_1 \times \ldots \times U'_d$. By condition (1), $U \times U'$ is an open neighbourhood of $(p_1 + \ldots + p_d, p'_1 + \ldots + p'_d)$ on $C^{(d)} \times C^{(d)}$. As the Abel-Jaacobi map is an isomorphism near $(p_1 + \ldots + p_d, p'_1 + \ldots + p'_d)$, as $(u(D), u(D')) \in W^d d_{\text{smooth}} \times W^d d_{\text{smooth}}$. We take $V = \beta^{-1} \cap (U \times U') \cap U_E$. It is easy to see that this works.

It follows from theorem 7.6, of [3], that β is a local isomorphism at $(u(D), u(D'))$ since this point is in the smooth locus of $\mathrm{E}(d, g)$ by lemma (6.1). It remains to show that B contains an open dense subset of C^*.

By (7.1) there exists an open subset $V \subseteq C^*$ such that for each $H \in V$,

$$H.C = 2p_1 + p_2 + \ldots + p_{2g-3},$$

with the $p_1, \ldots, p_{2g-3}$ are distinct. Since $g \neq 0$ and

$$K \sim 2p_1 + p_2 + \ldots + p_{2g-3},$$

we have that $H = \overline{\phi_K(p_1 + p_2 + \ldots + p_{2g-3})}$. (Notice that there is no 2 in front of p_1 in the last statement.) After reindexing we may assume that $p_1, p_2, \ldots, p_{g-1}$ span H and the tangent line at p_1 to C lies inside H. Let

$$D = q_1 + q_2 + \ldots + q_d \quad \text{and} \quad D' = q'_1 + q'_2 + \ldots + q'_d$$

where $q_i = p_i$ for $1 \leq i \leq d$ and $q'_i = p_{g-i}$ for $1 \leq i \leq d$. So $(u(D), u(D')) \in U_{\mathrm{E}}$. Let z_i (resp. z'_i) be local coordinates centred at q_i (resp. q'_i). Since $q_i \neq q_j$ (resp. $q'_i \neq q'_j$) for $i \neq j$, we have local coordinates $(z_1, z_2, \dots, z_d)$ (resp. $(z'_1, z'_2, \dots, z'_d)$ on $C^{(d)}$ centred at $(q_1, q_2, \dots, q_d)$ (resp. $(q'_1, \dots, q'_d)$). As u is an isomorphism around D (resp. D'), by (2.1) and (2.3) and as $\dim \overline{\phi_K(D)} = d-1$ (resp. $\dim \overline{\phi_K(D')} = d-1$), we have that $((z_1, z_2, \dots z_d), (z'_1, z'_2, \dots z'_d))$ descend to local coordinates on $W^d \times W^d$ centred at $(u(D), u(D'))$.

Choose a basis $\omega_1, \dots, \omega_g$ for $\mathrm{H}^0(C, \Omega^1_C)$ and write $\omega_i = \Omega_{ij}(z_j)dz_j$ (resp. $\omega_i = \Omega'_{ij}(z'_j)dz'_j$). Let

$$M(z) = \begin{pmatrix} \Omega_{11}(z_1) & \Omega_{12}(z_2) & \dots & \Omega_{1,d}(z_d) & \Omega'_{11}(z'_1) & \Omega'_{12}(z'_2) & \dots & \Omega'_{1,d}(z'_d) \\ \Omega_{21}(z_1) & \Omega_{22}(z_2) & \dots & \Omega_{2,d}(z_d) & \Omega'_{21}(z'_1) & \Omega'_{22}(z'_2) & \dots & \Omega'_{2,d}(z'_2) \\ \dots & \dots & & \dots & \dots & \dots & & \dots \\ \Omega_{g1}(z_1) & \Omega_{g2}(z_2) & \dots & \Omega_{g,d}(z_d) & \Omega'_{g1}(z'_1) & \Omega'_{g2}(z'_2) & \dots & \Omega'_{g,d}(z'_d) \end{pmatrix}$$

and let

$$M'(z) = \begin{pmatrix} \frac{\partial \Omega_{11}(z_1)}{\partial z_1} & \Omega_{12}(z_2) & \dots & \Omega_{1,d}(z_d) & \Omega'_{11}(z'_1) & \Omega'_{12}(z'_2) & \dots & \Omega'_{1,d}(z'_d)z_1 \\ \frac{\partial \Omega_{21}(z_1)}{\partial z_1} & \Omega_{22}(z_2) & \dots & \Omega_{2,d}(z_d) & \Omega'_{21}(z'_1) & \Omega'_{22}(z'_2) & \dots & \Omega'_{2,d}(z'_2) \\ \dots & \dots & & \dots & \dots & \dots & & \dots \\ \frac{\partial \Omega_{g1}(z_1)}{\partial z_1} & \Omega_{g2}(z_2) & \dots & \Omega_{g,d}(z_d) & \Omega'_{g1}(z'_1) & \Omega'_{g2}(z'_2) & \dots & \Omega'_{g,d}(z'_d) \end{pmatrix}$$

By definition of D and D' the first $g-1$ columns of $M(z)$ are linearly independent in a neighbourhood of $(u(D), u(D'))$. So $\mathrm{E}(d, g)$ is defined by

$$f_i = \det(M(z)_{1,2,\dots,g-1,i}),$$

where $g \le i \le 2d$, in a neighbourhood of $(u(D), u(D'))$. (To see this, use (4.1) as in (6.1)) Now since the tangent line to C at p_1 is inside H we have

$$\frac{\partial f_i}{\partial z_1} = \det(M'(z)_{1,2,\dots,g-1,i})|_{(u(D),u(D'))} = 0.$$

So the Jacobian matrix, as in the proof of (6.1), reduces to

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \frac{\partial f_g}{\partial z_2} & \frac{\partial f_{g+1}}{\partial z_2} & \cdots & \frac{\partial f_{2d}}{\partial z_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_g}{\partial z_d} & \frac{\partial f_{g+1}}{\partial z_d} & \cdots & \frac{\partial f_{2d}}{\partial z_d} \\ \frac{\partial f_g}{\partial z'_1} & \frac{\partial f_{g+1}}{\partial z'_1} & \cdots & \frac{\partial f_{2d}}{\partial z'_1} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_g}{\partial z'_{g-1-d}} & \frac{\partial f_{g+1}}{\partial z'_{g-1-d}} & \cdots & \frac{\partial f_{2d}}{\partial z'_{g-1-d}} \\ \frac{\partial f_g}{\partial z'_{g-d}} & 0 & \cdots & 0 \\ 0 & \frac{\partial f_{g+1}}{\partial z'_{g-d+1}} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \frac{\partial f_2 d}{\partial z'_d} \end{pmatrix}$$

Arguing as in (6.1) we find that $(u(D), u(D'))$ is a smooth point of $\mathrm{E}(d, g)$. We also see that $\frac{\partial}{\partial z_1}|_{(u(D),u(D'))}$ is in the null space of the above Jacobian. Hence $\frac{\partial}{\partial z_1}|_{(u(D),u(D'))}$ is in fact a tangential to $\mathrm{E}(d, g)$ at $(u(D), u(D'))$. In order to show that $H \in B$ it will suffice to show that $\frac{\partial}{\partial z_1}|_{(u(D),u(D'))}$ maps to zero under the morphism of tangent space induced by β. Let

$$N(z) = \begin{pmatrix} \Omega_{11}(z_1) & \cdots & \Omega_{1d}(z_d) & \Omega'_{11}(z'_1) & \cdots & \Omega_{1,g-1-d}(z_{g-1-d}) \\ \vdots & & \vdots & \vdots & & \vdots \\ \Omega_{g1}(z_1) & \cdots & \Omega_{gd}(z_d) & \Omega'_{g1}(z'_1) & \cdots & \Omega_{g,g-1-d}(z_{g-1-d}) \end{pmatrix}.$$

So $N(z)$ is just the first $g-1$ columns of $M(z)$. In a neighbourhood of $(u(D), u(D'))$ the morphism $\beta\colon U \to (\mathbb{P}^{g-1})^*$ is given by $z \mapsto$ col. space$N(z)$. Identify $(\mathbb{P}^{g-1})^* \cong \mathbb{P}(\bigwedge^{g-1}\mathbb{C}^g)$ we see that β is the morphism

$$z \mapsto [\det(N(z)_1) : \det(N(z)_2) : \ldots : \det(N(z)_g)].$$

Recall that $N(z)_i$ is the submatrix of $N(z)$ obtained by deleting the ith row. We may assume that $\det(N(z)_1) \neq 0$. So we need to show that

$$\frac{\partial}{\partial z_1}|_{(u(D),u(D'))}(\frac{\det(N(z)_i)}{\det(N(z)_1)} = 0.$$

That is

$$\frac{\partial \det(N(z)_1)}{\partial z_1}.\det(N(z)_i) = \frac{\partial \det(N(z)_i}{\partial z_1}.\det(N(z)_1)$$

after evaluation at $(u(D), u(D'))$. Let

$$\frac{\partial N(z)}{\partial z_1}$$

be the matrix obtained from $N(z)$ by differentiating the first column with respect z_1. Observe that

$$\text{col. space}\frac{\partial N(z)}{\partial z_1}|_{(u(D),u(D'))} \subseteq \text{col. space}N(z)|_{(u(D),u(D'))}$$

as the tangent line at p_1 lies inside H. It is a general fact from linear algebra that given two $g \times (g-1)$ matrices M, N with col. space$M \subseteq$ col. space$N$ then for each i j in the range $1 \leq i, j \leq g$ we have

$$\det(M_i)\det(N_j) = \det(M_j)\det(N_i).$$

We will include the proof of this statement at the end of this proof for completeness. This shows that $\bar{B} = C^*$.

Now we treat the case that C is a hyperelliptic curve. We show that $\bar{B} = C^* \cup b_1^* \cup b_2^* \cup \ldots \cup b_{2g+2}^*$ where the b_i are the branch points of the canonical morphism $\phi_K \colon C \to (\mathbb{P}^{g-1})^*$. The proof is almost identical to the above. Here are a few details. The same proof as in the non-hyperelliptic case shows that $\bar{B} \subseteq C^* \cup b_1^* \cup b_2^* \cup \ldots \cup b_{2g+2}^*$, and similarly we show that $\bar{B} \supseteq C^*$. To show that $\bar{B} \supseteq b_i^*$ proceed as follows. From (7.1) we know that for a generic $H \in b_i^*$ that

$$H.C = 2p_1 + p_2 + \ldots + p_{2g-3}$$

where the p_i are distinct and $p_1 = b_1$. As above we form, after appropriate reindexing,

$$D = q_1 + q_2 + \ldots + q_d \quad \text{and} \quad D' = q_1' + q_2' + \ldots + q_d'.$$

Note, these two divisors are defined exactly as they were before. Also define, as before, z_i, z_i', $M(z)$, $M'(z)$ and f_i. To see that

$$\frac{\partial f_i}{\partial z_1}|_{(u(D),u(D'))} = 0,$$

first observe that since p_1 is a branch point, $J(\phi_K)|_{q_1} = 0$. Around q_1,

$$\phi_K = [\Omega_{11}(z_1) : \ldots : \Omega_{g1}(z_1)].$$

We may assume that $\Omega_{11}(z_1) \neq 0$. Since the Jacobian at q_1 vanishes we see that

$$\Omega_{11}(q_1)\frac{\partial \Omega_{1j}(q_1)}{\partial z_1}|_{q_1} = \frac{\partial \Omega_{11}(z_1)}{\partial z_1}|_{q_1}\Omega_{1j}(q_1),$$

which in turn implies

$$\phi_K(q_1) = [\Omega_{11}(q_1) : \ldots : \Omega_{g1}(q_1)] = [\frac{\partial \Omega_{11}}{\partial z_1} : \ldots : \frac{\partial \Omega_{g1}}{\partial z_1}] \mid_{q_1} .$$

So

$$\frac{\partial f_i}{\partial z_1}|_{(u(D),u(D'))} = f_i|_{(u(D),u(D'))} = 0.$$

Now proceed as before.

□

Here is the linear algebra result that was needed before.

(8.2). Lemma. *Let M, N be two $g \times (g-1)$ matrices over $\mathbb{C}$. If*

$$\text{col.space}M \subseteq \text{col.spaceN}$$

then

$$\det M_i \det N_j = \det M_j \det N_i, \tag{1}$$

for each i, j with $1 \leq i, j \leq g$. Recall that M_i is the submatrix of M obtained by deleting the ith row.

Proof. Firstly if rank$M < g-1$ then both sides of (1) vanish. So we may assume M, N are of maximal rank and that there column spaces are equal. So $N = M.H$ for some $H \in \text{Gl}(g-1, \mathbb{C})$. The result follows from the observation $(M.H)_i = M_i.H$. □

(8.3). Corollary. Let C and C' be two smooth projective curves and let d be an integer less than or equal to the genus of C. If $C^{(d)} \cong C'^{(d)}$ then $C \cong C'$.

Proof. This is because the Albanese varaiety Alb$(C^{(d)})$ is isomorphic to $J(C)$ and the image of $C^{(d)}$ under the Albanese map is W^d. □

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