Transversality theory, cobordisms, and invariants of symplectic quotients

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Introduction

Symplectic quotients and their invariants

This paper gives methods for understanding invariants of symplectic quotients. The symplectic quotients that we consider are compact symplectic manifolds (or more generally orbifolds), which arise as the symplectic quotients of a symplectic manifold by a compact torus. A companion paper [23] examines symplectic quotients by a nonabelian group, showing how to reduce to the maximal torus.

Throughout this paper we assume X is a symplectic manifold, and that a compact torus $T \cong S^1 \times \ldots \times S^1$ acts on X, preserving the symplectic form, and having moment map $\mu : X \to \mathfrak{t}^*$, where $\mathfrak{t}^*$ denotes the dual of the Lie algebra of T. We assume that μ is a proper map. (For definitions and our sign conventions see the notation section at the end of this introduction).

For every regular value $p \in \mathfrak{t}^*$ of the moment map, the inverse image $\mu^{-1}(p)$ is a compact submanifold of X which is stable under T, and on which the T-action is locally free (that is, every point in $\mu^{-1}(p)$ has finite stabilizer subgroup). The *symplectic quotient*, which we denote $X/\!\!/T(p)$, is defined by taking the topological quotient by T

$$X/\!\!/T(p) := \frac{\mu^{-1}(p)}{T},$$

and is a compact orbifold (it is a manifold if the stabilizer subgroup is the same for every point in $\mu^{-1}(p)$). Moreover the symplectic form on X defines in a natural way a symplectic form on $X/\!\!/T(p)$.

Many celebrated theorems in this field relate invariants of the triple (X, T, μ) to invariants of the quotients $X/\!\!/T(p)$. For example, the **Duistermaat-Heckman theorem** [8] relates a certain oscillatory integral over X to the volumes of the symplectic quotients $X/\!\!/T(p)$. Another example is the **Guillemin-Sternberg quantization theorem** [12], which relates the 'geometric quantization' of X to that of its symplectic quotients[1]. A third example is the **Atiyah-Guillemin-Sternberg convexity theorem**, which relates a very simple invariant of (X, T, μ), namely the convex hull of the finite set of points $\mu(X^T)$, to an even simpler invariant of $X/\!\!/T(p)$, namely whether it is empty. One common feature of these results is that the relevant invariants of (X, T, μ) can be calculated in terms of data localized at the T-fixed points $X^T \subset X$.

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[1] the geometric quantization is the index of a certain naturally-defined Dirac operator; in the case of a Kähler manifold this equals the space of holomorphic sections of a certain holomorphic line bundle

The scope of this paper

This paper provides results concerning a larger class of invariants, including the integrals of arbitrary cohomology classes (thus generalizing the volume) and the indexes of arbitrary elliptic differential operators (generalizing the geometric quantization). In order to describe this class of invariants, we first note that any invariant of $X/\!\!/T(p)$ is also an invariant of the pair $(\mu^{-1}(p), T)$ (the converse is of course not true). The easiest way to describe the results of this paper is in terms of the submanifolds $\mu^{-1}(p)$, for p any regular value of μ.

The submanifold $\mu^{-1}(p)$ defines an equivalence class $[\mu^{-1}(p)]$, defined in terms of certain equivariant cobordisms, and the invariants accessible by the methods of this paper are those invariants that only depend on the class $[\mu^{-1}(p)]$.

Explicitly, let $X' \subset X$ denote the subset consisting of those points whose stabilizer subgroup is finite. Then the submanifold $\mu^{-1}(p) \subset X'$ defines the cobordism class

$$[\mu^{-1}(p)] \in \mathcal{U}_T^*(X'),$$

where representatives of $\mathcal{U}_T^*(X')$ are given by T-equivariant maps of oriented manifolds to X', and equivalences are given by the boundaries of T-equivariant maps of oriented manifolds-with-boundary. Explicitly, if $W \hookrightarrow X'$ is any oriented manifold-with-boundary mapped T-equivariantly to X', then $[\partial W \hookrightarrow X'] = 0 \in \mathcal{U}_T^*(X')$. Note that since X' has a locally free T-action, every manifold and cobordism must also have a locally free T-action.

An example of an invariant that only depends on the class $[\mu^{-1}(p)]$ is described in terms of the natural ring homomorphism

$$\kappa : \mathrm{H}_T^*(X; \mathbb{Q}) \twoheadrightarrow \mathrm{H}_T^*(X/\!\!/T(p); \mathbb{Q})$$

defined by restriction, followed by the natural identification of the equivariant cohomology of $\mu^{-1}(p)$ with the regular cohomology of its quotient. This map is often referred to as the 'Kirwan map', and is known to be surjective [21]. Given classes $a, b \in \mathrm{H}_T^*(X)$, then Stokes's theorem implies that the 'cohomology pairing'

$$\begin{aligned} \mathrm{H}_T^*(X) \otimes \mathrm{H}_T^*(X) &\to \mathbb{Q} \\ a, b &\mapsto \int_{X/\!\!/T(p)} \kappa(a) \smile \kappa(b) \end{aligned}$$

is an invariant of the equivalence class $[\mu^{-1}(p)]$ (Stokes's theorem is also valid for orbifolds, as we explain in appendix A). A similar map exists in K-theory, and again only depends on the class $[\mu^{-1}(p)]$

The main result of this paper

We now describe the main topological result of this paper: theorem C (which appears in section 8). Theorem C describes a cobordism between $\mu^{-1}(p)$ and a collection of submanifolds of X that lie near the T-fixed points:

Theorem C (Approximate version). *Suppose the fixed point set X^T is finite. Then for every regular value p of the moment map,*

$$[\mu^{-1}(p)] = \sum_{i \in I} [\mathcal{S}(F_i)];$$

where each $F_i \in X^T$ is a fixed point, and $\mathcal{S}(F_i)$ is a d-fold product of odd-dimensional spheres, lying in a small neighbourhood of F_i, with $d = \dim T$.

In general, $\mathcal{S}(F_i)$ is a d-fold fibre product of sphere bundles over a connected component $F_i \subset X^T$ of the fixed point set. Recall that, by definition, the equivalence class $\mu^{-1}(p)$ is defined in terms of submanifolds on which T has a locally free action. The quotient $\mathcal{S}(F_i)/T$ is an orbifold, and can be described as a d-fold 'tower' of weighted projective bundles over F_i.

By describing the submanifolds $\mathcal{S}(F_i)$ explicitly, we can calculate the cohomology pairings described above in terms of data localized at the fixed points. Theorem D carries this out, giving cohomological formulae in terms of characteristic classes.

It is also possible, by applying techniques in K-theory, to derive formulae for the indices of elliptic operators: these formulae will appear in another paper.

Overview of the paper

This paper has four main results, theorems A, B, C, and D. Their logical relationship is as follows (the numbers indicate sections)

$$\begin{array}{ccccc}
 & \textit{Topology} & & \textit{Cohomology} \\[2ex]
\textit{Walls} & \begin{array}{c}\text{Theorem A (1–4)}\\ \text{wall-crossing-cobordism}\end{array} & \Longrightarrow & \begin{array}{c}\text{Theorem B (5–6)}\\ \text{wall-crossing formula}\end{array} \\
 & \Downarrow & & \Downarrow \\
\textit{Fixed points} & \begin{array}{c}\text{Theorem C (7–8)}\\ \text{fixed point cobordism}\end{array} & \Longrightarrow & \begin{array}{c}\text{Theorem D (9)}\\ \text{fixed point formula.}\end{array}
\end{array}$$

Theorem A is the main topological construction in this paper. Theorems A and C each give a cobordism between $\mu^{-1}(p)$ and a collection of 'simpler' spaces: in theorem C each such space is a d-fold fibre product of sphere bundles over a component of X^T, where $d = \dim T$; in theorem A each such space is a sphere bundle over a submanifold of a manifold X^H, where $H \subset T$ is a 1-dimensional subtorus. In fact X^H is a symplectic manifold, with an action of the $(d-1)$-torus T/H, and having a moment map μ'. The submanifold of X^H which appears in theorem A is $\mu'^{-1}(q)$, for q some regular value of μ'. Theorem A forms the inductive step in the proof of theorem C, and the induction is carried out in sections 7 and 8. The main techniques used in the proofs of theorems A and C are transversality theory, and general results in the theory of Lie group actions on manifolds. The symplectic geometry which is used boils down to a single fact, fact 1.1, which is illustrated in figure 1.

Theorems B and D result from applying cohomological techniques to the cobordisms constructed in theorems A and C. Whereas a naive application of Stokes's theorem would result in formulae which were computable in principle, but unwieldy in practise, the real content of theorems B and D is to show how such formulae can be reduced to computable formulae, eventually in terms of only the fixed points of X. This is explained in more detail at the beginning of section 5. In the proofs of theorems B and D, fairly extensive use is made of techniques in equivariant cohomology. We also use various facts about orbifolds, which are explained in appendix A, as well as formulae which calculate integrals over the fibres of weighted projective bundles. These formulae are proved in appendix B, and generalize classical formulae involving Chern classes and Segre classes.

Finally, sections 11 and 12 calculate some explicit examples. In section 11 we study the n-fold product of 2-spheres $(S^2)^n$. This is a symplectic manifold, with a Hamiltonian action

of $SO(3)$, and the symplectic quotient $(S^2)^n /\!\!/ SO(3)(0)$ is a manifold when n is odd. These symplectic quotients have been studied extensively, beginning with Kirwan's determination of the Betti numbers [21, 18, 13]. We use theorem B, together with an integration formula which allows us to reduce from a symplectic quotient by $SO(3)$ to a symplectic quotient by the maximal torus S^1 (proved in a companion paper [23]) to give the following formula for integrals of arbitrary cohomology classes on the symplectic quotient $(S^2)^n /\!\!/ SO(3)(0)$, for n odd:

$$\int_{(S^2)^n /\!\!/ SO(3)(0)} v_1^{l_1} \smile v_2^{l_2} \smile \ldots \smile v_n^{l_n} = -\frac{1}{2}(-1)^{\frac{n-1}{2}} \sum_{\substack{K \subset \{1\ldots n-1\} \\ |K| = \frac{n-1}{2}}} (-1)^{|K \cap \{1\ldots m\}|}$$

where $\sum_i l_i = n-3$ and m is equal to the number of odd l_i, and v_i is the natural degree 2 cohomology class arising from the i-th sphere in the product.

In section 12 we consider the space $(\mathbb{CP}^2)^n$. This has a Hamiltonian action of $SU(3)$, and we calculate the volume of the symplectic quotient $(\mathbb{CP}^2)^n /\!\!/ SU(3)(0)$ (the formula is not very enlightening, but the methods are an application of theorem D).

Relationship to other results

There are a number of relationships between the cohomological formulae proved in this paper (theorems B and D) and results of other authors.

The mathematics in this paper was worked out in 1994, in Oxford and at the Newton Institute in Cambridge. The intervening years have been partly spent trying (possibly unsuccessfully) to understand how to turn raw mathematics into a comprehensible manuscript. However, this is a first attempt at writing mathematics, and so I beg the readers indulgence in judging it.

The nonabelian localization formula of Witten [29] and Jeffrey-Kirwan [14] gives an alternative way of calculating cohomology pairings on symplectic quotients, involving residues when $T = S^1$, and a multidimensional generalization of the residue when $\dim T > 1$. An alternative approach to the Witten-Jeffrey-Kirwan cohomology formula was taken by Guillemin and Kalkman [10], following from earlier independent work of Kalkman [17]. Guillemin and Kalkman use 'symplectic cutting' and 'reduction in stages', but the geometric arguments bear a strong resemblance to some of the arguments of this paper.

Jeffrey and Kirwan used the wall-crossing formula (theorem B in this paper, also the main result in Guillemin-Kalkman [10]), together with results in the companion paper [23] to give a mathematically rigorous proof of Witten's formulae for cohomology pairings on the moduli spaces of stable holomorphic bundles over a Riemann surface described above.

Some independent results on cobordisms of symplectic manifolds have also been announced by Ginzburg, Guillemin and Karshon [9].

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Contents

Notation and conventions

Fixed through the entire paper, are the following:

- X — is a fixed smooth symplectic manifold (with symplectic form ω);
- T — $\cong S^1 \times \ldots \times S^1$ is a compact torus acting smoothly on X, preserving ω;
- $\mathfrak{t}, \mathfrak{t}^*$ — are the Lie algebra of T and its dual, respectively;
- $\mu : X \to \mathfrak{t}^*$ — is a moment map for the T action on X (we will assume throughout that μ is proper).

We will use the following notational conventions:

- $X/\!\!/T(p)$ — $= \mu^{-1}(p)/T$ denotes the 'symplectic quotient of X by T at p';
- X^H — denotes the subset of points fixed by the subgroup $H \subset T$;
- $\mathrm{H}^*(-)$ — will always denote cohomology with rational coefficients;
- $\mathrm{H}^*_G(-)$ — denotes G-equivariant cohomology (rational coefficients) for G a group;
- $\kappa : \mathrm{H}^*_T(X) \to \mathrm{H}^*(X/\!\!/T(p))$ — for p a regular value of the moment map, denotes the natural map given by first restricting to $\mu^{-1}(p)$, and then applying the natural isomorphism $\mathrm{H}^*_T(\mu^{-1}(p)) \cong \mathrm{H}^*(X/\!\!/T(p))$ (the point p will always be clear from the context). κ is often referred to as the Kirwan map.

Sign conventions for the moment map

Different authors use varying sign conventions for the moment map. Ours will be as follows. Given a symplectic manifold (X, ω) with an action of a torus $T \cong S^1 \times \ldots \times S^1$ by symplectomorphisms, let $V : \mathfrak{t} \to \Gamma(TX)$ be the infinitesimal action map, taking an element ξ of the Lie algebra of T to the corresponding vector field $V(\xi)$ on X. Then $\mu : X \to \mathfrak{t}^*$ is a **moment map** if it intertwines the T-action on X and the coadjoint action of T on $\mathfrak{t}^*$ (which is trivial in our case, since T is abelian), and which satisfies

$$\langle d\mu_x(v), \xi\rangle = \omega_x(V(\xi), v), \qquad \forall x \in X, v \in T_xX, \xi \in \mathfrak{t}. \tag{0.1}$$

An almost complex structure $J : TX \to TX$ is **compatible** with ω if

$$g(\cdot,\cdot) := \omega(\cdot, J\cdot) \tag{0.2}$$

defines a Riemannian metric on X (i.e. if g is symmetric and positive-definite).

In the case of $S^1 \subset \mathbb{C}^*$ acting on $\mathbb{C}$ by multiplication, our conventions boil down to the following. Letting $z = x + iy$, and choosing the symplectic form

$$\omega = dx \wedge dy,$$

then the standard complex structure on $\mathbb{C}$ is compatible with ω, and a moment map for the S^1-action is given by

$$\mu(z) = -\frac{1}{2}|z|^2.$$

Finally, we recall the standard orientation of a complex vector space, as defined in algebraic geometry: if $\{e_1, \ldots, e_n\}$ is a complex basis, then

$$\{e_1, ie_1, e_2, ie_2, \ldots e_n, ie_n\} \tag{0.3}$$

is a real oriented basis. Thus, if X is a symplectic manifold and J is a compatible almost complex structure, the orientation induced by J agrees with the orientation given by the top power of the symplectic form.

1. Constructing the wall-crossing-cobordism

This section contains the main construction of the paper: the construction of the 'wall-crossing-cobordism'. The tools needed for this construction comprise one fact from symplectic geometry, and some transversality theory. We begin by stating the fact from symplectic geometry, and illustrating it with a simple example. We then go on to the main construction.

Prelude: the geometry of the moment map

We begin by explaining the key fact from symplectic geometry that we use in this paper: this fact relates submanifolds defined by the group action to submanifolds defined by critical points of the moment map.

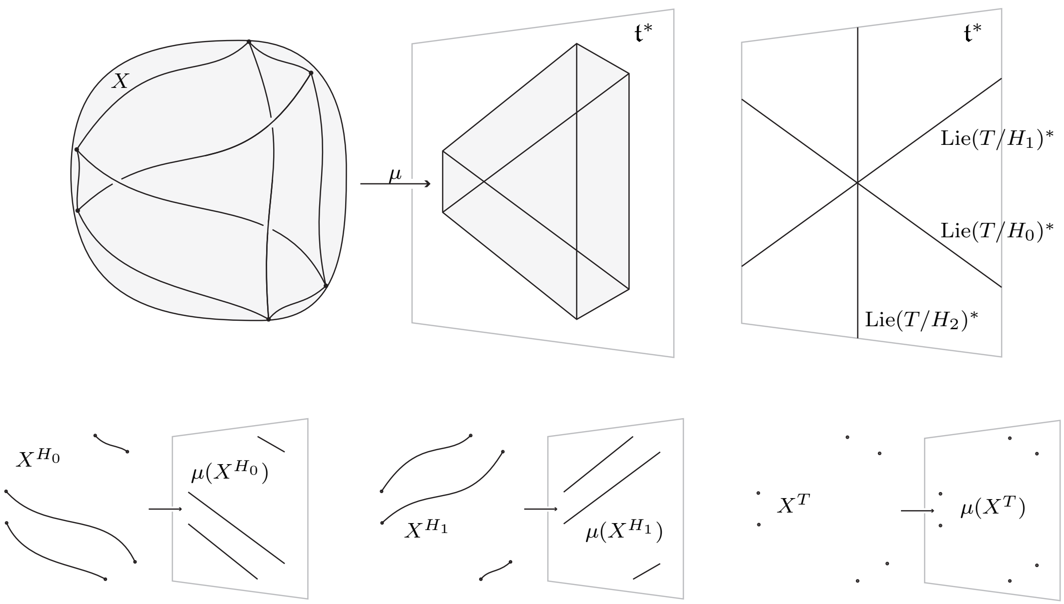


Figure 1: A moment map, and its restriction to various submanifolds: illustrating fact 1.2. Here T is 2-dimensional, and the subgroups H_i are 1-dimensional subtori. The manifold X and its submanifolds are only represented schematically: in the concrete example from which this illustration is derived, X is 6-dimensional, and each component of X^{H_i} (represented by a curved line in X) is a 2-sphere (explained in example 1.3).

Recall that X denotes a symplectic manifold, acted on by a torus $T \cong S^1 \times \ldots \times S^1$, with associated moment map $\mu : X \to \mathfrak{t}^*$, where $\mathfrak{t}^*$ denotes the dual of the Lie algebra of T. Let $\tau \subset T$ be a subtorus. Then the short exact sequence of groups $\tau \hookrightarrow T \twoheadrightarrow T/\tau$ induces the following exact sequences of Lie algebras and their duals

$$\mathrm{Lie}(\tau) \hookrightarrow \mathfrak{t} \twoheadrightarrow \mathrm{Lie}(T/\tau)$$
$$\mathrm{Lie}(\tau)^* \twoheadleftarrow \mathfrak{t}^* \hookleftarrow \mathrm{Lie}(T/\tau)^*.$$

Hence for any subtorus τ we will consider $\mathrm{Lie}(T/\tau)^*$ to be a subspace of $\mathfrak{t}^*$ (of codimension $\dim \tau$).

The key fact concerning the geometry of μ describes the way that the derivative of μ encodes information about the T action. For any point $x \in X$, letting $d\mu : T_xX \to \mathfrak{t}^*$ denote the derivative, we have

Fact 1.1 (Infinitesimal version). *A subtorus $\tau \subset T$ fixes x if and only if*

$$d\mu(T_x X) \subset \mathrm{Lie}(T/\tau)^*.$$

For example, if the action is locally free at x, then $d\mu_x$ must be onto, and hence if $p \in \mathfrak{t}^*$ is a regular value of μ, then the action on T on $\mu^{-1}(p)$ is locally free.

The above fact has a global consequence. If $\tau \subset T$ is a subtorus, then we denote by X^τ the set of points fixed by τ: a local-coordinate argument shows that X^τ is a closed submanifold of X (and an averaging argument shows that X^τ is a symplectic submanifold of X).

Fact 1.2 (Global version). *The moment map μ maps each component of X^τ to an affine translate of* $\mathrm{Lie}(T/\tau)^*$ *in* $\mathfrak{t}^*$*.*

For example, fixing a 1-dimensional subtorus $H \cong S^1$ of T, then μ maps each connected component of X^H to an affine hyperplane in $\mathfrak{t}^*$, parallel to $\mathrm{Lie}(T/H)^*$. The images of such submanifolds X^H, as H varies through all 1-dimensional subtori of T, form 'walls' which separate regions of regular values in $\mu(X)$. At the other extreme, μ maps each connected component of X^T to a point in $\mathfrak{t}^*$.

Example 1.3. Let X be the set of 3×3 Hermitian matrices with eigenvalues $0, 1$ and 4, and let $T \subset SU(3)$ be the maximal torus. Then T acts on X by conjugation, and a moment map for this action is given by sending a matrix to its diagonal entries. Figure 1 illustrates some of the features of the moment map in this case (the image of the moment map is accurate, but the illustration of X is schematic: X is 6-dimensional). The details in this illustration are explained below.

We describe X and T explicitly as follows. Let T be the diagonal matrices in $SU(3)$, that is, $T = \{\mathrm{diag}(e^{i\theta_0}, e^{i\theta_1}, e^{i\theta_2}) \mid \theta_1 + \theta_2 + \theta_3 = 0\}$, and let $t \in T$ act on a matrix $A \in X$ by $A \mapsto tAt^{-1}$. The map which takes $A \in X$ to its diagonal entries (a_{11}, a_{22}, a_{33}) takes values in a 2-dimensional hyperplane in $\mathbb{R}^3$ (since $a_{11} + a_{22} + a_{33} = \mathrm{tr}A = 5$), and this hyperplane can then be identified with $\mathfrak{t}^*$ to give a moment map for the T-action (the symplectic form on X is defined by identifying X with a certain coadjoint orbit[2]).

The set of T-fixed points in X are the diagonal matrices: the diagonal entries must be $0, 1, 4$ in some order, and so there are 6 such matrices. That is, X^T consists of 6 isolated points. These points and their images under μ are depicted in the lower right part of figure 1. The Atiyah-Guillemin-Sternberg convexity theorem states that the image $\mu(X)$ equals the convex hull of the image $\mu(X^T)$ of these points. Note that the example we are considering is atypical, because each point of $\mu(X^T)$ defines a vertex of the polyhedron $\mu(X)$. In general, not every point in $\mu(X^T)$ defines a vertex: some may map to the interior of $\mu(X)$.

Now consider the 1-dimensional subtorus $H_0 := \{\mathrm{diag}(e^{i\theta_0}, e^{-i\theta_0/2}, e^{-i\theta_0/2}\} \subset T$. Then H_0 fixes the 'block-diagonal' matrices of the form

$$\begin{pmatrix} b & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

The entry b must be one of the eigenvalues $0, 1$ or 4, and the remaining 2×2 block has eigenvalues given by the other two. Thus X^{H_0} is made up of three components (each such component turns out to be a 2-sphere). A similar analysis holds for the subtori

[2]The map $A \mapsto iA$ identifies X with an adjoint orbit of $U(3)$; using an invariant inner product to identify $\mathrm{Lie}(U(3)) \cong \mathrm{Lie}(U(3))^*$ then identifies X with a coadjoint orbit, on which there is a natural symplectic form. A moment map for the T-action is then given by the composition $X \hookrightarrow \mathrm{Lie}(U(3))^* \twoheadrightarrow \mathrm{Lie}(T)^*$.

$H_1 := \{\mathrm{diag}(e^{-i\theta_1/2}, e^{i\theta_1}, e^{-i\theta_1/2}\}$ and $H_2 := \{\mathrm{diag}(e^{-i\theta_2/2}, e^{-i\theta_2/2}, e^{i\theta_2}\}$. There are infinitely many 1-dimensional subtori of T: all the others have as their fixed points only the points X^T.

In figure 1 the subspaces $\mathrm{Lie}(T/H_i) \subset \mathfrak{t}^*$ are shown (here $0 \leq i \leq 2$). Since each H_i has dimension 1, these subspaces have *codimension* 1: they are hyperplanes. Each submanifold X^{H_i} has three components, each of which maps to an affine translate of $\mathrm{Lie}(T/H_i)$ (shown for $i = 0, 1$, the picture for $i = 2$ is similar).

The main lemma, and the resulting construction

Definition 1.4. Let p_0 and p_1 be regular values of the moment map $\mu : X \to \mathfrak{t}^*$. A **transverse path** is a one-dimensional submanifold $Z \subset \mathfrak{t}^*$, with boundary $\{p_0, p_1\}$, such that Z is transverse to μ.

It follows from transversality theory that $\mu^{-1}(Z)$ is a submanifold of X, with boundary $\mu^{-1}(p_0) \sqcup \mu^{-1}(p_1)$ (the boundary of Z is a submanifold of $\mathfrak{t}^*$ which is also transverse to μ). The wall-crossing-cobordism, which we define in 1.6, is constructed from the submanifold $\mu^{-1}(Z)$. This construction is made possible by the following result.

Proposition 1.5. *For any $x \in \mu^{-1}(Z)$, the stabilizer subgroup of x is either finite or 1-dimensional. If $H \subset T$ is any subgroup isomorphic to S^1 then the submanifold X^H of points fixed by H is transverse to $\mu^{-1}(Z)$.*

We prove proposition 1.5 below. First, we use this result to define the wall-crossing-cobordism:

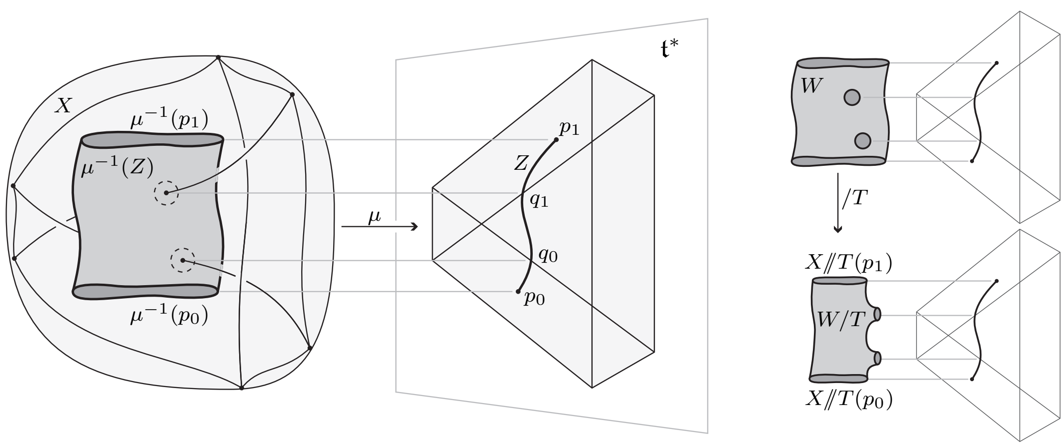


Figure 2: A transverse path Z and the resulting wall-crossing-cobordism W/T. In the diagram on the left, the submanifolds X^{H_i} intersect $\mu^{-1}(Z)$, and the dashed circles indicate open tubular neighbourhoods of these intersections: removing these open neighbourhoods from $\mu^{-1}(Z)$ results in W.

Definition 1.6. Let $W \subset X$ be the manifold-with-boundary given by removing open subsets of $\mu^{-1}(Z)$ as follows. Fix a T-invariant metric on $\mu^{-1}(Z)$, and set

$$W := \mu^{-1}(Z) \setminus \bigsqcup_{H \cong S^1} N_\epsilon(X^H) \cap \mu^{-1}(Z)$$

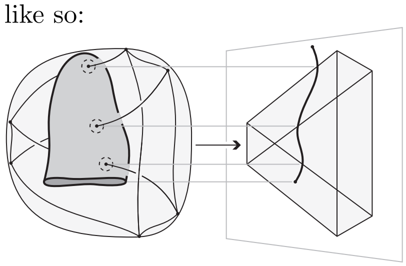
where H runs through all S^1-subgroups of T, and $N_\epsilon(X^H)$ is the open ϵ-tubular neighbourhood of X^H. We choose an ϵ small enough to ensure that these subsets of $\mu^{-1}(Z)$ have

disjoint closures (it follows from proposition 1.5 that this is possible). We define the **wall-crossing-cobordism** to be the quotient orbifold-with-boundary W/T. This procedure is illustrated in figure 1.

Remarks 1.7.

1. Only finitely many subgroups $H \cong S^1$ actually contribute in the above definition. This is because $\mu^{-1}(Z)$ is a compact T-manifold (since the moment map is assumed to be proper), and thus only finitely many subgroups of T can occur as stabilizer subgroups [5, 19].

2. We may choose p_0 or p_1 outside the image of μ, in which case the corresponding boundary component will be empty. For example, moving p_1 to lie outside the image of μ removes the boundary component $\mu^{-1}(p_1)$, but introduces an extra wall-crossing, like so:



A combinatorial characterization of transverse paths, and the proof of Proposition 1.5

Definition 1.8. We define a **wall** in $\mathfrak{t}^*$ to be a connected component of the image of $\mu(X^H)$, for some $H \cong S^1$. We define the **interior** of a wall to be the set of points q in the wall such that every point in $\mu^{-1}(q)$ has stabilizer subgroup which is either 0- or 1-dimensional.

For example, in figure 1 there are 9 walls in total. The arrangement of walls in $\mathfrak{t}^*$ completely characterizes the set of transverse paths:

Lemma 1.9 (Geometry of Z in $\mathfrak{t}^*$). *A path Z is transverse to μ if and only if it intersects each wall transversely in its interior.*

Proof. We must show that, for every $x \in \mu^{-1}(Z)$, the tangent space $T_{\mu(x)}\mathfrak{t}^*$ is spanned by $d\mu(T_xX)$ and $T_{\mu(x)}Z$:

$$T_{\mu(x)}\mathfrak{t}^* = d\mu(T_xX) + T_{\mu(x)}Z. \tag{1.10}$$

We will use the natural identification $T_{\mu(x)}\mathfrak{t}^* \cong \mathfrak{t}^*$. Let $\tau \subset T$ denote the subtorus given by the identity component of the stabilizer subgroup of x: that is, τ is the maximal subtorus which fixes x. Then fact 1.1 implies that $d\mu(T_xX) = \mathrm{Lie}(T/\tau)^*$. Since Z is 1-dimensional, in order for (1.10) to hold τ must be either 0- or 1-dimensional. This immediately implies that every point of Z must be either a regular value of μ or lie in the interior of any wall which it is in. If τ is 0-dimensional then $d\mu(T_xX)$ already spans $\mathfrak{t}^*$. If τ is 1-dimensional, then in order for (1.10) to hold, $T_{\mu(x)}Z$ must be complementary to $\mathrm{Lie}(T/\tau)^*$. Applying fact 1.1, this is the assertion that Z is transverse to the wall $\mu(X^\tau)$ at $\mu(x)$. □

Proof of Proposition 1.5. In the course of proving lemma 1.9, we have already seen that, for every point $x \in \mu^{-1}(Z)$, the stabilizer subgroup of x must be either 0- or 1-dimensional.

The statement that Z is transverse to $\mu(X^H)$ (lemma 1.9) is equivalent to the statement that the composition

$$T_xX^H \xrightarrow{d\mu} T_q\mathfrak{t}^* \to \nu_q Z \tag{1.11}$$

is surjective, for every q in $Z \cap \mu(X^H)$, and for every $x \in \mu^{-1}(q) \cap X^H$. Using the natural identification, via the pullback, of the normal bundles:

$$\mu^* : \nu Z \xrightarrow{\cong} \nu\mu^{-1}(Z),$$

then the composition (1.11) can be factored

$$T_xX^H \hookrightarrow T_xX \to \nu_x\mu^{-1}(Z) \xrightarrow{\cong} \nu_q Z.$$

Since this map is surjective, it follows that the composition $T_xX^H \to \nu_x\mu^{-1}(Z)$ is surjective, for every $q \in Z \cap \mu(X^H)$, and for every $x \in \mu^{-1}(q) \cap X^H$, which gives the result. $\square$

2. The data of a path, and how it describes the boundary of the wall-crossing-cobordism

The data associated to a transverse path

Definition 2.1. Associated to each transverse path $Z \subset \mathfrak{t}^*$ is a finite set $\mathrm{data}(Z)$, which we refer to as the **wall-crossing data for** Z. We define $\mathrm{data}(Z)$ to be the set of pairs (H, q), such that $H \cong S^1$ is an oriented subgroup of T, and $q \in Z \cap \mu(X^H)$. The orientation of H is defined by the direction of the wall-crossing: we orient Z so that the positive direction goes from p_0 to p_1; then a positive tangent vector in T_qZ, thought of as an element of $\mathfrak{t}^*$, defines a linear functional on $\mathfrak{t}$, and this restricts to a nonzero functional on $\mathfrak{h}$; and we orient H to be positive with respect to this functional.

Remarks 2.2.

1. We may also apply the above definition to a closed 1-manifold $Z \subset \mathfrak{t}^*$, as long as Z is oriented and transverse to μ. The wall-crossing data has a nontrivial interpretation in this case, too.
2. We give an example to illustrate the orientation of H. Suppose our torus T is the standard circle $T = S^1 = \mathbb{R}/\mathbb{Z}$, with Lie algebra and its dual identified with $\mathbb{R}$ in the standard manner. In this case p_0 and p_1 are real numbers. If $p_0 < p_1$, then Z must be the interval $[p_0, p_1]$, and each wall-crossing induces the positive (i.e. standard) orientation on S^1. If $p_1 < p_0$, then Z must be the interval $[p_1, p_0]$, and each wall-crossing induces the negative orientation on S^1.)
3. It is not possible for the same pair (H, q) to appear twice in the wall-crossing data, however we may have pairs (H_0, q_0) and (H_1, q_1) with $H_0 = H_1$ while $q_0 \neq q_1$: since Z may cross the same wall more than once; or Z may cross different walls which are parallel and thus correspond to the same subgroup. And it is also possible for q_0 to equal q_1 (with $H_0 \neq H_1$). This is because a point q may lie in the interior of two different walls simultaneously. This happens when components of the submanifolds X^{H_0} and X^{H_1} are disjoint in X, while their images under μ both contain $q_0 = q_1$. There are three points in figure 1 with this property.

The boundary of the wall-crossing-cobordism

The wall-crossing data indexes the boundary components of W:

Proposition 2.3. *The submanifold $W \subset X$ has boundary*

$$\mu^{-1}(p_0) \quad \sqcup \quad \mu^{-1}(p_1) \quad \sqcup \quad \bigsqcup_{(H,q)\in \mathrm{data}(Z)} S_{(H,q)}$$

where

$$S_{(H,q)} := S(\nu X^H)\big|_{X^H \cap \mu^{-1}(q)}.$$

Here $S(\nu X^H)$ denotes the unit sphere bundle in the normal bundle of X^H in X. Note that $S_{(H,q)}$ need not be connected: its components correspond to the connected components of $X^H \cap \mu^{-1}(q)$.

Proof. By proposition 1.5, each X^H is transverse to $\mu^{-1}(Z)$. Hence the intersection $N_\epsilon(X^H) \cap \mu^{-1}(Z)$ gives a tubular neighbourhood of $X^H \cap \mu^{-1}(Z)$ in $\mu^{-1}(Z)$. Similarly, the normal bundle to $X^H \cap \mu^{-1}(Z)$ in $\mu^{-1}(Z)$ is the restriction of the normal bundle to X^H in X. By scaling, the unit sphere bundle is equivalent to the ϵ-sphere bundle. ☐

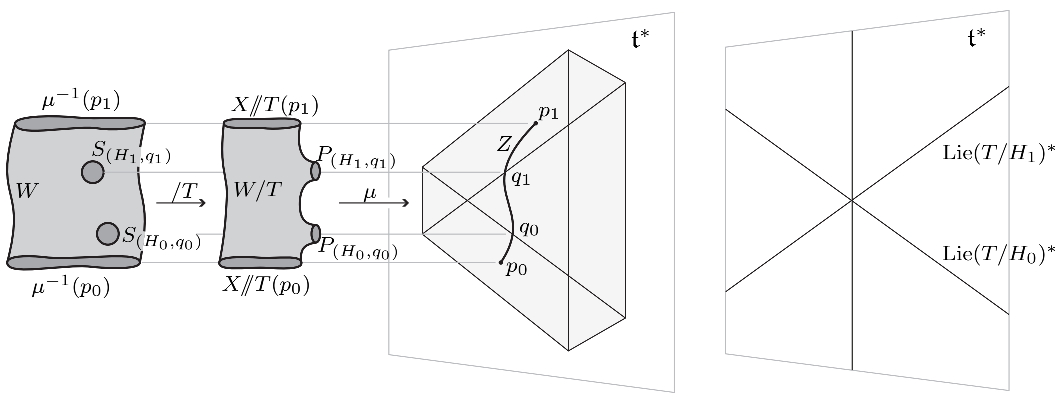


Figure 3: A transverse path Z with $\mathrm{data}(Z) = \{(H_0, q_0), (H_1, q_1)\}$, and the corresponding boundary components of W and of the wall-crossing-cobordism W/T.

Taking the quotient by T (which has a locally free action on W), we thus have a description of the boundary of the wall-crossing-cobordism:

$$\partial(W/T) \quad \cong \quad X/\!\!/T(p_0) \quad \sqcup \quad X/\!\!/T(p_1) \quad \sqcup \quad \bigsqcup_{(H,q)\in \mathrm{data}(Z)} P_{(H,q)}, \tag{2.4}$$

where

$$P_{(H,q)} := S_{(H,q)}/T. \tag{2.5}$$

The boundary components as weighted projective bundles

The rest of this section is devoted to giving a more explicit description of the boundary components $P_{(H,q)}$.

The projection of the fibre bundle

$$\pi : S_{(H,q)} \to X^H \cap \mu^{-1}(q) \tag{2.6}$$

is a T-equivariant map, and by construction, T acts with at most finite stabilizers on the total space $S_{(H,q)}$. The subgroup H acts trivially on the base, so that the T-action descends to an action of T/H, and it follows from proposition 1.5 that this T/H-action on the base is locally free.

We will first consider the quotient of the base of the fibre bundle (2.6), and then we will state a proposition which describes the quotient of the total space.

The submanifold $X^H \subset X$ is a closed symplectic submanifold, stable under T, and the restriction of the moment map μ to X^H gives a moment map for the T-action on X^H. Hence the quotient of the base can be described as a symplectic quotient

$$\left(X^H \cap \mu^{-1}(q)\right)/T = X^H /\!\!/ T(q).$$

This looks like a singular kind of quotient: q is not a regular value of μ, for instance. But the appearance of singularity is an illusion: we know a priori that H acts trivially on the manifold X^H, and so by Fact 1.2 we know that the image under μ of each component of X^H must lie in some affine hyperplane $\mathcal{S} \subset \mathfrak{t}^*$ (parallel to $\mathrm{Lie}(T/H)^*$). Now q *is* a regular value in $\mathcal{S}$ for the restriction of μ, thought of as a map to $\mathcal{S}$ (the fact that q is a regular value in this sense is equivalent to the condition that Z cross each wall in its interior). Hence $\mu^{-1}(q) \cap X^H$ is a compact closed submanifold of X^H, and its quotient $X^H /\!\!/ T(q)$ is a compact symplectic orbifold. This kind of symplectic quotient is explained in more detail in section 7.

Proposition 2.7. *There exists a complex vector orbibundle*

$$\nu \to X^H /\!\!/ T(q)$$

together with an action of H on ν, covering the trivial action on $X/\!\!/T(q)$, and such that the set of fixed points equals the zero section, such that

$$P_{(H,q)} \cong S(\nu)/H \to X^H /\!\!/ T(q).$$

Here $S(\nu)$ denotes the unit sphere bundle in ν (relative to a choice of invariant metric). In the case that the symplectic quotient $X/\!\!/T(q)$ is a free quotient, ν is a vector bundle, induced by the normal bundle νX^H.

The vector bundle $\nu \to X/\!\!/T(q)$ is not uniquely defined: to defined it we choose a complementary subgroup $T' \subset T$ so that $T = T' \times H$. Then T' defines a lift of the action of T/H on X^H to its normal bundle νX^H, and we let this action define the induced vector orbibundle (as defined in appendix A) over $X^H /\!\!/ T(q)$. Then the H-action on νX^H naturally descends to an action on ν, and we will show that $S_{(H,q)}/T = S(\nu)/H$.

Proof of Proposition 2.7. The space $S_{(H,q)}$ is formed from νX^H by the operations of taking the sphere bundle, restriction, and taking the quotient by T. By decomposing T as $T' \times H$ we can take the quotient by T in two stages. The proof then amounts to permuting the order of these operations (and seeing that the result is indepent of the order of operations). Explicitly, $S_{(H,q)}/T = \left(S(\nu X^H|_{X^H \cap \mu^{-1}(q)})/T'\right)/H = S(\nu)/H$.

The complex structure on ν is induced by fixing an invariant almost complex structure on X, compatible with the symplectic form ω (see e.g. [25, Proposition 2.48]). It follows by T-invariance that the normal bundle νX^H is an invariant complex vector bundle, so that the complex structure descends to the quotient ν. ∎

Remarks 2.8. 1. In general H will act on the fibres of ν with both positive and negative weights (recall that H is oriented, and so has a natural identification with S^1) and we can thus decompose ν into the positive and negative weight subbundles $\nu = \nu^+ \oplus \nu^-$. Letting $\bar{\nu}^-$ denote the same underlying real vector bundle as ν^-, but with the conjugate complex structure (that is, with multiplication by i replaced by multiplication by $-i$), then $S(\nu)/H$ can be identified with a weighted projectivization of $\nu^+ \oplus \bar{\nu}^-$. Although this describes the diffeomorphism type of $S(\nu)/H$, the natural orientation of $S(\nu)/H$ (definition 3.5) is not given by this description.

2. The vector bundle ν depends on the choice of T'. However the quotient $S(\nu)/H$ is independent of this choice, as we can see from its description as $S_{(H,q)}/T$. Changing the choice of T' has the effect of tensoring ν with a certain line bundle, but this change doesn't affect the quotient $S(\nu)/H$. This can be seen as a generalization of the fact that the projectivization of a complex vector bundle bundle is invariant under tensoring the vector bundle with a line bundle.

3. The orbifold singularities and orientation of the wall-crossing-cobordism

Orbifold singularities in the wall-crossing-cobordism

We now address the question of the orbifold singularities in the wall-crossing-cobordism W/T. These arise from points in W whose stabilizer subgroup is nontrivial. To be more precise, since we allow for the possibility that there is some finite subgroup of T which stabilizes every point in X, the orbifold singularities arise from points in W whose stabilizer subgroup is larger than the generic one.

Lemma 3.1. *Let $F \subset T$ be a finite subgroup, and $X^F \subset X$ the subset of points fixed by F. Then X^F is a closed symplectic submanifold of X, transverse to ∂W, and also to the interior of W. It follows that the wall-crossing-cobordism W/T is an orbifold-with-boundary.*

This lemma gives both coarse information, and very fine information. The coarse information provided by this lemma is that the wall-crossing-cobordism is an orbifold-with-boundary, which we will see is oriented, and hence satisfies Stokes's theorem.

However, this lemma actually makes it possible to determine the structure of the orbifold singularities quite accurately. This is because each X^F is a closed symplectic submanifold of X, and it follows that the restriction of the moment map μ to X^F gives a moment map for the action of T on X^F, where the walls and chambers of the image of μ for X^F being a subset of the corresponding walls and chambers for X. Thus we can treat all the arguments in this paper as applying simultaneously to X and to X^F: each symplectic quotient $X/\!\!/T(p)$ contains the symplectic quotient $X^F/\!\!/T(p)$, as does each wall-crossing-cobordism, and so on. (We won't have cause to carry out such a detailed analysis in this paper.)

Proof of lemma 3.1. The fact that X^F is a closed manifold is a standard result of the theory of compact group actions on manifolds (proved using an equivariant exponential map, see e.g. [5]), and an easy averaging argument shows that the restriction of the symplectic form ω to X^F is nondegenerate.

Now, let $X^* \subset X$ denote the set of points with finite stabilizer subgroup. Then $W \subset X^*$, by construction. Any $p \in \mathfrak{t}^*$ is a regular value for $\mu|_{X^*}$, and using the same argument as in the proof of Proposition 1.5 we see that $\mu^{-1}(p) \cap X^*$ is transverse to X^F. It follows X^F is transverse to W, and to the boundary components $\mu^{-1}(p_0)$ and $\mu^{-1}(p_1)$.

It remains to show transversality to the boundary components $S_{(H,q)}$. The description of $S_{(H,q)}$ as the sphere bundle in a vector bundle makes it clear that the stabilizer doesn't depend on the radius of the sphere.

By varying the radius of the sphere, we can foliate W locally by a one-parameter family of submanifolds. Since X^F is transverse to W, to show transversality to one of the leaves of this foliation, we must simply show that for every point in the intersection with X^F, there is a tangent vector to X^F which is transverse to the leaves of the foliation. But this follows from the fact that the stabilizer subgroup is independent of the radius of the sphere. $\square$

Orienting the wall-crossing-cobordism

In this subsection we define an orientation on the wall-crossing-cobordism W/T. We then calculate the induced orientations on its various boundary components.

Definition 3.2. The orientation is extremely easy from a conceptual point of view: W/T is foliated by symplectic orbifolds $X/\!\!/T(p) \cap W/T$, for $p \in Z$, and the normal bundle to this foliation is identified with TZ by the moment map. Thus the symplectic orientation of the leaves, combined with the orientation of Z in which the positive direction goves from p_0 to p_1, gives an orientation of the wall-crossing-cobordism W/T.

To carry this out explicitly, we begin by fixing a metric on W/T. Let x be a point in W, denote by $[x]$ the corresponding point in W/T, and set $p = \mu(x)$. We assume for simplicity that $[x]$ is a smooth point of W/T (but by using orbifold metrics and orbifold differential forms, as described in appendix A, this construction also works at the orbifold points). By construction, $d\mu$ is surjective at x (fact 1.1). Moreover, since μ is T-invariant, it descends to a map from W/T to Z. We can thus decompose the tangent space $T_{[x]}(W/T)$ into the kernel and the cokernel of $d\mu$. Identifying these spaces explicitly gives us

$$T_{[x]}(W/T) \cong T_{[x]}X/\!\!/T(p) \oplus T_pZ.$$

Now $X/\!\!/T(p)$ is a symplectic orbifold, and we denote its symplectic form by ω_p. Using the above decomposition, we can extend ω_p to $T_{[x]}(W/T)$. denoting the extension by $\widetilde{\omega}_p$ (this 2-form will not necessarily be closed, but it will be nondegenerate on the tangent spaces to the leaves). Let Z be parametrized by the variable t, with $t = 0$ at p_0 and $t = 1$ at p_1. Then

$$\widetilde{\omega}_p^k \wedge \mu^* dt$$

(where $\dim X/\!\!/T(p) = 2k$) defines a top-degree form, and hence an orientation of W/T at $[x]$. But the above construction can be simultaneously applied to every smooth point of W/T, with the resulting form varying smoothly, hence orienting W/T.

Remark 3.3. In fact, the above definition can be enhanced in a straighforward manner to define a 'complex orientation' of W/T. We won't need it in this paper, however.

The rest of this section is taken up with describing in a precise way an orientation on the wall-crossing boundary components of W/T, and then stating the result that this orientation equals the induced boundary orientation. We give two definitions, and then state this result (the proof of which is given in appendix C).

Definition 3.4. Let V be an oriented real vector space, and suppose the oriented group $H \cong S^1$ acts on V, fixing only the origin. We define the **induced orientation** of $S(V)/H$

(where $S(V)$ denotes the unit sphere in V relative to an invariant metric). Given a point $v \in S(V)$, denote by $H \cdot v \in S(V)/H$ the associated H-orbit. There is a natural isomorphism

$$T_{H\cdot v}(S(V)/H) \oplus \mathbb{R}^+ \cdot v \oplus \mathfrak{h} \cong T_v V \cong V,$$

where $\mathbb{R}^+ \cdot v$ denotes the ray from the origin through v. We define the orientation of $S(V)/H$ to be that orientation which is compatible with the above isomorphism together with the given orientations of $\mathbb{R}^+$, $\mathfrak{h}$, and V.

For example, let $V = \mathbb{C}^n$, and let $H \cong S^1$ act with weight 1. Then $S(V)/H$ is naturally identified with complex projective space, and the orientation we have defined agrees with the orientation induced by the complex structure. Similarly, if H acts with positive weights, then $S(V)/H$ is a weighted projective space, and the above-defined orientation again agrees with the orientation induced by the complex structure (see appendix B for more details).

We now define an orientation on the boundary components of the wall-crossing-cobordism corresponding to wall-crossings. We then prove that this agrees with the induced boundary orientation.

Definition 3.5. Recall that proposition 2.7 identifies the boundary component of the wall-crossing-cobordism corresponding to the pair (H, q) as the total space of the bundle

$$S_{(H,q)}/T = S(\nu)/H \to X^H /\!\!/ T(q).$$

where $\nu \to X^H /\!\!/ T(q)$ is a vector bundle induced by the normal bundle νX^H and a decompostion of T as $T' \times H$. We orient this space as follows. Since X^H is a symplectic submanifold, the symplectic orientations of X and of X^H induce a natural orientation on the normal bundle νX^H, which descends (by invariance) to the induced bundle V. Combining this with definition 3.4 and the orientation of H given in definition 2.1 gives an orientation of the fibres of the bundle $S(\nu)/H \to X^H /\!\!/ T(q)$. The base is a symplectic quotient, and we orient it by its symplectic form. We then orient the total space $S(\nu)/H$ by the product orientation. (the order is irrelevant, since both the base and fibre are even-dimensional).

Lemma 3.6. *Let the wall-crossing-cobordism W/T be oriented as in definition 3.2. Then the induced boundary orientation of $X /\!\!/ T(p_0)$ is $-(\omega_{p_0}^k)$, and of $X /\!\!/ T(p_1)$ is $\omega_{p_1}^k$ (where ω_{p_i} denote the respective induced symplectic forms), and the induced boundary orientation of each $P_{(H,q)}$ is equal to the product orientation defined in 3.5 above.*

The proof is conceptually rather simple, but keeping track of the various vector spaces involved in a comprehensible way makes it quite long, and it has been relegated to appendix C.

4. Theorem A: a summary of the existence and properties of the wall-crossing-cobordism.

Theorem A. *Suppose $p_0, p_1 \in \mathfrak{t}^*$ are regular values of the moment map μ, and let $Z \subset \mathfrak{t}^*$ be path joining p_0 and p_1 which is transverse to μ. There there are two objects naturally associated to Z. The first is a finite set* data(Z)*, consisting of pairs (H, q), where $H \cong S^1$ is a subgroup of T, and q is a point in $\mathfrak{t}^*$. And the second object naturally associated to Z is an oriented cobordism, whose boundary equals*

$$-X /\!\!/ T(p_0) \quad \sqcup \quad X /\!\!/ T(p_1) \quad \sqcup \quad \bigsqcup_{(H,q)\in \mathrm{data}(Z)} P_{(H,q)}.$$

For each pair $(H, q) \in$ data(Z) the space $P_{(H,q)}$ is the total space of a bundle over the compact symplectic orbifold $X^H /\!\!/ T(q)$, whose fibres are weighted projective spaces.

Moreover

1. The cobordism arises as the quotient, by T, of a submanifold-with-boundary $W \subset X$, such that the T-action on W is locally free.

2. The points p_0 and p_1 need not lie in the image of μ. If either lies outside the image of μ, then the associated boundary component is empty.

3. The boundary component $-X /\!\!/ T(p_0)$ denotes $X /\!\!/ T(p_0)$ with the negative of its symplectic orientation.

4. Each space $P_{(H,q)}$ can be described as follows. There exists a complex vector orbibundle $\nu \to X^H /\!\!/ T(q)$, with an action of H on the fibres, such that

$$P_{(H,q)} = S(\nu)/H \to X^H /\!\!/ T(q).$$

   The bundle $\nu \to X^H /\!\!/ T(q)$ is induced by the normal bundle νX^H, and depends on the choice of a complement to H in T; however the bundle $P_{(H,q)} \to X^H /\!\!/ T(q)$ is independent of this choice.

5. The induced orientations on the spaces $P_{(H,q)}$ are given by the product of the symplectic orientation of $X^H /\!\!/ T(q)$ and a natural orientation on the fibres, defined in terms of the oriented group H, and the oriented fibres of ν.

6. The wall-crossing-data data(Z) is determined by the arrangement of walls in $\mathfrak{t}^*$ (which can be deduced from the fixed point data of (X, T, μ)), together with the path Z.

5. The localization map and the wall-crossing formula

In this section we fix our attention on a single wall-crossing. Fixing notation, we suppose $p_0, p_1 \in \mathfrak{t}^*$ are regular values of μ, joined by a transverse path Z having a single wall-crossing at q, and we let $H \cong S^1$ be the oriented subgroup associated to the wall.

Theorem A says, roughly, that the symplectic quotients $X /\!\!/ T(p_0)$ and $X /\!\!/ T(p_1)$ are in some way related by the symplectic quotient $X^H /\!\!/ T(q)$. Theorem B gives a cohomologically precise version of this.

Theorem B. *There is a map*

$$\lambda_H : \mathrm{H}^*_T(X) \to \mathrm{H}^*_{T/H}(X^H)$$

such that, for any $a \in \mathrm{H}^*_T(X)$,

$$\int_{X /\!\!/ T(p_0)} \kappa(a) - \int_{X /\!\!/ T(p_1)} \kappa(a) = \int_{X^H /\!\!/ T(q)} \kappa(\lambda_H(a|_{X^H})).$$

(The maps κ on the left hand side are the natural maps $\mathrm{H}^*_T(X) \to \mathrm{H}^*(X /\!\!/ T(p_i))$ and on the right hand side is the natural map $\mathrm{H}^*_{T/H}(X^H) \to \mathrm{H}^*(X^H /\!\!/ T(q))$.)

Moreover, for any component $X^H_i \subset X^H$, the restriction of $\lambda_H(a)$ to X^H_i only depends on the restriction of a to X^H_i.

Recall that $X^H /\!\!/ T(q)$ can be considered to be a symplectic quotient of X^H by the quotient group T/H (expained in section 2); and the various maps denoted by κ are defined by restriction to the relevant submanifold, followed by the natural identification of the equivariant cohomology of this manifold with the rational cohomology of its quotient.

We call λ_H the **localization map**: we first define λ_H, and then we prove theorem B. In the next section we give an explicit formula for λ_H in terms of characteristic classes. The localization map is the key to an inductive process, which will allow us to localize calculations to the fixed points X^T. We will carry out the induction in section 8.

Definition 5.1. The localization map λ depends on the triple (X, T, H), where X is a symplectic manifold, T is a compact torus which acts on X (preserving the symplectic form), and $H \cong S^1$ is an oriented subgroup of T. In this section, X and T will be fixed, and we will write λ_H to denote the dependence on the oriented subgroup H (in later sections will decorate the symbol λ with any data that is not obvious from the context.)

Given X and T, then λ_H is the (degree-lowering) map

$$\lambda_H : \mathrm{H}^*_T(X) \to \mathrm{H}^*_{T/H}(X^H)$$

defined as follows. Let $S(\nu X^H)$ denote the sphere bundle in the normal bundle νX^H to X^H in X. We then denote by p and π the projections

$$\begin{tikzcd} S(\nu X^H) \arrow[rr, "/H"] \arrow[dr, "p"'] & & S(\nu X^H)/H \arrow[dl, "\pi"] \\ & X^H & \end{tikzcd}$$

Let π_* denote integration over the fibres of π (where the fibres are oriented according the definition 3.4, using the symplectic orientation of the normal bundle to X^H). Then we let λ_H equal the composition

$$\begin{tikzcd} & \mathrm{H}^*_T(S(\nu X^H)) \arrow[r, "/H", "\cong"'] & \mathrm{H}^*_{T/H}(S(\nu X^H)/H) \arrow[d, "\pi_*"] \\ \mathrm{H}^*_T(X) \arrow[r, "i^*"] & \mathrm{H}^*_T(X^H) \arrow[u, "p^*"] & \mathrm{H}^*_{T/H}(X^H) \end{tikzcd}$$

where $i : X^H \hookrightarrow X$ denotes the inclusion, and the map $\mathrm{H}^*_T(S(\nu X^H)) \xrightarrow[\cong]{/H} \mathrm{H}^*_{T/H}(S(\nu X^H)/H)$ is the natural map on equivariant cohomology induced by the locally free quotient (see for example [1]).

Proof of theorem B. The proof is a straightforward exercise involving identifying the various maps involved, and repeatedly using the fact that integration over the fibre commutes with restriction (together with some general facts about equivariant cohomology.)

Let $j : W \hookrightarrow X$ denote the inclusion. Then, for any $a \in \mathrm{H}^*_T(X)$, we have $j^*(a) \in \mathrm{H}^*_T(W)$, and we write

$$j^*(a)/T \in \mathrm{H}^*(W/T),$$

for the corresponding naturally induced class (recall that the T-action is locally free on W, and we are taking cohomology with rational coefficients).

Since the wall-crossing-cobordism W/T is an oriented orbifold-with-boundary, it follows that the boundary is homologous to zero (fact A.5), and hence

$$\int_{\partial(W/T)} j^*(a)/T = 0.$$

Using the identification of the boundary of W/T (theorem A), we thus get

$$-\int_{X/\!\!/T(p_0)} j^*(a)/T + \int_{X/\!\!/T(p_1)} j^*(a)/T + \int_{P_{(H,q)}} j^*(a)/T = 0.$$

We rewrite this, letting $i : S_{(H,q)} \hookrightarrow X$ denote the inclusion, and identifying the maps κ:

$$-\int_{X/\!\!/T(p_0)} \kappa(a) + \int_{X/\!\!/T(p_1)} \kappa(a) + \int_{P_{(H,q)}} i^*(a)/T = 0.$$

Letting π denote the projection

$$\pi : P_{(H,q)} \to X_{(H,q)} = X^H/\!\!/T(q)$$

and π_* denote integration over the fibres of π, then we have

$$\int_{P_{(H,q)}} i^*(a)/T = \int_{X_{(H,q)}} \pi_*(i^*(a)/T).$$

Thus we have been reduced to proving

$$\pi_*(i^*(a)/T) = \kappa(\lambda_H(a)). \tag{5.2}$$

We will now use two naturality properties of integration over the fibre, for maps in the commutative diagram

$$\begin{tikzcd}
S_{(H,q)} \arrow[rr, hook] \arrow[d, "/H"] & & S(\nu X^H) \arrow[d, "/H"] & \\
S_{(H,q)}/H \arrow[rr, hook] \arrow[dr, "\widetilde{\pi}"] \arrow[dd, "/(T/H)"] & & S(\nu X^H)/H \arrow[dr, "\widetilde{\widetilde{\pi}}"] & \\
& \mu^{-1}(q) \cap X^H \arrow[rr, hook] \arrow[dd, "/(T/H)"] & & X^H \\
P_{(H,q)} \arrow[dr, "\pi"] & & & \\
& X^H/\!\!/T(q) & &
\end{tikzcd} \tag{5.3}$$

Letting $i : S_{(H,q)} \hookrightarrow X$ and $\widetilde{\widetilde{i}} : S(\nu X^H) \hookrightarrow X$ denote the inclusions, we have

$$\pi_*(i^*(a)/T) = \widetilde{\pi}_*(i^*(a)/H)/(T/H).$$

This is because of the first naturality property of integration over the fibre: it commutes with simultaneous quotient of the base and the total space.

The second naturality property of integration over the fibre is that it 'commutes with restriction'. Concretely, in our case, this gives

$$\widetilde{\pi}_*(i^*(a)/H) = \widetilde{\widetilde{\pi}}_*(\widetilde{\widetilde{i}}^*(a)/H)\Big|_{\mu^{-1}(q)\cap X^H}.$$

Now, in the diagram

$$\begin{tikzcd}
S(\nu X^H) \arrow[rr, hook, "\widetilde{\widetilde{i}}"] \arrow[dr, "p"'] & & X \\
& X^H \arrow[ur, hook, "k"'] &
\end{tikzcd}$$

an easy scaling argument shows that $\tilde{\tilde{i}}$ is equivariantly homotopic to $k \circ p$. Hence

$$\begin{aligned} \tilde{\tilde{\pi}}_*(\tilde{\tilde{i}}^*(a)/H) &= \tilde{\tilde{\pi}}_*(p^*(k^*(a))/H) \\ &= \lambda_H(a) \end{aligned}$$

since this turns out to be precisely the definition of λ_H, with the data (X, T, H).

Putting all this together, we thus have

$$\begin{aligned} \pi_*(i^*(a)/T) &= \left(\lambda_H(a)|_{\mu^{-1}(q)\cap X^H} \right) /(T/H) \\ &= \kappa(\lambda_H(a)) \end{aligned}$$

by definition of κ. But this proves equation (5.2), and hence, by the arguments preceding equation (5.2), we have completed the proof. ☐

6. The wall-crossing formula in terms of characteristic classes

By giving an explicit formula for the localization map in terms of characteristic classes, we can restate a more explicit version of the wall-crossing formula (which we call theorem B′.)

We can give an explicit formula for the localization map λ_H using the definitions and results of appendix B. Using this explicit formula, we can then recast theorem B in a more explicit form. Before carrying this out, we must give a definition, which will help us account for the possibility that X^H has a number of components, and describe the way a decomposition of T induces a decomposition of a cohomology class.

Definition 6.1. Let Y be a connected manifold with an action of T. We define $o_T(Y)$ to be the order of the maximal subgroup of T which stabilizes every point in Y (so $o_T(Y) = 1$ if and only if T acts effectively on Y). (In every case which we consider, this number will be finite). We extend this definition to the case in which Y may have a number of components by defining $o_T(Y)$ to be the degree-0 cohomology class which restricts to give this number on each component.

Now suppose $T' \subset T$ is a complement to H, so that $T = T' \times H$. Then the restriction of any class $a \in \mathrm{H}^*_T(X)$ to X^H decomposes

$$a|_{X^H} = \sum_{i\geq 0} a_i \otimes u^i \tag{6.2}$$

according to the natural isomorphism

$$\mathrm{H}^*_T(X^H) \cong \mathrm{H}^*_{T'}(X^H) \otimes \mathrm{H}^*(BH),$$

where $u \in \mathrm{H}^2(BH)$ is the positive generator (with respect to the orientation of H defined in 2.1).

We then have

Proposition 6.3. *Let $a \in \mathrm{H}^*_T(X)$, and suppose $T' \subset T$ is a complement to H, so that $T = T' \times H$. Then*

$$\lambda_H(a) = \frac{o_T(X)}{o_{T/H}(X^H)} \sum_{i\geq 0} a_i \smile s^w_{i-r+1},$$

where the classes $a_i \in \mathrm{H}^*_{T'}(X^H)$ are defined by the natural decomposition of a given in equation (6.2) above, and s^w_i denotes the i-th T'-equivariant weighted Segre class of $(\nu X^H, H)$ (definitions B.6 and B.12), and r is the function, constant on connected components of X^H, such that $2r = \mathrm{rk}(\nu X^H)$.

Proof. We will show how this proposition follows from the integration formula proved in appendix B, namely proposition B.8 (together with its 'equivariant enhancement', equation (B.14)).

Explicitly, we are using the vector bundle $\nu X^H \to X^H$ and the groups H and T' in place of the vector bundle $V \to Y$ and the groups S^1 and G in appendix B.

We need first to give the normal bundle νX^H a complex structure compatible with its symplectic form, so that the definition of the orientation of $S(\nu X^H)/H$ used in appendix B agrees with its natural orientation (definition 3.5). And second, we must show that our factor $o_T(X)/o_{T/H}(X^H)$ is equal, on each component of X^H, to the factor k in the appendix.

Firstly, general principles in symplectic topology imply that there exists a T-invariant almost complex structure $J : TX \to TX$, compatible with the symplectic form ω, and such that TX^H is stable under J (see, for example, McDuff and Salamon [25, Proposition 2.48]). Such an almost complex structure gives the normal bundle νX^H a complex structure, and the orientations induced by the complex structure and the symplectic form agree (equation (0.3)), and thus we can apply Proposition B.8 with this complex structure.

Secondly, we need to show that, for each component X^H_i of X^H, we have

$$k = o_T(X)/o_{T/H}(X^H_i),$$

where k is the greatest common divisor of the weights of the H-action on the fibres of $\nu X^H_i \to X^H_i$. But using the decomposition $T = T' \times H$, together with lemma 3.1, it is clear that $k = o_H(\nu X^H_i)$, and that $o_T(X) = o_{T' \times H}(\nu X^H_i) = o_{T'}(X^H_i) \cdot o_H(\nu X^H_i)$. □

We can now rewrite theorem B using this explicit identification.

Theorem B′. *Suppose $p_0, p_1 \in \mathfrak{t}^*$ are regular values of μ, joined by a transverse path Z which has a single wall-crossing, at q. Let $H \cong S^1$ be the subgroup associated to the wall, and choose $T' \subset T$ so that $T = T' \times H$.*

Then there are characteristic classes $s^w_i \in \mathrm{H}^{2i}_{T'}(X^H)$ (the equivariant weighted Segre classes of νX^H, as defined in B.6) such that, for any $a \in \mathrm{H}^*_T(X)$,

$$\int_{X/\!\!/T(p_0)} \kappa(a) - \int_{X/\!\!/T(p_1)} \kappa(a) = \int_{X^H/\!\!/T'(q)} \kappa\left(\frac{o_T(X)}{o_{T/H}(X^H)} \textstyle\sum_{i\geq 0} a_i \smile s^w_{i-r+1}\right).$$

where r is the function, constant on connected components of X^H, such that $2r = \mathrm{rk}(\nu X^H)$; and the classes $a_i \in \mathrm{H}^*_{T'}(X^H)$ are defined by restricting a to X^H and decomposing, as in equation (6.2) above. (The map κ on the left hand side of the main equation is the natural map $\mathrm{H}^*_T(X) \to \mathrm{H}^*(X/\!\!/T(p_i))$ and on the right hand side is the natural map $\mathrm{H}^*_{T'}(X^H) \to \mathrm{H}^*(X^H/\!\!/T'(q))$.)

7. A generalization of a transverse path and its data

This is the first of three sections in which we apply the preceding results inductively, ending up with results concerning the T-fixed points of X. In this section we generalize the notion of a transverse path, and the associated data. In section 8 we show how this generalized data corresponds to cobordisms involving the fixed points X^T, and in section 9 we show how this generalized data governs integration formulae localized at the fixed points.

A τ-transverse path and its data

We begin with a straightforward generalization of the notion of a transverse path, and its associated data. Recall that X is a symplectic manifold, with an action of the torus T, and

with associated moment map $\mu : X \to \mathfrak{t}^*$. Let $\tau \subset T$ be a subtorus. In section 1 we saw how $\mathrm{Lie}(T/\tau)^*$ can be considered to be a subspace of $\mathfrak{t}^*$ via a natural embedding (it is a subspace of dimension $\dim T - \dim \tau$). Recall also that X^τ, the set of points fixed by τ, is a closed symplectic submanifold of X.

Fact 7.1. *If X_i^τ is any connected component of X^τ, then we have:*

1. *The restriction of μ to X_i^τ gives a moment map for the T-action on X_i^τ;*
2. *The image $\mu(X_i^\tau)$ lies in an affine translate $\mathcal{S} \subset \mathfrak{t}^*$ of $\mathrm{Lie}(T/\tau)$;*
3. *The T-action on X_i^τ descends to a T/τ-action; and*
4. *Composing the restriction of μ with an identification of $\mathcal{S}$ with $\mathrm{Lie}(T/\tau)^*$ gives a moment map for the T/τ-action on X_i^τ.*

Hence we define, in analogy with section 1

Definition 7.2. Given $q \in \mathfrak{t}^*$, set $\mathcal{S} := q + \mathrm{Lie}(T/\tau)^*$. We say q is **τ-regular** if μ maps some component of X^τ to $\mathcal{S}$, and for each such component, the point q is regular value for the restriction of μ, thought of as a map to $\mathcal{S}$.

For example, using the notions of 'wall' and 'interior' from definition 1.8, if $H \cong S^1$ is a subgroup of T, and if q lies in a wall corresponding to H, then q is H-regular iff q lies in the interior of this wall.

Definition 7.3. Let $\mathcal{S}$ be an affine translate of $\mathrm{Lie}(T/\tau)^*$, and suppose $q_0, q_1 \in \mathcal{S}$ are τ-regular values. Then a path $Z \subset \mathfrak{t}^*$ from q_0 to q_1 is **τ-transverse** if it is contained in the subspace $\mathcal{S}$, and for each component of X^τ which μ maps to $\mathcal{S}$, the path Z is transverse to the restriction of μ, thought of as a map to $\mathcal{S}$.

Definition 7.4. Suppose $Z \subset \mathcal{S}$ is a τ-transverse path, with endpoints the τ-regular values q_0 and q_1. We define the **wall-crossing data** for Z to be the set

$$\mathrm{data}(Z) := \{(H, q) \mid H \text{ is a subtorus of } T \text{ with } \tau \subset H, \text{ and } q \in Z \cap \mu(X^H)\}$$

Applying proposition 1.5, it follows that $H/\tau \cong S^1$, and we orient H/τ as in definition 2.1, that is, we orient Z so that the positive direction goes from q_0 to q_1, and we orient H/τ compatibly.

The module of relations

We now define a module which records the data from all possible τ-transverse paths simultaneously.

Definition 7.5. An **oriented τ-flag of subtori** in T is a collection of subtori

$$\Theta = (1 = H_0 \subset H_1 \subset H_2 \subset \ldots \subset H_k = \tau \subset T),$$

such that H_i is an i-torus, and each $H_i/H_{i-1} \cong S^1$ is given an orientation.

Definition 7.6. We define the Z-module $\mathcal{A}$ by

$$\mathcal{A} := \bigoplus_{\tau \subset T} \mathcal{A}_\tau,$$

as τ runs through all subtori of T, where

$$\mathcal{A}_\tau := \bigoplus \mathbb{Z}(\Theta, q)$$

is the set of formal linear combinations of pairs (Θ, q), where q is τ-regular and Θ is an oriented τ-flag of subtori.

Note that $\mathcal{A}_\tau$ will be nontrivial for only finitely many τ, namely those for which there exists a τ-regular value. These correspond to the τ such that there is some point $x \in X$ whose stabilizer subgroup has identity component τ (the fact that there are only finitely many such τ is a standard fact in the theory of group actions on manifolds [5, 19]). We also note that $\mathcal{A}_T$ corresponds to the T-fixed points of X: if $(\Theta, q) \in \mathcal{A}_T$ then $q \in \mathfrak{t}^*$ is one of the finite set of points in the set $\mu(X^T) \subset \mathfrak{t}^*$.

Definition 7.7. We now define the submodule of relations $\mathcal{R} \subset \mathcal{A}$. There are two kinds of generators of $\mathcal{R}$. The first kind comes from a pair consisting of a τ-transverse path Z and an oriented τ-flag of subtori Θ, for any choice of subtorus τ. The associated generator of $\mathcal{R}$ is the sum

$$-(\Theta, q_0) + (\Theta, q_1) + \sum_{(H,r) \in \mathrm{data}(Z)} (\Theta \cup H, r),$$

where q_0 and q_1 are the endpoints of Z, and $\Theta \cup H$ denotes the oriented H-flag defined by concatenating Θ and H, with H/τ oriented as in $\mathrm{data}(Z)$. The second kind of generator of $\mathcal{R}$ corresponds to points which are outside the image of μ: for any subtorus $\tau \subset T$, suppose q is a τ-regular value and let Θ be an oriented τ-flag. If $q \notin \mu(X^\tau)$ then

$$(\Theta, q)$$

is a generator of $\mathcal{R}$. Finally, given $(\Theta, q) \in \mathcal{A}$, we write $[\Theta, q]$ for its equivalence class in the quotient module $\mathcal{A}/\mathcal{R}$.

Since X is compact, for any regular value $p_0 \in \mathfrak{t}^*$, there is a path Z starting at p_0 and ending outside the image of the moment map. The corresponding fact is true for each $X^\tau \subset X$. Hence

Lemma 7.8. *For any $(\Theta, q) \in \mathcal{A}$ we have*

$$[\Theta, q] = \sum_{i \in I} [\Theta_i, v_i]$$

in $\mathcal{A}/\mathcal{R}$, where I is a finite indexing set, and each $(\Theta_i, v_i) \in \mathcal{A}_T$.

8. Cobordisms between symplectic quotients and bundles over the fixed points

In this section we show how the relations defined in the previous section correspond to cobordisms. We begin by defining, for each generator (Θ, q) of $\mathcal{A}$, a space $P_{(\Theta,q)}$. We will then show how 'relations', i.e. finite sums in the submodule $\mathcal{R}$, correspond to cobordisms between these spaces. The constructions in this section are illustrated in figure 8.

The spaces involved

For every pair (Θ, q), where Θ is a τ-flag and $q \in \mathfrak{t}^*$ is a τ-regular value, we will define an associated space $P_{(\Theta,q)}$. We first describe $P_{(\Theta,q)}$ in two special cases, and then give the general definition. In the case that $\tau = \{1\}$ is the trivial group, then the only τ-flag is the trivial flag, which we denote by $1 \subset T$, and a τ-regular value is just a regular value of the moment map $\mu : X \to \mathfrak{t}^*$. In this case

$$P_{(1\subset T,q)} = X/\!\!/T(q).$$

If $Z \subset \mathfrak{t}^*$ is a transverse path, and $(H, q) \in \text{data}(Z)$ is one of its wall-crossing pairs, then it follows that q is an H-regular value, and $H \cong S^1$ defines the oriented H-flag $1 \subset H \subset T$, and we have

$$P_{(1\subset H\subset T,q)} = P_{(H,q)},$$

where the space on the right is the wall-crossing space defined in equation (2.5).

Definition 8.1. Suppose the torus τ acts on the complex vector space V, with $0 \in V$ the only point fixed by τ. Then associated to every flag of subtori of τ is a submanifold of V on which the τ-action is locally free (this submanifold may be empty). To define the submanifold, we first define a canonical decomposition of V. Let $\Theta = (1 = H_0 \subset H_1 \subset \ldots \subset H_k = \tau)$ be a τ-flag, that is, a full flag of subtori of τ. There is an associated flag of subspaces of V, stable under the τ-action:

$$V = V^{H_0} \supset V^{H_1} \supset \ldots \supset V^{H_k} = \{0\}$$

where V^{H_i} is the subspace fixed by H_i. We define $V_i \subset V$ to be the orthogonal complement to V^{H_i} in $V^{H_{i-1}}$, relative to a τ-invariant metric, for $1 \leq i \leq k$. Then $V_i \cong V^{H_{i-1}}/V^{H_i}$, and these subspaces define a decomposition of V into subrepresentations

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_k.$$

We set

$$S_\Theta(V) := S(V_1) \times S(V_2) \times \ldots \times S(V_k) \quad \subset V$$

where $S(V_i)$ is the unit sphere, relative to an invariant metric. Note that $S_\Theta(V)$ will be nonempty precisely when each V_i is nontrivial, that is, when each inclusion is strict in the flag of subspaces $V^{H_0} \supset V^{H_1} \supset \ldots \supset V^{H_k}$.

Finally, we define

$$P_\Theta(V) := S_\Theta(V)/\tau.$$

This is a locally free quotient, and hence has the structure of an orbifold. An orientation of V induces an orientation on $P_\Theta(V)$ as follows. We fix an orientation of each V_i so that the product orientation equals the given orientation of V. We then orient each $S(V_i)/T_i$ by applying the formula of definition 3.4, and give $P_\Theta(V)$ the induced product orientation (see the end of this section, where the structure of $P_\Theta(V)$ is described in more detail).

Remarks 8.2. 1. To see that the τ-action is locally free on $S_\Theta(V)$ we choose a decomposition of τ which is compatible with Θ, that is

$$\tau = T_1 \times T_2 \times \ldots \times T_k,$$

where each $T_i \cong H_i/H_{i-1} \cong S^1$. Then the above decomposition of V has the property that the T_i-action on V_i leaves only $0 \in V_i$ fixed, so that the T_i-action on $S(V_i)$ is locally free.

2. The quotient $P_\Theta(V)$ can be described as a k-fold 'tower' of weighted projective bundles, where $k = \dim \tau$. We make some remarks about this at the end of this section.

Definition 8.3. We now observe that we can apply the above construction both fibrewise and equivariantly. Suppose $T \supset \tau$ acts on a manifold Y, and the action lifts to a complex vector bundle $V \to Y$. Moreover, suppose that the stabilizer subgroup of each point $y \in Y$ is τ. Then each fibre V_y is a τ-representation and, if $0 \in V_y$ is the only point fixed by τ, we define the submanifold $S_\Theta(V_y) \subset V_y$ by applying the above construction. Applying this to each fibre simultaneously, relative to a T-invariant metric, gives a submanifold

$$S_\Theta(V) \subset V$$

which is stable under the action of T.

We now apply this fibrewise construction to the symplectic manifold X, with T-moment map μ. Given a pair (Θ, q), where Θ is a τ-flag and $q \in \mathfrak{t}^*$ is a τ-regular value, we let $S_{(\Theta,q)}$ be the result of applying the above construction with $Y := X^\tau \cap \mu^{-1}(q)$ and $V := \nu X^\tau|_Y$, with a T-invariant almost complex structure, compatible with the symplectic form, giving V the structure of a complex vector bundle. That is

$$S_{(\Theta,q)} := S_\Theta \left(\nu X^\tau|_{X^\tau \cap \mu^{-1}(q)} \to X^\tau \cap \mu^{-1}(q) \right).$$

Using an equivariant exponential map to identify a neighbourhood of the zero-section of νX^τ with a neighbourhood of X^τ in X we can consider $S_{(\Theta,q)}$ to be a submanifold of X. It follows from the above construction and the fact that q is τ-regular that the T-action is locally free on $S_{(\Theta,q)}$. We then define

$$P_{(\Theta,q)} := S_{(\Theta,q)}/T$$

which we see is the total space of a bundle over the symplectic quotient $X^\tau /\!/ T(q)$ with fibre $P_\Theta(\nu_x X^\tau)$. We note that in the case that the symplectic quotient $X^\tau /\!/ T(q)$ is smooth, this is an honest fibre bundle, but in general, the symplectic quotient $X^\tau /\!/ T(q)$ may have orbifold singularities, in which case the above construction defines $P_{(\Theta,q)} \to X^\tau /\!/ T(q)$ as an orbibundle.

The cobordism theorem

Theorem C. *Suppose*

$$\sum_i c_i[\Theta_i, q_i] \; = \; 0 \; \in \; \mathcal{A}/\mathcal{R}, \qquad c_i \in \mathbb{Z}.$$

Then there exists an oriented manifold W, with a locally free action of T, and a T-equivariant map

$$W \to X$$

such that

$$\partial(W/T) \; \cong \; \bigsqcup_i c_i P_{(\Theta_i, q_i)}.$$

In particular, for any regular value $p \in \mathfrak{t}^*$ of the moment map, the symplectic quotient $X/\!/T(p)$ is cobordant in the above sense to a union of spaces $P_{(\Theta_i, v_i)}$, for $(\Theta_i, v_i) \in \mathcal{A}_T$, and such spaces can be described as towers of weighted projective bundles over components of the fixed points X^T.

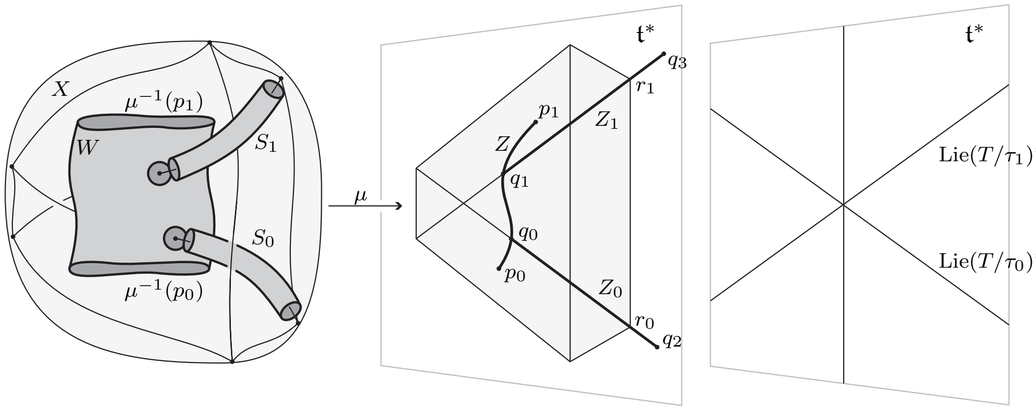


Figure 4: The definitions of this section: Z_0 is a τ_0-transverse path, with endpoints the τ_0-regular values q_0, q_2. Since τ_0 is a 1-torus, there is only one τ_0-flag, namely $\Theta_0 := (1 \subset \tau_0)$. The wall-crossing data of Z_0 is the pair (T, r_0). Now associated to Z_0 is a submanifold-with-boundary $W_0 \subset X^{\tau_0}$, and the space labelled S_0 is $S_{\Theta_0}(\nu X^{\tau_0}|_{W_0})$ (as described in the proof of theorem C). An analogous description holds for Z_1.

Proof. Since we can glue together oriented cobordisms along their boundaries, it is enough to show the above result in the case that $\sum_i c_i(\Theta_i, q_i)$ is one of the relations which generate $\mathcal{R}$.

Each such relation comes from a τ-transverse path Z, and a choice of τ-flag Θ, and so we fix such a Z and Θ. Then we wish to find a manifold W with a locally free T-action, together with an equivariant map $W \to X$, such that

$$\partial(W/T) \cong -P_{(\Theta,q_0)} + P_{(\Theta,q_1)} + \sum_{(H,r)\in \mathrm{data}(Z)} P_{(\Theta\cup H,r)}.$$

In fact we can construct a submanifold $W \subset X$ with this property. The first step is to apply theorem A to Z. Explicitly, Z lies in a subspace $\mathcal{S} \subset \mathfrak{t}^*$, which we can identify with $\mathrm{Lie}(T/\tau)^*$. We then apply theorem A, where the symplectic manifold consists of those components of X^τ which μ maps to $\mathcal{S}$, the torus is T/τ, and the moment map is given by μ with the identification of $\mathcal{S}$ with $\mathrm{Lie}(T/\tau)^*$. This gives a submanifold-with-boundary $W' \subset X^\tau$, with a locally free action of T/τ, and with boundary

$$-X^\tau \cap \mu^{-1}(q_0) \sqcup X^\tau \cap \mu^{-1}(q_1) \sqcup \bigsqcup_{(H,r)\in \mathrm{data}(Z)} S(\nu X^H : X^\tau)\big|_{X^H\cap\mu^{-1}(r)}$$

where $\nu X^H : X^\tau$ denotes the normal bundle to X^H in X^τ, and q_0, q_1 are the endpoints of Z.

But, since $W' \subset X^\tau$ is a submanifold-with-boundary, with a locally free action of T/τ, it follows that

$$W := S_\Theta\left(\nu X^\tau|_{W'} \to W'\right)$$

defines a submanifold of X with a locally free action of T, and $\partial W = S_\Theta\left(\nu X^\tau|_{\partial W'} \to \partial W'\right)$.

Finally, using the fact that

$$S_\Theta\left(\nu X^\tau|_{S(\nu X^H : X^\tau)}\right) = S_{\Theta\cup H}(\nu X^H),$$

we see that W/T has the desired boundary, thus proving the result. $\square$

The structure of the spaces $P_{(\Theta,q)}$

Let $(\Theta, q) \in \mathcal{A}_\tau$, that is, q is a τ-regular value and Θ is a τ-flag.

Proposition 8.4. *The space $P_{(\Theta,q)}$ is the total space of a tower*

$$P_{(\Theta,q)} = P_1 \xrightarrow{\pi_1} P_2 \xrightarrow{\pi_2} \dots \xrightarrow{\pi_{k-1}} P_k \xrightarrow{\pi_k} X^\tau /\!\!/ T(q)$$

where $k = \dim \tau$, and each π_i is an orbibundle projection with fibre a weighted projective space.

We can identify the spaces P_i explicitly (see below). The explicit formulae for cohomology pairings in the next section follow from these identifications (although they can also be deduced by inductively applying theorem B).

Proof. For simplicity of notation we treat explicitly the case in which $\tau = T$, so that $P_{(\Theta,q)}$ is a bundle over certain components of the fixed point set, and we assume such components consist of a single point. Adapting these arguments to deal with the general case is straightforward.

Letting x be the point in question, we set $V = T_x X$, so that V is a complex representation of T.

We choose a decomposition of $\tau = T$ which is compatible with Θ, that is

$$\tau = T_1 \times T_2 \times \ldots \times T_k,$$

where each $T_i \cong H_i/H_{i-1} \cong S^1$.

Then, tracing through the definitions, we see that

1. For $1 \leq i, j \leq k$, each T_i acts on each V_j;
2. If $j > i$ then T_i acts trivially on V_j;
3. The T_i action on V_i leaves only 0 fixed.

We now note the following general fact.

Fact: Suppose $Y_1 \times Y_2$ is acted on by $T_1 \times T_2$, such that the T_1-action is free on Y_1 and trivial on Y_2, and the T_2-action is free on Y_2. Then the projection $Y_1 \times Y_2$ descends to a projection

$$(Y_1 \times Y_2)/(T_1 \times T_2) \to Y_2/T_2$$

with fibre Y_1/T_1.

Hence, defining

$$\begin{aligned} S_i &:= S(V_i) \times S(V_{i+1}) \times \ldots \times S(V_k), \quad \text{and} \\ P_i &:= S_i/(T_i \times T_{i+1} \times \ldots \times T_k), \end{aligned}$$

we see that the natural projection $S_i \to S_{i+1}$ descends to a projection $\pi_i : P_i \to P_{i+1}$, with fibre $S(V_i)/T_i$. As in Proposition 2.7, we can thus express $\pi_i : P_i \to P_{i+1}$ as the weighted projectivization of the complex vector bundle induced by $V_i \times S_{i+1} \to S_{i+1}$. $\square$

Definition 8.5. We can use the above description to orient $P_{(\Theta,q)}$. Recall that Θ is an *oriented* flag: this is equivalent to the statement that each $T_i \cong S^1$ is oriented. Since each V_i is a complex subrepresentation of V, each V_i has an orientation. We thus use the formula of definition 3.4 to orient each $S(V_i)/T_i$, and we give $P_{(\Theta,q)}$ the induced product orientation.

9. Localizating integration formulae to the fixed points

In this section we show how the relations defined in section 7 correspond to integration formulae. We begin by defining a map which generalizes the localization map λ_H defined in section 5. We then state theorem D in terms of this map. We then give an explicit formula for this localization map in terms of characteristic classes.

Definition 9.1. Let τ be a subtorus of T, and let Θ be an oriented τ-flag. Then we define the map

$$\lambda_\Theta : \mathrm{H}^*_T(X) \to \mathrm{H}^*_{T/\tau}(X^\tau)$$

as follows. Firstly, in the case that $\tau = \{1\}$ is the trivial subtorus, so that $\Theta = (1 \subset T)$ is the trivial flag, then we define λ_Θ to be the identity map. Otherwise we set

$$\lambda_\Theta := \lambda_{H_k/H_{k-1}} \circ \dots \lambda_{H_2/H_1} \circ \lambda_{H_1}.$$

Here H_i is the subtorus in the flag Θ:

$$\Theta = (1 = H_0 \subset H_1 \subset H_2 \subset \dots \subset H_k = \tau \subset T),$$

and

$$\lambda_{H_i/H_{i-1}} : \mathrm{H}^*_{T/H_{i-1}}(X^{H_{i-1}}) \to \mathrm{H}^*_{T/H_i}(X^{H_i})$$

is the localization map of definition 5.1, with data consisting of the triple $(X^{H_{i-1}}, T/H_{i-1}, H_i/H_{i-1})$. Recall that $H_i/H_{i-1} \cong S^1$ is assumed to be oriented.

After stating theorem D we will give an explicit formula for λ_Θ using a decomposition of T and characteristic classes.

Note that λ_Θ can equivalently be defined via integration over the fibre of the bundle

$$P_\Theta(\nu X^\tau) \to X^\tau$$

in an analogous way to the definition of λ_H (definition 5.1).

Theorem D. *Suppose*

$$\sum_i c_i[\Theta_i, q_i] \;=\; 0 \;\in\; \mathcal{A}/\mathcal{R}, \quad c_i \in \mathbb{Z}.$$

Then for any $a \in \mathrm{H}^*_T(X)$*,*

$$\sum_i c_i \int_{X^{\tau_i} /\!\!/ T(q_i)} \kappa(\lambda_{\Theta_i}(a)) \;=\; 0.$$

where, for each i*, the flag* Θ_i *is a* τ_i*-flag, and where* κ *is the relevant natural map from the equivariant cohomology of a manifold to the ordinary cohomology of its symplectic quotient, as described in the notation section of the Introduction.*

Moreover, for each flag Θ_i*, the class* $\lambda_{\Theta_i}(a)$ *only depends on the restriction of* a *to the submanifold* X^{τ_i}*.*

The proof consists of straightforward unwinding of the definitions, and can be seen to either follow from theorem C, or from theorem B, using inductive arguments analogous to those in the proof of theorem C. We give a concrete application of this theorem in section 12, in which we calculate some cohomology pairings on the symplectic reduction of products of $\mathbb{CP}^2$.

A formula for λ_Θ in terms of characteristic classes

Suppose Θ is an (oriented) T-flag of subtori (that is, we suppose $\tau = T$). We consider the map

$$\lambda_\Theta : \mathrm{H}^*_T(X) \to \mathrm{H}^*(X^T).$$

We first observe that, for any component $F \subset X^T$ of the fixed point set and any class $a \in \mathrm{H}^*_T(X)$, the restriction of $\lambda_\Theta(a)$ to F only depends on the restriction of a to F (this follows from the definition of λ_H).

Since T acts trivially on F, we have $\mathrm{H}^*_T(F) \cong \mathrm{H}^*(F) \otimes \mathrm{H}^*_T(\mathrm{pt})$. We choose a decomposition

$$T = T_1 \times T_2 \times \ldots \times T_d$$

compatible with the flag Θ, that is, where each $T_i \cong H_i/H_{i-1} \cong S^1$. This gives a set of generators $\{u_1, u_2, \ldots, u_d\}$ of $\mathrm{H}^*_T(\mathrm{pt})$ so that

$$\mathrm{H}^*_T(F) \cong \mathrm{H}^*(F) \otimes \mathbb{Q}[u_1, u_2, \ldots, u_d].$$

Explicitly, u_i is the equivariant first Chern class of the representation of T on $\mathbb{C}$ where T_i acts with weight 1 (recall T_i is oriented), and the other T_j act trivially.

We now define the map

$$\begin{aligned} \ell_i : \mathbb{Q}[u_i] &\to \mathrm{H}^*(F) \otimes \mathbb{Q}[u_{i+1}, \ldots, u_d], \quad \text{by} \\ u_i^{j+k_i} &\mapsto s_j^{T_{i+1}\times\ldots\times T_d}(V_i, T_i) \end{aligned}$$

where $k_i + 1 = \operatorname{rk} V_i$ and $s_j^{T_{i+1}\times\ldots\times T_d}(V_i, T_i)$ is the equivariant weighted Segre class (equivariant with respect to $T_{i+1} \times \ldots \times T_d$) of the bundle $V_i \to F$.

Then ℓ_i extends to a map

$$\tilde{\ell}_i : \mathrm{H}^*(F) \otimes \mathbb{Q}[u_i, \ldots, u_d] \to \mathrm{H}^*(F) \otimes \mathbb{Q}[u_{i+1}, \ldots, u_d]$$

by tensoring with the identity map on the complement of $\mathbb{Q}[u_i]$. Thus $\tilde{\ell}_i$ is a homomorphism of $\mathrm{H}^*(F) \otimes \mathbb{Q}[u_{i+1}, \ldots, u_d]$-modules.

Now for any $a \in \mathrm{H}^*_T(X)$, the restriction $a|_F$ can be decomposed

We then have

Proposition 9.2.

$$\lambda_\Theta(a) = o_T(X) \cdot \tilde{\ell}_d \circ \tilde{\ell}_{d-1} \circ \ldots \circ \tilde{\ell}_1(a|_F).$$

where $o_T(X)$ is the order of the maximal subgroup of T which fixes every point in X.

We will use this formula in the explicit calculations of section 12.

Proof. This follows by repeatedly applying proposition 6.3, using explicit identifications coming from the choice of decomposition of T. For example, we have

$$H_i = T_1 \times T_2 \times \ldots \times T_i$$

and so on. ∎

10. A more refined look at the module of relations

In section 7, we gave a number of definitions, culminating in the definitions of the modules $\mathcal{A}$ and $\mathcal{R}$. The aim of those definitions was to keep track of the relations arising from paths as simply as possible. In this section we give 'improved' versions of these definitions. The result of these improved definitions will be that $\mathcal{A}$ and $\mathcal{A}/\mathcal{R}$ will be much smaller, and should have properties which more accurately reflect the manifold X. The cost of this improvement is that the definitions are somewhat more subtle.

This section contains no new results: its only aim is to give alternative definitions which may be useful in some applications. Theorems C and D are still true with the improved definitions given in this section.

Definition 10.1. We say an action of a Lie group G on a manifold Y is **locally effective** if there is some point in Y whose stabilizer subgroup is finite.

Definition 10.2. Let $\tau \subset T$ be a subtorus. We denote by $X^{[\tau]} \subset X^\tau$ the connected components of X^τ on which the T/τ-action is locally effective.

Note that there are only finitely many subtori τ for which $X^{[\tau]}$ is nonempty.

Given this definition, we redefine the notions of a **τ-regular** point q, a **τ-transverse** path Z, and the **wall-crossing data** of a τ-transverse path by substituting $X^{[\tau]}$ for X^τ in definitions 7.2, 7.3 and 7.4.

Definition 10.3. Let V be a τ-representation, such that $0 \in V$ is the only fixed point. A τ-flag $\Theta = (1 = H_0 \subset H_1 \subset H_2 \subset \ldots \subset H_k = \tau)$ is called **V-admissible** if the associated flag in V

$$V = V^{H_0} \supset V^{H_1} \supset \ldots \supset V^{H_k} = \{0\}$$

has each inclusion a strict inclusion.

Definition 10.4. Let τ be a subtorus of T, and let Θ be a τ-flag and q a τ-regular value (using the version of τ-regular defined in this section). We say the pair (Θ, q) is **admissible** if there is some point $x \in X^{[\tau]} \cap \mu^{-1}(q)$ such that Θ is $\nu_x X^{[\tau]}$-admissible.

It is easy to see that the admissible pairs are precisely those pairs (Θ, q) for which the space $P_{(\Theta,q)}$ is nonempty.

Definition 10.5. We now redefine $\mathcal{A}$ to have generators the set of admissible pairs (Θ, q). We will redefine the submodule of relations $\mathcal{R}$ to come from the data for τ-transverse paths, as τ runs through all subtori, in the same way as before. However there is one difference: some of the pairs which arise from the data of a path may not be admissible, and we simply discard these pairs and construct relations from the pairs that remain. (The point is that these pairs correspond to empty spaces, so there is no harm in discarding them). Explicitly, if $\tau \subset T$ is a subtorus, Z is a τ-transverse path and Θ is an oriented τ-flag, then we take the sum

$$-(\Theta, q_0) + (\Theta, q_1) + \sum_{(H,r)\in \mathrm{data}(Z)} (\Theta \cup H, r),$$

and throw out any terms in this sum which are not admissible pairs. We then define the resulting sum to be a generator of $\mathcal{R}$. (Here, as before, q_0 and q_1 are the endpoints of Z, and $\Theta \cup H$ denotes the oriented H-flag defined by concatenating Θ and H, with H/τ oriented as in data(Z).)

The statement that every element of $\mathcal{A}$ can be localized to the fixed points becomes

Proposition 10.6. *$\mathcal{A}/\mathcal{R}$ is generated by $\mathcal{A}_T/(\mathcal{R}\cap\mathcal{A}_T)$.*

I conjecture

Conjecture 10.7. *For any $0 \le i \le \dim T$, let $\mathcal{A}_i = \bigoplus_{\dim \tau = i} \mathcal{A}_\tau$. Then $\mathcal{A}/\mathcal{R}$ is generated by $\mathcal{A}_i/(\mathcal{R}\cap\mathcal{A}_i)$.*

Question 10.8. Using the 'improved' definitions of this section, is the following 'converse' to theorem D true: Given $a \in \mathrm{H}^*_T(X^T)$, suppose that, for every relation $r \in \mathcal{R}\cap\mathcal{A}_T$, the sum of integrals of classes induced by a vanishes (as in theorem D). Then does a extend to a class $\tilde{a} \in \mathrm{H}^*_T(X)$?

11. Calculations I: cohomology pairings on symplectic quotients of $(S^2)^n$

Consider the unit sphere $S^2 \subset \mathbb{R}^3$. The Euclidean volume on $\mathbb{R}^3$ restricts to give a symplectic form on S^2 (with respect to which its volume is 4π). $SO(3)$ acts naturally on S^2, and this action is Hamiltonian (it is possible to identify $\mathbb{R}^3$ with $\mathrm{Lie}(SO(3))^*$ such that the inclusion of S^2 is a moment map). We choose a maximal torus $S^1 \subset SO(3)$ to be the subgroup which fixes the north and south poles, and normalize so that the positive direction in S^1 rotates the sphere counterclockwise, as seen from the north pole.

Let $X = (S^2)^n$, the n-fold product, with the diagonal action of $SO(3)$, and hence S^1. The symplectic form on X is given by the direct sum of the symplectic forms on the factors. We will fix n to be odd, and calculate cohomology pairings on $X/\!\!/S^1(0)$. We will also invoke a formula proved in [23] to use these pairings to determine cohomology pairings on $(S^2)^n/\!\!/SO(3)$.

The moment map

The action of S^1 on S^2 is Hamiltonian, with moment map given by the height function

$$\begin{aligned} \mu : S^2 &\to \mathbb{R} \\ x &\mapsto \mathrm{ht}(x). \end{aligned}$$

We have $\mu(S^2) = [-1, 1]$, $\mu(\text{north pole}) = 1$, $\mu(\text{south pole}) = -1$. Choosing a compatible almost complex structure (for example the standard one), the weight of the action on the tangent space at the north pole is 1, and the weight at the south pole is -1.

A point in $(S^2)^n$ is given by an n-tuple $(x_1, \dots, x_n)$ where each $x_i \in S^2$. The action of S^1 on $(S^2)^n$ has moment map given by summing the heights on each of the factors:

$$\mu(x_1, \dots, x_n) = \mathrm{ht}(x_1) + \ldots + \mathrm{ht}(x_n).$$

Thus $\mu((S^2)^n) = [-n, n]$.

A point in X is fixed if and only if each x_i is either the north pole or the south pole.

Definition 11.1. Let I be any subset of the set $\{1, \dots, n\}$. Then we define the point $f_I \in (S^2)^n$ by setting x_i to be the south pole if $i \in I$ and the north pole otherwise.

This defines a one-to-one correspondence between the fixed points and the subsets of $\{1, \dots, n\}$. In particular, the fixed points are isolated, and we have

$$\mu(f_I) = n - 2|I|.$$

Hence 0 is a regular value of μ when n is odd.

The integration formula relating the symplectic quotients by a nonabelian group and by its maximal torus

Suppose X is a symplectic manifold with a Hamiltonian action of the nonabelian Lie group G, having moment map $\mu_G : X \to \mathrm{Lie}(G)^*$. The inclusion $T \hookrightarrow G$ induces a projection $\mathrm{Lie}(G)^* \twoheadrightarrow \mathfrak{t}^*$, and composing of μ_G with this projection gives a moment map $\mu_T : X \to \mathfrak{t}^*$ for the action of T on X. In the companion paper [23] the following formula is proven, relating integrals on $X/\!\!/G(0) = \frac{\mu_G^{-1}(0)}{G}$ to integrals on $X/\!\!/T(0) = \frac{\mu_T^{-1}(0)}{T}$.

Proposition 11.2. *For any $a \in \mathrm{H}_G^*(X)$,*

$$\int_{X/\!\!/G(0)} \kappa(a) = \frac{1}{|W|} \int_{X/\!\!/T(0)} \kappa(a) \smile \prod_{\alpha \in \Delta} c_1(L_\alpha).$$

Here $|W|$ denotes the order of the Weyl group, and $\Delta \subset \mathfrak{t}^*$ denotes the set of roots of G (both positive and negative). Given a root α, then the complex line bundle $L_\alpha \to X/\!\!/T(0)$ is the line bundle associated to the fibering $\mu_T^{-1}(0) \to X/\!\!/T(0)$ and the 1-dimensional T-representation of weight α.

The volume of the symplectic quotient

Definition 11.3. The **symplectic volume** of a compact symplectic manifold (or orbifold) (M^{2n}, ω) is the integral $\int_M \omega^n/n!$. From now on we will refer to the symplectic volume as simply the **volume**.

We will now go through the calculations necessary to prove

Proposition 11.4. *For n odd,*

$$\mathrm{vol}((S^2)^n /\!\!/ S^1(0)) = \frac{(2\pi)^{n-1}}{(n-1)!} \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{k} (n-2k)^{n-1},$$

and

$$\mathrm{vol}((S^2)^n /\!\!/ SO(3)) = -\frac{(2\pi)^{n-3}}{(n-3)!} \frac{1}{2} \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{k} (n-2k)^{n-3}.$$

X is endowed with a line bundle $\mathcal{L} \to X$ (known in the literature as the prequantum line bundle), with a connection whose curvature is $-i\omega$. Hence $c_1(\mathcal{L}) = \left[\frac{\omega}{2\pi}\right]$. The action on X lifts to an action on $\mathcal{L}$. Hence the volume of $X/\!\!/S^1(0)$ is given by

$$\mathrm{vol}(X/\!\!/S^1(0)) = \frac{(2\pi)^{n-1}}{(n-1)!} \left\langle \kappa(c_1^{S^1}(\mathcal{L})^{n-1}), [X/\!\!/S^1(0)] \right\rangle$$

where $c_1^{S^1}$ denotes the equivariant first Chern class (and, as usual, $\kappa : \mathrm{H}_T^*(X) \to \mathrm{H}^*(X/\!\!/S^1)$ is the map described in the introduction.)

In order to evaluate classes on $X/\!\!/SO(3)$ we use the integration formula, proposition 11.2 of the companion paper [23]. We first need a definition. Let α be a weight of S^1. Then we denote by $\mathbb{C}_{(\alpha)}$ the representation induced by α, and set $\underline{\mathbb{C}}_{(\alpha)} := X \times \mathbb{C}_{(\alpha)}$, thought of as an equivariant line bundle over X. $\underline{\mathbb{C}}_{(\alpha)}$ induces a line bundle on the quotient $X/\!\!/S^1(0)$, which we denote by $L_{(\alpha)}$. Applying the integration formula, proposition 11.2, we have

$$\mathrm{vol}(X/\!\!/SO3)(0)) =$$

$$\frac{(2\pi)^{n-3}}{(n-3)!} \frac{1}{2} \int_{X/\!\!/S^1(0)} \kappa(c_1^{S^1}(\mathcal{L})^{n-3} \smile c_1^{S^1}(\underline{\mathbb{C}}_{(1)}) \smile c_1^{S^1}(\underline{\mathbb{C}}_{(-1)})).$$

The calculation

We now go through the steps necessary to evaluate cohomology classes on $X/\!\!/T(0)$. Steps 1-3 are independent of the particular class we wish to evaluate, and steps 4 and 5 depend on the class.

Step 1: Fix Z, and enumerate the components $X_{q,i}$. We fix our submanifold Z to be the interval $[0, n+1]$, with $p_0 = 0$ and $p_1 = n+1$, which is outside the image of μ. Then $Z \cap \{\text{walls}\} = \{n - 2k \mid k = 0 \dots \left[\frac{n}{2}\right]\}$. For $q = n - 2k$,

$$\{X_{q,i}\} = \{f_I \mid |I| = k\}.$$

(There are $\binom{n}{k}$ such points.)

Step 2: Identify $\nu X_{q,i}$. Our submanifolds are the points f_I. The normal bundle to f_I is the direct sum of copies of $T_{\underline{n}}S^2$ and $T_{\underline{s}}S^2$. Hence, with $k = |I|$,

$$\nu f_I \cong \mathbb{C}^{n-k}_{(1)} \oplus \mathbb{C}^{k}_{(-1)}$$

Here $\mathbb{C}^m_{(w)}$ denotes $\mathbb{C}^m$ with the S^1-action with weight w. Note that the weights are determined by the isomorphism $S^1 \xrightarrow{\cong} S^1$ induced by the orientation of Z; in our case this is the identity isomorphism. (If we had instead chosen $Z = [-n-1, 0]$, we would have the orientation-reversing isomorphism.)

Step 3: Calculate $s^w(\nu X_{q,i})$. The weighted Segre class lies in $\mathrm{H}^*_{T/H}(X_{q,i})$. In our case this is just $\mathrm{H}^*(f_I)$. We have

$$s^w_j(\nu f_I) = \begin{cases} (\prod\{\text{wts}\})^{-1} = (-1)^k & j = 0, \\ 0 & j > 0. \end{cases}$$

Since all the weights are ± 1, we have $\mathrm{hcf}\{|\text{wts}|\} = 1$.

Step 4: Calculate a in terms of local bases.

$$\mathcal{L}|_{f_I} \cong \mathbb{C}_{(\mu(f_I))} = \mathbb{C}_{(n-2k)}.$$

Hence

$$c_1^{S^1}(\mathcal{L})|_{f_I} = (n-2k)u,$$

where u is the positive generator of $\mathrm{H}^*_{S^1}(\mathrm{pt})$. And

$$c_1^{S^1}(\underline{\mathbb{C}}_{(w)})|_{f_I} = wu, \qquad w \in \mathbb{Z}.$$

The two classes we wish to evaluate are

$$a_1 := c_1^{S^1}(\mathcal{L})^{n-1}$$

which gives us the degree of $X/\!\!/S^1(0)$, and

$$a_2 := \frac{1}{2} c_1^{S^1}(\mathcal{L})^{n-3} \smile c_1^{S^1}(\underline{\mathbb{C}}_{(1)}) \smile c_1^{S^1}(\underline{\mathbb{C}}_{(-1)})$$

which gives the degree of $X/\!\!/SO(3)$. We thus have

$$a_1|_{f_I} = (n-2k)^{n-1} u^{n-1}$$

and

$$a_2|_{f_I} = -\frac{1}{2}(n-2k)^{n-3} u^{n-1}.$$

Step 5: Apply the formula. Using the Segre classes calculated above, we have

$$\lambda_{S^1}(a_1)|_{f_I} = (-1)^k (n-2k)^{n-1}$$

and

$$\lambda_{S^1}(a_2)|_{f_I} = -\frac{1}{2}(-1)^k (n-2k)^{n-3}$$

Hence, summing over the $\binom{n}{k}$ points f_I with $|I| = k$, and letting k run from 0 to $\left[\frac{n}{2}\right]$, we get proposition 11.4.

Cohomology classes on $(S^2)^n$.

We wish to describe cohomology classes on X, and in particular understand their restrictions to the fixed points. There are some standard results which will help us greatly. We first recall these general results.

Let G be a compact Lie group, with $T \subset G$ the maximal torus, and W the Weyl group. For any G-space Y, there is a natural action of W on $\mathrm{H}^*_T(Y)$, and the natural map $\mathrm{H}^*_G(X) \to \mathrm{H}^*_T(X)$ defines an isomorphism

$$\mathrm{H}^*_G(X) \xrightarrow{\cong} \mathrm{H}^*_T(X)^W$$

(see e.g. [1, Equation 2.11]).

Suppose X_1 and X_2 are symplectic manifolds, with Hamiltonian actions of the group G. Then the homotopy quotients $(X_1)_G$ and $(X_2)_G$ are cohomologially trivial as bundles over BG [21, Proposition 5.8]. This means that the Serre spectral sequence of the fibering $(X_i)_G \to BG$ degenerates at the E_2 term. We give the product $X_1 \times X_2$ the diagonal G-action. Then it follows that

$$\mathrm{H}^*_G(X_1 \times X_2) \cong \mathrm{H}^*_G(X_1) \otimes_{\mathrm{H}^*(BG)} \mathrm{H}^*_G(X_2). \tag{11.5}$$

In order to see this, we first note that $X_1 \times X_2$ is both a Hamiltonian G-manifold (with moment map given by the sums of the respective moment maps), and a Hamiltonian $G \times G$ manifold. Now consider the diagonal map

$$j : BG \hookrightarrow BG \times BG.$$

This induces the ring homomorphism

$$\begin{aligned} \mathrm{H}^*((X_1)_G) \otimes_{\mathrm{H}^*(BG)} \mathrm{H}^*((X_2)_G) &\to \mathrm{H}^*((X_1)_G \times_{BG} (X_2)_G) \\ a, b &\mapsto j^*(a \otimes b). \end{aligned}$$

But by degeneracy of the relevant spectral sequences, this must be an isomorphism of groups, and hence an isomorphism of rings. Thus we have equation (11.5) above.

This means we can represent an equivariant cohomology class on $X_1 \times X_2$ as a sum of tensor products of equivariant classes on X_1 and X_2. Now for our calculations we will only need to know the restriction of a class to the fixed points. However restriction commutes with the above tensor product (and the fixed point set of $X_1 \times X_2$ is the product of the fixed points of X_1 with the fixed points of X_2).

We now specialize to the case at hand. Let $\underline{n}$ denote the north pole and $\underline{s}$ the south pole of S^2. The restriction map

$$\mathrm{H}^*_{S^1}(S^2) \to \mathrm{H}^*_{S^1}(\{\underline{n}, \underline{s}\})$$

is injective. We set

$$\mathrm{H}^*_{S^1}(\{\underline{n},\underline{s}\}) = \mathbb{Q}[u_{\underline{n}}] \oplus \mathbb{Q}[u_{\underline{s}}]$$

so that for example $u_{\underline{n}}$ is the degree-2 generator of the equivariant cohomology of $\{\underline{n}\}$.

Then the image of the restriction consists of those pairs of polynomials whose degree-zero terms agree. (One can see this e.g. by thinking about the topology of the homotopy quotients.)

The $SO(3)$-equivariant cohomology is the subring invariant under the Weyl group, in this case $\mathbb{Z}/2\mathbb{Z}$. The nontrivial element of W permutes the north and south poles and simultaneously acts via the involution on S^1. This identifies $u_{\underline{n}}$ with $-u_{\underline{s}}$. Hence, fixing a normalization,

$$\mathrm{H}^*_{SO(3)}(S^2) \cong \mathbb{Q}[v]$$

with the inclusion given by

$$\begin{aligned} \mathbb{Q}[v] &\hookrightarrow \mathbb{Q}[u_{\underline{n}}] \oplus \mathbb{Q}[u_{\underline{s}}] \\ a(v) &\mapsto a(u_{\underline{n}}) \oplus a(-u_{\underline{s}}). \end{aligned}$$

Applying equation 11.5, we represent a class on X as a sum of tensor products of classes on S^2. Hence, consider the class

$$a^{(1)} \otimes a^{(2)} \otimes \ldots \otimes a^{(n)}$$

where $a^{(i)}$ is an equivariant cohomology class on the i-th copy of S^2. We represent $a^{(i)}$ as the pair of polynomials $(a^{(i)}_{\underline{n}}, a^{(i)}_{\underline{s}})$. A fixed point f_I is an element of the set $\{\underline{n},\underline{s}\}^n$, and the restriction of $a^{(1)} \otimes a^{(2)} \otimes \ldots \otimes a^{(n)}$ to f_I is simply the product of the appropriate polynomials. (For example if $I = \varnothing$, then $f_I = (\underline{n}, \ldots, \underline{n})$, and the restriction to f_I is $\prod_{i=1}^n a^{(i)}_{\underline{n}}$.)

We shall concentrate on evaluating classes on $X/\!\!/SO(3)$. Any such class is a linear combination of classes of the form

$$a = v_1^{l_1} \otimes v_2^{l_2} \otimes \ldots \otimes v_n^{l_n}.$$

a is of top degree when $l_1 + \ldots + l_n = n-3$. We now calculate $a|_{f_I}$. Define, for $i \in \{1 \ldots n\}$,

$$\sigma(i) = \begin{cases} 1, & i \notin I; \\ -1, & i \in I. \end{cases}$$

Then

$$\begin{aligned} a|_{f_I} &= (\sigma(1)u)^{l_1} \cdot \ldots \cdot (\sigma(n)u)^{l_n} \\ &= u^{\sum l_i} \prod_{i=1}^n \sigma(i)^{l_i}. \end{aligned}$$

Applying the integration formula, and assuming $\sum l_i = n-3$,

$$\int_{X/\!\!/SO(3)} a = -\frac{1}{2} \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \sum_{|I|=k} \left(\prod_{i=1}^n \sigma(i)^{l_i} \right).$$

It is clear that this expression only depends on the parity of the l_i, and is invariant under permuting the S^2 factors. This will allow us to deduce quite a lot, but for the moment we will press on and derive an explicit formula. Define $J \subset \{1 \dots n\}$ by

$$i \in J \iff l_i \text{ odd}$$

and set $m = |J|$. Then

$$\begin{aligned}\int_{X/\!\!/SO(3)} a &= -\frac{1}{2}\sum_{k=0}^{\frac{n-1}{2}}(-1)^k \sum_{|I|=k}(-1)^{|J\cap I|} \\ &= -\frac{1}{2}\sum_{|I|\le\frac{n-1}{2}}(-1)^{|I|}(-1)^{|J\cap I|}.\end{aligned}$$

Now since $\sum l_i = n-3$, at least one $l_i = 0$. By invariance we may as well assume $J = \{1 \dots m\}$, and hence $l_n = 0$. We can split the above sum into those I which contain n and those which don't. The resulting cancellations leave us with

$$\begin{aligned}\int_{X/\!\!/SO(3)} a &= -\frac{1}{2}\sum_{\substack{K\subset\{1\dots n-1\}\\ |K|=\frac{n-1}{2}}}(-1)^{|K|}(-1)^{|K\cap\{1\dots m\}|} \\ &= -\frac{1}{2}(-1)^{\frac{n-1}{2}}\sum_{\substack{K\subset\{1\dots n-1\}\\ |K|=\frac{n-1}{2}}}(-1)^{|K\cap\{1\dots m\}|}.\end{aligned}$$

From this description one can easily derive explicit computational formulæ. Alternatively, using the easily-described product structure in $\mathrm{H}^*_{SO(3)}(X)$ and Poincaré duality in the symplectic quotient, we can see some classes whose image must vanish on the symplectic quotient.

Proposition 11.6. *Using the identification described above*

$$\mathrm{H}^*_{SO(3)}((S^2)^n) \cong \mathbb{Q}[v_1, v_2, \dots, v_n]$$

and the natural ring homomorphism $\kappa : \mathrm{H}^*_{SO(3)}((S^2)^n) \to \mathrm{H}^*((S^2)^n//SO(3))$, we have

$$\begin{aligned}&\int_{(S^2)^n/\!\!/SO(3)} \kappa(v_1^{l_1}v_2^{l_2}\dots v_n^{l_n}) \\ &\qquad = -\frac{1}{2}(-1)^{\frac{n-1}{2}}\sum_{\substack{K\subset\{1\dots n-1\}\\ |K|=\frac{n-1}{2}}}(-1)^{|K\cap\{1\dots m\}|} \\ &\qquad = \frac{1}{2}(-1)^{\frac{n-1}{2}}\left(\binom{n-1}{\frac{n-1}{2}} - 2\sum_{j=0}^{\frac{m}{2}}\binom{m}{2j}\binom{n-1-m}{\frac{n-1}{2}-2j}\right)\end{aligned}$$

where $\sum_i l_i = n-3$ and m is equal to the number of odd l_i.

It follows, for example, that the ideal $\ker(\kappa)$ contains the elements $v_i^2 - v_j^2$.

12. Calculations II: volume of the symplectic quotient of $(\mathbb{CP}^2)^n$

Generalities on $\mathbb{CP}^{k-1}$

Consider the defining representation of U_k on $\mathbb{C}^k$.

The maximal torus $T^k \subset U_k$ consists of the diagonal matrices

$$\{\begin{pmatrix} e^{it_1} & & & \\ & e^{it_2} & & \\ & & \ddots & \\ & & & e^{it_k} \end{pmatrix} \mid t_i \in \mathbb{R}\}.$$

The moment map for the action of the maximal torus on $\mathbb{C}^k$ is

$$\mu(z_1, \ldots z_k) = -\frac{1}{2}(|z_1|^2, \ldots, |z_k|^2).$$

The centre

$$Z(U_k) = \{\begin{pmatrix} e^{it} & & \\ & \ddots & \\ & & e^{it} \end{pmatrix} \mid t \in \mathbb{R}\}$$

acts, with moment map

$$\mu_Z(z) = -\frac{1}{2}\sum |z_i|^2.$$

We can take the symplectic quotient of $\mathbb{C}^k$ by $Z(U_k)$ at any negative value, to get $\mathbb{CP}^{k-1}$ (with a scaled symplectic form). Henceforth, we let $\mathbb{CP}^{k-1}$ denote the symplectic manifold $\mathbb{C}^k /\!\!/ Z(U_k)(-k)$. This is endowed with prequantum line bundle $\mathcal{L} \to \mathbb{CP}^{k-1}$ of degree k. PU_k acts on $\mathbb{CP}^{k-1}$, and the action lifts to $\mathcal{L}$.

Let T denote the $(k-1)$-torus. We identify T with the maximal torus of PU_k via the inclusion into U_k

$$T := \{\begin{pmatrix} e^{it_1} & & \\ & \ddots & \\ & & e^{it_{k-1}} \end{pmatrix}\} \hookrightarrow \{\begin{pmatrix} e^{it_1} & & & \\ & \ddots & & \\ & & e^{it_{k-1}} & \\ & & & 1 \end{pmatrix}\}$$

The image of this inclusion is a *slice*: every element of T^k decomposes in a unique way as a product of elements of $Z(U_k)$ and T, thus identifying T with the maximal torus of PU_k.

Let $\mathfrak{t}^* \cong \mathbb{R}^{k-1}$ have standard basis $\{e_1, \ldots, e_{k-1}\}$. Then e_j corresponds to the representation

$$e_j : \begin{pmatrix} e^{it_1} & & \\ & \ddots & \\ & & e^{it_{k-1}} \end{pmatrix} \mapsto e^{it_j}$$

The T-action on $\mathbb{CP}^{k-1}$ has fixed points $\{F_j \mid j = 1 \ldots k\}$, where F_j denotes the point $[0 : \ldots : 1 : \ldots : 0]$ (only the j-th coordinate nonzero). We henceforth let μ denote the moment map for the action of T on $\mathbb{CP}^{k-1}$. The image of μ is the convex hull of the points $\mu(F_j)$. And

$$\mu(F_j) = \begin{cases} \left(\sum_{i=1}^{k-1} e_i\right) - ke_j, & j < k \\ \sum_{i=1}^{k-1} e_i & j = k \end{cases}$$

The walls for μ have corresponding subgroups $H_j \cong S^1$, for $j = 1 \dots k$. H_j stabilizes the wall equal to the convex hull of the points $\{\mu(F_i) \mid i \neq j\}$. We have

$$H_j = \{ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & e^{it} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, t \in \mathbb{R}\} \qquad \text{for } j = 1 \dots k-1$$

and

$$H_k = \{ \begin{pmatrix} e^{-it} & & \\ & \ddots & \\ & & e^{-it} \end{pmatrix}, t \in \mathbb{R}\}$$

We let H_j denote the above subgroup, with the isomorphism $H_j \xrightarrow{\cong} S^1$ implied by the above coordinates on H_j. We write $\bar{H}_j$ for the subgroup with the opposite isomorphism with S^1.

A set of positive roots for PU_k is given by $\{e_i - e_j \mid i < j \leq k-1\} \cup \{e_i \mid i = 1 \dots k-1\}$.

The action of T on the normal bundle to the fixed point F_j is given by

$$\nu F_j \cong \begin{cases} \bigoplus_{i \neq j} \mathbb{C}_{(e_i - e_j)} \oplus \mathbb{C}_{(-e_j)}, & j \leq k-1 \\ \bigoplus_{i \neq k} \mathbb{C}_{(e_i)}, & j = k \end{cases}$$

We now consider the diagonal action of PU_k on $(\mathbb{CP}^{k-1})^n$, and hence of $T \subset PU_k$. The fixed points under the T-action are simply elements of the n-fold product of the fixed points in $\mathbb{CP}^{k-1}$. Thus they correspond to partitions

$$\{1 \dots n\} = I_1 \sqcup I_2 \sqcup \dots \sqcup I_k$$

in the obvious way. We denote such a fixed point by

$$F_{I_1, \dots, I_k} \in (\mathbb{CP}^{k-1})^n$$

Calculations on $(\mathbb{CP}^2)^n$

We now specialize to $\mathbb{CP}^2$, setting $X = (\mathbb{CP}^2)^n$. We will calculate invariants of $X /\!\!/ T(0)$ and $X /\!\!/ PU_3$, for n not a multiple of 3.

The fixed points correspond to partitions

$$\{1 \dots n\} = I_1 \sqcup I_2 \sqcup I_3.$$

For $n = 1$ we have

$$\begin{aligned} \mu(F_1) &= e_2 - 2e_1 \\ \mu(F_2) &= e_1 - 2e_2 \\ \mu(F_3) &= e_1 + e_2 \end{aligned}$$

Hence, in general

$$\mu(F_{I_1, I_2, I_3}) = (-2|I_1| + |I_2| + |I_3|)e_1 + (|I_1| - 2|I_2| + |I_3|)e_2.$$

It follows that 0 is a regular value as long as n is not a multiple of 3.

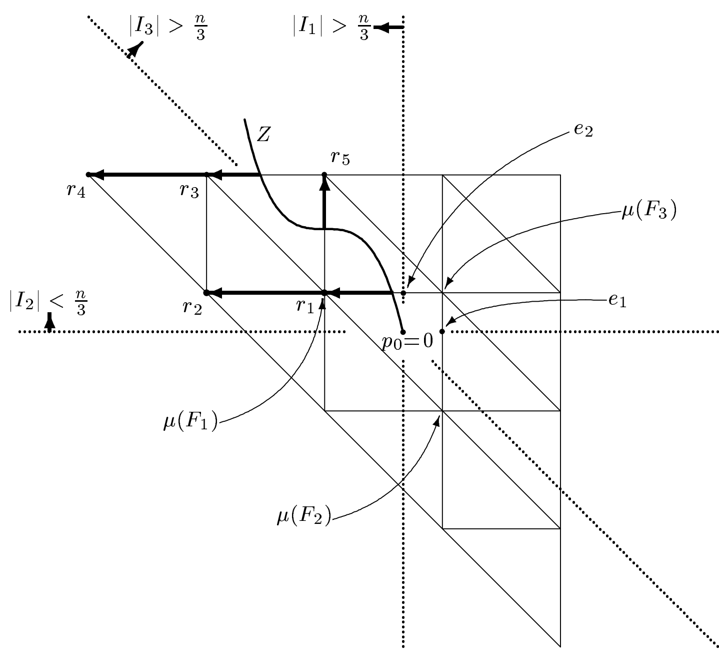


Figure 5: The moment map for $(\mathbb{CP}^2)^4$, showing the transverse paths used in the calculation, with their wall-crossings.

We start by taking the path Z, as depicted in figure 12. In the case of $(\mathbb{CP}^2)^4$, Z has 3 wall-crossings. The horizontal walls (in the figure) correspond to the subgroup

$$H_2 = \{ \begin{pmatrix} 1 & \\ & e^{it} \end{pmatrix} \mid t \in \mathbb{R} \}$$

Z crosses these walls in in the same direction as e_2, so that the isomorphism $H_2 \xrightarrow{\cong} S^1$ is the standard one for H_2, as described above. On the other hand, the vertical walls correspond to the subgroup H_1, and the direction of the crossing by Z corresponds to the oriented subgroup $\bar{H}_1$.

Let Θ_1 denote the flag

$$\Theta_1 = (H_2, H_2 \times \bar{H}_1)$$

and let Θ_2 denote the flag

$$\Theta_2 = (\bar{H}_1, \bar{H}_1 \times H_2)$$

We then have, in the case $n = 4$,

$$p_0 \sim (r_1, \Theta_1) + (r_2, \Theta_1) + (r_3, \Theta_1) + (r_4, \Theta_1) + (r_5, \Theta_2)$$

In general, let R_1 denote the set of vertices corresponding to fixed points F_{I_1,I_2,I_3} with $|I_1| > \frac{n}{3}$ and $|I_3| > \frac{n}{3}$, and R_2 the vertices corresponding to fixed points F_{I_1,I_2,I_3} with $|I_2| < \frac{n}{3}$ and $|I_3| < \frac{n}{3}$. We then have

$$p_0 \sim \sum_{r \in R_1} (r, \Theta_1) + \sum_{r \in R_2} (r, \Theta_2). \tag{12.1}$$

Fixing attention on the point F_{I_1,I_2,I_3}, we now calculate the maps

$$\lambda_{\Theta_i} : \mathrm{H}^*_T(\mathrm{pt}) = \mathbb{Q}[u_1, u_2] \to \mathrm{H}^*(\mathrm{pt}) = \mathbb{Q}$$

where u_1 and u_2 are the generators corresponding to H_1 and H_2. We have

$$V := \nu F_{I_1,I_2,I_3} \cong \mathbb{C}^{|I_1|}_{(e_2-e_1)} \oplus \mathbb{C}^{|I_1|}_{(-e_1)} \oplus \mathbb{C}^{|I_2|}_{(e_1-e_2)} \oplus \mathbb{C}^{|I_2|}_{(-e_2)} \oplus \mathbb{C}^{|I_3|}_{(e_1)} \oplus \mathbb{C}^{|I_3|}_{(e_2)}.$$

The subbundle stabilized by H_2 is

$$V^{H_2} = \mathbb{C}^{|I_1|}_{(-e_1)} \oplus \mathbb{C}^{|I_3|}_{(e_1)}.$$

Hence, to calculate λ_{Θ_1} we need the equivariant weighted Segre classes of V/V^{H_2}.

$$V/V^{H_2} \cong \mathbb{C}^{|I_1|}_{(e_2-e_1)} \oplus \mathbb{C}^{|I_2|}_{(e_1-e_2)} \oplus \mathbb{C}^{|I_2|}_{(-e_2)} \oplus \mathbb{C}^{|I_3|}_{(e_2)}.$$

Now the weighted Chern class

$$c_1^{H_1}(\mathbb{C}_{(ke_1+le_2)}) = ku_1$$

and hence

$$s^w_{H_1}(\mathbb{C}_{(ke_1+le_2)}) = (l + ku_1)^{-1}$$

Therefore

$$\begin{aligned} s^w_{H_1}(V/V^{H_2}) &= (1-u_1)^{-|I_1|}(u_1-1)^{-|I_2|}(-1)^{-|I_2|}(1)^{-|I_3|} \\ &= (1-u_1)^{-(|I_1|+|I_2|)} \end{aligned}$$

and $\mathrm{rk}(V/V^{H_2}) = |I_1| + 2|I_2| + |I_3|$. Hence, setting $k = \mathrm{rk}(V/V^{H_2})$, and $l = |I_1| + |I_2|$,

$$\lambda(H_2, H_1, V/V^{H_2}) : u_2^j \mapsto \begin{cases} 0, & j < k-1 \\ \binom{j+1-k+l-1}{l-1} u_1^{j+1-k}, & j \geq k-1 \end{cases}$$

By the functorial properties of integration over the fibre, this map commutes with multiplication by $u_1^{j_1}$. To get λ_{Θ_1} we must compose with $\lambda(\bar{H}_1, 1, V^{H_2})$, which is equal to $-\lambda(H_1, 1, V^{H_2})$. We have

$$\lambda(H_1, 1, V^{H_2}) : u_1^j \mapsto \begin{cases} (-1)^{|I_1|}, & j = |I_1| + |I_3| - 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\lambda_{\Theta_1}(u_1^{j_1} u_2^{j_2}) = \begin{cases} (-1)^{|I_1|+1} \binom{j_2 - |I_2| - |I_3|}{|I_1|+|I_2|-1}, & \substack{j_2 \geq |I_1|+2|I_2|+|I_3|-1 \\ \text{and } j_1+j_2=2m-2} \\ 0 & \text{otherwise.} \end{cases}$$

And similarly

$$\lambda_{\Theta_2}(u_1^{j_1} u_2^{j_2}) = \begin{cases} (-1)^{|I_2|+1} \binom{j_1 - |I_1| - |I_3|}{|I_1|+|I_2|-1}, & \substack{j_1 \geq 2|I_1|+|I_2|+|I_3|-1 \\ \text{and } j_1+j_2=2m-2} \\ 0 & \text{otherwise.} \end{cases}$$

The Volume

We can now easily write down formulæ for the evaluation of cohomology classes on $(\mathbb{CP}^2)^n/\!\!/T$. And, by applying the integration formula from the companion paper [23] relating evaluation of classes on G-symplectic-quotients to evaluation on T-symplectic-quotients, we can write down formulæ for the evaluation of classes on $(\mathbb{CP}^2)^n/\!\!/PU_3$. As an example, we will give a formula for the volume of $(\mathbb{CP}^2)^n/\!\!/PU_3$.

As usual, the 'prequantum line bundle' $\mathcal{L} \to (\mathbb{CP}^2)^n$, which has first Chern class equal to $\left[\frac{\omega}{2\pi}\right]$, descends to a line bundle over the symplectic quotient, which we also denote by $\mathcal{L}$. The dimension of $(\mathbb{CP}^2)^n/\!\!/PU_3$ is $4n-16$, and hence the volume is equal to the evaluation of the class $\frac{(2\pi c_1(\mathcal{L}))^{2n-8}}{(2n-8)!}$ against the fundamental class.

We define

$$a := \frac{1}{6(2n-8)!}(2\pi c_1^{S^1}(\mathcal{L}))^{2n-8} \smile \prod_{\alpha\in\Delta} c_1^{S^1}(\mathbb{C}_{(\alpha)}),$$

where $\Delta = \{e_1-e_2, e_1, e_2, e_2-e_1, -e_1, -e_2\}$ is the set of roots. Then we have

$$\mathrm{vol}((\mathbb{CP}^2)^n/\!\!/PU_3) = \int_{(\mathbb{CP}^2)^n/\!\!/T(0)} a$$

and, applying Theorem D and equation (12.1),

$$\mathrm{vol}((\mathbb{CP}^2)^n/\!\!/PU_3) = \sum_{\substack{(I_1,I_2,I_3)\\ |I_1|>\frac{n}{3},|I_3|>\frac{n}{3}}} \lambda_{\Theta_1}(a|_{F_{I_1,I_2,I_3}}) + \sum_{\substack{(I_1,I_2,I_3)\\ |I_2|<\frac{n}{3},|I_3|<\frac{n}{3}}} \lambda_{\Theta_2}(a|_{F_{I_1,I_2,I_3}})$$

where the triples (I_1, I_2, I_3) run through partitions of $\{1\dots n\}$. Given such a partition, set $i_1 = |I_1|, i_2 = |I_2|, i_3 = |I_3|$. Then

$$\begin{aligned}\mathcal{L}|_{F_{I_1,I_2,I_3}} &\cong \mathbb{C}_{(\mu(F_{I_1,I_2,I_3}))}\\ &= \mathbb{C}_{((-2i_1+i_2+i_3)e_1+(i_1-2i_2+i_3)e_2)}.\end{aligned}$$

Hence

$$c_1^{S^1}(\mathcal{L})\Big|_{F_{I_1,I_2,I_3}} = (-2i_1+i_2+i_3)u_1 + (i_1-2i_2+i_3)u_2,$$

so that

$$a|_{F_{I_1,I_2,I_3}} = ((-2i_1+i_2+i_3)u_1 + (i_1-2i_2+i_3)u_2)^{2n-8} \cdot (2u_1^3u_2^3 - u_1^4u_2^2 - u_1^2u_2^4).$$

Applying the formulæ for λ_{Θ_1} and λ_{Θ_2}, and using the identity $2\binom{m}{k} - \binom{m+1}{k} - \binom{m-1}{k} =$

$-\binom{m-1}{k-2}$, we easily derive the unilluminating but nontheless computable formula

$$\frac{(2n-8)!}{(2\pi)^{2n-8}}\operatorname{vol}((\mathbb{CP}^2)^n/\!\!/PU_3) =$$

$$\sum_{\substack{i_1>\frac{n}{3},i_3>\frac{n}{3}\\ i_1+i_3\le n}} \frac{n!(-1)^{i_1+1}}{i_1!i_3!(n-i_1-i_3)!}\cdot$$

$$\left(\binom{2n-8}{i_1+i_3-4}(n-3i_1)^{i_1+i_3-4}(3i_1+3i_3-n)^{2n-4-i_1-i_3}(2+i_3-n)-\right.$$

$$\binom{2n-8}{i_1+i_3-3}(n-3i_1)^{i_1+i_3-3}(3i_1+3i_3-n)^{2n-5-i_1-i_3}-$$

$$\left.\sum_{j=0}^{i_1+i_3-5}\binom{n+i_1-6-j}{n-i_3-3}\binom{2n-8}{j}(n-3i_1)^j(3i_1+3i_3-n)^{2n-8-j}\right)$$

$$+\sum_{\substack{i_1<\frac{n}{3},i_2<\frac{n}{3}\\ i_1+i_2\le n}} \frac{n!(-1)^{i_1+1}}{i_1!i_2!(n-i_1-i_2)!}\cdot$$

$$\left(\binom{2n-8}{i_1+i_2-4}(n-3i_1)^{i_1+i_2-4}(3i_1+3i_2-n)^{2n-4-i_1-i_2}(2+i_2-n)-\right.$$

$$\binom{2n-8}{i_1+i_2-3}(n-3i_1)^{i_1+i_2-3}(3i_1+3i_2-n)^{2n-5-i_1-i_2}-$$

$$\left.\sum_{j=0}^{i_1+i_2-5}\binom{n+i_1-6-j}{n-i_2-3}\binom{2n-8}{j}(n-3i_1)^j(3i_1+3i_2-n)^{2n-8-j}\right).$$

A. Orbifolds, orbifold-fibre-bundles, and integration over the fibre

The purpose of this appendix is to collect together a number of facts about orbifolds which we use in the paper. These are all straightforward generalizations of standard results.

An orbifold is a generalization of a manifold, and can roughly be thought of as follows: whereas an n-dimensional manifold is locally modelled on $\mathbb{R}^n$, an n-dimensional orbifold is locally modelled on the quotient of $\mathbb{R}^n$ by a finite group. Orbifolds were first defined and studied by Satake in his announcement [26] and his paper [27] (Satake used the term 'V-manifold'; the term 'orbifold' is due to Thurston). Our interest in orbifolds comes from the fact that the wall-crossing-cobordism and its boundary are in general orbifolds (even if we are interested in a symplectic quotient which is smooth, we may encounter orbifold singularities after crossing a wall).

In this appendix we collect together facts involving orbifolds which we need in the rest of the paper. These facts all involve integration on orbifolds, in one form or another, and can be seen as straightforward generalizations of standard facts involving manifolds. These generalizations exist because an orbifold is a 'rational (co)homology manifold', which basically means that, if we take rational coefficients, it possesses the same homological and cohomological properties as a manifold.

We begin by giving Satake's definition of an orbifold, as well as his generalizations to oriented and symplectic orbifolds. We then state the various facts involving orbifolds, and indicate how these facts follow from results in the literature.

The definition of an orbifold

We now give Satake's definitions. We do this to set up notation which we refer to in the rest of the appendix, but also to make explicit some of the subtleties in the definition. These subtleties are necessary for orbifolds to have the good properties that we need (such as a rational fundamental class).

Definition A.1 (Satake [26, 27]). Let M be a Hausdorff topological space. A (C^∞) **orbifold structure** on M consists of a covering $\mathcal{U}$ of M by open sets, and for each open set $U \in \mathcal{U}$, an associated triple $(\tilde{U}, G_U, \varphi_U)$, where

$\tilde{U}$ is a connected open subset of $\mathbb{R}^n$;

G_U is a finite group of linear transformations mapping $\tilde{U}$ to itself, such that the set of points fixed by G_U has codimension ≥ 2; and

φ_U is a continuous map $\tilde{U} \to U$ such that, for every $x \in \tilde{U}$ and $g \in G_U$, $\varphi_U(gx) = \varphi_U(x)$. We assume that the induced map $G_U \backslash \tilde{U} \to U$ is a homeomorphism.

Moreover, if $U, V \in \mathcal{U}$ are open sets such that $U \subset V$, then we are given an injective group homomorphism $\beta_{UV} : G_U \hookrightarrow G_V$, and an inclusion $i_{UV} : \tilde{U} \hookrightarrow \tilde{V}$ which is a diffeomorphism onto its image, and which is equivariant with respect to the action of G_U (and its image in G_V), and such that $\varphi_U = \varphi_V \circ i_{UV}$. Finally, we assume that the open sets in $\mathcal{U}$ form a basis for the topology of M. (It is fairly standard to refer to $\tilde{U}$ as a local cover, G_U as a local group, and φ_U as a local covering map.)

An **orbifold**, then, is a space M together with an equivalence class of orbifold structures on M (see Satake [26] for details of the straightforward notion of when two such sets of data define the same orbifold structure).

By enhancing the definition of an orbifold structure, we can define an **oriented orbifold**: we ask that each $\tilde{U}$ be given an orientation which is preserved by the action of the group G_U, and that such orientations be compatible with the inclusions $i_{UV} : \tilde{U} \hookrightarrow \tilde{V}$.

Similarly, we define a **symplectic orbifold** by asking that each $\tilde{U}$ be given a symplectic form, with the same invariance and compatibility conditions.

Definition A.2. A point x of an orbifold M is a **smooth point** if there exists some open set $U \in \mathcal{U}$ containing x, and such that the associated group G_U is the trivial group. The set of points which are not smooth points are called **singular points**.

Remark A.3. The set of smooth points of an orbifold M is connected (within each component of M). More precisely, given any open set $U \in \mathcal{U}$ with associated triple $(\tilde{U}, G_U, \varphi_U)$, then the set of singular points in U is the image, under φ_U, of a finite union of submanifolds of $\tilde{U}$ having codimension ≥ 2. Each of these submanifolds is the submanifold of points fixed by some nontrivial element $g \in G_U$. (A straightforward argument by contradiction shows that the codimension restriction on the fixed points of each local group G_U implies the same restriction for each nontrivial subgroup of G_U, and hence for each nontrivial $g \in G_U$).

The fundamental class of an oriented orbifold

Fact A.4. *Let M be an n-dimensional compact oriented orbifold (without boundary). Then the orientation defines a rational fundamental class $[M] \in \mathrm{H}_n(M)$ (recall that we are taking homology and cohomology with rational coefficients throughout this paper). Moreover, M satisfies rational Poincaré duality, which can be expressed as the fact that the pairing*

$\mathrm{H}^i(M) \times \mathrm{H}^{n-i}(M) \to \mathbb{Q}$ *given by* $(a,b) \mapsto \int_M a \smile b$ *is a dual pairing on the rational cohomology of* M*.*

The relationship between the orientation and the fundamental class is as follows. At any smooth point $x \in M$, we use the orientation to define a generator $1_x \in \mathrm{H}_n(M, M \setminus \{x\})$ via the identification with $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \mathbb{Q}$ given by excision (using an oriented chart). Then the fundamental class is the unique class $[M] \in \mathrm{H}_n(M)$ whose image under the natural map $\mathrm{H}_n(M) \to \mathrm{H}_n(M, M \setminus \{x\})$ has image 1_x, for each smooth point x. (Since the set of smooth points is connected, we actually only need to use one smooth point for each component of M to get the right normalization.)

Sketch of proof. There are two different approaches to the proof. Satake's approach [26, 27] is to define an orbifold version of the de Rham complex[3] and to prove de Rham's theorem: that the orbifold de Rham cohomology is canonically isomorphic to the singular cohomology of M (with real coefficients). The fundamental class is then defined in terms of integration.

The other approach is to use the notion of a 'rational homology manifold', as described by Borel in [4, chapters I–II][4]. An orbifold is a rational homology manifold, and Borel shows how various properties of the homology of manifolds go over to rational homology manifolds, including the existence of a (rational) fundamental class and (rational) Poincaré duality. ☐

Oriented orbifolds with boundary and Stokes's theorem

Satake defines an orbifold-with-boundary in [27, section 3.4]. His definition is equivalent to modifying the definition of orbifold by allowing the open covers $\tilde{U}$ to be open subsets of $\mathbb{R}^n$ or of the halfspace $\mathbb{R}^{n-1} \times [0, \infty)$ (but keeping the same conditions with respect to G_U and φ_U). We then have

Fact A.5. *Let* M *be an* n*-dimensional compact oriented orbifold-with-boundary. Then the boundary* ∂M *is an* $(n-1)$*-dimensional orbifold, with a natural orientation induced from the orientation of* M*, and* ∂M *is null-homologous in* M *(that is, the image of the fundamental class* $[\partial M]$ *is zero in* $\mathrm{H}_{n-1}(M)$*).*

Sketch of proof. In the language of differential forms, this is just Stokes's theorem, and the standard local argument applies (e.g. [3, theorem 3.5]). Alternatively, using the rational (co)homology manifold approach, this fact follows from Poincaré-Lefschetz duality [4, chapter II]. ☐

Orbibundles and integration over the fibre

An 'orbibundle' is the natural orbifold version of a fibre bundle. Satake defined orbibundles (he called them V-bundles).

Definition A.6 (Satake [27]). Let M be an orbifold, with orbifold structure defined by the open cover $\mathcal{U}$. An **orbibundle** over M is defined by giving, for each open set $U \in \mathcal{U}$

[3] A differential form on an orbifold M is a collection of differential forms on the sets $\tilde{U}$, invariant under the local groups G_U, and compatible with the inclusion maps in the obvious way; integration is defined using a partition of unity and adding up integrals on sets $\tilde{U}$ multiplied by the factors $1/|G_U|$.

[4] A **rational homology** n**-manifold** is a space whose local homology, with rational coefficients, agrees with that of an n-manifold (where the local homology at $x \in M$ is $\mathrm{H}_*(M, M \setminus \{x\})$. It's an easy calculation to show that an orbifold is a rational homology manifold. The construction of the rational fundamental class of a rational homology manifold mimcs the usual construction: one shows that an orientation gives a constant section of the local homology sheaf, and then applies a Mayer-Vietoris patching argument,(as in [3, section 5] or [28, section 6.3]).

(with associated triple $(\tilde{U}, G_U, \varphi_U)$) a G_U-equivariant fibre bundle $\tilde{E} \to \tilde{U}$. (Each inclusion i_{UV} must lift to a G_U-equivariant bundle map, which is an isomorphism on the fibres.) Given an orbibundle over M, there is an associated topological space (which we will refer to as the **total space**) E with a map $E \xrightarrow{\pi} M$ defined so that $\pi^{-1}(U) = \tilde{E}/G_U$. An orbibundle is **oriented** if the fibres of each bundle $\tilde{E} \to \tilde{U}$ are oriented (these orientations must be preserved by the local groups G_U and compatible with inclusion maps).

Remarks A.7.

1. Although the fibre of an orbibundle may be any space, in our applications the fibre will always be an orbifold.
2. The total space E of an orbibundle $E \xrightarrow{\pi} M$ is not in general a fibre bundle: if x is a smooth point of M then $\pi^{-1}(x)$ will be a copy of the fibre F, but if x is an orbifold point of M, then $\pi^{-1}(x)$ may be the quotient of F by a finite group.

We will describe the properties of a map on cohomology known as 'integration over the fibre', but in order to do this, we must define the notion of a suborbifold.

Definition A.8. Given an orbifold M, then a **suborbifold** M' of M is defined by giving a submanifold of each $\tilde{U}$, stable under G_U and compatible with the inclusion maps, and such that the restriction of the orbifold structure on M defines an orbifold structure on M' (in particular, in each submanifold the set of points fixed by G_U should have codimension ≥ 2).

It is important to note that, with this definition, a suborbifold M' of M consists mainly of smooth points of M (more precisely, those points of M' which are smooth in M make up a dense open subset of M'). This is consistent with Satake's definition of an orbifold, which forces most points of an orbifold to be smooth points[5].

Fact A.9. *Let $E \xrightarrow{\pi} M$ be an orbibundle, with fibre the compact oriented orbifold F. Then there is a map*

$$\pi_* : \mathrm{H}^*(E) \to \mathrm{H}^{*-\dim F}(M)$$

known as **integration over the fibre**[6] *having the following properties:*

1. *Integration over the fibre is a module homomorphism of $\mathrm{H}^*(M)$-modules (the module structure is given by pullback via π followed by cup product). This is equivalent to the 'push-pull formula'*

$$\pi_*(\pi^*(a) \smile b) = a \smile \pi_*(b), \qquad \forall a \in \mathrm{H}^*(E), b \in \mathrm{H}^*(M).$$

2. *Let $i : M' \hookrightarrow M$ be the inclusion of a suborbifold of M , and let $E' \xrightarrow{\pi'} M'$ denote the orbibundle over M' defined by the restriction of E. Then the following square commutes:*

$$\begin{tikzcd} \mathrm{H}^*(E') \arrow[d, "\pi'_*"] & \mathrm{H}^*(E) \arrow[l, "\tilde{i}^*"'] \arrow[d, "\pi_*"] \\ \mathrm{H}^{*-\dim F}(M') & \mathrm{H}^{*-\dim F}(M). \arrow[l, "i^*"'] \end{tikzcd}$$

[5]It would be possible to give an alternative definition of an orbifold which removed these restrictions. Specifically, given a local triple $(\tilde{U}, G_U, \varphi_U)$, we could remove the restriction that the set of points fixed by the G_U-action on $\tilde{U}$ have codimension ≥ 2, and alter the rest of the definition in a compatible manner. This alternative definition would be more natural in some respects, but it would also be more involved, since we would then need to take into account various numerical factors.

[6]often referred to as the **Gysin map** (it generalizes the Gysin map defined for a sphere bundle) or, in a more general setting, the **pushforward**.

(where $\tilde{i} : E' \to E$ is the lift of i).

3. *If E, M and F are compact oriented orbifolds, and the orientation of E equals the product of the orientations of M and F, then for any class $a \in \mathrm{H}^*(E)$ we have*

$$\int_E a = \int_M \pi_*(a).$$

Sketch of proof. We again indicate two different proofs. Using differential forms, the usual formula for integration over the fibre is well-defined on the local bundles $\tilde{E} \to \tilde{U}$ (this was defined by Lichnerowicz [22], and is also explained by Bott and Tu [3, p. 61]; of course we are using fact A.4, allowing us to integrate over the orbifold fibres). It is easy to check that this gives G_U-invariant differential forms on the sets $\tilde{U}$, and hence orbifold differential forms on M (see fotnote 3). The advantage of this approach is that the three properties we have listed above follow immediately from the definition.

Alternatively, in the manifold case, integration over the fibre can be defined using the Leray-Serre spectral sequence of the fibration (described for sphere bundles quite explicitly in Bott and Tu [3, pp. 177–179]). For an orbibundle $E \xrightarrow{\pi} M$ we use the Leray spectral sequence (with rational coefficients) of the map π [3, pp. 179–182], trivializing the the top cohomology sheaf of the fibres by the rational fundamental classes on the local covers $\tilde{E} \to \tilde{U}$. Finally, the algebraic and naturality properties of the Leray-Serre spectral sequence which imply properties 1-3 above also carry over to the Leray spectral sequence (see e.g. McCleary [24]). ☐

Remark A.10. We also need a related result concerning integration over the fibre: this time for an (honest) fibre bundle $E \xrightarrow{\pi} B$, with fibre an oriented orbifold F, but where the base space B may be any CW-complex. Using the same arguments as above, it is easy to show that integration over the fibre π_* is well-defined for such bundles, and satisifes properties 1 and 2.

How orbifold-fibre-bundles can arise as locally free quotients of manifolds

Fact A.11. *Suppose the compact connected Lie group G acts on a compact oriented manifold N with a locally free action (that is, the stabilizer subgroup of each point is finite). Then the quotient space N/G can be given an oriented orbifold structure (the orientation is fixed by orienting G).*

This orbifold structure on N/G is constructed by taking local slices for the action (for the existence and properties of local slices, see for example Bredon [5, Chapter IV], Kawakubo [19, section 4.4], or the chapter by Palais [4, chapter VIII]). Specifically, given a point $x \in N$, then there exists a **linear slice** for the G-action at x: a submanifold $S \subset N$ which is transverse to the G-orbits, is mapped to itself by the stabilizer subgroup G_x, and is equivariantly identified with an open subset of $\mathbb{R}^n$ with respect to a linear action of G_x on $\mathbb{R}^n$. Letting F denote the subgroup of G_x which fixes every point in S, then the triple $(S, G_x/F, \varphi)$ defines the orbifold structure at $[x] \in N/G$ (where φ maps S to $S \cdot G \subset N/G$).

The following existence facts follow easily from the definition of orbifold together with simple arguments involving local slices.

Facts A.12. 1. *If N' is an oriented submanifold of N, stable under G and transverse to the submanifolds N^H, for each finite subgroup $H \subset G$, then N'/G is a suborbifold of N.*

2. *If N is an oriented manifold-with-boundary on which the compact connected Lie group G acts, with a locally free action, then N/G is an oriented orbifold-with-boundary.*
3. *Suppose E and N are oriented manifolds, and $E \to N$ is a fibre bundle. Then if G and H are compact connected Lie groups, and $G \times H$ acts on E, covering an action of H on N, and these actions are locally free, then $E/(G \times H) \to N/H$ is an orbibundle.*

B. Cohomology and integration formulae for weighted projective bundles

The purpose of this appendix is to give generalizations of two classical formulae concerning projective bundles. Let Y be a CW-complex, let $V \to Y$ be a complex vector bundle, and let $\mathbb{P}(V) \to Y$ be its projectivization [3, p. 269]. The first classical formula describes the cohomology of $\mathbb{P}(V)$, and the second (and perhaps less well-known) calculates integrals over the fibres of the bundle $\mathbb{P}(V) \to Y$.

The generalizations we give apply to bundles constructed as follows. Let $V \to Y$ be a complex vector bundle, and suppose S^1 acts on V, such that the action is linear on the fibres of V (that is, the action covers the trivial action on Y), and such that the set of fixed points equals the zero section. We consider the bundle $S(V)/S^1 \xrightarrow{\pi} Y$, where $S(V)$ denotes the unit sphere bundle in V, relative to some invariant metric.

These bundles can be considered as generalizations of projective bundles in the following sense. If S^1 acts with 'weight one' on the fibres (i.e. the standard multiplication action of $S^1 \subset \mathbb{C}^\times$), then each S^1-orbit lies in precisely one line in V, and identifying S^1-orbits with lines induces a isomorphism $S(V)/S^1 \cong \mathbb{P}(V)$. The general case that we consider allows any combination of positive and negative weights. This general case includes 'weighted projectivizations' which correspond to S^1 actions having only positive weights (Kawasaki calculates the cohomology of weighted projective spaces in [20]; for some definitions and results in algebraic geometry on weighted projective spaces, see [7]).

We begin by reviewing the cohomology and integration formulae in the case of projective bundles. We then state and prove the general cohomology formula, followed by the general integration formula. Finally, using the homotopy quotient construction, we will observe that all the definitions, formulae, and proofs naturally extend to the case in which an auxilliary group G acts on V and Y, commuting with the S^1-action and with the projection.

Projective bundles

The projectivization $\mathbb{P}(V)$ possesses a distinguished cohomology class $h \in \mathrm{H}^2(\mathbb{P}(V))$, which is usually defined as follows. Let $S \to \mathbb{P}(V)$ denote the **tautological line bundle** (where the fibre of S over a point is just the corresponding line in V), and define h to be the first Chern class of the dual line bundle, $h = c_1(S^*)$.

Then the cohomology of $\mathbb{P}(V)$ is given by the formula[7]

$$\mathrm{H}^*(\mathbb{P}(V)) \cong \frac{\mathrm{H}^*(Y)[h]}{\langle c_0(V)h^r + c_1(V)h^{r-1} + \ldots + c_r(V)\rangle}.$$

where $c_i(V) \in \mathrm{H}^{2i}(Y)$ is the i-th Chern class, and $r = \mathrm{rk}(V)$. In this formula the product ah^i (where $a \in \mathrm{H}^*(Y)$) is identified with the class $(\pi^*a)h^i \in \mathrm{H}^*(\mathbb{P}(V))$.

The vector bundle $V \to Y$ has associated Segre classes $s_i(V) \in \mathrm{H}^{2i}(Y)$. The total Chern class and the total Segre class are multiplicative inverses to each other (in the cohomology

[7] Bott and Tu [3, pp. 269-271] describe the projectivization and the tautological line bundle, and following Grothedieck, they define the Chern classes in terms of the cohomology formula.

ring of Y), that is

$$c(V)s(V) = 1,$$

and this can be used to define the Segre classes in terms of the Chern classes. (As an example, consider the tautological line bundle $S \to \mathbb{CP}^n$. Then $c(S) = 1 - h$, where h is the generator of $\mathrm{H}^*(\mathbb{CP}^n)$, and $s(S) = (1-h)^{-1} = 1 + h + h^2 + \ldots + h^n$.)

The integration formula expresses integrals over the fibres in terms of Segre classes:

$$\pi_*(h^i) = \begin{cases} 0 & i < \mathrm{rk}(V) - 1, \\ s_{i-\mathrm{rk}(V)+1}(V), & i \geq \mathrm{rk}(V) - 1, \end{cases}$$

where π_* denotes integration over the fibre (see fact A.9). (This formula is sufficient to calculate the integral over the fibres of any class on $\mathbb{P}(V)$, since every class can be expressed in the form $(\pi^* a)h^i$, and we have $\pi_*((\pi^* a)h^i) = a\pi_*(h^i)$.)[8]

Weighted Chern classes and the cohomology formula

We now return to the general case: $V \to Y$ is a complex vector bundle, with an action of S^1 on V, covering the trivial action on Y, and such that the set of fixed points equals the zero section.

We first define the weighted Chern class of the pair (V, S^1) (although we will sometimes abuse notation and simply refer to this as the weighted Chern class of V). We will then state and prove a formula for the cohomology of the total space of the bundle $S(V)/S^1 \xrightarrow{\pi} Y$ (where $S(V)$ denotes the unit sphere bundle in V relative to some invariant metric).

Definition B.1. The quick definition of the weighted Chern class is this: the weighted Chern class c^w is multiplicative under direct sum of bundles, and commutes with pullbacks (so that the splitting principle applies), and for a line bundle L acted on with weight i, is given by $c^w(L) = i + c_1(L)$ (where $c_1(L)$ is the regular first Chern class).

Explicitly, under the S^1 action, V splits into 'isotypic' subbundles

$$V \cong \bigoplus_{i \in \mathbb{Z}} V_i,$$

where S^1 acts with weight i on V_i (that is, $\lambda \in S^1$ acts on V_i by multiplying the fibre coordinates by λ^i). Then the **weighted Chern class** of (V, S^1), which we denote $c^w(V) \in \mathrm{H}^*(Y)$, is the product

$$c^w(V) := \prod_i c^w(V_i),$$

where, setting r equal to the rank of V_i,

$$c^w(V_i) = i^r + i^{r-1}c_1(V_i) + i^{r-2}c_2(V_i) + \ldots + c_r(V_i)$$

(here $c_j(V_i)$ is the regular j-th Chern class of V_i). It follows from the properties of the regular Chern class that the weighed Chern class is natural with respect to pullbacks, and

[8] The integration formula might appear to be overkill: since it follows from the cohomology formula that every class on $\mathbb{P}(V)$ can be expressed as $(\pi^* a)h^i$ for $0 \leq i \leq \mathrm{rk}(V) - 1$, in fact we only need to observe that $\pi_*(h^i) = 0$ for $0 \leq i \leq \mathrm{rk}(V) - 1$, and $\pi_*(h^{\mathrm{rk}(V)-1}) = 1$. However in applications we are often given a class on $\mathbb{P}(V)$ expressed as $(\pi^* a)h^i$ where i is not necessarily in this range. Using the cohomology formula, we could rewrite such a class in terms of the cohomology of Y and the classes $\{1, h, h.., h^{\mathrm{rk}(V)-1}\}$, in which case the integral over the fibres would be the coefficient of $h^{\mathrm{rk}(V)-1}$. The integration formula is simply the answer one gets by following this process.

multiplicative with respect to direct sum (it is easiest to think of the S^1-action as simply decomposing V into a direct sum of subbundles, each of which is labelled with an integer, and to note that this decomposition commutes with pullback and direct sum in an obvious way).

Proposition B.2. *Let $V \to Y$ be a complex vector bundle with an action of S^1 as above. Define $h \in \mathrm{H}^2(S(V)/S^1)$ to be the first Chern class of the principal orbifold bundle $S(V) \to S(V)/S^1$ (see remark B.3 below). Then there is a ring isomorphism*

$$\mathrm{H}^*(S(V)/S^1) \cong \frac{\mathrm{H}^*(Y)[h]}{\langle c_0^w(V)h^r + c_1^w(V)h^{r-1} + \ldots + c_r^w(V)\rangle},$$

induced by identifying a product ah^i, where $a \in \mathrm{H}^*(Y)$, with the class $(\pi^* a)h^i \in \mathrm{H}^*(S(V)/S^1)$.

Remark B.3. Suppose S^1 acts with weight one on the fibres, so that we have a natural isomorphism $S(V)/S^1 \cong \mathbb{P}(V)$. Then the two definitions of the class h agree: the classical definition, as the first Chern class of the dual of the tautological line bundle over $\mathbb{P}(V)$, and the definition in the above proposition. (The above definition of h is equivalent to defining h as the first Chern class of the associated orbifold line bundle $S(V) \times_{S^1} \mathbb{C}_{(1)} \to S(V)/S^1$, where $\mathbb{C}_{(1)}$ denotes $\mathbb{C}$ with the weight one action of S^1. In the classical case, it is easy to show that this associated line bundle is isomorphic to the dual of the tautological line bundle.)

Proof of Proposition B.2. This proof comprises two steps. We first identify the weighted Chern classes of (V, S^1) as certain coefficients of an equivariant Euler class. We then show how this equivariant Euler class appears in a standard long exact sequence, and how the properties of this long exact sequence give us the proposition.

Step 1: Relating $c^w(V)$ to an equivariant Euler class. The S^1-equivariant bundle $V \to Y$ has an S^1-equivariant Euler class

$$e_{S^1}(V) \in \mathrm{H}^*_{S^1}(Y) \cong \mathrm{H}^*(Y) \otimes \mathrm{H}^*(BS^1),$$

which we claim is given by

$$e_{S^1}(V) = c_0^w(V)u^r + c_1^w(V)u^{r-1} + \ldots + c_r^w(V), \tag{B.4}$$

where $u \in \mathrm{H}^2(BS^1)$ denotes the positive integral generator. (We briefly recall the definition of the equivariant Euler class. The equivariant cohomology of Y is defined to be the regular cohomology of the homotopy quotient $Y_{S^1} = (Y \times ES^1)/S^1$. An equivariant vector bundle $V \to Y$ pulls back to an equivariant vector bundle over $Y \times ES^1$, and by the quotient construction induces a regular vector bundle over Y_{S^1}; the equivariant Euler class of V is defined to be the regular Euler class of this induced bundle.)

To show the above relationship between $e_{S^1}(V)$ and $c^w(V)$, we first show that it holds for line bundles. We then appeal to the splitting principle to extend this to vector bundles.

Suppose $L \to Y$ is a complex line bundle, possessing an action of S^1 covering a trivial action on Y. Let $i \in \mathbb{Z}$ equal the weight of the action of S^1 on the fibres of L. Let $L_{(0)} \to Y$ denote the same line bundle, but with a trivial action of S^1, and let $\underline{\mathbb{C}}_{(i)} \to Y$ denote the trivial line bundle with a weight-i action of S^1. Then

$$L \cong L_{(0)} \otimes \underline{\mathbb{C}}_{(i)}$$

(as S^1-equivariant line bundles). Hence, since Euler classes add when we tensor line bundles,

$$\begin{aligned} e_{S^1}(L) &= e_{S^1}(L_{(0)}) + e_{S^1}(\underline{\mathbb{C}}_{(i)}) \\ &= c_1(L) + iu \\ &= c_1^w(L) + c_0^w(L)u. \end{aligned}$$

This proves our claim (equation (B.4)) for line bundles, and the general case follows from the splitting principle, together with the observation that both sides of equation (B.4) are multiplicative with respect to direct sum of the vector bundles we are considering.

Step 2: The map $\mathrm{H}^*_{S^1}(Y) \to \mathrm{H}^*(S(V)/S^1)$. Let p and π denote the maps

$$\begin{tikzcd} S(V) \arrow[rr, "/S^1"] \arrow[dr, "p"'] & & S(V)/S^1 \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

and let $/S^1$ denote the natural identification in equivariant cohomology $\mathrm{H}^*_{S^1}(S(V)) \xrightarrow[\cong]{/S^1} \mathrm{H}^*(S(V)/S^1)$.

Then, by naturality of this isomorphism, together with the definition of h, we have

$$(p^*(au^i))/S^1 = (\pi^* a)h^i$$

for any $a \in \mathrm{H}^*(Y)$.

But the natural map $(p^* \cdot)/S^1$ fits into a short exact sequence of $\mathrm{H}^*(Y)$-modules

$$0 \longrightarrow \langle e_{S^1}(V)\rangle \hookrightarrow \mathrm{H}^*_{S^1}(Y) \xrightarrow{(p^*\cdot)/S^1} \mathrm{H}^*(S(V)/S^1) \longrightarrow 0, \tag{B.5}$$

where $\langle e_{S^1}(V)\rangle \subset \mathrm{H}^*_{S^1}(Y)$ denotes the ideal generated by $e_{S^1}(V)$.

These properties follow from the existence of the long exact sequence in equivariant cohomology for the pair $(V, S(V))$, together with the following identifications:

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{H}^*_{S^1}(V, S(V)) \arrow[r] & \mathrm{H}^*_{S^1}(V) \arrow[r] \arrow[d, "\cong"'] & \mathrm{H}^*_{S^1}(S(V)) \arrow[r] \arrow[d, "\cong"', "/S^1"] & \cdots \\ & \mathrm{H}^*_{S^1}(Y) \arrow[u, "\cong", "\smile\Phi"'] \arrow[r, "\smile e_{S^1}(V)"] & \mathrm{H}^*_{S^1}(Y) \arrow[r] \arrow[ur, "p^*"] & \mathrm{H}^*(S(V)/S^1) & \end{tikzcd}$$

Here the leftmost identification (denoted $\smile \Phi$) is the Thom isomorphism in equivariant cohomology, with Φ the Thom class (see [1, section 2] for more on this identification); the next identification is induced by restriction to the zero-section of V, and is an isomorphism because of the homotopy equivalence between V and Y. The restriction of the Thom class Φ to the zero section equals the equivariant Euler class $e_{S^1}(V)$, and hence the composition of the Thom isomorphism with the restriction is given by multiplication by the equivariant Euler class on $\mathrm{H}^*_{S^1}(Y)$. The remaining maps are easily identified as labelled. Finally, using our explicit identification of the Euler class $e_{S^1}(V)$, we see that multiplication by this Euler class is injective, and thus the sequence is short exact.

Hence we have

$$\mathrm{H}^*(S(V)/S^1) \cong \frac{\mathrm{H}^*_{S^1}(Y)}{\langle e_{S^1}(V)\rangle},$$

and, substituting our formula for $e_{S^1}(V)$, we have proven the proposition. $\square$

Weighted Segre classes and the integration formula

We now prove a formula which calculates integrals over the fibres of the bundle $S(V)/S^1 \xrightarrow{\pi} Y$. This formula involves the 'weighted Segre classes' of the pair (V, S^1), which we define. We must also define an orientation of the fibres of π, so that integration over the fibre is well-defined. In the case that $S(V)/S^1$ can be naturally identified with a weighted projective bundle (i.e. if the weights of the S^1-action are all positive) this orientation agrees with the standard orientation induced by the complex structure on the fibres.

Definition B.6. Let $V \to Y$ be a complex vector bundle with an action of S^1 as above. The condition that the set of points fixed by the action equals the zero section is equivalent to the condition that no subbundle of V be acted on with weight zero. It follows that the total weighted Chern class of (V, S^1) is invertible in the rational cohomology ring of Y (since the degree-zero component is nonzero), and we define the **weighted Segre class** to be its multiplicative inverse:

$$s^w(V)c^w(V) = 1.$$

Definition B.7. Given any point $y \in Y$, let V_y denote the fibre of V over the point y. Then, for any $v \in S(V_y)$, we have the isomorphism

$$T_{S^1 \cdot v}(S(V_y)/S^1) \oplus \mathbb{R}^+ \cdot v \oplus \mathfrak{s}^1 \cong V_y,$$

where $\mathbb{R}^+ \cdot v \subset V_y$ denotes the ray from the origin through v, and $\mathfrak{s}^1$ is the Lie algebra of S^1, identified with $\mathbb{R}$ in the standard way. We define the orientation of $S(V_y)/S^1$ to be that orientation which is compatible with the above isomorphism together with the given orientations of $\mathbb{R}^+$, $\mathfrak{s}^1$, and V_y (where V_y has the standard orientation defined by its complex structure, as in equation (0.3)).

Proposition B.8. *Let Y be connected and $V \to Y$ be a complex vector bundle with an action of S^1 as above. Consider the bundle $S(V)/S^1 \xrightarrow{\pi} Y$, and define $h \in \mathrm{H}^2(S(V)/S^1)$ to be the first Chern class of the principal orbifold bundle $S(V) \to S(V)/S^1$ as in proposition B.2 above. Then, for any $a \in \mathrm{H}^*(Y)$,*

$$\pi_*\left((\pi^* a) \smile h^i\right) = \begin{cases} 0 & i < \mathrm{rk}(V) - 1, \\ ka \smile s^w_{i-\mathrm{rk}(V)+1}(V), & i \geq \mathrm{rk}(V) - 1. \end{cases}$$

Here π_* denotes integration over the fibre with respect to the orientation defined above, and k is the greatest common divisor of the absolute values of the weights appearing in the S^1 action on the fibres of V.

Proof. This proof consists of two steps. In step 1 we relate the rational fundamental class of the fibres with the fundamental class of complex projective space. Then, in step 2, we use the formula from proposition B.2 above.

Step 1: The rational fundamental class of the fibres of $S(V)/S^1 \to Y$. Given $y \in Y$, let V_y denote the fibre of V over the point y. Then S^1 acts on V_y, and we can make an S^1-equivariant identification

$$V_y \cong \mathbb{C}^r_{(i_1, i_2, \ldots, i_r)},$$

where $\mathbb{C}^r_{(i_1,i_2,\dots,i_r)}$ denotes $\mathbb{C}^r$ with the weight-$(i_1,i_2,\dots,i_r)$ action of S^1 (that is, $\lambda \in S^1 \subset \mathbb{C}^\times$ acts by $\lambda \cdot (z_1,\dots,z_r) = (\lambda^{i_1} z_1,\dots,\lambda^{i_r} z_r)$.) Moreover, we can arrange that $i_1,\dots,i_n < 0$ and $i_{n+1},\dots,i_r > 0$. Then the map

$$\begin{aligned} \tilde{\varphi} : \mathbb{C}^r_{(1,1,\dots,1)} &\to V_y = \mathbb{C}^r_{(i_1,i_2,\dots,i_r)} \\ (z_1,\dots,z_r) &\mapsto (\bar{z}_1^{|i_1|},\dots,\bar{z}_n^{|i_n|}, {z_{n+1}}^{i_{n+1}}, {z_r}^{i_r}) \end{aligned}$$

is smooth and intertwines the S^1-actions.

There is an obvious S^1-invariant metric on $V_y = \mathbb{C}^r_{(i_1,i_2,\dots,i_r)}$ such that $\tilde{\varphi}$ maps the standard unit sphere in $\mathbb{C}^r_{(1,1,\dots,1)}$ to the unit sphere in V_y.

Hence $\tilde{\varphi}$ descends to a map

$$\varphi : S(\mathbb{C}^r_{(1,1,\dots,1)})/S^1 = \mathbb{CP}^{r-1} \to S(V_y)/S^1.$$

We can now relate the rational fundamental class of $S(V_y)/S^1$ to the fundamental class of $\mathbb{CP}^{r-1}$ by calculating the oriented degree of φ (that is, the topological degree of φ, multiplied by ± 1 according to whether φ preserves or reverses orientation).

We easily see that the oriented degree of $\tilde{\varphi}$ equals $\prod_{j=1}^r i_j = c_0^w(V)$ (and this also equals the oriented degree of the restriction of $\tilde{\varphi}$ to the unit sphere). To calculate the degree of φ, we must divide this number by the degree with which a generic S^1-orbit in $S(\mathbb{C}^r_{(1,1,\dots,1)})$ covers its image. It is easy to see that this degree equals the greatest common divisor of the absolute values of the i_j. Hence, setting

$$k := \gcd(|i_1|,|i_2|,\dots,|i_r|)$$

then the oriented degree of φ is given by

$$\deg(\varphi) = k^{-1} c_0^w(V). \tag{B.9}$$

Now consider the maps

$$\begin{tikzcd} \mathbb{CP}^{r-1} \arrow[r, "\varphi"] & S(V_y)/S^1 \arrow[r, hook, "\psi"] \arrow[d, "\pi'"'] & S(V)/S^1 \arrow[d, "\pi"] \\ & y \arrow[r, hook] & Y \end{tikzcd}$$

We have defined the class $h \in \mathrm{H}^*(S(V)/S^1)$ to be the first Chern class of the orbifold S^1-bundle $S(V) \to S(V)/S^1$. (Or equivalently, h is the first Chern class of the associated orbifold line bundle $S(V) \times_{S^1} \mathbb{C}_{(1)} \to S(V)/S^1$, where $\mathbb{C}_{(1)}$ denotes $\mathbb{C}$ with the weight-one action of S^1.) By naturality of this definition, we see that h pulls back to the integral generator of the cohomology of $\mathbb{CP}^{r-1}$, so that

$$\int_{\mathbb{CP}^{r-1}} (\varphi^* \psi^* h)^{r-1} = 1.$$

Using the degree of φ, we thus have

$$\pi'_*((\psi^* h)^{r-1}) = k c_0^w(V)^{-1},$$

and hence, since integration over the fibre commutes with restriction, and the result is a degree-zero cohomology class, we have

$$\pi_*(h^{r-1}) = k c_0^w(V)^{-1} \qquad \in \mathrm{H}^0_G(Y). \tag{B.10}$$

(Of course $\pi_*(h^i) = 0$ if $i < r-1$, for degree reasons.)

Step 2: Using the relation in cohomology to extend this formula to all powers of h. We now calculate

$$\pi_* \left((\pi^* c^w(V)) \smile (1 + h + h^2 + h^3 + \dots) \right). \tag{B.11}$$

Since π_* lowers degree by $2r-2$, we only need to consider terms in the product $(\pi^* c^w(V)) \smile (1 + h + h^2 + h^3 + \dots)$ of degree $2r-2$ and greater. The degree $2r-2$ term is

$$\pi^* c^w_{r-1}(V) + \pi^* c^w_{r-2}(V) h + \dots + \pi^* c^w_0(V) h^{r-1},$$

and applying π_* to this term gives the coefficient of h^{r-1}, multiplied by $\pi_*(h^{r-1})$ (which we have calculated in equation (B.10) above). (We are using the fact that π_* is a homomorphism of $\mathrm{H}^*(Y)$-modules.) Hence, the integral over the fibre of the degree $2r-2$ term of the product (B.11) equals $k \in \mathrm{H}^0(Y)$.

The degree $2r$ term of the product (B.11) is

$$\pi^* c^w_r(V) + \pi^* c^w_{r-1}(V) h + \dots + \pi^* c^w_0(V) h^r.$$

Comparing with our explicit identification of $e_{S^1}(V)$ in equation (B.4) above, we see that this term is exactly the class $p^* \left(e_{S^1}(V)\right)/S^1$, and, using the short exact sequence (B.5), this term vanishes. Similarly, the degree $2(r+j)$ term equals $p^* \left(u^j e_{S^1}(V)\right)/S^1$ and hence also vanishes. Thus we have

$$\pi_* \left((\pi^* c^w(V)) \smile (1 + h + h^2 + h^3 + \dots) \right) = k$$

and hence (using the module-homomorphism property of π_*)

$$\pi_*(1 + h + h^2 + h^3 + \dots) = k c^w(V)^{-1} = k s^w(V).$$

The proposition now follows by identifying terms by degree. □

Equivariant weighted Segre classes, and the equivariant integration formula

Suppose $V \to Y$ is a vector bundle, with an action of S^1 as above, and suppose moreover that an auxilliary group G acts on V and Y, commuting with the projection and with the action of S^1. Then we can generalize the definition of weighted Chern classes and weighted Segre classes to the G-equivariant case as follows.

Recall that the homotopy quotient construction replaces a G-space Y with the space $Y_G := EG \times_G Y$, and the equivariant cohomology of Y is defined to be the ordinary cohomology of Y_G. Given a G-equivariant vector bundle $V \to Y$ the same construction gives a vector bundle $V_G \to Y_G$ (this is explained by Atiyah and Bott in [1, section 2: equation (2.1) and remark (1)]), and the G-equivariant characteristic classes of V are then taken to be the ordinary characteristic classes of $V_G \to Y_G$, which thus take values in the G-equivariant cohomology of Y.

Definition B.12. In our case, in the presence of a commuting S^1-action, the bundle $V_G \to Y_G$ has an induced S^1-action, and we define the **G-equivariant weighted Chern classes** and the **G-equivariant weighted Segre classes** of the pair (V, S^1) to be the weighted Chern classes and weighted Segre classes of the pair $(V_G \to Y_G, S^1)$ (definitions B.1 and B.6).

Applying the cohomology formula (Proposition B.2) to the bundle $V_G \to Y_G$ and making the obvious identifications, we thus get the equivariant version of the cohomology formula:

$$\mathrm{H}^*_G(S(V)/S^1) \cong \frac{\mathrm{H}^*_G(Y)[h]}{\langle c^w_0(V)h^r + c^w_1(V)h^{r-1} + \ldots + c^w_r(V)\rangle}, \tag{B.13}$$

where $c^W_i(V)$ now denotes the i-th G-equivariant weighted Chern class of (V, S^1). Similarly, the integration formula (Proposition B.8) applied to $V_G \to Y_G$ gives the (formally identical) equivariant formula:

$$\pi_*\left((\pi^* a) \smile h^i\right) = \begin{cases} 0 & i < \mathrm{rk}(V) - 1, \\ ka \smile s^w_{i-\mathrm{rk}(V)+1}(V), & i \geq \mathrm{rk}(V) - 1, \end{cases} \tag{B.14}$$

where $s^W_i(V)$ now denotes the i-th G-equivariant weighted Segre class of (V, S^1).

C. Proof of the orientation lemma

We now give the proof of lemma 3.6, which describes the orientations of the boundary components of the wall-crossing-cobordism.

Lemma C.1 (Lemma 3.6). *Let the wall-crossing-cobordism W/T be oriented as in definition 3.2. Then the induced boundary orientation of $X/\!\!/T(p_0)$ is $-(\omega^k_{p_0})$, and of $X/\!\!/T(p_1)$ is $\omega^k_{p_1}$ (where ω_{p_i} denote the respective induced symplectic forms), and the induced boundary orientation of each $P_{(H,q)}$ is equal to the product orientation defined in 3.5 above.*

Proof. Before beginning the proof proper, we fix three conventions which will hold throughout the proof.

1. Most of the steps in this proof consist of exhibiting isomorphisms of the form

$$V_1 \oplus V_2 \cong V_3,$$

where the V_i are vector spaces. For each such isomorphism, we will be using orientations of two of the vector spaces to induce an orientation on the third, in the obvious manner (explicitly: so that concatenating oriented bases for V_1 and V_2 gives an oriented basis for V_3).

2. When we decompose tangent spaces, we will assume without explicit mention that these decompositions are orthogonal decompositions relative to some choice of invariant metric; and it will always be the case that the induced orientations are independent of the choices made.

3. Finally, given a symplectic form (that is, a nondegenerate 2-form) on any vector space, then the **symplectic orientation** of that vector space will mean the orientation defined by the top power of the symplectic form.

We break the proof up into two steps. In step 1 we assume that T is 1-dimensional, and in step 2 we reduce the general case to to the case treated in step 1.

Step 1: Assuming T is 1-dimensional. If T is 1-dimensional, then Z is just a closed subinterval of $\mathfrak{t}$, bounded by p_0 and p_1. The orientation of Z (definition 2.1) and of each wall-crossing subgroup (which is just T itself) orients $\mathfrak{t}$ so that $p_0 < p_1$. We orient $\mathfrak{t}^*$ compatibly (that is, so that the duality pairing between a positive vector in $\mathfrak{t}$ and a positive vector in $\mathfrak{t}^*$ is positive).

Then, restating definition 3.2 in this case, we have oriented the wall-crossing-cobordism W/T via the isomorphism

$$T_{[x]}(W/T) \cong T_{[x]}X/\!\!/T(p) \oplus \mathfrak{t}^* \tag{C.2}$$

for each $x \in W$ (with $p = \mu(x)$), relative to the orientation of $\mathfrak{t}^*$ described above and the symplectic orientation of $X/\!\!/T(p)$.

The key calculation in step 1 is to compare this orientation of W/T with the symplectic orientation of X. Now at each $x \in W$, we can decompose $T_xW = T_xX$ into the orbit direction and its orthgonal complement. Using the natural identifications, then we claim that

$$T_xX = T_xW \cong T_{[x]}(W/T) \oplus \bar{\mathfrak{t}}, \tag{C.3}$$

is orientation-preserving, where $\bar{\mathfrak{t}}$ denotes $\mathfrak{t}$ with the opposite orientation. Using equation (C.2) above, this is equivalent to showing that the isomorphism

$$T_xX \cong T_{[x]}X/\!\!/T(p) \oplus \mathfrak{t}^* \oplus \bar{\mathfrak{t}} \tag{C.4}$$

is orientation-preserving. Here we think of the spaces on the right as subspaces of T_xX. Explicitly, we let

$$\begin{aligned} h : T_{[x]}X/\!\!/T(p) &\hookrightarrow T_{[x]}X, \\ i : \mathfrak{t}^* &\hookrightarrow T_xX, \quad \text{and} \\ j : \mathfrak{t} &\hookrightarrow T_xX \end{aligned} \tag{C.5}$$

denote these identifications (so that $i^{-1} = d\mu|_{\mathrm{im}(i)}$ and j is given by the infinitesimal action of T at x, and h is the identification of a complement to $j(\mathfrak{t})$ in $\mu^{-1}(p)$ with a slice at x). Let $\psi \in \mathfrak{t}^*$ and $\xi \in \mathfrak{t}$ be positive with respect to the orientations we have chosen for $\mathfrak{t}^*$ and $\mathfrak{t}$ (so that ξ is negative with respect to the orientation of $\bar{\mathfrak{t}}$).

Then our sign convention for the moment map condition (equation (0.1)) implies that

$$\begin{aligned} \omega(i(\psi), -j(\xi)) &= \omega(j(\xi), i(\psi)) \\ &= \langle d\mu(i(\psi)), \xi \rangle \\ &= \langle \psi, \xi \rangle \\ &> 0. \end{aligned} \tag{C.6}$$

This means that $(i(\psi), -j(\xi))$ is a positively oriented basis of $i(\mathfrak{t}^*) \oplus j(\bar{\mathfrak{t}}) \subset T_xX$ with respect to the restriction of the symplectic form on X.

Now, recall that the symplectic form on $X/\!\!/T(p)$ is induced by restricting the symplectic form on X to $\mu^{-1}(p)$, where it is degenerate in the orbit directions, and hence descends to $X/\!\!/T(p)$. In terms of the maps in (C.5) above, this means that the symplectic form on $T_{[x]}X/\!\!/T(p)$ agrees with the pullback, via h, of the symplectic form on T_xX.

Thus we have shown that the identification in equation (C.4) is orientation-preserving. (Being completely explicit: if $(v_1, \ldots, v_k)$ is an oriented basis of $T_{[x]}X/\!\!/T(p)$, then $(v_1, \ldots, \mathfrak{v}_k, \psi, -\xi)$ is an oriented basis of the right-hand-side of equation (C.4), and $(h(v_1), \ldots, h(v_k), i(\psi), j(-\xi))$ is an oriented basis of the left-hand-side.)

Having derived this alternative description of the orientation of W/T, we can now calculate the induced orientations of the various boundary components. Now W/T is an odd-dimensional manifold, and hence the induced boundary orientation is defined to be that orientation of $\partial(W/T)$ which is compatible with the isomorphism

$$T_{[x]}(W/T) \cong T_{[x]}\partial(W/T) \oplus \mathbb{R} \cdot \nu_{\mathrm{out}}$$

where ν_{out} is an outward-pointing normal vector. (Here we are using the convention that makes Stokes's theorem sign-free, as explained in [3, page 31]; for the boundary of an even-dimensional manifold we would need to use the inward-pointing normal vector in the above equation.) Combining this with equation (C.3), this means that we can calculate the orientation of $\partial(W/T)$ via the isomorphism

$$T_x X \cong T_{[x]}\partial(W/T) \oplus \mathbb{R}\cdot\nu_{\mathrm{out}} \oplus \bar{\mathfrak{t}},$$

and the orientations of $\mathbb{R}$, $\bar{\mathfrak{t}}$, and the symplectic orientation of $T_x X$.

For the boundary component $X/\!\!/T(p_0)$, the isomorphism $\mathbb{R}\cdot\nu_{\mathrm{out}} \cong \overline{\mathfrak{t}^*}$ is orientation preserving. Thus we use the isomorphism

$$T_x X \cong T_{[x]} X/\!\!/T(p_0) \oplus \overline{\mathfrak{t}^*} \oplus \bar{\mathfrak{t}}$$

together with the orientations of $\overline{\mathfrak{t}^*}, \bar{\mathfrak{t}}$, and $T_x X$ to orient this boundary component. Using the reasoning in equations (C.6), this gives

$$-(\omega_{p_0})^k.$$

A similar argument (except with $\mathfrak{t}^*$ replacing $\overline{\mathfrak{t}^*}$) shows that the induced boundary orientation of $X/\!\!/T(p_1)$ is equal to its symplectic orientation:

$$\omega_{p_1}^k.$$

Finally we come to $P_{(H,q)}$, which in our case is $P_{(T,q)}$, since $H = T$. To conform with the notation of definition 3.5, let $x \in X^T = X_{(T,q)}$ and $v \in S(\nu_x X^T)$, where we identify the point (x, v) with a point in X via an equivariant exponential map. Then $P_{(T,q)}$ is oriented by the isomorphism

$$T_{(x,v)}X \cong T_{[x,v]}P_{(T,q)} \oplus \mathbb{R}\cdot(-v) \oplus \bar{\mathfrak{t}}$$

(with the symplectic orientation of $T_{(x,v)}X$). Since X^T is a symplectic submanifold, we can decompose the symplectic form according to the isomorphism

$$T_{(x,v)}X \cong T_x X^T \oplus \nu_x X^T,$$

so that this can be viewed as an orientation-preserving isomorphism with respect to the induced symplectic forms on all three spaces. Then, since $\mathbb{R}\cdot(-v) \oplus \bar{\mathfrak{t}}$ gives the same orientation as $\mathbb{R}\cdot v \oplus \mathfrak{t}$, we have recovered the orientation of definitions 3.4 and 3.5.

Step 2: Reducing the general case to the case of step 1. In step 1 we assumed that the torus T was 1-dimensional. It is easy to reduce the general case to the case of step 1, as follows.

We first observe that the orientation of W/T is locally defined (in terms of a codimension-1 foliation by symplectic orbifolds). In order to reduce the general orientation calculation, we need only consider the wall-crossing-cobordism in a neighbourhood of a boundary component. We will describe the construction for the boundary components $P_{(H,q)}$; the case of the components $X/\!\!/T(p_i)$ is analogous.

We fix attention on a single wall-crossing, with wall-crossing data (H, q), and associated boundary $P_{(H,q)}$. Choose $T' \subset T$ so that $T = T' \times H$. Then we have the following inclusions and associated dual projections

$$\begin{array}{rlcrl} & \mathfrak{t}' \hookrightarrow \mathfrak{t} & \qquad\qquad & & \mathfrak{h} \hookrightarrow \mathfrak{t} \\ \phi: & \mathfrak{t}^* \twoheadrightarrow \mathfrak{t}'^* & & \psi: & \mathfrak{t}^* \twoheadrightarrow \mathfrak{h} \end{array}$$

We define $q' := \phi(q)$, and $\mu' := \phi \circ \mu$, so that μ' is a moment map for the action of T'.

Now suppose that, in some neighbourhood of q, Z is parallel to $\mathfrak{t}'^{\perp}$ (this can easily be arranged by deforming Z). Then

$$\mu^{-1}(Z) = \mu'^{-1}(q')$$

in a neighbourhood of $\mu^{-1}(q)$. Now the T-action on X descends to an action of H on $X/\!\!/T'(q')$, with moment map given by the restriction of $\psi \circ \mu$.

It is now easy to see that, in a neighbourhood of $P_{H,q}$, the wall-crossing-cobordism $W(X, T, \mu, Z)$ constructed from the data X, T, μ, Z coincides with the wall-crossing-cobordism $W(X/\!\!/T'(q'), H, \psi \circ \mu, \psi(Z))$. These are foliated by the same symplectic suborbifolds, since

$$X/\!\!/T(p) \cong (X/\!\!/T'(q'))/\!\!/H(\psi(p))$$

is an isomorphism of symplectic stratified spaces.

Since Z is transverse to μ at q, it follows that $X/\!\!/T'(q')$ is a symplectic orbifold in a neighbourhood of $\mu^{-1}(q)$, with a Hamiltonian action of the 1-dimensional torus H, and we have thus reduced our calculation to the case of step 1. $\square$

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