

Quantum Anti-Cloning

David D. Song¹ and Lucien Hardy²

*Centre for Quantum Computation
Clarendon Laboratory, University of Oxford
Parks Road, Oxford OX1 3PU, U.K.*

ABSTRACT

We derive the transformation for the optimal universal quantum anti-cloner which produces two anti-parallel outputs for a single input state. The fidelity is shown to be $2/3$ which is same as the measurement fidelity. We consider a probabilistic quantum anti-cloner and show quantum states can be anti-cloned exactly with non-zero probability and its efficiency is higher than the efficiency of distinguishing between the two states.

¹**E-mail:** d.song@qubit.org

²**E-mail:** l.hardy@qubit.org

1 Introduction

Possessing superposition and entanglement properties, quantum information has revealed many interesting features which classical information has no counterpart. Unlike classical information, quantum information cannot be duplicated, i.e. unknown quantum state cannot be copied exactly [1]. However a universal quantum (approximate) cloner has been introduced [2, 3] which takes an unknown quantum state and generates multiple copies, with fidelity $5/6$ in case of two copies as outputs regardless of an input. Another feature of quantum information has been discovered recently [4] where it was shown that more quantum information can be gained from two anti-parallel spins than from two parallel ones, i.e. one can measure the spin direction $|\mathbf{n}\rangle$ with better fidelity when two qubits are in anti-parallel, $|\mathbf{n}, -\mathbf{n}\rangle$, than in parallel, $|\mathbf{n}, \mathbf{n}\rangle$.

In this paper, we consider a universal quantum anti-cloner which takes an unknown quantum state just as in quantum cloner but its output as one with the same copy while the second one with opposite spin direction to the input state. For the Bloch vector, an input \mathbf{n} , quantum anti-cloner would have the input as $\frac{1}{2}(\mathbf{1} + \mathbf{n} \cdot \boldsymbol{\sigma})$, then it generates two outputs, $\frac{1}{2}(\mathbf{1} + \eta \mathbf{n} \cdot \boldsymbol{\sigma})$ and $\frac{1}{2}(\mathbf{1} - \eta \mathbf{n} \cdot \boldsymbol{\sigma})$, where $0 \leq \eta \leq 1$ is the shrinking factor and the fidelity is defined as $F = \langle \mathbf{n} | \rho^{(\text{out})} | \mathbf{n} \rangle = \frac{1}{2}(1 + \eta)$. If spin flipping were allowed then anti-cloner would have the same fidelity as the regular cloner since one could clone first then flip the spin of the second copy. However spin flipping of an unknown state is not allowed in quantum mechanics. Consider a spin-flipping of an unknown state,

$$\begin{pmatrix} e^{\frac{-i\varphi}{2}} \cos \frac{\theta}{2} \\ e^{\frac{i\varphi}{2}} \sin \frac{\theta}{2} \end{pmatrix} \rightarrow \begin{pmatrix} -e^{\frac{-i\varphi}{2}} \sin \frac{\theta}{2} \\ e^{\frac{i\varphi}{2}} \cos \frac{\theta}{2} \end{pmatrix} \quad (1)$$

This transformation can be done only by an anti-unitary operation where the anti-unitary transformation, V , satisfies the following two conditions

$$\begin{aligned} (i) \quad & |\langle \psi | \phi \rangle| = |\langle \psi' | \phi' \rangle| \\ (ii) \quad & V(a|0\rangle + b|1\rangle) = a^*V|0\rangle + b^*V|1\rangle \end{aligned} \quad (2)$$

where $V|\psi\rangle \rightarrow |\psi'\rangle$ and $V|\phi\rangle \rightarrow |\phi'\rangle$.

In sect. 2, we derive a unitary transformation for a optimal universal quantum anti-cloner where we obtain $2/3$ for fidelity. This value is equal to the fidelity of measurement which is for a given single unknown state, how precisely one can determine its state [6]. In sect. 3, we show that the quantum state can be anti-cloned exactly with non-zero probability. In case of two states, the probability of exact anti-cloning is higher than the probability of distinguishing between the two states. We conclude with discussions on further prospects on related issues.

2 Universal quantum anti-cloning

In this section, we study the unitary transformation with optimal fidelity for a universal quantum anti-cloner. Let us consider an input state $|\mathbf{n}\rangle = \alpha|0\rangle + \beta|1\rangle$ such that for an input density matrix,

$$\rho^{(\text{in})} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{pmatrix} = \begin{pmatrix} 1 + n_z & n_x - in_y \\ n_x + in_y & 1 - n_z \end{pmatrix} \quad (3)$$

the output density matrix yields the first particle same as the input while the second one with opposite spin direction as follows

$$\rho_1^{(\text{out})} = \frac{\mathbf{1} + \eta\mathbf{n} \cdot \boldsymbol{\sigma}}{2} = \frac{1}{2} \begin{pmatrix} 1 + \eta n_z & \eta(n_x - in_y) \\ \eta(n_x + in_y) & 1 - \eta n_z \end{pmatrix} \quad (4)$$

$$\rho_2^{(\text{out})} = \frac{\mathbf{1} - \eta\mathbf{n} \cdot \boldsymbol{\sigma}}{2} = \frac{1}{2} \begin{pmatrix} 1 - \eta n_z & -\eta(n_x - in_y) \\ -\eta(n_x + in_y) & 1 + \eta n_z \end{pmatrix} \quad (5)$$

We want to consider the constraints in order to satisfy the output density matrices (4,5) with maximum fidelity, i.e. η . The conditions (4) and (5) imply that the two output density matrices are symmetric except for their spin direction which are opposite to each other. We also impose the universality constraint that the fidelity does not depend on the input state $|\mathbf{n}\rangle$.

Let us consider the following general transformation,

$$\begin{aligned}
|0\rangle|Q_{23}\rangle &\rightarrow a|00\rangle|A\rangle + b|01\rangle|B\rangle + c|10\rangle|C\rangle + d|11\rangle|D\rangle \\
|1\rangle|Q_{23}\rangle &\rightarrow \tilde{a}|11\rangle|\tilde{A}\rangle + \tilde{b}|10\rangle|\tilde{B}\rangle + \tilde{c}|01\rangle|\tilde{C}\rangle + \tilde{d}|00\rangle|\tilde{D}\rangle
\end{aligned} \tag{6}$$

where $|Q_{23}\rangle$ is the state to be anti-cloned and the initial ancilla state and the ancillas, $|A\rangle, \dots, |\tilde{D}\rangle$, are normalised but not necessarily orthogonal. After following the transformation (6) for the input state $|\mathbf{n}\rangle$, we have the following reduced density matrices after tracing out 23 and 13, respectively,

$$\begin{aligned}
\rho_1 = & \{(|a|^2 + |b|^2)|\alpha|^2 + (a\tilde{d}^*\langle\tilde{D}|A\rangle + b\tilde{c}^*\langle\tilde{C}|B\rangle)\alpha\beta^* \\
& + (\tilde{c}b^*\langle B|\tilde{C}\rangle + \tilde{d}a^*\langle A|\tilde{D}\rangle)\beta\alpha^* + (|\tilde{c}|^2 + |\tilde{d}|^2)|\beta|^2\}|0\rangle\langle 0| \\
& + \{(ac^*\langle C|A\rangle + bd^*\langle D|B\rangle)|\alpha|^2 + (a\tilde{b}^*\langle\tilde{B}|A\rangle + b\tilde{a}^*\langle\tilde{A}|B\rangle)\alpha\beta^* \\
& + (\tilde{c}d^*\langle D|\tilde{C}\rangle + \tilde{d}c^*\langle C|\tilde{D}\rangle)\beta\alpha^* + (\tilde{c}\tilde{a}^*\langle\tilde{A}|\tilde{C}\rangle + \tilde{d}\tilde{b}^*\langle\tilde{B}|\tilde{D}\rangle)|\beta|^2\}|0\rangle\langle 1| \\
& + \{(ca^*\langle A|C\rangle + db^*\langle B|D\rangle)|\alpha|^2 + (c\tilde{d}^*\langle\tilde{D}|C\rangle + d\tilde{c}^*\langle\tilde{C}|D\rangle)\alpha\beta^* \\
& + (\tilde{a}b^*\langle B|\tilde{A}\rangle + \tilde{b}a^*\langle A|\tilde{B}\rangle)\beta\alpha^* + (\tilde{a}\tilde{c}^*\langle\tilde{C}|\tilde{A}\rangle + \tilde{b}\tilde{d}^*\langle\tilde{D}|\tilde{B}\rangle)|\beta|^2\}|1\rangle\langle 0| \\
& + \{(|c|^2 + |d|^2)\alpha^2 + (c\tilde{b}^*\langle\tilde{B}|C\rangle + d\tilde{a}^*\langle\tilde{A}|D\rangle)\alpha\beta^* \\
& + (\tilde{a}d^*\langle D|\tilde{A}\rangle + \tilde{b}c^*\langle C|\tilde{B}\rangle)\beta\alpha^* + (|\tilde{a}|^2 + |\tilde{b}|^2)|\beta|^2\}|1\rangle\langle 1|
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
\rho_2 = & \{(|a|^2 + |c|^2)|\alpha|^2 + (a\tilde{d}^*\langle\tilde{D}|A\rangle + c\tilde{b}^*\langle\tilde{B}|C\rangle)\alpha\beta^* \\
& + (\tilde{b}c^*\langle C|\tilde{B}\rangle + \tilde{d}a^*\langle A|\tilde{D}\rangle)\beta\alpha^* + (|\tilde{b}|^2 + |\tilde{d}|^2)|\beta|^2\}|0\rangle\langle 0| \\
& + \{(ab^*\langle B|A\rangle + cd^*\langle D|C\rangle)|\alpha|^2 + (a\tilde{c}^*\langle\tilde{C}|A\rangle + c\tilde{a}^*\langle\tilde{A}|C\rangle)\alpha\beta^* \\
& + (\tilde{b}d^*\langle D|\tilde{B}\rangle + \tilde{d}b^*\langle B|\tilde{D}\rangle)\beta\alpha^* + (\tilde{b}\tilde{a}^*\langle\tilde{A}|\tilde{B}\rangle + \tilde{d}\tilde{c}^*\langle\tilde{C}|\tilde{D}\rangle)|\beta|^2\}|0\rangle\langle 1| \\
& + \{(ba^*\langle A|B\rangle + dc^*\langle C|D\rangle)|\alpha|^2 + (b\tilde{d}^*\langle\tilde{D}|B\rangle + d\tilde{b}^*\langle\tilde{B}|D\rangle)\alpha\beta^* \\
& + (\tilde{a}c^*\langle C|\tilde{A}\rangle + \tilde{c}a^*\langle A|\tilde{C}\rangle)\beta\alpha^* + (\tilde{a}\tilde{b}^*\langle\tilde{B}|\tilde{A}\rangle + \tilde{c}\tilde{d}^*\langle\tilde{D}|\tilde{C}\rangle)|\beta|^2\}|1\rangle\langle 0| \\
& + \{(|b|^2 + |d|^2)|\alpha|^2 + (b\tilde{c}^*\langle\tilde{C}|B\rangle + d\tilde{a}^*\langle\tilde{A}|D\rangle)\alpha\beta^* \\
& + (\tilde{a}d^*\langle D|\tilde{A}\rangle + \tilde{c}b^*\langle B|\tilde{C}\rangle)\beta\alpha^* + (|\tilde{a}|^2 + |\tilde{c}|^2)|\beta|^2\}|1\rangle\langle 1|
\end{aligned} \tag{8}$$

We want to consider constraints for ρ_1 and ρ_2 in (7,8) to be same as $\rho_1^{(\text{out})}$ and $\rho_2^{(\text{out})}$ in (4,5) with maximum value for η . Let us write the coefficients as follows

$$a = |a|e^{i\delta_a}, \quad b = |b|e^{i\delta_b}, \quad c = |c|e^{i\delta_c}, \quad d = |d|e^{i\delta_d} \quad (9)$$

and likewise for tilded cases. Also we could write $\langle A|B \rangle = |\langle A|B \rangle|e^{i\delta_{AB}}$ and others are similarly defined. First, there are normalisation conditions to be satisfied for the transformation (6),

$$\begin{aligned} |a|^2 + |b|^2 + |c|^2 + |d|^2 &= 1 \\ |\tilde{a}|^2 + |\tilde{b}|^2 + |\tilde{c}|^2 + |\tilde{d}|^2 &= 1 \end{aligned} \quad (10)$$

and the orthogonality

$$a^*\tilde{d}\langle A|\tilde{D} \rangle + c^*\tilde{b}\langle C|\tilde{B} \rangle + b^*\tilde{c}\langle B|\tilde{C} \rangle + d^*\tilde{a}\langle D|\tilde{A} \rangle = 0 \quad (11)$$

Comparing n_z terms in $\rho_1^{(\text{out})}$ and $\rho_2^{(\text{out})}$, we get the following constraints from (7,8)

$$|a| = |d|, \quad |\tilde{a}| = |\tilde{d}| \quad (12)$$

$$a\tilde{d}^*\langle \tilde{D}|A \rangle + b\tilde{c}^*\langle \tilde{C}|B \rangle - \tilde{c}b^*\langle \tilde{B}|C \rangle - d\tilde{a}^*\langle \tilde{A}|D \rangle = 0 \quad (13)$$

and

$$\eta = |b|^2 - |c|^2 = 2|b|^2 + 2|a|^2 - 1 \quad (14)$$

where the last relation in (14) results from (10) and (12) Next, Comparing n_x and n_y terms yields

$$\eta = \text{Re}[a^*\tilde{b}\langle A|\tilde{B} \rangle + b^*\tilde{a}\langle B|\tilde{A} \rangle] \quad (15)$$

$$= \text{Re}[\tilde{c}a^*\langle A|\tilde{C} \rangle + \tilde{a}c^*\langle C|\tilde{A} \rangle] \quad (16)$$

and also the following must be satisfied.

$$\text{Im}[a^*\tilde{b}\langle A|\tilde{B} \rangle + b^*\tilde{a}\langle B|\tilde{A} \rangle] = 0 \quad (17)$$

$$\text{Im}[\tilde{c}a^*\langle A|\tilde{C}\rangle + \tilde{a}c^*\langle C|\tilde{A}\rangle] = 0 \quad (18)$$

$$b\tilde{d}^*\langle \tilde{D}|B\rangle + d\tilde{b}^*\langle \tilde{B}|D\rangle = 0 \quad (19)$$

$$ca^*\langle A|C\rangle + db^*\langle B|D\rangle = 0 \quad (20)$$

$$\tilde{a}\tilde{c}^*\langle \tilde{C}|\tilde{A}\rangle + \tilde{b}\tilde{d}^*\langle \tilde{D}|\tilde{B}\rangle = 0 \quad (21)$$

$$c\tilde{d}^*\langle \tilde{D}|C\rangle + d\tilde{c}^*\langle \tilde{C}|D\rangle = 0 \quad (22)$$

$$a^*b\langle A|B\rangle + c^*d\langle C|D\rangle = 0 \quad (23)$$

$$\tilde{b}^*\tilde{a}\langle \tilde{B}|\tilde{A}\rangle + \tilde{d}^*\tilde{c}\langle \tilde{D}|\tilde{C}\rangle = 0 \quad (24)$$

For the transformation (6), we could also impose the constraint such that the output reduced density matrices do not change under $|0\rangle \leftrightarrow |1\rangle$, then the following is true,

$$|a| = |\tilde{a}| \quad , \quad |b| = |\tilde{b}| \quad , \quad |c| = |\tilde{c}| \quad (25)$$

From (14,15,16)

$$\eta = |a||b|\text{Re}[e^{i(\delta_a - \delta_b + \delta_{AB})}|\langle A|\tilde{B}\rangle| + e^{i(\delta_b - \delta_a + \delta_{BA})}|\langle B|\tilde{A}\rangle|] \quad (26)$$

$$= -|a||c|\text{Re}[e^{i(\delta_c - \delta_a + \delta_{AC})}|\langle A|\tilde{C}\rangle| + e^{i(\delta_a - \delta_c + \delta_{CA})}|\langle C|\tilde{A}\rangle|] \quad (27)$$

then the maximum η can be obtained when Re part in (27) is maximum, i.e. 2. Therefore with (14), following conditions can be obtained,

$$|a|^2 + |c|^2 = \frac{1 - \eta}{2} \quad , \quad |a||c| = \frac{\eta}{2} \quad (28)$$

then

$$\begin{aligned} (|a| - |c|)^2 &= |a|^2 + |c|^2 - 2|a||c| \\ &= \frac{1 - \eta}{2} - \eta \geq 0 \\ \Rightarrow \eta &\leq \frac{1}{3} \end{aligned} \quad (29)$$

Therefore the maximum of η is $1/3$. For $\eta = \frac{1}{3}$,

$$|\tilde{a}| = |a| = \sqrt{\frac{1}{6}} \quad , \quad |\tilde{b}| = |b| = \sqrt{\frac{1}{2}} \quad (30)$$

$$|\tilde{c}| = |c| = \sqrt{\frac{1}{6}} \quad , \quad |\tilde{d}| = |d| = \sqrt{\frac{1}{6}} \quad (31)$$

and the minus sign for (27) and (26) can be satisfied with the following phase choice.

$$\delta_c = \delta_{\bar{c}} = \pi \quad , \quad \delta_b = \delta_{\bar{b}} = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad (32)$$

while all other phases, $\delta_a, \dots, \delta_{A\bar{B}}, \dots$, vanish. The ancillas satisfying the constraint (11),(13),(17-24) can be of the following form,

$$|A\rangle = (1, 0, 0, 0) \quad , \quad |\tilde{A}\rangle = (0, 1, 0, 0) \quad (33)$$

$$|B\rangle = (0, 1, 0, 0) \quad , \quad |\tilde{B}\rangle = (1, 0, 0, 0) \quad (34)$$

$$|C\rangle = (0, 1, 0, 0) \quad , \quad |\tilde{C}\rangle = (1, 0, 0, 0) \quad (35)$$

$$|D\rangle = (0, 0, 1, 0) \quad , \quad |\tilde{D}\rangle = (0, 0, 0, 1) \quad (36)$$

with the usual basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. Then the fidelity $\frac{2}{3}$ can be obtained with the following transformation

$$|0\rangle|Q\rangle \rightarrow \sqrt{\frac{1}{6}}|0000\rangle + \sqrt{\frac{1}{2}}\exp[i\cos^{-1}(\frac{1}{\sqrt{3}})]|0101\rangle - \sqrt{\frac{1}{6}}|1001\rangle + \sqrt{\frac{1}{6}}|1110\rangle \quad (37)$$

$$|1\rangle|Q\rangle \rightarrow \sqrt{\frac{1}{6}}|1101\rangle + \sqrt{\frac{1}{2}}\exp[i\cos^{-1}(\frac{1}{\sqrt{3}})]|1000\rangle - \sqrt{\frac{1}{6}}|0100\rangle + \sqrt{\frac{1}{6}}|0011\rangle \quad (38)$$

Note that the fidelity of universal anti-cloner, F_{OACM} , is same as the measurement fidelity, which is $2/3$. Hence one way to implement optimal anti-cloning is to measure the unknown input state and prepare two qubits with opposite spin directions. It is also implied that fidelity of spin flipping also should be bounded from below by $2/3$, i.e. $F_{OSFM} \geq F'_{SFM} = \frac{2}{3}$, since after anti-cloning, one can throw away the first qubit and will be left with the second qubit which has opposite direction to the input state. Therefore, the following holds,

$$F_{OACM} \leq F_{OSFM} \quad (39)$$

In [4, 5]³, it was claimed that the optimal spin flipping is achieved with $2/3$ fidelity including classical measurement. Due to this classical information, one can prepare additional qubit as the original input state (i.e. opposite to the output) which implies

³ In [5], the term *Universal NOT-gate* was used rather than spin flipping

$F_{OACM} \geq F'_{ACM} = \frac{2}{3}$, therefore

$$F_{OACM} \geq F_{OSFM} \quad (40)$$

Therefore, equality between F_{OACM} and F_{OSFM} holds.

3 Probabilistic quantum anti-cloning

There is another type of imperfect cloning, a probabilistic cloner. Duan and Guo [7] showed that there can be a unitary transformation such that linearly independent states can be cloned perfectly, with non-zero probability. Can anti-cloning be done probabilistically, i.e. can we find a unitary transformation such that $|\mathbf{m}_i\rangle|0\rangle \xrightarrow{\text{Prob} \neq 0} |\mathbf{m}_i\rangle|-\mathbf{m}_i\rangle$, $i = 1, \dots, n$, can be achieved. In order to show it, we follow Duan and Guo's method [7] with the following transformation,

$$U(|\mathbf{m}_i\rangle|0\rangle|Y_0\rangle) = \sqrt{f}|\mathbf{m}_i\rangle|-\mathbf{m}_i\rangle|Y_0\rangle + \sum_{j=1}^n a_{ij}|Q_{12}^{(j)}\rangle|Y_j\rangle \quad (41)$$

where $|Y_0\rangle$ and $|Y_j\rangle$ are orthonormal probe, such that whether cloning was successful or failed can be known, and $|Q_{12}^{(j)}\rangle$ are normalised. Taking inner product of (41), we get

$$[\langle\mathbf{m}_i|\mathbf{m}_j\rangle] = f[\langle\mathbf{m}_i|\mathbf{m}_j\rangle\langle-\mathbf{m}_i|-\mathbf{m}_j\rangle] + [a_{ij}][a_{ji}^*] \quad (42)$$

where we take $[\cdot]$ to be a matrix. For any n -vector $\mathbf{k} = (k_1, \dots, k_n)$, we can write $\mathbf{k}[\langle\mathbf{m}_i|\mathbf{m}_j\rangle]\mathbf{k}^\dagger = \langle K|K\rangle$ where $|K\rangle \equiv k_1|\mathbf{m}_1\rangle + \dots + k_n|\mathbf{m}_n\rangle$. Since $|K\rangle$ is a quantum state (linear combination of $|\mathbf{m}_i\rangle$ s), its norm is always greater than or equal to zero. It is zero only when $|K\rangle$ itself is zero. If $|\mathbf{m}_1\rangle, \dots, |\mathbf{m}_n\rangle$ are linearly independent, then $|K\rangle$ is never zero for any n -vector (k_1, \dots, k_n) . Therefore when $|\mathbf{m}_i\rangle$ are linearly independent, $[\langle\mathbf{m}_i|\mathbf{m}_j\rangle]$ is positive definite. Due to continuity, $[\langle\mathbf{m}_i|\mathbf{m}_j\rangle] - f[\langle\mathbf{m}_i|\mathbf{m}_j\rangle\langle-\mathbf{m}_i|-\mathbf{m}_j\rangle]$ is also positive definite with sufficiently small f . Therefore $[\langle\mathbf{m}_i|\mathbf{m}_j\rangle] - f[\langle\mathbf{m}_i|\mathbf{m}_j\rangle\langle-\mathbf{m}_i|-\mathbf{m}_j\rangle]$ can be diagonalised and $[a_{ij}][a_{ji}^*]$ can be chosen such that (42) is satisfied. Therefore there exists a unitary operator U such that (41) is satisfied.

Consider the following general unitary transformation,

$$\begin{aligned} U(|\mathbf{m}_1\rangle) &= \sqrt{f}|\mathbf{m}_1\rangle|-\mathbf{m}_1\rangle|Y_0\rangle + \sqrt{1-f}|Q\rangle|Y_1\rangle \\ U(|\mathbf{m}_2\rangle) &= \sqrt{f}|\mathbf{m}_2\rangle|-\mathbf{m}_2\rangle|Y_0\rangle + \sqrt{1-f}|Q\rangle|Y_1\rangle \end{aligned} \quad (43)$$

where $|Y_0\rangle$ and $|Y_1\rangle$ are orthonormal and $|Q\rangle$ are normalised. Then a cloning efficiency f for probabilistic quantum anti-cloner can be obtained as follows,

$$f \leq \frac{1 - |\langle \mathbf{m}_1 | \mathbf{m}_2 \rangle|}{1 - |\langle \mathbf{m}_1 | \mathbf{m}_2 \rangle| |\langle -\mathbf{m}_1 | -\mathbf{m}_2 \rangle|} = \frac{1 - |\langle \mathbf{m}_1 | \mathbf{m}_2 \rangle|}{1 - |\langle \mathbf{m}_1 | \mathbf{m}_2 \rangle|^2} \quad (44)$$

The equality in (44) holds if $\langle \mathbf{m}_1 | \mathbf{m}_2 \rangle$ and $\langle -\mathbf{m}_1 | -\mathbf{m}_2 \rangle$ are real and positive which can be achieved by redefining these states by multiplying them by a phase. Therefore, the probabilistic anti-cloner has the same efficiency as in the Duan and Guo's regular cloner. One can see that the above probabilistic quantum anti-cloner can be generalised to clone $\mu = (L, M)$ copies for L regular copies and M copies of opposite spin direction. For $n = 2$ case, as $\mu \rightarrow \infty$, the bound (44) approaches the probability of distinguishability given by $1 - |\langle \mathbf{m}_1 | \mathbf{m}_2 \rangle|$ for given two states $|\mathbf{m}_1\rangle$ and $|\mathbf{m}_2\rangle$ [8]. In [9], it was shown that the no-signalling condition restricts the number of states that can be cloned in a given Hilbert space. Following the same argument, one can show that if PQACM can clone $N + 1$ or more states in a N -dimensional Hilbert space, then faster-than-light signalling can be achieved. Therefore no-signalling condition imposes a constraint such that probabilistic quantum anti-cloner cannot clone more than N states.

Let us consider the following simple example. We take $|\mathbf{m}_1\rangle = |0\rangle$, $|\mathbf{m}_2\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$, $|Y_0\rangle = |0\rangle$, $|Y_1\rangle = |1\rangle$, $|Q_{12}\rangle = |00\rangle$ then with maximum efficiency (44) of

$$f = \frac{1 - \cos\theta}{1 - \cos^2\theta} \quad (45)$$

we can find the unitary operator $U = \sum_{i=1}^8 |N_i\rangle\langle M_i|$ where $|M_1\rangle = |000\rangle$, $|M_2\rangle = |001\rangle$, \dots , $|M_8\rangle = |111\rangle$ and $|N_i\rangle$'s are as follows

$$|N_1\rangle = \frac{1}{\sqrt{1 + \cos\theta}}|010\rangle + \frac{\sqrt{\cos\theta}}{\sqrt{1 + \cos\theta}}|001\rangle \quad (46)$$

$$\begin{aligned}
|N_2\rangle = & -\frac{\cos\theta}{\sqrt{1+\cos\theta}}|000\rangle + \frac{\cos\theta(\cos\theta-1)}{\sin\theta\sqrt{1+\cos\theta}}|010\rangle - \frac{\sin\theta}{\sqrt{1+\cos\theta}}|100\rangle \\
& + \frac{\cos\theta}{\sqrt{1+\cos\theta}}|110\rangle + \frac{\sqrt{\cos\theta}(1-\cos\theta)}{\sin\theta\sqrt{1+\cos\theta}}|001\rangle
\end{aligned} \tag{47}$$

and $|N_3\rangle, \dots, |N_8\rangle$ are chosen as orthonormal states to (46) and (47). One can see $UU^\dagger = U^\dagger U = \mathbf{1}$ and can easily check that U yields $|0\rangle|1\rangle$ with the maximum efficiency given in (44).

With a similar argument, one can show the spin flipping, $|\mathbf{m}\rangle \rightarrow |-\mathbf{m}\rangle$, can be done probabilistically, i.e.

$$\begin{aligned}
U|\mathbf{m}_1\rangle|B_0\rangle &= \sqrt{F}|-\mathbf{m}_1\rangle|B_1\rangle + \sqrt{1-\xi_1}|Q\rangle \\
U|\mathbf{m}_2\rangle|B_0\rangle &= \sqrt{F}|-\mathbf{m}_2\rangle|B_2\rangle + \sqrt{1-\xi_2}|Q\rangle
\end{aligned} \tag{48}$$

can be shown to exist. When $|B_1\rangle$ and $|B_2\rangle$ are orthogonal, one can identify $|-\mathbf{m}_1\rangle$ and $|-\mathbf{m}_2\rangle$ and can prepare as many states as one wants and its efficiency bound is same as distinguishability between the two states.

4 Discussions

We have considered two types of quantum cloning for two anti-parallel outputs. In probabilistic cloning, for two input states, the anti-cloning efficiency is higher than the efficiency of distinguishing between the two states. On the other hand, in case of deterministic cloning, the fidelity of universal anti-cloner and the fidelity of measurement are above equal to $2/3$. In other words, one could measure the input state and prepare two anti-parallel qubits (or as many as one wants) and this would have the same fidelity as in universal anti-cloner. The question of why universal anti-cloner has the same fidelity as the fidelity of measurement does not seem to have an immediate explanation.

In [10], it was shown that, unlike as in the classical case, quantum conditional entropy, which is the information about B which cannot be gained by measuring A, can have

negative values. This negativity of entropy has been puzzling and its exact physical meaning has been questioned. In an analogy with particle physics, it has been suggested [10] that anti-qubits may be useful in describing quantum information processes where anti-qubits were introduced as qubits traveling backward in time [11].

References

- [1] W.K. Wootters and W.H. Zurek, *Nature* 299 (1982) 802.
- [2] V. Buzek and M. Hillery, *Phys. Rev. A* 54 (1996) 1844.
- [3] D. Bruss, D.P. DiVincenzo, A. Ekert, C.A. Fuchs, C. Macchiavello and J.A. Smolin, *Phys. Rev. A* 57 (1998) 2368
- [4] N. Gisin and S. Popescu, *Phys. Rev. Lett.* 83 (1999) 432.
- [5] V. Buzek, M. Hillery and R.F. Werner, *Phys. Rev. A* 60 (1999) R2626; *J. Mod. Opt.* 47 (2000) 211.
- [6] S. Massar and S. Popescu, *Phys. Rev. Lett.* 74 (1995) 1259.
- [7] L.-M. Duan and G.-C. Guo, *Phys. Rev. Lett.* 80 (1998) 4999; *Phys. Lett. A* 243 (1998) 261.
- [8] D. Dieks, *Phys. Lett. A* 126 (1988) 303; A. Peres, *Phys. Lett. A* 128 (1988) 19.
- [9] L. Hardy and D.D. Song, *Phys. Lett. A* 259 (1999) 331.
- [10] N.J. Cerf and C. Adami, *Phys. Rev. Lett.* 79 (1997) 5194.
- [11] C.H. Bennett and S.J. Wiesner, *Phys. Rev. Lett.* 69 (1992) 2881.