MULTIPARTICLE SUSY QUANTUM MECHANICS AND THE REPRESENTATIONS OF PERMUTATION GROUP.

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Abstract. The method of multidimensional SUSY Quantum Mechanics is applied to the investigation of supersymmetrical $N-$particle systems on a line for the case of separable center-of-mass motion. New decomposition of the superhamiltonian into block-diagonal form with elementary matrix components is constructed. Matrices of coefficients of these minimal blocks are shown to coincide with matrices of irreducible representations of permutation group S_N, which correspond to the Young tableaux $(N-M, 1^M)$. The connections with known generalizations of $N-$particle Calogero and Sutherland models are established.

1. Introduction

One of the most natural generalizations of the standard [1] 1-dimensional Supersymmetrical Quantum Mechanics (SUSY QM) concerns the systems in the spaces of arbitrary dimension d [2]. It was shown for such systems that superhamiltonian is a matrix $(2^d \times 2^d)$ block-diagonal operator with $(d+1)$ components on the diagonal. These components of the superhamiltonian are Schrödinger-type operators with matrix $(C_d^n \times C_d^n)$ potentials (C_d^n- binomial coefficients, $n = 0, 1, ..., d$). Supersymmetry of the system leads to important SUSY intertwining relations between neighbouring components of the superhamiltonian and provides definite connections between their spectra and eigenfunctions. More definitely, for each component its spectrum consists of the eigenvalues which coincide with a part of eigenvalues of neighbouring components of superhamiltonian. Corresponding eigenfunctions are connected with each other by the action of supercharge operators (see details in [2]). This approach was successfully used for some 2- and 3-dimensional physical systems [3].

It would be interesting to apply this method for the systems with another possible interpretation of several degrees of freedom in the superhamiltonian. Namely, it seems to be useful in the description of supersymmetrical systems of N interacting quantum particles on a line.

The supersymmetric generalization of a known exactly solvable [4], [5] N-particle Calogero model was considered for the first time in the paper [6], where its spectrum was found (see also papers [7], [8], [9], [10], [11]). In the paper [7] the hypothesis was put forward (but not proved), that the (super)Calogero, Sutherland and some other models possess shape-invariance [12], which could help us find the spectrum of the models [a].

In the present paper the most general variety of supersymmetrical $N-$particle systems on a line will be considered using the generalization of method [2]. The only restriction, introduced in Sect. 2, is the condition of separability of center-of-mass motion (CMM) [7] in the superpotential which seems to be very natural for such systems. Introducing usual bosonic Jacobi coordinates and their fermionic analogues, we derive the superhamiltonian and SUSY intertwining relations for systems with separable CMM. This block-diagonal superhamiltonian has the same matrix dimension $2^N \times 2^N$, as in [2], but more detailed structure: $2N$ blocks $C^M_{N-1} \times C^M_{N-1}$ instead of $N+1$ blocks $C^M_N \times C^M_N$ in [2].

In Sect. 3 the internal structure of the blocks on the diagonal of the superhamiltonian is considered. It is shown that for any M the coefficients $B^{(M)}_{ij}$ in matrix potentials coincide with the matrices of irreducible representation of permutation group S_N, which is characterized by the Young tableau $(N-M, 1^M)$. This statement provides that these matrix potentials are elementary blocks of the superhamiltonian, i.e. they cannot be further decomposed into the block-diagonal form. At the end of Sect. 3 the SUSY intertwining relations are built in terms of Jacobi coordinates, using the Clebsh-Gordan coefficients for the corresponding irreducible representations of S_N. Two examples are considered in Sect. 4. For the case $N = 3$ with particular choice of superpotential our approach gives a part of the spectrum of 2×2 matrix Hamiltonian. The class of superpotentials corresponding to $N-$particle models with pairwise interactions (including Calogero and Sutherland models) is considered in the second example. The connections with known generalizations [6],[13],[9],[11] of $N-$particle Calogero and Sutherland models are established. The proof of the Theorem of Sect. 3 can be found in Appendix.

[a] However, some kind of shape-invariance for the Calogero model was constructed in [8] using the Dunkl operators [14].

2. Systems with a separable centre-of-mass motion

The supersymmetric quantum system for arbitrary number of dimensions N consists [2] of the superhamiltonian and the supercharges[b]:

$$H_S = \frac{1}{2}(-\Delta + \sum_{i=1}^{N}(\partial_i W)^2 - \Delta W) + \sum_{i,j=1}^{N} \psi_i^+ \psi_j \partial_i \partial_j W; \qquad \Delta \equiv \sum_{i=1}^{N} \partial_i \partial_i; \quad \partial_i \equiv \partial/\partial x_i; \quad (1)$$

$$Q^{\pm} \equiv \frac{1}{\sqrt{2}} \sum_{j=1}^{N} \psi_j^{\pm}(\pm\partial_j + \partial_j W); \qquad (2)$$

with the algebra

$$H_S = \{Q^+, Q^-\}, \qquad (3)$$

$$(Q^+)^2 = (Q^-)^2 = 0, \qquad (4)$$

$$[H_S, Q^{\pm}] = 0, \qquad (5)$$

where $\psi_i,\ \psi_i^+$ are fermionic operators:

$$\{\psi_i, \psi_j\} = 0, \qquad \{\psi_i^+, \psi_j^+\} = 0, \qquad \{\psi_i, \psi_j^+\} = \delta_{ij}. \qquad (6)$$

The dynamics of a SUSY QM system is determined by a superpotential W, depending on N real coordinates $(x_1, \dots, x_N)$.

For N-particle systems on a line it is natural to consider potentials with a separable centre-of-mass motion (CMM). Therefore in this paper we restrict ourselves by considering the superpotentials [c]:

$$W(x_1, ..., x_N) = w(x_1, \dots, x_N) + W_C\Big(\frac{x_1 + ... + x_N}{\sqrt{N}}\Big)\ ; \quad \sum_{j=1}^{N} \partial_j w(x_1, \dots, x_N) = 0, \qquad (7)$$

allowing a separation of CMM motion ($w(x_1, \dots, x_N)$ does not depend [d] on $x_1 + ... + x_N$).

Let us introduce the operator:

$$\hat{K}_{ij} \equiv \psi_i^+\psi_j + \psi_j^+\psi_i - \psi_i^+\psi_i - \psi_j^+\psi_j + 1 = 1 - (\psi_i^+ - \psi_j^+)(\psi_i - \psi_j) = \hat{K}_{ji} = (\hat{K}_{ij})^{\dagger}, \quad (8)$$

[b] Here and below the indices $i, j, k, \dots$ range from 1 to N.

[c] The usefulness of the factor $1/\sqrt{N}$ will be explained later.

[d] The equation (7) is equivalent to the condition: $\sum_k \partial_k(\partial_i - \frac{1}{N}\sum_l \partial_l)w = 0$ for every $i = 1, \dots, N$.

which plays the role of the fermionic permutation operator

$$\begin{aligned} \hat{K}_{ij}\psi_i^+ &= \psi_j^+\hat{K}_{ij}, & (9)\\ \hat{K}_{ij}\psi_k^+ &= \psi_k^+\hat{K}_{ij}, \quad k \neq i,j. & (10) \end{aligned}$$

In the fermionic Fock space

$$\psi_{i_1}^+ \dots \psi_{i_M}^+|0> \equiv |i_1 \dots i_M>; \quad \psi_i|0>=0 \qquad i, i_1 \dots i_M, M \leq N \tag{11}$$

this operator acts as:

$$\begin{aligned} \hat{K}_{ij}|i_1 \dots i \dots j \dots i_M> &= |i_1 \dots j \dots i \dots i_M> \\ \hat{K}_{ij}|i_1 \dots i \dots i_M> &= |i_1 \dots j \dots i_M> \\ \hat{K}_{ij}|i_1 \dots i_M> &= |i_1 \dots i_M> \qquad \text{for} \quad i_1 \dots i_M \neq i,j. \end{aligned} \tag{12}$$

Let us rewrite H_S for the superpotentials (7) using $\hat{K}_{ij}$. We will take into account the following equations:

$$\sum_{i,j=1}^{N} \psi_i^+\psi_j\partial_i\partial_j w = \frac{1}{2}\sum_{i,j=1}^{N}\hat{K}_{ij}\partial_i\partial_j w = \frac{1}{2}\sum_{i\neq j}^{N}\hat{K}_{ij}\partial_i\partial_j w + \frac{1}{2}\sum_{i=1}^{N}\partial_i^2 w, \tag{13}$$

$$\sum_{i,j=1}^{N}\psi_i^+\psi_j\partial_i\partial_j W_C = \frac{1}{N}\Big(\sum_{i=1}^{N}\psi_i^+\Big)\Big(\sum_{j=1}^{N}\psi_j\Big)W_C'', \tag{14}$$

$$\sum_{j=1}^{N}\Big(\partial_j(w+W_C)\Big)^2 = \sum_{j=1}^{N}\Big(\partial_j w\Big)^2 + (W_C')^2 \tag{15}$$

to obtain

$$\begin{aligned} H_S = -\frac{1}{2}\Delta + \frac{1}{2}\sum_{j=1}^{N}(\partial_j w)^2 + \frac{1}{2}\sum_{i\neq j}^{N}\hat{K}_{ij}\partial_i\partial_j w + \\ +\frac{1}{2}((W_C')^2 - W_C'') + \frac{1}{N}\Big(\sum_{i=1}^{N}\psi_i^+\Big)\Big(\sum_{j=1}^{N}\psi_j\Big)W_C''. \end{aligned} \tag{16}$$

For the superpotentials (7) with a separable CMM it is natural to go to the well-known Jacobi coordinates[e] [15]:

$$\begin{aligned} y_b &= \frac{1}{\sqrt{b(b+1)}}(x_1 + \dots + x_b - bx_{b+1}) \\ y_N &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N} x_i, \end{aligned} \tag{17}$$

[e] From this moment on, the variables denoted by letters $a, b, c, \dots$ range from 1 to $N-1$.

or $y_k = \sum_{l=1}^{N} R_{kl} x_l$, where the matrix R is determined by (17). The derivatives are connected by the same matrix: $\partial/\partial y_k = \sum_{l=1}^{N} R_{kl}\partial/\partial x_l$, because R is an orthogonal matrix.

For the supersymmetric systems it is natural to introduce also the fermionic analogues of the Jacobi variables:

$$\begin{aligned}
\phi_b &= \frac{1}{\sqrt{b(b+1)}}(\psi_1 + \ldots + \psi_b - b\psi_{b+1});\\
\phi_N &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\psi_i,
\end{aligned}$$

or $\phi_k = \sum_{l=1}^{N} R_{kl}\psi_l$, where the matrix R is the same as in (17). These variables also satisfy the standard anticommutation relations:

$$\{\phi_k, \phi_l\} = 0, \qquad \{\phi_k^+, \phi_l^+\} = 0, \qquad \{\phi_k, \phi_l^+\} = \delta_{kl}. \tag{18}$$

In terms of the Jacobi variables the supercharges (2) can be rewritten as:

$$Q^{\pm} = Q_C^{\pm} + q^{\pm};$$

$$Q_C^{\pm} \equiv \frac{1}{\sqrt{2}}\phi_N^{\pm}\Big(\pm\frac{\partial}{\partial y_N} + W_C'\Big); \qquad q^{\pm} \equiv \frac{1}{\sqrt{2}}\sum_{b=1}^{N-1}\phi_b^{\pm}\Big(\pm\frac{\partial}{\partial y_b} + \frac{\partial}{\partial y_b}w\Big).$$

Because

$$\{Q_C^{\pm}, q^{\pm}\} = 0, \tag{19}$$

the superhamiltonian H_S, acting in a N- dimensional superspace, describes two non-interacting supersymmetric quantum systems:

$$H_S = \{Q^+, Q^-\} = \{q^+, q^-\} + \{Q_C^+, Q_C^-\} \equiv h + H_C, \tag{20}$$

where

$$h = \frac{1}{2}\sum_{b=1}^{N-1}\Big(-\frac{\partial^2}{\partial y_b^2} + \Big(\frac{\partial w}{\partial y_b}\Big)^2 - \frac{\partial^2 w}{\partial y_b^2}\Big) + \sum_{b,c=1}^{N-1}\phi_b^+\phi_c\frac{\partial^2 w}{\partial y_b \partial y_c}, \tag{21}$$

$$H_C = \frac{1}{2}\Big(-\frac{\partial^2}{\partial y_N^2} + (W_C')^2 - W_C''\Big) + \phi_N^+\phi_N W_C'' \tag{22}$$

are $(N-1)$- and 1-dimensional SUSY Hamiltonians, respectively.

The Hamiltonian h acting in the fermionic Fock space :

$$\phi_{b_1}^+ \ldots \phi_{b_M}^+ |0>; \qquad b_i < b_j \text{ for } i < j; \quad M < N, \tag{23}$$

generated by fermionic creation operators ϕ_b^+, conserves the corresponding fermionic number. Therefore, in the basis (23) it has [2] a block-diagonal form:
$h = diag(h^{(0)}, ..., h^{(N-1)})$, where matrix operator $h^{(M)}$ of dimension $C^M_{N-1} \times C^M_{N-1}$ is the component of h in the subspace with fixed fermionic number M.

In the same basis the supercharge q^+ changes [2] the fermionic number from M to $M+1$ and has the following under-diagonal structure:

$$q^+ = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ q^+_{(0,1)} & 0 & \dots & 0 & 0 \\ 0 & q^+_{(1,2)} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & q^+_{(N-2,N-1)} & 0 \end{pmatrix}. \tag{24}$$

Similarly, $q^- = (q^+)^\dagger$ is an over-diagonal matrix with nonzero elements
$q^-_{(M+1,M)} = \left(q^+_{(M,M+1)}\right)^\dagger$. Analogously, H_C has a diagonal form $H_C = diag(H_C^{(0)}, H_C^{(1)})$ in a basis $(|0>, \phi_N^+|0>)$ and conserves the number of fermions ϕ_N. In this case $H_C^{(0),(1)}$ are scalar (non-matrix) Hamiltonians. The one-dimensional supercharges Q_C^+, Q_C^- are a partial case of (24) with off-diagonal components $Q^+_{C(0,1)}, Q^-_{C(1,0)}$. Superinvariance (5) of the superhamiltonian corresponds, in components, to the intertwining relations [2] which can now be decomposed as:

$$hq^+ = q^+h \;\;\Leftrightarrow\;\; h^{(M+1)}q^+_{(M,M+1)} = q^+_{(M,M+1)}h^{(M)} \tag{25}$$

$$q^-h = hq^- \;\;\Leftrightarrow\;\; q^-_{(M+1,M)}h^{(M+1)} = h^{(M)}q^-_{(M+1,M)} \tag{26}$$

$$H_CQ_C^+ = Q_C^+H_C \;\;\Leftrightarrow\;\; H_C^{(1)}Q^+_{C(0,1)} = Q^+_{C(0,1)}H_C^{(0)} \tag{27}$$

$$Q_C^-H_C = H_CQ_C^- \;\;\Leftrightarrow\;\; Q^-_{C(1,0)}H_C^{(1)} = H_C^{(0)}Q^-_{C(1,0)} \tag{28}$$

These intertwining relations lead [2] to the important connections between spectra and eigenfunctions of "neighbouring" Hamiltonians, whose fermionic numbers differ by 1. In particular, $q^+_{(M,M+1)}(q^-_{(M,M-1)})$ maps eigenfunctions of $h^{(M)}$ onto those of $h^{(M+1)}(h^{(M-1)})$ with the same energy[f] (see details in [2]).

In the total fermionic Fock space

$$\phi^+_{b_1}\dots\phi^+_{b_M}|0>, \quad \phi^+_{b_1}\dots\phi^+_{b_M}\phi^+_N|0>;\; M<N \tag{29}$$

the superhamiltonian H_S commutes with the operators $\sum_{b=1}^{N-1}\phi_b^+\phi_b$ and $\phi_N^+\phi_N$ and therefore conserves the fermionic numbers of both ϕ_b and ϕ_N, separately. Therefore, in this basis it has a block-diagonal form, too:

[f] Let us note that the eigenfunctions of $h^{(M)}$ and $h^{(M+2)}$ are not connected directly by supercharges $q^\pm$, contrary to the hypothesis of the paper [7] in the context of Calogero-like models.

$$H_S = \begin{pmatrix} h^{(0)}+H_C^{(0)} & & & & & \\ & \ddots & & & & \\ & & h^{(N-1)}+H_C^{(0)} & & & \\ & & & h^{(0)}+H_C^{(1)} & & \\ & & & & \ddots & \\ & & & & & h^{(N-1)}+H_C^{(1)} \end{pmatrix} \qquad (30)$$

where $h^{(M)},\ H_C^{(0),(1)}$ were determined above.

It is important that, due to (19), in the intertwining relations (25)-(28) one can replace $h^{(M)},\ H_C^{(0),(1)}$ by the components (30) of H_S.

3. Internal structure of the components of the superhamiltonian

In the Eq. (16) the superhamiltonian H_S was written in the coordinates (x_i, ψ_j) in terms of the fermionic permutation operator $\hat{K}_{ij}$. In this section the structure of the blocks of H_S (30) in the basis (29) will be investigated. In this basis the components of H_S have the form:

$$H_S^{(M)} = -\frac{1}{2}\Delta + \frac{1}{2}\sum_{j=1}^{N}\Big(\frac{\partial w}{\partial x_j}\Big)^2 + \frac{1}{2}\sum_{i\neq j}^{N} B_{ij}^{(M)}\partial_i\partial_j w + \frac{1}{2}((W_C')^2 \mp W_C''), \qquad (31)$$

where the matrices $B_{ij}^{(M)}$ represent[g] the operator $\hat{K}_{ij}$ in the Fock subspace with fermionic number M. Signs $\mp$ in (31) correspond to the components $(h^{(M)} + H_C^{(0),(1)})$ of the superhamiltonian. From this moment on we will consider the components of H_S only in the form (31), i.e., in terms of the coordinates (x_i, ϕ_i^+). We prefer ϕ_i^+ to ψ_i^+, because in terms of ϕ_i^+ the block structure of H_S is more detailed($2N$ blocks instead of $N+1$). The variables x_i are preferable to y_i, because x_i represent the coordinates of physical particles. It is especially important in the cases when the particles are identical or the interaction is pairwise.

We move on to determining the matrices[h] $B_{ij}^{(M)}$. The relations (9),(10) can be rewritten as

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[g] It can be checked that $\hat{K}_{ij}$ conserves the fermionic numbers of both ϕ_b and ϕ_N, so it has the same block-diagonal structure in the basis (29) as H_S.

[h] Let us stress that $B_{ij}^{(M)}$ is not a matrix element, but a whole matrix of dimension $C_{N-1}^{M} \times C_{N-1}^{M}$.

$$\hat{K}_{ij}\psi_k^+ = \sum_l T_{(ij)lk}\psi_l^+\hat{K}_{ij}, \tag{32}$$

where

$$T_{(ij)lk} \equiv \delta_{lk} - \delta_{li}\delta_{ki} - \delta_{lj}\delta_{kj} + \delta_{li}\delta_{kj} + \delta_{lj}\delta_{ki}. \tag{33}$$

Applying this commutation rule to a state (11) M times, we get:

$$\hat{K}_{ij}\psi_{k_1}^+ \dots \psi_{k_M}^+|0>= \sum_{l_1,\dots,l_M} T_{(ij)l_1k_1}\dots T_{(ij)l_Mk_M}\psi_{l_1}^+\dots\psi_{l_M}^+|0> . \tag{34}$$

From the partial case of (34) with $M=1$ one can see that $T_{(ij)lk}$ is a matrix element of the permutation operator $\hat{K}_{ij}$ between one-fermionic states.

Thus, $\hat{K}_{ij}$ realizes a tensor representation of rank M of the symmetric group S_N of permutations of ψ_i^+ on the states (11) with fixed fermionic number M. These states are obviously antisymmetric.

Substituting $\psi_l^+ = \sum_k R_{kl}\phi_k^+$ into (34), one obtains:

$$\hat{K}_{ij}\phi_{n_1}^+ \dots \phi_{n_M}^+|0>= \sum_{m_1,\dots,m_M} \tilde{T}_{(ij)m_1n_1}\dots \tilde{T}_{(ij)m_Mn_M}\phi_{m_1}^+\dots\phi_{m_M}^+|0>, \tag{35}$$

where [i]

$$\tilde{T}_{(ij)mn} \equiv \sum_{k,l} R_{mk}T_{(ij)kl}R_{nl}. \tag{36}$$

Note that $\hat{K}_{ij}\phi_N^+ = \phi_N^+\hat{K}_{ij}$, so it is enough to consider $\hat{K}_{ij}$ in the subspace

$$\phi_{b_1}^+\dots\phi_{b_M}^+|0>;\ M<N. \tag{37}$$

It is therefore possible to rewrite (35) as

$$\hat{K}_{ij}\phi_{a_1}^+ \dots \phi_{a_M}^+|0>= \sum_{b_1,\dots,b_M} \tilde{T}_{(ij)b_1a_1}\dots \tilde{T}_{(ij)b_Ma_M}\phi_{b_1}^+\dots\phi_{b_M}^+|0> . \tag{38}$$

Thus, in the space spanned by the states (37) the operators K_{ij} and matrices $B_{ij}^{(M)}$ also realize a tensor representation of rank M of the symmetric group S_N of permutations of ψ_i^+ (the states (37) are also antisymmetric).

In the Appendix we prove by induction the

Theorem: the representation (38) of the group S_N of permutations of ψ_i^+ is irreducible and corresponds to the Young tableau[j] $(N-M,1,\dots,1)\equiv(N-M,1^M)$.

[i] It should be stressed that $\hat{K}_{ij}$ in (35) permutes ψ_k^+, not ϕ_k^+.

[j] The standard notation [16] for a Young tableau containing λ_i cells in the i-th line is $(\lambda_1,\dots,\lambda_n)$; if the tableau contains m identical lines with μ cells, it is denoted by $(\dots,\mu^m,\dots)$.

It is clear that the $2N$ blocks $(h^{(M)}+H_C^{(0),(1)})$ in (30) are "subblocks" of the $N+1$ components of H_S that would be obtained if we developed a standard supersymmetric formalism [2] for H_S in the coordinates $x_i,\ \psi_i^+$. The natural question is whether there exist even smaller "subblocks", or the blocks (31) are "elementary". Because for any M, due to the statement of the Theorem, the matrices $B_{ij}^{(M)}$ realize an irreducible representation of S_N, they cannot be simultaneously made block-diagonal by a change of basis. In general case, when all their coefficients $\partial_i\partial_j w$ are independent, it means that $H_S^{(M)}$ cannot in turn be made block-diagonal by any change of basis. Thus, the blocks (31) are "elementary".

According to (25)-(28), these elementary blocks of H_S are intertwined by the components $q^+_{(M,M+1)}, q^-_{(M,M+1)}, Q^+_{(0,1)}, Q^-_{(1,0)}$ of supercharges. Let us now consider the properties of $q^+_{(M,M+1)}, q^-_{(M,M+1)}$ in the context of permutation group S_N.

Being the components of

$$q^+ = \sum_{b=1}^{N-1} \phi_b^+ a_b^+; \qquad a_b^+ \equiv \frac{1}{\sqrt{2}}(\partial/\partial y_b + \partial w/\partial y_b), \tag{39}$$

the operators $q^+_{(M,M+1)}$ map the eigenfunctions $\Psi^{(M)}$ of the subspace with fermionic number M into the eigenfunctions $\Psi^{(M+1)}$ of the subspace with fermionic number $M+1$. Let $\Psi_\nu^{(M)}$ be the components of $\Psi^{(M)}$ in some basis of the subspace (37). The exact form of the basis is unimportant, though it is necessary that the matrices $B_{ij}^{(M)}$ and the Clebsh-Gordan coefficients introduced below be defined in this basis too. Because the dimension of the subspace (37) is C_{N-1}^M, ν ranges from 1 to C_{N-1}^M.

Then the operator $q^+_{(M,M+1)}$ takes a matrix form:

$$\Psi_\mu^{(M+1)} = (q^+_{(M,M+1)}\Psi^{(M)})_\mu = \sum_{\nu=1}^{C_{N-1}^M} (q^+_{(M,M+1)})_{\mu\nu}\Psi_\nu^{(M)}. \tag{40}$$

In the same way,

$$(\phi_b^+\Psi^{(M)})_\mu = \sum_{\nu=1}^{C_{N-1}^M} (\phi_b^+)_{\mu\nu}\Psi_\nu^{(M)}. \tag{41}$$

Therefore (39) can be rewritten as

$$\left(q^+_{(M,M+1)}\right)_{\mu\nu} = \sum_{b=1}^{N-1} a_b^+(\phi_b^+)_{\mu\nu}. \tag{42}$$

We know that ϕ_b^+ is transformed under permutations of ψ_i^+ as an irreducible representation of S_N with a Young tableau $(N-1,1)$.Therefore, the matrix element

$(\phi_b^+)_{\mu\nu} = (M\ \mu, 1\ b|M+1\ \nu)$, where $(M\ \nu, 1\ b|M+1\ \mu)$ are Clebsh-Gordan coefficients which correspond to the transition[k]: $(N-M, 1^M) \times (N-1, 1) \to (N-M-1, 1^{M+1})$. In the notations of [16] the symbols $M, M+1, 1$ in the Clebsh-Gordan coefficients correspond to the representations of S_N with the Young tableaux $(N-M, 1^M), (N-M-1, 1^{M+1}), (N-1, 1)$, respectively. Finally,

$$\left(q^+_{(M,M+1)}\right)_{\mu\nu} = \sum_{b=1}^{N-1} a_b^+ \cdot (M\ \nu, 1\ b|M+1\ \mu). \tag{43}$$

Similarly,

$$\left(q^-_{(M+1,M)}\right)_{\rho\sigma} = \sum_{b=1}^{N-1} a_b^- \cdot (M+1\ \sigma, 1\ b|M\ \rho); \qquad a_b^- = (a_b^+)^\dagger \tag{44}$$

In the x_i coordinates, $a_b^\pm = \sum_{l=1}^N R_{bl} A_l^\pm$ where $A_l^\pm \equiv \frac{1}{\sqrt{2}}(\pm\partial_l + \partial_l w)$.

Substituting $H_S^{(M)}$ and (43)-(44) into the intertwining relations (25)-(26), one obtains:

$$\begin{aligned}
&\sum_{b=1}^{N-1} \sum_{\mu=1}^{C_{N-1}^{M+1}} \sum_{l=1}^{N} H_{\rho\mu}^{(M+1)} R_{bl} A_l^+ (M\ \sigma, 1\ b|M+1\ \mu) = \\
&= \sum_{c=1}^{N-1} \sum_{\nu=1}^{C_{N-1}^{M}} \sum_{m=1}^{N} R_{cm} A_m^+ (M\ \nu, 1\ c|M+1\ \rho) H_{\nu\sigma}^{(M)};
\end{aligned} \tag{45}$$

$$\begin{aligned}
&\sum_{b=1}^{N-1} \sum_{\sigma=1}^{C_{N-1}^{M+1}} \sum_{l=1}^{N} R_{bl} A_l^- (M+1\ \sigma, 1\ b|M\ \rho) H_{\sigma\epsilon}^{(M+1)} = \\
&= \sum_{c=1}^{N-1} \sum_{\delta=1}^{C_{N-1}^{M}} \sum_{m=1}^{N} H_{\rho\delta}^{(M)} R_{cm} A_m^- (M+1\ \epsilon, 1\ c|M\ \delta).
\end{aligned} \tag{46}$$

Now it follows from (45)-(46) that the eigenfunctions $\Psi^{(M)}$ and $\Psi^{(M+1)}$ are connected by

$$\Psi_\mu^{(M+1)} = \sum_{b=1}^{N-1} \sum_{l=1}^{N} \sum_{\nu=1}^{C_{N-1}^{M}} R_{bl} A_l^+ (M\ \nu, 1\ b|M+1\ \mu) \Psi_\nu^{(M)}; \tag{47}$$

$$\Psi_\rho^{(M)} = \sum_{c=1}^{N-1} \sum_{m=1}^{N} \sum_{\sigma=1}^{C_{N-1}^{M+1}} R_{cm} A_m^- (M+1\ \sigma, 1\ c|M\ \rho) \Psi_\sigma^{(M+1)}. \tag{48}$$

The relations (27),(28) remain unchanged, because the supercharges $Q^+_{(0,1)}, Q^-_{(1,0)}$ are scalar operators.

[k] $\times$ denotes the interior product of Young tableaux.

4. Examples.

4.1. 3-particle supersymmetrical system

In the case of $N = 2$ the superhamiltonian h corresponds to the standard one-dimensional SUSY QM. For the simplest non-trivial case of $N = 3$ the superhamiltonian (30) consists of six components: four scalar and two 2×2 matrix Schrödinger-type operators. Their spectra and eigenfunctions are connected with each other due to SUSY intertwining relations.

Let us consider a simple system, generated by the superpotential

$$W = -\ln\Big(3 + a\sum_{j<k}^{3}(x_j - x_k)^2\Big) - \frac{a}{2}\sum_{i=1}^{3} x_i^2; \quad a > 0. \tag{49}$$

This superpotential allows the separability of CMM (7) with

$$W_C = -\frac{a}{2} y_3^2; \tag{50}$$

$$w = -\ln(3 + a\sum_{j<k}^{3}(x_j - x_k)^2) - \frac{a}{6}\sum_{j<k}^{3}(x_j - x_k)^2 \tag{51}$$

Two of scalar blocks (31) of the corresponding superhamiltonian will then take the form of 3-dimensional harmonic oscillator with well-known spectrum:

$$H_S^{(0)} = \frac{1}{2}\Big(\sum_{i=1}^{3}\partial_i^2 + a^2\sum_{i=1}^{3} x_i^2\Big) \pm \frac{a}{2}, \tag{52}$$

Two other scalar components of (30) are not exactly solvable:

$$H_S^{(2)} = \frac{1}{2}\Big(-\Delta + a^2\sum_{i=1}^{3} x_i^2\Big) - \frac{36a}{(3 + a\sum_{j<k}^{3}(x_j - x_k)^2)^2} \pm \frac{a}{2}. \tag{53}$$

Nevertheless, SUSY intertwining relations (45) - (46) give us the opportunity to find the part[l] of spectrum of both matrix components (31):

$$H_S^{(1)} = \frac{1}{2}\Big(-\Delta + a^2\sum_{i=1}^{3} x_i^2\Big) - \frac{18a + 12a^2\sum_{i<j}^{3} B_{ij}^{(1)}(x_i - x_j)^2}{(3 + a\sum_{j<k}^{3}(x_j - x_k)^2)^2} \pm \frac{a}{2}, \tag{54}$$

[l] In some sense, this situation resembles so called quasi-exactly-solvable models [17], for which only a part of eigenstates and eigenfunctions is known.

where matrices $B_{ij}^{(1)}$ realize a simple irreducible representation of group S_3 :

$$B_{12}^{(1)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad B_{23}^{(1)} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}; \qquad B_{13}^{(1)} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}.$$

These Hamiltonians (52)-(53) are intertwined (see (45), (46)) by the supercharges (43),(44), where

$$A_l^{\pm} = \frac{1}{\sqrt{2}}\Big(\pm\partial_l - \frac{2a(3x_l - \sqrt{3}y_3)}{3 + a\sum_{j<k}^{3}(x_j - x_k)^2} - a(x_l - \sqrt{3}/3y_3)\Big). \tag{55}$$

The Clebsh-Gordan coefficients can be found, for example, in [16]. In the case of $N = 3$ one can write them explicitly:

$$\begin{aligned} &(0\ \nu, 1\ b|1\ \mu) = (1\ \mu, 1\ b|0\ \nu) = \delta_\mu^1\delta_b^2 + \delta_\mu^2\delta_b^1 \text{ where } \ \nu = 1; \\ &(2\ \sigma, 1\ b|1\ \rho) = (1\ \rho, 1\ b|2\ \sigma) = \delta_\rho^1\delta_b^2 - \delta_\rho^2\delta_b^1 \text{ where } \ \sigma = 1. \end{aligned} \tag{56}$$

In these expressions $\nu = 1$ and $\sigma = 1$ means that these indices span the basis of the representations of S_3, corresponding to the Young tableaux (3) and (1^3), respectively. These representations are one-dimensional.

The generalization onto higher N is straightforward. Let us note, that the above approach can also be applied without any change to the systems, which are not symmetric under permutations of x_i.

4.2. N-particle supersymmetric systems with a pairwise interaction.

If we are interested in the scalar and matrix Hamiltonians $H_S^{(M)}$ with a pairwise interaction[m], it is necessary (but insufficient) to consider superpotentials such that [n]

$$\partial_i\partial_j w = f(x_i - x_j);\ i \neq j, \tag{57}$$

where $f(x)$ is some real function. It means that

$$w = \sum_{i<k}^{N} U(x_i - x_j) + \sum_{j=1}^{N} h(x_j), \tag{58}$$

where $U(x), h(x)$ are also real functions.

[m] We restrict ourselves here by considering the superpotentials w that are symmetric under permutations of x_i, though this approach admits direct generalization to non-symmetric superpotentials with separable CMM.

[n] We imply here that $W_C = 0$, except for the well-known Calogero model (see below).

We will restrict ourselves futher by considering only the case of

$$w = \sum_{i<j}^{N} U(x_i - x_j); \qquad U(x) = U(-x). \tag{59}$$

For such w components $H_S^{(M)}$ (see (31)) have the form

$$H_S^{(M)} = \frac{1}{2}\Big[-\Delta + \sum_{i\neq l}^{N}(U'_{il})^2 + \sum_{i\neq l_1\neq l_2\neq i}^{N} U'_{il_1}U'_{il_2} - \sum_{i\neq j}^{N} B_{ij}^{(M)}U''_{ij}\Big], \tag{60}$$

where $U_{ij} \equiv U(x_i - x_j)$.

These matrix Hamiltonians are intertwined by relations (45), (46), where

$$A_l^{\pm} = \frac{1}{\sqrt{2}}(\pm\partial_l + \sum_{k\neq l}^{N} U'(x_l - x_k)). \tag{61}$$

For H_S^M to be pairwise, it is also necessary [5],[7], that $\sum_{i\neq l_1\neq l_2\neq i}^{N} U'_{il_1}U'_{il_2}$ decompose into a sum of pairwise terms. Therefore there should exist a real function $v_0(x)$:

$$U'(A)U'(B) + U'(A)U'(C) + U'(B)U'(C) = v_0(A) + v_0(B) + v_0(C). \tag{62}$$

Then the scalar term in (60) has the form:

$$\sum_{i\neq l}^{N}\Big[(U'(x_i - x_l))^2 + v_0(x_i - x_l)\Big] \equiv \sum_{i\neq l}^{N} V(x_i - x_l).$$

All solutions of (62) were found by Calogero [18]. They are given in the Table (See [7]). So, the components (60) are now pairwise and take the form:

$$H_S^{(M)} = \frac{1}{2}\Big[-\Delta + \sum_{i\neq l}^{N} V(x_i - x_l) - \sum_{i\neq j}^{N} B_{ij}^{(M)}U''_{ij}\Big]. \tag{63}$$

Let us note, that to obtain standard Calogero model [4] ($U(x) = ax^2/2 + b\ln x$), we have to add into (63) the terms:

$$\frac{1}{2}(W_C'^2 \mp W_C'') = \frac{1}{2}(a^2N^2y_N^2 \mp aN),$$

corresponding to nonzero $W_C(y_N) = \frac{1}{2}aNy_N^2$.

In the cases of superpotentials from the Table, which correspond to Calogero and Sutherland models, spectrum of the superhamiltonian H_S was obtained in [6], [13]. For the same superpotentials, components (63) of the superhamiltonian coincide with (also exactly solvable) matrix generalizations of Calogero and Sutherland models [9],[11] in partial case of the Young tableaux $(N-M, 1^M)$. These tableaux were obtained in the Theorem in the present paper (see Appendix).

Table

$U(x)$	$V(x)$	$U''(x)$
$\frac{1}{2}ax^2 + b\ln(x)$	$\frac{b(b+1)}{x^2} + \frac{3}{2}a^2x^2 + 3ab + a$	$a - \frac{b}{x^2}$
$a\lvert x\rvert$	$a^2 sgn^2 x + a\delta(x) - \frac{1}{3}a^2$	$2a\delta(x)$
$a\ln\sin x$	$\frac{a(a-1)}{\sin^2 x} - \frac{4}{3}a^2$	$-\frac{a}{\sin^2 x}$
$a\ln\sinh x$	$\frac{a(a-1)}{\sinh^2 x} + \frac{2}{3}a^2$	$-\frac{a}{\sinh^2 x}$
$\ln\lvert\theta_1(\frac{\pi x}{2\omega}\vert\frac{ir}{2\omega})\rvert$	$(a+\frac{1}{2})P(x) + \frac{\zeta(\omega)}{\omega}$	$aP(x) - \frac{\zeta(\omega)}{\omega}$

$\zeta(x)$ is the Weierstrass ζ function; $\zeta'(x) = -P(x)$ and ω is half-period [5].

5. Appendix

In this Appendix, we will prove the following

Theorem: the operator $\hat{K}_{ij}$ realizes on the states (37) the irreducible representation of the group S_N of permutations of ψ_i^+, corresponding to the Young tableau $(N-M, 1^M)$.

At first, it is necessary to prove the Lemma, corresponding to the partial case of the Theorem for $M = 1$:

Lemma: the operator $\hat{K}_{ij}$ (and matrices $\tilde{T}_{(ij)}$ (36)), acting on the states $\phi_b^+|0>$, realize the irreducible representation of the group S_N of permutations of ψ_i^+, corresponding to the Young tableau $(N-1, 1)$.

Proof: by induction on N.

For $N = 2$ we have only one Jacobi coordinate: $\phi_1^+ = \frac{1}{\sqrt{2}}(\psi_1^+ - \psi_2^+)$. Obviously, the state $\phi_1^+|0>$ is transformed as a representation of S_2, with a Young tableau (1^2).

Let us suppose that $\phi_1^+|0>, \ldots, \phi_{N-2}^+|0>$ form a representation of the group S_{N-1} of permutations[o] of $\psi_{i'}^+$; $i' < N$, corresponding to a Young tableau $(N-2, 1)$. It is to be proved that $\phi_1^+|0>, \ldots, \phi_{N-1}^+|0>$ form a representation of the group S_N of permutations of ψ_i^+, corresponding to a Young tableau $(N-1, 1)$.

(i) We follow the method from the book [16] of construction of the irreducible representations of S_N when the representations of S_{N-1} are known[p]. It uses the fact

[o] Here and below, the (physical) coordinates $x_{i'}, \psi_{i'}$ of the first $N-1$ particles are denoted by indices $i', j', k', \ldots$. Let us stress that they are not the Jacobi coordinates y_a, ϕ_a.

[p] In [16] another numeration of basis vectors was used.

that arbitrary permutation $\hat{K}_{ij}$ of ψ_i^+, ψ_j^+ is a combination of the permutations $\hat{K}_{i'j'}$ where $i', j' < N$ and $\hat{K}_{N-1,N}$. The only nontrivial case here is

$$\hat{K}_{j'N} = \hat{K}_{N-1,N}\hat{K}_{j',N-1}\hat{K}_{N-1,N}. \tag{64}$$

The method of [16], applied to a representation with a Young tableau $(N-1,1)$(we do not reproduce here the long proof) yields the following result: the subgroup S_{N-1} of S_N consisting of the permutations $\hat{K}_{i'j'}$ is realized in this representation by $(N-1)\times(N-1)$ matrices

$$\begin{pmatrix} U_{(i'j')} & 0 \\ 0 & 1 \end{pmatrix}, \tag{65}$$

where $U_{(i'j')}$ are the matrices of representation of S_{N-1} corresponding to a Young tableau $(N-2,1)$; their components are $U_{(i'j')ab}$, but here and below we will not write the indices a, b for brevity. The permutation $\hat{K}_{N-1,N}$ is realized in this representation by the $(N-1)\times(N-1)$ matrix

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \frac{1}{N-1} & \frac{\sqrt{N(N-2)}}{N-1} \\ & & & \frac{\sqrt{N(N-2)}}{N-1} & -\frac{1}{N-1} \end{pmatrix}. \tag{66}$$

(ii) It is easy to show that in the terms of the Jacobi variables the permutations of $\psi_{i'}^+$ are realized by the following matrices:

$$\begin{pmatrix} \tilde{T}^{(N-1)}_{(i'j')} & 0 \\ 0 & 1 \end{pmatrix}, \tag{67}$$

where $\tilde{T}^{(N-1)}_{(i'j')}$ is equal to $\tilde{T}_{(ij)}$ from the previous step of induction. Because of the assumption of induction, $\tilde{T}^{(N-1)}_{(i'j')} = U_{(i'j')}$.

The permutation $\hat{K}_{N-1,N}$ is realized (see (36)) by the following matrix: $\tilde{T}_{(N-1,N)bc} \equiv \sum_{j,l=1}^{N} R_{bj}T_{(N-1,N)jl}R_{cl}$. Substituting $T_{(N-1,N)jl}$ from (33) (R is the same as above) we obtain:

$$\tilde{T}_{(N-1,N)} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \frac{1}{N-1} & \frac{\sqrt{N(N-2)}}{N-1} \\ & & & \frac{\sqrt{N(N-2)}}{N-1} & -\frac{1}{N-1}. \end{pmatrix} \tag{68}$$

Permutation $\hat{K}_{bN}$ can be decomposed into a combination of these. Thus, the proof of the Lemma is completed.

Now we can prove the Theorem by induction on M.

For $M = 1$ the statement of the Theorem is satisfied due to Lemma.

Let us suppose that, for some fixed M, on the states $\phi^+_{b_1} \dots \phi^+_{b_M}|0>$ the irreducible representation of the group S_N of permutations of ψ^+_i with $i \leq N$, corresponding to the Young tableau $(N-M, 1^M)$, is realized.

Then we use this assumption to prove that on the states $\phi^+_{b_1} \dots \phi^+_{b_{M+1}}|0>$ the irreducible representation of the group S_N, corresponding to the Young tableau $(N-M-1, 1^{M+1})$, is realized .

(i) As we have mentioned above, (37) is transformed under permutations from S_N as an antisymmetric tensor. Such tensor can be provided with a Young tableau, describing the symmetry of its indices, namely, $[1^M]$(we deliberately use different brackets[q]).

The assumption of induction affirms that (37) transforms under S_N as an irreducible representation with a Young tableau $(N-M, 1^M)$.

Let us consider a tensor product of the states (37) and $\phi^+_{b_{M+1}}|0>$ and show that it contains $\phi^+_{b_1} \dots \phi^+_{b_{M+1}}|0>$.

The tensor product will contain the tensors whose index structure corresponds to the Young tableaux, contained in the so-called exterior product [16] of Young tableaux, corresponding to the index structure of the states (37) and $\phi^+_{b_{M+1}}|0>$, namely[r], $[1^M] \otimes [1]$.

If some Young tableau $[D]$ is contained in $[1^M] \otimes [1]$, then, the corresponding tensor representation of S_N is contained in the tensor product of the tensor representation of S_N corresponding to (37) and one corresponding to $\phi^+_{b_{M+1}}|0>$. But these representations can also be considered as irreducible representations of S_N with Young tableaux $(N-M, 1^M)$ and $(N-1, 1)$. The tensor representations from the tensor product may also be decomposed into a direct sum of irreducible representations of S_N corresponding to some Young tableaux. The last Young tableaux form a so-called interior product: $(N-M, 1^M) \times (N-1, 1)$.

So the Young tableaux obtained after decomposition of a tensor corresponding to $[D]$ are contained in $(N-M, 1^M) \times (N-1, 1)$.

[q] Let us stress that every state (37) is described by two principally different Young tableaux: $[1^M]$, denoting its index structure as antisymmetric tensor, and $(\lambda_1, \dots, \lambda_N)$, describing it as an irreducible representation of the symmetric group S_N (the notations are taken from [16]).

[r] $\otimes$, $\times$ denote the exterior and interior products of Young tableaux, respectively.

(ii) It is shown in [16] that for any Young tableau $[\lambda_1 \dots \lambda_k]$

$$[\lambda_1 \dots \lambda_k] \otimes [1] = \sum_{i=1}^{k} [\lambda_1 \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1} \dots \lambda_k]. \tag{69}$$

So, for the state (37) we get:

$$[1^M] \otimes [1] = [1^{M+1}] \oplus [2, 1^{M-1}]. \tag{70}$$

The state $\phi^+_{b_1} \dots \phi^+_{b_{M+1}} |0>$ has an index structure described by the Young tableau $[1^{M+1}]$ and is a direct sum of irreducible representations of S_N corresponding to some Young tableaux. We will further denote the set of these tableaux as (A). Taking into account the considerations from the point (ii), we can state that $(N - M, 1^M) \times (N - 1, 1) = (A) \oplus \dots$.

(iii) It is known [16] that the interior product of every Young tableau with $(N-1, 1)$ contains only the Young tableaux differing from the initial one by no more than the position of one cell. (Let us remind that all the Young tableaux describing the irreducible representations of S_N contain exactly N cells each.) Therefore, $(N-M, 1^M) \times (N-1, 1)$ may contain only the following Young tableaux: $(N-M, 1^M)$, $(N - (M+1), 1^{M+1})$, $(N - M, 2, 1^{M-2})$, $(N - (M+1), 2, 1^{M-1})$.

(iv)It is useful here to consider the state $\phi^+_{b_1} \dots \phi^+_{b_{N-1}} |0>$. Its index structure corresponds to a Young tableau $[1^{N-1}]$. As a representation of S_N, this state corresponds to a Young tableau (1^N),because

$$\begin{aligned} \phi^+_1 \dots \phi^+_{N-1} |0> = \phi^+_1 \dots \phi^+_{N-1} \phi_N \phi^+_N |0> = (-1)^{N-1} \phi_N \phi^+_1 \dots \phi^+_N |0> = \\ = C \phi_N \psi^+_1 \dots \ \psi^+_N |0>, \end{aligned} \tag{71}$$

where C is some nonzero constant. It is obvious that $\hat{K}_{ij}$, acting upon (71), changes its sign.

(v) Let us consider a tensor product of $\phi^+_{b_1} \dots \phi^+_{b_M} |0>$ with $\phi^+_{b_{M+1}} |0>$, then once more a tensor product of the result with $\phi^+_{b_{M+2}} |0>$, ... , then a tensor product of the result with $\phi^+_{b_{N-1}} |0>$, altogether $(N - M - 2)$ times. Reproducing the considerations from the point (i), especially the formulae (69),(70) we can see that this product contains a tensor with the index structure described by $[1^{N-1}]$:

$$[1^M] \otimes [1] \otimes \dots \otimes [1] = [1^{N-1}] \oplus \dots \ . \tag{72}$$

In terms of irreducible representations of S_N, (72) can be rewritten as: $(A) \times (N - 1, 1) \times \dots \times (N-1, 1) = (B) \oplus \dots$ where (B) denotes the Young tableaux ,corresponding to the state $\phi^+_1 \dots \phi^+_{N-1} |0>$, and (A) is defined in (ii).

Let (A) not contain the tableau $(N-(M+1), 1^{M+1})$. As mentioned in point (iii), the interior product of every Young tableau with $(N-1,1)$ contains only the Young tableaux differing from the initial one by no more than the position of one cell. Therefore, the interior product of every Young tableau $(N-M-2)$ times with $(N-1,1)$contains only the Young tableaux differing from the initial one by no more than the position of $N-M-2$ cells. The rest three tableaux in (iii) differ from (1^N) by more than $N-M-2$ cells because they all have less than $M+2$ cells in the first column. So, (72) is not satisfied, unless (A) contains the tableau $(N-(M+1), 1^{M+1})$.

(vi) The dimension of the space of states $\phi^+_{b_1} \ldots \phi^+_{b_{M+1}}|0>$ is equal to C^{M+1}_{N-1}, because it is a dimension of a subspace with a fermionic number equal to $M+1$, when the total fermionic number is equal to $N-1$.

The dimension of the Young tableau $(N-(M+1), 1^{M+1})$ (and hence the dimension of the corresponding irreducible representation of S_N) is also equal to C^{M+1}_{N-1}, because [16] the dimension is the number of different ways of placing the integer numbers from 1 to N consecutively into the N cells of the tableau, so that the number of the occupied cells not increase with the number of the line, and, placing each number into the line, we place it as close to its left end as possible.

At first, we place 1 into the corner cell. Then, the distribution of the numbers in the cells is determined unambiguously by deciding what numbers we place into the $N-M-2$ lateral cells. It can be done in $C^{N-M-2}_{N-1} = C^{M+1}_{N-1}$ ways.

So, (A) contains the tableau $(N-(M+1), 1^{M+1})$ and nothing more. Therefore, the state $\phi^+_{b_1} \ldots \phi^+_{b_{M+1}}|0>$ corresponds to an irreducible representation of S_N with this Young tableau .

The proof is now completed.

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