Poisson Algebras associated with Constrained Dispersionless Modified KP Hierarchies

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Abstract

We investigate the bi-Hamiltonian structures associated with constrained dispersionless modified KP hierarchies which are constructed from truncations of the Lax operator of the dispersionless modified KP hierarchy. After transforming their second Hamiltonian structures to those of Gelfand-Dickey type, we obtain the Poisson algebras of the coefficient functions of the truncated Lax operators. Then we study the conformal property and free-field realizations of these Poisson algebras. Some examples are worked out explicitly to illustrate the obtained results.

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I. INTRODUCTION

The dispersionless integrable hierarchies can be viewed as the quasi-classical limit of the ordinary integrable systems [1]. A typical example is the dispersionless Kadomtsev-Petviashvili (dKP) hierarchy which has played an important role in theoretical and mathematical physics (see, for example, [2] and references therein). The Lax formulation of the dKP hierarchy can be constructed by replacing the pseudo-differential Lax operator of KP with the corresponding Laurent series. On the other hand, an analogue construction can be made for the modified KP (mKP) hierarchy and thus leads to the dmKP hierarchy.

In the previous work [3], we established the Miura map between the dKP and dmKP hierarchies, which turns out to be canonical in the sense that the bi-Hamiltonian structure of the dmKP hierarchy [4] is mapped to the bi-Hamiltonian of the dKP hierarchy [4,5]. We also studied the solution structure of the dmKP hierarchy using the twistor construction [2]. In this paper we turn to the Poisson algebras of the bi-Hamiltonian structures associated with the dmKP hierarchy and, particularly, its reductions. For the ordinary mKP hierarchy the reductions are quite limited [6–8]. However, in the dispersionless limit, we show that the Lax operator of the dmKP hierarchy can be truncated to any finite order and their associated bi-Hamiltonian structures can be obtained via the Dirac reduction. To proceed the formulation of the dmKP hierarchy, we recall some basic facts about the algebra of Laurent series in the following.

Let Λ be an algebra of Laurent series of the form

$$\Lambda = \{A | A = \sum_{i=-\infty}^{N} a_i p^i\},$$

with coefficients a_i depending on an infinite set of variables $t_1 \equiv x, t_2, t_3, \cdots$. The algebra Λ can be decomposed into the subalgebras as

$$\Lambda = \Lambda_{\geq k} \oplus \Lambda_{<k},$$

where

$$\Lambda_{\geq k} = \{A \in \Lambda | A = \sum_{i \geq k} a_i p^i\}$$
$$\Lambda_{<k} = \{A \in \Lambda | A = \sum_{i<k} a_i p^i\}$$

and using the notations : $\Lambda_+ = \Lambda_{\geq 0}$ and $\Lambda_- = \Lambda_{<0}$ for short. Although Λ form a commutative and associative algebra under multiplication, we can define a Lie-bracket associated with Λ such that

$$[[A, B]] = \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial B}{\partial p}, \qquad A, B \in \Lambda$$

which can be regarded as the Poisson bracket defined in the 2-dimensional phase space (x, p). For a given Laurent series A we define its residue as

$$\mathrm{res} A = a_{-1}$$

and its trace as

$$\mathrm{tr}A = \int \mathrm{res}A.$$

For any two Laurent series $A = \sum_i a_i p^i$ and $B = \sum_i b_i p^i$ we have

$$\mathrm{res}\,[[A, B]] = \sum_i i(a_i b_{-i})'$$

which implies

$$\mathrm{tr}\,[[A, B]] = 0,$$

and

$$\mathrm{tr}(A\,[[B, C]]) = \mathrm{tr}([[A, B]]\,C).$$

Finally, given a functional $F(A) = \int f(a)$ we define its gradient as

$$\frac{\delta F}{\delta A} = \sum_i \frac{\delta f}{\delta a_i} p^{-i-1}$$

where the variational derivative is defined by

$$\frac{\delta f}{\delta a_k} = \sum_i (-1)^i \left(\partial^i \frac{\partial f}{\partial a_k^{(i)}}\right),$$

with $a_k^{(i)} \equiv (\partial^i a_k), \partial \equiv \partial/\partial x$.

This paper is organized as follows: In section II, we will derive the bi-Hamiltonian structures of constrained dmKP hierarchies from the one of the dmKP hierarchy by the Dirac reduction. In section III, we will investigate the conformal property of the second Poisson brackets associated with constrained dmKP hierarchies. In section IV, the free-field realizations of these Poisson algebras will be given through the corresponding Kupershmidt-Wilson (KW) theorem. We will give some examples to illustrate the obtained results in section V. Section VI contains some concluding remarks.

II. BI-HAMILTONIAN STRUCTURES

The (generalized) dmKP hierarchy is defined by the Lax operator of the form

$$K_n = p^n + v_{n-1} p^{n-1} + \cdots, \qquad (n > 0)$$

which satisfies the equations of motion

$$\frac{dK_n}{dt_k} = [[B_k, K_n]]\,, \qquad B_k = (K_n^{k/n})_{\geq 1} \tag{2.1}$$

or zero-curvature conditions

$$\frac{\partial B_k}{\partial t_l} - \frac{\partial B_l}{\partial t_k} + [[B_k, B_l]] = 0. \tag{2.2}$$

For $n = 1$, the first nontrivial flows $(t_2 = y, t_3 = t)$ of (2.2) are given by

$$\begin{aligned} v_{-1x} &= \frac{3}{2}v_{0y} - \frac{3}{2}(v_0^2)_x, \\ v_{-1y} &= 2v_{0t} - \frac{3}{2}(v_0^2)_y - 2v_{-1}v_{0x}. \end{aligned}$$

which, by eliminating v_{-1}, yields the dmKP equation

$$v_{0t} = -\frac{3}{2}v_0^2 v_{ox} + \frac{3}{2}v_{0x}\partial_x^{-1}v_{0y} + \frac{3}{4}\partial_x^{-1}v_{0yy}.$$

The Hamiltonian structures associated with K_n have been obtained by Li [4] using the classical r-matrix formulation. Especially, the second structure can be expressed as

$$\{F, G\}(K_n) = \int \mathrm{res}\left(J_2^{(n)}\left(\frac{\delta F}{\delta K_n}\right)\frac{\delta G}{\delta K_n}\right)$$

where the Hamiltonian map $J_2^{(n)}$ is defined by

$$J_2^{(n)}(X) = [[K_n, X]]_{\geq -1}\, K_n - [[K_n, (K_n X)_{\geq 1}]] \tag{2.3}$$

with $X = \sum_i x_i p^{-i-1}$. It is quite natural to define the conserved quantities as

$$H_k = \frac{n}{k}\mathrm{tr}K_n^{k/n},$$

then the Lax flows (2.1) can be described by the Hamiltonian equations

$$\frac{dK_n}{dt_k} = \{H_k, K_n\}_2^{(n)}(K_n) = J_2^{(n)}\left(\frac{\delta H_k}{\delta K_n}\right).$$

Based on the above results, we would like to consider reductions of the dmKP hierarchy and their associated Hamiltonian structures. Let us consider truncations of the Lax operator K_n as follows

$$K_{(n,m)} = p^n + v_{n-1}p^{n-1} + \cdots + v_{-m}p^{-m}, \qquad m \in Z/\{0\}. \tag{2.4}$$

It is quite easy to show that these are consistent truncations with respect to the Lax flows (2.1). Thus for each pair (n, m), the Lax operator K_n with infinitely many coefficient functions is reduced to a finite-dimensional one $-K_{(n,m)}$ which we refer to the constrained dmKP hierarchy. However, the Hamiltonian map (2.3) can not preserve the form of $dK_{(n,m)}/dt_k$ since the lowest order term of $J_2^{(n)}(\delta H/\delta K_{(n,m)})$ in p is p^{-m-1}. Hence we shall consider the Lax operator $\bar{K}_{(n,m)} = K_{(n,m)} + \mu p^{-m-1}$ and then impose the constraint $\mu = 0$ by the Dirac reduction. It turns out that Hamiltonian flows for $\bar{K}_{(n,m)}$ under the condition $\mu = 0$ gives the second class constraint:

$$\left(\text{res}\left[\left[\bar{K}_{(n,m)}, \frac{\delta H}{\delta \bar{K}_{(n,m)}}\right]\right]\right)_{\mu=0} = 0 \tag{2.5}$$

where

$$\frac{\delta H}{\delta \bar{K}_{(n,m)}} = \frac{\delta H}{\delta K_{(n,m)}} + \frac{\delta H}{\delta \mu} p^m.$$

That means the function $\delta H/\delta\mu$ should be in terms of $\delta H/\delta v_i, i = -m, -m+1, \cdots, n-1$. Solving the constraint (2.5), we obtain

$$v_{-m}\frac{\delta H}{\delta \mu} = \frac{1}{m}\int^x \text{res}\left[\left[K_{(n,m)}, \frac{\delta H}{\delta K_{(n,m)}}\right]\right]$$

which implies

$$\begin{aligned}
J_2^{(n)}\left(\frac{\delta H}{\delta \bar{K}_{(n,m)}}\right) &= \left[\left[K_{(n,m)}, \frac{\delta H}{\delta \bar{K}_{(n,m)}}\right]\right]_{\geq -1} K_{(n,m)} - \left[\left[K_{(n,m)}, \left(K_{(n,m)}\frac{\delta H}{\delta \bar{K}_{(n,m)}}\right)_{\geq 1}\right]\right], \\
&= \left[\left[K_{(n,m)}, \frac{\delta H}{\delta K_{(n,m)}}\right]\right]_{\geq -1} K_{(n,m)} - \left[\left[K_{(n,m)}, \left(K_{(n,m)}\frac{\delta H}{\delta K_{(n,m)}}\right)_{\geq 1}\right]\right] \\
&\quad + \left[\left[K_{(n,m)}, v_{-m}\frac{\delta H}{\delta \mu}\right]\right], \\
&= \left[\left[K_{(n,m)}, \frac{\delta H}{\delta K_{(n,m)}}\right]\right]_{+} K_{(n,m)} - \left[\left[K_{(n,m)}, \left(K_{(n,m)}\frac{\delta H}{\delta K_{(n,m)}}\right)_{+}\right]\right] \\
&\quad + \left[\left[K_{(n,m)}, \left(K_{(n,m)}\frac{\delta H}{\delta K_{(n,m)}}\right)_{0}\right]\right] + \left[\left[K_{(n,m)}, \frac{\delta H}{\delta K_{(n,m)}}\right]\right]_{-1} K_{(n,m)} \\
&\quad + \frac{1}{m}\left[\left[K_{(n,m)}, \int^x \text{res}\left[\left[K_{(n,m)}, \frac{\delta H}{\delta K_{(n,m)}}\right]\right]\right]\right], \\
&\equiv J_2^{(n,m)}\left(\frac{\delta H}{\delta K_{(n,m)}}\right).
\end{aligned} \tag{2.6}$$

We note that the above modified Hamiltonian map for $m = \pm 1$ are just the classical limit of the second structures of mKP hierarchies obtained in [6–8]. Besides, when $m \to \infty$, $J_2^{(n,\infty)}$ recovers the Hamiltonian map $J_2^{(n)}$, as expected.

Finally, we would like to remark that the first Poisson structure of the constrained dmKP hierarchies can be defined as a deformation of $J_2^{(n,m)}$ by shifting $K_{(n,m)} \mapsto K_{(n,m)} + \lambda$ by a constant parameter λ. Then the second structure induces a linear term $J_2^{(n,m)} \mapsto J_2^{(n,m)} + \lambda J_1^{(n,m)}$ with

$$J_1^{(n,m)}\left(\frac{\delta H}{\delta K_{(n,m)}}\right) = \left[\left[K_{(n,m)}, \frac{\delta H}{\delta K_{(n,m)}}\right]\right]_{\geq -1} - \left[\left[K_{(n,m)}, \left(\frac{\delta H}{\delta K_{(n,m)}}\right)_{\geq 1}\right]\right] \tag{2.7}$$

which, by definition, is compatible with the second structure and is a Laurent series of order at most $n-1$. It turns out that (2.7) is nothing but the first Poisson structure defined in

[4]. Note that the Hamiltonian map $J_1^{(n,m)}$ is consistent with the Lax flows (2.1) for $m > 0$ but not for $m < 0$ due to the fact that the lowest order term of $J_1^{(n,m)}(\delta H/\delta K_{(n,m)})$ in p is p^{-1}, not $p^{|m|}$. Hence, just like the second structure, we require the use of Dirac's theory of constraints to obtain the consistent result. This will be done in the next section.

III. POISSON ALGEBRAS

Having constructed the Hamiltonian map of the constrained dmKP hierarchies, we are now ready to calculate the Poisson brackets of the coefficient functions v_i in (2.4). Before doing that, we would like to show that the complicated form of the modified Hamiltonian map $J_2^{(n,m)}$ defined by $K_{(n,m)}$ can be transformed to the familiar Gelfand-Dickey (GD) type structures via the following identification

$$K_{(n,m)} = L_{n+m}p^{-m} = (p^{n+m} + u_{n+m-1}p^{n+m-1} + \cdots + u_0)p^{-m}$$

where

$$v_i = u_{i+m}, \qquad i = -m, -m+1, \cdots, n-1. \tag{3.1}$$

On the other hand, from the variation

$$\delta F = \int \mathrm{res}\left(\delta K_{(n,m)}\frac{\delta F}{\delta K_{(n,m)}}\right) = \int \mathrm{res}\left(\delta L_{n+m}\frac{\delta F}{\delta L_{n+m}}\right)$$

we have

$$\frac{\delta F}{\delta K_{(n,m)}} = p^m\frac{\delta F}{\delta L_{n+m}}.$$

Using the above relations, some terms in (2.6) can be rewritten as

$$\begin{aligned}
\left[\left[K_{(n,m)}, \frac{\delta F}{\delta K_{(n,m)}}\right]\right]_+ K_{(n,m)} &= p^{-m}\left[\left[L_{n+m}, \frac{\delta F}{\delta L_{n+m}}\right]\right]_+ L_{n+m} \\
&\quad -mK_{(n,m)}p^{-1}\left(K_{(n,m)}\frac{\delta F}{\delta K_{(n,m)}}\right)'_+ \\
&\quad -mK_{(n,m)}p^{-1}\left(K_{(n,m)}\frac{\delta F}{\delta K_{(n,m)}}\right)'_0, \\
\left[\left[K_{(n,m)}, \left(K_{(n,m)}\frac{\delta F}{\delta K_{(n,m)}}\right)_+\right]\right] &= p^{-m}\left[\left[L_{n+m}, \left(L_{n+m}\frac{\delta F}{\delta L_{n+m}}\right)_+\right]\right] \\
&\quad -mp^{-1}K_{(n,m)}\left(K_{(n,m)}\frac{\delta F}{\delta K_{(n,m)}}\right)'_+, \\
\frac{1}{m}\left[\left[K_{(n,m)}, \int^x \mathrm{res}\left[\left[K_{(n,m)}, \frac{\delta F}{\delta K_{(n,m)}}\right]\right]\right]\right] &= \frac{1}{m}p^{-m}\left[\left[L_{n+m}, \int^x \mathrm{res}\left[\left[L_{n+m}, \frac{\delta F}{\delta L_{n+m}}\right]\right]\right]\right] \\
&\quad -p^{-1}K_{(n,m)}\mathrm{res}\left[\left[K_{(n,m)}, \frac{\delta F}{\delta K_{(n,m)}}\right]\right]
\end{aligned}$$

$$
\begin{aligned}
&-\left[\left[K_{(n,m)}, \left(K_{(n,m)} \frac{\delta F}{\delta K_{(n,m)}}\right)_0\right]\right] \\
&-mp^{-1} K_{(n,m)} \left(K_{(n,m)} \frac{\delta F}{\delta K_{(n,m)}}\right)_0',
\end{aligned}
$$

which imply

$$
\begin{aligned}
\{F, G\}_2^{(n,m)}(K_{(n,m)}) &= \int \mathrm{res}\left(J_2^{(n,m)}\left(\frac{\delta F}{\delta K_{(n,m)}}\right) \frac{\delta G}{\delta K_{(n,m)}}\right), \\
&= \int \mathrm{res}\left(\Theta_{2+3}^{GD}\left(\frac{\delta F}{\delta L_{n+m}}\right) \frac{\delta G}{\delta L_{n+m}}\right), \\
&= \{F, G\}_{2+3}^{GD}(L_{n+m}) \qquad (3.2)
\end{aligned}
$$

where the Hamiltonian map $\Theta_{2+3}^{GD} \equiv \Theta_2^{GD} + \frac{1}{m}\Theta_3^{GD}$ with

$$
\Theta_2^{GD}(X) = [[L_{n+m}, X]]_+ L_{n+m} - [[L_{n+m}, (L_{n+m}X)_+]], \qquad (3.3)
$$

$$
\Theta_3^{GD}(X) = \left[\left[L_{n+m}, \int^x \mathrm{res}\,[[L_{n+m}, X]]\right]\right]. \qquad (3.4)
$$

Besides the standard second GD structure Θ_2^{GD}, (3.4) is called the third GD bracket which is compatible with the second one. Hence, under the identification (3.1), the modified Hamiltonian structure (2.6) has been mapped to the sum of the second and the third GD structures defined by the polynomial L_{n+m}.

Since the Poisson algebras associated with the second GD structure have been obtained [5], we only need to treat the third one. Therefore, by (3.2), we can now use (3.3) and (3.4) instead of (2.6) to read off the Poisson brackets $\{v_i(x), v_j(y)\}_2^{(n,m)} = -\left(J_2^{(n,m)}(v)\right)_{ij} \delta(x-y)$ where the operators $\left(J_2^{(n,m)}(v)\right)_{ij}$ are taken at the point x. After some straightforward algebras we have

$$
\begin{aligned}
\left(J_2^{(n,m)}\right)_{n-1,n-1} &= \frac{n(n+m)}{m}\partial, \\
\left(J_2^{(n,m)}\right)_{i,n-1} &= \frac{n(i+m+1)}{m} v_{i+1}\partial, \\
\left(J_2^{(n,m)}\right)_{n-1,j} &= \frac{n(j+m+1)}{m} \partial v_{j+1}, \\
\left(J_2^{(n,m)}\right)_{i,j} &= (n-i-1)v_{i+j+2-n}\partial + (n-j-1)\partial v_{i+j+2-n} \\
&+ \sum_{k=j+2}^{n-1} [(k-i-1)v_{i+j+2-k}\partial v_k + (k-j-1)v_k \partial v_{i+j+2-k}] \\
&+ \frac{(i+m+1)(j+1)}{m} v_{i+1}\partial v_{j+1} \qquad (3.5)
\end{aligned}
$$

where $i, j = -m, -m+1, \cdots, n-2$ and $v_{i<-m} = 0$. We refer the above Poisson algebra to $w^{(n,m)}$-algebra.

For the Poisson algebra associated with the first structure, it can be directly obtained from the Hamiltonian map (2.7) for the case of $m > 0$ as follows:

$$\left(J_1^{(n,m>0)}\right)_{ij} = \begin{cases} (i+1)v_{i+j+2}\partial + (j+1)\partial v_{i+j+2}, & -1 \le i,j \le n-1 \\ -(i+1)v_{i+j+2}\partial - (j+1)\partial v_{i+j+2}, & -m \le i,j \le -2 \\ 0 & \text{otherwise.} \end{cases}$$

However the case for $m < 0$ requires the Dirac reduction and turns out to be

$$\left(J_1^{(n,m<0)}\right)_{ij} = \left(J_1^{(n,1)}\right)_{ij} - \sum_{k,l=-1}^{|m|-1} \left(J_1^{(n,1)}\right)_{ik} \left(J_1^{(n,1)}\right)_{kl}^{-1} \left(J_1^{(n,1)}\right)_{lj}, \qquad |m| \le i,j \le n-1.$$

Note that the bi-Hamiltonian structures of constrained dmKP hierarchies can be cast into the following recursive formula:

$$\left(J_1^{(n,m)}\right)_{ij} \frac{\delta H_{k+n}}{\delta v_j} = \left(J_2^{(n,m)}\right)_{ij} \frac{\delta H_k}{\delta v_j}.$$

Next, let us focus on the algebraic structures of the $w^{(n,m)}$-algebra (3.5). The first few of them are

$$\begin{aligned} \{v_{n-1}(x), v_{n-1}(y)\}_2^{(n,m)} &= -\frac{n(n+m)}{m}\partial \cdot \delta(x-y), \\ \{v_{n-1}(x), v_{n-2}(y)\}_2^{(n,m)} &= -\frac{n(n+m-1)}{m}\partial v_{n-1}(x) \cdot \delta(x-y), \\ \{v_{n-2}(x), v_{n-2}(y)\}_2^{(n,m)} &= -\left[v_{n-2}(x)\partial + \partial v_{n-2}(x) + \frac{(n-1)(n+m-1)}{m} v_{n-1}(x)\partial v_{n-1}(x)\right] \cdot \delta(x-y). \end{aligned} \tag{3.6}$$

In spite of the fact that v_{n-1} satisfies the $U(1)$-*Kac-Moody* algebra, the algebraic structure shown above is still unclear. However, if we define

$$w_2(x) = v_{n-2}(x) - \frac{n-1}{2n} v_{n-1}^2(x) \tag{3.7}$$

then the second and the third equations in (3.6) can be rewritten as

$$\begin{aligned} \{v_{n-1}(x), w_2(y)\}_2^{(n,m)} &= -[v_{n-1}(x)\partial + v_{n-1}'(x)] \cdot \delta(x-y), \\ \{w_2(x), w_2(y)\}_2^{(n,m)} &= -[2w_2(x)\partial + w_2'(x)] \cdot \delta(x-y) \end{aligned}$$

where w_2, being a generator, is a DiffS^1 tensor with weight 2 and v_{n-1} a tensor of weight 1. In fact, using (3.5) and (3.7) we have

$$\{v_{n-i}(x), w_2(y)\}_2^{(n,m)} = -[iv_{n-i}(x)\partial + v_{n-i}'(x)] \cdot \delta(x-y)$$

that means, except v_{n-2}, each coefficient v_{n-i} in the Lax operator $K_{(n,m)}$, with respect to the generator w_2, is already a DiffS^1 tensor with weight i. Hence the Poisson algebra $w^{(n,m)}$ defined in (3.5) is isomorphic to $w_{(n+m)}$-$U(1)$-Kac-$Moody$-algebra generated by the primary fields

$$w_1 \equiv v_{n-1}, \quad w_2 \equiv v_{n-2} - \frac{n-1}{2n} v_{n-1}^2, \quad w_i \equiv v_{n-i}, \quad (3 \le i \le n+m)$$

Note that the DiffS^1 flows can be viewed as the Hamiltonian flows generated by the Hamiltonian $\int \epsilon(x) h_1$ due to the fact that $h_1 = n\text{res}(K_{(n,m)})^{1/n} = w_2$. That is the reason why the w-algebraic structure is encoded in the constrained dmKP hierarchy. Finally when we take the limit $m \to \infty$, (3.7) still holds and the Poisson algebra (3.5) recovers to $w^{(n,\infty)} \equiv w_{dmKP}^{(n)}$ defined by the Lax operator K_n.

IV. KW THEOREM AND FREE-FIELD REALIZATIONS

It has been shown [5,9,10] that the second GD structure defined by (3.3) has nice properties with respect to the factorization of the associated Lax operator. For example, if we factorize $L = L_1 L_2$, then

$$\{F, G\}_2^{GD}(L) = \{F, G\}_2^{GD}(L_1) + \{F, G\}_2^{GD}(L_2). \tag{4.1}$$

On the other hand, if $L = L_1^\alpha, \alpha \in N$ then we have

$$\{F, G\}_2^{GD}(L) = \frac{1}{\alpha}\{F, G\}_2^{GD}(L_1). \tag{4.2}$$

Eqs.(4.1) and (4.2) are just the corresponding KW theorem for the classical limit of the second GD bracket. More generally, we shall consider the factorization of the polynomial L_{n+m} of the form

$$L_{n+m} = \prod_{i=1}^{l}(p + \phi_i)^{\alpha_i}, \qquad \sum_{i=1}^{l}\alpha_i = n + m, \tag{4.3}$$

where the Miura variables ϕ_i are zeros of L_{n+m} with multiplicities α_i, then (4.1) and (4.2) imply that

$$\{F, G\}_2^{GD}(L_{n+m}) = -\sum_{i=1}^{l}\frac{1}{\alpha_i}\int\left(\frac{\delta F}{\delta\phi_i}\right)'\left(\frac{\delta G}{\delta\phi_i}\right). \tag{4.4}$$

To complete the discussion of the KW theorem we have to treat the third GD structure under the factorization (4.3). In fact, we can show that the third GD structure enjoys the following property [for the proof, see Appendix A]:

$$\{F, G\}_3^{GD}(L_1^\alpha L_2) = \{F, G\}_3^{GD}(L_1 L_2). \tag{4.5}$$

That means the multiplicities α_i do not involve in the KW theorem with respect to the third GD structure. Hence,

$$\begin{aligned}\{F, G\}_3^{GD}(L_{n+m}) &= \{F, G\}_3^{GD}\Big(\prod_{i=1}^{l}(p + \phi_i)\Big),\\ &= \sum_{i,j=1}^{l}\int\left(\frac{\delta F}{\delta\phi_i}\right)'\left(\frac{\delta G}{\delta\phi_j}\right).\end{aligned} \tag{4.6}$$

Combining (4.4) and (4.6) we have

$$\{F, G\}_{2+3}^{GD}(L_{n+m}) = -\sum_{i,j=1}^{l}\left(\frac{1}{\alpha_i}\delta_{ij} - \frac{1}{m}\right)\int\left(\frac{\delta F}{\delta\phi_i}\right)'\left(\frac{\delta G}{\delta\phi_j}\right).$$

Among other things, the fundamental brackets for the Miura variables ϕ_i are

$$\begin{aligned}\{\phi_i(x), \phi_j(y)\}_2^{(n,m)}(K_{(n,m)}) &= \{\phi_i(x), \phi_j(y)\}_{2+3}^{GD}(L_{n+m}) \\ &= \left(\frac{1}{\alpha_i}\delta_{ij} - \frac{1}{m}\right) \partial \cdot \delta(x-y) \end{aligned} \tag{4.7}$$

where $i, j = 1, 2, ..., l$.

Since the above Poisson matrix is symmetric and hence can be diagonalized by linearly combining the Miura variables ϕ_i to obtain the free fields. For example, suppose all zeros are simple, i.e. $\alpha_i = 1, i = 1, 2, \cdots, n+m$, then $v_i = S_{n-i}(\phi_j)$ being the symmetric functions of $\{\phi_j\}$ and the Poisson matrix (4.7) becomes $P_{2+3} = \mathbf{1} - \frac{1}{m}\mathbf{h}\mathbf{h}^T$ where T denotes the transpose operation, $\mathbf{1}$ is a $(n+m) \times (n+m)$ identity matrix and $\mathbf{h}^T = (1, \cdots, 1)$. It is quite easy to construct $n+m-1$ orthonormal eigenvectors $\mathbf{h}_i$ as follows

$$\begin{aligned}\mathbf{h}_1^T &= (1, -1, 0 \cdots, 0)/\sqrt{2}, \\ \mathbf{h}_2^T &= (1, 1, -2 \cdots, 0)/\sqrt{6}, \\ &\cdots \\ \mathbf{h}_{n+m-1}^T &= (1, 1, \cdots, 1, -n-m+1)/\sqrt{(n+m)(n+m-1)}\end{aligned}$$

which satisfy $P_{2+3}\mathbf{h}_i = \mathbf{h}_i$ and hence have eigenvalue $+1$. Finally, from orthogonality, the remaining orthonormal eigenvector has the form

$$\mathbf{h}_{n+m}^T = (1, 1, \cdots, 1)/\sqrt{n+m}$$

with eigenvalue $-n/m$. Now if we rewrite the Miura variables $\phi^T = (\phi_1, \phi_2, \cdots, \phi_{n+m})$ as $\phi = \mathbf{He}$ where $\mathbf{H}$ is a $(n+m) \times (n+m)$ matrix defined by $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_{n+m}]$, then

$$\{e_i(x), e_j(y)\}_2^{(n,m)} = \lambda_i \partial \cdot \delta(x-y) \tag{4.8}$$

with $\lambda_i = 1$ $(i = 1, 2, \cdots, n+m-1)$, $\lambda_{n+m} = -n/m$. Therefore (4.8) provides a free-field realization of the $w^{(n,m)}$-algebras (3.5) and the Lax operator $K_{(n,m)}$ can be expressed as

$$K_{(n,m)} = \prod_{i=1}^{n+m} (p + (\mathbf{He})_i) p^{-m},$$

where the free fields e_i satisfy the Hamiltonian flows

$$\frac{\partial e_i}{\partial t_k} = -\lambda_i \left(\frac{\delta H_k}{\delta e_i}\right)'.$$

In the case of $m \to \infty$, the Poisson matrix of (4.7) becomes diagonal, which provides the free-field realization of the $w_{dmKP}^{(n)}$-algebra.

V. EXAMPLES

Example 1 : For the Lax operator $K_{(2,1)} = p^2 + v_1 p + v_0 + v_{-1} p^{-1}$ the first nontrivial equations are given by

$$\frac{d}{dt_2}\begin{pmatrix} v_1 \\ v_0 \\ v_{-1} \end{pmatrix} = \begin{pmatrix} 2v_{0x} \\ v_1 v_{0x} + 2v_{-1x} \\ (v_1 v_{-1})_x \end{pmatrix},$$

$$8\frac{d}{dt_3}\begin{pmatrix} v_1 \\ v_0 \\ v_{-1} \end{pmatrix} = \begin{pmatrix} -3v_1^2 v_{1x} + 12(v_0 v_1)_x + 24v_{-1x} \\ 12v_{1x}v_{-1} + 24v_1 v_{-1x} + 12v_0 v_{0x} + 3v_1^2 v_{0x} \\ 12(v_0 v_{-1})_x + 3(v_1^2 v_{-1})_x \end{pmatrix}$$

which are first equations of the generalized Benney hierarchy. The first Hamiltonians of these hierarchy flows are given by

$$\begin{aligned}
H_1 &= \int \left(v_0 - \frac{1}{4}v_1^2\right), \\
H_2 &= \int v_{-1}, \\
H_3 &= \int \left(\frac{1}{2}v_1 v_{-1} + \frac{1}{4}v_0^2 - \frac{1}{8}v_0 v_1^2 + \frac{1}{64}v_1^4\right), \\
H_4 &= \int v_0 v_{-1}, \\
H_5 &= \int \left(-\frac{1}{512}v_1^6 + \frac{3}{128}v_1^4 v_0 - \frac{1}{16}v_1^3 v_{-1} - \frac{3}{32}v_1^2 v_0^2 + \frac{3}{4}v_1 v_0 v_{-1} + \frac{1}{8}v_0^3 + \frac{3}{4}v_{-1}^2\right).
\end{aligned}$$

Then the Lax flows can be rewritten as Hamiltonian flows as follows:

$$\begin{aligned}
\frac{d}{dt_2}\mathbf{v} &= J_1^{(2,1)}\frac{\delta H_4}{\delta \mathbf{v}} = J_2^{(2,1)}\frac{\delta H_2}{\delta \mathbf{v}} \\
\frac{d}{dt_3}\mathbf{v} &= J_1^{(2,1)}\frac{\delta H_5}{\delta \mathbf{v}} = J_2^{(2,1)}\frac{\delta H_3}{\delta \mathbf{v}}
\end{aligned}$$

where $\mathbf{v}^T = (v_1, v_0, v_{-1})$, $(\delta H_i/\delta \mathbf{v})^T = (\delta H_i/\delta v_1, \delta H_i/\delta v_0, \delta H_i/\delta v_{-1})$ and

$$J_1^{(2,1)} = \begin{pmatrix} 0 & 0 & 2\partial \\ 0 & 2\partial & v_1\partial \\ 2\partial & \partial v_1 & 0 \end{pmatrix},$$

$$J_2^{(2,1)} = \begin{pmatrix} 6\partial & 4\partial v_1 & 2\partial v_0 \\ 4v_1\partial & v_0\partial + \partial v_0 + 2v_1\partial v_1 & 2\partial v_{-1} + v_{-1}\partial + v_1\partial v_0 \\ 2v_0\partial & \partial v_{-1} + 2v_{-1}\partial + v_0\partial v_1 & \partial v_1 v_{-1} + v_1 v_{-1}\partial \end{pmatrix}.$$

On the other hand, the Lax operator $K_{(2,1)}$ can be expressed in terms of primary fields as

$$K_{(2,1)} = p^2 + w_1 p + (w_2 + \frac{1}{4}w_1^2) + w_3 p^{-1}$$

where w_i satisfy the w_3-*U(1)-Kac-Moody*-algebra

$$
\begin{aligned}
\{w_1(x), w_1(y)\}_2^{(2,1)} &= -6\partial \cdot \delta(x-y), \\
\{w_1(x), w_2(y)\}_2^{(2,1)} &= -[w_1(x)\partial + w_1'(x)] \cdot \delta(x-y), \\
\{w_1(x), w_3(y)\}_2^{(2,1)} &= -[(2w_2(x) + \frac{1}{2}w_1^2(x))\partial + (2w_2(x) + w_1^2(x))'] \cdot \delta(x-y), \\
\{w_2(x), w_2(y)\}_2^{(2,1)} &= -[2w_2(x)\partial + w_2'(x)] \cdot \delta(x-y), \\
\{w_3(x), w_2(y)\}_2^{(2,1)} &= -[3w_3(x)\partial + w_3'(x)] \cdot \delta(x-y), \\
\{w_3(x), w_3(y)\}_2^{(2,1)} &= -[2w_1(x)w_3(x)\partial + (w_1(x)w_3(x))'] \cdot \delta(x-y).
\end{aligned}
$$

The free-field realization of the above algebra is given by

$$
\begin{aligned}
w_1 &= \sqrt{3}e_3, \\
w_2 &= e_3^2 - \frac{1}{2}e_2^2 - \frac{1}{2}e_1^2, \\
w_3 &= -\frac{1}{2\sqrt{3}}(e_1^2 + e_2^2)e_3 + \frac{1}{\sqrt{6}}(e_1^2 - \frac{1}{3}e_2^2)e_2 + \frac{1}{3\sqrt{3}}e_3^3
\end{aligned}
$$

with

$$
\begin{aligned}
\{e_1(x), e_1(y)\}_2^{(2,1)} &= \{e_2(x), e_2(y)\}_2^{(2,1)} = \partial \cdot \delta(x-y), \\
\{e_3(x), e_3(y)\}_2^{(2,1)} &= -2\partial \cdot \delta(x-y).
\end{aligned}
$$

Example 2 : For the Lax operator $K_{(3,-1)} = p^3 + v_2p^2 + v_1p$, the first nontrivial Lax equations are

$$
\begin{aligned}
3\frac{d}{dt_2}\begin{pmatrix} v_2 \\ v_1 \end{pmatrix} &= \begin{pmatrix} 6v_{1x} - 2v_2v_{2x} \\ 2v_2v_{1x} - 2v_{2x}v_1 \end{pmatrix}, \\
81\frac{d}{dt_4}\begin{pmatrix} v_2 \\ v_1 \end{pmatrix} &= \begin{pmatrix} (5v_2^4 - 36v_2^2v_1 + 54v_1^2)_x \\ -4v_2^3v_{1x} + 12v_2^2v_{2x}v_1 - 36v_{2x}v_1^2 \end{pmatrix}
\end{aligned}
$$

which are first equations of the dispersionless modified KdV hierarchy. The Hamiltonian flows are defined by

$$
\begin{aligned}
\frac{d}{dt_2}\mathbf{v} &= J_1^{(3,-1)}\frac{\delta H_5}{\delta \mathbf{v}} = J_2^{(3,-1)}\frac{\delta H_2}{\delta \mathbf{v}} \\
\frac{d}{dt_3}\mathbf{v} &= J_1^{(3,-1)}\frac{\delta H_7}{\delta \mathbf{v}} = J_2^{(3,-1)}\frac{\delta H_4}{\delta \mathbf{v}}
\end{aligned}
$$

with the first Hamiltonians

$$
\begin{aligned}
H_1 &= \int \left(v_1 - \frac{1}{3}v_2^2\right), \\
H_2 &= \int \left(\frac{2}{27}v_2^3 - \frac{1}{3}v_2v_1\right), \\
H_4 &= \int \left(-\frac{2}{243}v_2^5 + \frac{5}{81}v_2^3v_1 - \frac{1}{9}v_2v_1^2\right), \\
H_5 &= \int \left(\frac{7}{2187}v_2^6 - \frac{7}{243}v_2^4v_1 + \frac{2}{27}v_2^2v_1^2 - \frac{1}{27}v_1^3\right), \\
H_7 &= \int \left(-\frac{11}{19683}v_2^8 + \frac{44}{6561}v_2^6v_1 - \frac{20}{729}v_2^4v_1^2 + \frac{10}{243}v_2^2v_1^3 - \frac{1}{81}v_1^4\right)
\end{aligned}
$$

and

$$J_1^{(3,-1)} = \begin{pmatrix} 9v_2v_1^{-1}\partial v_1^{-1} + 9v_1^{-1}\partial v_2 v_1^{-1} & -9v_1^{-1}\partial + 6v_2v_1^{-1}\partial v_2 v_1^{-1} + 6v_1^{-1}\partial v_2^2 v_1^{-1} \\ -9\partial v_1^{-1} + 6v_2v_1^{-1}\partial v_2 v_1^{-1} + 6v_2^2 v_1^{-1}\partial v_1^{-1} & -6\partial v_2 v_1^{-1} - 6v_2 v_1^{-1}\partial + 8v_2v_1^{-1}\partial v_2 v_1^{-1} \end{pmatrix},$$

$$J_2^{(3,-1)} = \begin{pmatrix} -6\partial & -3\partial v_2 \\ -3v_2\partial & v_1\partial + \partial v_1 - 2v_2\partial v_2 \end{pmatrix}.$$

Rewriting the Lax operator $K_{(3,-1)}$ in terms of w_i yields

$$K_{(3,-1)} = p^3 + w_1 p^2 + (w_2 + \frac{1}{3}w_1^2)p$$

where w_1 and w_2 satisfy the (centerless-) *Virasoro-U(1)-Kac-Moody* algebra, namely,

$$\begin{aligned} \{w_1(x), w_1(y)\}_2^{(3,-1)} &= 6\partial \cdot \delta(x-y), \\ \{w_1(x), w_2(y)\}_2^{(3,-1)} &= -[w_1(x)\partial + w_1'(x)] \cdot \delta(x-y), \\ \{w_2(x), w_2(y)\}_2^{(3,-1)} &= -[2w_2(x)\partial + w_2'(x)] \cdot \delta(x-y). \end{aligned}$$

The free-field realization of the above algebra can be easily obtained as

$$\begin{aligned} w_1 &= \sqrt{2}e_2, \\ w_2 &= -\frac{1}{2}e_1^2 - \frac{1}{6}e_2^2 \end{aligned}$$

with

$$\begin{aligned} \{e_1(x), e_1(y)\}_2^{(3,-1)} &= \partial \cdot \delta(x-y), \\ \{e_2(x), e_2(y)\}_2^{(3,-1)} &= 3\partial \cdot \delta(x-y). \end{aligned}$$

VI. CONCLUDING REMARKS

We have studied the constrained dmKP hierarchies from the dmKP hierarchy by truncating the Lax operator K_n to any finite order. We have obtained the compatible bi-Hamiltonian structures of constrained dmKP hierarchies via the Dirac reduction and written down their associated Poisson algebras explicitly. We show that the second Poisson algebra $w^{(n,m)}$ turns out to be the $w_{(n+m)}$-$U(1)$-*Kac-Moody*-algebra. Its free-field realization can be obtained via the corresponding KW theorem. Several examples including the generalized Benney hierarchy have been used to illustrate the obtained results.

We would like to remark that the bi-Hamiltonian structures obtained in this paper are of hydrodynamic type [11], i.e. the Hamiltonian operators can be expressed as [for convention, we have to rewrite the Hamiltonian operators as contravariant tensors]

$$J^{ij}(v) = g^{ij}(v)\partial - \Gamma_k^{ij}(v)v_x^k$$

where, under the non-degenerate condition $\det(g_{ij}(v)) \neq 0$, $g_{ij}(v) \equiv (g^{ij})^{-1}$ can be viewed as a (pseudo-) Riemannian metric and $\Gamma_{ij}^k(v) \equiv g_{il}\Gamma_j^{kl}$ are the components of the Levi−Cività connection defined by $g_{ij}(v)$. Moreover, the Jacobi identity of the Hamiltonian structures implies that the metric is flat. This can be easily checked for the illustrated examples. On the other hand, it was pointed out [4] that the third (or higher) Hamiltonian structures may induce non-local Hamiltonian operators which also possess nontrivial geometrical interpretations [12] and thus deserve more investigations.

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APPENDIX A: A PROOF OF (4.5)

Let $L = L_1^\alpha L_2$ then the variation

$$\delta F = \int \mathrm{res}\left(\delta L \frac{\delta F}{\delta L}\right) = \int \mathrm{res}\left(\delta L_1 \frac{\delta F}{\delta L_1} + \delta L_2 \frac{\delta F}{\delta L_2}\right)$$

gives the relations

$$\frac{\delta F}{\delta L_1} = \alpha L_1^{\alpha-1} L_2 \frac{\delta F}{\delta L}, \qquad \frac{\delta F}{\delta L_2} = L_1^\alpha \frac{\delta F}{\delta L}.$$

Hence

$$\begin{aligned}
\{F,G\}_3^{GD}(L) &= \int \mathrm{res}\left(\left[\left[L_1^\alpha L_2, \int^x \mathrm{res}\left[\left[L_1^\alpha L_2, \frac{\delta F}{\delta L}\right]\right]\right]\right] \frac{\delta G}{\delta L}\right), \\
&= \int \mathrm{res}\left(\left[\left[L_1^\alpha L_2, \int^x \mathrm{res}\left(\left[\left[L_1, \frac{\delta F}{\delta L_1}\right]\right] + \left[\left[L_2, \frac{\delta F}{\delta L_2}\right]\right]\right)\right]\right] \frac{\delta G}{\delta L}\right), \\
&= \int \mathrm{res}\left(\left[\left[L_1, \int^x \mathrm{res}\left(\left[\left[L_1, \frac{\delta F}{\delta L_1}\right]\right] + \left[\left[L_2, \frac{\delta F}{\delta L_2}\right]\right]\right)\right]\right] \frac{\delta G}{\delta L_1}\right) + (1 \leftrightarrow 2).
\end{aligned} \tag{A1}$$

Now define $\hat{L} = L_1 L_2$ then

$$\frac{\delta F}{\delta L_1} = \frac{\delta F}{\delta \hat{L}} L_2, \qquad \frac{\delta F}{\delta L_2} = \frac{\delta F}{\delta \hat{L}} L_1$$

and

$$\begin{aligned}
(\mathrm{A1}) &= \int \mathrm{res}\left(L_2 \left[\left[L_1, \int^x \mathrm{res}\left[\left[\hat{L}, \frac{\delta F}{\delta \hat{L}}\right]\right]\right]\right] \frac{\delta G}{\delta \hat{L}}\right) + (1 \leftrightarrow 2), \\
&= \int \mathrm{res}\left(\left[\left[\hat{L}, \int^x \mathrm{res}\left[\left[\hat{L}, \frac{\delta F}{\delta \hat{L}}\right]\right]\right]\right] \frac{\delta G}{\delta \hat{L}}\right), \\
&= \{F,G\}_3^{GD}(L_1 L_2). \qquad \square
\end{aligned}$$

REFERENCES

[1] D. Lebedev and Yu. Manin, Phys. Lett. A **74**, 154 (1979).
[2] K. Takasaki and T. Takebe, Rev. Math. Phys. **7**, 743 (1995).
[3] Jen-Hsu Chang and Ming-Hsien Tu, solv-int/9912016.
[4] Luen-Chau Li, Commun. Math. Phys. **203**, 573 (1999).
[5] J.M. Figueroa-O'Farrill and E. Ramos, Phys. Lett. B **282**, 357 (1992).
[6] B. Konopelchenko and W. Oevel, Publ. RIMS, Kyoto Univ. **29**, 581 (1993).
[7] W. Oevel and W. Strampp, Comm. Math. Phys. **157**, 51 (1993).
[8] W. Oevel, Phys. Lett. A **186**, 79 (1994).
[9] J. Mas and E. Ramos, Phys. Lett. B **351**, 194 (1995).
[10] Y. Cheng and Z.F. Li, Lett. Math. Phys. **42**, 73 (1997).
[11] B. Dubrovin and S.P. Novikov, Russ. Math. Surv. **44**, 35 (1989).
[12] E.V. Ferapontov, in *Topics in Topology and Mathematical Physics*, Amer. Math. Soc. Transl. Ser2. **170**, 33 (1995).