Realistic Ghost State : Pauli Forbidden State from Rigorous Solution of the α Particle

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Abstract

The antisymmetrization of the composite particles in cluster model calculations manifests itself in Pauli forbidden states (ghost states), if one restricts oneself on undeformed cluster in the low energy region. The resonating group method and the generating coordinate method rely on a property of the norm kernel, which introduces some of the ghost states. The norm kernel has been usually calculated under the assumption that the inner wavefunctions have a simple Gaussian form. It is the first time that this assumption is tested by the rigorous way. In the ^{4}He+N system, we demonstrate a ghost state, which is calculated from a rigorous solution of Yakubovsky equations for the α particle. The ghost states calculated by rigorous and approximate methods turn out to have a very similar form. It is analytically proved that the trace of the norm kernel does not depend on the inner wavefunction we choose.

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Since 1937 [1], when the resonating group method (RGM) was established, it has been successfully applied to many light nuclei systems. The method is essentially based on the variational principle under the conditions that the clusters remain in the ground state in the low energy region, and the total wave function is totally antisymmetrized because of the Pauli exclusion principle. A typical example is the two α model [2,3] of ^{8}Be. In 1970's the RGM had great successes [4] and the method was extended to the generating coordinate method (GCM) [5], the orthogonality condition model (OCM) [6], the fish-bone optical model (FBOM) [7], etc. The Pauli exclusion principle plays an important role in the relative motion part of the wave function because it rules out part of the model space by an orthogonality condition. The Pauli forbidden states (ghost states) are generated by diagonalization of an integral kernel in the RGM. The integral kernel is known as the norm kernel (NK) $\mathcal{N}$ which is defined, for example, in the two cluster model as

$$\mathcal{N}(\vec{r},\vec{r'}) \equiv < \phi_1\phi_2\delta(\vec{r})|(1-\mathcal{A})|\phi_1\phi_2\delta(\vec{r'}) > \tag{1}$$

where ϕ_1, ϕ_2 and $\mathcal{A}$ are two inner cluster wavefunctions of the system and an antisymmetrizer for all nucleons, respectively. r is the relative motion coordinate. Namely, the ghost states $u_n(\vec{r})$ are eigenstates of $\mathcal{N}$ with the eigenvalue $\gamma_n = 1$.

$$\int d\vec{r'}\mathcal{N}(\vec{r},\vec{r'})u_n(\vec{r'}) = \gamma_n u_n(\vec{r}) \tag{2}$$

The total wavefunction Ψ of the system

$$|\Psi > \equiv \mathcal{A}|\phi_1\phi_2\chi > \tag{3}$$

is orthogonal to the ghost states $|\phi_1\phi_2 u_n >$

$$< \Psi|\phi_1\phi_2 u_n > = < \phi_1\phi_2\chi|\mathcal{A}|\phi_1\phi_2 u_n > = < \phi_1\phi_2\chi|(1-1)|\phi_1\phi_2 u_n > = 0, \tag{4}$$

where χ is the relative wavefunction between the two clusters.

Beyond ^{8}Be the α cluster model has been studied in ^{12}C [8–10], ^{16}O [11], etc. The correct treatment of the Pauli forbidden states is essential even in the case of bound states of clusters, where the neglect of the Pauli principle leads to an extreme overbinding. However, even with the condition they are over-bound, which is still a pending problem [12–14] in the α cluster model.

It is analytically proved [15] that if the inner wavefunctions are simple products of Gaussian functions, then the eigenvectors u_n of (2) become familiar harmonic oscillator functions. For the sake of simplicity four spinless cluster (α particle) system in ^{16}O has been studied [11] using the Yakubovsky equations [16]. Nowadays, it is possible to obtain rigorous solutions of the α particle wavefunction using realistic potentials [17]. Recent Progress of 4N scattering state is reviewed in [18]. Therefore, it becomes possible to compare the ghost states which one obtains from the rigorous Yakubovsky solution of the α particle.

In this letter we would like to choose the most simple case, the α + N system.

The first 4 nucleons build up the ground state of the α particle, while the fifth particle is the spectator. The nucleons are identical particles, furthermore, the wavefunction ϕ_α of α particle is normalized to $< \phi_\alpha|\phi_\alpha > = 1$ and NK is defined as Eq. (1),

$$\mathcal{N} = 4 < \phi_\alpha\phi_n\delta|P_{45}|\phi_\alpha\phi_n\delta > . \tag{5}$$

P_{45} is the particle exchange operator (4 and 5). Fig.1 suggests the picture of three cluster system (3N + N +N). The calculation of the NK is done with Jacobi coordinates which we show in Fig. 2. The relative Jacobi momenta are prepared and the relations between the Jacobi coordinates are

$$\vec{p}_1 = \vec{q}_2 + \frac{1}{4}\vec{q}_1, \quad \vec{p}_2 = \vec{q}_1 + \frac{1}{4}\vec{q}_2 \tag{6}$$

where p_i and q_i $(i = 1, 2)$ are Jacobi momenta of 3N+N relative motion and 4N + N one, respectively.

Our way of calculating the NK is very similar to the calculation of a leading (Born) term of the Alt-Grassberger-Sandhas equations [19,20]. For example, in the text [21] they treat the Born term Z by the partial wave representation with a function $F^{\mathcal{L}}$. In our calculation it is simply replaced as

$$F^{\mathcal{L}}_{N_1N_2}(q_1, q_2) = \frac{4}{2}\int_{-1}^{1} d\cos\theta \left[\int_0^\infty dx \int_0^\infty dy \phi_{\alpha:N_1}(x, y, |\vec{p}_1|)\phi_{\alpha:N_2}(x, y, |\vec{p}_2|)\right] P_{\mathcal{L}}(\cos\theta) \tag{7}$$

where the angle θ is between the vectors $\vec{q}_1$ and $\vec{q}_2$, N_1 and N_2 the state channels of the partial wave, and $P_{\mathcal{L}}$ the Legendre function. x and y are rests of Jacobi momenta which describe the motion of particles (1,2 and 3) inside of the α particle. The numerator 4 derives from (5).

As an example, the α wavefunction [17] of the Argonne potential (AV14) [22] is applied, and we take the case of total spin J = $1/2^+$. For the sake of simplicity we assume the spin j of 3N is almost $1/2^+$ (in fact, 94.9% for the case of AV14 potential), therefore, the angular momentum between clusters α and neutron is S-wave. This leads to $\mathcal{L}$=0.

Under this choice of the partial waves the recoupling coefficient $A^{\mathcal{L}}_{N_1,N_2}$ [21] is 1/4 and one gets

$$\mathcal{N}_0(q_1, q_2) = \frac{1}{2}\int_{-1}^{1} d\cos\theta \left[\int dx \int dy \phi_{\alpha:[1/2^+]}(x, y, p_1)\phi_{\alpha:[1/2^+]}(x, y, p_2)\right] \equiv \frac{1}{2}\int_{-1}^{1} d\cos\theta \tilde{\mathcal{N}}(p_1, p_2) \tag{8}$$

where the subscript "0" of the norm kernel means the angular momentum of $\vec{q}$ and the kernel $\tilde{\mathcal{N}}$ will be used later.

The ghost state is shown in Fig.3. For our calculation the eigen value γ_0 of Eq. (2) is not exactly equal to one, but 0.937 (if it is renormalized by the abovementioned 94.9%, γ_0= 0.987). The solid line is the ghost state u^Y_0 calculated from our Yakubovsky solution ϕ_α, comparing to the dashed line from usual Gaussian function,

$$u^G_0(r) = \left(\frac{128 \ \ \omega^3_{\alpha N}}{\pi}\right)^{1/4} \exp(-\omega_{\alpha N} r^2) \tag{9}$$

with $\omega_{\alpha N} = \Omega \times (4 \cdot 1)/(4 + 1)$ where Ω is a common shell model mode (0.275 fm $^{-2}$) [3]. They are normalized by $\int_0^\infty u^2_n(r)r^2dr = 1$. In the short range our ghost state u^Y_0 is smaller than u^G_0. The repulsive core of realistic potentials reflects in this range. This behavior is similar to that of correlation functions [23]. Beyond 4 fm our u^Y_0 is bigger than u^G_0 because in general the Gaussian function is more quickly decreasing than exponential one. We also show them in the momentum space (see Fig.4). Here the repulsive core manifests itself by a node at $\approx$ 2 fm $^{-1}$ which is absent in u^G_0. Overall they agree well.

To find the most realistic width parameter Ω we optimize $R = | < u_0^Y | u_0^G\{\Omega\} > |^2 \times 100$ [%] in Fig.(5). We could recommend $\Omega = 0.24[\text{fm}^{-2}]$ of the Gaussian width parameter which is similar to Ω=0.275 [fm^{-2}] [3].

In table I we summerize the biggest eigenvalues in Eq. (2). Analytically we find in the Gaussian case eigenvalues $\gamma_n = (-4)^{-(2n+0)}$, $n = 0, 1, 2, \ldots$ [15]. The realistic NK has got a similar spectrum. We compare the states u_1^Y and u_1^G for n=1 in Fig. 6. It is remarkable that in this case the realistic ghost state has more structure though the eigenvalues are very similar.

The matrix traces $Tr[\mathcal{N}_0]$ are given

$$
\begin{aligned}
Tr[\mathcal{N}_0]^G &= \sum_{n=0}^{\infty} \gamma_n = \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n = \frac{16}{15} = 1.0666\ldots \\
Tr[\mathcal{N}_0]^Y &= \int_0^{\infty} \mathcal{N}_0(q,q) q^2 dq = 1.0125
\end{aligned}
\tag{10}
$$

If the wavefunction of α particle is renormalized by only $j = 1/2^+$, $\int_0^\infty p^2 dp \tilde{\mathcal{N}}(p,p) = 1$ (0.949 : original norm) we get $Tr[\mathcal{N}_0]^Y$= 1.0666 which must exactly be the number of the Gaussian form. Because it is analytically proved that the trace $Tr[\mathcal{N}_0]$ does not depend what kinds of the α wave function we choose:

$$
\begin{aligned}
Tr[\mathcal{N}_0] &= \int_0^{\infty} \left[\frac{1}{2} \int_{-1}^{1} d\cos\theta \tilde{\mathcal{N}}\left(\sqrt{\frac{17}{16} + \frac{\cos\theta}{2}} q, \sqrt{\frac{17}{16} + \frac{\cos\theta}{2}} q\right) \right] q^2 dq \\
&= \frac{1}{2} \int_{-1}^{1} d\cos\theta \frac{1}{\sqrt{\frac{17}{16} + \frac{\cos\theta}{2}}^3} = \frac{16}{15}.
\end{aligned}
\tag{11}
$$

We illustrate both NKs (Fig. 7 for $\mathcal{N}_0^Y$ and Fig. 8 for $\mathcal{N}_0^Y$). The Gaussian case is analytically given,

$$
\mathcal{N}_0(q,q') = \frac{32}{qq'} \sqrt{\frac{1}{6\pi\Omega}} \exp\left[\frac{17}{48\Omega}(q^2 + q'^2)\right] \sinh\left[\frac{1}{3\Omega} qq'\right]
\tag{12}
$$

The shape is so similar that the difference $(\mathcal{N}_0^Y - \mathcal{N}_0^G)$ is also shown in Fig. 9.

Although there is only a single ghost state in α-N system, in general, the cluster-cluster effective interaction in light nuclei has a lot of ghost states. In this simple case we could find some remarkable differences in the eigen state (n=1) and the eigenvalue for n=2 which might effect RGM calculations of systems with A $>$ 5. For a most probable case such a Pauli blocking will be applied to the α-n-n three-body model system. There are already some applications [24,25] by using some Pauli methods.

It will be important benchmark calculations for more nucleons system to look into the ghost states using rigorous solutions [26] from Few-Body Physics.

Note that here we discuss the Pauli forbidden state which is different from the spurious state of the Faddeev calculations [27,28]. The naming of spurious state has been used a lot in many places, even if a cluster model has no inner structure the ghost states appear in the model and they are interpreted kinds of spurious states. We should not confuse spurious states caused from Faddeev decomposition [27–29]. In this paper we simply take the physical Yakubovsky solution of α particle to test quantitatively how precise the former Gaussian norm kernel is.

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REFERENCES

[1] J. A. Wheeler, Phys. Rev. **52**, 1083 (1937); 1107 (1937).

[2] D. R. Thompson, Y. C. Tang, R. E. Brown, Phys. Rev. **C5**, 56 (1974); V. I. Kukulin and V. G. Neudatchin, Nucl. Phys. **A157**, 609 (1970).

[3] R. Kircher, H. Kamada and S. Oryu, Prog. Theor. Phys. **73**, 1442 (1985).

[4] K. Wildermuth and Y. C. Tang, "A Unified Theory of the Nucleus" (Vieweg, Braunsceig, Germany, 1977); K. Ikeda, H. Horiuchi, S. Saito, Suppl. Prog. Theor. Phys. **68**, 1 (1980); Y. Fujiwara, H. Horiuchi, K. Ikeda, M. Kamimura, K. Kato, Y. Suzuki, E. Uegaki, Suppl. Prog. Thoer. Phys. **68**, 29 (1980); as recent article, ed. S. Ohkubo, Suppl. Prog. Theor. Phys. **132**, (1998).

[5] D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1106 (1953); J. J. Griffin and J. A. Wheeler, Phys. Rev. **108**, 311 (1957).

[6] S. Saito, Prog. Theor. Phys. **41**, 705 (1969).

[7] E. W. Schmid, Z Phys. **A297**, 105 (1980).

[8] M. Kamimura and Y. Fukushima, Proc. INS Int. Symp. on Nuclear Direct Reaction Mechanism, Fukuoka, 1978, eds. M. Tanifuji and K. Yazaki, p 409.

[9] Y. Fujiwara, R. Tamagaki, Prog. Theor. Phys. **56**, 1503 (1976).

[10] H. Kamada and S. Oryu, Prog. Theor. Phys. **76**, 1260 (1986); S. Oryu and H. Kamada, Nucl. Phys. **A493**, 91 (1989); S. Oryu, K. Samata, T. Suzuki, S. Nakamura and H. Kamada, Few-Body Syst. **17**, 185 (1994).

[11] S. Oryu, H. Kamada, H. Sekine, T. Nishino, H. Sekiguchi, Nucl. Phys. **A534**, 221 (1991).

[12] H. Walliser and S. Nakaichi-Maeda, Nucl. Phys. **A464**, 366 (1987).

[13] H. Kamada, S. Oryu, Nucl. Phys. **A463**, 347c (1987); ref. [10].

[14] K. Fukatsu, K. Kato and H. Tanaka, Prog. Theor. Phys. **81**, 738 (1989); E. Hiyama, M. Kamimura, T. Motoba and Y. Yamamoto, Prog. Theor. Phys. **97**, 881 (1997).

[15] D. A. Zaikin, Nucl. Phys. **A170**, 584 (1970).

[16] O. A. Yakubovsky, Sov. J. Nucl. Phys. **5** , 937 (1967).

[17] H. Kamada and W. Glöckle, Nucl. Phys. **A548**, 205 (1992); W. Glöckle and H. Kamada, Phys. Rev. Lett. **71**, 971 (1993).

[18] A. C. Fonseca, Phys. Rev. Lett. **83**, 4021 (1999).

[19] E. O. Alt, P. Grassberger, W. Sandhas, Nucl. Phys. **B2**, 167 (1967).

[20] W. Glöckle, "The Quantum-Mechanical Few-Body Problem" (Springer Verlag, Berlin, Heidelberg, 1983).

[21] A. W. Thomas, "Modern Three-Hadron Physics" Topics Current Physics, (Springer Verlag, Berlin, 1977).

[22] R. B. Wiringa, R.A. Smith, and T.L. Ainsworth, Phys. Rev. **D17**, 768 (1978).

[23] A. Nogga, D. Hüber, H. Kamada, W. Glöckle, Phys. Lett. B **409**, 19 (1997).

[24] I. J. Thompson, B. V. Danilin, V. D. Efros, J. S. Vaagen, J. M. Bang, M. V. Zhukov, Phys. Rev. **C 61**, 024318 (2000).

[25] Prog. Theor. Phys. **93**, 99 (1995).

[26] For example, J. Carlson, R. Schiavilla, Rev. of Mod. Phys. **70**, 743 (1998); K. Varga, Y. Suzuki, R. G. Lovas, Nucl. Phys. **A 571**, 447 (1994).

[27] A. S. Jensen, E. Garrido, D. V. Fedorov, Few-Body Systems **22**, 193 (1997).

[28] P. Navrátil, B. R. Barrett, and W. Glöckle, Phys. Rev. **C59**, 611 (1999).

[29] S. Oryu, H. Kamada, H. Sekine, H. Yamashita, M. Nakazawa, Few-Body Systems **27**, 1 (2000).

FIGURES

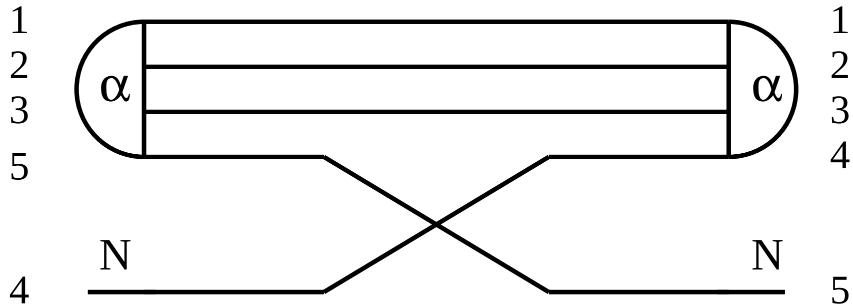

FIG. 1. The diagram of the norm kernel.

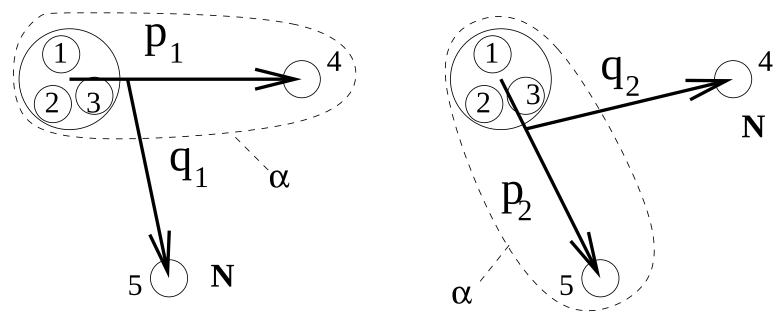

FIG. 2. Jacobi momenta

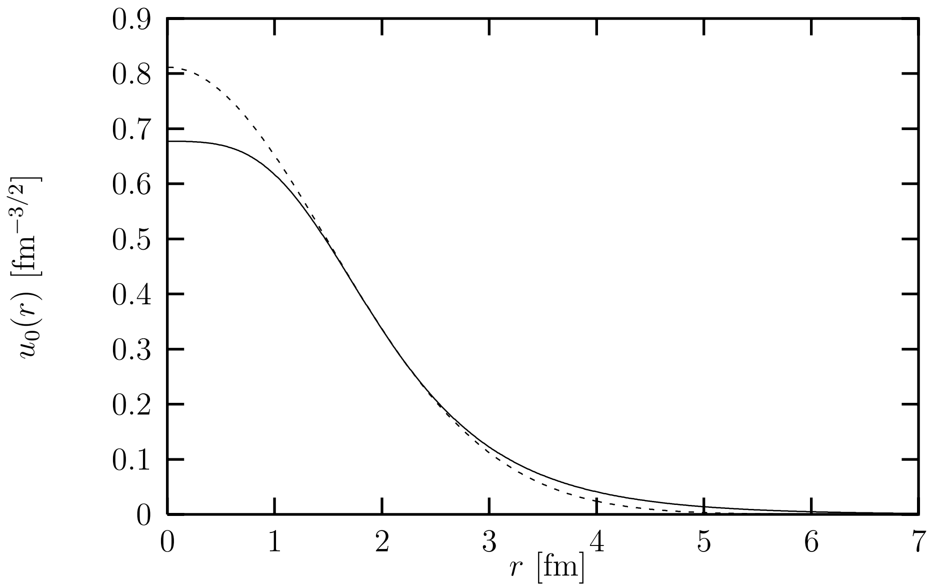

FIG. 3. The ghost state in coordinate space. The solid (dashed) line is $u_0^Y(r)$ ($u_0^G(r)$).

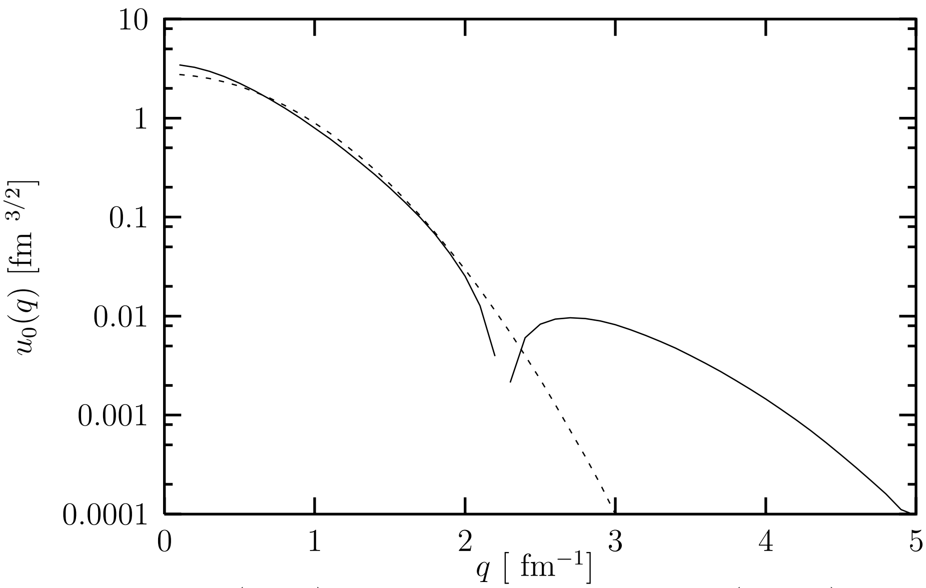

FIG. 4. The ghost state ($n = 0$) in momentum space. The solid (dashed) line is $u_0^Y(q)$ ($u_0^G(q)$). The disconnection of the solid line causes from change of sign.

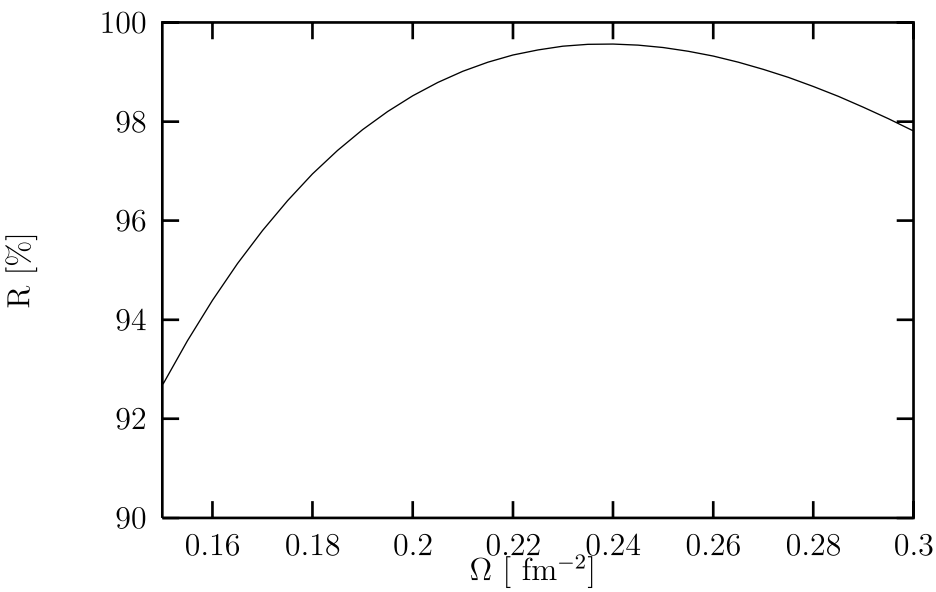

FIG. 5. The percentage R of the ghost state as a function of oscillator parameter Ω.

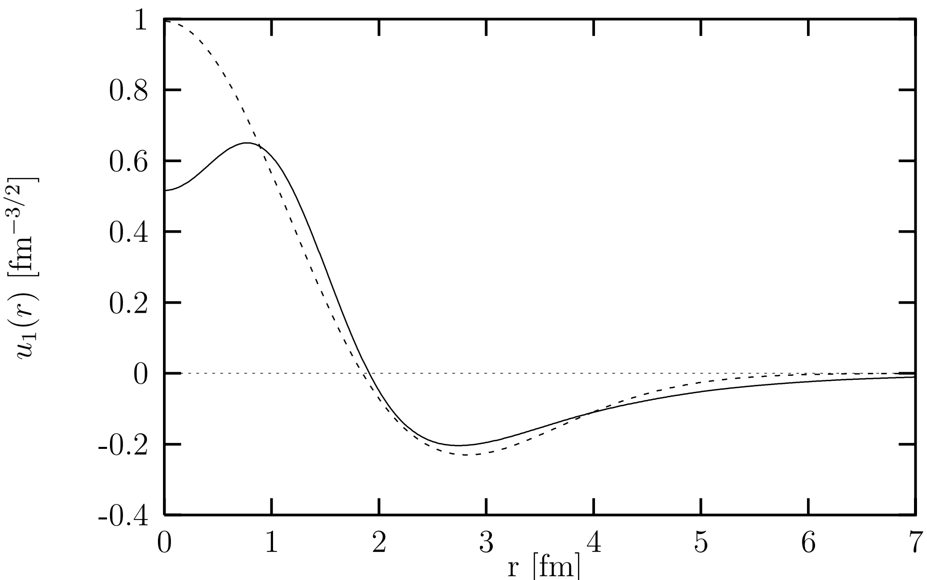

FIG. 6. The eigen state (n=1) of the Eq. (2) in coordinate space. The solid (dashed) line is $u_1^Y(r)$ ($u_1^G(r)$).

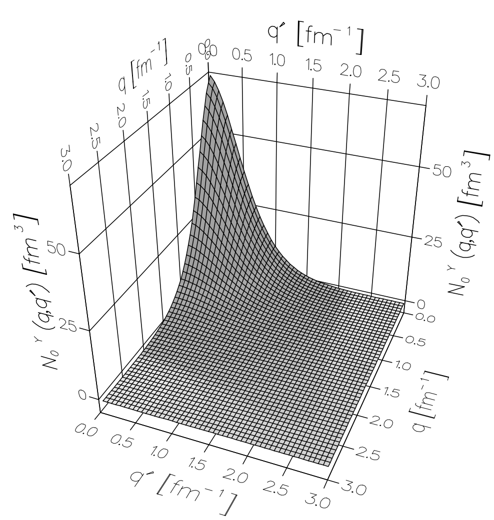

FIG. 7. Norm kernel $\mathcal{N}_0^Y$.

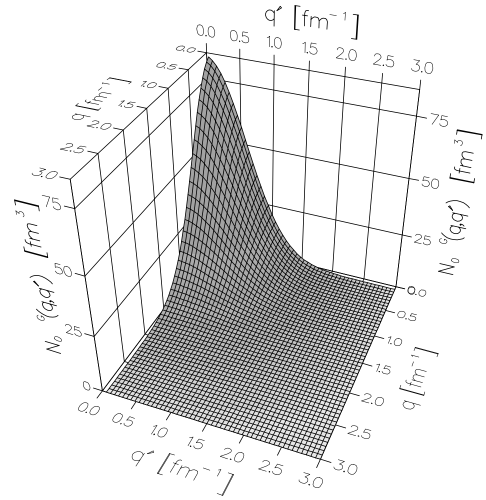

FIG. 8. Norm kernel $\mathcal{N}_0^G$.

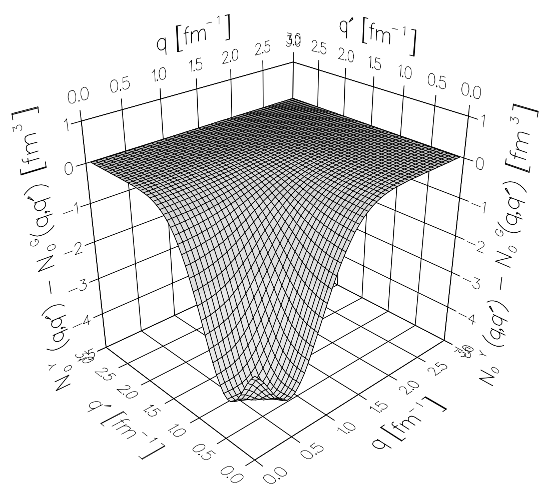

FIG. 9. The difference of the norm kernels($\mathcal{N}_0^Y - \mathcal{N}_0^G$).

TABLES

TABLE I. Eigen values the norm kernel.

n	γ_n	(γ_n^{-1}) of u_n^Y	γ_n	(γ_n^{-1}) of u_n^G [15]
0	0.937	(1.068)	1.00000	(1)
1	0.0663	(15.09)	0.06666	(16)
2	0.00753	(132.)	0.00391	(256)