Reductions of the Volterra and Toda chains

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Abstract

The Volterra and Toda chains equations are considered. A class of special reductions for these equations are derived.

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The Volterra and Toda chains are known to have numerous applications. In particular, they appear in connection with continuum two-boson KP hierarchies [1] and their discrete symmetries (or auto-Bäcklund transformations). It is worthing also to mention the works [4], [5] where integrable chains serve as discrete symmetries of differential evolution equations.

In this note we establish finite-dimensional systems arising in process of special reductions of Volterra and Toda lattices. Translations on the lattices, as in the case of two-boson KP hierarchies, yield groups of discrete symmetries for these finite-dimensional systems.

Consider the Volterra lattice in the form

$$\dot{r}(i) = r(i)\left\{r(i-1) - r(i+1)\right\}, \quad i \in \mathbf{Z}. \tag{1}$$

For any fixed $n \in \mathbf{N}$, impose a constraint in the form of the following algebraic equations:

$$r(i) + ... + r(i+n-1) = r(i-1)r(i)...r(i+n), \quad i \in \mathbf{Z}. \tag{2}$$

We are going to show that infinite collection of equations (1) restricted by relations (2) are equivalent to finite-dimensional systems complemented by auto-Bäcklund transformations (ABT).

Fix any $i = i_0 \in \mathbf{Z}$ and define finite collection of functions $\mathbf{y} = (y_1(t), ..., y_{n+1}(t))$ by

$$y_1 = r(i_0),\ y_2 = r(i_0+1), ..., y_{n+1} = r(i_0+n).$$

Taking into account (2), we easy extract finite-dmensional systems

$$\begin{aligned}
\dot{y}_1 &= \frac{y_1 + ... + y_n}{y_2 \cdots y_{n+1}} - y_1 y_2, \\
\dot{y}_k &= y_k(y_{k-1} - y_{k+1}), \quad k = 2, ..., n, \\
\dot{y}_{n+1} &= y_n y_{n+1} - \frac{y_2 + ... + y_{n+1}}{y_1 \cdots y_n}.
\end{aligned} \tag{3}$$

With any solution of Volterra chain constrained by (2), for different i_0, we should have respectively different solutions of the system (3). If the constraint (2) is consistent with Volterra chain (1) then the shift $i_0 \to i_0 + 1$ must yield invertible ABT for the system (3). Define "new" functions

$$\tilde{y}_1 = r(i_0+1),\ \tilde{y}_2 = r(i_0+2), ..., \tilde{y}_{n+1} = r(i_0+n+1).$$

From (2), we easy obtain

$$\tilde{y}_1 = y_2, ..., \tilde{y}_n = y_{n+1}, \;\; \tilde{y}_{n+1} = \frac{y_2 + ... + y_{n+1}}{y_1 y_2 ... y_{n+1}}. \tag{4}$$

By straightforward but tedious calculations, one can verify that if $(y_1(t), ..., y_{n+1}(t))$ is some solution of (3) then, by virtue (4), $(\tilde{y}_1(t), ..., \tilde{y}_{n+1}(t))$ also will be solution of (3). So we can conclude that the mapping (4), for the system (3), serve as discrete symmetry. It is easy to deduce inverse to (4). We have

$$y_1 = \frac{\tilde{y}_1 + ... + \tilde{y}_n}{\tilde{y}_1 \tilde{y}_2 ... \tilde{y}_{n+1}}, \; y_2 = \tilde{y}_1, ..., y_{n+1} = \tilde{y}_n.$$

One defines

$$a_0(i) = -r(2i-1) - r(2i), \;\; a_1(i) = r(2i-2)r(2i-1).$$

These relations are known to realize the map from the solution space of (1) to that of Toda chain equations [2], [3]

$$\dot{a}_0(i) = a_1(i+1) - a_1(i), \;\; \dot{a}_0(i) = a_1(i)\left\{a_0(i) - a_0(i-1)\right\}, \; i \in \mathbf{Z}. \tag{5}$$

Now observe that, for even n, replacing in (2) $i \to 2i-1$, these equations can be rewritten in the form

$$-a_0(i) - ... - a_0(i+m-1) = a_1(i)a_1(i+1)...a_1(i+m), \; i \in \mathbf{Z} \tag{6}$$

where $m = n/2$.

It is naturally to suppose that constraint (6) also will lead to finite-dimensional systems. To write Toda lattice (5) in more familiar form, one introduces the functions $\{u_i(t), \; i \in \mathbf{Z}\}$ by

$$a_0(i) = -\dot{u}_i, \;\; a_1(i) = e^{u_{i-1} - u_i}.$$

After that we have

$$\ddot{u}_i = e^{u_{i-1} - u_i} - e^{u_i - u_{i+1}}, \;\; i \in \mathbf{Z}. \tag{7}$$

The constraint (6) becomes

$$\dot{u}_i + ... + \dot{u}_{i+m-1} = e^{u_{i-1} - u_{i+m}}, \;\; i \in \mathbf{Z}. \tag{8}$$

Observe that, in terms of τ-function defined through

$$e^{u_i} = \frac{\tau_i}{\tau_{i+1}},$$

the equation (8) takes simple form. Namely, we have bilinear equation

$$D_t\tau_i\cdot\tau_{i+m}=\tau_{i-1}\tau_{i+m+1}.$$

Recall that Hirota's D-operator is defined by

$$D_t^l f\cdot g=(\partial_{t_1}-\partial_{t_2})^l f(t_1)g(t_2)|_{t_1=t_2=t}.$$

By analogy with Volterra lattice situation, introduce finite collection of functions $\mathbf{q}=(q_1(t),...,q_{m+1}(t))$ identifying

$$q_1=u_{i_0},\ q_2=u_{i_0+1},...,q_{m+1}=u_{i_0+m}.$$

Taking into account the relation (8), we easy obtain finite-dimensional systems

$$\begin{aligned}
\ddot{q}_1&=(\dot{q}_1+...+\dot{q}_m)e^{q_{m+1}-q_1}-e^{q_1-q_2},\\
\ddot{q}_k&=e^{q_{k-1}-q_k}-e^{q_k-q_{k+1}},\quad k=2,...,m,\\
\ddot{q}_{m+1}&=e^{q_m-q_{m+1}}-(\dot{q}_2+...+\dot{q}_{m+1})e^{q_{m+1}-q_1}.
\end{aligned}\tag{9}$$

Define

$$\tilde{q}_1=u_{i_0+1},\ \tilde{q}_2=u_{i_0+2},...,\tilde{q}_{m+1}=u_{i_0+m+1}.$$

From (8), we easy deduce the relations

$$\tilde{q}_1=q_2,...,\tilde{q}_m=q_{m+1},\ \tilde{q}_{m+1}=q_1-\ln\left[\dot{q}_2+...+\dot{q}_{m+1}\right].\tag{10}$$

By straightforward calculations, one can check that transformation (10), for any $m\in\mathbf{N}$, realize ABT for corresponding system. It is easy to write down inverse to (10). We have

$$q_1=\tilde{q}_{m+1}+\ln\left[\dot{\tilde{q}}_1+...+\dot{\tilde{q}}_m\right],\quad q_2=\tilde{q}_1,...,q_{m+1}=\tilde{q}_m.$$

We summarize above in the following statement.

Proposition 1. The equations of Volterra and Toda lattices restricted, respectively, by relations (2) and (8) are equivalent to the system (3) and (9) complemented by ABT (4) and (10).

It is easy to get the relations connecting the solution spaces of the systems (3) and (9), in the case when $n=2m$. We have

$$y_{2k-1}+y_{2k}=\dot{q}_k,\quad k=1,...,m,$$

$$y_{2m+1}+\frac{y_2+...+y_{2m+1}}{y_1y_2...y_{2m+1}}=\dot q_{m+1},$$
$$y_{2k}y_{2k+1}=e^{q_k-q_{k+1}},\ \ k=1,...,m.$$

Observe that any system (9) can be cast into Lagrangian formalism.

Proposition 2. The systems (9) admit Lagrangian formulation with appropriate Lagrangian

$$\mathcal{L}=\sum_{i<j}\dot q_i\dot q_j-\sum_{i=1}^{m}e^{q_i-q_{i+1}}-(\frac{1}{2}\dot q_1+\sum_{j=2}^{m}\dot q_j+\frac{1}{2}\dot q_{m+1})e^{q_{m+1}-q_1}.$$

Proof. Calculate Euler derivatives of $\mathcal{L}$

$$E_{q_i}\mathcal{L}=\frac{\partial\mathcal{L}}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot q_i}\right).$$

We have

$$E_{q_1}\mathcal{L}=-\sum_{j=2}^{m+1}\ddot q_j-e^{q_1-q_2}+\left(\sum_{j=2}^{m+1}\dot q_j\right)e^{q_{m+1}-q_1},$$
$$E_{q_k}\mathcal{L}=-\sum_{j\neq k}\ddot q_j-e^{q_k-q_{k+1}}+e^{q_{k-1}-q_k}+$$
$$+(\dot q_{m+1}-\dot q_1)e^{q_{m+1}-q_1},\ \ k=2,...,m,$$
$$E_{q_{m+1}}\mathcal{L}=-\sum_{j=1}^{m}\ddot q_j+e^{q_m-q_{m+1}}-\left(\sum_{j=1}^{m}\dot q_j\right)e^{q_{m+1}-q_1},$$

Equating $E_{q_i}\mathcal{L}$ to zero results in the system of the form

$$\sum_{j=1}^{m+1}A_{ij}\ddot q_j=f_i(q_k,\dot q_k)\tag{11}$$

where the matrix $A=(A_{ij})$ consists of zeros in main diagonal and units elsewhere. It is easy to see that A is nondegenerate. Moreover it can be checked that A^{-1} consists of $(2-m-1)/m$'s in main diagonal and $1/m$'s elsewhere. One can verify that substituting right-hand sides of the system (9) in (11) gives identity. So we can conclude that (9) and (11) are equivalent. □

It is simple exercise, using Legendre transformation, to cast any system (9) in Hamiltonian setting. In the simplest case $m=1$, (9) becomes

$$\ddot q_1=\dot q_1e^{q_2-q_1}-e^{q_1-q_2},\ \ \ddot q_2=e^{q_1-q_2}-\dot q_2e^{q_2-q_1}.\tag{12}$$

Its ABT reads

$$\tilde q_1=q_2,\ \ \tilde q_2=q_1-\ln(\dot q_2).\tag{13}$$

Note two first integrals for the system (12). They are

$$P = -\dot{q}_1 - \dot{q}_2 - e^{q_2 - q_1}, \;\; E = -\dot{q}_1\dot{q}_2 + e^{q_1 - q_2}.$$

It can be checked that P and E are invariant with respect to ABT (13).

Lagrangian in this case looks as

$$\mathcal{L} = -\dot{q}_1\dot{q}_2 - e^{q_1 - q_2} - \frac{1}{2}(\dot{q}_1 + \dot{q}_2)e^{q_2 - q_1}.$$

We easy obtain

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = -\dot{q}_2 - \frac{1}{2}e^{q_2 - q_1}, \;\; p_2 = \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = -\dot{q}_1 - \frac{1}{2}e^{q_2 - q_1},$$

$$\mathcal{H} = \dot{q}_1\frac{\partial \mathcal{L}}{\partial \dot{q}_1} + \dot{q}_2\frac{\partial \mathcal{L}}{\partial \dot{q}_2} - \mathcal{L} = -(p_1 + \frac{1}{2}e^{q_2 - q_1})(p_2 + \frac{1}{2}e^{q_2 - q_1}) + e^{q_1 - q_2}$$

It is easy to check that the functions $\mathcal{H}$ and $P = p_1 + p_2$ are in involution with respect to standard Poisson bracket. So we can conclude that the system (12) is integrable due to Liouville's theorem. It is naturally to suppose that all systems (9) are completely integrable in the sense of Liouville's theorem [6].

References

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