An Algebraic Duality Theory for Multiplicative Unitaries

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Abstract

Multiplicative unitaries are described in terms of a pair of commuting shifts of relative depth two. They can be generated from ambidextrous Hilbert spaces in a tensor C^*–category. The algebraic analogue of the Takesaki-Tatsuuma Duality Theorem characterizes abstractly C^*–algebras acted on by unital endomorphisms that are intrinsically related to the regular representation of a multiplicative unitary. The relevant C^*–algebras turn out to be simple and indeed separable if the corresponding multiplicative unitaries act on a separable Hilbert space. A categorical analogue provides internal characterizations of minimal representation categories of a multiplicative unitary. Endomorphisms of the Cuntz algebra related algebraically to the grading are discussed as is the notion of braided symmetry in a tensor C^*-category.

1   Introduction

An interesting open problem in the duality theory of tensor C^*–categories is to decide which ones admit a (tensor–preserving) embedding into the tensor C^*-category of Hilbert spaces. A first positive result in this direction is the duality theorem of [7] asserting that symmetric tensor C^*–categories with conjugates and irreducible tensor unit admit such an embedding. On the other hand, the theory of dimension introduced in [13] allows one to see that certain tensor C^*–categories with conjugates and irreducible tensor unit cannot be embedded. For such tensor C^*–categories admit an intrinsic dimension function defined on objects. Under certain circumstances, the embedding functor must preserve dimensions. This is well known to be so in the rational case, i.e. when the set of equivalence classes of irreducibles is finite, since it is a simple consequence of the Perron–Frobenius Theorem. However, it is also true in the amenable case[13]. We are in the amenable case whenever the category admits a unitary braiding, see Theorem 5.31 of [13]. This means that the tensor C^*–categories with conjugates and a braiding appearing in low dimensional quantum field theory cannot be embedded whenever the dimensions are non–integral. On the other hand, the finite dimensional unitary representation theory of compact quantum groups provides us with examples of tensor C^*–categories with conjugates and irreducible tensor units with non–integral dimensions which are embeddable by construction. The duality theorem of Woronowicz [18] generalizing Tannaka–Krein to compact quantum groups provides a way of recognizing such categories if the embedding is given.

We will not treat the embedding problem here in full generality; we shall instead present a positive solution for abstract tensor C^*–categories with specific additional structure. Such structure is present in a class of model tensor C^*–categories namely the minimal tensor C^*–categories generated by the regular representation of locally compact groups or of multiplicative unitaries in the sense of Baaj and Skandalis [1]. We prove that abstract tensor C^*–categories with this additional structure are isomorphic to a model tensor C^*–category and are hence embeddable.

Our result, Thm. 6.13, is thus a duality theorem for multiplicative unitaries and is hence applicable to the case of locally compact quantum groups [11]. Multiplicative unitaries have already played a role in Tatsuuma's duality theorem for locally compact groups [17] where the group elements are identified in the regular representation using the multiplicative unitary and in Takesaki's Hopf von Neumann algebra version of duality [16]. Multiplicative unitaries express

the fundamental property of the regular representation V, namely that $V \times V$ is equivalent to a multiple of V. This can also be expressed through the existence of a remarkable Hilbert space H of intertwiners from V to $V \times V$ (cf also [6]). Our minimal model is the smallest tensor C^*–category containing the regular representation as an object and H as a subspace of arrows.

A closely related result, Thm. 6.11, characterizes a class of C^*–algebras $\mathcal{A}$ acted on by an endomorphism ρ encoding the regular representation. Intertwiners between powers of this endomorphism encode intertwiners between tensor powers of the regular representation. The minimal model here is obtained as follows: take the regular representation V considered as an object in the tensor C^*–category of representations of the multiplicative unitary and associate to it as in [7] the C^*–algebra $\mathcal{O}_V$ and its endomorphism ρ_V. Then the minimal model is the smallest ρ_V–stable C^*–subalgebra containing H acted on by the restriction of ρ_V. The C^*–algebra obtained in this way is simple and is separable if and only if the given multiplicative unitary acts on a separable Hilbert space. Further variants of these duality results are Theorems 6.5, 6.6, 6.7 and 6.12.

In connection with the above results, attention should be drawn to Longo's characterization of actions of finite dimensional Hopf algebras [12] which has a similar algebraic and categorical flavour. However, his axiomatic structure involving Q–systems is quite distinct from those used here. Despite this, the Hilbert space H puts in an appearance here too and a multiplicative unitary appears in his proof.

The principal results in the remaining sections may be summarized as follows. Section 2 gathers together some elementary results on categories of Hilbert spaces. The discussion centres round the concept of *shift*, a category of Hilbert spaces with objects labelled by the integers, 0 being irreducible, and equipped with a normal *–functor adding one on objects. Such a structure is isomorphic to the category of tensor powers of a given Hilbert space K with the functor being tensoring on the right by 1_K.

In Section 3, it is shown how two commuting shifts of relative depth two are equivalent to giving a multiplicative unitary for the tensor product defined by one of the commuting shifts. Duality for multiplicative unitaries reflects the symmetry between the commuting shifts. The representation category for a multiplicative unitary finds a natural expression within this framework.

In Section 4, it is shown how two commuting shifts of relative depth two and hence multiplicative unitaries arise naturally in terms of ambidextrous Hilbert spaces in a tensor W^*–category.

Section 5 is devoted to studying Hilbert spaces in and endomorphism of the Cuntz algebra which are algebraic with respect to the natural grading. We show, for example, how the problem of determining intertwiners between algebraic endomorphisms can be reduced to a purely algebraic problem. These results are used in the final section to arrive at the duality results already announced earlier in the introduction. The paper concludes with an appendix on braided symmetry.

In this paper we prefer to work with strictly associative tensor products and a simple way of achieving this is to use as the underlying Hilbert spaces the Hilbert spaces in some fixed von Neumann algebra since these are objects in a strict tensor W^*–category. We will be concerned here with the representation categories of multiplicative unitaries and recall the basic definitions from [1]. If K is such a Hilbert space then a unitary V on the tensor square K^2 is said to be multiplicative if

$$V_{12}V_{13}V_{23} = V_{23}V_{12},$$

where we use the usual convention regarding indices and tensor products. A representation of V on a Hilbert space H is a unitary $W \in (HK, HK)$ such that

$$W_{12}W_{13}V_{23} = V_{23}W_{12}, \quad \text{on} \quad HK^2.$$

If W and W' are representations of V on H and H' respectively, we say that $T \in (H, H')$ intertwines W and W' and write $T \in (W, W')$ if $T \times 1_K W = W'T \times 1_K$. We define the tensor product of W and W' to be the representation $W \times W'$ on HH' given by $W \times W' := W_{13}W'_{23}$. The usual tensor product of intertwiners is again an intertwiner and in this way we get a strict tensor W^*–category $\mathcal{R}(V)$ of representations of V. In fact this assertion does not depend on V being multiplicative. When it is then V itself is a representation of V called the regular representation.

A corepresentation of V on H is a unitary $W \in (KH, KH)$ such that

$$V_{12}W_{13}W_{23} = W_{23}V_{12} \quad \text{on} \quad K^2H.$$

If W and W' are corepresentations on H and H' respectively, we say that $T \in (H, H')$ intertwines W and W' and write $T \in (W, W')$ if $1_K \times TW = W'1_K \times T$. The tensor product $W \times W'$ of corepresentations is defined by $W \times W' := W_{12}W'_{13}$. Just as in the case of representations we get a strict tensor W^*–category now denoted by $\mathcal{C}(V)$. If $\vartheta = \vartheta_{K,K}$ denotes the flip on K^2 then $\vartheta V^*\vartheta$ is again a multiplicative unitary and the mapping $W \mapsto \tilde{W} := \vartheta_{H,K}W^*\vartheta_{K,H}$ defines a 1–1 correspondence between representations of V and

corepresentations of $\vartheta V^* \vartheta$. However, it does not define an isomorphism of tensor W^*–categories since $W \times W' \mapsto \tilde{W}'_{13} \tilde{W}_{12}$ and so leads to an alternative definition of the tensor product of corepresentations. In fact the two expressions for the tensor product will be equal if and only if $\vartheta_{W,W'} \in (W \times W', W' \times W)$, and this corresponds to the case of a group cf. [19], Prop. 2.5. Thus exchanging the definitions of tensor product corresponds to exchanging representations of a multiplicative unitary and corepresentations of the dual multiplicative unitary.

2   Preliminaries on Categories of Hilbert Spaces

We begin our considerations with a simple but useful lemma on natural transformations in the context of W^*–categories. Elementary results and definitions on W^*–categories can be found in [9] and Lemma 2.1 below is just a slight generalization of Corollary 7.4 in [9]. The notion of direct sum will be used in the Hilbert space sense rather than in the purely algebraic sense. Thus A is a direct sum of objects B_i, $i \in I$, if there are isometries $W_i \in (B_i, A)$ such that $\sum_i W_i \circ W_i^* = 1_A$, where the convergence is in, say, the s–topology. In particular if the $B_i = B$ for all i, the condition amounts to saying that there is a Hilbert space of support 1_A in (B, A). The isometries W_i form an orthonormal basis of such a Hilbert space. An object B has *central support one* or is a *generator* if given any object A there are partial isometries $W_{i,A} \in (B, A)$ such that $\sum_i W_{i,A} \circ W_{i,A}^* = 1_A$, see Proposition 7.3 of [9].

2.1 Lemma *Let $\mathcal{T}$ and $\mathcal{K}$ be W^*–categories and E and F be normal *–functors from $\mathcal{K}$ to $\mathcal{T}$. Suppose $\mathcal{K}$ has a object B of central support one. Then a natural transformation t from E to F has form*

$$t_A = \sum_i F(W_{i,A}) \circ T \circ E(W_{i,A})^*,$$

where the sum is taken over partial isometries $W_{i,A}$ of (B, A) with $\sum_i W_{i,A} \circ W_{i,A}^* = 1_A$ and T is an arbitrary element of $(E(B), F(B))$ satisfying the intertwining relation $TE(S) = F(S)T$, $S \in (B, B)$. t is automatically bounded and $t_B = T$.

Proof. The sum defining t_A converges in the s–topology, say, and $\|t_A\| \leq \|T\|$. (Consider a finite sum and use the C^*–property of the norm.) Noting that $t_B = T$, we conclude that $\|t\| = \|T\|$. Pick $Y \in (A, C)$, then

$$t_C \circ E(Y) = \sum_{i,j} F(W_{i,C}) \circ T \circ E(W_{i,C}^* \circ Y \circ W_{j,A} \circ W_{j,A}^*).$$

But $W_{i,B}^* \circ Y \circ W_{j,A} \in (B,B)$ so using the intertwining property of T, we deduce that $t_C E(Y) = F(Y) t_A$ and we have a bounded natural transformation. Conversely, suppose $t \in (E,F)$ and $W \in (B,A)$ then $t_A \circ E(W) = F(W) \circ t_B$ implying that t is obtained by the above construction with $T = t_B$.

Note that if $E(B) = F(B)$ and B is irreducible, then we even have a canonical natural transformation $t \in (E,F)$, satisfying $t_B = 1_{E(B)}$. We use the notation (F,T,E) for the natural transformation constructed as above from $T \in (E(B), F(B))$ satisfying the intertwining relation. The usual operations on natural transformations have simple expressions in this notation. Thus composition of natural transformations corresponds to composing these symbols in the obvious way:

$$(F,T,E) \circ (E,S,D) = (F, T \circ S, D).$$

In fact, the map $t \mapsto t_B$ is a full and faithful *–functor from the category of normal *–functors from $\mathcal{K}$ to $\mathcal{T}$ to the category of normal representations of (B,B). If G is a normal *–functor from $\mathcal{T}$, then acting on (F,T,E) on the left with G gives $(GF, G(T), GE)$. If D is a normal *–endofunctor of $\mathcal{K}$ then acting on (F,T,E) on the right by D gives (FD, S, ED), where $S = (F,T,E)_{D(B)}$.

2.2 Lemma *With the above notation, suppose that $GE = ED$ and $GF = FD$. Then if $G(T) = S$, $G \times (F,T,E) = (F,T,E) \times D$. If $G \times (F,T,E)$ is invertible, then*

$$(F,T,E) \times D = G \times (F,T,E) \circ (ED, G(T)^{-1} \circ S, ED).$$

Proof. As the natural transformations are uniquely determined by their values in B, the result follows by evaluating in B.

By a category of Hilbert spaces, we mean a W^*–category whose objects are Hilbert spaces and whose mappings are all bounded linear mappings between these Hilbert spaces. Any object in such a category has central support one. Any W^*–category with an irreducible object of central support one is a category of Hilbert spaces in a natural way.

We now consider a category $\mathcal{K}$ of Hilbert spaces whose objects are labelled by $\mathbb{N}_0$ with $(0,0) = \mathbb{C}$ and equipped with a normal *–functor F from $\mathcal{K}$ to $\mathcal{K}$ such that the action of F on objects is given by $F(n) = n+1$, $n \in \mathbb{N}_0$.

Since $(0,0) = \mathbb{C}$, $(0,n)$ is a Hilbert space of support 1 for each n and we let $\psi_{i,n}$, $i \in I_n$, be an orthonormal basis of $(0,n)$ and set

$$\hat{F}(X) := \sum_i F^n(\psi_{i,1}) \circ X \circ F^m(\psi_{i,1}^*), \quad X \in (m,n).$$

Then, with the notation of Lemma 2.1, $\hat{F}(X) = (F^n, X, F^m)_1$ and $\hat{F}$ is another normal *-functor with $\hat{F}(n) = n + 1$, $n \in \mathbb{N}_0$.

It is obvious that F and $\hat{F}$ commute. If we iterate $\hat{F}$, we find that

$$\hat{F}^k(X) = \sum_i F^n(\psi_{i,k}) \circ X \circ F^m(\psi_{i,k}^*) = (F^n, X, F^m)_k.$$

Since $\hat{F}$ has the properties assumed for F, we can form $\hat{\hat{F}}$. Calculating, we find that

$$\hat{\hat{F}}(X) = \sum_{i,j,k} F(\psi_{j,m}) \circ \psi_{i,1} \circ \psi_{j,n}^* \circ X \circ \psi_{k,m} \circ \psi_{i,1}^* \circ F(\psi_{k,m})^*$$

$$= \sum_{j,k} F(\psi_{j,m}) \circ F(\psi_{j,n}^* \circ X \circ \psi_{k,m}) \circ F(\psi_{k,m})^* = F(X).$$

Thus the operation $\hat{\,}$ is involutive.

From Lemma 2.1, we know that the natural transformations between powers of the functor F are automatically bounded, and furthermore, that a natural transformation $t \in (F^r, F^s)$ has the form $t_n = \hat{F}^n(T)$ with $T \in (r, s)$. Finally, we know that $T \mapsto t$ is an isomorphism of W^*–categories between $\mathcal{K}$ and the category of natural transformations between the powers of F. The latter category is, however, a tensor W^*–category, so we may use the isomorphism to equip $\mathcal{K}$ with a tensor product making it into a tensor W^*–category. We compute this tensor product. If $y \in (F^r, F^s)$ and $y' \in (F^{r'}, F^{s'})$, then $y \times y' = y \times F^{s'} \circ F^r \times y'$. In other words, $(y \times y')_n = y_{n+s'} \circ F^r(y'_n) = \hat{F}^{n+s'}(Y) \circ F^r \hat{F}^n(Y')$. Setting $n = 0$, we see that the tensor product in $\mathcal{K}$ is given by

$$Y \times Y' := \hat{F}^{s'}(Y) \circ F^r(Y').$$

Thus $\hat{F}$ is the functor of tensoring on the right by the object 1 and F the functor of tensoring on the left by the same object.

The above result can also be seen in a different way: there is a functor $\mathcal{F}$ from $\mathcal{K}$ into the category of endofunctors of $\mathcal{K}$ defined on objects by $\mathcal{F}(n) = \hat{F}^n$ and on arrows by $\mathcal{F}(X)_r := F^r(X)$, $X \in (m, n)$. It combines the operations. In fact if T is any arrow of $\mathcal{K}$,

$$\mathcal{F} \times 1_{\mathcal{F}(1)} = \mathcal{F}F(T), \quad 1_{\mathcal{F}(1)} \times \mathcal{F}(T) = \mathcal{F}\hat{F}(T).$$

In view of this result, we refer to a normal *–endofunctor on $\mathcal{K}$ with $F(n) = n+1$ as being a *shift* on $\mathcal{K}$.

As is well known, any two definitions of the tensor product on a category of Hilbert spaces are equivalent, i.e. the identity functor extends to a relaxed

tensor functor. To be specific in the case at hand, if we have two shifts F and G on $\mathcal{K}$ and we define $V_{m,n} := (\hat{G}^n, 1_n, \hat{F}^n)_m$ then

$$V_{m',n'} \circ \hat{F}^{n'}(Y) \circ F^m(Z) = \hat{G}^{n'}(Y) \circ G^m(Z) \circ V_{m,n},$$

where $Y \in (m, m')$ and $Z \in (n, n')$. Furthermore,

$$\hat{G}^p(V_{m,n}) \circ V_{m+n,p} = G^m(V_{n,p}) \circ V_{m,n+p}.$$

If we now define inductively $K := (0, 1)$ and $K^n := F(K^{n-1}) \circ K$, where the norm closed linear span is understood, then the above result shows that $K^n = (0, n)$.

We now make some remarks on the automorphisms of $\mathcal{K}$. Such an automorphism will be a normal *–functor Γ and we suppose, as part of the definition that Γ leaves the objects fixed. Since an automorphism of a Hilbert space is given by a unitary operator, any automorphism of $\mathcal{K}$ will be inner, meaning that there is a unitary natural transformation u from the identity functor to Γ. Lemma 2.1 tells us that u is determined by $u_0 \in \mathbb{C}$. Since we are interested in Γ rather than u, we fix the free phase by requiring that $u_0 = 1$. An inner automorphism is then determined by a sequence of unitaries $u_n \in (n, n)$ with $u_0 = 1$. We now look for the inner automorphisms which commute with F. They must therefore commute with $\hat{F}$ and preserve the tensor product structure determined by F. Applying Lemma 2.2, we derive the condition $u_{n+1} = F(u_n)\hat{F}^n(u_1)$. Solving the recurrence relation gives $u_{n+1} = F^n(u_1)F^{n-1}\hat{F}(u_1) \cdots \hat{F}^n(u_1)$. In terms of the tensor product structure determined by F, this means that the u_n are just tensor powers of u_1. If u_1 is a phase, we get automorphisms which commute with every shift and are the analogues of the grading automorphisms of the Cuntz algebra. Of course, if Γ does not commute with F it maps the tensor product structure determined by F onto that determined by $\Gamma F \Gamma^{-1}$.

Our next goal is to characterize all normal *–functors on $\mathcal{K}$ that commute with the given functor F. Note first that the action of such a functor G on objects must be of the form $G(n) = r + n = F^r(n)$, $n \in \mathbb{N}_0$, for some $r \in \mathbb{N}_0$. Thus $(G, 1_r, F^r)$ will be a natural unitary transformation from F^r to G and since F commutes with these two functors, we may apply the second identity of Lemma 2.2 to deduce that

$$R_{n+1} = F(R_n) \circ \hat{F}^n(R_1),$$

where we have written R for $(F^r, 1_r, G)$.

Conversely, given a unitary $R_1 \in (r+1, r+1)$, take R_0 to be the unit on r and define R_n, $n > 1$, inductively using the above formula. Finally, define

$$G(X) = R_n \circ F^r(X) \circ R_m^*.$$

Then G is obviously a normal *–functor with $G(n) = n + r$, $n \in \mathbb{N}_0$. Furthermore, if $X \in (m, n)$,

$$GF(X) = R_{n+1} \circ F^{r+1}(X) \circ R_{m+1}^* = F(R_n) \circ \hat{F}^n(R_1) \circ F^{r+1}(X) \circ R_{m+1}^*$$

$$= F(R_n) \circ F^{r+1}(X) \circ F(R_m)^* = FG(X).$$

Thus F and G commute and we have proved the following result.

2.3 Proposition *Normal *–functors G commuting with F on $\mathcal{K}$ are of the form*

$$G(\psi) = R \circ F^r(\psi), \quad \psi \in (0, 1), \ R \in (r+1, r+1).$$

r is called the rank of G.

Note that since we have already computed all natural transformations between the tensor powers of F, we have implicitly computed all natural transformations betwen two functors G and G' commuting with F. Note, too, that G may be obtained from F^r by acting on the left with an inner automorphism.

It is easy to compute what composition of functors means for the corresponding unitary operators. If we use the notation G_R to denote the functor corresponding to the unitary $R \in (r+1, r+1)$, then $G_R G_S = G_T$, where $T = G_R(S) \circ F^s(R)$. Note that Proposition 2.2 is closely related to Cuntz's result[4] characterizing the endomorphisms of the Cuntz algebra in terms of unitary operators.

We now consider another category $\mathcal{H}$ of Hilbert spaces whose objects will be denoted H_n with $n \in \mathbb{N}_0$. Suppose we have a normal *–functor H from $\mathcal{K}$ to $\mathcal{H}$ whose action on objects takes n to H_n. Then we may define a normal *-endofunctor A on $\mathcal{H}$ by setting

$$A(X) := \sum_i H\hat{F}^n(\psi_{i,1}) \circ X \circ H\hat{F}^m(\psi_{i,1})^* = (H\hat{F}^n, X, H\hat{F}^m)_1, \quad X \in (H_m, H_n),$$

where, as before, the sum runs over an orthonormal basis. We see at once that, with this definition, $AH = HF$. H is to be thought of as tensoring on the left by H_0 and A as tensoring on the right by the object 1 of $\mathcal{K}$. An easy calculation now shows that

$$A^s(X) \circ H\hat{F}^m(Y) = H\hat{F}^n(Y) \circ A^r(X), \quad X \in (H_m, H_n), \ Y \in (r, s).$$

These equations show that we may define a functor $\mathcal{H}$ from $\mathcal{K}$ to $\text{End}\mathcal{H}$ by setting $\mathcal{H}(r) = A^r$ and $\mathcal{H}(Y)_n = H\hat{F}^n(Y)$ for $Y \in (r,s)$.

2.4 Lemma *There is a $1-1$ correspondence between normal *–functors H from $\mathcal{K}$ to $\mathcal{H}$ with $H(n) = H_n$ and normal *–functors $\mathcal{H}$ from $\mathcal{K}$ to $\text{End}\mathcal{H}$ such that $\mathcal{H}(Y) \times \hat{F} = \mathcal{H}\hat{F}(Y)$ given by $H(Y) := \mathcal{H}(Y)_0$.*

Proof. We have seen above how to construct the functor $\mathcal{H}$ from H.

$$(\mathcal{H}(Y) \times F)_r = \mathcal{H}(Y)_{r+1} = HF^{n+1}(Y) = (\mathcal{H}F(Y))_r.$$

Conversely, given $\mathcal{H}$, we obtain H by evaluating in 0: $H(Y) := \mathcal{H}(Y)_0$. Now $HF^n(Y) = \mathcal{H}(F^n(Y))_0 = \mathcal{H}(Y)_n$. Thus $\mathcal{H}$ is the functor associated with H.

We have seen how $\mathcal{K}$ with the functor F is isomorphic to the category of tensor powers of $(0,1)$ such that F becomes the functor of tensoring on the left by the object 1. There is a similar result for $\mathcal{H}$ with the functor H.

2.5 Proposition *Let $\mathcal{H}$ be a category of Hilbert spaces and $H : \mathcal{K} \to \mathcal{H}$ a normal *–functor with $H(r) = H_r$. Then there is a unique isomorphism Φ of $(H_0, H_0) \otimes \mathcal{K}$ into $\mathcal{H}$ such that $\Phi(T \otimes Y) = A^s(T)H(Y)$, $Y \in (r,s)$.*

Proof. Noting that $A^s(T)H(Y) = H(Y)A^r(T)$, we see that Φ extends uniquely to the algebraic tensor product. Now given $X \in (H_r, H_s)$, write

$$X = \sum_{i,j} H(\psi_{i,s}) \circ X_{ij} \circ H(\psi_{j,r})^*,$$

where $X_{ij} := H(\psi_{i,s})^* \circ X \circ H(\psi_{j,r}) \in (H_0, H_0)$. Hence $X = \sum_{i,j} A^s(X_{ij}) \circ H(\psi_{i,s} \circ \psi_{j,r}^*)$ showing that Φ extends by continuity to a full functor. But any normal *–functor from $\mathcal{K}$ is faithful hence Φ is an isomorphism.

In view of the above results, given a shift F on $\mathcal{K}$, we may say that a normal *–functor H from $\mathcal{K}$ to $\mathcal{H}$ with $H(r) = H_r$ determines an *action* of $(\mathcal{K}, F)$ on $\mathcal{H}$ via Lemma 2.4 making $\mathcal{H}$ into a (right) $(\mathcal{K}, F)$–module.

We clearly should be able to define the tensor product of two $\mathcal{K}$–modules $\mathcal{H}$ and $\mathcal{H}'$ and it should just involve replacing (H_0, H_0) and (H_0', H_0') by $(H_0, H_0) \otimes (H_0', H_0')$. Let H and H' be actions of $\mathcal{K}$ on $\mathcal{H}$ and $\mathcal{H}'$, respectively. Then the tensor product of the two actions is an action $H \otimes H'$ on a category of Hilbert spaces $\mathcal{H} \otimes \mathcal{H}'$ together with normal *–functors $D : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}'$ and $D' : \mathcal{H}' \to \mathcal{H} \otimes \mathcal{H}'$ such that $H \otimes H' = DH = D'H'$ and $DA = A \otimes A'D$ and $D'A' = A \otimes A'D'$, where $A \otimes A'$ is the endofunctor on $\mathcal{H} \otimes \mathcal{H}'$ associated with $H \otimes H'$. In restriction to (H_0, H_0) and (H_0', H_0'), we require that D and D' should define $((H \otimes H')_0, (H \otimes H')_0)$ as a tensor product of the von Neumann algebras (H_0, H_0) and (H_0', H_0'). It should be noted that D and D' are uniquely

determined by their values on (H_0, H_0) and (H_0', H_0'), respectively. In fact, we have seen that with a suitable definition of tensor product the action H becomes $1_{(H_0,H_0)}\otimes : \mathcal{K} \to (H_0, H_0)\otimes\mathcal{K}$ with a similar expression for H'. Taking $\mathcal{H} \otimes \mathcal{H}' = (H_0, H_0) \otimes (H_0', H_0') \otimes \mathcal{K}$ and $H \otimes H'(Y) := 1_{(H_0,H_0)\otimes(H_0',H_0')} \otimes Y$ and defining D to be the normal *–functor such that $D(T \otimes Y) := T \otimes 1_{(H_0',H_0')} \otimes Y$, for $T \in (H_0, H_0)$ and $Y \in (r, s)$, with a similar expression for D' we do get a tensor product of H and H'.

In fact, every tensor product of actions is of this form, since, writing H'' for $H \otimes H'$, we may use D and D' to realize (H_0'', H_0'') as a tensor product of (H_0, H_0) and (H_0', H_0'). Then since H'' is an action, we have an isomorphism Φ'' from $H'' \otimes \mathcal{K}$ to $\mathcal{H} \otimes \mathcal{H}'$ and Φ and Φ' from $H \otimes \mathcal{K}$ and $H' \otimes \mathcal{K}$ to $\mathcal{H}$ and $\mathcal{H}'$, respectively. If we then define $d(T \otimes Y) := T \otimes 1_{(H_0',H_0')} \otimes Y$ and $d'(T' \otimes Y) := 1_{(H_0,H_0)} \otimes Y$ using the tensor product structure on (H_0'', H_0'') coming from D and D', we have $\Phi'' d = D\Phi$ and $\Phi'' d' = D'\Phi$. Thus the action H'' is ismorphic to an explicit tensor product of actions.

We now look for generalizations of our result characterizing categories $\mathcal{K}$ of Hilbert spaces with $(0, 0) = \mathbb{C}$ which admit a shift by dropping the condition that 0 should be irreducible.

2.6 Lemma *Let H and K be Hilbert spaces and F a unital normal homomorphism from (H, H) to (K, K). Then we can pick matrix units E_{ab} in (H, H) and $E_{ar,bs}$ in (K, K) such that $F(E_{bc}) = \sum_r E_{br,cr}$.*

Proof. Pick matrix units E_{ab} in (H, H) then for a fixed a, $F(E_{aa})$ is a non-zero projection and we may pick matrix units $E_{ar,as}$ for the Hilbert space $F(E_{aa})K$. Thus, in particular, $F(E_{aa}) = \sum_r E_{ar,ar}$. Now define

$$E_{br,cs} := F(E_{ba})E_{ar,as}F(E_{ac}).$$

A routine computation shows that

$$E_{br,cs}E_{dt,ev} = \delta_{cd}\delta_{st}E_{br,ev}$$

$$\sum_r E_{br,cr} = F(E_{bc}) = I.$$

Thus we have defined a set of matrix units with the required properties.

2.7 Lemma *Let H and K be Hilbert spaces and F a unital normal homomorphism from (H, H) to (K, K) then there is a Hilbert space L of support one in (H, K) such that*

$$\psi T = F(T)\psi, \quad \psi \in L, \ T \in (H, H).$$

Proof. Pick orthonormal bases e_a and e_{ar} in H and K in such a way that the corresponding matrix units are as in Lemma 2.6. Define $\psi_r \in (H, K)$ by $\psi_r e_a := e_{ar}$ and a computation shows that ψ_r is a basis of a Hilbert space L of support one in (H, K). Now $\psi_r E_{bc} e_d = \delta_{cd} e_{br}$ and $F(E_{bc}) \psi_r e_d = \sum_s E_{bs,cs} e_{dr} = \delta_{cd} e_{br}$, completing the proof.

Now let $\mathcal{H}$ be a category of Hilbert spaces with objects labelled by $\mathbb{N}_0$ and F a normal *-functor from $\mathcal{H}$ to $\mathcal{H}$ acting on objects as $F(n) = n + 1$. We have analyzed the situation when 0 is irreducible and we begin a similar analysis in the general case.

By Lemma 2.7, there is a Hilbert space L of support one in $(0, 1)$ inducing F on $(0, 0)$. We now define as before

$$\hat{F}(X) = \sum_i F^n(\psi_i) \circ X \circ F^m(\psi_i^*).$$

$\hat{F}$ will be a normal *–functor with $\hat{F}(n) = n + 1$ and commuting with F and $\hat{\hat{F}} = F$. The

$$F^{n-1}(\psi_{i_1}) \circ \cdots \circ F(\psi_{n-1}) \circ \psi_{i_n}$$

form an orthonormal basis $\psi_{i,n}$ of a Hilbert space of support one in $(0, n)$. We find

$$\hat{F}^k(X) = \sum_i F^n(\psi_{i,k}) \circ X \circ F^m(\psi_{i,k}^*).$$

We can interchange the role of F and $\hat{F}$. But care is required as we need another Hilbert space of support one in $(0, n)$. It follows from Lemma 2.7 that $\hat{F}(T) = F(T)$ for $T \in (0, 0)$ so that L remains the correct Hilbert space in $(0, 1)$. In $(0, n)$ we need a Hilbert space with orthonormal basis

$$\hat{F}^{n-1}(\psi_{i_1}) \circ \cdots \circ \hat{F}(\psi_{n-1}) \circ \psi_{i_n}$$

denoted $\hat{\psi}_{i,n}$. Then

$$F^k(X) = \sum_i \hat{F}^n(\hat{\psi}_{i,k}) \circ X \circ \hat{F}^m(\hat{\psi}_{i,k}^*).$$

We now define a W^*–subcategory $\mathcal{K}$ of $\mathcal{H}$ whose arrows consist of those $S \in (m, n)$ such that $S \circ F^m(T) = F^n(T) \circ S$. This subcategory is invariant under F and the object 0 is now irreducible. By Lemma 2.7, L is a Hilbert space of support one in this category so that this is actually a category of Hilbert spaces. So our previous result is applicable and $\mathcal{K}$ is isomorphic to the

category of tensor powers of a Hilbert space K in such a way that F is the functor of tensoring on the right by 1_K.

But the inclusion functor of $\mathcal{K}$ in $\mathcal{H}$ is now a normal $*$–functor which is the identity on objects. The associated endofunctor A is just F. Our previous analysis yields the following result.

2.8 Proposition *Let $\mathcal{H}$ be a category of Hilbert spaces with objects labelled by $\mathbb{N}_0$ and F a normal $*$-functor from $\mathcal{H}$ to $\mathcal{H}$ acting on objects as $F(n) = n + 1$. Then $\mathcal{H}$ is isomorphic to the category of Hilbert spaces whose objects are of the form $H \otimes K^n$, $n \in \mathbb{N}_0$, for Hilbert spaces of H and K in such a way that F becomes the functor of tensoring on the right by 1_K.*

3   Shifts and Multiplicative Unitaries

We used above the characterization of endomorphisms of the Cuntz algebra in terms of unitaries given by Cuntz who also showed[5] that if K is a finite-dimensional Hilbert space then a unitary $V \in (K^2, K^2)$ is a multiplicative unitary if the endomorphism λ_R of the Cuntz algebra $\mathcal{O}_K$ satisfies

$$\lambda_R \lambda_R = \rho \lambda_R,$$

where $R = V\theta$, θ being the flip on K^2, and ρ the endomorphism generated by the defining Hilbert space K. This result remains valid in the extended Cuntz algebra if K is infinite dimensional[15]. The multiplicativity of V is alternatively expressed by the identity $\lambda_{V^*}\lambda_R = \rho\lambda_{V^*}$. The corresponding identities for normal $*$-functors on $\mathcal{K}$ commuting with F imply that a unitary $V \in (2, 2)$ is multiplicative for the tensor structure induced by $\hat{F}$, i.e. where F corresponds to tensoring on the right and $\hat{F}$ to tensoring on the left by the object 1. The role of the endomorphism ρ of the Cuntz algebra is played by $\hat{F} = G_\theta$. In other words, the following result holds.

3.1 Proposition *Let F and G be two commuting shifts on $\mathcal{K}$. Let $R \in (2, 2)$ be the unitary such that $G(\psi) = R \circ F(\psi)$, $\psi \in (0, 1)$. Set $V := R \circ \theta$, where θ is the flip on $(2, 2)$ derived from the tensor structure induced by $\hat{F}$, then the following conditions are equivalent.*

a) *V is a multiplicative unitary.*

b) *$GG = \hat{F}G$.*

b') *$FF = \hat{G}F$.*

c) $n \mapsto F^{n+1}(\psi)$ is a natural transformation from G to GG, $\psi \in (0,1)$.

c') $n \mapsto G^{n+1}(\psi)$ is a natural transformation from F to FF, $\psi \in (0,1)$.

d) $GG(\psi) \circ F(\psi') = FF(\psi') \circ G(\psi)$, $\psi, \psi' \in (0,1)$.

e) $G_{V^*}G = \hat{F}G_{V^*}$.

f) $n \mapsto F^{n+1}(\psi)$ is a natural transformation from G_{V^*} to $G_{V^*}G$, $\psi \in (0,1)$.

g) $G_{V^*}G(\psi) \circ F(\psi') = FF(\psi') \circ G_{V^*}(\psi)$, $\psi, \psi' \in (0,1)$.

Proof. The equivalence of a), b) and e) is a simple computation whose origins were explained above. Suppose a) is valid and pick $X \in (m,n)$ then

$$\sum_i F^{n+1}(\psi_{i,1}) \circ G(X) \circ F^{m+1}(\psi_{i,1})^* = \hat{F}G(X) = GG(X).$$

Hence $F^{n+1}(\psi) \circ G(X) = G(X) \circ F^{m+1}(\psi)$ for $\psi \in (0,1)$, giving c). d) follows as a special case. But if d) holds, then

$$\hat{F}G(\psi) = \sum_i F^2(\psi_{i,1})G(\psi)F(\psi_{i,1})^* = GG(\psi).$$

But the set of arrows X in $\mathcal{K}$ such that $\hat{F}G(X) = GG(X)$, is a W^*–subcategory of $\mathcal{K}$ which is invariant under the action of F, seeing that F commutes with $\hat{F}$ and G. Thus d) implies b). In the same way, we can prove that e), f) and g) are equivalent. Finally, the symmetry between F and G, visible in d), shows that b') and c') are also equivalent to the remaining conditions.

We have seen that the situation in Proposition 3.1 is symmetric in the two commuting shifts F and G. Interchanging F and G obviously corresponds to replacing R by R^{-1} and hence $V := R \circ \theta$ by $\hat{V} = R^{-1} \circ \theta$, the dual multiplicative unitary. The multiplicative unitaries on $(0,2)$ for the tensor structure determined by F are clearly in $1 - 1$ correspondence with the shifts commuting with F and satisfying the equivalent conditions of Proposition 3.1. Equivalent multiplicative unitaries correspond to shifts conjugate under (inner) automorphisms of $\mathcal{K}$ commuting with F.

We can also characterize the intertwining operators between the tensor powers of V, regarded as an object in the category of representations of V, in terms of the commuting shifts.

3.2 Lemma *Let F and G be commuting shifts on $\mathcal{K}$ with $GG = \hat{F}G$ and let V be the associated multiplicative unitary. Set $G' := G_{\vartheta V}$ then*

$$(V^r, V^s) = \{Y \in (r,s) : G'(Y) = \hat{F}(Y)\}.$$

Proof. Interpreting F as tensoring on the right by 1, the condition $G'(Y) = \hat{F}(Y)$ reads

$$(\vartheta V)_s \circ Y \times 1 \circ (\vartheta V)_r^* = \vartheta_s \circ Y \times 1 \circ \vartheta_r^*.$$

Hence, it suffices to show that $\vartheta_s^* \circ (\vartheta V)_s = V_{1s+1} V_{2s+1} \cdots V_{ss+1}$, where we have used the index notation on the right hand side. However, this may be proved by induction. In fact

$$\vartheta_s^* \circ (\vartheta V)_s = 1_{s-1} \times \vartheta \circ \vartheta_{s-1}^* \times 1 \circ (\vartheta V)_{s-1} \times 1 \circ 1_{s-1} \times (\vartheta V)$$

$$= \vartheta_{ss+1} V_{1s} V_{2s} \cdots V_{s-1s} \vartheta_{ss+1} V_{ss+1} = V_{1s+1} V_{2s+1} \cdots V_{ss+1}.$$

3.3 Corollary *Let F and G be two commuting shifts on $\mathcal{K}$ such that $GG = \hat{F}G$ and V the corresponding multiplicative unitary. Let $G' := G_{\vartheta V}$ and consider a sequence $t_n \in (r+n, s+n)$ that defines simultaneously a natural transformation t in $(\hat{G}'^r, \hat{G}'^s)$ and $(\hat{F}^r, \hat{F}^s)$. Such natural transformations form a tensor W^*–subcategory of the tensor W^*–category of all natural transformations between the powers of $\hat{G}'$. Evaluating t in 0 establishes an isomorphism with the tensor W^*–category of intertwiners between powers of V.*

Proof. If $t \in (\hat{G}'^r, \hat{G}'^s) \cap (\hat{F}^r, \hat{F}^s)$, then, by Lemma 2.1, $t_n = G'^n(t_0) = F^n(t_0)$. Thus t is uniquely determined by t_0 and, as G' and F commute, the only constraint on t_0 is that $G'(t_0) = F(t_0)$. Lemma 3.2 shows that we have an isomorphism of W^*–categories and a computation shows that it preserves tensor products.

We now comment on the role of multiplicative unitaries, seen in this light. As we have seen, F determines a tensor product structure on $\mathcal{K}$ and G determines another so it is natural to interpret R as describing the transition from one tensor product to another. However, any unitary $R \in (2, 2)$ would serve here. We do not need $V := R\theta$ to be a multiplicative unitary. Although it is perfectly correct to say that F determines a tensor product structure on $\mathcal{K}$, this structure really involves two commuting shifts F and $\hat{F}$ which we interpret as tensoring on the right by 1 and on the left by 1. Thus taking two commuting shifts F and G can be regarded as generalizing the idea of a tensor product. It is less symmetric in that θ has been replaced by R and $\hat{F}$ by G and does not give rise to a bifunctor as a true tensor product unless $G = \hat{F}$. Looked at this way, the condition of V being multiplicative being equivalent to b) has a simple interpretation. It is a 'depth 2' condition: it is not necessary to apply G more than once since in successive applications it can always be replaced by $\hat{F}$. The justification for adopting this terminology from the theory of subfactors is Lemma 6.3 of [12].

We shall say that the two commuting shifts have relative depth two to stress that the notion involves the two shifts symmetrically.

Let us look at some examples of actions of $\mathcal{K}$ on $\mathcal{H}$. A first example is suggested directly by Proposition 2.5. Given a Hilbert space H, take $\mathcal{H} := H \otimes \mathcal{K}$ and then define $H : \mathcal{K} \to \mathcal{H}$ by $H(Y) := 1_H \otimes Y$. With this definition we find $A(X) = X \otimes 1_K$. We get other normal *–functors from $\mathcal{K}$ to $\mathcal{H}$ by picking for each r a unitary $W_r \in (H_r, H_r)$ and then defining $E(Y) := W_s \circ 1_H \otimes Y \circ W_r^*$, $Y \in (r, s)$. In fact, we are in the situation of Lemma 2.1 and $W_s = (E, 1_{H_0}, H)_s$. A computation shows that in this case the corresponding endofunctor B, say, is given by

$$B(X) = W_{n+1} \circ A(W_n^* \circ X \circ W_m) \circ W_{m+1}^*, \ \ X \in (H_m, H_n).$$

In particular, suppose we start from a multiplicative unitary V on K^2 and a representation W on H. We can define H and E as above but making the particular choice

$$W_s := W_{12} W_{13} \ldots W_{1s+1},$$

using index notation.

3.4 Lemma *Let W be a representation of a multiplicative unitary then defining functors $E, H : \mathcal{K} \to \mathcal{H}$, as above and letting G be the endofunctor on $\mathcal{K}$ defined by $R := V\theta$, as before, we have $EG = HG$.*

Proof. It suffices to show that $EG(\psi) = HG(\psi)$ for each $\psi \in (0, 1)$. Now $HG(\psi) = H(R)HF(\psi)$, whereas

$$EG(\psi) = E(R) \circ EF(\psi) = E(R) \circ W_{12} W_{13} HF(\psi) W_{12}^* = E(R) \circ W_{12} HF(\psi).$$

However $E(R) = W_{12} W_{13} R_{23} W_{13}^* W_{12}^*$ and $H(R) = R_{23}$, so the identity in question follows from the definition of a representation W of V, $W_{12} W_{13} V_{23} = V_{23} W_{12}$.

We can therefore adopt the following point of view. Regard a multiplicative unitary as a category $\mathcal{K}$ of Hilbert spaces, as above, equipped with two commuting shifts F and G satisfying $GG = \hat{F}G$. A representation of such a multiplicative unitary is a category of Hilbert spaces $\mathcal{H}$, as above, equipped with two normal *–functors E and H from $\mathcal{K}$ to $\mathcal{H}$ of rank zero such that $EG = HG$. An intertwining operator between two representations is a bounded linear operator $T \in (H_0, H_0')$ such that, setting

$$A(X) := \sum_i H' \hat{F}^n (\psi_{i,1}) \circ X \circ H \hat{F}^m (\psi_{i,1})^*, \quad X \in (H_m, H_n'),$$

$r \mapsto A^r(T)$ defines a natural transformation from E to E'. In fact, if we pick $\psi \in (0,1)$, this gives

$$A(T)W1_H \otimes \psi = W'1_{H'} \otimes \psi T = W'A(T)1_{H'} \otimes \psi.$$

Since $A(T) = T \otimes 1_K$, $T \in (W, W')$. On the other hand, if $T \in (W, W')$, then $A^r(T)W_r = W'_r A^r(T)$ and this implies that $r \mapsto A^r(T)$ is a natural transformation from E to E'. Note, however, that $A^r(T) = (H', T, H)_r$, so that $T \in (H_0, H'_0)$ is in (W, W') if and only if $r \mapsto A^r(T)$ defines an element of $(H, H') \cap (E, E')$.

Superficially, the relation $GG = \hat{F}G$ looks like a special case of $EG = HG$. However, E and H have rank zero, whereas G and $\hat{F}$ have rank one. However, if we delete the object 0 from $\mathcal{K}$ to give a full subcategory $\mathcal{K}_+$ and let G_+ and $\hat{F}_+$ denote the functors obtained by restricting the range to $\mathcal{K}_+$, the relation $G_+G = \hat{F}_+G$ is a special case of $EG = HG$ and we are considering V as an object in its category of representations.

The relation $EG = HG$ is symmetrical in E and H. Exchanging E and H corresponds to replacing W by W^{-1} which will not, in general be a representation of the multiplicative unitary V. However, the tensor product notation involved in the definition of representation refers to H.

H is to be interpreted here as the trivial representation on the same space H_0 as E. Since $(E, E') \cap (H, H') \subset (H, H')$, the category of representations of V automatically comes equipped with a faithful functor into the subcategory of trivial representations.

When we have two commuting shifts and two actions E and H, as above, it seems appropriate to refer to $\mathcal{H}$ as a $\mathcal{K}$–bimodule. We have already considered the tensor product of $\mathcal{K}$–modules in the last section and we now extend these considerations to bimodules. Given therefore two action E' and H' on $\mathcal{H}'$, we form $\mathcal{H} \otimes \mathcal{H}'$ and $H \otimes H'$. Of course, we could equally well have formed $E \otimes E'$ instead, but what we really need is the relation between the two actions. We therefore use the functors D and D', expressing $\mathcal{H} \otimes \mathcal{H}'$ as a tensor product and set
$$(E \otimes E')(Y) = D(W_s) \circ D'E'(Y) \circ D(W^*_r), \quad Y \in (r,s)$$

where $W_r = (E, 1_{H_0}, H)_r$. It is easy to check that $E \otimes E'$ is a normal *–functor and that this definition applied to H and H' gives $H \otimes H'$. Expressing E' in terms of H' using $W'_r = (E', 1_{H'_0}, H')_r$, the definition is equivalent to

$$(E \otimes E', 1_{(H \otimes H')_0}, H \otimes H')_r = D(W_r) \circ D'(W'_r).$$

There is obviously a convention involved here because we have chosen to write the primed terms to the right of the unprimed terms. However, this convention is consistent with that used for multiplicative unitaries in that it corresponds to taking the tensor product of representations W and W' as $W_{13}W'_{23}$. It is easily checked that $EG = HG$ and $E'G = H'G$ imply $E \otimes E'G = H \otimes H'G$.

4   Multiplicative Unitaries and Tensor Categories

We now exhibit a mechanism leading from objects in a tensor W^*–category to multiplicative unitaries. It will involve a category of Hilbert spaces $\mathcal{K}$, equipped with two commuting shifts F and G. Let ρ be an object in a strict tensor W^*–category and suppose that K is a Hilbert space of support one contained in (ρ, ρ^2). We then define inductively $K^n := K^{n-1} \times 1_\rho \circ K$, where the norm closed linear span is understood. Then K^n is a Hilbert space of support 1 in (ρ, ρ^{n+1}). K^0 will denote $\mathbb{C}1_\rho$. We now set

$$(K^m, K^n) := \{X \in (\rho^{m+1}, \rho^{n+1}) : X \circ K^m \subset K^n\}.$$

We see that $K^n = (K^0, K^n)$ and thus we have defined a W^*–subcategory $\mathcal{K}$ of Hilbert spaces of the tensor W^*–category $\mathcal{T}_\rho$ whose objects are the tensor powers of ρ. We claim that $\mathcal{K}$ is invariant under tensoring on the right by 1_ρ. In fact, it suffices to show that $K \times 1_\rho \subset (K, K^2)$, i.e. that $K \times 1_\rho \circ K \subset K^2$ but this is true by construction. If F denotes the restriction of $\times 1_\rho$ to $\mathcal{K}$, $\mathcal{K}$ has unique structure of tensor W^*–category such that F becomes the functor of tensoring on the right by 1_K.

4.1 Lemma *Let $K \subset (\rho, \rho^2)$ be a Hilbert space of support one in a tensor W^*–category and let $\mathcal{K}$ be the subcategory of Hilbert spaces defined as above, then the following conditions are equivalent.*

a) $1_\rho \times K \subset (K, K^2)$.

b) $\mathcal{K}$ is an invariant subcategory for $1_\rho \times$.

c) $K^n = 1_\rho \times K^{n-1} \circ K$, $n \in \mathbb{N}$.

d) $K^2 = 1_\rho \times K \circ K$.

Proof. $\mathcal{K}$ is the smallest W^*–subcategory containing K and invariant under $\times 1_\rho$. Furthermore $1_\rho \times$ and $\times 1_\rho$ commute, thus a) implies b). Given b), we know from Proposition 2.3 that $1_\rho \times \psi = R_n \circ \psi \times 1_\rho$, for $\psi \in K^n$ with R_n unitary, giving c). c) implies d), trivially and d) implies a).

We call a Hilbert space K satisfying the equivalent conditions of Lemma 4.1 *ambidextrous.*

4.2 Theorem *Let $K \subset (\rho, \rho^2)$ be an ambidextrous Hilbert space of support one in (ρ, ρ^2). Let F and G denote the restrictions of $\times 1_\rho$ and $1_\rho \times$ to $\mathcal{K}$ then there is a unique $V \in (K^2, K^2)$ such that*

$$G(\psi) = V \circ \hat{F}(\psi), \quad \psi \in K.$$

V is a multiplicative unitary.

Proof. V is unique and is unitary because it is given by

$$V = \sum_i G(\psi_{i,1}) \hat{F}(\psi_{i,1})^*$$

where the sum is taken over an orthonormal basis. Now since $K \in (\rho, \rho^2)$, for $X \in (K^m, K^n)$,

$$\hat{F}G(X) = \sum_i F^{n+1}(\psi_{i,1}) \circ 1_\rho \times X \circ F^{m+1}(\psi_{i,1})^* = 1_{\rho^2} \times X = G^2(X).$$

Thus $G^2 = \hat{F}G$ and V is a multiplicative unitary by Proposition 2.3.

Every multiplicative unitary can be realized in this manner.

4.3 Proposition *Let V be a multiplicative unitary $V \in (K^2, K^2)$, then $G_{V^*}(K)$ is an ambidextrous Hilbert space $H \subset (K, K^2)$ of support one and if $U \in (K, H)$ is the unitary taking ψ to $G_{V^*}(\psi)$ the multiplicative unitary defined by H is $U \times U \circ V \circ (U \times U)^{-1}$.*

Proof. $H := V^* \circ K \times 1_K = G_{V^*}(K)$ is obviously a Hilbert space of support one in (K, K^2). To verify that H is ambidextrous, it suffices by Proposition 2.4 to verify that

$$\psi_2^* \times 1_K \circ V \circ 1_K \times \psi_1^* \circ 1_K \times V \circ V^* \times 1_K \circ \psi_3 \times 1_{K^2} \circ V^* \circ \psi_4 \times 1_K$$

is a multiple of 1_K for any choice of $\psi_i \in K$, $i = 1, 2, 3, 4$. Using the multiplicativity of V, this reduces to

$$(\psi_2^* \times \psi_1^* \circ V^* \circ \psi_3 \times \psi_4) \times 1_K$$

and hence is a multiple of 1_K. It remains to compute the operator S defined by

$$S \circ V^* \times 1_K \circ \psi \times 1_{K^2} = 1_K \times V^* \circ 1_K \times \psi \times 1_K, \quad \psi \in K.$$

Writing $\phi_i \in H$ for $V^* \circ \psi_i \times 1_K$ and computing using the multiplicativity of V, we find

$$\phi_1^* \times \phi_2^* \circ S \circ \phi_3 \times \phi_4 = ((\psi_1 \times \psi_2)^* \circ R \circ \psi_3 \times \psi_4) \times 1_K.$$

Thus the multiplicative unitary associated with H is as asserted.

An analogous computation shows that we may replace G_{V^*} by G_R in Proposition 4.3.

We now ask how the multiplicative unitary depends on the choice of ambidextrous Hilbert space $H \subset (\rho, \rho^2)$. Let H' be another such Hilbert space then there is a unitary $U \in (\rho^2, \rho^2)$ such that $U \circ H = H'$. Let V and V' denote the multiplicative unitary operators associated with H and H' and set $R := V\theta$, $R' := V'\theta'$, where θ and θ' are the flips on H^2 and H'^2, respectively. We compute the relation between R and R'. Let $\psi \in H$, then

$$R' \circ (U \circ \psi) \times 1_\rho = 1_\rho \times (U \circ \psi) = 1_\rho \times U \circ R \circ \psi \times 1_\rho.$$

Thus $R' = 1_\rho \times U \circ R \circ U^* \times 1_\rho$. This is in contrast to the transformation law of θ, namely $\theta' = u_2 \circ \theta \circ u_2{}^*$ where $u_2 := U \times 1_H \circ 1_H \times U$ is the tensor power of U. It should be remembered however that R' is intrinsically determined by H' whereas U is not. The interesting question is whether the associated multiplicative unitaries V and V' are necessarily equivalent and for which unitaries U, $U \circ H$ is ambidextrous.

We present an example. Let $\mathcal{K}$ denote the W^*–tensor category of tensor powers of a Hilbert space K. Let $V \in (K^2, K^2)$ be a multiplicative unitary, then, as we have seen, $H := V^* \circ K$ is an ambidextrous Hilbert space in (K, K^2) whose associated multiplicative unitary is equivalent to V. Since we are free to choose any multiplicative unitary V, this makes it clear that the associated multiplicative unitaries can depend in an essential way on the ambidextrous Hilbert space.

We may sum up the results to date in this section as follows.

4.4 Theorem *Let $\mathcal{K}$ be a category of Hilbert spaces with objects K_n, $n \in \mathbb{N}_0$ and $(K_0, K_0) = \mathbb{C}$ equipped with commuting shifts F and G such that $GG = \hat{F}G$. Let $V \in (K_2, K_2)$ be the multiplicative unitary such that $G(\psi) = V \circ \hat{F}(\psi)$, $\psi \in K_1$, then $H := V^* \circ F(K_1)$ is an ambidextrous Hilbert space. Let $\mathcal{H}$ be the resulting category of Hilbert spaces with commuting shifts D and E obtained by restricting F and $\hat{F}$ to $\mathcal{H}$, then the shift G^* on $\mathcal{K}$ defined by $G^*(\psi) = V^* \circ F(\psi)$, $\psi \in K_1$ yields an isomorphism of $\mathcal{K}$, F, G with $\mathcal{H}$, D, E.*

Proof. We know from Proposition 3.1 that V is a multiplicative unitary and from Proposition 4.3 that $G^*(K_1)$ is an ambidextrous Hilbert space. The resulting category $\mathcal{H}$ of Hilbert spaces is thus $G^*(\mathcal{K})$. Since G^* commutes with F and by Proposition 3.1, $G^*G = \hat{F}G^*$, G^* does yield the desired isomorphism.

Remark. The construction of this section deriving a multiplicative unitary from an ambidextrous Hilbert space is invariant under tensor *–functors since the image of an ambidextrous Hilbert space of support one is again such a Hilbert space and a tensor *–functor commutes with tensoring on the right and left.

We have seen in Theorem 4.2 how an ambidextrous Hilbert space leads to a multiplicative unitary. There is a variant of this result which instead yields a representation of a multiplicative unitary. To motivate this result, we let V be a multiplicative unitary on the tensor power of a Hilbert space L and W a representation of V on a Hilbert space M. Then V and W are objects of the tensor W^*–category $\mathcal{R}(V)$ and, as we know, $K := V^* \circ L \times 1_L$ is an ambidextrous Hilbert space in (V, V^2). However, W being a representation of V, $H_0 := W^* \circ M \times 1_L$ is a Hilbert space of support one in (V, WV). Hence $H_n := H_{n-1} \times 1_V \circ K$ is a Hilbert space of support one in (V, WV^{n+1}). Thus just as we have a category $\mathcal{K}$ of Hilbert spaces associated with K, there is a category $\mathcal{H}$ associated with H_0. We claim that tensoring on the left with 1_W restricts to a *–functor from $\mathcal{K}$ to $\mathcal{H}$. It suffices to show that $1_W \times K \circ H_0 \subset H_1$ and, expressing K and H_0 in terms of L and M, this is again a consequence of W being a representation of V. We have here an obvious generalization of the notion of ambidextrous Hilbert space involving K and H_0.

We use this as the basis of a definition. Let $\mathcal{T}$ be a tensor W^*–category and ρ and σ objects of $\mathcal{T}$. Let $K \subset (\rho, \rho^2)$ and $H_0 \subset (\rho, \sigma\rho)$ be Hilbert spaces of support one and $\mathcal{K}$ and $\mathcal{H}$ the corresponding categories of Hilbert spaces with objects $K^n := K^{n-1} \times 1_\rho \circ K$ and $H_n := H_{n-1} \times 1_\rho \circ K$, $n \in \mathbb{N}_0$, respectively. We say that H is K–ambidextrous if $1_\sigma \times$ restricts to a *–functor from $\mathcal{K}$ to $\mathcal{H}$.

Lemma 4.5 *Let $\mathcal{T}$ be a tensor W^*–category and $K \subset (\rho, \rho^2)$ and $H \subset (\rho, \sigma\rho)$ be Hilbert spaces of support one then the following conditions are equivalent.*

a) $1_\sigma \times K \subset (H_0, H_1)$,

b) H is K–ambidextrous,

c) $H_n = 1_\sigma \times K^n \circ H_0$,

d) $H_1 = 1_\sigma \times K \circ H_0$.

Proof. $\mathcal{K}$ is the smallest W^*–subcategory containing K and invariant under $\times 1_\rho$. Furthermore $1_\sigma \times$ and $\times 1_\rho$ commute, thus a) implies b). Since both sides of c) are Hilbert spaces of support one, b) implies c). c) implies d), trivially and d) implies a).

4.6 Theorem *Let $K \subset (\rho, \rho^2)$ be an ambidextrous Hilbert space of support one in (ρ, ρ^2) and H a K–ambidextrous Hilbert space in $(\rho, \sigma\rho)$. Let F, G, E and H denote the restrictions of $\times 1_\rho$, $1_\rho \times$, $1_\sigma \times$ and $1_{H_0} \times$, respectively, to $\mathcal{K}$ then there is a unique $V \in (K^2, K^2)$ such that*

$$G(\psi) = V \circ \hat{F}(\psi), \quad \psi \in K,$$

and a unique $W \in (H_1, H_1)$ such that

$$E(\psi) = W \circ H(\psi), \quad \psi \in K.$$

V is a multiplicative unitary and W is a representation of V on H_0.

Proof. In view of Theorem 4.2, we need only prove the assertions relating to W. W is unique and is unitary because it is given by

$$W = \sum_i E(\psi_{i,1}) \circ H(\psi_{i,1})^*$$

where the sum is taken over an orthonormal basis of K. Now since $H_0 \in (\rho, \sigma\rho)$, for $X \in (K^m, K^n)$,

$$HG(X) = \sum_i F^{n+1}(\phi_i) \circ 1_\rho \times X \circ F^{m+1}(\phi_i)^* = 1_{\sigma\rho} \times X = EG(X),$$

where ϕ_i is an orthonormal basis of H_0. Thus $HG = EG$ and W is a representation of V by the discussion following Lemma 3.4.

Theorem 4.6 does not really refer to the whole tensor W^*–category $\mathcal{T}$ but only to the full subcategory with objects ρ^n and $\sigma\rho^n$ and the structures induced on this subcategory by tensoring on the left and right by 1_ρ and on the left by 1_σ. Introducing $\mathcal{T}$ enables us to avoid spelling out the structures.

We now show how a representation of multiplicative unitaries gives rise to an interchange law.

4.7 Proposition *Let F and G be two commuting shifts on $\mathcal{K}$ with $GG = \hat{F}G$ and E and H normal *–functors from $\mathcal{K}$ to $\mathcal{H}$ with $EG = HG$, then given $X \in (H_m, H_n)$ and $Y \in (p, q)$,*

$$B^{q+1}(X) \circ EG^{m+1}(Y) = EG^{n+1}(Y) \circ B^{p+1}(X),$$

where B is the normal *–functor on $\mathcal{H}$ associated with E:

$$B(X) : \sum_i EF^n(\psi_{i,1}) \circ X \circ EF^m(\psi_{i,1})^*, \ X \in (H_m, H_n).$$

Thus defining $X \times' Y := B^{q+1}(X) \circ EG^{m+1}(Y)$ gives an action of the tensor W^*–category $\mathcal{K}^+$ on the category $\mathcal{H}$ of Hilbert spaces. If $\mathcal{K}$ is given the tensor structure determined by F then there is a unique normal tensor $*$–functor G^* from $\mathcal{K}^+$ to $\mathcal{K}$ such that $G^*(\psi) := V^* \circ F(\psi)$, $\psi \in K$, where V is the multiplicative unitary associated with F and G.

Proof. To prove the interchange law, write $EG^{m+1} = H\hat{F}^m G$ and $EG^{n+1} = H\hat{F}^n G$ and use the interchange law between A and $H\hat{F}$ discussed before Lemma 2.4. The remarks above show that $\mathcal{K}^+$ is a tensor W^*–category and allow us to check that G^* is a tensor $*$–functor using Proposition 3.1e.

As we shall be considering Hilbert spaces $L \subset (K^r, K^{r+g})$ in Section 5, it is natural to ask to what extent the results can be generalized to Hilbert spaces K of support one in (ρ^r, ρ^{r+g}), where we suppose, of course, that $g \neq 0$. The initial construction can be easily modified. We define inductively $K^n := K^{n-1} \times 1_{\rho^g} \circ K$, the norm closed linear span being understood. K^n is a Hilbert space of support 1 in (ρ^r, ρ^{r+ng}). We now set

$$(K^m, K^n) := \{X \in (\rho^{r+mg}, \rho^{r+ng}) : X \circ K^m \subset K^n\}.$$

In this way we have a W^*–subcategory $\mathcal{K}$ of Hilbert spaces whose objects are of the form ρ^{r+ng}, $n \in \mathbb{N}_0$. $\mathcal{K}$ is now invariant under tensoring on the right by 1_{ρ^g}. Letting F be the restriction of this functor to $\mathcal{K}$ we have a shift on $\mathcal{K}$ that can be regarded as tensoring on the right by 1_K. The analogue of Lemma 4.1 now holds if we consider tensoring on the left by 1_{ρ^g} and can be used to define the notion of ambidextrous Hilbert space. Thus we are again led to a category of Hilbert spaces $\mathcal{K}$ equipped with two commuting shifts. At this point there is the essential difference: we cannot use the exchange law in $\mathcal{T}_\rho$ to get an analogue of Theorem 4.2 unless $r \leq g$.

4.8 Theorem *Let K be an ambidextrous Hilbert space of support one in (ρ^r, ρ^{r+g}) where $r \leq g$. Let F and G denote the restrictions of $\times 1_{\rho^g}$ and $1_{\rho^g} \times$ to $\mathcal{K}$ then there is a unique $V \in (K^2, K^2)$ such that*

$$G(\psi) = V \circ \hat{F}(\psi), \quad \psi \in K.$$

V is a multiplicative unitary.

The proof follows that of Theorem 4.2, bearing in mind that $r \leq g$. If $r > g$, there is the possibility of starting with the ambidextrous Hilbert space K^n, for n sufficiently large.

There is another interesting general result involving Hilbert spaces in $\mathcal{T}_\rho$.

4.9 Theorem *Let K be a Hilbert space of support one in (ρ^r, ρ^{r+g}) and define the associated category of Hilbert spaces $\mathcal{K}$ as above. Suppose that*

$$(\rho^r, \rho^r) \times 1_{\rho^{rg}} \subset (K^r, K^r).$$

Then

$$(\rho^{r+mg}, \rho^{r+ng}) \times 1_{\rho^{rg}} \subset (K^{r+m}, K^{r+n}).$$

Hence defining, for $X \in (\rho^{r+mg}, \rho^{r+ng})$,

$$F(X)\psi := X \times 1_{\rho^{rg}} \circ \psi, \quad \psi \in K^{r+m},$$

*F is a faithful *–functor from the full subcategory of $\mathcal{T}_\rho$ whose objects are ρ^{r+ng}, $n \in \mathbb{N}_0$ to the category $\mathcal{K}$ of Hilbert spaces and*

$$F(X \times 1_{\rho^{rg}}) = F(X) \times 1_K.$$

Proof. $K \subset (\rho^r, \rho^{r+g})$ implies $K^n \subset (\rho^r, \rho^{r+ng})$ and $K^{*n} \circ (\rho^{r+mg}, \rho^{r+ng}) \circ K^m \subset (\rho^r, \rho^r)$. Since $(\rho^r, \rho^r) \times 1_{\rho^{rg}} \subset (K^r, K^r)$, tensor the above on the right with $1_{\rho^{rg}}$ and compose on the right with K^r and on the left with K^{*r} to conclude that

$$K^{*r+n} \circ (\rho^{r+mg}, \rho^{r+ng}) \times 1_{\rho^{rg}} \circ K^{r+m} \subset K^{*r} \circ (K^r, K^r) \circ K^r \subset \mathbb{C}1_{\rho^r}.$$

Since K^{r+m} has support one, we conclude that

$$(\rho^{r+mg}, \rho^{r+ng}) \times 1_{\rho^{rg}} \subset (K^{r+m}, K^{r+n}).$$

The remaining assertions are now obvious.

We show that the hypotheses of Theorem 4.9 with $r = g = 1$ are fulfilled, when V is a multiplicative unitary, considered as an object of $\mathcal{R}(V)$. In fact, $V \in (V^2, \iota(V)V)$, so $V^* \circ (\mathbb{C}, \iota(V)) \times 1_V$ is a Hilbert space H of support one in (V, V^2). Using the condition for $T \in (K, K)$ to be an arrow of (V, V), we get

$$T \times 1_V \circ V^* = V^* \circ \iota(T) \times 1_V,$$

showing that $(V, V) \times 1_V \circ H \subset H$, as required. The hypothesis of Theorem 4.3 does imply for the tensor unit ι that $(\iota, \iota) = \mathbb{C}$.

Theorem 4.9 raises some interesting questions. Let $\mathcal{T}_{r,g}$ denote the full subcategory of $\mathcal{T}_\rho$ whose objects are of the form ρ^{r+ng} with $n \in \mathbb{N}_0$. Then we have *–functors $X \mapsto 1_{\rho^s} \times X \times 1_{\rho^{g-s}}$, $0 \leq s \leq g$, defined on $\mathcal{T}_{r,g}$ through the ambient tensor category $\mathcal{T}_\rho$. Restricting the domain of the functors to the subcategory

$\mathcal{K}$ and composing with the functor F of Theorem 4.9 gives us *–endofunctors of $\mathcal{K}$ taking K^m to K^{r+m}. This raises the question of whether the composition with F is necessary, or, more precisely, whether $1_{\rho^s} \times K \times 1_{\rho^{g-s}} \subset (K, K^2)$? In fact, we know that this is true by construction for $s = 0$ and the basis of the definition of ambidextrous for $s = g$. When it is valid for some $s > r$ there is some analogue of a multiplicative unitary.

We have seen how two commuting shifts F and G on $\mathcal{K}$ cannot be interpreted as tensoring on the right and tensoring on the left with the object 1 unless $G = \hat{F}$ because the interchange law would fail to hold. On the other hand, we have, in this section, been using the interchange law in a tensor category to produce commuting shifts tied to multiplicative unitaries. We want, now, to show how this process can be reversed. We first describe a tensor category in terms of tensoring on the right and tensoring on the left. Let $\mathcal{T}$ be a category and give for each object ρ endofunctors F_ρ and G_ρ such that for $X \in (\mu, \nu)$ and $Y \in (\pi, \rho)$,

$$F_\rho(X) \circ G_\mu(Y) = G_\nu(Y) \circ F_\pi(X),$$

where this identity defines $X \times Y$ and expresses the interchange law. It is understood to imply that $F_\rho(\mu) = G_\mu(\rho)$ for each pair ρ, μ of objects. The set of these endofunctors is supposed to commute pairwise and $F_{F_\rho(\sigma)} = F_\rho F_\sigma$ and $G_{G_\mu(\nu)} = G_\mu G_\nu$ implying that $\times$ is associative. If we further require that for some object ι F_ι and G_ι are the identity functors then $\mathcal{T}$ becomes a (strict) monoidal category with monoidal unit ι. A functor J between two such monoidal categories is a (strict) monoidal functor if for each object ρ of $\mathcal{T}$,

$$JF_\rho = F_{J(\rho)}J, \quad JG_\rho = G_{J(\rho)}J.$$

To have corresponding statements for tensor categories or tensor C^*–categories or tensor W^*–categories we need only add the obvious conditions that the functors involved be linear, *–preserving or normal as the case may be.

4.10 Proposition *Let F and G be two commuting shifts with $GG = \hat{F}G$, then given $X \in (m, n)$ and $Y \in (p, q)$,*

$$F^{q+1}(X) \circ G^{m+1}(Y) = G^{n+1}(Y) \circ F^{p+1}(X).$$

Thus defining $X \times' Y := F^{q+1}(X) \circ G^{m+1}(Y)$ gives a tensor W^*–category $\mathcal{K}^+$ after adjoining an irreducible tensor unit $\mathbb{C}$ with no arrows to any other object. If $\mathcal{K}$ is given the tensor structure determined by F then there is a unique normal tensor *–functor G^* from $\mathcal{K}^+$ to $\mathcal{K}$ such that $G^*(\psi) := V^* \circ F(\psi)$, $\psi \in K$, where V is the multiplicative unitary associated with F and G.

Proof. Write $G^{m+1} = \hat{F}^m G$ and $G^{n+1} = \hat{F}^n G$ and use the interchange law between F and $\hat{F}$. The remarks above show that $\mathcal{K}^+$ is a tensor W^*–category and allow us to check that G^* is a tensor *–functor using Proposition 3.1e.

Note that $\times'$ is not addition on the objects of $\mathcal{K}$. Instead we have $m \times n = m+n+1$. However as we have adjoined a tensor unit $\mathbb{C}$ to give $\mathcal{K}^+$, it is natural to renumber the objects by adding one and $\times'$ is then addition on the objects of $\mathcal{K}^+$.

We now generalize some of our results so that we can work with tensor C^*–categories rather than just tensor W^*–categories. We begin with an object ρ in a strict tensor C^*–category and a Hilbert space K in (ρ, ρ^2) such that $K \times 1_{\rho^m}$ has left annihilator zero for $m \in \mathbb{N}_0$. We can define K^n and (K^m, K^n) as at the beginnning of this section to get a C^*–category $\mathcal{K}$. $\mathcal{K}$ will now not be a category of Hilbert spaces but just some subcategory. $\mathcal{K}$ is obviously an invariant subcategory for $\times 1_\rho$. The category $\mathcal{K}$ can be completed in an obvious way to give a category of Hilbert spaces $\tilde{\mathcal{K}}$, say, by identifying $X \in (K^m, K^n)$ with the corresponding linear map $\phi \mapsto X \circ \phi$, $\phi \in K^m$. The concept of ambidextrous Hilbert space is dealt with in the following lemma.

4.11 Lemma *Let $K \subset (\rho, \rho^2)$ be a Hilbert space such that $K \times 1_{\rho^m}$ has left annihilator zero for $m \in \mathbb{N}_0$. Let $\mathcal{K}$ be the C^*–category defined above, then the following conditions are equivalent.*

a) $\mathcal{K}$ is an invariant subcategory for $1_\rho\times$.

b) $1_\rho \times K \subset (K, K^2)$.

c) $K^n = 1_{\rho^{n-1}} \times K \circ K^{n-1}$, $n \in N_0$.

d) $K^2 = 1_\rho \times K \circ K$.

Proof. If b) holds, then $1_\rho \times \psi \circ \psi'^* \times 1_\rho$ maps K^2 into K^2 if ψ, $\psi' \in K$. Hence $\sum_i 1_\rho \times \psi_i \circ \psi_i^* \times 1_\rho$ converges in, say, the s–topology to a unitary arrow U in $\tilde{\mathcal{K}}$, where the sum is taken over an orthonormal basis in K. But $U \circ \psi \times 1_\rho \circ \psi' = 1_\rho \times \psi \circ \psi'$ and this proves d). c) follows from d) by induction since

$$K \times 1_{\rho^{n-1}} \circ 1_{\rho^{n-2}} \times K = 1_{\rho^{n-1}} \times K \circ K \times 1_{\rho^{n-2}},$$

Composing on the right with K^{n-2} and using the induction hypothesis we obtain c). But just as $\mathcal{K}$ is invariant under $\times 1_\rho$, c) implies that it is invariant under $1_\rho\times$. But b) implies a), trivially.

We can now prove an analogue of Theorem 4.2.

4.12 Theorem *Let ρ be an object in a tensor C^*–category and K an ambidextrous Hilbert space in (ρ, ρ^2) such that $K \times 1_{\rho^m}$ has left annihilator zero for $m \in \mathbb{N}_0$. Let $\mathcal{K}$ be as above and $\tilde{\mathcal{K}}$ its completion to a category of Hilbert spaces, then the endofunctors on $\mathcal{K}$ determined by $\times 1_\rho$ and $1_\rho \times$ extend uniquely to commuting shifts F and G on $\tilde{\mathcal{K}}$ with $G^2 = \hat{F}G$ and there is a multiplicative unitary $V \in (\tilde{K}_2, \tilde{K}_2)$, such that $G(\psi) = V \circ \hat{F}(\psi)$, $\psi \in K$.*

Proof. The functor $\times 1_\rho$ defines each K^n as a tensor power of K and hence there is a unique shift F on $\tilde{\mathcal{K}}$ such that $F(\psi_n) = \psi_n \times 1_\rho$, for $\psi_n \in K_n$ and $n \in \mathbb{N}_0$. If $X \in (K^m, K^n)$, then

$$X \times 1_\rho \circ F(\psi_m) \circ \psi = X \times 1_\rho \circ \psi_m \times 1_\rho \circ \psi = F(X \circ \psi_m)\psi.$$

Thus $F(X) = X \times 1_\rho$. In the same way, in view of Lemma 4.11, there is a unique shift G on $\tilde{\mathcal{K}}$ with $G(X) = 1_\rho \times X$. F and G obviously commute and, as in Theorem 4.2, we see that $GG = \hat{F}G$. Thus there is a multiplicative unitary V with the properties claimed.

The asymmetry in the formulation of Theorem 4.12 is only apparent: its hypotheses imply that $1_{\rho^m} \times K$ has left annihilator zero for $m \in \mathbb{N}_0$.

5 Algebraic Endomorphisms of the Cuntz Algebra

As a preliminary to our main duality result, we present in this section results on Hilbert spaces in the Cuntz algebra and endomorphisms of that algebra that are 'algebraic' with respect to the natural grading of the Cuntz algebra.

When K is a finite dimensional Hilbert space, $\mathcal{O}_K$ will denote the Cuntz algebra, a simple unital C^*–algebra introduced by Cuntz[3]. When K is infinite dimensional, it denotes the extended Cuntz algebra, a simple C^*–algebra introduced in [2]. These algebras are special cases of a more general construction needed in §6, where the Hilbert space is replaced by an object in a tensor C^*–category. $\mathcal{O}_K$ has a $\mathbb{Z}$–grading derived from the automorphic action α of $\mathbb{T}$ with $\alpha_\lambda(\psi) = \lambda\psi$, $\lambda \in \mathbb{T}$. Thus the part of $\mathcal{O}_K$ of grade k is given by

$$\mathcal{O}_K^k := \{X \in \mathcal{O}_K : \alpha_\lambda(X) = \lambda^k X, \ \lambda \in \mathbb{T}\}.$$

It is this grading in the case of the Cuntz algebra $\mathcal{O}_K$ which will allow us to refer to certain Hilbert spaces in this algebra and endomorphisms of this algebra as being 'algebraic'. Here K can be a finite dimensional or infinite dimensional Hilbert space and in the latter case $\mathcal{O}_K$ is the extended Cuntz algebra

introduced in [2]. The results can be roughly summarized by saying that computations involving these algebraic objects reduce to problems involving linear operators between Hilbert spaces that can be identified a priori. These results will prove useful in other contexts when dealing with concrete endomorphisms on the Cuntz algebra.

The problem has its origins in the relation between the Cuntz algebra $\mathcal{O}_K$ and the tensor W^*–category of bounded linear mappings between tensor powers of K. The Cuntz algebra is obtained from the category by factoring out the operation of tensoring on the right by 1_K whilst the operation of tensoring on the left is retained in the shape of the canonical endomorphism ρ_K. Then a direct sum is taken over the grading and finally the algebra is completed in the unique C^*–norm. This raises certain questions when proving results using the Cuntz algebra. A result may be purely algebraic in nature involving only the algebraic part of the Cuntz algebra. The manipulations may have been simplified by factoring out the operation of tensoring on the right by 1_K. At the same time their significance may have been obscured by the asymmetric treatment of tensoring on the two sides.

The problems treated in this section illustrate both the analytic and the algebraic aspects of the problem and we have had more success with the analytic aspects proving that the set of solutions of these problems involves only the algebraic part.

If H is a Hilbert space in a C^*–algebra $\mathcal{A}$, $Y \in \mathcal{A}$, and $L^1(H)$ denotes the trace class operators on H, then there is a unique continuous linear mapping $T \mapsto \mathrm{Tr}_H(YT)$ from $L^1(H)$ to $\mathcal{A}$ such that

$$\mathrm{Tr}_H(Y\psi'\psi^*) = \psi^* Y \psi', \quad \psi, \psi' \in M.$$

The norm of this mapping is $\leq \|Y\|$ and $\mathrm{Tr}_H(YT) = \mathrm{Tr}_H(TY)$. As the notation suggests, we are taking a partial trace relative to H.

5.1 Lemma. *Let $\mathcal{C}$ be a $\mathbb{Z}$–graded *–subalgebra of $\mathcal{O}_K$, i.e. $\mathcal{C}$ is generated by the subspaces $\mathcal{C}^k := \mathcal{C} \cap {\mathcal{O}_K}^k$. Then $\mathcal{C}^k \subset H$ for some k and some Hilbert space H in $\mathcal{O}_K$ of dimension > 1 implies $\mathcal{C}^{nk} = 0$ for $n \in \mathbb{Z} \neq 0$.*

Proof. If $n \in \mathbb{N}$, then $\mathcal{C}^{nk} \subset H^n$ so it suffices to show $\mathcal{C}^k = 0$. Let $\psi \in \mathcal{C}^k$, then $\psi\psi\psi^* \in \mathcal{C}^k \subset H$. Hence $\psi^*\psi\psi\psi^* \in \mathbb{C}I$, but $\psi\psi^*$ is a multiple of a minimal projection in (H, H) so $\psi = 0$.

In our applications, the $\mathbb{Z}$–grading will be that introduced above and $H = K^k$. We next give results on computing the relative commutant of certain Hilbert spaces of support I in the Cuntz algebra. This amounts to the same

thing as determining the fixed points under the inner endomorphism generated by the Hilbert spaces in question. If λ is an endomorphism of the Cuntz algebra, then (λ, λ) is just the relative commutant of $\lambda(K)$ and we give our result in a generality to include computing certain spaces of intertwining operators. We consider Hilbert spaces of support I that are *algebraic* in the sense that they are contained in some (K^r, K^{r+g}). g will be referred to as the grade of the Hilbert space and we shall only consider the case that $g \geq 1$. Indeed if K is finite–dimensional $g \geq 0$ and $g = 0$ implies that the Hilbert space has dimension one. The minimal value for r will be referred to as the rank of the Hilbert space. If K is finite dimensional, then every such Hilbert space is of the form WK, where $W \in (K^{r+1}, K^{r+g})$ is an isometry. In infinite dimensions, we need to consider coisometries W, too. As these Hilbert spaces have a fixed grade, the endomorphisms they generate commute with α_λ and their fixed–point algebras are graded C^*–subalgebras of $\mathcal{O}_K$. The basic observation is that if L and M are such Hilbert spaces of grade g and rank q and r, respectively, then

$$L^*(K^m, K^{m+k})M \subset (K^{m-1}, K^{m-1+k}), \quad m \geq q - k + g, r + g, {}^1$$

$$L^*(K^m, K^{m+k})M \subset (K^r, K^{r+k}), \quad r + g \geq m, q - r \leq k,$$

$$L^*(K^m, K^{m+k})M \subset (K^{q-k}, K^q), \quad q - k + g \geq m, q - r \geq k.$$

This will be used in the following way. Pick $\varphi \in L$ and $\psi \in M$ of norm ≤ 1 and consider the linear mapping $X \mapsto \varphi^* X \psi$ of norm ≤ 1 on $\mathcal{O}_K$. Let Φ denote a limit point of iterates of such mappings in the pointwise weak operator topology of some locally normal representation of $\mathcal{O}_K$ on some Hilbert space $\mathcal{H}$. A priori Φ maps $\mathcal{O}_K$ into $\mathcal{B}(\mathcal{H})$, however since the subspaces of the form (K^m, K^n) are weak operator closed in such representations[2], Φ will map $\mathcal{O}_K^k$ into (K^r, K^{r+k}) or (K^{q-k}, K^q) according as $k \geq (q - r)$ or $k \leq (q - r)$. Hence Φ maps $\mathcal{O}_K$ into itself.

We now consider the following intertwining problem. Given a bounded linear mapping $Y \in (M, L)$ of norm ≥ 1, find the set $\mathcal{C}$ of elements X of $\mathcal{O}_K$ satisfying one of the following equivalent conditions

a) $X\psi = Y\psi X, \quad \psi \in M,$

b) $\psi'^* X\psi = \psi'^* Y\psi X, \quad \psi \in M, \ \psi' \in L,$

c) $\psi'^* X = X\psi'^* Y, \quad \psi' \in L,$

---

[1] For convenience, we regard expressions involving the compositions of linear spaces as referring to the norm closed linear span.

d) $X^* \psi' = Y^* \psi' X^*, \quad \psi' \in L,$

e) $X = Y \rho_M(X).$

Notice that as we have chosen Hilbert spaces L and M of equal grade, $\mathcal{C}$ is stable under the automorphisms α_λ defining the $\mathbb{Z}$–grading. To compute $\mathcal{C}$ it therefore suffices to compute $\mathcal{C}^k$ for each k. The first step is to use an appropriate mapping Φ. If we can pick $\varphi \in L$ and $\psi \in M$ of norm ≤ 1 such that $\varphi^* Y \psi = I$, then by b), the map $X \mapsto \varphi^* X \psi$ leaves $\mathcal{C}$ pointwise invariant, and letting Φ be a limit point of iterates of this particular mapping, we conclude that

$$\mathcal{C}^k \subset (K^r, K^{r+k}), \quad k \geq (q - r),$$

$$\mathcal{C}^k \subset (K^{q-k}, K^q), \quad k \leq (q - r).$$

In general, we could define Φ to be a pointwise weak operator limit point of $X \mapsto \varphi_n^{*n} X \psi_n^n$, where the φ_n and ψ_n are chosen of norm ≤ 1 such that $(\varphi_n^* Y \psi_n)^n \to I$ as $n \to \infty$, this being possible since $\|Y\| \geq 1$ by assumption. But we want to go further and reduce the problem of computing $\mathcal{C}^k$ to a purely local problem.

5.2 Proposition *Let L and M be algebraic Hilbert spaces of equal grade and rank q and r, respectively and $Y \in (M, L)$ of norm ≥ 1. Let $\mathcal{C}$ denote the set of $X \in \mathcal{O}_K$ such that*

$$X \psi = Y \psi X, \quad \psi \in M,$$

then if $k \geq q - r$, $X \in \mathcal{C}^k$ if and only if $X \in (K^r, K^{r+k})$ and one of the following equivalent conditions hold

a) $X \vartheta(K^r, M) = Y \vartheta(K^{r+k}, M) X.$

b_n) $X \operatorname{Tr}_{M^n}(T Y^{\times n} \vartheta(K^r, M)) = \operatorname{Tr}_{L^n}(Y^{\times n} T Y \vartheta(K^{r+k}, M)) X, T \in L^1(L^n, M^n).$

Here L^1 is used to denote the set of trace class operators. If $k \leq q - r$, then $X \in (K^{q-k}, K^q)$ and we need only replace r by $q - k$ in the above.

Proof. We have already seen that $X \in (K^r, K^{r+k})$ if $k \geq q - r$ and a) now follows noting that:

$$X \vartheta(K^r, M) = Y \rho_M(X) \vartheta(K^r, M) = Y \vartheta(K^{r+k}, M) X.$$

Conversely, a) implies $X = Y \rho_M(X)$ since $X \in (K^r, K^{r+k})$. Now a) also implies that

$$X \psi'^* Y \vartheta(K^r, M) \psi = \psi'^* Y \vartheta(K^{r+k}, M) Y \psi X, \quad \psi' \in L, \ \psi \in M.$$

This is b_1) for rank one operators T and hence equivalent to b_1). On the other hand, a) follows from b_1) since M and L have support one. The same argument shows that b_n) is equivalent to b_{n-1}), completing the proof.

Let us comment on these conditions: a) is a simple canonical condition that already serves to make the basic point that $\mathcal{C}^k$ is determined by intertwining conditions between fixed tensor powers of K and is in this sense algebraic. However the permutation operators map between higher tensor powers of K than is really necessary if $X \in (K^r, K^{k+r})$ is to intertwine. By using the partial trace, we can reduce the powers of the tensor spaces involved at the cost of increasing the number of intertwining relations. In fact, for n sufficiently large, the operators involved on the left hand side are in (K^r, K^r) and those on the right hand side in (K^q, K^q).

In concrete cases, the following strategy for computing $\mathcal{C}^k$ proves useful. Let $X \in \mathcal{C}^k$ and $V \in (K, M)$; in practice, V can usually be picked unitary. Then

$$\vartheta(K^n, M)\rho^n(V) = V\vartheta(K^n, K), \quad n \in \mathbb{N}_0,$$

where we have written ρ for ρ_K. Since $X \in (K^r, K^{r+k})$, $X\rho^r(V) = \rho^{r+k}(V)X$, and using a) of Proposition 5.2, we get

$$XV\vartheta(K^r, K) = YV\vartheta(K^{r+k}, K)X.$$

If V has a right inverse, we can, conversely, deduce a) of Proposition 5.2 from this equation. If wished, the permutations operators can be eliminated in favour of the endomorphism ρ. In fact, since $\vartheta(K^{r+k}, K)X = \rho(X)\vartheta(K^r, K)$, we get

$$XV = YV\rho(X),$$

but this is best derived directly from d), above. Similarly, if $U \in (K, L)$, we may conclude that

$$UX = \rho(X)UY.$$

This is equivalent to a) of Proposition 5.2, if V has a left inverse.

After these results, let us try and clarify whether more might be expected by relating the set of solutions to questions posed entirely in terms of a category of Hilbert spaces and hence independent of the identifications used to define the Cuntz algebra.

In place of $\mathcal{O}_K$ we consider a category $\mathcal{K}$ of Hilbert spaces with objects $\mathbb{N}_0$ and equipped with a shift F to be thought of as the tensor powers of a Hilbert space K, as described in detail in Section 2. Instead of considering a

Hilbert spaces $L \subset (K^q, K^{q+g})$ and $M \subset (K^r, K^{r+g})$, we consider another such category $\mathcal{L}$ with a shift G and two normal $*$–functors J and J' from $\mathcal{L}$ to $\mathcal{K}$ such that $JG = F^g J$, $J'G = F^g J'$ and $J(0) = q$ and $J'(0) = r$. We consider natural transformations $t \in (J', J) \cap (\hat{F}, \hat{F})$. As we know from computations in Section 2, such a natural transformation is uniquely determined by $t_0 := X \in (K^q, K^r)$ satisfying

$$F^g(X) = \sum_j J(\psi_j) \circ X \circ J'(\psi_j)^*,$$

where the sum is taken over an orthonormal basis of M. If we write this in the Cuntz algebra we obtain our condition e), $X = Y \rho_M(X)$, where $Y = \sum_j J(\psi_j) J'(\psi_j)^*$ in the Cuntz algebra and is unitary. There is no difficulty in generalizing to include cases where Y is not unitary.

We learn from this that it is quite natural to expect solutions of grade $r - q$. Furthermore, by composing J or J' with tensoring on the right by 1_K, we can replace q by $q+1$ or r by $r+1$. Thus we have potential solutions for any grade. We see, therefore that the identifications involved in defining the Cuntz algebra mean that one problem at the level of the Cuntz algebra involves a countable set of problems at the level of $\mathcal{T}_K$.

We conclude that the results obtained using the spaces (K^m, K^n) in the Cuntz algebra are the best that can be expected in complete generality. However, we now show how the estimates on the localization of $\mathcal{C}^k$ can be improved under conditions involving the relative localization of YM and K^g or Y^*L and K^g. Note that $M^{n*} K^{gn} \subset (K^r, K^r)$ for all n.

5.3 Lemma *Let m denote the smallest integer $\geq \frac{r}{g}$ and ℓ the smallest integer $\geq \frac{q}{g}$. If the weak operator closed linear span of the $(L^*Y)^n K^{gn}$, $n \geq m$, in (K^r, K^r) contains I then $\mathcal{C}^k \subset K^k$ for $k \geq q$ and $\mathcal{C}^k \subset (K^{q-k}, K^q)$ for $q \geq k \geq q - r$. Similarly, if the weak operator closed linear span of the $(M^*Y^*)^n K^{gn}$, $n \geq \ell$, in (K^q, K^q) contains I then $\mathcal{C}^k \subset K^{*-k}$ for $k \leq -r$ and $\mathcal{C}^k \subset (K^r, K^{r+k})$ for $-r \leq k \leq q - r$.*

Proof. We know that $\mathcal{C}^k \subset (K^r, K^{r+k})$ if $k \geq q - r$. But L^n has rank q and grade ng and K^{ng} has rank 0 and grade ng. Thus from previous computations

$$L^{n*} \mathcal{C}^k K^{ng} \subset K^k, \quad k \geq q, \ ng \geq r,$$

$$L^{n*} \mathcal{C}^k K^{ng} \subset (K^{q-k}, K^q), \quad q \geq k \geq q - r, \ ng \geq r.$$

But if $X \in \mathcal{C}$,

$$L^{n*} X K^{gn} = X(L^*Y)^n K^{gn}.$$

Thus if the weak operator closed linear span of the $(L^*Y)^n K^{gn}$ contains I, X will be in the weak operator closed linear span of the $L^{n*}XK^{gn}$ and the first part follows. The second part can be proved similarly or deduced from the first by using X^* and Y^* in place of X and Y.

Recalling Lemma 5.1 at this point, we get the following corollary.

5.4 Corollary *Suppose $L = M$ and Y is a projection or a unitary. Let m denote the smallest integer $\geq \frac{q}{g}$ and suppose the weak operator closed linear span of the $(L^*Y)^n K^{gn}$, $n \geq m$, in (K^r, K^r) contains I, then $\mathcal{C}^k = 0$ for $k \geq q$ and $k \leq -q$ and $\mathcal{C}^k \subset (K^{q-k}, K^q)$ for $q \geq k \geq 0$.*

Proof. When Y is a projection, we need only remark that $\mathcal{C}$ is a $\mathbb{Z}$–graded *–subalgebra of $\mathcal{O}_K$. If Y is unitary, then $X \in \mathcal{C}^k$ implies $XXX^* \in \mathcal{C}^k$ and this is all that is used in Lemma 5.1.

To give a simple example, $\rho^r(K)$ is an algebraic Hilbert space of grade one and rank r. Its relative commutant is (ρ^r, ρ^r) which was shown in [8], using techniques similar to those above, to be equal to (K^r, K^r). In this case, $\rho^r(K^{n*})K^n$ is the space of finite rank operators on K^r for $n \geq r$ and the space of compact operators from $\rho^{r-n}(K^n)$ to K^n if $1 \leq n \leq r$.

The theory of multiplicative unitaries provides us with further examples of algebraic Hilbert spaces L and M of equal grade, where the weak operator closure of L^*M and hence of M^*L contains I. For example, if $V \in (K^2, K^2)$ is a regular multiplicative unitary then the weak operator closures of K^*VK, $K^*\vartheta V\vartheta K$ and $K^*V\vartheta K$ in (K, K) are even *–algebras containing the unit [1]. Of course, K could be replaced here by any other algebraic Hilbert space of support I. Note that if L_i and M_i are algebraic Hilbert spaces such that I is in the weak operator closure of $L_i^*M_i$, $i = 1, 2$, then I is also in the weak operator closure of $L_1^*L_2^*M_2M_1$.

Let us now return to the special case used to motivate our basic intertwining relation. If we take $Y = 1_M$ then $\mathcal{C}$ is just the relative commutant of M and it is of interest to ask when M has trivial relative commutant. $\mathcal{C}^0$ will reduce to the complex numbers by b_1) of Proposition 5.2 if $M^*\vartheta(K^r, M)M$ has trivial commutant in (K^r, K^r). We again have examples with $r = 1$ drawn from the theory of a regular multiplicative unitary and this leads to the following result.

5.5 Proposition *Let V be a regular multiplicative unitary in K^2, then the following Hilbert spaces have trivial relative commutant in $\mathcal{O}_K$: V^*K and $\vartheta V\vartheta K$. In the case of the Hilbert spaces ϑVK and $V^*\vartheta K$, the relative commutants are the commutants of K^*VK and $K^*\vartheta V^*\vartheta K$ in (K, K), respectively.*

Proof. We need only remark that in each case we know that $\mathcal{C}^k = 0$ for $k \geq 1$.

Furthermore, $\mathcal{C}^0$ is, in each case, as claimed since, for a Hilbert space of the form UK with $U \in (K^2, K^2)$ unitary, b_1 of Proposition 5.2 just reduces to saying that $\mathcal{C}^0$ is the commutant of the first component of $U\vartheta$.

Following [1], we denote K^*VK and $K^*\vartheta V^*\vartheta K$ by $\mathcal{A}(V)$ and $\hat{\mathcal{A}}(V)$, respectively. If V is a regular multiplicative unitary, these algebras are actually $*$–algebras [1].

We now come to the second application we had in mind, namely to study intertwiners between certain endomorphisms of the Cuntz algebra. We say that an endomorphism τ has grade g if $\tau(K)$ has grade $g+1$ and is algebraic of rank r if $\tau(K)$ is algebraic of rank r. There is a unique unitary V such that $\tau(\psi) = V\psi$, for $\psi \in K$ and τ has rank r if and only if r is the smallest integer such that $V \in (K^{r+1+g}, K^{r+1+g})$. If σ is an algebraic endomorphism of grade f and rank q then the composition $\sigma\tau$ is of grade $f + g$ and rank $\leq q + r + g + (r + g)f$.

Now suppose that we have endomorphisms σ and τ as above of equal grade g, then the space of intertwiners (τ, σ) is $\mathbb{Z}$–graded and if $X \in (\tau, \sigma)$, then

$$X\psi = Y\psi X, \quad \psi \in \tau(K),$$

where $Y \in (\tau(K), \sigma(K))$ is the unitary taking $\tau(\psi)$ to $\sigma(\psi)$ for each $\psi \in K$. Thus the analysis of Proposition 5.2 holds. In particular, we have

$$(\tau, \sigma)^k \subset (K^r, K^{r+k}), \quad k \geq q - r,$$

$$(\tau, \sigma)^k \subset (K^{q+k}, K^q), \quad k \leq q - r.$$

6 An Algebraic Version of Takesaki–Tatsuuma Duality

After these results on algebraic Hilbert spaces and endomorphisms, our aim is to describe an algebraic model for a dual of a multiplicative unitary. We present a duality result for 'locally compact' multiplicative unitaries in terms of the C^*–algebra generated by the regular representation considered as an object in the tensor C^*–category of representations of the multiplicative unitary. Thus we get, in particular, an algebraic version of a duality result for the representation categories of locally compact groups.

We recall that $\mathcal{O}_H$ is a simple C^*–algebra and every unitary operator $X \in (H, H')$ extends to a unital morphism $\mathcal{O}_H \to \mathcal{O}_{H'}$, which we denote by λ_X.

Let V be a regular multiplicative unitary acting on K^2 and W a representation contained in $M((H, H) \otimes \mathcal{A}(V))$, the multiplier algebra of the minimal

tensor product $(H, H) \otimes \mathcal{A}(V)$. Let us identify $\mathcal{A}(V)\rho_K(\mathcal{O}_H) = \mathcal{A}(V) \otimes \mathcal{O}_H$. Then

$$\lambda_{\vartheta_{H,K}W}(H^r, H^s)\mathcal{A}(V) + \mathcal{A}(V)\lambda_{\vartheta_{H,K}W}(H^r, H^s) \subset \mathcal{A}(V)\rho_K(H^r, H^s).$$

Here in the definition of $\lambda_{\vartheta_{H,K}W}$ we consider H and K as Hilbert spaces of support I in some $\mathcal{B}(\mathcal{H})$ and regard $\vartheta_{H,K}W$ as mapping H onto $\vartheta_{H,K}WH$. The arguments of [5] or [14]; Section 6 generalize to show that the monomorphism $\lambda_{\vartheta_{H,K}W} : \mathcal{O}_H \to M(\mathcal{A}(V) \otimes \mathcal{O}_H)$ defines a coaction of $\mathcal{A}(V)$ on $\mathcal{O}_H$, i.e. $\lambda_{\vartheta_{H,K}W}$ is a unital *–homomorphism satisfying

$$\delta \otimes i \circ \lambda_{\vartheta_{H,K}W} = i \otimes \lambda_{\vartheta_{H,K}W} \circ \lambda_{\vartheta_{H,K}W} \ ,$$

where δ is the coproduct of $\mathcal{A}(V)$ induced by the adjoint action of $\vartheta_{K,K}V$ [1]. The corresponding fixed point algebra is

$$\mathcal{O}^W = \{T \in \mathcal{O}_H : \lambda_{\vartheta_{H,K}W}(T) = \rho_K(T)\} \ .$$

Direct computations show that $\mathcal{O}^W \cap (H^r, H^s) = (W^{\times r}, W^{\times s})$. We recall that any object W in a tensor C^*–category $\mathcal{T}$ has a canonically associated C^*–algebra $\mathcal{O}_W$ with a unital endomorphism ρ_W[7]. This construction can be used in two quite different ways. On the one hand it provides one with a large class of model endomorphisms with rather well defined properties. In favourable cases, the associated *–functor F_W from $\mathcal{T}_W$, the tensor C^*–category whose objects are the tensor powers of W with arrows taken from $\mathcal{T}$, to the tensor C^*–category whose objects are the powers of ρ_W and whose arrows are intertwining operators is even an isomorphism. This illustrates the second aspect of the construction that it also encodes properties of the object W in question. W is C^*–amenable when F_W is an isomorphism, in the terminology of [13], where related notions of amenability are discussed.

Now $\mathcal{T}_W$ carries an automorphic action of the circle group $\mathbb{T}$ defined by

$$\alpha_\lambda(T) := \lambda^{s-r}T, \quad T \in (W^r, W^s),$$

and the induced automorphic action of $\mathbb{T}$ on $(\mathcal{O}_W, \rho_W)$ is also denoted by α. The spectral subspaces of the action make $\mathcal{O}_W$ into a $\mathbb{Z}$–graded C^*–algebra:

$$\mathcal{O}_W{}^k = \{T \in \mathcal{O}_W : \alpha_\lambda(T) = \lambda^k T \ , \quad \lambda \in \mathbb{T}\} \ .$$

The construction is functorial so, given a *–functor F from $\mathcal{T}$, it yields a morphism $F_* : \mathcal{O}_W \to \mathcal{O}_{F(W)}$ of C^*–algebras intertwining the canonical endomorphisms and the actions of $\mathbb{T}$. In particular, if we have a faithful functor

into a tensor C^*–category of Hilbert spaces as is the case for the categories $\mathcal{R}(V)$ or $\mathcal{C}(V)$, then it yields an inclusion $\mathcal{O}_W \subset \mathcal{O}_H$ of C^*–algebras such that $\rho_H \upharpoonright \mathcal{O}_W = \rho_W$. Here $H = F(W)$ is the Hilbert space of W. If $\mathcal{O}_W$ has trivial relative commutant in $\mathcal{O}_H$ then the group of automorphisms of $\mathcal{O}_H$ leaving $\mathcal{O}_W$ pointwise fixed can be identified with

$$G_W := \{U \in \mathcal{U}(H) : TU^{\times r} = U^{\times s}T, \quad T \in (W^{\times r}, W^{\times s}), \ r, s \in \mathbb{N}\}.$$

Furthermore,
$$(\rho_W^r, \rho_W^s) = (H^r, H^s) \cap \mathcal{O}_W.$$

Returning to the fixed point algebra under the above action, we have $\mathcal{O}_W \subseteq \mathcal{O}^W$. As in the case of a group action, equality follows from an amenability condition on V .

6.1 Theorem *Let $W \in M((H, H) \otimes \mathcal{A}(V))$ be a unitary representation of V and suppose that there is an invariant mean m on $(\mathcal{A}(V), \delta)$. Then there is a conditional expectation $E : \mathcal{O}_H \to \mathcal{O}^W$ satisfying $E(H^r, H^s) = (W^{\times r}, W^{\times s})$. In particular, $\mathcal{O}_W = \mathcal{O}^W$.*

Proof. We extend m to the multiplier algebra of $\mathcal{A}(V)$ via strict continuity. Let ω be a normal state of some faithful representation π of $\mathcal{O}_{\mathcal{H}}$ where $\pi(H)$ has support I. Then $i \otimes \omega$ induces a strictly continuous positive map from $M(\mathcal{A}(V) \otimes \mathcal{O}_H)$ to $M(\mathcal{A}(V))$, and setting

$$\omega(E(T)) := m \circ i \otimes \omega \circ \lambda_{\vartheta_{H,K}W}(T) , \quad T \in \mathcal{O}_H ,$$

gives a positive map E of norm one from $\mathcal{O}_H$ to $\mathcal{B}(\mathcal{H}_\pi)$ satisfying $E(AT) = AE(T)$, $A \in \mathcal{O}^W$, $T \in \mathcal{O}_H$. Now the arguments of [14]; Proposition 6.5, except that $\rho_K(H^s)^* \lambda_{\vartheta_{H,K}W}(H^r, H^s) \rho_K(H^r) \subseteq M(\mathcal{A}(V))$ here, show that E is the desired conditional expectation.

In particular, if V is compact and $T \in K$ is a fixed normalized vector then $m(A)I = T^*AT$, $A \in \mathcal{A}(V)$, is the unique Haar measure on $\mathcal{A}(V)$, the conditional expectation corresponding to the representation W is $E(X) = T^* \lambda_{\vartheta_{H,K}W}(X)T$.

6.2 Theorem *Let V be a regular multiplicative unitary, then V is C^*–amenable as an object of $\mathcal{R}(V)$, i.e.*

$$(\rho_V^r, \rho_V^s) = (V^r, V^s), \quad r, s \in \mathbb{N}_0.$$

Proof. The pentagon equation expressing the multiplicativity of V is equivalent to $V^*K \subset (V, V^2)$. V^*K has trivial relative commutant, by Proposition 5.5, if V is regular. Thus $\mathcal{O}'_V \cap \mathcal{O}_K = \mathbb{C}$ and, consequently,

$$(\rho^r_V, \rho^s_V) = (K^r, K^s) \cap \mathcal{O}_V.$$

On the other hand, a computation, cf. §6 of [15], shows that

$$(V^r, V^s) = (K^r, K^s) \cap \mathcal{O}^V,$$

where $\mathcal{O}^V = \{X \in \mathcal{O}_K : \lambda_{\vartheta V}(X) = \rho_K(X)\}$.

It is not clear whether V is C^*–amenable as an object of $\mathcal{C}(V)$, when V is regular. The analogous proof does not work. The pentagon equation is now equivalent to

$$V \rho_K(K) = V \vartheta K \subset (V, V^2).$$

But if V is regular, the relative commutant of $V \vartheta K$ is, by Proposition 5.5, the commutant of $K^* \vartheta V \vartheta K$, the first component of V, in (K, K). This difference between $\mathcal{C}(V)$ and $\mathcal{R}(V)$ relates to the alternative definition of tensor product pointed out in the introduction. With the alternative tensor product for $\mathcal{C}(V)$, we would find $\vartheta V \vartheta K \subset (V, V^2)$ and $\vartheta V \vartheta K$ does have trivial relative commutant in $\mathcal{O}_K$.

In virtue of Theorem 6.2, the model endomorphism $(\mathcal{O}_V, \rho_V)$ is a natural candidate for a dual of V. So we pose the following question. When does a pair $(\mathcal{A}, \hat{\rho})$ consisting of a unital C^*–algebra and a unital endomorphism arise from a system of the form $(\mathcal{O}_V, \rho_V)$?

At the same time we have seen that the Cuntz algebra allows a simple description of a large variety of interesting model endomorphisms, giving rise to systems of the form $(\mathcal{O}_K, \lambda_R)$, where $\mathcal{O}_K$ is the (extended) Cuntz algebra over the Hilbert space K and λ_R the algebraic endomorphism determined by the unitary operator R, $R \in (K^{r+1}, K^{r+1+g})$ for an algebraic endomorphism of grade g and rank r. In fact, we would like our model systems to combine two features: they should be of the form $(\mathcal{O}_\rho, \hat{\rho})$, where ρ is an object in a tensor C^*–category. This means that $\mathcal{O}_\rho$ is generated by the intertwiners between the powers of $\hat{\rho}$. The second feature is that there should be a Hilbert space K of intertwiners between powers of $\hat{\rho}$.

Now the Cuntz algebra and the extended Cuntz algebra $\mathcal{O}_K$ are derived from the tensor W^*-category of Hilbert spaces whose objects are the tensor powers of K. If we begin, as in previous sections, simply with a W^*–category of Hilbert spaces $\mathcal{K}$ we need a shift F to give $\mathcal{O}_K$. Giving a second commuting

shift G amounts to giving an algebraic endomorphism λ_R of grade zero and rank one. Here $R \in (K^2, K^2)$ is determined by $G(\psi) = R \circ F(\psi)$, $\psi \in K$. Natural transformations yield intertwiners of endomorphisms: more precisely, if $t \in (G^r, G^s) \cap (\hat{F}^r, \hat{F}^s)$ then $t_0 \in (\lambda_R^r, \lambda_R^s)$ but it is not clear whether all intertwiners arise in this way.

Looking at Theorem 4.4 in this light gives us the following result.

6.3 Theorem *Let V be a multiplicative unitary on K^2, viewed as an element of $\mathcal{O}_K$. Let $H := V^* K$ and let λ^* be the induced isomorphism of $\mathcal{O}_K$ onto $\mathcal{O}_H \subset \mathcal{O}_K$, then $\rho \lambda^* = \lambda^* \lambda_R$, where $R = V\theta$ and ρ is the canonical endomorphism of $\mathcal{O}_K$.*

Remark In fact, $\mathcal{O}_H = \mathcal{O}_V$ since $(V^r, V^s) \times 1_K \subset (H^r, H^s)$ for $r, s > 0$ but this still leaves the finer details open on how the intertwiner spaces (V^r, V^s) sit in $\mathcal{O}_H$.

We now take up the situation of Theorem 4.9. Thus we suppose that $\mathcal{T}_\rho$ is a tensor W^*–category whose objects are the powers of ρ with a Hilbert space K of support one in (ρ^r, ρ^{r+g}) and suppose that $(\rho^r, \rho^r) \times 1_{\rho^{rg}} \subset (K^r, K^r)$. We now compare C^*–algebra $\mathcal{O}_\rho$ with the Cuntz algebra $\mathcal{O}_K$. Obviously, in constructing $\mathcal{O}_K$, we use only spaces of arrows between objects of the form ρ^{r+ng} with $n \in \mathbb{N}_0$. But these spaces define the C^*–algebra $\mathcal{O}_{\rho^g}$. If we use $\hat{\rho}$ to denote the endomorphism of $\mathcal{O}_{\rho^g}$ induced by tensoring on the left by 1_ρ, then the estimates of Theorem 4.9 show that $\hat{\rho}(K) \subset (K^{r+1}, K^{r+2})$ in $\mathcal{O}_{\rho^g}$ and that $\mathcal{O}_{\rho^g}$ can be identified canonically with $\mathcal{O}_K$. Hence there is a unitary $R \in (K^{r+2}, K^{r+2})$ such that $R\psi = \hat{\rho}(\psi)$, $\psi \in K$. In other words $\hat{\rho}$ can be identified with λ_R, an algebraic endomorphism of grade zero and rank $\leq r + 1$. This estimate on the rank is only an upper bound. In fact, if K is ambidextrous, then $\hat{\rho}^g(K) \subset (K, K^2)$. We summarize the discussion as follows.

6.4 Proposition *Let ρ be an object in a tensor W^*–category and K a Hilbert space of support one in (ρ^r, ρ^{r+g}) with $(\rho^r, \rho^r) \times 1_{\rho^{rg}} \subset (K^r, K^r)$. Then $(\mathcal{O}_{\rho^g}, \hat{\rho}) = (\mathcal{O}_K, \lambda_R)$, where $R \in (K^{r+2}, K^{r+2})$ is the unitary operator in $\mathcal{O}_\rho$ such that $R\psi = \hat{\rho}(\psi)$, for all $\psi \in K$.*

Returning to the question of when a pair $(\mathcal{A}, \hat{\rho})$ consisting of a unital C^*–algebra and a unital endomorphism arises from a system of the form $(\mathcal{O}_V, \rho_V)$, where V is a multiplicative unitary, we shall see that the following conditions are necessary and sufficient:

a) $\mathcal{A}$ is generated by a tensor W^*–category which is a tensor subcategory of the category of intertwiners between the powers of $\hat{\rho}$. The generating object will be denoted by ρ;

b) there is a Hilbert space of support I, $K \subset (\rho, \rho^2)$;

c) $\rho(K) \subset (K, K^2)$;

d) $(\rho, \rho) \subset (K, K)$.

We shall see in Theorem 6.11 that $\mathcal{O}_\rho$ is simple, so, in virtue of a), $(\mathcal{A}, \hat{\rho})$ is the system $(\mathcal{O}_\rho, \hat{\rho})$ associated with the object ρ of the tensor W^*–category.

We recognize that c) just says that λ_R of Proposition 6.4 has grade zero and rank ≤ 1, or equivalently that $R \in (K^2, K^2)$. b), on the other hand, tells us that $\lambda_R \lambda_R = \rho_K \lambda_R$, or, equivalently that $V := R\vartheta$ is a multiplicative unitary. If V is even regular then c) can be strengthened to $K^* \rho(K) = KK^*$.

Notice that, as a consequence of Theorem 6.3 and the following remark, $(\mathcal{O}_V, \rho_V)$ satisfies a) to d) above relative to the Hilbert space $H := V^*K$.

As Theorem 6.3 establishes the isomorphism of $(\mathcal{O}_V, \rho_V)$ and $(\mathcal{O}_K, \lambda_R)$, $(\mathcal{O}_K, \lambda_R)$ satisfies a) to d), above, too. Indeed, except for d), this is easily seen directly. However, (λ_R, λ_R) being just the relative commutant of the Hilbert space $\lambda_R(K) = RK$, we do get $(\lambda_R, \lambda_R) \subset (K, K)$ by Corollary 5.4 if V is regular. We now prove our first duality result.

6.5 Theorem *Let $(\mathcal{A}, \hat{\rho})$ satisfy a) to d) then there is a unique multiplicative unitary V on the Hilbert space K^2, such that $(\mathcal{A}, \hat{\rho}, K) = (\mathcal{O}_V, \rho_V, V^*K)$, where V is regarded as a representation of V. Two systems of the form $(\mathcal{O}_V, \rho_V, V^*K)$ for multiplicative unitaries V on K^2 are isomorphic if and only if the multiplicative unitaries are equivalent.*

Proof. By Proposition 6.4, there is a unitary $V \in \mathcal{A}$ such that $(\mathcal{A}, \hat{\rho}, K) = (\mathcal{O}_K, \lambda_{V\vartheta_{K,K}}, K)$. But c) implies that $V \in (K^2, K^2)$ and b) may be read as $\hat{\rho}^2 = \rho_K \hat{\rho}$ so that V is a multiplicative unitary on K^2 [5],[15]. By Theorem 6.3 again, we can consider isomorphisms of systems of the form $(\mathcal{O}_K, \lambda_R, K)$. So if τ is an isomorphism from $\mathcal{O}_K$ to $\mathcal{O}_{K'}$ with $\tau(K) = K'$ and $\tau \circ \lambda_R = \lambda_{R'} \circ \tau$. Then $\tau(R) = R'$ and, since $\tau(\vartheta_{K,K}) = \vartheta_{K',K'}$, $\tau(V) = V'$. So if U is the unitary from K to K' such that $\tau(\psi) = U\psi$, for $\psi \in K$, its second tensor power will intertwine V and V', realizing the desired equivalence. The converse is obvious.

In the case of a regular multiplicative unitary we can even obtain a categorical rather than an algebraic duality theorem. The following result characterizes tensor W^*–categories of the form $\mathcal{T}_V$ for a regular multiplicative unitary V.

6.6 Theorem *Given a tensor W^*–category $\mathcal{T}_\rho$ whose objects are the tensor powers of a C^*-amenable object ρ, suppose there is an ambidextrous Hilbert*

space K of support one in (ρ, ρ^2) with $(\rho, \rho) \times 1_\rho \subset (K, K)$ and

$$K^* \times 1_\rho \circ 1_\rho \times K = K \circ K^*$$

then the unitary V on K^2 defined by

$$V \circ \psi_1 \times 1_\rho \circ \psi_2 = 1_\rho \times \psi_2 \circ \psi_1, \quad \psi_1, \psi_2 \in K,$$

is multiplicative and $\mathfrak{T}_\rho$ and $\mathfrak{T}_V$ are isomorphic.

Proof. The fact that V is multiplicative follows from Theorem 4.2 and the condition $K^* \times 1_\rho \circ 1_\rho \times K = K \circ K^*$ just says that K is regular. We now consider the image of $\mathfrak{T}_\rho$ in $\mathcal{O}_\rho$ under the canonical map, then the conditions a) to d) above are satisfied by $\mathcal{O}_\rho$ and we conclude from Theorem 6.5 that $(\mathcal{O}_\rho, \hat{\rho}, K) = (\mathcal{O}_V, \rho_V, V^*K)$. Now ρ is C^*–amenable by assumption and V is C^*–amenable by Theorem 6.2 and, with an obvious notation, $\mathfrak{T}_{\hat\rho}$ and $\mathfrak{T}_{\rho_V}$ are isomorphic hence $\mathfrak{T}_\rho$ and $\mathfrak{T}_V$ are isomorphic.

We now want to give an algebraic characterization of the situation where the multiplicative unitary V on K^2 is endowed with a standard braided symmetry ε, a concept explained in the appendix. In this case the system $(\mathcal{O}_{\hat{V}}, \rho_{\hat{V}}, \hat{V}^*K)$, with $\varepsilon = V \vartheta_{K,K} \hat{V}$, satisfies conditions a) to d). As pointed out in the appendix, $(\mathcal{O}_{\hat{V}}, \rho_{\hat{V}}) = (\mathcal{O}_V, \rho_V)$, where V is the regular corepresentation and $\hat{V}$ the regular representation. A direct computation shows that the condition for being standard, namely that $\hat{V}$ and V_{23} commute on (K^3, K^3), is equivalent to $\rho_V(\varepsilon\psi) = \rho_H(\varepsilon\psi), \psi \in H := \hat{V}^*K$. We start with the following result.

6.7 Theorem *Let $(\mathcal{A}, \rho)$ be a pair consisting of a C^*–algebra and a unital endomorphism satisfying a) to d). Let ε be a braided symmetry for ρ satisfying*

e) $\rho(\varepsilon\psi) = \rho_K(\varepsilon\psi)$, $\quad \psi \in K$.

Then there is a multiplicative unitary V on a Hilbert space H whose regular corepresentation V is endowed with a standard braided symmetry ε_V of V and an isomorphism $\Phi : \mathcal{A} \to \mathcal{O}_V$ such that

1) $\Phi \circ \rho = \rho_V \circ \Phi$;

2) $\Phi(\varepsilon) = \varepsilon_V$;

3) $\Phi(K) = \hat{V}^*H$, *where $\varepsilon_V = V \vartheta \hat{V}$* .

The multiplicative unitary V is determined, up to equivalence, by the above conditions. If ε is a permutation symmetry then $(\mathcal{A}, \rho)$ correponds to a locally compact group G and ε to the usual permutation symmetry.

Proof. By Theorem 6.5 we can find a regular representation $\hat{V}$ acting on a Hilbert space H such that the triples $(\mathcal{A}, \rho, K)$ and $(\mathcal{O}_{\hat{V}}, \rho_{\hat{V}}, \hat{V}^*H)$ are isomorphic via an isomorphism Φ . If we write $\varepsilon_V := \Phi(\varepsilon) = V\vartheta\hat{V}$ then by e) V_{23} and $\hat{V}$ commute on H^3 , so as shown in the appendix, V is multiplicative and ε_V is a standard braided symmetry of V . It will follow from the next proposition that V is unique up to equivalence. If ε is a permutation symmetry then, by Proposition A.4, V is cocommutative, thus coming from a locally compact group G [1].

6.8 Proposition. *Let V and V' be multiplicative unitaries on Hilbert spaces H^2 and H'^2 endowed with standard braided symmetries $\varepsilon_V = V\vartheta_{H,H}\hat{V}$ and $\varepsilon_{V'} = V'\vartheta_{H',H'}\hat{V}'$ respectively. If there is an isomorphism $\Phi : \mathcal{O}_V \to \mathcal{O}_{V'}$ satisfying*

a) $\Phi \circ \rho_V = \rho_{V'} \circ \Phi$;

b) $\Phi(\varepsilon_V) = \varepsilon_{V'}$;

c) $\Phi(K) = K'$, *where $K = \hat{V}^*H$ and $K' = \hat{V}'^*H'$* ,

then V and V' are unitarily equivalent.

Proof. By c), $\Phi \circ \rho_K = \rho_{K'} \circ \Phi$ on $\mathcal{O}_V$. Thus by a) and b) $\Phi \circ \lambda_{\hat{V}^*}(V) = \lambda_{\hat{V}'^*}(V')$ since $\lambda_{\hat{V}^*}(V) = \rho_V(\varepsilon_V^*)\rho_K(\varepsilon_V)$ with a similar result for V' . Now by c) there is a unitary operator $U : H \to H'$ extending to an isomorphism $\alpha : \mathcal{O}_H \to \mathcal{O}_{H'}$ such that $\Phi \circ \lambda_{\hat{V}^*} = \lambda_{\hat{V}'^*} \circ \alpha$, on $\mathcal{O}_H$. It follows that $\alpha(\hat{V}) = \hat{V}'$. Let us define $\tilde{\varepsilon}_V = \hat{V}\varepsilon_V\hat{V}^* = \hat{V}V\vartheta_{H,H}$. Then $\varepsilon_V = \lambda_{\hat{V}^*}(\tilde{\varepsilon}_V)$ since $(V^{\times 2}, V^{\times 2})$ is generated by $K(V,V)K^*$ as a weakly closed subspace and $\hat{V}$ commutes with (V, V) . If we define the operator $\tilde{\varepsilon}_{V'}$ corresponding to V' , as above, then we deduce $\alpha(\hat{V}V\vartheta_{H,H}) = \hat{V}'V'\vartheta_{H',H'}$, from b) and the previous relation intertwining Φ and α. Clearly $\alpha(\vartheta_{H,H}) = \vartheta_{H',H'}$, so $\alpha(V) = V'$. Now the adjoint action of $U \times U$ on (H^2, H^2) implements α, completing the proof.

Remark. We can now complement the discussion following Theorem 6.2. With a standard braided symmetry, $\mathcal{C}(V)$ and $\mathcal{R}(V)$ are isomorphic as tensor W^*–categories embedded in Hilbert spaces. Hence by Theorem 6.2, a V with a standard braided symmetry and $\hat{V}$ regular is C^*–amenable in $\mathcal{C}(V)$. Indeed $\hat{V}^*K \subset (V, V^2)$ has trivial relative commutant in $\mathcal{O}_K$.

We now examine model endomorphisms which are more C^*–algebraic in nature. In fact, when K is infinite dimensional, the above systems are not really C^*–algebraic in nature since $\mathcal{O}_K$ is the norm closure of subspaces endowed with a W^*–topology and is not even separable when K is an infinite dimensional

separable Hilbert space. To cure this defect, we might replace $(\mathcal{O}_K, \lambda_R)$ by $(\mathcal{P}_K, \tau_R)$, where $\mathcal{P}_K$ is the smallest C^*–subalgebra of $\mathcal{O}_K$ containing K and stable under λ_R and τ_R denotes the restriction of λ_R to $\mathcal{P}_K$. Note that, since $K \subset \mathcal{P}_K$,

$$(\tau_R^m, \tau_R^n) = (\lambda_R^m, \lambda_R^n) \cap \mathcal{P}_K.$$

At the same time, we would now like our model endomorphisms to have the form $(\mathcal{O}_\rho, \hat{\rho})$ where ρ is an object in a tensor C^*-category. We still want a Hilbert space K of intertwiners between powers of ρ. We say that a Hilbert space $H \subset (\rho, \sigma)$ in a C^*–category has *zero left annihilator* if $X \circ \psi = 0$ for all $\psi \in H$ implies $X = 0$. It obviously suffices to take X to be a positive element of (σ, σ).

We first prove a result that will imply that the C^*–algebras of interest are simple C^*–algebras.

6.9 Theorem *Let $\mathcal{T}_\rho$ be a tensor C^*–category whose objects are the powers of the object ρ. Let $K \in (\rho^r, \rho^{r+g})$ be a Hilbert space such that $K \times 1_{\rho^m}$ has left annihilator zero for $m \in \mathbb{N}_0$ and suppose that*

$$(\rho^r, \rho^r) \times 1_{\rho^r} \subset (K^r, K^r).$$

Then $\mathcal{O}_\rho$ is a simple C^*–algebra.

Proof. The proof follows that of the simplicity of the extended Cuntz algebra Theorem 3.1 of [2]. Obviously, K must now play the role of the generating Hilbert space. We identify the arrows of $\mathcal{T}_\rho$ with their images in $\mathcal{O}_\rho$. It suffices to show that any non-degenerate representation π of $\mathcal{O}_\rho$ is faithful. π is trivially isometric on Hilbert spaces in $\mathcal{O}_\rho$ and in particular on K^n, $n \in \mathbb{N}_0$. But then π is also isometric on (K^m, K^n), defined as in the discussion preceding Lemma 4.11. Given $k \in \mathbb{Z}$, we let

$$^o\mathcal{O}_\rho{}^k := \cup_k (K^r, K^{r+k}), \quad r \geq 0, \ r + k \geq 0$$

and let $^o\mathcal{O}_\rho$ denote the *–subalgebra of $\mathcal{O}_\rho$ obtained by taking finite sums of elements from the $^o\mathcal{O}_\rho{}^k$. We have seen that π is isometric on each $^o\mathcal{O}_\rho^k$. If K is infinite dimensional, we can continue as in the proof of Theorem 3.1 of [2] to show that π is isometric on $^o\mathcal{O}_\rho$. However this algebra is dense in $\mathcal{O}_\rho$, since arguing as in Theorem 4.9, we have $(\rho^{r+m}, \rho^{r+n}) \times 1_{\rho^r} \subset (K^{r+m}, K^{r+n})$. Hence π is isometric and $\mathcal{O}_\rho$ is simple. If K is finite dimensional, we need only remark that (K^m, K^n) is the set of all linear mappings from K^m to K^n, so that, as a C^*–algebra, $\mathcal{O}_\rho = \mathcal{O}_K$ which is a simple C^*–algebra.

Thus instead of considering $(\mathcal{O}_V, \rho_V)$, we consider the smallest C^*–subalgebra $\mathcal{A}_V$ of $\mathcal{O}_V$ containing $H := V^*K$ and stable under ρ_V, equipped with the endomorphism σ_V obtained by restricting ρ_V. The system $(\mathcal{A}_V, \sigma_V)$ is then a natural candidate for such a minimal C^*–model system and we therefore address the question of finding necessary and sufficient conditions on a system $(\mathcal{A}, \hat{\sigma})$ for it to be of the form $(\mathcal{A}_V, \sigma_V)$. We shall only discuss the case where V is its regular representation, as the case of a corepresentation can, as before, be reduced to this case, for systems endowed with a standard braided symmetry. (This is not completely trivial, in that the concept of a braided symmetry not taking values in the intertwiner spaces, but just in their weak closures in some Hilbert space representation with support I has to be formalized.)

We start by pointing out that the smallest tensor C^*–subcategory $\mathcal{S}_V$ of $\mathcal{T}_V$ containing V and the intertwining space $H = V^* \circ K \times 1_K$ is, up to tensoring on the right by 1_V, W^*–dense in $\mathcal{T}_V$. More precisely, $(V^r, V^s) \times 1_V$ is contained in the W^*–closure of $H^s H^{r*}$ in (V^{r+1}, V^{s+1}). Note that $(\mathcal{A}_V, \sigma_V)$ is the canonical system derived from $\sigma = V$ regarded as an object of the tensor C^*–category $\mathcal{S}_V$. It is therefore canonically associated with $\mathcal{T}_V$. The following simple result relates the systems associated with V considered as an object of $\mathcal{S}_V$ and $\mathcal{T}_V$, respectively.

6.10 Proposition *For $r, s = 0, 1, 2 \ldots$, $(\sigma_V{}^r, \sigma_V{}^s) = (\rho_V{}^r, \rho_V{}^s) \cap \mathcal{A}_V$*

Proof. As H generates $\mathcal{O}_V$, $T \in (\rho_V{}^r, \rho_V{}^s)$ if it intertwines the restrictions of the corresponding endomorphisms to the space of intertwiners $H = V^*K$. Now this space is contained in $\mathcal{A}_V$, therefore $(\rho_V{}^r, \rho_V{}^s) \cap \mathcal{A}_V \supset (\sigma_V{}^r, \sigma_V{}^s)$, and the reverse inclusion is obvious.

Summarizing, the above discussion and conditions a)–d) lead to the following necessary conditions for $(\mathcal{A}, \hat{\sigma})$ to be of the form $(\mathcal{A}_V, \sigma_V)$ with V a *regular* multiplicative unitary:

a′) $\mathcal{A}$ is the smallest $\hat{\sigma}$–stable C^*–subalgebra containing a Hilbert space K with zero left annihilator,

b′) $K \subset (\hat{\sigma}, \hat{\sigma}^2)$,

c′) $K^* \hat{\sigma}(K) = KK^*$,

d′) $(\hat{\sigma}, \hat{\sigma})K = K$.

To see that these conditions are necessary, we need only remark that c′) just expresses the regularity of the multiplicative unitary and ensures that the hypotheses of Lemma 5.3 hold for $L = M = \hat{\sigma}(K)$ as we have already remarked.

d$'$) is now a consequence of Corollary 5.4. The sufficiency of these conditions follows from the next result.

6.11 Theorem *Let* $(\mathcal{A}, \hat{\sigma}, K)$ *satisfy conditions* a$'$) *to* d$'$) *then* $\mathcal{A}$ *is simple and there is a regular multiplicative unitary* V, *unique up to equivalence, such that* $(\mathcal{A}, \hat{\sigma}, K)$ *is isomorphic to the model dual object* $(\mathcal{A}_V, \sigma_V, V^*K)$, *where* V *is regarded as a representation of* V.

Proof. $\mathcal{A}$ is simple by Theorem 6.9. From condition c$'$) it follows that

$$K^* K^* \hat{\sigma}(K) K \subset \mathbb{C}$$

and hence, since K has left annihilator zero that $\hat{\sigma}(K) \subset (K, K^2)$. We now may apply Lemma 4.11 and Theorem 4.12 to the tensor C^*–category of intertwiners between the powers of $\hat{\sigma}$ to conclude that there is a multiplicative unitary V on K^2 with $V\theta\psi = \hat{\sigma}(\psi)$, $\psi \in K$. V is regular by c$'$), therefore the remaining conclusions follow using arguments similar to those of Theorem 6.5.

We can also give necessary and sufficient conditions for $(\mathcal{A}, \hat{\sigma})$ to be of the form $(\mathcal{A}_V, \sigma_V)$ for a general multiplicative unitary. We retain a$'$) and b$'$) above and replace c$'$) and d$'$) by

c$''$) $\hat{\sigma}(K) \subset (K, K^2)$,

d$''$) If we consider the tensor C^*–subcategory $\mathcal{T}$ of the tensor C^*–category of intertwiners between the powers of $\hat{\sigma}$ generated by K and denote its objects by σ^n, where $n \in \mathbb{N}_0$, then $(\sigma, \sigma)K = K$.

6.12 Theorem *Let* $(\mathcal{A}, \hat{\sigma}, K)$ *satisfy conditions* a$'$), b$'$), c$''$) *and* d$''$) *then* $\mathcal{A}$ *is simple and there is a multiplicative unitary* V, *unique up to equivalence, such that* $(\mathcal{A}, \hat{\sigma}, K)$ *is isomorphic to the model dual object* $(\mathcal{A}_V, \sigma_V, V^*K)$, *where* V *is regarded as a representation of* V.

The proof is a simplified version of that of the previous theorem seeing that regularity now plays no role. Finally, as a pendant to Theorem 6.6, we give a characterization of tensor C^*–categories of the form $\mathcal{S}_V$ for a multiplicative unitary V

6.13 Theorem *Let* $\mathcal{T}_\rho$ *be a tensor* C^*–*category whose objects are the tensor powers of an object* ρ *and suppose* $\mathcal{T}_\rho$ *is generated by an ambidextrous Hilbert space* K *in* (ρ, ρ^2) *such that* $K \times 1_{\rho^m}$ *has left annihilator zero for* $m \in \mathbb{N}_0$. *Let* V *be the associated multiplicative unitary of Theorem 4.12, then* $\mathcal{T}_\rho$, K *is isomorphic to* $\mathcal{S}_V$, $V^* \circ K \times 1_\rho$.

Proof. We let $\tilde{\mathcal{K}}$ be the category of Hilbert spaces with commuting shifts F and G associated with the ambidextrous space $K \in (\rho, \rho^2)$ as in Theorem 4.12.

Then the functor G_{V^*} not only commutes with F but satisfies $G_{V^*}G = \hat{F}G_{V^*}$ by Proposition 3.1e. If we interpret G_{V^*} as a tensor $*$–functor as in Proposition 4.10, we recognize that the image of $\mathcal{T}_\rho$ is the tensor C^*–subcategory generated by $H := V^* \circ K \times 1_\rho \in \mathcal{T}_V$. But this is, by definition, the minimal C^*–subcategory $\mathcal{S}_V$.

7 Appendix. Braided Symmetry

This appendix is devoted to the notion of braided symmetry. This evolved from a notion of the same name introduced in [2] in the context of endomorphisms of C^*–algebras to generalize a previous more restricted notion of permutation symmetry in [7]. Expressed in the context of a general tensor category, this notion can be expressed as follows. Let σ denote the endomorphism of the braid group $\mathbb{B}_\infty = \cup \mathbb{B}_n$ that shifts the braids $b \in \mathbb{B}_n$ on n threads to the right. By a braided symmetry for an object V in a tensor category $\mathcal{T}$ we mean a representation ε of $\mathbb{B}_\infty$ in $\mathcal{T}$ such that

$$\varepsilon(b) \in (V^{\times n}, V^{\times n}) \ , \quad b \in \mathbb{B}_n \ ,$$
$$\varepsilon(\sigma(b)) = 1_V \times \varepsilon(b) \ , \quad b \in \mathbb{B}_\infty = \cup \mathbb{B}_n \ ,$$
$$\varepsilon(s,1) \circ X \times 1_V = 1_V \times X \circ \varepsilon(r,1) \ , \quad X \in (V^{\times r}, V^{\times s}) \ ,$$

where $(1,1) = b_1$ is the braid on the first two threads and $(s,1) = b_1 \sigma(b_1) \ldots \sigma^{s-1}(b_1)$.

Obviously, if the full subcategory whose objects are the tensor powers of V can be made into a braided tensor category then this braiding does define a braided symmetry. However, not all braided symmetries arise in this way. An application of this notion of braided symmetry is Theorem 5.31 of [13] relating notions of amenability to the existence of a unitary braided symmetry.

Whilst this definition, with a view to simplicity, focused attention on the full subcategory whose objects are the tensor powers of V, we here need to consider the full tensor category and we will hence call $\mathcal{T}$ braided relative to a distinguished object $V \in \mathcal{T}$ if for any object W in $\mathcal{T}$ there is an invertible arrow $\varepsilon_W \in (W \times V, V \times W)$ such that

$$\varepsilon_{W \times W'} = \varepsilon_W \times 1_{W'} \circ 1_W \times \varepsilon_{W'} \ ,$$
$$\varepsilon_{W'} \circ T \times 1_V = 1_V \times T \circ \varepsilon_W \ , \quad T \in (W, W') \ .$$

The second equation just says that ε is a natural transformation from the functor of tensoring on the right by V to that of tensoring on the left by V. The first

equation implies in particular that this natural transformation takes the value 1_V on the tensor unit. If ε is a braided symmetry for V then we may define braided symmetries for the tensor powers $V^{\times n}$ of V inductively, setting

$$\varepsilon_W^{\times n} := 1_V \times \varepsilon_W^{\times n-1} \circ \varepsilon_W \times 1_{V^{n-1}}.$$

It is easy to see that if ε_W is defined on a subset of objects, closed under tensor products, and such that every object of $\mathcal{T}$ is a subobject of a (finite) direct sum of objects from the subset and the equations are satisfied for W and W' in the subset, then ε_W extends uniquely to a braided symmetry on the whole category. This remains true if $\mathcal{T}$ is a W^*–category and we allow infinite direct sums.

Braided symmetries in this sense have appeared as the starting point of the centre construction in tensor categories, see eg. [10] where references to the original articles are given. But in view of its simplicity, the notion may well have appeared in other contexts, still unknown to the authors. We recall, however, the notion of an arrow between braided symmetries. If ε and ε' are braided symmetries for V and V', respectively, then an arrow $T \in (\varepsilon, \varepsilon')$ is an arrow $T \in (V, V')$ such that

$$\varepsilon'_W \circ 1_W \times T = T \times 1_W \circ \varepsilon_W,$$

for each object W of $\mathcal{T}$.

When does an arrow $\varepsilon_V \in (V \times V, V \times V)$ define a braided symmetry? To give some kind of answer, we consider a strict tensor category $\mathcal{T}$ and an object V with the property that the functor of tensoring on the right by V is faithful and such that $W \times V$ is a direct sum of copies of V for each object W of $\mathcal{T}$. This property is related to the notion *right regular* representation. The corresponding property of $V \times W$ being a direct sum of copies of V is similarly related to the notion of *left regular* representation.

Given an invertible $\varepsilon_V \in (V \times V, V \times V)$ such that

$$\varepsilon_V \circ T \times 1_V = 1_V \times T \circ \varepsilon_V, \quad T \in (V, V),$$

there is, for each object W of $\mathcal{T}$ a unique invertible

$$\varepsilon_{W \times V} \in (W \times V \times V, V \times W \times V)$$

such that

$$\varepsilon_{W \times V} \circ S \times 1_V = 1_V \times S \circ \varepsilon_V, \quad S \in (V, W \times V).$$

In fact, we have only to pick $X_i \in (V, W \times V)$ and $Y_i \in (W \times V, V)$ such that $\sum_i X_i Y_i = 1_{W \times V}$ and we see that we have no option but to set

$$\varepsilon_{W \times V} := \sum_i 1_V \times X_i \circ \varepsilon_V \circ Y_i \times 1_V.$$

A routine computation shows that

$$\varepsilon_{W' \times V} \circ S \times 1_V = 1_V \times S \circ \varepsilon_{W \times V}, \quad S \in (W \times V, W' \times V).$$

Now consider the set Σ of objects W such that

$$\varepsilon_{W \times V} \circ 1_W \times \varepsilon_V^{-1} \in (W \times V, V \times W) \times 1_V$$

and define ε_W by

$$\varepsilon_W \times 1_V = \varepsilon_{W \times V} \circ 1_W \times \varepsilon_V^{-1}.$$

If W and W' are in Σ and $T \in (W, W')$,

$$\varepsilon_{W'} \times 1_V \circ T \times 1_{V \times V} = \varepsilon_{W' \times V} \circ 1_{W'} \times \varepsilon_V^{-1} \circ T \times 1_{V \times V}$$

$$= \varepsilon_{W' \times V} \circ T \times 1_{V \times V} \circ 1_W \times \varepsilon_V^{-1} = 1_V \times T \times 1_V \circ \varepsilon_{W \times V} \circ 1_W \times \varepsilon_V^{-1},$$

so that

$$\varepsilon_{W'} \circ T \times 1_V = 1_V \times T \circ \varepsilon_W, \quad T \in (W, W').$$

The set Σ trivially contains the tensor unit. Suppose both W and W' are in Σ then

$$\varepsilon_{W \times W' \times V} \circ 1_W \times \varepsilon_{W' \times V}^{-1} \circ 1_{W \times V} \times S = \varepsilon_{W \times W' \times V} \circ 1_W \times S \times 1_V \circ 1_W \times \varepsilon_V^{-1}$$

$$= 1_V \times 1_W \times S \circ \varepsilon_{W \times V} \circ 1_W \times \varepsilon_V^{-1}$$

$$= 1_V \times 1_W \times S \circ \varepsilon_W \times 1_V = \varepsilon_W \times 1_{W' \times V} \circ 1_W \times 1_V \times S,$$

where $S \in (V, W' \times V)$. Now since $W' \times V$ is a direct sum of copies of V we conclude that

$$\varepsilon_{W \times W' \times V} \circ 1_W \times \varepsilon_{W' \times V}^{-1} = \varepsilon_W \times 1_{W' \times V}.$$

But since $W' \in \Sigma$, $\varepsilon_{W' \times V} = \varepsilon_{W'} \times 1_V \circ 1_{W'} \times \varepsilon_V$. Thus

$$\varepsilon_{W \times W' \times V} = \varepsilon_W \times 1_{W' \times V} \circ 1_W \times \varepsilon_{W'} \times 1_V,$$

so that $W \times W' \in \Sigma$. Furthermore,

$$\varepsilon_{W \times W'} = \varepsilon_W \times 1_{W'} \circ 1_W \times \varepsilon_{W'}.$$

It is also easy to see that Σ is closed under subobjects and direct sums.

We have now proved the following result.

A.1 Proposition *Let $\mathfrak{T}$ be a tensor category, V an object such that $W \times V$ is a direct sum of copies of V for each W. Let $\varepsilon_V \in (V \times V, V \times V)$ be a unitary such that*

$$\varepsilon_V \circ T \times 1_V = 1_V \times T \circ \varepsilon_V, \quad T \in (V, V),$$

$$\varepsilon_V \times 1_V \circ 1_V \times \varepsilon_V \circ S \times 1_V = 1_V \times S \circ \varepsilon_V, \quad S \in (V, V \times V).$$

then there is a unique maximal tensor subcategory with a braided symmetry ε whose value at V coincides with the given invertible ε_V. This is a full subcategory closed under subobjects and direct sums.

Proof. We need only remark that the second equation above implies that

$$\varepsilon_{V \times V} = \varepsilon_V \times 1_V \circ 1_V \times \varepsilon_V$$

and hence V is in Σ and the two definitions of ε_V coincide.

We now consider the category $\mathcal{C}(V)$ of corepresentations of a multiplicative unitary V with its forgetful functor ι into the underlying category of Hilbert spaces. We let ϑ denote the braided symmetry relative to $\iota(V)$ derived from the symmetry on the category of Hilbert spaces but write ϑ_W in place of $\vartheta_{\iota(W)}$. The relevance of braided symmetries to this paper lies in the fact that there are many cases where $\mathcal{C}(V)$ admits a braided symmetry ε relative to V with the further property that $\hat{V}$ defined by $V \vartheta \hat{V} = \varepsilon_V$ is another multiplicative unitary on the same space. Such a braided symmetry will be called *standard*. $\hat{V}$ determines and is uniquely determined by ε.

Theorem A.2 *A braided symmetry ε on $\mathcal{C}(V)$ relative to V is standard if and only if*

$$\hat{V}_{12} V_{23} = V_{23} \hat{V}_{12}.$$

If ε is standard and W is a corepresentation of V, then $\hat{W}$ defined by

$$W \vartheta_{W,K} \hat{W} = \varepsilon_W$$

is a representation of $\hat{V}$ and for any pair W, W' of corepresentations, we have

$$\hat{W}_{12} W'_{23} = W'_{23} \hat{W}_{12},$$

$$\widehat{W \times W'} = \hat{W} \times \hat{W}'.$$

Proof. Whether ε is standard or not, a simple calculation shows that $T \in (W, W')$ if and only if $T \in (\hat{W}, \hat{W}')$. Since $\hat{W} \in (W \times V, \iota(W) \times V)$, this yields

$$\hat{W}_{12} \widehat{W \times V} = \iota(\widehat{W}) \times V \hat{W}_{12}.$$

Now

$$\hat{W}_{13}\hat{W}'_{23} = \hat{W}_{13}\vartheta_{23}W'^{-1}_{23}\varepsilon_{23}$$

$$= \vartheta_{23}\hat{W}_{12}W'^{-1}_{23}\varepsilon_{23}.$$

If $\hat{W}_{12}W'_{23} = W'_{23}\hat{W}_{12}$ then we get

$$\hat{W}_{13}\hat{W}'_{23} = \vartheta_{23}W'^{-1}_{23}\vartheta^{-1}_{12}W^{-1}_{12}\varepsilon_{W\times W'} = \vartheta^{-1}_{W\times W'}W'^{-1}_{13}W^{-1}_{12}\varepsilon_{W\times W'}.$$

So $\hat{W}_{13}\hat{W}'_{23} = \widehat{W\times W'}$. Thus if $\hat{W}_{12}$ and V_{23} commute, we could conclude from our first identity that

$$\hat{W}_{12}\hat{W}_{13}\hat{V}_{23} = \hat{V}_{23}\hat{W}_{12},$$

and, in particular, if $\hat{V}_{12}$ and V_{23} commute that $\hat{V}$ is a multiplicative unitary. We now show that $\hat{V}_{12}V_{23} = V_{23}\hat{V}_{12}$ implies $\hat{W}_{12}W'_{23} = W'_{23}\hat{W}_{12}$. Now

$$V_{23}\hat{V}_{12}W_{24}W_{34} = \hat{V}_{12}V_{23}W_{24}W_{34} = \hat{V}_{12}W_{34}V_{23},$$

$$V_{23}W_{24}\hat{V}_{12}W_{34} = V_{23}W_{24}W_{34}\hat{V}_{12} = W_{34}V_{23}\hat{V}_{12},$$

where we have used the multiplicativity of V. But these expressions are equal, so cancelling, we find $\hat{V}_{12}W_{24} = W_{24}\hat{V}_{12}$ and we have replaced V by W. The proof is completed by a similar step using the multiplicativity of $\hat{V}$.

$$\hat{W}_{12}\hat{W}_{13}W'_{34}\hat{V}_{23} = \hat{W}_{12}\hat{W}_{13}\hat{V}_{23}W'_{34} = \hat{V}_{23}\hat{W}_{12}W'_{34},$$

$$\hat{W}_{12}W'_{34}\hat{W}_{13}\hat{V}_{23} = W'_{34}\hat{W}_{12}\hat{W}_{13}\hat{V}_{23} = W'_{34}\hat{V}_{23}\hat{W}_{12}.$$

Again these two expressions are equal and cancelling gives $\hat{W}_{13}W'_{34} = W'_{34}\hat{W}_{13}$, as claimed. Hence, our previous computation shows that $\hat{W}$ is a representation of $\hat{V}$, completing the proof.

Remark. Since a braided symmetry relative to V might not be defined on the whole of $\mathcal{C}(V)$, it is worth remarking that $\mathcal{C}(V)$ can be replaced by a full tensor subcategory in the above theorem.

Note that the above theorem shows that when ε is a standard braided symmetry, $\mathcal{C}(V)$ and $\mathcal{R}(\hat{V})$ are canonically isomorphic as tensor W^*–categories. Of course, we could start with the multiplicative unitary $\hat{V}$ and then use ε to define V and V would again be multiplicative if and only if $\hat{V}_{12}V_{23} = V_{23}\hat{V}_{12}$. We conclude by showing that the interesting standard braided symmetries cannot be permutation symmetries.

A.3 Lemma *Let $\varepsilon = V\vartheta\hat{V}$ define a standard braided symmetry of V . Then* $\hat{V}^{-1}\hat{V}^{-1}_{23}\hat{V} = \varepsilon^{-1}_{23}\hat{V}^{-1}\varepsilon_{23}\hat{V}^{-1}_{23}$.

Proof.

$$\varepsilon_{23}^{-1}\hat{V}^{-1}\varepsilon_{23}^{-1}\hat{V}_{23} = (V\vartheta\hat{V})_{23}^{-1}\hat{V}^{-1}(V\vartheta)_{23} =$$

$$\hat{V}_{23}^{-1}\hat{V}_{13}^{-1} = \hat{V}_{12}^{-1}\hat{V}_{23}^{-1}\hat{V}_{12} \ ,$$

since V_{23} and $\hat{V}$ commute and $\hat{V}$ is multiplicative.

A.4 Proposition *Let* $\varepsilon = V\vartheta\hat{V}$ *be a standard braided symmetry for* V *. Then* $\vartheta = \varepsilon\hat{V}_{23}^{-1}\varepsilon_{23}\hat{V}(\varepsilon_{23}^{-1})^2\hat{V}^{-1}\varepsilon_{23}\hat{V}_{23}$ *. In particular if* ε *is a permutation symmetry then* $\varepsilon = \vartheta$ *and* V *is cocommutative.*

Proof. $(V^{\times 2}, V^{\times 2})$ is generated, as a weakly closed subspace, by $H(V,V)H^*$, with $H = \hat{V}^{-1}K$, thus setting $\tilde{\varepsilon} = \hat{V}\varepsilon\hat{V}^{-1} = \hat{V}V\vartheta$ then $\varepsilon = \hat{V}^{-1}\hat{V}_{23}^{-1}\tilde{\varepsilon}\ \hat{V}_{23}\hat{V}$. So ε commutes with $\hat{V}^{-1}\hat{V}_{23}^{-1}\hat{V}$. Thus by the previous lemma

$$\varepsilon = \hat{V}_{23}\varepsilon_{23}^{-1}\hat{V}\varepsilon_{23}\varepsilon\varepsilon_{23}^{-1}\hat{V}^{-1}\varepsilon_{23}\hat{V}_{23}^{-1} \ .$$

Now ε defines a braided symmetry for V, thus for any $\psi = \hat{V}^{-1}\varphi \in H \subset (V, V^{\times 2})$, $\varepsilon\varepsilon_{23}\psi\varepsilon^{-1} = (\hat{V}^{-1}\varphi)_{23} = \hat{V}^{-1}\hat{V}_{23}^{-1}\hat{V}\hat{V}_{13}\vartheta\varphi$ since $\hat{V}$ is multiplicative, and, again by Lemma A.3, $\varepsilon_{23}^{-1}\hat{V}^{-1}\varepsilon_{23}\hat{V}_{23}^{-1}\vartheta\hat{V}_{23} = \varepsilon\varepsilon_{23}\hat{V}^{-1}\varepsilon_{23}^{-1}$ hence

$$\vartheta = \hat{V}_{23}\varepsilon_{23}^{-1}\hat{V}\varepsilon_{23}\varepsilon\varepsilon_{23}\hat{V}^{-1}\varepsilon_{23}^{-1}\hat{V}_{23}^{-1}$$

and the conclusion follows.

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