RANDOM ALMOST HOLOMORPHIC SECTIONS OF AMPLE LINE BUNDLES ON SYMPLECTIC MANIFOLDS

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ABSTRACT. The spaces $H^0(M, L^N)$ of holomorphic sections of the powers of an ample line bundle L over a compact Kähler manifold (M, ω) have been generalized by Boutet de Monvel and Guillemin to spaces $H^0_J(M, L^N)$ of 'almost holomorphic sections' of ample line bundles over an almost complex symplectic manifold (M, J, ω). We consider the unit spheres $SH^0_J(M, L^N)$ in the spaces $H^0_J(M, L^N)$, which we equip with natural inner products. Our purpose is to show that, in a probabilistic sense, almost holomorphic sections behave like holomorphic sections as $N \to \infty$. Our first main result is that almost all sequences of sections $s_N \in SH^0_J(M, L^N)$ are 'asymptotically holomorphic' in the Donaldson-Auroux sense that $||s_N||_\infty/||s_N||_2 = O(\sqrt{\log N})$, $||\bar{\partial}s_N||_\infty/||s_N||_2 = O(\sqrt{\log N})$ and $||\partial s_N||_\infty/||s_N||_2 = O(\sqrt{N \log N})$. Our second main result concerns the joint probability distribution of the random variables $s_N(z^p)$, $\nabla s_N(z^p)$, $1 \le p \le n$, for n distinct points $z^1, \ldots, z^n$ in a neighborhood of a point $P_0 \in M$. We show that this joint distribution has a universal scaling limit about P_0 as $N \to \infty$. In particular, the limit is precisely the same as in the complex holomorphic case. Our methods involve near-diagonal scaling asymptotics of the Szegö projector Π_N onto $H^0_J(M, L^N)$, which also yields proofs of symplectic analogues of the Kodaira embedding theorem and Tian asymptotic isometry theorem.

INTRODUCTION

This paper is concerned with asymptotically holomorphic sections of ample line bundles over almost-complex symplectic manifolds (M, J, ω). Such line bundles and sections are symplectic analogues of the usual objects in complex algebraic geometry. Interest in their properties has grown in recent years because of their use by Donaldson [Don1, Don2], Auroux [Aur1, Aur2] and others [AuKa, Sik] in proving symplectic analogues of standard results in complex geometry. These results involve properties of asymptotically holomorphic sections of high powers of the bundle, particularly those involving their zero sets and the maps they define to projective space.

We take up the study of asymptotically holomorphic sections from the viewpoint of the microlocal analysis of the $\bar{\partial}$ operator on a symplectic almost-complex manifold, and define the class of 'almost holomorphic sections' by a method due to Boutet de Monvel and Guillemin [Bout, BoGu]. Sections of powers of a complex line bundle $L^N \to M$ over M are identified with equivariant functions s on the associated S^1-bundle X, and the $\bar{\partial}$ operator is identified with the $\bar{\partial}_b$ operator on X. In the non-integrable almost-complex symplectic case there are in general no solutions of $\bar{\partial}_b s = 0$. To define an 'almost holomorphic section' s, Boutet de Monvel and Guillemin define a certain (pseudodifferential) $\bar{D}_j$-complex over X [BoGu]

[BoSj]. The space $H^0_J(M, L^N)$ of almost holomorphic sections is then defined as the space of sections corresponding to solutions of $\bar{D}_0 s = 0$. The operator $\bar{D}_0$ is not uniquely or even canonically defined, and it is difficult to explicitly write down these almost holomorphic sections. The importance, and we hope usefulness, of these sections lies in the fact that they typically have the properties of asymptotically holomorphic sections as defined by Donaldson and Auroux, as we describe below. We use the term 'almost holomorphic' to emphasize that a priori, they are distinct from 'asymptotically holomorphic' sections.

Our main results involve the 'typical' behavior of almost holomorphic sections in a probabilistic sense, as in our work with Bleher [BSZ1, BSZ2] and our prior work [ShZe] on holomorphic sections. A wide variety of measures could be envisioned here, and much of what we do is independent of the precise choice of measure. However, the simplest measures are the Haar measures on the unit spheres in the spaces $H^0_J(M, L^N)$. To be precise, we use a hermitian metric h on L and the volume form $dV = \frac{\omega^m}{m!}$ on M to endow $H^0_J(M, L^N)$ with an $\mathcal{L}^2$ inner product. We denote by $SH^0_J(M, L^N)$ the elements of unit norm in $H^0_J(M, L^N)$ and by ν_N the Haar probability measure on the sphere $SH^0_J(M, L^N)$. We also consider the essentially equivalent Gaussian measures on $H^0_J(M, L^N)$. The theme of our work is to obtain results about almost holomorphic sections by calculating asymptotically (as $N \to \infty$) the probabilities that sections $s_N \in H^0_J(M, L^N)$ do various things. This theme has a variety of potential applications in geometry, which we hope to pursue in the future.

In this article we focus on two applications. Our first main result gives estimates on various norms of a typical sequence of almost holomorphic sections of growing degree. Let us recall that a sequence of sections s_N is called asymptotically holomorphic by Donaldson and Auroux [Don1, Aur1] if

$$\|s_N\|_\infty + \|\bar{\partial} s_N\|_\infty = O(1), \ \|\nabla s_N\|_\infty + \|\nabla \bar{\partial} s_N\|\|_\infty = O(\sqrt{N}), \ \|\nabla\nabla s_N\|_\infty = O(N).$$

We will prove that almost every sequence $\{s_N\}$ of $\mathcal{L}^2$-normalized ($\|s_N\|_2 = 1$) almost holomorphic sections in the Boutet de Monvel -Guillemin sense is close to being asymptotically holomorphic in the Donaldson sense. We also let $\nabla : \mathcal{C}^\infty(M, L^N \otimes (T^*M)^{\otimes k}) \to \mathcal{C}^\infty(M, L^N \otimes (T^*M)^{\otimes(k+1)})$ denote the connection, and we write $\nabla^k = \nabla \circ \cdots \circ \nabla : \mathcal{C}^\infty(M, L^N) \to \mathcal{C}^\infty(M, L^N \otimes (T^*M)^{\otimes k})$. We also have the decomposition $\nabla = \partial + \bar{\partial}$; note that here $\bar{\partial}$ depends on the choice of connection.

THEOREM **0.1.** *Endow the infinite product* $\Pi^\infty_{N=1} SH^0_J(M, L^N)$ *with the product spherical measure* $\nu_\infty := \Pi^\infty_{N=1} \nu_N$. *Then* ν_∞-*almost every sequence* $\{s_N\}$ *of sections satisfies the following estimates:*

$$\|s_N\|_\infty = O(\sqrt{\log N}), \qquad \|\nabla^k s_N\|_\infty = O(N^{k/2}\sqrt{\log N}),$$

$$\|\bar{\partial} s_N\|_\infty = O(\sqrt{\log N}), \qquad \|\nabla^k \bar{\partial} s_N\|_\infty = O(N^{k/2}\sqrt{\log N}), \quad (k \geq 1).$$

Our second main result concerns the joint probability distribution

$$\mathbf{D}^N_{(z^1,\ldots,z^n)} = D^N(x, \xi; z^1, \ldots, z^n) dx d\xi$$

of the random variables

$$x^p = s_N(z^p), \ \xi^p = N^{-\frac{1}{2}} \nabla s_N(z^p) \qquad (1 \leq p \leq n) \tag{1}$$

on $SH^0(M, L^N)$, for n distinct points $z^1, \ldots, z^n \in M$. We prove that upon rescaling, this joint probability distribution has a universal limit which agrees with that of the holomorphic case determined in [BSZ2].

THEOREM **0.2.** *Let L be a pre-quantum line bundle over a $2m$-dimensional compact integral symplectic manifold (M, ω). Let $P_0 \in M$ and choose complex local coordinates $\{z_j\}$ centered at P_0 so that $\omega|_{P_0}$ and $g|_{P_0}$ are the usual Euclidean Kähler form and metric respectively. Then*

$$\mathbf{D}^N_{(z^1/\sqrt{N}, \ldots, z^n/\sqrt{N})} \longrightarrow \mathbf{D}^\infty_{(z^1, \ldots, n^n)}$$

where $\mathbf{D}^\infty_{(z^1, \ldots, z^n)}$ is a universal Gaussian measure supported on the holomorphic 1-jets.

A technically interesting novelty in the proof is the role of the $\bar{\partial}$ operator. In the holomorphic case, $\mathbf{D}^N_{(z^1, \ldots, z^n)}$ is supported on the subspace of sections satisfying $\bar{\partial}s = 0$. In the almost complex case, sections do not satisfy this equation, so $\mathbf{D}^N_{(z^1, \ldots, z^n)}$ is a measure on a higher-dimensional space of jets. However, Theorem 0.2 says that the mass in the '$\bar{\partial}$-directions' shrinks to zero as $N \to \infty$.

An alternate statement of Theorem 0.2 involves equipping $H^0(M, L^N)$ with a Gaussian measure, and letting $\widetilde{\mathbf{D}}^N_{(z^1, \ldots, z^n)}$ be the corresponding joint probability distribution on $H^0_J(M, L^N)$, which is a Gaussian measure on the complex vector space of 1-jets of sections. We show (Theorem 5.4) that these Gaussian measures $\widetilde{\mathbf{D}}^N$ also have the same scaling limit $\mathbf{D}^\infty$, so that asymptotically the probabilities are the same as in the holomorphic case, as established in [BSZ2]. To be more precise, recall that a Gaussian measure on $\mathbb{R}^n$ is a measure of the form

$$\gamma_\Delta = \frac{e^{-\frac{1}{2}\langle \Delta^{-1}x, x \rangle}}{(2\pi)^{n/2}\sqrt{\det \Delta}}dx \,,$$

where Δ is a positive definite symmetric $n \times n$ matrix. It is then easy to see that $\widetilde{\mathbf{D}}^N_{(z^1, \ldots, z^n)} = \gamma_{\Delta^N}$ where Δ^N is the covariance matrix of the random variables in (1). To deal with singular measures, we introduce in §5 generalized Gaussians whose covariance matrices are only semi-positive definite. A generalized Gaussian is simply a Gaussian supported on the subspace corresponding to the positive eigenvalues of the covariance matrix. The main step in the proof is to show that the covariance matrices Δ^N underlying $\widetilde{\mathbf{D}}^N$ tend in the scaling limit to a semi-positive matrix Δ^∞. It follows that the scaled distributions $\widetilde{\mathbf{D}}^N$ tend to a generalized Gaussian γ_{Δ^∞} 'vanishing in the $\bar{\partial}$-directions.'

In joint work with Bleher [BSZ3], we use this result to prove universality of scaling limits of correlations of zeros in the setting of almost holomorphic sections over almost-complex symplectic manifolds. The analysis underlying Theorem 0.2 should also be useful for calculating many other kinds of probabilities in the setting of asymptotically holomorphic sections. For instance, we believe it should be useful for proving existence results for asymptotically holomorphic sections satisfying transversality conditions.

These results are based on two essential analytical results which have an independent interest and which we believe will have future applications. The first is the scaling asymptotics of the Szegö kernels $\Pi_N(z, w)$, i.e. the orthogonal projections onto $H^0_J(M, L^N)$. To be more precise, we lift the Szegö kernels to X and the asymptotics are as follows:

Choose local coordinates $\{z_j\}$ centered at a point $P_0 \in M$ as in Theorem 0.2 and choose a 'preferred' local frame for L, which together with the coordinates on M give us 'Heisenberg coordinates' on X (see §1.2). We then have

$$N^{-m}\Pi_N(P_0 + \tfrac{u}{\sqrt{N}}, \tfrac{\theta}{N}; P_0 + \tfrac{v}{\sqrt{N}}, \tfrac{\varphi}{N})$$

$$= \tfrac{1}{\pi^m} e^{i(\theta-\varphi)+u\cdot\bar{v}-\frac{1}{2}(|u|^2+|v|^2)} \left[1 + \sum_{r=1}^{K} N^{-\frac{r}{2}} b_r(P_0, u, v) + N^{-\frac{K+1}{2}} R_K(P_0, u, v, N)\right], \tag{2}$$

where $\|R_K(P_0, u, v, N)\|_{\mathcal{C}^j(\{|u|+|v|\leq\rho\})} \leq C_{K,j,\rho}$ for $j = 1, 2, 3, \ldots$.

A more precise statement will be given in Theorem 2.3. As more or less immediate corollaries of these scaling asymptotics, we prove symplectic analogues of the holomorphic Kodaira embedding theorem and Tian almost-isometry theorem [Tian]; these two results have previously been proved by Borthwick-Uribe [BoUr1, BoUr2] using a related microlocal approach. The Borthwick-Uribe proof of the almost-complex Tian theorem was in turn motivated by a similar proof in the holomorphic case in [Zel].

The proof of (2) is based on our second analytic result: the construction of explicit parametrices for Π and its Fourier coefficients Π_N. These parametrices closely resemble those of Boutet de Monvel - Sjöstrand [BoSj] in the holomorphic case. The construction is new but closely follows the work of Menikoff and Sjöstrand [MenSj, Sjö] and of Boutet de Monvel and Guillemin [Bout, BoGu]. For the sake of completeness, we will give a fairly detailed exposition of the construction of the zeroth term of the $\bar{D}_j$ complex and of the Szegö kernel.

GUIDE FOR THE READER

For the readers' convenience, we provide here a brief outline of the paper. We begin in §1 by first describing some terminology from symplectic geometry and then giving an outline of Boutet de Monvel and Guillemin's construction [Bout, BoGu] of a complex of pseudodifferential operators, which replaces the $\bar{\partial}_b$ complex in the symplectic setting. The zeroth term of this complex is used to define sequences of almost holomorphic sections and Szegö projectors analogous to the integrable complex case (§1.3). In §2, we show that the Szegö projectors Π_N are complex Fourier integral operators of the same type as in the holomorphic case, and we use this formulation to obtain the scaling asymptotics of $\Pi_N(P_0 + \tfrac{u}{\sqrt{N}}, \tfrac{\theta}{N}; P_0 + \tfrac{v}{\sqrt{N}}, \tfrac{\varphi}{N})$. Section 3 gives two applications of these asymptotics: a proof of a 'Kodaira embedding theorem' (using global almost holomorphic sections) for integral symplectic manifolds, and a generalization of the asymptotic expansion theorem of [Zel] to symplectic manifolds. Section 4 uses the scaling asymptotics to prove that sequences of almost holomorphic sections are almost surely (in the probabilistic sense) asymptotically close to holomorphic (Theorem 0.1). Finally, in §5, we determine the joint probability distributions $\mathbf{D}^N$, $\widetilde{\mathbf{D}}^N$ and again apply the scaling asymptotics to prove Theorem 0.2.

The following chart shows the interdependencies of the sections:

$$
\begin{array}{ccccc}
 & & \S1 & & \\
 & & \Downarrow & & \\
\S3 & \Longleftarrow & \S2 & \Longrightarrow & \S4 \\
 & & \Downarrow & & \\
 & & \S5 & &
\end{array}
$$
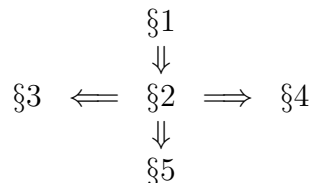
We advise the reader who wishes to proceed quickly to the applications in §§3–5 that these sections depend only on the scaling asymptotics of the Szegö kernel stated in Theorem 2.3 and the notation and terminology given in §§1.1–1.3.

CONTENTS

1. CIRCLE BUNDLES AND ALMOST CR GEOMETRY

We denote by (M, ω) a compact symplectic manifold such that $[\frac{1}{\pi}\omega]$ is an integral co-homology class. As is well known (cf. [Woo, Prop. 8.3.1]; see also [GuSt]), there exists a hermitian line bundle $(L, h) \rightarrow M$ and a metric connection ∇ on L whose curvature Θ_L satisfies $\frac{i}{2}\Theta_L = \omega$. We denote by L^N the Nth tensor power of L. The 'quantization' of (M, ω) at Planck constant $1/N$ should be a Hilbert space of polarized sections of L^N ([GuSt, p. 266]). In the complex case, polarized sections are simply holomorphic sections. The notion of polarized sections is problematic in the non-complex symplectic setting, since the Lagrangean subbundle $T^{1,0}M$ defining the complex polarization is not integrable and there usually are no 'holomorphic' sections. A subtle but compelling replacement for the notion of polarized section has been proposed by Boutet de Monvel and Guillemin [Bout, BoGu], and it is this notion which we describe in this section. For the asymptotic analysis, it is best to view sections of L^N as functions on the unit circle bundle $X \subset L^*$; we shall describe the 'almost CR geometry' of X in §1.2 below.

1.1. **Almost complex symplectic manifolds.** We begin by reviewing some terminology from almost complex symplectic geometry. An almost complex symplectic manifold is a symplectic manifold (M, ω) together with an almost complex structure J satisfying the compatibility condition $\omega(Jv, Jw) = \omega(v, w)$ and the positivity condition. $\omega(v, Jv) > 0$. We give M the Riemannian metric $g(v, w) = \omega(v, Jw)$. We denote by $T^{1,0}M$, resp. $T^{0,1}M$, the holomorphic, resp. anti-holomorphic, sub-bundle of the complex tangent bundle TM; i.e., $J = i$ on $T^{1,0}M$ and $J = -i$ on $T^{0,1}M$. We give M local coordinates $(x_1, y_1, \ldots, x_m, y_m)$, and we write $z_j = x_j + iy_j$. As in the integrable (i.e., holomorphic) case, we let $\{\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}\}$ denote the dual frame to $\{dz_j, d\bar{z}_j\}$. Although in our case, the coordinates z_j are not holomorphic and consequently $\frac{\partial}{\partial z_j}$ is generally not in $T^{1,0}M$, we nonetheless have

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\frac{\partial}{\partial x_j} - \frac{i}{2}\frac{\partial}{\partial y_j}, \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\frac{\partial}{\partial x_j} + \frac{i}{2}\frac{\partial}{\partial y_j}.$$

At any point $P_0 \in M$, we can choose a local frame $\{\bar{Z}_1^M, \ldots, \bar{Z}_m^M\}$ for $T^{0,1}M$ near P_0 and coordinates about P_0 so that

$$\bar{Z}_j^M = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=1}^m B_{jk}(z)\frac{\partial}{\partial z_k}, \quad B_{jk}(P_0) = 0, \tag{3}$$

and hence $\partial/\partial z_j|_{P_0} \in T^{1,0}(M)$. This is one of the properties of our 'preferred coordinates' defined below.

Definition: Let $P_0 \in M$. A coordinate system $(z_1, \ldots, z_m)$ on a neighborhood U of P_0 is *preferred* at P_0 if

$$\sum_{j=1}^m dz_j \otimes d\bar{z}_j = (g - i\omega)|_{P_0}.$$

In fact, the coordinates $(z_1, \ldots, z_m)$ are preferred at P_0 if an only if any two of the following conditions (and hence all three) are satisfied:

i) $\partial/\partial z_j|_{P_0} \in T^{1,0}(M)$, for $1 \le j \le m$,
ii) $\omega(P_0) = \omega_0$,
iii) $g(P_0) = g_0$,

where ω_0 is the standard symplectic form and g_0 is the Euclidean metric:

$$\omega_0 = \frac{i}{2}\sum_{j=1}^m dz_j \wedge d\bar{z}_j = \sum_{j=1}^m (dx_j \otimes dy_j - dy_j \otimes dx_j), \quad g_0 = \sum_{j=1}^m (dx_j \otimes dx_j + dy_j \otimes dy_j).$$

(To verify this statement, note that condition (i) is equivalent to $J(dx_j) = -dy_j$ at P_0, and use $g(v, w) = \omega(v, Jw)$.) Note that by the Darboux theorem, we can choose the coordinates so that condition (ii) is satisfied on a neighborhood of P_0, but this is not necessary for our scaling results.

1.2. **The circle bundle and Heisenberg coordinates.** We now let (M, ω, J) be a compact, almost complex symplectic manifold such that $[\frac{1}{\pi}\omega]$ is an integral cohomology class, and we choose a hermitian line bundle $(L, h) \to M$ and a metric connection ∇ on L with $\frac{i}{2}\Theta_L = \omega$. In order to simultaneously analyze sections of all positive powers L^N of the line bundle L, we work on the associated principal S^1 bundle $X \to M$, which is defined as

follows: let $\pi : L^* \to M$ denote the dual line bundle to L with dual metric h^*, and put $X = \{v \in L^* : \|v\|_{h^*} = 1\}$. We let α be the the connection 1-form on X given by ∇; we then have $d\alpha = \pi^* \omega$, and thus α is a contact form on X, i.e., $\alpha \wedge (d\alpha)^m$ is a volume form on X.

We let $r_\theta x = e^{i\theta} x$ $(x \in X)$ denote the S^1 action on X and denote its infinitesimal generator by $\frac{\partial}{\partial \theta}$. A section s of L determines an equivariant function $\hat{s}$ on L^* by the rule $\hat{s}(\lambda) = (\lambda, s(z))$ $(\lambda \in L_z^*, z \in M)$. It is clear that if $\tau \in \mathbb{C}$ then $\hat{s}(z, \tau\lambda) = \tau\hat{s}$. We henceforth restrict $\hat{s}$ to X and then the equivariance property takes the form $\hat{s}(r_\theta x) = e^{i\theta}\hat{s}(x)$. Similarly, a section s_N of L^N determines an equivariant function $\hat{s}_N$ on X: put

$$\hat{s}_N(\lambda) = \left(\lambda^{\otimes N}, s_N(z)\right) , \quad \lambda \in X_z , \tag{4}$$

where $\lambda^{\otimes N} = \lambda \otimes \cdots \otimes \lambda$; then $\hat{s}_N(r_\theta x) = e^{iN\theta}\hat{s}_N(x)$. We denote by $\mathcal{L}_N^2(X)$ the space of such equivariant functions transforming by the Nth character.

In the complex case, X is a CR manifold. In the general almost-complex symplectic case it is an almost CR manifold. The *almost CR structure* is defined as follows: The kernel of α defines a horizontal hyperplane bundle $H \subset TX$. Using the projection $\pi : X \to M$, we may lift the splitting $TM = T^{1,0}M \oplus T^{0,1}M$ to a splitting $H = H^{1,0} \oplus H^{0,1}$. The almost CR structure on X is defined to be the splitting $TX = H^{1,0} \oplus H^{0,1} \oplus \mathbb{C}\frac{\partial}{\partial\theta}$. We also consider a local orthonormal frame $Z_1, \ldots, Z_n$ of $H^{1,0}$, resp. $\bar{Z}_1, \ldots, \bar{Z}_m$ of $H^{0,1}$, and dual orthonormal coframes $\vartheta_1, \ldots, \vartheta_m$, resp. $\bar{\vartheta}_1, \ldots, \bar{\vartheta}_m$. On the manifold X we have $d = \partial_b + \bar{\partial}_b + \frac{\partial}{\partial\theta} \otimes \alpha$, where $\partial_b = \sum_{j=1}^m \vartheta_j \otimes Z_j$ and $\bar{\partial}_b = \sum_{j=1}^m \bar{\vartheta}_j \otimes \bar{Z}_j$. We define the almost-CR $\bar{\partial}_b$ operator by $\bar{\partial}_b = df|_{H^{1,0}}$. Note that for an $\mathcal{L}^2$ section s^N of L^N, we have

$$\left(\nabla_{L^N} s^N\right)\widehat{} = d^h \hat{s}^N , \tag{5}$$

where $d^h = \partial_b + \bar{\partial}_b$ is the horizontal derivative on X.

Our near-diagonal asymptotics of the Szegö kernel (§2.2) are given in terms of the Heisenberg dilations, using local 'Heisenberg coordinates' at a point $x_0 \in X$. To describe these coordinates, we first need the concept of a 'preferred frame':

Definition: A *preferred frame* for $L \to M$ at a point $P_0 \in M$ is a local frame e_L in a neighborhood of P_0 such that

   i) $\|e_L\|_{P_0} = 1$;
   ii) $\nabla e_L|_{P_0} = 0$;
   iii) $\nabla^2 e_L|_{P_0} = -(g + i\omega) \otimes e_L|_{P_0} \in T_M^* \otimes T_M^* \otimes L$.

(A preferred frame can be constructed by multiplying an arbitrary frame by a function with specified 2-jet at P_0; any two such frames agree to third order at P_0.) Once we have property (ii), property (iii) is independent of the choice of connection on T_M^* used to define $\nabla : \mathcal{C}^\infty(M, L \otimes T_M^*) \to \mathcal{C}^\infty(M, L \otimes T_M^* \otimes T_M^*)$. In fact, property (iii) is a necessary condition for obtaining universal scaling asymptotics, because of the 'parabolic' scaling in the Heisenberg group. Note that if e_L is a preferred frame at P_0 and if $(z_1, \ldots, z_m)$ are preferred coordinates at P_0, then we compute the Hessian of $\|e_L\|$:

$$\left(\nabla^2 \|e_L\|_h\right)_{P_0} = \Re\left(\nabla^2 e_L, e_L\right)_{P_0} = -g(P_0) ;$$

thus if the preferred coordinates are 'centered' at P_0 (i.e., $P_0 = 0$), we have

$$\|e_L\|_h = 1 - \frac{1}{2}|z|^2 + O(|z|^3) . \tag{6}$$

Remark: Recall ([BSZ2, §1.3.2]) that the Bargmann-Fock representation of the Heisenberg group acts on the space of holomorphic functions on $(M, \omega) = (\mathbb{C}^m, \omega_0)$ that are square integrable with respect to the weight $h = e^{-|z^2|}$. We let $L = \mathbb{C}^m \times \mathbb{C}$ be the trivial bundle. Then the trivializing section $e_L(z) := (z, 1)$ is a preferred frame at $P_0 = 0$ with respect to the Hermitian connection ∇ given by

$$\nabla e_L = \partial \log h \otimes e_L = - \sum_{j=1}^m \bar{z}_j dz_j \otimes e_L \,.$$

Indeed, the above yields $\nabla^2 e_L|_0 = - \sum d\bar{z}_j \otimes dz_j \otimes e_L(0) = -(g_0 + i\omega_0) \otimes e_L(0)$.

The preferred frame and preferred coordinates together give us 'Heisenberg coordinates':

Definition: A *Heisenberg coordinate chart* at a point x_0 in the principal bundle X is a coordinate chart $\rho : U \approx V$ with $0 \in U \subset \mathbb{C}^m \times \mathbb{R}$ and $\rho(0) = x_0 \in V \subset X$ of the form

$$\rho(z_1, \ldots, z_m, \theta) = e^{i\theta} a(z)^{-\frac{1}{2}} e_L^*(z) \,, \tag{7}$$

where e_L is a preferred local frame for $L \to M$ at $P_0 = \pi(x_0)$, and $(z_1, \ldots, z_m)$ are preferred coordinates centered at P_0. (Note that P_0 has coordinates $(0, \ldots, 0)$ and $e_L^*(P_0) = x_0$.)

We now give some computations using local coordinates $(z_1, \ldots, z_m, \theta)$ of the form (7) for a local frame e_L. (For the moment, we do not assume they are Heisenberg coordinates.) We write

$$a(z) \;=\; \|e_L^*(z)\|_{h^*}^2 \;=\; \|e_L(z)\|_h^{-2} \,,$$

$$\alpha \;=\; d\theta + \beta \,, \qquad \beta = \sum_{j=1}^m (A_j dz_j + \bar{A}_j d\bar{z}_j) \,,$$

$$\nabla e_L \;=\; \varphi \otimes e_L \,, \qquad \text{hence} \quad \nabla e_L^{\otimes N} \;=\; N\varphi \otimes e_L^{\otimes N} \,.$$

We let $\frac{\partial^h}{\partial z_j} \in H^{1,0} X$ denote the horizontal lift of $\frac{\partial}{\partial z_j}$. The condition $\left(\frac{\partial^h}{\partial z_j}, \alpha \right) = 0$ yields

$$\frac{\partial^h}{\partial z_j} = \frac{\partial}{\partial z_j} - A_j \frac{\partial}{\partial \theta} \,, \quad \frac{\partial^h}{\partial \bar{z}_j} = \frac{\partial}{\partial \bar{z}_j} - \bar{A}_j \frac{\partial}{\partial \theta} \,. \tag{8}$$

Suppose $s_N = f e_L^{\otimes N}$ is a local section of L^N. Then by (4) and (7),

$$\hat{s}_N(z, \theta) = f(z) a(z)^{-\frac{1}{2}} e^{iN\theta} \,. \tag{9}$$

Differentiating (9) and using (5), we conclude that

$$\begin{aligned}
\varphi \;&=\; -\frac{1}{2} d \log a - i\beta \\
&=\; -\sum_{j=1}^m \left(\frac{1}{2} \frac{\partial \log a}{\partial z_j} + i A_j \right) dz_j - \sum_{j=1}^m \left(\frac{1}{2} \frac{\partial \log a}{\partial \bar{z}_j} + i \bar{A}_j \right) d\bar{z}_j \,. \tag{10}
\end{aligned}$$

Now suppose that $(z_1, \ldots, z_m, \theta)$ are Heisenberg coordinates at P_0; i.e., e_L is a preferred frame at P_0 and $(z_1, \ldots, z_m)$ are preferred coordinates centered at P_0 (with $P_0 = 0$). By property (ii) of preferred frames, we have $\varphi(0) = 0$, and hence by (10)

$$da|_0 = d \log a|_0 = 0, \tag{11}$$

$$A_j(0) = 0, \quad (1 \le j \le m). \tag{12}$$

By differentiating (10) and applying the properties of preferred coordinates and frames, we further obtain

$$\sum_{j=1}^{m} d\bar{z}_j \otimes dz_j = -\nabla\varphi = \sum_{j=1}^{m} d\left(\frac{1}{2}\frac{\partial \log a}{\partial z_j} + iA_j\right) \otimes dz_j + \sum_{j=1}^{m} d\left(\frac{1}{2}\frac{\partial \log a}{\partial \bar{z}_j} + i\bar{A}_j\right) \otimes d\bar{z}_j \text{ at } 0.$$

Thus the following four equations are satisfied at $P_0 = 0$:

$$
\begin{aligned}
\frac{1}{2}\frac{\partial^2 \log a}{\partial z_j \partial z_k} + i\frac{\partial A_j}{\partial z_k} &= 0, & \frac{1}{2}\frac{\partial^2 \log a}{\partial z_j \partial \bar{z}_k} + i\frac{\partial A_j}{\partial \bar{z}_k} &= \delta_k^j, \\
\frac{1}{2}\frac{\partial^2 \log a}{\partial \bar{z}_j \partial z_k} + i\frac{\partial \bar{A}_j}{\partial z_k} &= 0, & \frac{1}{2}\frac{\partial^2 \log a}{\partial \bar{z}_j \partial \bar{z}_k} + i\frac{\partial \bar{A}_j}{\partial \bar{z}_k} &= 0,
\end{aligned}
\tag{13}
$$

at P_0. Solving (13) and recalling that $a(0) = 1$, $da|_0 = 0$, we obtain

$$\frac{\partial^2 a}{\partial z_j \partial z_k}(0) = 0, \qquad \frac{\partial^2 a}{\partial z_j \partial \bar{z}_k}(0) = \delta_k^j, \tag{14}$$

$$\frac{\partial A_j}{\partial z_k}(0) = 0, \qquad \frac{\partial A_j}{\partial \bar{z}_k} = -\frac{i}{2}\delta_k^j. \tag{15}$$

Hence $A_j = -\frac{i}{2}\bar{z}_j + O(|z|^2)$ and

$$\frac{\partial^h}{\partial z_j} = \frac{\partial}{\partial z_j} + \left[\frac{i}{2}\bar{z}_j + O(|z|^2)\right]\frac{\partial}{\partial \theta}, \qquad \frac{\partial^h}{\partial \bar{z}_j} = \frac{\partial}{\partial \bar{z}_j} - \left[\frac{i}{2}z_j + O(|z|^2)\right]\frac{\partial}{\partial \theta}. \tag{16}$$

1.3. **The $\bar{D}$ complex and Szegö kernels.** In the complex case, a holomorphic section s of L^N lifts to a $\hat{s} \in \mathcal{L}_N^2(X)$ which satisfying $\bar{\partial}_b \hat{s} = 0$. The operator $\bar{\partial}_b$ extends to a complex satisfying $\bar{\partial}_b^2 = 0$, which is a necessary and sufficient condition for having a maximal family of CR holomorphic coordinates. In the non-integrable case $\bar{\partial}_b^2 \neq 0$, and there may be no solutions of $\bar{\partial}_b f = 0$. To define polarized sections and their equivariant lifts, Boutet de Monvel [Bout] and Boutet de Monvel - Guillemin [BoGu] defined a complex $\bar{D}_j$, which is a good replacement for $\bar{\partial}_b$ in the non-integrable case. Their main result is:

THEOREM **1.1.** (see [BoGu], Lemma 14.11 and Theorem A 5.9) *There exists an S^1-invariant complex of first order pseudodifferential operators $\bar{D}_j$ over X*

$$0 \to C^\infty(\Lambda_b^{0,0}) \xrightarrow{\bar{D}_0} C^\infty(\Lambda_b^{0,1}) \xrightarrow{\bar{D}_1} \cdots \xrightarrow{\bar{D}_{m-1}} C^\infty(\Lambda_b^{0,m}) \to 0,$$

where $\Lambda_b^{0,j} = \Lambda^j(H^{0,1}X)^*$, such that:
   i) *$\sigma(\bar{D}_j) = \sigma(\bar{\partial}_b)$ to second order along $\Sigma := \{(x, r\alpha_x) : x \in X, r > 0\} \subset T^*X$;*
   ii) *The orthogonal projector $\Pi : \mathcal{L}^2(X) \to \mathcal{H}^2(X)$ onto the kernel of $\bar{D}_0$ is a complex Fourier integral operator which is microlocally equivalent to the Cauchy-Szegö projector of the holomorphic case;*
   iii) *$(\bar{D}_0, \frac{\partial}{\partial \theta})$ is jointly elliptic.*

The results stated here use only the $\bar{D}_0$ term of the complex; its kernel consists of the spaces of almost holomorphic sections of the powers L^N of the line bundle L, as explained below. The complex $\bar{D}_j$ was used by Boutet de Monvel -Guillemin [BoGu, Lemma 14.14] to show that the dimension of $H^0_J(M, L^N)$ or $\mathcal{H}^2_N(X)$ is given by the Riemann-Roch formula (for N sufficiently large). For our results, we need only the leading term of Riemann-Roch, which we obtain as a consequence of Theorem 3.1(a). (The reader should be warned that the symbol is described incorrectly in Lemma 14.11 of [BoGu]. However, it is correctly described in Theorem 5.9 of the Appendix to [BoGu] and also in [GuUr]).

We refer to the kernel $\mathcal{H}^2(X) = \ker \bar{D}_0 \cap \mathcal{L}^2(X)$ as the Hardy space of square-integrable 'almost CR functions' on X. The $\mathcal{L}^2$ norm is with respect to the inner product

$$\langle F_1, F_2 \rangle = \frac{1}{2\pi} \int_X F_1 \overline{F_2} dV_X, \quad F_1, F_2 \in \mathcal{L}^2(X), \tag{17}$$

where

$$dV_X = \frac{1}{m!} \alpha \wedge (d\alpha)^m = \alpha \wedge \pi^* dV_M. \tag{18}$$

The S^1 action on X commutes with $\bar{D}_0$; hence $\mathcal{H}^2(X) = \bigoplus_{N=0}^{\infty} \mathcal{H}^2_N(X)$ where $\mathcal{H}^2_N(X) = \{F \in \mathcal{H}^2(X) : F(r_\theta x) = e^{iN\theta} F(x)\}$. We denote by $H^0_J(M, L^N)$ the space of sections which corresponds to $\mathcal{H}^2_N(X)$ under the map $s \mapsto \hat{s}$. Elements of $H^0_J(M, L^N)$ are the *almost holomorphic sections* of L^N. (Note that products of almost holomorphic sections are not necessarily almost holomorphic.) We henceforth write $\hat{s} = s$ and identify $H^0_J(M, L^N)$ with $\mathcal{H}^2_N(X)$. Since $(\bar{D}_0, \frac{\partial}{\partial \theta})$ is a jointly elliptic system, elements of $H^0_J(M, L^N)$ and $\mathcal{H}^2_N(X)$ are smooth. In many other respects, $H^0_J(M, L^N)$ is analogous to the space of holomorphic sections in the complex case. Subsequent results will bear this out.

We let $\Pi_N : \mathcal{L}^2(X) \to \mathcal{H}^2_N(X)$ denote the orthogonal projection. The level N Szegö kernel $\Pi_N(x, y)$ is defined by

$$\Pi_N F(x) = \int_X \Pi_N(x, y) F(y) dV_X(y), \quad F \in \mathcal{L}^2(X). \tag{19}$$

It can be given as

$$\Pi_N(x, y) = \sum_{j=1}^{d_N} S_j^N(x) \overline{S_j^N(y)}, \tag{20}$$

where $S_1^N, \ldots, S_{d_N}^N$ form an orthonormal basis of $\mathcal{H}^2_N(X)$.

1.4. **Construction of the Szegö kernels.** In this section, we will sketch the construction of the operator $\bar{D}_0$ of Theorem 1.1 in the special setting of almost complex manifolds, and in so doing we will describe the symbol of the complex in more detail. This will require the introduction of many objects from symplectic geometry and from the microlocal analysis of $\bar{\partial}_b$. We will need this material later on in the construction of a parametrix for the Szegö kernel.

1.4.1. *The characteristic variety of $\bar{\partial}_b$.* In general, we denote by σ_A the principal symbol of a pseudodifferential operator A. To describe the principal symbol of $\bar{\partial}_b$, we introduce convenient local coordinates and frames. Recalling that $HX = H^{1,0}X \oplus H^{0,1}X$, we again consider local orthonormal frames $Z_1, \ldots, Z_n$ of $H^{1,0}X$, resp. $\bar{Z}_1, \ldots, \bar{Z}_m$ of $H^{0,1}X$, and dual orthonormal coframes $\vartheta_1, \ldots, \vartheta_m$, resp. $\bar{\vartheta}_1, \ldots, \bar{\vartheta}_m$. Then we have $\bar{\partial}_b = \sum_{j=1}^m \bar{\vartheta}_j \otimes \bar{Z}_j$. Let us define complex-valued functions on T^*X by:

$$p_j(x,\xi) = \langle Z_j(x), \xi \rangle, \quad \bar{p}_j(x,\xi) = \langle \bar{Z}_j(x), \xi \rangle.$$

Then

$$\sigma_{\bar{\partial}_b}(x,\xi) = \sum_{j=1}^m p_j(x,\xi)\epsilon(\bar{\vartheta}_j)$$

where ϵ denotes exterior multiplication. We note that $\{\bar{p}_j, \bar{p}_k\} = \langle [\bar{Z}_j, \bar{Z}_k], \xi \rangle$.

To state results, it is convenient to introduce the operator $\Box_b := \bar{\partial}_b^* \bar{\partial}_b = \sum_{j=1}^m \bar{Z}_j^* \bar{Z}_j$ where $\bar{Z}_j^*$ is the adjoint of the vector field regarded as a linear differential operator. To conform to the notation of [BoGu] we also put $q = \sigma(\Box_b) = \sum_{j=1}^m |\bar{p}_j|^2$. The characteristic variety $\Sigma = \{q = 0\}$ of $\bar{\partial}_b$ is the same as that of $\Box_b$, namely the vertical sub-bundle of $T^*X \to M$. It is the conic submanifold of T^*X parametrized by the graph of the contact form, $\Sigma = \{(x, r\alpha_x) : r > 0\} \sim X \times \mathbb{R}^+$. It follows that Σ is a symplectic submanifold. It is the dual (real) line bundle to the vertical subbundle $V \subset TX$, since $\alpha(X) = G(X, \frac{\partial}{\partial \theta})$.

1.4.2. *The positive Lagrangean ideal I.* To construct the $\bar{D}_j$-complex replacing the $\bar{\partial}_b$-complex in the non-integrable case, and to construct the Szegö kernel, we will need to study a positive Lagrangean ideal I whose generators will define the principal symbol of $\bar{D}_0$. For background on positive Lagrangean ideals, see [Hör].

PROPOSITION **1.2.** *There exists a unique positive Lagrangean ideal I with respect to Σ containing q. That is, there exists a unique ideal $I \subset I_\Sigma$ (where I_Σ is the ideal of functions vanishing on Σ) satisfying:*

- *I is closed under Poisson bracket;*
- *Σ is the set of common zeros of $f \in I$;*
- *There exist local generators $\zeta_1, \ldots, \zeta_m$ such that the matrix $\left(\frac{1}{i}\{\zeta_j, \bar{\zeta}_k\}\right)$ is positive definite on Σ and that $q = \sum_{j,k} \lambda_{j\bar{k}} \zeta_j \bar{\zeta}_k$, where $\{\lambda_{j\bar{k}}\}$ is a hermitian positive definite matrix of functions.*

Proof. In the holomorphic case, I is generated by the linear functions $\zeta_j(x,\xi) = \langle \xi, \bar{Z}_j \rangle$. In the general almost complex (or rather almost CR) setting, these functions do not Poisson commute and have to be modified. Since the deviation of an almost complex structure from being integrable (i.e. a true complex structure) is measured by the Nijenhuis bracket, it is not surprising that the generators ζ_j can be constructed from the linear functions $\langle \xi, \bar{Z}_j \rangle$ and from the Nijenhuis tensor. We now explain how to do this, basically following the method of [BoGu].

As a first approximation to the ζ_j we begin with the linear functions $\zeta_j^{(1)} = \bar{p}_j$ on T^*X. As mentioned above, the $\zeta_j^{(1)}$ do not generate a Lagrangean ideal in the non-integrable almost

complex case, indeed

$$\{\zeta_j^{(1)}, \zeta_k^{(1)}\} = \langle \xi, [\bar{Z}_j(x), \bar{Z}_k(x)] \rangle .$$ (21)

However we do have that

$$\{\zeta_j^{(1)}, \zeta_k^{(1)}\} = \{\langle \xi, \bar{Z}_j(x) \rangle, \langle \xi, \bar{Z}_k(x) \rangle\} = 0 \text{ on } \Sigma.$$

Indeed, for $(x, \xi) \in \Sigma$, we have $\xi = r\alpha_x$ for some $r > 0$ so that

$$\{\langle \xi, \bar{Z}_j(x) \rangle, \langle \xi, \bar{Z}_k(x) \rangle\} = r\alpha_x([\bar{Z}_j(x), \bar{Z}_k(x)])$$
$$= rd\alpha_x(\bar{Z}_j(x), \bar{Z}_k(x)) = r\pi^*\omega(\bar{Z}_j(x), \bar{Z}_k(x)) = 0$$ (22)

since $\{\bar{Z}_j\}$ forms a Lagrangean subspace for the horizontal symplectic form $\pi^*\omega$. Here, $\pi : X \to M$ is the natural projection. Moreover if we choose the local horizontal vector fields Z_j to be orthonormal relative to $\pi^*\omega$, then we also have:

$$\{\zeta_j^{(1)}, \bar{\zeta}_k^{(1)}\}(x, \xi) = \langle \xi, [\bar{Z}_j(x), Z_k(x)] \rangle = r\pi^*\omega(\bar{Z}_j(x), Z_k(x))$$
$$= ir\delta_j^k = i\delta_j^k p_\theta(x, \xi), \qquad (x, \xi) \in \Sigma.$$ (23)

Here, $p_\theta(x, \xi) = \langle \xi, \frac{\partial}{\partial \theta} \rangle$.

Finally, we have

$$q = \sum_{j=1}^m |\langle \xi, Z_j \rangle|^2 = \sum_{j=1}^m |\zeta_j^{(1)}|^2 .$$

Hence the second and third conditions on the ζ_j are satisfied by the functions $\zeta_j^{(1)}$. Furthermore, equation (21) tells us that the first condition is satisfied to zero-th order for the ideal $I_1 = (\zeta_1^{(1)}, \ldots, \zeta_m^{(1)})$. In fact, let us precisely describe the error. We consider the orthonormal (relative to ω) vector fields $Z_j^M = \pi_* Z_j$ of type (1,0) on M. Recall that the Nijenhuis tensor is given by

$$N(V, W) = \frac{1}{2}\big([JV, JW] - [V, W] - J[V, JW] - J[JV, W]\big) .$$

Hence,

$$N(Z_j^M, Z_k^M) = (-1 - iJ)[Z_j^M, Z_k^M] = -2[Z_j^M, Z_k^M]_{(0,1)} \stackrel{\text{def}}{=} \sum_{p=1}^m N_{jk}^p \bar{Z}_p^M .$$ (24)

We note that by definition,

$$N_{jk}^p = N_{kj}^p .$$ (25)

Furthermore, by the Jacobi identity

$$\{\{\zeta_j, \zeta_k\}, \zeta_p\} + \{\{\zeta_p, \zeta_j\}, \zeta_k\} + \{\{\zeta_k, \zeta_p\}, \zeta_j\} = 0$$

applied to $(x, \alpha_x) \in \Sigma$, we have

$$N_{jk}^p + N_{pj}^k + N_{kp}^j = 0 .$$ (26)

By (22) and (24), we have

$$\{\zeta_j^{(1)}, \zeta_k^{(1)}\} = \sum_{p=1}^m f_p^1 \zeta_p^{(1)} + \sum_{p=1}^m \bar{N}_{jk}^p \bar{\zeta}_p^{(1)} .$$ (27)

We now argue, following [BoGu], that these functions can be successively modified to satisfy the same conditions to infinite order on Σ. The next step is to modify the functions $\zeta_j^{(1)}$ by quadratic terms so that they satisfy the conditions $\{\zeta_j, \zeta_k\} \in I$ to first order and the condition $q = \sum_j |\zeta_j|^2$ to order 3 on Σ. So we try to construct new functions

$$\zeta_p^{(2)} = \zeta_p^{(1)} + R_p \,, \quad R_p = \sum_{j,k} \nu_p^{jk} \bar{\zeta}_j^{(1)} \bar{\zeta}_k^{(1)}$$

so that

$$\{\zeta_j^{(2)}, \zeta_k^{(2)}\} = \sum_p f_p^2 \zeta_p^{(2)} + \sum_{\alpha_1, \alpha_2} \mu_{jk}^{\alpha_1 \alpha_2} \bar{\zeta}_{\alpha_1}^{(2)} \bar{\zeta}_{\alpha_2}^{(2)} \,; \tag{28}$$

$$q = \sum_p v_p^2 \zeta_p^{(2)} + \sum_\alpha \varphi_p^\alpha \bar{\zeta}_{\alpha_1}^{(2)} \bar{\zeta}_{\alpha_2}^{(2)} \bar{\zeta}_{\alpha_3}^{(2)} \bar{\zeta}_{\alpha_4}^{(2)} \,, \quad (\alpha = (\alpha_1, \ldots, \alpha_4)) \,. \tag{29}$$

Let us now solve (28)–(29) for the ν_p^{jk}. First of all, we choose $\nu_p^{jk} = \nu_p^{kj}$. We have

$$\{\zeta_j^{(2)}, \zeta_k^{(2)}\} = \sum_{p=1}^m f_p^1 \zeta_p^{(1)} + \sum_{p=1}^m \bar{N}_{jk}^p \bar{\zeta}^{(1)} + \{\zeta_j^{(1)}, R_k\} - \{\zeta_k^{(1)}, R_j\} \quad \mathrm{mod}\ I_\Sigma^2 \,.$$

By (23), we have

$$\{\zeta_j^{(1)}, \bar{\zeta}_k^{(1)}\} = i\delta_j^k p_\theta \quad \mathrm{mod}\ I_\Sigma \,, \tag{30}$$

and thus

$$\{\zeta_j^{(1)}, R_k\} = \sum_{p=1}^m 2i\nu_k^{pj} p_\theta \bar{\zeta}_p^{(1)} \quad \mathrm{mod}\ I_\Sigma^2 \,.$$

Therefore,

$$\{\zeta_j^{(2)}, \zeta_k^{(2)}\} = \sum_{p=1}^m f_p^1 \zeta_p^{(2)} + \sum_{p=1}^m \left(\bar{N}_{jk}^p + 2i(\nu_k^{pj} - \nu_j^{pk}) p_\theta \right) \bar{\zeta}_p^{(1)} \quad \mathrm{mod}\ I_\Sigma^2 \,. \tag{31}$$

Hence

$$\bar{N}_{jk}^p = 2i(\nu_j^{pk} - \nu_k^{pj}) p_\theta \quad \mathrm{on}\ \Sigma \,,$$

or equivalently,

$$\nu_j^{pk} - \nu_k^{pj} = \frac{i}{2p_\theta} \bar{N}_{jk}^p \quad \mathrm{mod}\ I_\Sigma \,. \tag{32}$$

On the other hand,

$$q = \sum_p |\zeta_p^{(2)} - R_p|^2 = \sum_p v_p^2 \zeta_p^{(2)} - R_p \bar{\zeta}_p^{(2)}$$

$$= \sum_p v_p^2 \zeta_p^{(2)} - \sum_{j,k,p} \nu_p^{jk} \bar{\zeta}_j^{(2)} \bar{\zeta}_k^{(2)} \bar{\zeta}_p^{(2)} + \sum_\alpha \varphi_p^\alpha \bar{\zeta}_{\alpha_1}^{(2)} \bar{\zeta}_{\alpha_2}^{(2)} \bar{\zeta}_{\alpha_3}^{(2)} \bar{\zeta}_{\alpha_4}^{(2)} \,.$$

Hence (29) is equivalent to

$$\nu_p^{jk} + \nu_k^{pj} + \nu_j^{kp} = 0 \,. \tag{33}$$

Using (25)–(26), we can solve the equations (30) and (32) to obtain

$$\nu_p^{jk} = \frac{i}{6p_\theta}\left(\bar{N}_{pj}^k + \bar{N}_{pk}^j\right).\tag{34}$$

Indeed, the solution (34) is unique (modulo I_Σ) and hence the R_p are unique modulo I_Σ^3. In summary,

$$\zeta_p^{(2)} = \zeta_p^{(1)} + \frac{i}{3p_\theta}\sum_{j,k}\bar{N}_{pj}^k\bar{\zeta}_j^{(1)}\bar{\zeta}_k^{(1)}.\tag{35}$$

The passage from the nth to the $(n+1)^{\text{st}}$ step is similar, and we refer to [BoGu, pp. 147–149]. □

Remark: Define $p_\theta(x,\xi) = \langle\xi,\frac{\partial}{\partial\theta}\rangle$. Since the joint zero set of $\{\zeta_1,\ldots,\zeta_m\}$ equals Σ and since $p_\theta \neq 0$ on $\Sigma - 0$ it follows that $\{\zeta_1,\ldots,\zeta_m,p_\theta\}$ is an elliptic system of symbols.

1.4.3. *The complex canonical relation.* Our eventual goal is to prove that Π is a complex Fourier integral operator and to construct a parametrix for it. As a preliminary step we need to construct and describe the complex canonical relation C underlying Π. As is typical with complex Fourier integral operators, C does not live in $T^*X \times T^*X$ but rather in its almost analytic extension $T^*\tilde{X} \times T^*\tilde{X}$. Here, $\tilde{N}$ denotes the almost analytic extension of a C^∞ manifold N. Although the language of almost analytic extensions may seem heavy, it is very helpful if one wishes to understand the full (complex) geometry of C. When N is real analytic, $\tilde{N}$ is the usual complexification of X, i.e. a complex manifold in which N sits as a totally real submanifold. The reader may find it simpler to make this extra assumption. For background on almost analytic extensions, we refer to [MelSj, MenSj].

Since $\pi : X \to M$ is an S^1 bundle over M, its complexification $\tilde{\pi} : \tilde{X} \to \tilde{M}$ defines a $\mathbb{C}^*$ bundle over $\tilde{M}$. The connection form α has an (almost) analytic continuation to a connection $\tilde{\alpha}$ to this bundle and we may split $T\tilde{X} = \tilde{H} \oplus \tilde{V}$, where $\tilde{V} \to T\tilde{M}$ is the vertical subbundle of the fibration $\tilde{X} \to \tilde{M}$ and where $\tilde{H} \to T\tilde{M}$ is the kernel of $\tilde{\alpha}$.

The (almost) complexification of T^*X is of course $T^*(\tilde{X})$. We denote the canonical symplectic form on T^*X by σ and that on $T^*(\tilde{X}$ by $\tilde{\sigma}$; the notation is consistent because it is the complexification of σ. The symplectic cone Σ complexifies to $\tilde{\Sigma}$ and it remains symplectic with respect to $\tilde{\sigma}$. It is given by $\{(\tilde{x},\tilde{\lambda}\tilde{\alpha}_{\tilde{x}}) : \tilde{\lambda} \in \mathbb{C}^*\}$. We have a natural identification $L^* \iff \Sigma$ given by $rx \to (x,r\alpha_x)$. We further note that the $\mathbb{C}^*$ bundle $L^* \to M$ is the fiberwise complexification of the S^1 bundle $X \to M$, hence $L^* \to M$ is the restriction of $\tilde{\pi}$ to $\tilde{\pi}^{-1}(M)$. We will therefore view L^* as a submanifold of $\tilde{X}$.

1.4.4. *Definition of C.* Let $\tilde{\zeta}_j$ be the almost analytic extensions of the functions ζ_j. Then put

$$\mathcal{J}_+ = \{(\tilde{x},\tilde{\xi}) \in T^*\tilde{X} : \tilde{\zeta}_j = 0\ \forall j\}.\tag{36}$$

It is an involutive manifold of $T^*(\tilde{X})$ with the properties:

$$
\begin{aligned}
(i) \quad & (\mathcal{J}_+)_{\mathbb{R}} = \Sigma \\
(ii) \quad & q|_{\mathcal{J}_+} \sim 0 \\
(iii) \quad & \tfrac{1}{i}\sigma(u, \bar{u}) > 0, \forall u \in T(\mathcal{J}_+)^{\perp} \\
(iv) \quad & T_\rho(\mathcal{J}_+) = T_\rho\tilde{\Sigma} \oplus \Lambda_\rho^+.
\end{aligned}
\tag{37}
$$

Here, $\Lambda_\rho^{\pm}$ is the sum of the eigenspaces of F_ρ, the normal Hessian of q, corresponding to the eigenvalues $\{\pm i\lambda_j\}$. The null foliation of $\mathcal{J}_+$ is given by the joint Hamilton flow of the $\tilde{\zeta}_j$'s.

The following proposition, proved in [MenSj] and in ([BoGu],Appendix, Lemma 4.5) defines the complex canonical relation C:

PROPOSITION **1.3.** *There exists a unique strictly positive almost analytic canonical relation C satisfying*

$$
\mathrm{diag}(\Sigma) \subset C \subset \mathcal{J}_+ \times \overline{\mathcal{J}_+}.
\tag{38}
$$

Indeed,

$$
C = \{(\tilde{x}, \tilde{\xi}, \tilde{y}, \tilde{\eta}) \in \mathcal{J}_+ \times \overline{\mathcal{J}_+} : (\tilde{x}, \tilde{\xi}) \sim (\tilde{y}, \tilde{\eta})\},
\tag{39}
$$

where $\sim$ is the equivalence relation of 'belonging to the same leaf of the null foliation of $\mathcal{J}_+$. Thus, C is the flow-out of its real points, $\mathrm{diag}(\Sigma)$, under the joint Hamilton flow of the $\tilde{\zeta}_j$'s. It is clear from the description that $C \circ C = C^* = C$, i.e. that C is an idempotent canonical relation. It follows that $I^*(X \times X, C)$ is a $*$-algebra.

1.4.5. *Definition of the Szegö projector.* Having constructed C, we define a Szegö projector Π associated to Σ and C to be a self-adjoint projection Π in the Fourier integral operator class $I^*(X \times X, C)$ with principal symbol 1 (relative to the canonical 1/2-density of C).

It is simple to prove the existence of such a projection (see [BoGu], Appendix A.4): Since $I^*(X \times X, C)$ is a $*$-algebra, there exists an element $A \in I^*(X \times X, C)$ with $\sigma_A = 1$ on $\mathrm{diag}(\Sigma)$, or more precisely with σ_A equal to a projection onto a prescribed 'vacuum state'. The principal symbols of $A^2 - A$ and $A - A^*$ then vanish, so these operators are of negative order. It follows that the spectrum of A is concentrated near $\{0, 1\}$. Hence there exists an analytic function in a neighborhood of the spectrum such that $F(A) := \Pi$ is a true projection. Since $I^*(X \times X, C)$ is closed under functional calculus, this projection lies in that algebra.

We note that Π is far from unique; given any Π one could set $\Pi' = e^{iA}\Pi e^{-iA}$ where A is a pseudodifferential operator of order -1. We just fix one choice in what follows.

Remark: In [BoGu] the term Szegö projector (or Toeplitz structure) is used for a projection operator with wave front set on Σ which is microlocally equivalent to the following model case on $\mathbb{R}^{2m+2} \times \mathbb{R}^{2m}$ ([Bout, Sec. 5], [BoGu]).

Let us use coordinates $y \in \mathbb{R}^{2m+2}, t \in \mathbb{R}^{2m}$, let η, τ be the symplectically dual coordinates and consider the operators

$$
A_j := D_{y_j} + iy_j|D_t|, \qquad \bar{A}_j = D_{y_j} - iy_j|D_t|.
$$

Here, $D_x = \frac{\partial}{i\partial x}$ and $|D_t|$ is Fourier multiplication by $|\tau|$. The operators A_j, resp. $\bar{A}_j$, are what are familiarly known as creation operators, resp. annihilation, operators in the representation theory of the Heisenberg group. The characteristic variety of the system $\{\bar{A}_j\}$ is given by $\Sigma^0 = \{t = \tau = 0\} \equiv \mathbb{R}^{2m+2}$. The Hardy space is given by $\mathcal{H}^2 = \{f : \bar{A}_j f = 0, \quad \forall j\}$ and the Szegö kernel is given by the complex Fourier integral kernel

$$\Pi^0(t, y, t', y') = C_m \int_{\mathbb{R}^m} e^{i\Phi} |\tau|^m d\tau, \quad \Phi = \langle t - t', \tau \rangle + i|\tau|(|y|^2 + |y'|^2).$$

The positive Lagrangean ideal I is generated by the symbols $\sigma(A_j) = \zeta_j = \eta_j + i|\tau|y_j$.

1.4.6. *Construction of the complex.* Having defined Π, one first constructs $\bar{D}_0$ so that $\bar{D}_0\Pi = 0$. In terms of a local frame $\bar{\vartheta}_j$ of horizontal (0,1)-forms on X, we may write

$$\bar{D}_0 f = \sum_{j=1}^{m} \hat{\zeta}_j(x, D) f \bar{\vartheta}_j. \tag{40}$$

The coefficient operators $\hat{\zeta}_j(x, D)$ are first order pseudodifferential operators with principal symbols equal to ζ_j and satisfying

$$\hat{\zeta}_j \Pi \sim 0$$

modulo smoothing operators. That is, one 'quantizes' the ζ_j's as first order pseudodifferential operators which annihilate Π. Let us briefly summarize their construction (following [BoGu, Appendix]).

We begin with any S^1-equivariant symmetric first order pseudodifferential operator $\bar{D}_0'$ with principal symbol equal to $\sum_{j=1}^{m} \zeta_j \bar{\vartheta}_j$. Then $\bar{D}_0'\Pi$ is of order ≤ 0 so one may find a zeroth order pseudodifferential operator Q_0 such that $\bar{D}_0'\Pi \sim Q_0\Pi$ (modulo smoothing operators). Then put: $\bar{D}_0 = (\bar{D}_0' - Q_0) - (\bar{D}_0' - Q_0)\Pi$. Clearly, $\bar{D}_0\Pi = 0$ and $\sigma(\bar{D}_0) = \sigma(\bar{D}_0') = \sum_{j=1}^{m} \zeta_j \bar{\vartheta}_j$. The characteristic variety of $\bar{D}_0$ is then equal to Σ. Since p_θ is the symbol of $\frac{\partial}{\partial \theta}$ and since the system $\{\sigma_{\bar{D}_0}, p_\theta\}$ has no zeros in $T^*X - 0$ it follows that $\{\bar{D}_0, \frac{\partial}{\partial \theta}\}$ is an elliptic system.

Remark: One can then construct the higher $\bar{D}_j$ recursively so that $\bar{D}_j \bar{D}_{j-1} = 0$. We refer to [BoGu], Appendix §5, for further details.

2. PARAMETRIX FOR THE SZEGÖ PROJECTOR

In [BSZ2, Theorem 3.1], we showed that for the complex case, the scaled Szegö kernel Π_N near the diagonal is asymptotic to the Szegö kernel $\Pi_1^{\mathbf{H}}$ of level one for the reduced Heisenberg group, given by

$$\Pi_1^{\mathbf{H}}(z, \theta; w, \varphi) = \frac{1}{\pi^m} e^{i(\theta-\varphi)+i\Im(z\cdot\bar{w})-\frac{1}{2}|z-w|^2} = \frac{1}{\pi^m} e^{i(\theta-\varphi)+z\cdot\bar{w}-\frac{1}{2}(|z|^2+|w|^2)}. \tag{41}$$

The method was to apply the Boutet de Monvel-Sjöstrand oscillatory integral formula

$$\Pi(x, y) = \int_0^\infty e^{it\psi(x,y)} s(x, y, t) dt \tag{42}$$

arising from a parametrix construction ([BoSj, Th. 1.5 and §2.c]). Let us recall the construction of $\psi(x, y)$ in the integrable complex case. Fix a local holomorphic section e_L of L over $U \subset M$ and define $a \in C^\infty(U)$ by $a = |e_L|_h^{-2}$. Since $L^*|_U \approx U \times \mathbb{C}$ we can define

local coordinates on L^* by $(z, \lambda) \approx \lambda e_L(z)$. Then a defining function of $X \subset L^*$ is given by $\rho(z, \lambda) = 1 - |\lambda|^2 a(z)$. Define the function $a(z, w)$ as the almost analytic extension of $a(z)$, i.e. the solution of $\bar{\partial}_z a = 0 = \partial_w a$, $a(z, z) = a(z)$ and put $\psi(x, y) = i(1 - \lambda \bar{\mu} a(z, w))$. Then $t\psi$ is a phase for Π.

The object of this section is to show that the universal asymptotic formula of [BSZ2] for the near-diagonal scaled Szegö kernel holds for the symplectic case (Theorem 2.3). To do this, we first show that the Boutet de Monvel-Sjöstrand construction can be extended to the symplectic almost-complex case. Indeed we will obtain (Theorem 2.1) an integral formula of the form (42) for the symplectic case. In fact, our local phase function ψ will be shown to be of the form $\psi(x, y) = i(1 - \lambda \bar{\mu} a(z, w))$, where $\overline{a(w, z)} = a(z, w)$ and hence $\psi(y, x) = \overline{-\psi(x, y)}$.

2.1. Oscillatory integral for Π.

In order to obtain our integral formula, we first recall the notion of parametrizing an almost analytic Lagrangean Λ by a phase function. We assume $\varphi(x, \theta)$ is a regular phase function in the sense of ([MelSj, Def. 3.5]), i.e. that it has no critical points, is homogeneous of degree one in θ, that the differentials $d\frac{\partial \varphi}{\partial \theta_j}$ are linearly independent over $\mathbb{C}$ on the set

$$C_{\varphi \mathbb{R}} = \{(x, \theta) : d_\theta \varphi = 0\}$$

and such that $\Im \varphi \geq 0$. We then let $\tilde{\varphi}(\tilde{x}, \tilde{\theta})$ be an almost analytic extension, put

$$C_{\tilde{\varphi}} = \{(\tilde{x}, \tilde{\theta}) : d_{\tilde{\theta}} \tilde{\varphi} = 0\}$$

and define the Lagrange immersion

$$\iota_{\tilde{\varphi}} : (\tilde{x}, \tilde{\theta}) \in C_{\tilde{\varphi}} \to (\tilde{x}, d_{\tilde{x}} \tilde{\varphi}(\tilde{x}, \tilde{\theta})).$$

The phase φ parametrizes Λ if Λ is the image of this map.

The parametrix is an explicit construction of $\Pi(x, y)$ as a complex Lagrangean kernel. What we wish to prove now is that C can be parametrized, exactly as in the CR case, by a phase $\lambda \psi(x, y)$ defined on $\mathbb{R}^+ \times X \times X$. This is helpful in analyzing the scaling limit of $\Pi_N(x, y)$. In the following we use local coordinates (z, λ) on L^* coming from a choice of local coordinates z on M and a local frame $e_L(z)$ of L, and a corresponding local trivialization $(\tilde{z}, \lambda)$ of $\tilde{X} \to \tilde{M}$. As before, we let $a = \|e_L^*\|^2$.

THEOREM 2.1. *Let $\Pi(x, y) : \mathcal{L}^2(X) \to \mathcal{H}^2(X)$ be the Szegö kernel. Then there exists a unique regular phase function $it\psi(x, y) \in C^\infty(\mathbb{R}^+ \times X \times X)$ of positive type and a symbol $s \in S^m(X \times X \times \mathbb{R}^+)$ of the type*

$$s(x, y, t) \sim \sum_{k=0}^{\infty} t^{m-k} s_k(x, y)$$

such that $id_x \psi|_{x=y} = -id_y \psi|_{x=y} = \alpha$ and

$$\Pi(x, y) = \int_0^\infty e^{it\psi(x,y)} s(x, y, t) dt.$$

Furthermore, the almost analytic extension $\tilde{\psi} \in C^\infty(\tilde{X} \times \tilde{X})$ of ψ has the form $\tilde{\psi}(\tilde{x}, \tilde{y}) = i(1 - \lambda \bar{\mu} \tilde{a}(\tilde{z}, \tilde{w}))$ with $\tilde{a}(z, z) = a(z)$ and $\tilde{a}(\tilde{z}, \tilde{w}) = \overline{\tilde{a}(\tilde{w}, \tilde{z})}$.

Proof. We need to construct a function $a(z, w)$ so that $it\psi$ as above parametrizes the canonical relation C, i.e that C is the image of the Lagrange immersion

$$\iota_{\tilde{\psi}} : C_{t\tilde{\psi}} = \mathbb{R}^+ \times \{\tilde{\psi} = 0\} \to T^*(\tilde{X} \times \tilde{X})$$

$$(t, \tilde{x}, \tilde{y}) \mapsto (\tilde{x}, td_{\tilde{x}}\tilde{\psi}; \tilde{y}, -td_{\tilde{y}}\tilde{\psi})$$

$$(43)$$

Since C is the unique canonical relation satisfying $\mathrm{diag}(\Sigma) \subset C \subset \mathcal{J}_+ \times \overline{\mathcal{J}_+}$, the conditions that $\tilde{\psi}$ parametrize C are the following:

  i) $\{(x, y) \in X \times X : \psi(x, y) = 0\} = \mathrm{diag}(X)$;
  ii) $d_x\psi|_{x=y} = -d_y\psi|_{x=y} = r\alpha$ for $x, y \in X$ and for some function $r(x) > 0$;
  iii) $\tilde{\zeta}_j(\tilde{x}, d_{\tilde{x}}\tilde{\psi}) = 0 = \tilde{\zeta}_j(\tilde{y}, d_{\tilde{y}}\tilde{\psi})$ on $\{\tilde{\psi} = 0\}$.

Such a $\tilde{\psi}$ is not unique, so we require that $r \equiv 1$ in condition (ii), i.e.,

$$d_x\psi|_{x=y} = -d_y\psi|_{x=y} = \alpha\,.$$

Suppose we have $\tilde{\psi}(\tilde{x}, \tilde{y}) = i(1 - \lambda\bar{\mu}\tilde{a}(\tilde{z}, \tilde{w}))$. We observe that

$$\tilde{\psi} = 0 \iff \tilde{a}(\tilde{z}, \tilde{w}) = (\lambda\bar{\mu})^{-1},$$

and hence

$$id_{\tilde{x}}\tilde{\psi} = \bar{\mu}\tilde{a}(\tilde{z}, \tilde{w})d\lambda + \lambda\bar{\mu}d_{\tilde{z}}\pi^*\tilde{a}(\tilde{z}, \tilde{w})$$

$$= \lambda^{-1}d\lambda + \tilde{a}^{-1}d_{\tilde{z}}\tilde{a}(\tilde{z}, \tilde{w}) \iff \tilde{\psi} = 0.$$

$$(44)$$

The conditions on a are therefore:

$$\begin{cases} a(z, w)\lambda\bar{\mu} = 1 \iff (z, \lambda) = (w, \mu) \in X; \\[2mm] (a^{-1}d_za + \lambda^{-1}d\lambda)|_{\mathrm{diag}(X)} = -(a^{-1}d_wa + \lambda^{-1}d\lambda)|_{\mathrm{diag}(X)} = \alpha \\[2mm] \tilde{\zeta}_j\left(\tilde{z}, \lambda, \lambda^{-1}d\lambda + \tilde{a}^{-1}d_{\tilde{z}}\tilde{a}(\tilde{z}, \tilde{w})\right) = 0 = \tilde{\zeta}_j\left(\tilde{w}, \tilde{\mu}, \mu^{-1}d\mu + \tilde{a}^{-1}d_{\tilde{w}}\tilde{a}(\tilde{z}, \tilde{w})\right), \ \forall(z, w, \lambda, \mu) \end{cases}$$

A solution $a(z, w)$ satisfying the first condition must satisfy $a(z, z)|\lambda|^2 = 1$ on X, so that $a(z, z)|\lambda|^2$ is the local hermitian metric on L^* with unit bundle X, i.e. $a(z, z) = a(z)$.

We now prove that these conditions have a unique solution near the diagonal. We do this by reducing the canonical relation C by the natural S^1 symmetry. The reduced relation C_r has a unique generating function $\log a$; the three conditions above on a will follow automatically from this fact.

The S^1 action of X lifts to T^*X as the Hamiltonian flow of the function $p_\theta(x, \xi) := \langle \xi, \frac{\partial}{\partial\theta} \rangle$. The ζ_j are invariant under this S^1 action, hence

$$\{p_\theta, \zeta_j\} = 0 \ \ \forall j. \tag{45}$$

Now consider the level set $\{p_\theta = 1\} \subset T^*X$. Dual to the splitting $TX = H \oplus V$ we get a splitting $T^*X = H^* \oplus V^*$, where

$$V^*(X) = \mathbb{R}\alpha = H^o, \quad H^*(X) = V^o$$

where E^o denotes the annihilator of a subspace E, i.e. the linear functionals which vanish on E. Thus, $p_\theta = 0$ on the horizontal space $H^*(X)$ and $p_\theta(\alpha) = 1$. Since p_θ is linear on the fibers of T^*X, the set $\{p_\theta = 1\}$ has the form $\{\alpha + h : h \in H^*(X)\}$. We also note that

$p_\theta(d\theta) = 1$ in the local coordinates (z, θ) on X defined by $\lambda = e^{i\theta}$. Hence $\{p_\theta = 1\}$ may also be identified with $\{d\theta + h : h \in H^*(X)\}$.

Since $\{p_\theta = 1\}$ is a hypersurface, its null-foliation is given by the orbits of the Hamiltonian flow of p_θ, i.e. by the S^1 action. We use the term 'reducing by the S^1-action' to mean setting $p_\theta = 1$ and then dividing by this action. The reduction of T^*X is thus defined by $(T^*X)_r = p_\theta^{-1}(1)/S^1$. Since $p_\theta^{-1}(1)$ is an affine bundle over X with fiber isomorphic to $H^*(X) \approx T^*M$, it is clear that $(T^*X)_r \approx T^*M$ as vector bundles over M. We can obtain a symplectic equivalence using the local coordinates (z, θ) on X. Let (p_z, p_θ) be the corresponding symplectically dual coordinates, so that the natural symplectic form σ_{T^*X} on T^*X is given by $\sigma_{T^*X} = dz \wedge dp_z + d\theta \wedge dp_\theta$. The notation p_θ is consistent with the above. Moreover, the natural symplectic form on T^*M is given locally by $\sigma_{T^*M} = dz \wedge dp_z$. Now define the projection

$$\chi : p_\theta^{-1}(1) \to T^*M, \quad \chi(z, p_z, 1, p_\theta) = (z, p_z).$$

This map commutes with the S^1 action and hence descends to the quotient to define a local map over U, still denoted χ, from $(T^*X)_r \to T^*M$. Clearly χ is symplectic.

We now reduce the canonical relation C. Thus we consider the $\mathbb{C}^* \times \mathbb{C}^*$ action on $T^*\tilde{X} \times T^*\tilde{X} - 0$ generated by $p_\theta(x, \xi), p_\theta(y, \eta)$. The reduction of C is given by

$$C_r = C \cap (p_\theta \times p_\theta)^{-1}(1, 1)/\mathbb{C}^* \times \mathbb{C}^*.$$

We then use $\chi \times \chi$ to identify C_r with a (non-homogeneous) positive canonical relation in $T^*(\tilde{M} \times \tilde{M})$. Thus in coordinates,

$$C_r = \{(\tilde{z}, \tilde{p}_z, \tilde{w}, \tilde{p}_w) \in T^*(\tilde{M} \times \tilde{M}) : \exists \lambda, \mu, (\tilde{z}, \lambda, \tilde{p}_z, 1; \tilde{w}, \mu, \tilde{p}_w, 1) \in C\}. \tag{46}$$

Since reduction preserves real points, it is clear that

$$(C_r)_\mathbb{R} = C_\mathbb{R} \cap (p_\theta \times p_\theta)^{-1}(1, 1)/\mathbb{C}^* \times \mathbb{C}^*$$

$$= \{(z, p_z, z, p_z) \in \text{diag}(T^*(M \times M)) : \exists \theta \text{ such that } \alpha_{z, e^{i\theta}} = d\theta + p_z\}.$$

Let us denote by $\tilde{\zeta}_{jr}$ the reductions of the functions $\tilde{\zeta}_j$ by the S^1 symmetry. Then $\tilde{\zeta}_{jr} = 0$ on either pair of cotangent vectors in C_r. Moreover, by the uniqueness statement on C it follows that C_r is the unique canonical relation in $T^*(\tilde{M} \times \tilde{M})$ with the given set of real points and in the zero set of the $\tilde{\zeta}_{jr}$'s.

We now observe that C_r has, at least near the diagonal, a unique global generating function. This holds because the natural projection

$$C_r \subset T^*(\tilde{M} \times \tilde{M}) \to \tilde{M} \times \tilde{M} \tag{47}$$

is a local diffeomorphism near the diagonal. Indeed, its derivative gives a natural isomorphism

$$T_{\rho, \rho} C_r \approx H^* \oplus H^* \approx T(\tilde{M} \times \tilde{M}). \tag{48}$$

Therefore, there exists a global generating function $\log \tilde{a} \in C^\infty(\tilde{M} \times \tilde{M})$ i.e.

$$C_r = \{(\tilde{z}, d_{\tilde{z}} \log \tilde{a}, \tilde{w}, d_{\tilde{w}} \log \tilde{a}), \tilde{z}, \tilde{w} \in \tilde{M}\}. \tag{49}$$

Since $C^* = C$ it follows that $C_r^* = C^r$ and hence that $a(w, z) = \overline{a(z, w)}$.

Working backwards, we find that the function $\tilde{\psi}(\tilde{x}, \tilde{y}) = i(1 - \lambda\bar{\mu}\tilde{a}(\tilde{z}, \tilde{w}))$ satisfies the equations $\tilde{\zeta}_j(\tilde{x}, d_{\tilde{x}}\tilde{\psi}) = \tilde{\zeta}_j(\tilde{y}, d_{\tilde{y}}\tilde{\psi}) = 0$ on $\tilde{\psi} = 0$. Therefore the Lagrange immersion

$$i_{\tilde{\psi}} : C_{t\tilde{\psi}} = \mathbb{R}^+ \times \{\tilde{\psi} = 0\} \to T^*(\tilde{X} \times \tilde{X})$$

$$(t, \tilde{x}, \tilde{y}) \to (\tilde{x}, td_{\tilde{x}}\tilde{\psi}; \tilde{y}, -td_{\tilde{y}}\tilde{\psi})$$

$$(50)$$

takes its image inside $\mathcal{J}_+ \times \overline{\mathcal{J}_+}$ and reduces to C_r under the S^1-symmetry. To conclude the proof it is only necessary to show that the real points of the image of $i_{\tilde{\psi}}$ equal diag(Σ). We know however that these real points reduce to $(C_r)_{\mathbb{R}}$ and hence that $z = w$ at real points. But we have

$$1 = \lambda\bar{\mu}a(z, w) = e^{i(\theta - \varphi)} \frac{a(z, w)}{\sqrt{a(z)}\sqrt{a(w)}}, \quad \text{on } \{\tilde{\psi} = 0\}$$

hence when $z = w$ we have $e^{i(\theta - \varphi)} = 1$ and hence $x = y$. Since $d_{\tilde{x}}\tilde{\psi}(x, y)|_{x=y} = \alpha_x$, it follows that the real points indeed equal diag(Σ). Therefore $t\tilde{\psi}$ parametrizes C and hence there exists a classical symbol for which $\Pi(x, y)$ has the stated oscillatory integral representation.

To show that the phase is of positive type, we need to describe the asymptotics of $a(z, w)$ near the diagonal. Note that in the almost-complex case, we cannot describe $a(z, w)$ as the almost analytic extension of $a(z, z)$. (Of course, $\tilde{a}(\tilde{z}, \tilde{w})$ is the almost analytic extension of $a(z, w)$, by definition.) For our near-diagonal asymptotics in the nonintegrable case, we instead use the following second order expansion of a at points on the diagonal:

LEMMA **2.2.** *Suppose that* $(z_1, \ldots, z_m)$ *are preferred coordinates and* e_L *is a preferred frame at a point* $P_0 \in M$. *Then the Taylor expansion of* $a(z, w)$ *at* $z = w = 0$ *is*

$$a(z, w) = 1 + z \cdot \bar{w} + \cdots .$$

Proof. To begin, we recall that $a(0, 0) = a(0) = \|e_L^*(P_0)\|^2 = 1$. To compute the first and second order terms, we return to the equation

$$\zeta_j\left(z, \lambda, \frac{d\lambda}{\lambda} + d_z \log a(z, w)\right) = 0, \quad \forall(z, \lambda; w) \in X \times M. \tag{51}$$

Let us write $\zeta_j = \zeta_j^{(1)} + R_j^{(2)}$, where $R_j^{(2)}$ vanishes to second order on Σ and we recall that $\zeta_j^{(1)}(\xi) = (\bar{Z}_j, \xi)$. Let us also Taylor expand $\log a$:

$$\log a = L(z, w) + Q(z, w) + \cdots ,$$

where L is linear and Q is quadratic. Since e_L is a preferred frame at P_0, it follows from (6) that $a(z, z) = 1 + |z|^2 + \cdots$ and hence

$$L(z, z) = 0, \qquad Q(z, z) = |z|^2 . \tag{52}$$

Since $d_z \log a|_{z=w} + \frac{d\lambda}{\lambda} = \alpha \in \Sigma$, it follows from (51) that

$$\zeta_j^{(1)}\left(z, \lambda, \frac{d\lambda}{\lambda} + d_z \log a\right) = -R_j^{(2)}\left(z, \lambda, \frac{d\lambda}{\lambda} + d_z \log a\right) = O(|z - w|^2). \tag{53}$$

Since $a(z,w) = \overline{a(w,z)}$, we can write

$$L(z,w) = \sum_{j=1}^{m} (b_j z_j + c_j \bar{z}_j + \bar{c}_j w_j + \bar{b}_j \bar{w}_j) \,.$$

Since the z_j are preferred coordinates and e_L is a preferred frame at P_0, we can choose the $\bar{Z}_j$ so that $\bar{Z}_j(0) = \frac{\partial}{\partial \bar{z}_j}$ and hence by (53),

$$0 = \zeta_j^{(1)}\left(z, \lambda, \frac{d\lambda}{\lambda} + d_z \log a\right)\Bigg|_{z=w=0,\lambda=1} = \left(\frac{\partial}{\partial \bar{z}_j}, d_z \log a\right)\Bigg|_{(0,0)} = c_j \ \forall j \,.$$

Since $L(z,z) = 0$, we have $b_j + \bar{c}_j = 0$, and hence $L = 0$.

To investigate the quadratic term Q in (52), we write

$$(\frac{d\lambda}{\lambda} + d_z \log a)|_{(z,w)} = \alpha_z + \sum_{j=1}^{m} \left[z_j U_j' + \bar{z}_j U_j'' + w_j V_j' + \bar{w}_j V_j''\right] + O(|z|^2 + |w|^2) \,, \qquad (54)$$

where

$$U_j' = \sum_{k=1}^{m}\left(\frac{\partial^2 Q}{\partial z_j \partial z_k} dz_k + \frac{\partial^2 Q}{\partial z_j \partial \bar{z}_k} d\bar{z}_k\right), \qquad U_j'' = \sum_{k=1}^{m}\left(\frac{\partial^2 Q}{\partial \bar{z}_j \partial z_k} dz_k + \frac{\partial^2 Q}{\partial \bar{z}_j \partial \bar{z}_k} d\bar{z}_k\right),$$

$$V_j' = \sum_{k=1}^{m}\left(\frac{\partial^2 Q}{\partial w_j \partial z_k} dz_k + \frac{\partial^2 Q}{\partial w_j \partial \bar{z}_k} d\bar{z}_k\right), \qquad V_j'' = \sum_{k=1}^{m}\left(\frac{\partial^2 Q}{\partial \bar{w}_j \partial z_k} dz_k + \frac{\partial^2 Q}{\partial \bar{w}_j \partial \bar{z}_k} d\bar{z}_k\right).$$

Applying $\zeta_k^{(1)}$ to (54) and using (53) and the fact that $\zeta_k^{(1)}(\alpha_z) = 0$, we have

$$\sum_{j=1}^{m}\left[z_j(\bar{Z}_k|_z, U_j') + \bar{z}_j(\bar{Z}_k|_z, U_j'') + w_j(\bar{Z}_k|_z, V_j') + \bar{w}_j(\bar{Z}_k|_z, V_j'')\right] = O(|z|^2 + |w|^2) \,. \qquad (55)$$

By (3) and (8),

$$\bar{Z}_k|_z = \frac{\partial}{\partial \bar{z}_k} + \sum_{l=1}^{m} B_{kl}(z)\frac{\partial}{\partial z_l} + C_k(z)\frac{\partial}{\partial \theta} \,, \quad B_{kl}(0) = 0 \,.$$

Hence by (55),

$$\frac{\partial^2 Q}{\partial z_j \partial \bar{z}_k} = \left(\frac{\partial}{\partial \bar{z}_k}, U_j'\right) = (\bar{Z}_k|_0, U_j') = 0 \,.$$

Similarly, $\frac{\partial^2 Q}{\partial \bar{z}_j \partial \bar{z}_k} = \frac{\partial^2 Q}{\partial w_j \partial \bar{z}_k} = \frac{\partial^2 Q}{\partial \bar{w}_j \partial \bar{z}_k} = 0$. Thus $Q(z,w)$ has no terms containing $\bar{z}_k$. Since $Q(z,w) = \overline{Q(w,z)}$, the quadratic function Q also has no terms containing w_k, so we can write

$$Q(z,w) = B(z,z) + H(z,\bar{w}) + \overline{B(w,w)} \,,$$

where B, resp. H, is a bilinear, resp. hermitian, form on $\mathbb{C}^m$. Since $Q(z,z) = |z|^2$ (recall (52)), we conclude that $B(z,z) = 0$ and hence $Q(z,w) = H(z,\bar{w}) = z \cdot \bar{w}$. $\qquad\square$

To complete the proof of Theorem 2.1, it remains to show that the phase is of positive type; i.e., $\Im\psi \geq 0$ on some neighborhood of the diagonal in $X \times X$. Let $x \in X$ be arbitrary

and choose Heisenberg coordinates (z, θ) at $P_0 = \pi(x)$ (so that x has coordinates $(0,0)$). Recalling that $\lambda = a(z)^{-\frac{1}{2}} e^{i\theta}$ on X, we have by Lemma 2.2,

$$\frac{1}{i} \psi(0, 0; z, \theta) = 1 - \frac{a(0, z)}{\sqrt{a(z)}} e^{-i\theta} = (1 - e^{-i\theta}) + e^{-i\theta} \left[\frac{1}{2} |z|^2 + O(|z|^3) \right] \, .$$

Thus,

$$\Re \left[\frac{1}{i} \psi(0, 0; z, \theta) \right] \geq 0 \quad \text{for } |\theta| < \frac{\pi}{2}, \ |z| < \varepsilon \, ,$$

where ε is independent of the point $P_0 \in M$.   □

2.2. **Scaling limit of the Szegö kernel.** The Szegö kernels Π_N are the Fourier coefficients of Π defined by:

$$\Pi_N(x, y) = \int_0^\infty \int_0^{2\pi} e^{-iN\theta} e^{it\psi(r_\theta x, y)} s(r_\theta x, y, t) d\theta dt \tag{56}$$

where r_θ denotes the S^1 action on X. Changing variables $t \mapsto Nt$ gives

$$\Pi_N(x, y) = N \int_0^\infty \int_0^{2\pi} e^{iN(-\theta + t\psi(r_\theta x, y))} s(r_\theta x, y, Nt) d\theta dt \, . \tag{57}$$

We now determine the scaling limit of the Szegö kernel by the argument of [BSZ2]. For the sake of completeness, we provide the details of the argument and add some new details on homogeneities, which are useful in applications. To describe the scaling limit at a point $x_0 \in X$, we choose a Heisenberg chart $\rho : U, 0 \to X, x_0$ centered at $P_0 = \pi(x_0) \in M$. Recall (§1.2) that choosing ρ is equivalent to choosing preferred coordinates centered at P_0 and a preferred local frame e_L at P_0. We then write the Szegö kernel Π_N in terms of these coordinates:

$$\Pi_N^{P_0}(u, \theta; v, \varphi) = \Pi_N(\rho(u, \theta), \rho(v, \varphi)) \, ,$$

where the superscript P_0 is a reminder that we are using coordinates centered at P_0. (We remark that the function $\Pi_N^{P_0}$ depends also on the choice of preferred coordinates and preferred frame, which we omit from the notation.) The first term in our asymptotic formula below says that the Nth scaled Szegö kernel looks approximately like the Szegö kernel of level one for the reduced Heisenberg group (recall (41)):

$$\Pi_N^{P_0}\left(\frac{u}{\sqrt{N}}, \frac{\theta}{N}; \frac{v}{\sqrt{N}}, \frac{\varphi}{N}\right) \approx \Pi_1^{\mathbf{H}}(u, \theta; v, \varphi) = \frac{1}{\pi^m} e^{i(\theta - \varphi) + i\Im(u \cdot \bar{v}) - \frac{1}{2}|u - v|^2} \, .$$

In the following, we shall denote the Taylor series of a $\mathcal{C}^\infty$ function f defined in a neighborhood of $0 \in \mathbb{R}^K$ by $f \sim f_0 + f_1 + f_2 + \dots$ where f_j is the homogeneous polynomial part of degree j. We also denote by $R_n^f \sim f_{n+1} + \cdots$ the remainder term in the Taylor expansion.

The following is our main result on the scaling asymptotics of the Szegö kernels near the diagonal. Since the result is of independent interest, we state our asymptotic formula in a more precise form than is needed for the applications in this paper.

THEOREM 2.3. *Let $P_0 \in M$ and choose a Heisenberg coordinate chart about P_0. Then*

$$N^{-m} \Pi_N^{P_0}\left(\tfrac{u}{\sqrt{N}}, \tfrac{\theta}{N}; \tfrac{v}{\sqrt{N}}, \tfrac{\varphi}{N}\right)$$

$$= \Pi_1^{\mathbf{H}}(u, \theta; v, \varphi) \left[1 + \sum_{r=1}^K N^{-r/2} b_r(P_0, u, v) + N^{-(K+1)/2} R_K(P_0, u, v, N) \right] \, ,$$

where:

- $b_r = \sum_{\alpha=0}^{2[r/2]} \sum_{j=0}^{[3r/2]} (\psi_2)^\alpha Q_{r,\alpha,3r-2j}$, *where $Q_{r,\alpha,d}$ is homogeneous of degree d and*

$$\psi_2(u,v) = u \cdot \bar{v} - \frac{1}{2}(|u|^2 + |v|^2)\,;$$

  in particular, b_r has only even homogeneity if r is even, and only odd homogeneity if r is odd;

- $\|R_K(P_0, u, v, N)\|_{\mathcal{C}^j(\{|u| \le \rho, \ |v| \le \rho\}} \le C_{K,j,\rho}$ *for $j \ge 0$, $\rho > 0$ and $C_{K,j,\rho}$ is independent of the point P_0 and choice of coordinates.*

Proof. We now fix P_0 and consider the asymptotics of

$$
\begin{aligned}
\Pi_N&(\frac{u}{\sqrt{N}}, 0; \frac{v}{\sqrt{N}}, 0) \\
&= N \int_0^\infty \int_0^{2\pi} e^{iN(-\theta + t\psi(\frac{u}{\sqrt{N}}, \theta; \frac{v}{\sqrt{N}}, 0))} s(\frac{u}{\sqrt{N}}, \theta; \frac{v}{\sqrt{N}}, 0), Nt) d\theta dt\,,
\end{aligned}
\tag{58}
$$

where ψ and s are the phase and symbol from Theorem 2.1 written in terms of the Heisenberg coordinates.

On X we have $\lambda = a(z)^{-\frac{1}{2}} e^{i\varphi}$. So for $(x,y) = (z, \varphi, w, \varphi') \in X \times X$, we have by Theorem 2.1,

$$\psi(z, \varphi, w, \varphi') = i\left[1 - \frac{a(z,w)}{\sqrt{a(z)}\sqrt{a(w)}} e^{i(\varphi - \varphi')}\right]\,.\tag{59}$$

It follows that

$$
\begin{aligned}
\psi&(\frac{u}{\sqrt{N}}, \theta; \frac{v}{\sqrt{N}}, 0) \\
&= i\left[1 - \frac{a(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}})}{\sqrt{a(\frac{u}{\sqrt{N}}, \frac{u}{\sqrt{N}})}\sqrt{a(\frac{v}{\sqrt{N}}, \frac{v}{\sqrt{N}})}} e^{i\theta}\right]\,.
\end{aligned}
\tag{60}
$$

We observe that the asymptotic expansion of a function $f(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}})$ in powers of $N^{-\frac{1}{2}}$ is just the Taylor expansion of f at $u = v = 0$. By Lemma 2.2 and the notational convention established above, we have

$$a(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}) = 1 + \frac{1}{N} u \cdot \bar{v} + R_3^a(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}})\,, \qquad R_3^a(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}) = O(N^{-3/2})\,.\tag{61}$$

The entire phase

$$t\psi(\frac{u}{\sqrt{N}}, \theta; \frac{v}{\sqrt{N}}, 0) - \theta = it\left[1 - \frac{a(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}})}{a(\frac{u}{\sqrt{N}}, \frac{u}{\sqrt{N}})^{\frac{1}{2}} a(\frac{v}{\sqrt{N}}, \frac{v}{\sqrt{N}})^{\frac{1}{2}}} e^{i\theta}\right] - \theta\tag{62}$$

then has the asymptotic N-expansion

$$it[1 - e^{i\theta}] - \theta - \frac{it}{N}\psi_2(u,v)e^{i\theta} + tR_3^\psi(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}})e^{i\theta}\,.\tag{63}$$

As in [BSZ2], we absorb $(i\psi_2 + NR_3^\psi)te^{i\theta}$ into the amplitude, so that $\Pi_N^{P_0}(\frac{u}{\sqrt{N}}, 0; \frac{v}{\sqrt{N}}, 0)$ is an oscillatory integral with phase

$$\Psi(t, \theta) := it(1 - e^{i\theta}) - \theta$$

and with amplitude

$$A(t, \theta; P_0, u, v) := N e^{te^{i\theta}\psi_2(u,v) + ite^{i\theta}NR_3^\psi(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}})} \sum_{k=0}^{\infty} N^{m-k} t^{m-k} s_k(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}, \theta) ;$$

i.e.,

$$\Pi_N(\frac{u}{\sqrt{N}}, 0; \frac{v}{\sqrt{N}}, 0) = \int_0^{\infty} \int_0^{2\pi} e^{iN\Psi(t,\theta)} A(t, \theta; P_0, u, v) d\theta dt \qquad (64)$$

Before proceeding, it is convenient to expand $\exp\left[ite^{i\theta}NR_3^\psi(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}})\right]$ in powers of $N^{-\frac{1}{2}}$ and to keep track of the homogeneity in (u, v) of the coefficients. We simplify the notation by writing $g(t, \theta) := ite^{i\theta}$. By definition,

$$R_3^\psi(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}) \sim N^{-3/2}\psi_3(u, v) + N^{-2}\psi_4(u, v) + \cdots + N^{-d/2}\psi_d(u, v) + \cdots .$$

We then have

$$e^{NgR_3^\psi(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}})} \sim \sum_{r=0}^{\infty} N^{-r/2} c_r(u, v) , \qquad (65)$$

where

$$c_r = \sum_{\lambda=1}^{r} c_{r,r+2\lambda}(u, v; t, \theta) , \ r \geq 1 , \quad c_0 = c_{00} = 1 , \qquad (66)$$

with c_{rd} homogeneous of degree d in u, v. (The explicit formula for c_{rd} is:

$$c_{rd} = \sum \left\{ \frac{g^n}{n!} \Pi_{j=1}^{n} \psi_{a_j}(u, v) : n \geq 1, a_j \geq 3, \sum_{j=1}^{n} a_j = d, \sum_{j=1}^{n}(a_j - 2) = r \right\} , \ r \geq 1 .$$

The range of d is determined by the fact that $d = \sum_{j=1}^{n} a_j = r + 2n$ with $0 \leq n \leq r$.)

We further decompose the factor $\sum_{k=0}^{\infty} N^{m-k} t^{m-k} s_k(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}, \theta)$ into the homogeneous terms $\sum_{k,\ell=0}^{\infty} N^{m-k-\ell/2} t^{m-k} s_{k\ell}(P_0, u, v)$ where $s_{k\ell}$ is the homogeneous term of degree ℓ of s_k. Finally, we have

$$A \ \sim \ N e^{gNR_3^\psi(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}, \theta)} \sum_{k=0}^{\infty} N^{m-k} t^{m-k} s_k(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}, \theta)$$

$$= \ N^{m+1} \sum_{n=0}^{\infty} N^{-n/2} f_n(u, v; t, \theta, P_0) ,$$

$$f_n = \sum_{r+\ell+2k=n} c_r s_{k\ell} = \sum_{k=0}^{[n/2]} t^{m-k} \left(s_{k,n-2k} + \sum_{r=1}^{n-2k} \sum_{\lambda=1}^{r} c_{r,r+2\lambda} s_{k,n-2k-r} \right) = \sum_{j=0}^{[3n/2]} f_{n,3n-2j}$$

where $f_{n,d}$ is homogeneous of degree d in (u, v). (The asymptotic expansion holds in the sense of semiclassical symbols, i.e. the remainder after summing K terms is a symbol of order $m - k - (K + 1)/2$.)

We now evaluate the integral for Π_N by the method of stationary phase as in [BSZ2]. The phase is independent of the parameters (u, v) and we have

$$\begin{aligned} \tfrac{\partial}{\partial t}\Psi &= i(1 - e^{i\theta}) \\ \tfrac{\partial}{\partial\theta}\Psi &= te^{i\theta} - 1 \end{aligned} \qquad (67)$$

so the critical set of the phase is the point $\{t = 1, \theta = 0\}$. The Hessian Ψ'' on the critical set equals

$$\begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$$

so the phase is non-degenerate and the Hessian operator L_Ψ is given by

$$L_\Psi = \langle \Psi''(1,0)^{-1}D, D \rangle = 2\frac{\partial^2}{\partial t\partial\theta} - i\frac{\partial^2}{\partial t^2} .$$

We smoothly decompose the integral into one over $|t - 1| < 1$ and one over $|t - 1| > \frac{1}{2}$. Since the only critical point of the phase occurs at $t = 1, \theta = 0$, the latter is rapidly decaying in N and we may assume the integrand to be smoothly cut off to $|t - 1| < 1$. It follows by the stationary phase method for complex oscillatory integrals ([Hör], Theorem 7.7.5) that

$$N^{-m}\Pi_N^{P_0}\left(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}, \theta\right) = C\sum_{j=0}^{J}\sum_{n=0}^{K} N^{-n/2-j}L_j[e^{-ig\psi_2}f_n]|_{t=1,\theta=0} + \widehat{R}_{JK}(P_0, u, v, N), \quad (68)$$

where

$$C = N\frac{1}{\sqrt{\det(N\Psi''(1,0)/2\pi i)}} = \sqrt{-2\pi i}$$

and L_j is the differential operator of order $2j$ in (t, θ) defined by

$$L_j f(t, \theta; P_0, u, v) = \sum_{\nu-\mu=j}\sum_{2\nu\geq 3\mu}\frac{1}{2^\nu i^j \mu!\nu!}L_\Psi^\nu[f(t, \theta; P_0, u, v)(R_3^\Psi)^\mu(t, \theta)] \qquad (69)$$

with $R_3^\Psi(t, \theta)$ the third order remainder in the Taylor expansion of Ψ at $(t, \theta) = (1, 0)$. Also, the remainder is estimated by

$$|\widehat{R}_{JK}(P_0, u, v, N)| \leq C'N^{-J-\frac{K+1}{2}}\sum_{n=0}^{K}\sum_{|\alpha|\leq 2J+2}\sup_{t,\theta}|D_{t,\theta}^\alpha e^{-ig\psi_2}f_n|. \qquad (70)$$

Since L_Ψ is a second order operator in (t, θ), we see that

$$L_j[e^{-ig\psi_2}f_n]|_{t=1,\theta=0} = e^{\psi_2}\sum_{\alpha\leq 2j}(\psi_2)^\alpha F_{nj\alpha} . \qquad (71)$$

Therefore

$$
\begin{aligned}
N^{-m}\Pi_N^{P_0}(\frac{u}{\sqrt{N}},\frac{v}{\sqrt{N}},\theta) \;\; &\sim \;\; e^{\psi_2}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\alpha=0}^{2j}(\psi_2)^{\alpha}N^{-\frac{n}{2}-j}F_{nj\alpha} \\
&\sim \;\; e^{\psi_2}\sum_{r=0}^{\infty}\sum_{j=0}^{[r/2]}\sum_{\alpha=0}^{2j}(\psi_2)^{\alpha}N^{-r/2}F_{r-2j,j,\alpha} \\
&\sim \;\; e^{\psi_2}\sum_{r=0}^{\infty}\sum_{\alpha=0}^{2[r/2]}(\psi_2)^{\alpha}N^{-r/2}Q_{r\alpha}\,.
\end{aligned}
\tag{72}
$$

Thus, as with f_n we have the homogeneous expansion:

$$
Q_{r\alpha}=\sum_{j=0}^{[3r/2]}Q_{r,\alpha,3r-2j}\,.
\tag{73}
$$

Here, $Q_{r,\alpha,d}$ is homogeneous of degree d in (u,v). (The term ψ_2 is distinguished by being 'holomorphic' in u and 'anti-holomorphic' in v in a sense to be elaborated below.) Thus we have the desired Taylor series. The estimate for the remainder follows from (70). $\square$

3. Kodaira embedding and Tian almost isometry theorem

Definition: By the Kodaira maps we mean the maps $\Phi_N : M \to PH^0(M,L^N)'$ defined by $\Phi_N(z) = \{s^N : s^N(z) = 0\}$. Equivalently, we can choose an orthonormal basis $S_1^N,\ldots,S_{d_N}^N$ of $H^0(M,L^N)$ and write

$$
\Phi_N : M \to \mathbb{CP}^{d_N-1}\,, \qquad \Phi_N(z) = \left(S_1^N(z):\cdots:S_{d_N}^N(z)\right).
\tag{74}
$$

We also define the lifts of the Kodaira maps:

$$
\widetilde{\Phi}_N : X \to \mathbb{C}^{d_N}\,, \qquad \widetilde{\Phi}_N(x) = (S_1^N(x),\ldots,S_{d_N}^N(x))\,.
\tag{75}
$$

Note that

$$
\Pi_N(x,y) = \widetilde{\Phi}_N(x)\cdot\overline{\widetilde{\Phi}_N(y)}\,;
\tag{76}
$$

in particular,

$$
\Pi_N(x,x) = \|\widetilde{\Phi}_N(x)\|^2\,.
\tag{77}
$$

We now prove the following generalization to the symplectic category of the asymptotic expansion theorem of [Zel] (also proved independently by [Cat] using the Bergman kernel in place of the Szegö kernel) and Tian's approximate isometry theorem [Tian]:

THEOREM 3.1. *Let $L \to (M,\omega)$ be the pre-quantum line bundle over a $2m$-dimensional symplectic manifold, and let $\{\Phi_N\}$ be its Kodaira maps. Then:*
(a) There exists a complete asymptotic expansion:

$$
\Pi_N(z,0;z,0) = a_0 N^m + a_1(z)N^{m-1} + a_2(z)N^{m-2} + \ldots
$$

for certain smooth coefficients $a_j(z)$ with $a_0 = \pi^{-m}$. Hence, the maps Φ_N are well-defined for $N \gg 0$.

(b) *Let ω_{FS} denote the Fubini-Study form on $\mathbb{CP}^{d_N-1}$. Then*

$$\|\frac{1}{N}\Phi_N^*(\omega_{FS}) - \omega\|_{\mathcal{C}^k} = O(\frac{1}{N})$$

for any k.

Proof. (a) Using the expansion of Theorem 2.3 with $u = v = 0$ and noting that $b_r(z,0,0) = 0$ for r odd, we obtain the above expansion of $\Pi_N(z,0;z,0)$ with $a_r(z) = b_{2r}(z,0,0)$. (The expansion also follows by precisely the same proof as in [Zel].)

(b) In the holomorphic case, (b) followed by differentiating (a), using that $\Phi_N^*(\partial\bar{\partial}\log|\xi|^2) = \partial\bar{\partial}\log|\Phi_N|^2$. In the almost complex case, Φ_N^* does not commute with the complex derivatives, so we need to modify the proof. To do so, we use the following notation: the exterior derivative on a product manifold $Y_1 \times Y_2$ can be decomposed as $d = d^1 + d^2$, where d^1 and d^2 denote exterior differentiation on the first and second factors, respectively. (This is formally analogous to the decomposition $d = \partial + \bar{\partial}$; e.g., $d^1d^1 = d^2d^2 = d^1d^2 + d^2d^1 = 0$.)

Recall that the Fubini-Study form ω_{FS} on $\mathbb{CP}^{m-1}$ is induced by the 2-form $\widetilde{\omega}_m = \frac{i}{2}\partial\bar{\partial}\log|\xi|^2$ on $\mathbb{C}^m \setminus \{0\}$. We consider the 2-form Ω on $(\mathbb{C}^m \setminus \{0\}) \times (\mathbb{C}^m \setminus \{0\})$ given by

$$\Omega = \frac{i}{2}\partial\bar{\partial}\log\zeta \cdot \bar{\eta} = \frac{i}{2}d^1d^2\log\zeta \cdot \bar{\eta}.$$

Note that Ω is smooth on a neighborhood of the diagonal $\{\zeta = \eta\}$, and

$$\Omega|_{\zeta=\eta} = \widetilde{\omega}_m$$

(where the restriction to $\{\zeta = \eta\}$ means the pull-back under the map $\zeta \mapsto (\zeta, \zeta)$).

It suffices to show that

$$\frac{1}{N}\widetilde{\Phi}_N^*\omega_{d_N} \to \pi^*\omega, \quad \pi : X \to M.$$

To do this, we consider the maps

$$\Psi_N = \widetilde{\Phi}_N \times \widetilde{\Phi}_N : X \times X \to \mathbb{C}^{d_N} \times \mathbb{C}^{d_N}, \quad \Psi_N(x,y) = (\widetilde{\Phi}_N(x), \widetilde{\Phi}_N(y)).$$

It is elementary to check that Ψ_N^* commutes with d^1 and d^2. By (76), we have

$$\Psi_N^*(\log\zeta \cdot \bar{\eta}) = (\log\zeta \cdot \bar{\eta}) \circ \Psi_N = \log\Pi_N.$$

Therefore,

$$\frac{1}{N}\widetilde{\Psi}_N^*\Omega_{d_N} = \frac{i}{2N}\Psi_N^*d^1d^2\log\zeta \cdot \bar{\eta} = \frac{i}{2N}d^1d^2\Psi_N^*\log\zeta \cdot \bar{\eta} = \frac{i}{2N}d^1d^2\log\Pi_N. \tag{78}$$

Restricting (78) to the diagonal, we then have

$$\frac{1}{N}\widetilde{\Phi}_N^*\omega_{d_N} = \frac{i}{2N}(d^1d^2\log\Pi_N)|_{x=y} = \mathrm{diag}^*(d^1d^2\log\Pi_N),$$

where $\mathrm{diag} : X \to X \times X$ is the diagonal map $\mathrm{diag}(x) = (x,x)$.

Using Heisenberg coordinates as in Theorem 2.3, we have by the near-diagonal scaling asymptotics

$$\frac{1}{N}\widetilde{\Phi}_N^*\omega_{d_N}\bigg|_{P_0} = \frac{i}{2N}\mathrm{diag}^*d^1d^2\log\Pi_N^{P_0}(\frac{u}{\sqrt{N}}, \frac{\theta}{N}; \frac{v}{\sqrt{N}}, \frac{\varphi}{N})\bigg|_0$$

$$= \frac{i}{2N}\mathrm{diag}^*d^1d^2\log\Pi_1^{\mathbf{H}}(u,\theta;v,\varphi)\bigg|_0 + O(N^{-\frac{1}{2}}).$$

Finally,

$$
\begin{aligned}
\frac{i}{2N}\mathrm{diag}^* d^1 d^2 \log \Pi_1^{\mathbf{H}}(u,\theta;v,\varphi)\Big|_0 &= \frac{i}{2N}\mathrm{diag}^* d^1 d^2 \left[i(\theta-\varphi)+u\cdot\bar{v}-\frac{1}{2}(|u|^2+|v|^2)\right] \\
&= \frac{i}{2N}\sum_{q=1}^m du_q \wedge d\bar{u}_q \;=\; \frac{i}{2}\sum_{q=1}^m dz_q \wedge d\bar{z}_q \;=\; \omega|_{P_0} \;(79)
\end{aligned}
$$

$\square$

Remark: A more explicit way to show (b) is to expand the Fubini-Study form:

$$
\widetilde{\omega}_m = \frac{i}{2}|\xi|^{-4}\left[|\xi|^2 \sum_{j=1}^m d\xi_j \wedge d\bar{\xi}_j - \sum_{j,k=1}^m \bar{\xi}_j \xi_k d\xi_j \wedge d\bar{\xi}_k\right].
$$

Then

$$
\frac{1}{N}\tilde{\Phi}_N^* \omega_{d_N} = \frac{i}{2}\Pi_N(x,x)^{-2}\{(\Pi_N(x,x)d^1 d^2 \Pi_N(x,y)-d^1 \Pi_N(x,y)\wedge d^2\Pi_N(x,y)\}|_{x=y},
$$

and (b) follows from a short computation using Theorem 2.3 as above.

It follows from Theorem 3.1(b) that Φ_N is an immersion for $N \gg 0$. Using in part an idea of Bouche [Bch], we give a simple proof of the 'Kodaira embedding theorem' for symplectic manifolds:

THEOREM **3.2.** *For N sufficiently large, Φ_N is an embedding.*

Proof. Let $\{P_N, Q_N\}$ be any sequence of distinct points such that $\Phi_N(P_N) = \Phi_N(Q_N)$. By passing to a subsequence we may assume that one of the following two cases holds:

  (i) The distance $r_N := r(P_N, Q_N)$ between P_N, Q_N satisfies $r_N\sqrt{N} \to \infty$;
  (ii) There exists a constant C independent of N such that $r_N \le C\sqrt{N}$.

To prove that case (i) cannot occur, we observe that

$$
\int_{B(P_N,r_N)} |N^{-m}\Pi_N^{P_N}|^2 dV \ge 1 - o(1)
$$

where $\Pi_N^{P_N}(x) = \Pi_N(\cdot, P_N)$ is the 'peak section' at P_N. The same inequality holds for Q_N. If $\Phi_N(P_N) = \Phi_N(Q_N)$ then the total $\mathcal{L}^2$-norm of $\Pi_N(x,\cdot)$ would have to be $\sim 2N^m$, contradicting the asymptotic $\sim N^m$ from Theorem 3.1(a).

To prove that case (ii) cannot occur, we assume on the contrary that $\Phi_N(P_N) = \Phi_N(Q_N)$, where $P_N = \rho_N(0)$ and $Q_N = \rho_N(\frac{v_N}{\sqrt{N}})$, $0 \ne |v_N| \le C$, using a Heisenberg coordinate chart ρ_N about P_N. We consider the function

$$
f_N(t) = \frac{|\Pi_N^{P_N}(0,\frac{tv_N}{\sqrt{N}})|^2}{\Pi_N^{P_N}(0,0)\Pi_N^{P_N}(\frac{tv_N}{\sqrt{N}},\frac{tv_N}{\sqrt{N}})}. \tag{80}
$$

Recalling that

$$
\Pi_N(x,y) = \tilde{\Phi}_N(x)\cdot\overline{\tilde{\Phi}_N(y)},
$$

we see that $f_N(0) = 1$, which is a global and strict local maximum of f_N; furthermore, since $\Phi_N(P_N) = \Phi_N(Q_N)$, we also have $f_N(1) = 1$. Thus for some value of t_N in the open interval $(0, 1)$, we have $f_N''(t_N) = 0$. By Theorem 2.3,

$$f_N(t) = e^{-|v_N|^2 t^2} \left[1 + N^{-1/2} \widetilde{R}_N(t v_N) \right] , \qquad (81)$$

where

$$\widetilde{R}_N(v) = R_1(P_N; 0, v, N) + R_1(P_N; v, 0, N) - R_1(P_N; v, v, N) - R_1(P_N; 0, 0, N) + O(N^{-1/2}) .$$

The estimate for R_1 yields:

$$\|\widetilde{R}_N\|_{\mathcal{C}^2\{|v| \le C\}} = O(1) \qquad (82)$$

Since $f_N(1) = 1$, it follows from (81)–(82) that $|v_N|^2 = O(N^{-1/2})$. (A more careful analysis shows that we can replace $N^{-1/2}$ with N^{-1} in (81) and thus $|v_N| = O(N^{-1/2})$.)

Write $e^x = 1 + x + x^2 \varphi(x)$. We then have

$$f_N(t) = 1 - |v_N|^2 t^2 + |v_N|^4 t^4 \varphi(|v_N|^2 t^2) + N^{-1/2} \widetilde{R}_N(t v_N) \left[1 - |v_N|^2 t^2 + |v_N|^4 t^4 \varphi(|v_N|^2 t^2) \right] .$$

Thus by (82),

$$f_N''(t) = -2|v_N|^2 + O(|v_N|^4) + O(N^{-1/2}|v_N|^2), \quad |t| \le 1 .$$

Since $|v_N| = o(1)$, it follows that

$$0 = f_N''(t_N) = (-2 + o(1))|v_N|^2 ,$$

which contradicts the assumption that $v_N \ne 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. ASYMPTOTICALLY HOLOMORPHIC VERSUS ALMOST HOLOMORPHIC SECTIONS

We now use the scaling asymptotics of Theorem 2.3 to prove Theorem 0.1, which states that ν_∞-almost every sequence $\{s_N\}$ of sections (with unit $\mathcal{L}^2$-norm) satisfies the sup-norm estimates

$$\|s_N\|_\infty + \|\bar\partial s_N\|_\infty = O(\sqrt{\log N}) ,$$
$$\|\nabla^k s_N\|_\infty + \|\nabla^k \bar\partial s_N\|_\infty = O(N^{\frac{k}{2}} \sqrt{\log N}) ,$$

for $k = 1, 2, 3, \dots$.

The following elementary probability lemma is central to our arguments:

LEMMA **4.1.** *Let $A \in S^{2d-1} \subset \mathbb{C}^d$, and give S^{2d-1} Haar probability measure. Then the probability that a random point $P \in S^{2d-1}$ satisfies the bound $|\langle P, A \rangle| > \lambda$ is $(1 - \lambda^2)^{d-1}$.*

Proof. We can assume without loss of generality that $A = (1, 0, \dots, 0)$. Let

$$V_\lambda = \text{Vol}\big(\{ P \in S^{2d-1} : |P_1| > \lambda \} \big) \qquad (0 \le \lambda < 1) ,$$

where Vol denotes $(2d - 1)$-dimensional Euclidean volume. Our desired probability equals V_λ / V_0. Let $\sigma_n = \text{Vol}(S^{2n-1}) = \frac{2\pi^n}{(n-1)!}$. We compute

$$
\begin{aligned}
V_\lambda &= \int_\lambda^1 \sigma_{d-1}(1 - r^2)^{\frac{2d-3}{2}} \frac{2\pi r \, dr}{\sqrt{1 - r^2}} = 2\pi \sigma_{d-1} \int_\lambda^1 (1 - r^2)^{d-2} r \, dr \\
&= \frac{\pi \sigma_{d-1}}{d - 1}(1 - \lambda^2)^{d-1} = \sigma_d (1 - \lambda^2)^{d-1} .
\end{aligned}
$$

Therefore $V_\lambda/V_0 = (1-\lambda^2)^{d-1}$. $\square$

4.1. **Notation.** For the readers' convenience, we summarize here our notation for the various differential operators that we use in this section and elsewhere in the paper:

a) Derivatives on M:
  - $\frac{\partial}{\partial z_j} = \frac{1}{2}\frac{\partial}{\partial x_j} - \frac{i}{2}\frac{\partial}{\partial y_j},\quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\frac{\partial}{\partial x_j} + \frac{i}{2}\frac{\partial}{\partial y_j};$
  - $Z_j^M = \frac{\partial}{\partial z_j} + \sum \bar{B}_{jk}(z)\frac{\partial}{\partial \bar{z}_k},\ \bar{Z}_j^M = \frac{\partial}{\partial \bar{z}_j} + \sum B_{jk}(z)\frac{\partial}{\partial z_k},\ B_{jk}(P_0) = 0,$
    $\{Z_1,\dots,Z_m\}$ is a local frame for $T^{1,0}M$.

b) Derivatives on X:
  - $\frac{\partial^h}{\partial z_j} = \frac{\partial}{\partial z_j} - A_j(z)\frac{\partial}{\partial \theta} =$ horizontal lift of $\frac{\partial}{\partial z_j},\ A_j(P_0) = 0;$
  - $Z_j =$ horizontal lift of Z_j^M;
  - $d^h = \partial_b + \bar{\partial}_b =$ horizontal exterior derivative on X.

c) Covariant derivatives on M:
  - $\nabla : \mathcal{C}^\infty(M, L^N \otimes (T^*M)^{\otimes k}) \to \mathcal{C}^\infty(M, L^N \otimes (T^*M)^{\otimes(k+1)});$
  - $\nabla^k = \nabla \circ \cdots \circ \nabla : \mathcal{C}^\infty(M, L^N) \to \mathcal{C}^\infty(M, L^N \otimes (T^*M)^{\otimes k});$
  - $\nabla = \partial + \bar{\partial},\ \ \bar{\partial} : \mathcal{C}^\infty(M, L^N) \to \mathcal{C}^\infty(M, L^N \otimes T^{*0,1}M).$

d) Derivatives on $X \times X$:
  - d_j^1, d_j^2: the operator $\frac{\partial^h}{\partial z_j}$ applied to the first and second factors, respectively;
  - Z_j^1, Z_j^2: the operator Z_j applied to the first and second factors, respectively.

4.2. **The estimate $\|s_N\|_\infty/\|s_N\|_2 = O(\sqrt{\log N})$ almost surely.** Throughout this section we assume that $\|s_N\|_{\mathcal{L}^2} = 1$. We begin the proof of Theorem 0.1 by showing that

$$\nu_N\left\{s^N \in SH_J^0(M, L^N) : \sup_M |s_N| > C\sqrt{\log N}\right\} < O\left(\frac{1}{N^2}\right),\qquad (83)$$

for some constant $C < +\infty$. (In fact, for any $k > 0$, we can bound the probabilities by $O(N^{-k})$ by choosing C to be sufficiently large.) The estimate (83) immediately implies that

$$\limsup_{N\to\infty} \frac{\sup_X |s^N|}{\sqrt{\log N}} \leq C \qquad \text{almost surely},$$

which gives the first statement of Theorem 2.3.

We now show (83), following an approach inspired by Nonnenmacher and Voros [NoVo]. Recalling (75), we note that

$$\Pi_N(x, y) = \sum_{j=1}^{d_N} S_j^N(x)\overline{S_j^N(y)} = \langle \tilde{\Phi}_N(x), \tilde{\Phi}_N(y)\rangle.\qquad (84)$$

Let $s^N = \sum_{j=1}^{d_N} c_j S_j^N$ $(\sum |c_j|^2 = 1)$ denote a random element of $SH^0(M, L^N) = S\mathcal{H}_N^2(X)$, and write $c = (c_1, \ldots, c_{d_N})$. Recall that

$$s^N(x) = \int_X \Pi_N(x, y) s^N(y) dy = \sum_{j=1}^{d_N} c_j S_j^N(x) = c \cdot \tilde{\Phi}_N(x). \tag{85}$$

Thus

$$|s^N(x)| = \|\tilde{\Phi}_N(x)\| \cos \theta_x, \quad \text{where} \ \cos \theta_x = \frac{\left| c \cdot \tilde{\Phi}_N(x) \right|}{\|\tilde{\Phi}_N(x)\|}. \tag{86}$$

(Note that θ_x can be interpreted as the distance in $\mathbb{CP}^{d_N - 1}$ between $[\bar{c}]$ and $\tilde{\Phi}_N(x)$.)

We have by Theorem 3.1(a),

$$\|\tilde{\Phi}_N(x)\| = \Pi_N(x, x)^{\frac{1}{2}} = N^{m/2} + O(N^{m/2-1}) = (1 + \varepsilon_N) N^{m/2}, \tag{87}$$

where ε_N denotes a term satisfying the uniform estimate

$$\sup_{x \in X} |\varepsilon_N(x)| \leq O\left(\frac{1}{N}\right). \tag{88}$$

Now fix a point $x \in X$. By Lemma 4.1,

$$\nu_N \left\{ s^N : \cos \theta_x \geq C N^{-m/2} \sqrt{\log N} \right\} = \left(1 - \frac{C^2 \log N}{N^m} \right)^{d_N - 1}$$

$$\leq \left(e^{-\frac{C^2 \log N}{N^m}} \right)^{d_N - 1} = N^{-C^2 N^{-m}(d_N - 1)}. \tag{89}$$

We can cover M by a collection of k_N balls $B(z^j)$ of radius

$$R_N := \frac{1}{N^{\frac{m+1}{2}}} \tag{90}$$

centered at points $z^1, \ldots, z^{k_N}$, where

$$k_N \leq O(R^{-2m}) \leq O(N^{m(m+1)}).$$

By (89), we have

$$\nu_N \left\{ s^N \in SH_J^0(M, L^N) : \max_j \cos \theta_{x^j} \geq C N^{-m/2} \sqrt{\log N} \right\} \leq k_N N^{-C^2 N^{-m}(d_N - 1)}, \tag{91}$$

where x^j denotes a point in X lying above z^j.

Equation (91) together with (86)–(87) implies (by the argument below) that the desired sup-norm estimate holds at the centers of the small balls with high probability. To complete the proof of (83), we first need to extend (91) to points within the balls. To do this, we consider an arbitrary point $w^j \in B(z^j)$, and choose points $y^j \in X$ lying above the points w^j. We must estimate the distance, which we denote by δ_N^j, between $\Phi_N(z^j)$ and $\Phi_N(w^j)$ in

$\mathbb{CP}^{d_N-1}$. Letting γ denote the geodesic in M from z^j to w^j, we conclude by Theorem 3.1(b) that

$$
\begin{aligned}
\delta_N^j &\leq \int_{\Phi_{N*}\gamma} \sqrt{\omega_{FS}} = \int_\gamma \sqrt{\Phi_N^* \omega_{FS}} \leq \sqrt{N} \int_\gamma (1+\varepsilon_N)) \sqrt{\omega} \\
&\leq (1+\varepsilon_N) N^{\frac{1}{2}} R_N = \frac{1+\varepsilon_N}{N^{m/2}}.
\end{aligned}
\tag{92}
$$

By the triangle inequality in $\mathbb{CP}^{d_N-1}$, we have $|\theta_{x^j} - \theta_{y^j}| \leq \delta_N^j$. Therefore by (92),

$$
\cos\theta_{x^j} \geq \cos\theta_{y^j} - \delta_N^j \geq \cos\theta_{y^j} - \frac{1+\varepsilon_N}{N^{m/2}}.
\tag{93}
$$

By (93),

$$
\cos\theta_{y^j} \geq \frac{(C+1)\sqrt{\log N}}{N^{m/2}} \Rightarrow \cos\theta_{x^j} \geq \frac{(C+1)\sqrt{\log N} - (1+\varepsilon_N)}{N^{m/2}} \geq \frac{C\sqrt{\log N}}{N^{m/2}}
$$

and thus

$$
\left\{ s^N \in SH_J^0(M, L^N) : \sup\cos\theta \geq (C+1)N^{-m/2}\sqrt{\log N} \right\}
$$
$$
\subset \left\{ s^N \in SH_J^0(M, L^N) : \max_j \cos\theta_{x^j} \geq C N^{-m/2}\sqrt{\log N} \right\}.
$$

Hence by (91),

$$
\nu_N \left\{ s^N \in SH_J^0(M, L^N) : \sup\cos\theta \geq (C+1)N^{-m/2}\sqrt{\log N} \right\} \leq k_N N^{-C^2 N^{-m}(d_N-1)}.
\tag{94}
$$

By the Riemann-Roch formula of Boutet de Monvel - Guillemin [BoGu, §14] (which is a consequence of Theorem 1.1), we have the estimate for the dimensions d_N:

$$
d_N = \frac{c_1(L)^m}{m!} N^m + O(N^{m-1}).
\tag{95}
$$

We can also obtain (95) from Theorem 3.1(a) as follows: We note first that

$$
\int_X \Pi_N(x,x) d\mathrm{Vol}_X = \int_X \sum_{j=1}^{d_N} |S_j^N(x)|^2 d\mathrm{Vol}_X = d_N.
$$

On the other hand, by Theorem 3.1(a),

$$
\int_X \Pi_N(x,x) d\mathrm{Vol}_X = \left[\frac{1}{\pi^m} N^m + O(N^{m-1}) \right] \mathrm{Vol}(X),
$$

where

$$
\mathrm{Vol}(X) = \mathrm{Vol}(M) = \int_M \frac{1}{m!} \omega^m = \frac{\pi^m}{m!} c_1(L)^m.
$$

Equating the above computations of the integral yields (95).

It follows from (86), (87), (94) and (95) that

$$
\nu_N \left\{ s^N \in SH_J^0(M, L^N) : \sup_M |s^N| \geq (C+2)\sqrt{\log N} \right\}
$$
$$
\leq k_N N^{-C^2 N^{-m}(d_N-1)} \leq O\left(N^{m(m+1) - \frac{C^2}{m!+1}} \right).
$$

Choosing $C = (m+1)\sqrt{m!+1}$, we obtain (83).

Remark: An alternate proof of this estimate, which does not depend on Tian's theorem, is given by the case $k = 0$ of the $\mathcal{C}^k$ estimate in the next section.

4.3. The estimate $\|\nabla^k s_N\|_\infty / \|s_N\|_2 = O(\sqrt{N^k \log N})$ almost surely.
The proof of this and the other assertions of Theorem 0.1 follow the pattern of the above sup-norm estimate. First we note a consequence (Lemma 4.3) of our near-diagonal asymptotics. Recall that a differential operator on X is horizontal if it is generated by horizontal vector fields. In particular the operators $\nabla^k : \mathcal{C}^\infty(M, L^N) \to \mathcal{C}^\infty(M, L^N \otimes (T^*M)^{\otimes k})$ are given by (vector valued) horizontal differential operators (independent of N) on X. By definition, horizontal differential operators on $X \times X$ are generated by the horizontal differential operators on the first and second factors. We begin with the following estimate:

LEMMA **4.2.** *Let P_k be a horizontal differential operator of order k on $X \times X$. Then*

$$P_k \Pi_N(x, y)|_{x=y} = O(N^{m+k/2}).$$

Proof. Let $x_0 = (P_0, 0)$ be an arbitrary point of X, and choose local **real** coordinates $(x_1, \ldots, x_{2m}, \theta)$ about $(P_0, 0)$ as in the hypothesis of Theorem 2.3 (with $z_q = x_q + ix_{m+q}$). We let $\frac{\partial^h}{\partial x_q}$ denote the horizontal lift of $\frac{\partial}{\partial x_q}$ to X:

$$\frac{\partial^h}{\partial x_q} = \frac{\partial}{\partial x_q} - \tilde{A}_q(x) \frac{\partial}{\partial \theta}, \quad \tilde{A}_q = (\alpha, \frac{\partial}{\partial x_q}).$$

Since $\frac{\partial}{\partial x_q}\Big|_{x_0}$ is assumed to be horizontal, we have $\tilde{A}_q(P_0) = 0$.

We let d_q^1, d_q^2 denote the operator $\frac{\partial^h}{\partial x_q}$ applied to the first and second factors, respectively, on $X \times X$. For this result, we need only the zeroth order estimate of Theorem 2.3:

$$\Pi_N\left(\frac{u}{\sqrt{N}}, \frac{s}{N}; \frac{v}{\sqrt{N}}, \frac{t}{N}\right) = N^m e^{i(s-t)+\psi_2(u,v)} \mathcal{R}(P_0, u, v, N), \tag{96}$$

where $\mathcal{R}(P_0, u, v, N)$ denotes a term satisfying the remainder estimate of Theorem 2.3: $\|\mathcal{R}(P_0, u, v, N)\|_{\mathcal{C}^j(\{|u| \le \rho, \ |v| \le \rho\})} \le C_{j,\rho}$ for $j \ge 0$, $\rho > 0$, where $C_{j,\rho}$ is independent of the point P_0 and choice of coordinates.

Differentiating (96) and noting that $\partial/\partial x_q = \sqrt{N} \partial/\partial u_q$, $\partial/\partial \theta = N \partial/\partial s$, we have

$$
\begin{aligned}
& d_q^1 \Pi_N\left(\frac{u}{\sqrt{N}}, \frac{s}{N}; \frac{v}{\sqrt{N}}, \frac{t}{N}\right) \\
&= \sqrt{N} \left(\frac{\partial}{\partial u_q} - \sqrt{N} \tilde{A}_q(P_0 + \frac{u}{\sqrt{N}}) \frac{\partial}{\partial s}\right) \left(N^m e^{i(s-t)+\psi_2(u,v)} \mathcal{R}\right) \\
&= N^{m+1/2} e^{i(s-t)+\psi_2(u,v)} \left\{\left[L_q(u,v) - i\sqrt{N} \tilde{A}_q(\tfrac{u}{\sqrt{N}})\right] + \frac{\partial}{\partial u_q} \mathcal{R}\right\} \\
&= N^{m+1/2} e^{i(s-t)+\psi_2(u,v)} \widetilde{\mathcal{R}} = O(N^{m+1/2}),
\end{aligned}
\tag{97}
$$

where $L_q := \frac{\partial \psi_2}{\partial u_q}$ is a linear function. The same estimate holds for $d_q^2 \Pi_N$. Indeed, the above computation yields:

$$d_q^j e^{i(s-t)+\psi_2(u,v)} \mathcal{R}(P_0, u, v, N) = \sqrt{N} e^{i(s-t)+\psi_2(u,v)} \widetilde{\mathcal{R}}(P_0, u, v, N), \tag{98}$$

for $j = 1, 2$, $q = 1, \ldots, 2m$. The desired estimate follows by iterating (98). $\qquad\square$

Remark: The assumption that P_k is horizontal in Lemma 4.2 is necessary, since the operator $\partial/\partial\theta$ multiplies the estimate by N instead of $\sqrt{N}$.

LEMMA **4.3.** *Let P_k be a horizontal differential operator of order k on X. Then*

$$\sup_X \|P_k\tilde{\Phi}_N\| = O(N^{\frac{m+k}{2}})\,.$$

Proof. Let P_k^1, P_k^2 denote the operator P_k applied to the first and second factors, respectively, on $X \times X$. Differentiating (84) and restricting to the diagonal, we obtain

$$P_k^1 \bar{P}_k^2 \Pi_N(x,x) = \left\|P_k\tilde{\Phi}_N(x)\right\|^2\,. \tag{99}$$

The conclusion follows from (99) and Lemma 4.2 applied to the horizontal differential operator (of order $2k$) $P_k^1\bar{P}_k^2$ on $X \times X$. $\qquad\square$

We are now ready to use the small-ball method of the previous section to show that $\|\nabla^k s_N\|_\infty/\|s_N\|_{\mathcal{L}^2} = O(\sqrt{N^k \log N})$ *almost surely.* It is sufficient to show that

$$\nu_N\left\{s^N \in SH_J^0(M,L^N) : \sup_M |\nabla^k s_N| > C\sqrt{N^k \log N}\right\} < O\left(\frac{1}{N^2}\right)\,, \tag{100}$$

for C sufficiently large. To verify (100), we may regard s_N as a function on X and replace ∇^k by a horizontal r_θ-invariant differential operator of order k on X.

As before, we let $s^N = \sum c_j s_j^N$ denote a random element of $S\mathcal{H}_N^2(X)$. By (85), we have

$$P_k s^N(x) = \int_X P_k^1 \Pi_N(x,y) s^N(y) dy = \sum_{j=1}^{d_N} c_j P_k S_j^N(x) = c \cdot P_k\tilde{\Phi}_N(x)\,. \tag{101}$$

We then have

$$|P_k s^N(x)| = \|P_k\tilde{\Phi}_N(x)\| \cos\theta_x\,, \quad \text{where } \cos\theta_x = \frac{\left|c \cdot P_k\tilde{\Phi}_N(x)\right|}{\|P_k\tilde{\Phi}_N(x)\|}\,. \tag{102}$$

Now fix a point $x \in X$. As before, (89) holds, and hence by Lemma 4.3 we have

$$\nu_N\left\{s^N \in S\mathcal{H}_N^2 : |P_k s^N(x)| \geq C'\sqrt{N^k \log N}\right\} \leq k_N N^{-C^2 N^{-m}(d_N-1)}\,, \tag{103}$$

where $C' = C \sup_{N,x} N^{-(m+k)/2}|P_k\tilde{\Phi}_N(x)|$.

We again cover M by a collection of k_N very small balls $B(z^j)$ of radius $R_N = N^{-\frac{m+1}{2}}$ and first show that the probability of the required condition holding at the centers of all the balls is small. Choosing points $x^j \in X$ lying above the centers z^j of the balls, we then have

$$\nu_N\left\{s^N \in S\mathcal{H}_N^2 : \max_j |P_k s^N(x_j)| \geq C'\sqrt{N^k \log N}\right\} \leq k_N N^{-C^2 N^{-m}(d_N-1)}\,. \tag{104}$$

Now suppose that w^j is an arbitrary point in $B(z^j)$, and let y^j be the point of X above w^j such that the horizontal lift of the geodesic from z^j to w^j connects x^j and y^j. Hence by Lemma 4.3, we have

$$\|P_k\tilde{\Phi}_N(x^j) - P_k\tilde{\Phi}_N(y^j)\| \leq \sup_M \|d^h(P_k\tilde{\Phi}_N)\|r_N = O(N^{\frac{m+k+1}{2}})r_N = O(N^{\frac{k}{2}})\,. \tag{105}$$

It follows as before from (104) and (105) that

$$\nu_N \left\{ s^N \in S\mathcal{H}_N^2 : \sup_X |P_k s^N| \geq (C'+1)\sqrt{N^k \log N} \right\}$$

$$\leq k_N N^{-C^2 N^{-m}(d_N-1)} \leq O\left(N^{m(m+1)-\frac{C^2}{m!+1}} \right).$$

(Here, we used the fact that $|P_k s^N|$ is constant on the fibers of $\pi : X \to M$.) Thus, (100) holds with C sufficiently large. $\qquad\square$

4.4. **The estimate** $\|\bar{\partial} s_N\|_\infty / \|s_N\|_2 = O(\sqrt{\log N})$ **almost surely.** The proof of the $\bar{\partial} s_N$ estimate follows the pattern of the above estimate. However, there is one crucial difference: we must show the following upper bound for the modulus of $\bar{\partial}_b \tilde{\Phi}_N$. This estimate is a factor of $\sqrt{N}$ better than the one for $d^h \tilde{\Phi}_N$ arising from Lemma 4.3; the proof depends on the precise second order approximation of Theorem 2.3.

LEMMA **4.4.** $\qquad \sup_X \|\bar{\partial}_b \tilde{\Phi}_N(x)\| \leq O(N^{m/2})$.

Proof. Let $x_0 = (P_0, 0)$ be an arbitrary point of X, and choose preferred local coordinates $(z_1, \ldots, z_m, \theta)$ about $(P_0, 0)$ as in the hypothesis of Theorem 2.3. We lift a local frame $\{\bar{Z}_q^M\}$ of the form (3) to obtain the local frame $\{\bar{Z}_1, \ldots, \bar{Z}_m\}$ for $H^{0,1}X$ given by

$$\bar{Z}_q = \frac{\partial^h}{\partial \bar{z}_q} + \sum_{r=1}^m B_{qr}(z) \frac{\partial^h}{\partial z_r}, \qquad B_{qr}(P_0) = 0. \tag{106}$$

It suffices to show that

$$N^{-m/2} |\bar{Z}_q \tilde{\Phi}_N(x_0)| \leq C, \tag{107}$$

where C is a constant independent of x_0.

By Theorem 2.3, we have

$$N^{-m} \Pi_N \left(\frac{u}{\sqrt{N}}, \frac{s}{N}; \frac{v}{\sqrt{N}}, \frac{t}{N} \right)$$

$$= \frac{1}{\pi^m} \varphi_0(u,s) \overline{\varphi_0(v,t)} e^{u \cdot \bar{v}} \left[1 + \frac{1}{\sqrt{N}} b_1(P_0, u, v) + \frac{1}{N} R_2(P_0, u, v, N) \right], \tag{108}$$

where

$$\varphi_0(z, \theta) = e^{i\theta - |z|^2/2}.$$

(The function φ_0 is the 'ground state' for the 'annihilation operators' $\bar{Z}_q$ in the Heisenberg model; see the remark in §1.2 and [BSZ2, §1.3.2]). In our case, $\bar{Z}_q \varphi_0$ does not vanish as in the model case, but instead satisfies the asymptotic bound (110) below.)

Recalling (8) and (16), we have

$$\frac{\partial^h}{\partial \bar{z}_q} = \frac{\partial}{\partial \bar{z}_q} + \left[-\frac{i}{2} z_q - R_1^{\bar{A}_q}(z) \right] \frac{\partial}{\partial \theta}, \tag{109}$$

where $R_1^{\bar{A}_q}(z) = O(|z|^2)$. Recalling that $z = u/\sqrt{N}$, $\theta = s/N$, we note that $\varphi_0(u,s) = e^{iN\theta - N|z|^2/2} = \varphi_0(z,\theta)^N$, and thus by (109),

$$\frac{\partial^h}{\partial \bar{z}_q} \varphi_0(u,s) = \frac{\partial^h}{\partial \bar{z}_q} e^{iN\theta - N|z|^2/2} = -iN R_1^{\bar{A}_q} \left(\frac{u}{\sqrt{N}} \right) \varphi_0(u,s) = \mathcal{R}(P_0, u, N) \varphi_0(u,s), \tag{110}$$

where as before $\mathcal{R}$ denotes a term satisfying the remainder estimate of Theorem 2.3.

We let Z_q^1, Z_q^2 denote the operator Z_q applied to the first and second factors, respectively, on $X \times X$; we similarly let d_q^1, d_q^2 denote the operator $\frac{\partial^h}{\partial z_q}$ applied to the factors of $X \times X$. Equation (99) tells us that

$$\|\bar{Z}_q \tilde{\Phi}_N(x)\|^2 = \bar{Z}_q^1 Z_q^2 \Pi_N(x,x) \,. \tag{111}$$

By (106),

$$\bar{Z}_q^1 Z_q^2 = \left(\overline{d_q^1} + \sum_{r=1}^m B_{qr}(z) d_r^1 \right) \left(d_q^2 + \sum_{\rho=1}^m \bar{B}_{q\rho}(w) \overline{d_\rho^2} \right) \,, \tag{112}$$

where we recall that $B_{qr}(P_0) = 0$.

Differentiating (108), again noting that $\partial/\partial z_q = \sqrt{N}\partial/\partial u_q$, $\partial/\partial w_q = \sqrt{N}\partial/\partial v_q$ and using (110), we obtain

$$N^{-m} \left(\overline{d_q^1} d_q^2 \Pi_N \right) \left(\frac{u}{\sqrt{N}}, \frac{s}{N}; \frac{v}{\sqrt{N}}, \frac{t}{N} \right)$$

$$= \frac{1}{\pi^m} \varphi_0(u,s) \overline{\varphi_0(v,t)} e^{u \cdot \bar{v}} \left[\sqrt{N} \frac{\partial^2}{\partial \bar{u}_q \partial v_q} b_1 + \widetilde{\mathcal{R}} \right] \,. \tag{113}$$

Since b_1 has no terms that are quadratic in $(u, \bar{u}, v, \bar{v})$, it follows from (112)–(113) that

$$N^{-m} \left| \bar{Z}_q^1 Z_q^2 \Pi_N(P_0, 0; P_0, 0) \right| = N^{-m} \left| \overline{d_q^1} d_q^2 \Pi_N(P_0, 0; P_0, 0) \right| = \frac{1}{\pi^m} \left| \widetilde{\mathcal{R}}(P_0, 0, 0, N) \right| \leq O(1) \,. \tag{114}$$

The desired estimate (107) now follows immediately from (111) and (114). $\square$

By covering M with small balls and repeating the argument of the previous section, using Lemma 4.4, we conclude that

$$\nu_N \left\{ s^N \in SH_J^0(M, L^N) : \sup_M |\bar{\partial} s_N| > C\sqrt{\log N} \right\} < O\left(\frac{1}{N^2} \right) \,. \tag{115}$$

Thus $\|\bar{\partial} s_N\|_\infty / \|s_N\|_2 = O(\sqrt{\log N})$ almost surely.

4.5. **The estimate $\|\nabla^k \bar{\partial} s_N\|_\infty / \|s_N\|_2 = O(\sqrt{N^k \log N})$ almost surely.** To obtain this final estimate of Theorem 0.1, it suffices to verify the probability estimate

$$\nu_N \left\{ s^N \in SH_J^0(M, L^N) : \sup_M |\nabla^k \bar{\partial} s_N| > C\sqrt{N^k \log N} \right\} < O\left(\frac{1}{N^2} \right) \,. \tag{116}$$

Equation (116) follows by again repeating the argument of §4.3, using the following lemma.

LEMMA **4.5.** *Let P_k be a horizontal differential operator of order k on X ($k \geq 0$). Then*

$$\sup_X |P_k \bar{\partial}_b \tilde{\Phi}_N| = O(N^{\frac{m+k}{2}}) \,.$$

Proof. It suffices to show that

$$\sup_{U} |P_k \bar{Z}_q^k \tilde{\Phi}_N| = O(N^{\frac{m+k}{2}}) \tag{117}$$

for a local frame $\{\bar{Z}_q\}$ of $T^{0,1}M$ over U. As before, we have

$$P_k^1 \bar{P}_k^2 \bar{Z}_q^1 Z_q^1 \Pi_N(x,x) = \left| P_k \bar{Z}_q^h \tilde{\Phi}_N(x) \right|^2 . \tag{118}$$

We claim that

$$N^{-m} \bar{Z}_q^1 Z_q^2 \Pi_N = \frac{1}{\pi^m} e^{i(s-t)+\psi_2(u,v)} \left[\sqrt{N} \frac{\partial^2}{\partial \bar{u}_q \partial v_q} b_1 + \mathcal{R}(P_0, u, v, N) \right] . \tag{119}$$

To obtain the estimate (119), we recall from (112) in the proof of Lemma 4.4 that

$$\bar{Z}_q^1 Z_q^2 = \overline{d_q^1} d_q^2 + \sum_{\rho=1}^{m} \bar{B}_{q\rho}(w) \overline{d_q^1 d_\rho^2} + \sum_{r=1}^{m} B_{qr}(z) d_r^1 d_q^2 + \sum_{r,\rho} B_{qr}(z) \bar{B}_{q\rho}(w) d_r^1 \overline{d_\rho^2} . \tag{120}$$

Equation (113) says that the first term of $N^{-m} \bar{Z}_q^1 Z_q^2 \Pi_N$ coming from the expansion (120) satisfies the estimate of (119). To obtain the estimate for the second term, we compute:

$$
\begin{aligned}
N^{-m} \overline{d_\rho^2} \Pi_N(\frac{u}{\sqrt{N}}, \frac{s}{N}; \frac{v}{\sqrt{N}}, \frac{t}{N}) &= \frac{\sqrt{N}}{\pi^m} e^{i(s-t)+\psi_2(u,v)} \\
&\quad \cdot \left(\left[\frac{\partial \psi_2}{\partial \bar{v}_\rho} + i\sqrt{N} A_\rho(\frac{v}{\sqrt{N}}) \right] \left[1 + \frac{1}{\sqrt{N}} b_1 + \frac{1}{N} R_2 \right] + \frac{1}{\sqrt{N}} \frac{\partial b_1}{\partial \bar{v}_\rho} + \frac{1}{N} \frac{\partial R_2}{\partial \bar{v}_\rho} \right) \\
&= \frac{\sqrt{N}}{\pi^m} \varphi_0(u,s) \overline{\varphi_0(v,t)} e^{u \cdot \bar{v}} \left[\frac{\partial \psi_2}{\partial \bar{v}_\rho} + L_\rho(v) + \frac{1}{\sqrt{N}} \tilde{\mathcal{R}} \right] ,
\end{aligned} \tag{121}
$$

where L_ρ is a linear function. Since $\partial^2 \psi_2 / \partial \bar{u}_q \partial \bar{v}_\rho \equiv 0$, it then follows that

$$N^{-m} \overline{d_q^1 d_\rho^2} \Pi_N = \frac{\sqrt{N}}{\pi^m} e^{i(s-t)+\psi_2(u,v)} \frac{\partial}{\partial \bar{u}_q} \tilde{\mathcal{R}}(P_0, u, v, N) . \tag{122}$$

The estimate (119) for the second term follows from (122), using the fact that $B_{q\rho}(\frac{v}{\sqrt{N}}) = \frac{1}{\sqrt{N}} L_{q\rho}(v) + \cdots$. The proofs of the estimate for the third and fourth terms are similar.

The desired estimate (117) follows as before from (118), (119), and (98), using the fact that $\frac{\partial^2}{\partial \bar{u}_q \partial v_q} b_1$ is linear. $\qquad\square$

5. The joint probability distribution

In this section, we shall use Theorem 2.3 and the methods of of [BSZ2] to prove Theorem 0.2 and its analogue for Gaussian measures (Theorem 5.4), which say that the joint probability distributions on almost complex symplectic manifolds have the same universal scaling limit as in the complex case.

5.1. **Generalized Gaussians.** Recall that a Gaussian measure on $\mathbb{R}^n$ is a measure of the form

$$\gamma_\Delta = \frac{e^{-\frac{1}{2}\langle \Delta^{-1}x,x\rangle}}{(2\pi)^{n/2}\sqrt{\det \Delta}}dx_1\cdots dx_n\,,$$

where Δ is a positive definite symmetric $n \times n$ matrix. The matrix Δ gives the second moments of γ_Δ:

$$\langle x_j x_k\rangle_{\gamma_\Delta} = \Delta_{jk}\,. \tag{123}$$

This Gaussian measure is also characterized by its Fourier transform

$$\widehat{\gamma_\Delta}(t_1,\ldots,t_n) = e^{-\frac{1}{2}\sum \Delta_{jk}t_j t_k}\,. \tag{124}$$

If we let Δ be the $n \times n$ identity matrix, we obtain the standard Gaussian measure on $\mathbb{R}^n$,

$$\gamma_n := \frac{1}{(2\pi)^{n/2}}e^{-\frac{1}{2}|x|^2}dx_1\cdots dx_n\,,$$

with the property that the x_j are independent Gaussian variables with mean 0 and variance 1. Hence

$$\langle \|x\|^2\rangle_{\gamma_n} = \sum_{j=1}^n \langle x_j^2\rangle_{\gamma_n} = n\,.$$

Since we wish to put Gaussian measures on the spaces $H_J^0(M, L^N)$ with rapidly growing dimensions, it is useful to consider the *normalized standard Gaussians*

$$\tilde{\gamma}_n := k_n e^{-\frac{n}{2}|x|^2}dx_1\cdots dx_n\,, \quad k_n = \left(\frac{n}{2\pi}\right)^{n/2}\,,$$

which have the property that

$$\langle \|x\|^2\rangle_{\tilde{\gamma}_n} = 1\,.$$

The push-forward of a Gaussian measure by a surjective linear map is also Gaussian. In the next section, we shall push forward Gaussian measures (on the spaces $H_J^0(M, L^N)$) by linear maps that are sometimes not surjective. Since these non-surjective push-forwards are singular measures, we need to consider the case where Δ is positive semi-definite. In this case, we use (124) to define a measure γ_Δ, which we call a *generalized Gaussian*. If Δ has null eigenvalues, then the generalized Gaussian γ_Δ is a Gaussian measure on the subspace $\Lambda_+ \subset \mathbb{R}^n$ spanned by the positive eigenvectors. (Precisely, $\gamma_\Delta = \iota_* \gamma_{\Delta|\Lambda_+}$, where $\iota : \Lambda_+ \hookrightarrow \mathbb{R}^n$ is the inclusion. For the completely degenerate case $\Delta = 0$, we have $\gamma_\Delta = \delta_0$.) Of course, (123) holds for semi-positive Δ. One useful property of generalized Gaussians is that the push-forward by a (not necessarily surjective) linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ of a generalized Gaussian γ_Δ on $\mathbb{R}^n$ is a generalized Gaussian on $\mathbb{R}^m$:

$$T_*\gamma_\Delta = \gamma_{T\Delta T^*} \tag{125}$$

Another useful property of generalized Gaussians is the following fact:

LEMMA **5.1.** *The map $\Delta \mapsto \gamma_\Delta$ is a continuous map from the positive semi-definite matrices to the space of positive measures on $\mathbb{R}^n$ (with the weak topology).*

Proof. Suppose that $\Delta^N \to \Delta^0$. We must show that $(\Delta^N, \varphi) \to (\Delta^0, \varphi)$ for a compactly supported test function φ. We can assume that φ is $\mathcal{C}^\infty$. It then follows from (124) that

$$(\gamma_{\Delta^N}, \varphi) = (\widehat{\gamma_{\Delta^N}}, \widehat{\varphi}) \to (\widehat{\gamma_{\Delta^0}}, \widehat{\varphi}) = (\gamma_{\Delta^0}, \varphi).$$

$\square$

We shall use the following general result relating spherical measures to Gaussian measures in order to prove Theorem 0.2 on asymptotics of the joint probability distributions for $SH^0_J(M, L^N)$.

LEMMA **5.2.** *Let $T_N : \mathbb{R}^{d_N} \to R^k$, $N = 1, 2, \ldots$, be a sequence of linear maps, where $d_N \to \infty$. Suppose that $\frac{1}{d_N} T_N T_N^* \to \Delta$. Then $T_{N*} \nu_{d_N} \to \gamma_\Delta$.*

Proof. Let V_N be a k-dimensional subspace of $\mathbb{R}^{d_N}$ such that $V_N^\perp \subset \ker T^N$, and let $p_N : \mathbb{R}^{d_N} \to V_N$ denote the orthogonal projection. We decompose $T_N = B_N \circ A_N$, where $A_N = d_N^{1/2} p_N : \mathbb{R}^{d_N} \to V_N$, and $B_N = d_N^{-1/2} T_N|_{V_N} : V_N \xrightarrow{\approx} \mathbb{R}^k$. Write

$$A_{N*} \nu_{d_N} = \alpha_N, \qquad T_{N*} \nu_{d_N} = B_{N*} \alpha_N = \beta_N.$$

We easily see that (abbreviating $d = d_N$)

$$\alpha_N = A_{N*} \nu_d = \psi_d dx, \qquad \psi_d = \begin{cases} \frac{\sigma_{d-k}}{\sigma_d d^k} [1 - \frac{1}{d}|x|^2]^{(d-k-2)/2} & \text{for } |x| < \sqrt{d} \\ 0 & \text{otherwise} \end{cases}, \qquad (126)$$

where dx denotes Lebesgue measure on V_N, and $\sigma_n = \text{Vol}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. (The case $k = 1$, $d = 3$ of (126) is Archimedes' formula [Arc].) Since $[1 - |x|^2/d]^{(d-k-2)/2} \to e^{-|x|^2/2}$ uniformly on compacta and $\frac{\sigma_{d-k}}{\sigma_d d^k} \to \frac{1}{(2\pi)^{k/2}}$, we conclude that $\alpha_N \to \gamma_k$. (This is the Poincaré-Borel Theorem; see Corollary 5.3 below.) Furthermore,

$$\left(1 - \frac{1}{d}|x|^2\right)^{(d-k-2)/2} \leq \exp\left(-\frac{d-k-2}{2d}|x|^2\right) \leq e^{\frac{k+2}{2}} e^{-\frac{1}{2}|x|^2} \qquad \text{for } d \geq k+2, \ |x| \leq \sqrt{d},$$

and hence

$$\psi_{d_N}(x) \leq C_k e^{-|x|^2/2}. \qquad (127)$$

Now let φ be a compactly supported continuous test function on $\mathbb{R}^k$. We must show that

$$\int \varphi d\beta_N \to \int \varphi d\gamma_\Delta. \qquad (128)$$

Suppose on the contrary that (128) does not hold. After passing to a subsequence, we may assume that $\int \varphi d\beta_N \to c \neq \int \varphi d\gamma_\Delta$. Since the eigenvalues of B_N are bounded, we can assume (again taking a subsequence) that $B_N \to B_0$, where

$$B_0 B_0^* = \lim_{N \to \infty} B_N B_N^* = \lim_{N \to \infty} \frac{1}{d_N} T_N T_N^* = \Delta.$$

Hence,

$$\int_{\mathbb{R}^k} \varphi d\beta_N = \int_{V_N} \varphi(B_N x) \psi_{d_N}(x) dx \to \int_{V_N} \varphi(B_0 x) \frac{e^{-|x|^2/2}}{(2\pi)^{k/2}} dx = \int_{V_N} \varphi(B_0 x) d\gamma_k(x),$$

where the limit holds by dominated convergence, using (127). By (125), we have $B_{0*}\gamma_k = \gamma_{B_0 B_0^*} = \gamma_\Delta$, and hence

$$\int_{V_N} \varphi(B_0 x) d\gamma_k(x) = \int_{\mathbb{R}^k} \varphi d\gamma_\Delta \,.$$

Thus (128) holds for the subsequence, giving a contradiction. $\square$

We note that the above proof began by establishing the *Poincaré-Borel Theorem* (which is a special case of the of Lemma 5.2):

COROLLARY **5.3.** (Poincaré-Borel) *Let* $P_d : \mathbb{R}^d \to \mathbb{R}^k$ *be given by* $P_d(x) = \sqrt{d}(x_1, \ldots, x_k)$. *Then*

$$P_{d*}\nu_d \to \gamma_k \,.$$

By a *generalized complex Gaussian measure* on $\mathbb{C}^n$, we mean a generalized Gaussian measure γ_Δ^c on $\mathbb{C}^n \equiv R^{2n}$ with moments

$$\langle z_j \rangle_\gamma = 0, \quad \langle z_j z_k \rangle_\gamma = 0, \quad \langle z_j \bar{z}_k \rangle_\gamma = \Delta_{jk}, \qquad 1 \le j, k \le n,$$

where $\Delta = (\Delta_{jk})$ is an $n \times n$ positive semi-definite hermitian matrix; i.e. $\gamma_\Delta^c = \gamma_{\frac{1}{2}\Delta^c}$, where Δ^c is the $2n \times 2n$ real symmetric matrix of the inner product on R^{2n} induced by Δ. As we are interested here in complex Gaussians, we shall henceforth drop the 'c' and write $\gamma_\Delta^c = \gamma_\Delta$. In particular, if Δ is a strictly positive hermitian matrix, then

$$\gamma_\Delta = \frac{e^{-\langle \Delta^{-1} z, \bar{z} \rangle}}{\pi^n \det \Delta} d\mathcal{L}(z) \,,$$

where $\mathcal{L}$ denotes Lebesgue measure on $\mathbb{C}^n$.

5.2. **Proof of Theorem 0.2.** We return to our complex Hermitian line bundle (L, h) on a compact almost complex $2m$-dimensional symplectic manifold M with symplectic form $\omega = \frac{i}{2}\Theta_L$, where Θ_L is the curvature of L with respect to a connection ∇. We now describe the n-point joint distribution arising from our probability space $(SH_J^0(M, L^N), \nu_N)$. Recalling (17), we have the Hermitian inner product on $H_J^0(M, L^N)$:

$$\langle s_1, s_2 \rangle = \int_M h^N(s_1, s_2) \frac{1}{m!} \omega^m \qquad (s_1, s_2 \in H_J^0(M, L^N)) \,,$$

and we write $\|s\|_2 = \langle s, s \rangle^{1/2}$. Recall that $SH_J^0(M, L^N)$ denotes the unit sphere $\{\|s\| = 1\}$ in $H_J^0(M, L^N)$ and ν_N denotes its Haar probability measure.

We let $J^1(M, L^N)$ denote the space of 1-jets of sections of L^N. Recall that we have the exact sequence of vector bundles

$$0 \to T_M^* \otimes L^N \xrightarrow{\iota} J^1(M, L^N) \xrightarrow{\rho} L^N \to 0 \,. \tag{129}$$

We consider the jet maps

$$J_z^1 : H_J^0(M, L^N) \to J^1(M, V)_z, \quad J_z^1 s = \text{ the 1-jet of } s \text{ at } z, \quad \text{for } z \in M \,.$$

The covariant derivative $\nabla : J^1(M, L^N) \to T_M^* \otimes L^N$ provides a splitting of (129) and an isomorphism

$$(\rho, \nabla) : J^1(M, L^N) \xrightarrow{\approx} L^N \oplus (T_M^* \otimes L^N) \,. \tag{130}$$

Definition: The *n-point joint probability distribution* at points $P^1, \ldots, P^n$ of M is the probability measure

$$\mathbf{D}_{(P^1,\ldots,P^n)}^N := (J_{P^1}^1 \oplus \cdots \oplus J_{P^n}^1)_* \nu_N \tag{131}$$

on the space $J^1(M, L^N)_{P^1} \oplus \cdots \oplus J^1(M, L^N)_{P^n}$.

Since we are interested in the scaling limit of $\mathbf{D}^N$, we need to describe this measure more explicitly: Suppose that $P^1, \ldots, P^n$ lie in a coordinate neighborhood of a point $P_0 \in M$ and choose preferred coordinates $(z_1, \ldots, z_m)$ and a preferred frame e_L at P_0. We let $z_1^p, \ldots, z_m^p$ denote the coordinates of the point P^p $(1 \le p \le n)$, and we write $z^p = (z_1^p, \ldots, z_m^p)$. (The coordinates of P_0 are 0.) We consider the $n(2m+1)$ complex-valued random variables x^p, ξ_q^p $(1 \le p \le n,\ 1 \le q \le 2m)$ on $S\mathcal{H}_N^2(X) \equiv SH_J^0(M, L^N)$ given by

$$x^p(s) = s(z^p, 0), \tag{132}$$

$$\xi_q^p(s) = \frac{1}{\sqrt{N}} \frac{\partial^h s}{\partial z_q}(z^p), \quad \xi_{m+q}^p(s) = \frac{1}{\sqrt{N}} \frac{\partial^h s}{\partial \bar{z}_q}(z^p) \qquad (1 \le q \le m), \tag{133}$$

for $s \in SH_J^0(M, L^N)$.

We now write

$$x = (x^1, \ldots, x^p), \quad \xi = (\xi_q^p)_{1 \le p \le n, 1 \le q \le 2m}, \quad z = (z^1, \ldots, z^n).$$

Using (130) and the variables x^p, ξ_q^p to make the identification

$$J^1(M, L^N)_{P^1} \oplus \cdots \oplus J^1(M, L^N)_{P^n} \equiv \mathbb{C}^{n(2m+1)}, \tag{134}$$

we can write

$$\mathbf{D}_z^N = D^N(x, \xi; z)dxd\xi,$$

where $dxd\xi$ denotes Lebesgue measure on $\mathbb{C}^{n(2m+1)}$.

Before proving Theorem 0.2 on the scaling limit of $\mathbf{D}_z^N$, we state and prove a corresponding result replacing $(SH_J^0(M, L^N), \nu_N)$ with the essentially equivalent Gaussian space $H_J^0(M, L^N)$ with the *normalized standard Gaussian measure*

$$\mu_N := \tilde{\gamma}_{2d_N} = k_{2d_N} e^{-d_N |c|^2} d\mathcal{L}(c), \qquad s = \sum_{j=1}^{d_N} c_j S_j^N, \tag{135}$$

where $\{S_j^N\}$ is an orthonormal basis for $H_J^0(M, L^N)$. Recall that this Gaussian is characterized by the property that the $2d_N$ real variables $\Re c_j, \Im c_j$ $(j = 1, \ldots, d_N)$ are independent random variables with mean 0 and variance $1/2d_N$; i.e.,

$$\langle c_j \rangle_{\mu_N} = 0, \quad \langle c_j c_k \rangle_{\mu_N} = 0, \quad \langle c_j \bar{c}_k \rangle_{\mu_N} = \frac{1}{d_N} \delta_{jk}. \tag{136}$$

Our normalization guarantees that the variance of $\|s\|_2$ is 1:

$$\langle \|s\|_2^2 \rangle_{\mu_N} = 1.$$

We then consider the *Gaussian joint probability distribution*

$$\widetilde{\mathbf{D}}_{(P^1,\ldots,P^n)}^N = \widetilde{D}^N(x, \xi; z)dxd\xi = (J_{P^1}^1 \oplus \cdots \oplus J_{P^n}^1)_* \mu_N. \tag{137}$$

Since μ_N is Gaussian and the map $J^1_{P_1} \oplus \cdots \oplus J^1_{P_n}$ is linear, it follows that the joint probability distribution is a generalized Gaussian measure of the form

$$D^N(x, \xi; z) dx d\xi = \gamma_{\Delta^N(z)} \,. \tag{138}$$

We shall see below that the covariance matrix $\Delta^N(z)$ is given in terms of the Szegö kernel.

We have the following alternate form of Theorem 0.2:

THEOREM **5.4.** *Let L, M, ω be as above and let $\{z_j\}$ be preferred coordinates centered at a point $P_0 \in M$. Then*

$$\widetilde{\mathbf{D}}^N_{(z^1/\sqrt{N}, \ldots, z^n/\sqrt{N})} \longrightarrow \mathbf{D}^\infty_{(z^1, \ldots, n^n)}$$

where $\mathbf{D}^\infty_{(z^1, \ldots, z^n)}$ is the universal Gaussian measure (supported on the holomorphic 1-jets) of Theorem 0.2.

Proof. The covariance matrix $\Delta^N(z)$ in (138) is a positive semi-definite $n(2m+1) \times n(2m+1)$ hermitian matrix. If the map $J^1_{z^1} \oplus \cdots \oplus J^1_{z^n}$ is surjective, then $\Delta^N(z)$ is strictly positive definite and $\widetilde{D}^N(x, \xi; z)$ is a smooth function. On the other hand, if the map is not surjective, then $\widetilde{D}^N(x, \xi; z)$ is a distribution supported on a linear subspace. For example, in the integrable holomorphic case, $\widetilde{D}^N(x, \xi; z)$ is supported on the holomorphic jets, as follows from the discussion below.

By (123), we have

$$\Delta^N(z) = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \,,$$

$$A = \left(A^p_{p'} \right) = \left\langle x^p \bar{x}^{p'} \right\rangle_{\mu_N}, \quad B = \left(B^p_{p'q'} \right) = \left\langle x^p \bar{\xi}^{p'}_{q'} \right\rangle_{\mu_N}, \quad C = \left(C^{pq}_{p'q'} \right) = \left\langle \xi^p_q \bar{\xi}^{p'}_{q'} \right\rangle_{\mu_N}, \quad (139)$$
$$p, p' = 1, \ldots, n, \quad q, q' = 1, \ldots, 2m \,.$$

(We note that A, B, C are $n \times n$, $n \times 2mn$, $2mn \times 2mn$ matrices, respectively; p, q index the rows, and p', q' index the columns.)

We now describe the the entries of the matrix Δ^N in terms of the Szegö kernel. We have by (136) and (139), writing $s = \sum_{j=1}^{d_N} c_j S^N_j$,

$$A^p_{p'} = \left\langle x^p \bar{x}^{p'} \right\rangle_{\mu_N} = \sum_{j,k=1}^{d_N} \left\langle c_j \bar{c}_k \right\rangle_{\mu_N} S^N_j(z^p, 0) \overline{S^N_k(z^{p'}, 0)} = \frac{1}{d_N} \Pi^{P_0}_N(z^p, 0; z^{p'}, 0) \,. \tag{140}$$

We need some more notation to describe the matrices B and C: Write

$$\nabla_q = \frac{1}{\sqrt{N}} \frac{\partial^h}{\partial z_q}, \quad \nabla_{m+q} = \frac{1}{\sqrt{N}} \frac{\partial^h}{\partial \bar{z}_q}, \quad 1 \le q \le m \,.$$

As in §4.1, we let ∇^1_q, resp. ∇^2_q, denote the differential operator on $X \times X$ given by applying ∇_q to the first, resp. second, factor $(1 \le q \le 2m)$. By differentiating (140), we obtain

$$B^p_{p'q'} = \frac{1}{d_N} \overline{\nabla}^2_{q'} \Pi^{P_0}_N(z^p, 0; z^{p'}, 0) \,, \tag{141}$$

$$C^{pq}_{p'q'} = \frac{1}{d_N} \nabla^1_q \overline{\nabla}^2_{q'} \Pi^{P_0}_N(z^p, 0; z^{p'}, 0) \,. \tag{142}$$

It follows from (140)–(142) and Theorem 2.3, recalling (16) and (95), that

$$\Delta^N(\frac{z}{\sqrt{N}}) \to \Delta^\infty(z) = \frac{m!}{c_1(L)^m} \begin{pmatrix} A^\infty(z) & B^\infty(z) \\ B^\infty(z)^* & C^\infty(z) \end{pmatrix} \tag{143}$$

uniformly, where

$$A^\infty(z)_{p'}^p \;=\; \Pi_1^{\mathbf{H}}(z^p,0;z^{p'},0) \;=\; \frac{1}{\pi^m}e^{\psi_2(z^p,z^{p'})},$$

$$B^\infty(z)_{p'q'}^p \;=\; \begin{cases} \frac{1}{\pi^m}(z_{q'}-w_{q'})e^{\psi_2(z^p,z^{p'})} & \text{for} \quad 1\le q\le m \\ 0 & \text{for} \quad m+1\le q\le 2m \end{cases},$$

$$C^\infty(z)_{p'q'}^{pq} \;=\; \begin{cases} \frac{1}{\pi^m}(\bar{w}_q-\bar{z}_q)(z_{q'}-w_{q'})e^{\psi_2(z^p,z^{p'})} & \text{for} \quad 1\le q,q'\le m \\ 0 & \text{for} \quad \max(q,q')\ge m+1 \end{cases}.$$

(Recall that $\psi_2(u,v) = u\cdot\bar{v} - \frac{1}{2}(|u|^2+|v|^2)$.) In other words, the coefficients of $\Delta^\infty(z)$ corresponding to the anti-holomorphic directions vanish, while the coefficients corresponding to the holomorphic directions are given by the Szegö kernel $\Pi_1^{\mathbf{H}}$ for the reduced Heisenberg group and its covariant derivatives.

Finally, we apply Lemma 5.1 to (138) and conclude that

$$\widetilde{\mathbf{D}}^N_{z/\sqrt{N}} = \gamma_{\Delta^N(z/\sqrt{N})} \to \gamma_{\Delta^\infty(z)}.$$

Thus Theorem 5.4 holds with $\mathbf{D}_z^\infty = \gamma_{\Delta^\infty(z)}$. $\qquad\square$

Proof of Theorem 0.2: The proof is similar to that of Theorem 5.4. This time we define

$$\Delta^N = \frac{1}{d_N}\mathcal{J}_N\mathcal{J}_N^* : H^0(M,L^N) \to \mathbb{C}^{n(2m+1)},$$

where $\mathcal{J}_N = J_{P^1}^1 \oplus \cdots \oplus J_{P^n}^1$ under the identification (134). We see immediately that Δ^N is given by (140)–(142) and the conclusion follows from Lemma 5.2 and (143). $\qquad\square$

Remark: There are other similar ways to define the joint probability distribution that have the same universal scaling limits. One of these is to use the (un-normalized) standard Gaussian measure γ_{2d_N} on $H_J^0(M,L^N)$ in place of the normalized Gaussian μ_N in Theorem 5.4 to obtain joint densities $D_\#^N(x,\xi;z) = D^N(\frac{x}{N^{m/2}},\frac{\xi}{N^{m/2}};z)$. Then we would have instead

$$D_\#^N(N^{m/2}x, N^{m/2}\xi; N^{-1/2}z)dxd\xi \to \gamma_{\Delta^\infty(z)}.$$

Another similar result is to let λ_N denote normalized Lebesgue measure on the unit ball $\{\|s\|\le 1\}$ in $H_J^0(M,L^N)$ and to let $\widehat{\mathbf{D}}_z^N = \mathcal{J}_{N*}\lambda_N$. By a similar argument as above, we also have $\widehat{\mathbf{D}}^N_{z/\sqrt{N}} \to \gamma_{\Delta^\infty(z)}$.

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