Localized Superluminal Solutions to Maxwell Equations propagating along a normal-sized waveguide [†]

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Abstract – We show that localized (non-evanescent) solutions to Maxwell equations exist, which propagate without distortion along normal waveguides with Superluminal speed.

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1. – Introduction: Localized solutions to the wave equations

Since 1915 Bateman[1] showed that Maxwell equations admit (besides of the ordinary planewave solutions, endowed in vacuum with speed c) of wavelet-type solutions, endowed in vacuum with group-velocities $0 \leq v \leq c$. But Bateman's work went practically unnoticed. Only few authors, as Barut et al.[2] followed such a research line; incidentally, Barut et al. constructed even a wavelet-type solution travelling with Superluminal group-velocity[3] $v > c$.

In recent times, however, many authors discussed the fact that all (homogeneous) wave equations admit solutions with $0 < v < \infty$: see, e.g., Donnelly & Ziolkowski[4], Esposito[4], Vaz & Rodrigues[4]. Most of those authors confined themselves to investigate (sub- or Super-luminal) *localized* non-dispersive solutions in vacuum: namely, those solutions that were called "undistorted progressive waves" by Courant & Hilbert. Among localized solutions, the most interesting appeared to be the so-called "X-shaped" waves, which —predicted even by Special Relativity in its extended version[5]— had been mathematically constructed by Lu & Greenleaf[6] for acoustic waves, and by Ziolkowski et al.[7], and later Recami[8], for electromagnetism.

Let us recall that such "X-shaped" localized solutions are Superluminal (i.e., travel with $v > c$ in the vacuum) in the electromagnetic case; and are "super-sonic" (i.e., travel with a speed larger than the sound-speed in the medium) in the acoustic case. The first authors to produce X-shaped waves *experimentally* were Lu & Greenleaf[9] for acoustics, and Saari et al.[10] for optics.

Notwithstanding all that work, still it is not yet well understood what solutions (let us now confine ourselves to Maxwell equations and to electromagnetic waves) have to enter into the play in many experiments.

2. – About evanescent waves

Most of the experimental results, actually, did not refer to the abovementioned localized, sub- or Super-luminal, solutions, which in vacuum are expected to propagate rigidly (or almost rigidly, when suitably truncated). The experiments most after fashion are, on

the contrary, those measuring the group-velocity of *evanescent waves*[cf., e.g., refs.11,12]. In fact, both Quantum Mechanics[13] and Special Relativity[5] had predicted tunnelling wavepackets (tunnelling photons too) and/or evanescent waves to be Superluminal.

For instance, experiments[12] with evanescent waves travelling down an under-sized *waveguide* revealed that evanescent modes are endowed with Superluminal group-velocities[14].

A problem arises in connection with the experiment[15] with two "barriers" 1 and 2 (i.e., segments of *undersized* waveguide). In fact, it has been found that *for suitable frequency bands* the wave coming out from barrier 1 goes on with practically infinite speed, crossing the intermediate *normal-sized* waveguide 3 in zero time. Even if this can be theoretically understood by looking at the relevant transfer function (see the computer simulations, based on Maxwell equations only, in refs.[16,17]), it is natural to wonder *what are* the solutions of Maxwell equations that can travel with Superluminal speed in a *normal* waveguide (where one normally meets ordinary propagating —and not evanescent— modes)...

Namely, the dispersion relation in undersized guides is $\omega^2 - k^2 = -\Omega^2$, so that the standard formula $v \simeq \mathrm{d}\omega/\mathrm{d}k$ yields a $v > c$ group-velocity[17,18]. However, in normal guides the dispersion relation becomes $\omega^2 - k^2 = +\Omega^2$, so that the same formula yields values $v < c$ only.

We are going to show that actually localized solutions to Maxwell equations propagating with $v > c$ do exist even in normal waveguides; but their group-velocity v *cannot* be given[#1] by the approximate formula $v \simeq \mathrm{d}\omega/\mathrm{d}k$. One of the main motivations of the present note is just contributing to the clarification of this question.

3. – About some localized solutions to Maxwell equations.

Let us start by considering localized solutions to Maxwell equations in vacuum. A *theorem* by Lu et al.[19] showed how to start from a solution holding in the *plane* (x, y) for constructing a threedimensional solution rigidly moving along the z-axis with Super-

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[#1] Let us recall that the group-velocity is well defined only when the pulse has a clear bump in space; but it can be calculated by the approximate, elementary relation $v \simeq \mathrm{d}\omega/\mathrm{d}k$ *only* when some extra conditions are satisfied (namely, when ω as a function of k is also clearly bumped).

luminal velocity v. Namely, let us assume that $\psi(\boldsymbol{\rho}; t)$, with $\boldsymbol{\rho} \equiv (x, y)$, is a solution of the 2-dimensional homogeneous wave equation:

$$\left(\partial_x^2 + \partial_y^2 - \tfrac{1}{c^2}\partial_t^2 \right) \, \psi(\boldsymbol{\rho}; t) \;=\; 0 \; . \tag{1}$$

By applying the *transformation* $\boldsymbol{\rho} \to \boldsymbol{\rho} \, \sin\theta; \quad t \to t - (\cos\theta/c)\, z$, the angle θ being fixed, with $0 < \theta < \pi/2$, one gets[19] that $\psi(\boldsymbol{\rho} \, \sin\theta; \; t - (\cos\theta/c)\, z)$ is a solution to the threedimensional homogeneous wave-equation

$$\left(\boldsymbol{\nabla}^2 - \tfrac{1}{c^2}\partial_t^2 \right) \, \psi\!\left(\boldsymbol{\rho} \, \sin\theta; \; t - \frac{\cos\theta}{c}\, z \right) = 0 \; . \tag{2}$$

The mentioned theorem holds for the vacuum case, and in general is not valid when introducing boundary conditions. However we discovered that, in the case of a bidimensional solution ψ valid on a circular domain of the (x, y) plane, such that $\psi = 0$ for $|\boldsymbol{\rho}| = 0$, the transformation above leads us to a (three-dimensional) *localized* solution rigidly travelling with Superluminal speed $v = c/\cos\theta$ inside a *cylindrical waveguide*; even if the waveguide radius r will be no longer a, but $r = a/\sin\theta > a$. We can therefore obtain an undistorted Superluminal solution propagating down cylindrical (metallic) *waveguides* for each (2-dimensional) solution valid on a circular domain. Let us recall that, as well-known, any solution to the scalar wave equation corresponds to solutions of the (vectorial) Maxwell equations (cf., e.g., ref.[8] and refs. therein).

For simplicity, let us put the origin O at the center of the circular domain $\mathcal{C}$, and choose a 2-dimensional solution that be axially symmetric $\psi(\rho; t)$, with $\rho = |\boldsymbol{\rho}|$, and with the initial conditions $\psi(\rho; t = 0) \;=\; \phi(\rho)$, and $\partial\psi/\partial t \;=\; \xi(\rho)$ at $t = 0$.

Notice that, because of the transformations

$$\rho \Longrightarrow \rho \, \sin\theta \tag{3a}$$

$$t \Longrightarrow t \; - \; \frac{\cos\theta}{c}\, z \; , \tag{3b}$$

the more the initial $\psi(\rho; t)$ is localized at $t = 0$, the more the (threedimensional) wave $\psi(\boldsymbol{\rho} \, \sin\theta; \; t - (\cos\theta/c)z$ will be localized around $z = vt$. It should be also emphasized

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that, because of transformation (3b), the velocity c goes into the velocity $v = c/\cos\theta > c$.

Let us start with the formal choice

$$\phi(\rho) = \frac{\delta(\rho)}{\rho} \; ; \qquad \xi(\rho) \equiv 0 \; . \tag{4}$$

In cylindrical coordinates the wave equation (1) becomes

$$\left(\frac{1}{\rho} \partial_\rho \rho \partial_\rho \; - \; \frac{1}{c^2} \partial_t^2 \right) \psi(\rho;t) \; = \; 0 \; , \tag{1'}$$

which exhibits the assumed axial symmetry. Looking for factorized solutions of the type $\psi(\rho;t) = R(\rho) \cdot T(t)$, one gets the equations $\partial_t^2 T = -\omega^2 T$ and $(\rho^{-1}\partial_\rho + \partial_\rho^2 + \omega^2/c^2)R = 0$, where the "separation constant" ω is a real parameter, which yield the solutions

$$T \; = \; A \cos\omega t \; + \; B \sin\omega t$$

$$\tag{5}$$

$$R \; = \; C \; J_0(\frac{\omega}{c} \rho) \; ,$$

where quantities A, B, C are real constants, and J_0 is the ordinary zero-order Bessel function (we disregarded the analogous solution $Y_0(\omega\rho/c)$ since it diverges for $\rho = 0$). Finally, by imposing the boundary condition $\psi = 0$ at $\rho = a$, one arrives at the base solutions

$$\psi(\rho;t) \; = \; J_0(\frac{k_n}{a}\rho) \; (A_n \cos\omega_n t \; + \; B_n \sin\omega_n t) \; ; \quad k \equiv \frac{\omega}{c} a \; , \tag{6}$$

the roots of the Bessel function being

$$k_n \; = \; \frac{\omega_n a}{c} \; .$$

The general solution for our bidimensional problem (with our boundary conditions) will therefore be the Fourier-type series

$$\Psi_{2D}(\rho;t) \; = \; \sum_{n=1}^{\infty} J_0(\frac{k_n}{a}\rho) \; (A_n \cos\omega_n t \; + \; B_n \sin\omega_n t) \; . \tag{7}$$

The initial conditions (4) imply that $\sum A_n J_0(k_n\rho/a) = \delta(\rho)/\rho$, and $\sum B_n J_0(k_n\rho/a) = 0$, so that all B_n must vanish, while $A_n = 2[a^2 J_1^2(k_n)]^{-1}$; and eventually one gets:

$$\Psi_{2D}(\rho;t) = \sum_{n=1}^{\infty} \left(\frac{2}{a^2 J_1^2(k_n)}\right) J_0(\frac{k_n}{a}\rho) \cos\omega_n t . \qquad (8) ,$$

where $\omega_n = k_n c/a$.

Let us explicitly notice that we can pass from such a formal solution to more physical ones, just by considering a finite number N of terms. In fact, each partial expansion will satisfy (besides the boundary condition) the second initial condition $\partial_t \psi = 0$ for $t = 0$, while the first initial condition gets the form $\phi(\rho) = f(\rho)$, where $f(\rho)$ will be a (well) localized function, but no longer a delta-type function. Actually, the "localization" of $\phi(\rho)$ increases with increasing N. We shall come back to this point below.

4. – Localized waves propagating Superluminally down (normal-sized) waveguides.

We have now to apply transformations (3) to solution (8), in order to pass to threedimensional waves propagating along a cylindrical (metallic) waveguide with radius $r = a/\sin\theta$. We obtain that Maxwell equations admit in such a case the solutions

$$\Psi_{3D}(\rho, z; t) = \sum_{n=1}^{\infty} \left(\frac{2}{a^2 J_1^2(k_n)}\right) J_0(\frac{k_n}{a}\rho\sin\theta) \cos\left[\frac{k_n\cos\theta}{a}\left(z - \frac{c}{\cos\theta}t\right)\right] \qquad (9)$$

where $\omega_n = k_n c/a$, which are sums over different propagating modes.

Such solutions propagate, down the waveguide, rigidly with Superluminal velocity[#2] $v = c/\cos\theta$. Therefore, (non-evanescent) solutions to Maxwell equations exist, that are waves propagating undistorted along *normal* waveguides with Superluminal speed (even if in normal-sized waveguides the dispersion relation for each mode, i.e. for each term of the Fourier-Bessel expansion, is the ordinary "subluminal" one, $\omega^2/c^2 - k_z^2 = +\Omega^2$).

It is interesting that our Superluminal solutions travel rigidly down the waveguide:

[#2] Let us stress that each eq.(9) represents a *multimodal* (but *localized*) propagation, as if the geometric dispersion compensated for the multimodal dispersion.

this is at variance with what happens for truncated (Superluminal) solutions[7-10], which travel almost rigidly only along their finite "field depth" and then abruptly decay.

Finally, let us consider a finite number of terms in eq.(8), at $t = 0$. We made a few numerical evaluations: let us consider the results for $N = 22$ (however, similar results can be already obtained, e.g., for $N = 10$). The first initial condition of eq.(4), then, is no longer a delta function, but results to be the (bumped) bidimensional wave represented in Fig.1.

The threedimensional wave, eq.(9), corresponding to it, i.e., with the same finite number $N = 22$ of terms, is depicted in Fig.2. It is still an exact solution of the wave equation, for a metallic (normal-sized) waveguide with radius $r = a/\sin\theta$, propagating rigidly with Superluminal group-velocity $v = c/\cos\theta$; moreover, it is now a *physical* solution. In Fig.2 one can see its central portion, while in Fig.3 it is shown the space profile along z, for $t = $ const., of such a propagating wave.

Figure Captions

Fig.1 – Shape of the bidimensional solution of the wave equation valid on the circular domain $\rho \leq a$; $a = 0.1$ mm of the (x, y) plane, for $t = 0$, corresponding to the sum of $N = 22$ terms in the expansion (8). It is no longer a delta function, but it is still very well peaked. By choosing it as the initial condition, instead of the first one of eqs.(4), one gets the threedimensional wave depicted in Figs.2 and 3. The normalization condition is such that $|\Psi_{2D}(\rho = 0; \ t = 0)|^2 = 1$.

Fig.2 – The (very well localized) threedimensional wave corresponding to the initial, bidimensional choice in Fig.1. It propagates rigidly (along the normal-sized circular waveguide with radius $r = a/\sin\theta$) with Superluminal speed $v = c/\cos\theta$. Quantity η is defined as $\eta \equiv (z - \frac{c}{\cos\theta}t)$. The normalization condition is such that $|\Psi_{3D}(\rho = 0; \ \eta = 0)|^2 = 1$.

Fig.3 – The shape along z, at $t = 0$, of the threedimensional wave whose main peak is shown in Fig.2.

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