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UNIQUENESS THEOREMS FOR INVERSE OBSTACLE SCATTERING PROBLEMS IN LIPSCHITZ DOMAINS*

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Abstract. For the Neumann and Robin boundary conditions the uniqueness theorems for inverse obstacle scattering problems are proved in Lipschitz domains. The role of non-smoothness of the boundary is analyzed.

1. Introduction.

Let $D \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with a sufficiently smooth boundary Γ, not necessarily connected, but consisting of a finitely many connected components. Let $D' := \mathbb{R}^n \setminus D$ be the exterior domain, $k > 0$ a fixed wavelength, $\alpha \in S^{n-1}$ a given unit vector, S^{n-1} the unit sphere. It is well known that the obstacle scattering problem:

$$\nabla^2 u + k^2 u = 0 \text{ in } D', \tag{1}$$

$$u_N = 0 \text{ on } \Gamma, \tag{2}$$

$$u = u_0 + v, \quad u_0 := \exp(ik\alpha \cdot x), \tag{3}$$

where v satisfies the radiation condition

$$\lim_{r\to\infty} \int_{|x|=r} |v_r - ikv|^2 ds = 0, \tag{4}$$

and N is the exterior unit normal to Γ has been studied intensively and there are many ways known for proving the existence and uniqueness of its solution which is called the scattering solution [3]. The function v has the following asymptotics

$$v = A(\alpha', \alpha, k)\gamma(r) + o(\frac{1}{r}) \text{ as } r \to \infty,\ \alpha' := x/r. \tag{5}$$

The coefficient $A(\alpha', \alpha, k)$ is called the scattering amplitude.

We also consider the Robin boundary condition in place of (2):

$$u_N + \sigma(s)u = 0 \text{ on } \Gamma, \tag{6}$$

where σ is a continuous real-valued function on Γ.

In what follows, we denote by a subindex zero the quantity which is fixed. The inverse obstacle scattering problems (IOSP1-5) can be stated as follows:

1) Given $A(\alpha', \alpha_0, k)$ $\forall \alpha' \in S^{n-1}, \forall k \in [a, b]$, $0 \leq a < b$, find Γ, or, if Robin's condition is assumed, find Γ and σ;

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2) Given $A(\alpha', \alpha, k_0)$ $\forall \alpha', \alpha \in S^{n-1}$, find Γ, or, if Robin's condition is assumed, find Γ and σ;

3) Given $A(\alpha', \alpha_0, k_0)$ $\forall \alpha' \in S^{n-1}$, find Γ, or, if Robin's condition is assumed, find Γ and σ;

4) Given $A(-\alpha, \alpha, k_0)$ $\forall \alpha \in S^{n-1}$, find Γ, or, if Robin's condition is assumed, find Γ and σ; (backscattering data)

5) Given $A(-\alpha, \alpha, k)$ $\forall \alpha \in S^{n-1}, \forall k \in [a, b]$, $0 \le a < b$, find Γ, or, if Robin's condition is assumed, find Γ and σ;

Of course, if IOSP4 is solved then IOSP5 is solved.

In all these problems one can assume for uniqueness studies that the data are given on open subsets of S^{n-1}, however small, since such data allow one to uniquely recover the data on all of S^{n-1} [3].

In this paper we discuss only IOSP1-2. Uniqueness of the solution to other three problems has been (and still is) an open problem for several decades, although for IOSP5 uniqueness for convex obstacles follows from the results in [3]. The reconstruction of Γ from the scattering data is not discussed here, see [3],[4] and references therein.

The history and various proofs of the uniqueness theorems for IOSP1,2 are given in [3],[4] and references therein, and a new method of proof and its applications are given in [5]-[8].

Uniqueness of the solution for IOSP1 was proved by M.Schiffer (1962) for the Dirichlet boundary condition, while for IOSP2 it was proved by A.G.Ramm (1985) for the Dirichlet, Neumann and Robin boundary conditions (see [3] for these proofs). In [5]-[7] a new method of proof was given.

In this paper we discuss the technical question: the role of smoothness of the boundary in the various proofs of the uniqueness theorems for IOSP1-2. We justify the applicability of Green's formula in the Schiffer's and other proofs and point out that the question of whether the Neumann Laplacian has a discrete spectrum in a certain domain with non-smooth boundary can be avoided completely. This question arises in the Schiffer's type of proofs.

Furthermore, we generalize the uniqueness results for Lipschitz domains, i.e., for domains with Lipschitz boundaries.

In section 2 the Schiffer's type proof and the proof from [6],[7] are presented, and the role of the non-smoothness of some of the domains, used in these proofs, is analyzed. An important role is played by the sets of finite perimeter and Green's formula for such sets. The related theory is discussed in [1],[2] and [9].

We first assume in this paper that the boundary Γ is sufficiently smooth and then show that our argument is valid in Lipschitz domains. So, this paper deals with the technical problems.

Recall that a Lipschitz domain is a bounded domain each point of whose boundary has a neighborhood in which the equation of the boundary in the local coordinates is given by a function satisfying a Lipschitz condition. Lipschitz domains are denoted as $C^{0,1}$ domains. In [10] the potential theory results are given for Lipschitz domains.

The definition of the solution to problem (1)-(3) in non-smooth domains is as follows:

A function $u \in H^2_{loc} \cap H^1(D'_R)$ solves (1)-(3), iff it satisfies conditions (3) and (4), and the following identity:

$$\int_{D'} (k^2 u\phi - \nabla u \nabla \phi) dx = 0 \quad \forall \phi \in H^2_{loc} \cap H^1_c(D'). \tag{7}$$

Here H^l is the Sobolev space, H^2_{loc} is the space of functions which are in $H^2(\tilde{D}')$ for any compact strictly inner subdomain $\tilde{D}'$ of D', $H^1_c(D')$ is the space of functions which vanish near infinity (but not necessarily near Γ), and $H^1(D'_R)$ is the space of functions which for any sufficiently large R belong to $H^1(D' \cap B_R)$, where B_R is the ball of radius R, centered at the origin. This definition does not require any smoothness of the boundary.

The solution to (1), (6), (3) is a function in $H^2_{loc} \cap H^1(D'_R)$ which satisfies conditions (3) and (4), and the identity

$$\int_{D'} (k^2 u\phi - \nabla u \nabla \phi) dx + \int_{\Gamma} \sigma u \phi ds = 0 \quad \forall \phi \in H^2_{loc} \cap H^1_c(D'). \tag{8}$$

Here the Lipschitz boundary Γ is admissible because the imbedding theorem holds for such a boundary.

In this paper we use the following notations: $D_{12} := D_1 \cup D_2$, $D^{12} := D_1 \cap D_2$, Γ_{12} is the boundary of D_{12}, Γ^{12} is the boundary of D^{12}, Γ'_1 is the part of Γ_1 which lies outside of D_2, and Γ'_2 is defined likewise, $\tilde{D}_1$ is a connected component of $D_1 \setminus D^{12}$, $D_3 := D_{12} \setminus D^{12}$.

2. Uniqueness results for IOSP with the Neumann and Robin boundary conditions.

2.1.Uniqueness for IOSP1.

Consider IOSP1 first. Let us outline a variant of the Schiffer's type of proof, which allows us to deal with non-smooth boundaries of the domains arising in the proof. Assume that there are two different obstacles, $D_j, j = 1, 2$, which generate the same scattering data for IOSP1. Let $w := u_1 - u_2$, where u_j are the corresponding scattering solutions. The function w solves equation (1) in D'_{12} and $w = o(1/r)$ because the scattering data are the same for D_1 and D_2. Thus, lemma [3,p.25] implies $w = 0$ in D'_{12}. Let $U := u_1 = u_2$ in D'_{12}. Then U can be continued analytically, as a solution to (1) , to the domains D_3 and $(D^{12})'$, because either u_1 or u_2 are defined in these domains and solve (1) there. We assume that $(D^{12})'$ is not empty. If it is, the argument is even simpler: $V := U - u_0$ solves equation (1) in $\mathbb{R}^n$ and satisfies the radiation condition; thus, $V = 0$ and $U = u_0$ in $\mathbb{R}^n$. Since u_0 does not satisfy the boundary condition (2), we have got a contradiciton. This contradiction proves that the assumption ($(D^{12})'$ is empty) is wrong.

The domain D_3 is bounded since both D_j are. The function U solves equation (1) and satisfies the homogeneous boundary condition (2) on its boundary Γ_3, except for, possibly, the set of $(n-1)$-dimensional Hausdorff measure, namely, except for the set of points which belong to the intersection of Γ_1 and Γ_2. Since the scattering solutions in domains with smooth boundaries are uniformly bounded functions whose first derivatives are smooth (Lipschitz are sufficient for our argument), the function U has the same properties. Therefore, for any $k \in [a, b]$, U is in $L^2(D_3)$, and, as we prove below, the functions U, corresponding to different k, are orthogonal in $L^2(D_3)$. Since this Hilbert space is separable, we arrive at a contradiction: existence of a continuum of orthogonal non-trivial elements in the separable Hilbert space $L^2(D_3)$. This contradiction proves that $D_1 = D_2$, and the uniqueness theorem is proved for IOSP1. The original Schiffer's argument presented in the literature, uses discreteness of the spectrum of the Laplacian, corresponding to a boundary condition, in a bounded domain. The discreteness of the spectrum holds for any bounded domain for the Dirichlet Laplacian, but not necessarily for the Neumann one. This is why we want to avoid the reference to the discreteness of the spectrum of the Neumann Laplacian or the Robin Laplacian. To complete the proof, it is sufficient to prove the claim about the orthogonality of U with different k. The proof of this claim goes along the usual line. The new point is the discussion of the applicability of Green's formula, used in the argument, for non-smooth domains. Let

$U_j := U(x, \alpha_0, k_j)$, $L := \nabla^2$, and let the overline denote complex conjugate. Then:

$$I := \int_{D_3} (U_1 L\overline{U_2} - \overline{U_2} L U_1) dx = (k_1^2 - k_2^2) \int_{D_3} U_1 \overline{U_2} dx. \tag{9}$$

We wish to prove that the right-hand side vanishes. This follows if $I = 0$. The integral I can be transformed formally by Green's formula and, using the boundary condition, one concludes that $I = 0$. The problem is to justify the applicability of Green's formula in the domain D_3 with non-smooth boundary. The remaining part of the proof contains such a justification.

Our starting point is the known (see [1],[2], [9]) result:

Green's formula holds for the domains with finite perimeter and functions whose first derivatives are in the space BV, provided that their rough traces are summable on the reduced boundary of the domain (in our case D_3 is the domain) with respect to $(n-1)$-dimensional Hausdorff measure.

Let $\Omega \in \mathbb{R}^n$ be a domain. Recall that the space $BV(\Omega)$ consists of functions whose first derivatives are signed measures locally in Ω ([2], [9]). A set D_j has finite perimeter if χ_j, the characteristic function of this set, belongs to $BV(\mathbb{R}^n)$. The reduced boundary, denoted by Γ^*, is the set of points at which the exterior normal in the sense of Federer exists (see [1], [2], or [9] for the definition of this normal and [2] for that of the rough trace). It is proved in [9], that for the sets with finite perimeter the reduced boundary has full $(n-1)$-dimensional Hausdorff measure, so that the normal in the sense of Federer is defined almost everywhere on Γ with respect to $(n-1)$-dimensional Hausdorff measure (we will write s-almost everywhere for brevity). What we need is to check that:

1) the set D_3 has finite perimeter,

2) the function $\nabla \cdot \psi$ is a measure in D_3, where $\psi := U_1 \nabla \overline{U_2} - \overline{U}_2 \nabla U_1$,

and

3) ψ has a summable rough trace on Γ_3^*, the reduced boundary of D_3.

Note that the integrand in the first integral in formula (9) is of the form $\nabla \cdot \psi$, and $\nabla \cdot \psi = (k_1^2 - k_2^2) U_1 \overline{U}_2$.

First, let us prove that D_3 has finite perimeter. Note , that D_3 is not necessarily a Lipschitz domain, although D_1 and D_2 are. Let us denote by $P(D)$ the perimeter of D and by $||\nabla \chi||$ the norm of χ in the space $BV(\mathbb{R}^n)$, that is, the total variation of the vector measure $\nabla \chi$. By definition, $P(D) = ||\nabla \chi||$.

Let $s(\Gamma)$ denote the $n-1$-dimensional Hausdorff measure of Γ. It is known that $P(D) \leq s(\Gamma)$ and the strict inequality is possible for non-smooth Γ. Also, it can happen that $P(D) < \infty$, but $s(\Gamma) = \infty$. If $P(D) < \infty$, then Γ^* is s-measurable and $s(\Gamma^*) = P(D)$, see [9, p.193].

The set D_3 has finite perimeter iff $||\nabla \chi_3|| < \infty$. Clearly:

$$\chi_3 = \chi_{12} - \chi^{12}, \tag{10}$$

$$\chi_{12} = \chi_1 + \chi_2 - \chi^{12}, \tag{11}$$

$$\chi^{12} = \chi_1 \chi_2, \tag{12}$$

where χ^{12}, e.g., is the characteristic function of the domain D^{12}. By the assumption, $||\nabla \chi_j|| < \infty$, $j = 1, 2$. The space BV is linear. Therefore, by formulas (10)-(12), it follows that $P(D_3) < \infty$, if one checks that the function $\chi_1 \chi_2 \in BV(\mathbb{R}^n)$. This, however, is a direct consequence of the known formula [9,p.189] for the derivative of the product of bounded BV functions: $\nabla(\chi_1 \chi_2) = \hat{\chi_2} \nabla \chi_1 + \hat{\chi_1} \nabla \chi_2$, where $\hat{\chi}$ denotes the averaged value of χ at the point x (see [9, p.189] for the derivation of this formula). Note that the usual formula for the derivative of the product (the formula without the averaged values) is not valid for BV functions, in particular, it is wrong for the characteristic functions.

Let us now check that the function $\nabla \cdot \psi$ is a signed measure in D_3. Since $\nabla \cdot \psi = (k_1^2 - k_2^2)U_1\overline{U}_2$ and the functions U_1, U_2 belong to $H^1(D_3)$, it follows that $U_1\overline{U}_2 \in L^1(D_3)$. Thus, $\nabla \cdot \psi$ is a signed measure in D_3.

Finally, ψ has a summable rough trace on Γ_3^*. In fact, more holds: the summable trace ψ^+ exists s-almost everywhere on Γ_3^* and this implies existence of the summable rough trace. Recall that the trace ψ^+ is defined at the point $x \in \Gamma$ as the following limit (if it exists):

$$\psi^+(x) = \lim_{r\to 0} \frac{1}{meas_n(D_r(x))} \int_{D_r(x)} \psi(y)dy,$$

where $D_r(x) := \{y : y \in D_3, |x - y| < r\}$. Existence of the summable trace of the function ψ on Γ_3^* follows from [10, lemma 5.7]. One can see that the trace exists in yet stronger sense: $U_j(x)$ and $\nabla U_j(x)$ have non-tangential limits as $x \to t \in \Gamma_3^*$, these limits are in $L^2(\Gamma_3^*, ds)$ and therefore their product is in $L^1(\Gamma_3^*, ds)$, that is , the trace of ψ is summable. This completes the proof of the uniqueness theorem for IOSP1. Let us formulate the result:

Theorem 1. *Assume that the obstacles D_j, $j = 1, 2$, have the following properties:*
1) they are Lipschitz domains,
2) $A_1(\alpha', \alpha_0, k) = A_2(\alpha', \alpha_0, k)$ $\forall\alpha' \in S^{n-1}, \forall k \in [a, b]$, $0 \le a < b$.
Then $D_1 = D_2$ and, in the case of Robin's boundary condition, $\sigma_1 = \sigma_2$.

Proof. Only the last statement is not yet proved. However, since we have already established that $D_1 = D_2 := D$ and $u_1 = u_2$ in D', it follows that

$$\sigma_1 = -\frac{u_{1N}}{u_1} = -\frac{u_{2N}}{u_2} = \sigma_2 \quad \text{on } \Gamma. \quad \Box$$

Another proof can be given. It is based on formula (13) and on the method, developed in section 2.2 below.

If Γ_j are Lipschitz boundaries, then the existence and uniqueness of the scattering solutions can be established as in [3] with the help of the potential theory for domains with Lipschitz boundaries [10]. The details of this theory will be published elsewhere.

In the next subsection we consider IOSP2 and use the method developed in [5]-[7] for the uniqueness proof.

2.2. Uniqueness for IOSP2.

The starting point is the identity first established in [5]:

$$4\pi(A_1 - A_2) = \int_{\Gamma_{12}} [u_1 u_{2N} - u_{1N} u_2] ds, \tag{13}$$

where $u_1 := u_1(x, \alpha, k)$, $u_2 := u_2(x, -\alpha', k)$, u_N denotes the normal derivative, as before, u_j and $A_j := A_j(\alpha', \alpha, k)$ are, respectively, the scattering solution and scattering amplitude, corresponding to the obstacle D_j, $j = 1, 2$. Applications of this useful formula are given in [5]-[8].

If $A_1 = A_2$ for the fixed energy data in IOSP2, then (13) yields:

$$0 = \int_{\Gamma_{12}} [u_1(s, \alpha)u_{2N}(s, -\alpha') - u_{1N}(s, \alpha)u_2(s, -\alpha')]ds, \quad \forall \alpha, \alpha' \in S^{n-1}, \tag{14}$$

where we have dropped the dependence on the fixed energy k_0.

Let $G_j := G_j(x,y,k)$ denote Green's function for the problem (1)-(3), or the exterior problem with the Robin boundary condition. It is proved in [3,p.46], that

$$G_j = \gamma(r)[u_j(x,\alpha,k) + O(\frac{1}{r})], \quad r \to \infty,\ y/r := -\alpha,\ r := |y|, \tag{15}$$

where $\gamma(r)$ is a known function (e.g., $\gamma = \frac{exp(ikr)}{4\pi r}$ if $n = 3$), and the coefficient u_j in (15) is the scattering solution.

Lemma 1. *Equation (14) implies:*

$$0 = \int_{\Gamma_{12}} [G_1(s,x)G_{2N}(s,y) - G_{1N}(s,x)G_2(s,y)]ds, \quad \forall x, y \in D'_{12}. \tag{16}$$

Proof. We give a proof for $n = 3$. For other n the proof is similar. First, let us derive the equation:

$$W(y) := \int_{\Gamma_{12}} [u_1(s,\alpha)G_{2N}(s,y) - u_{1N}(s,\alpha)G_2(s,y)]ds = 0, \quad \forall y \in D'_{12}, \forall \alpha \in S^{n-1}. \tag{17}$$

Indeed, $W(y)$ solves equation (1) in D'_{12} and $W = o(1/r)$, as follows from (14) and (15). Thus, $W = 0$ in D'_{12}, see [3,p.25].

Let us prove (16) now. Fix any $y \in D'_{12}$ and let w denote the integral in (16). Then w solves (1) in D'_{12} and $w = o(1/r)$, as follows from (15) and (17). Thus, (16) follows and Lemma 1 is proved. $\square$

We want to derive a contradiction from (16). This contradiction will prove that $D_1 = D_2$. According to the argument given in the section 2.1, the set D_{12} has finite perimeter, Green's formula is applicable to (16) in the domain D'_{12}, and we get the following equation:

$$0 = G_1(y,x) - G_2(x,y) \quad \forall x, y \in D'_{12}, \tag{18}$$

where the radiation condition for G_1 and G_2 was used: it allowed us to neglect the integral over the large sphere, which appeared in Green's formula.

We now want to derive a contradiction from (18). Note that $G_j(x,y) = G_j(y,x)$ and consider, for instance, the Neumann condition (2). The Robin condition is treated similarly. Differentiate (18) with respect to y along the normal $N_t, t \in \Gamma'_2$, and let $y \to t$. This yields:

$$0 = G_{1N_t}(t,x) \quad \forall x \in D'_{12}, t \in \Gamma'_2. \tag{19}$$

The point t belongs to D'_1. Therefore

$$|G_{1N_t}(t,x)| \to \infty \text{ as } x \to t. \tag{20}$$

Equation (20) contradicts (19). This contradiction proves that $D_1 = D_2$. We have proved the following result:

Theorem 2. *Let the assumption 1) of Theorem 1 hold and assume that*
2′) $A_1(\alpha',\alpha,k_0) = A_2(\alpha',\alpha,k_0)$, $\forall \alpha, \alpha' \in S^{n-1}$.
Then $D_1 = D_2$ and, in the case of Robin boundary condition, $\sigma_1 = \sigma_2$.

This completes the discussion of the uniqueness theorem for IOSP2 for the case of Neumann and Robin boundary conditions.

References

1. H.Federer, *Geometric Measure Theory*, Springer Verlag, New York, 1969.
2. V.Mazja, *Sobolev Spaces*, Springer Verlag, New York, 1985.
3. A.G.Ramm, *Scattering by obstacles*, Reidel, Dordrecht, 1986.
4. A.G.Ramm, *Multidimensional Inverse Scattering Problems*, Longman-Wiley, New York, 1992 (expanded Russian esition MIR, Moscow, 1994).
5. A.G.Ramm, *Scattering amplitude as a function of the obstacle*, Appl.Math.Lett. **6** (1993), no. 5, 85-87.
6. A.G.Ramm, *Multidimensional inverse scattering problems,*, Proc. of Dynamic. Systems and Applications, **1** (1994), 287-296 (eds G.Ladde and M.Sambandham).
7. A.G.Ramm, *A new method for proving uniqueness theorems for inverse obstacle scattering*, Appl. Math. Lett. **6** (1993), 85-87.
8. A.G.Ramm, *Continuous dependence of the scattering amplitude on the surface of the obstacle*, Math. Meth. in the Appl.Sci. **18** (1994), 121-126.
9. A.Volpert and S.Hudjaev, *Analysis in Classes of Discontinuous Functions and Equations of Mathematical Physics*, M.Nijhoff, Dordrecht, 1985.
10. G.Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. of Func. Anal. **59** (1984), 572-611.

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