Bubbling and bistability in two parameter discrete systems

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Abstract

We present a graphical analysis of the mechanisms underlying the occurrences of bubbling sequences and bistability regions in the bifurcation scenario of a special class of one dimensional two parameter maps. The main result of the analysis is that whether it is bubbling or bistability is decided by the sign of the third derivative at the inflection point of the map function.

1. Introduction.

The studies related to onset of chaos in one- dimensional discrete systems modeled by nonlinear maps, have been quite intense and exhaustive during the last two decades. Such a system normally supports a sequence of period doublings leading to chaos. It is also possible to take it back to periodicity through a sequence of period halvings by adding perturbations or modulations to the original system[1,2]. This has, most often, been reported as a mechanism for control of chaos. In addition, there are features like tangent bifurcations, intermittency, crises etc., that occur inside the chaotic regime and are not of immediate relevance to the present work. However, if the system is sufficiently nonlinear, there are other interesting phenomena like bubble structures and bistability that have invited comparatively less attention. The simplest cases where these are realised are maps with at least two control parameters, one that controls the nonlinearity and the other which is a constant additive one. i.e., maps of the type,

$$X_{n+1} = f(X_n, a, b) = f_1(X_n, a) + b \tag{1}$$

In these maps, if a is varied for a given b, the usual period doubling route to chaos is observed. But when a is kept at a point beyond the first period doubling point a_1, and b is varied, the first period doubling is followed by a period halving forming a closed loop like structure called the primary bubble in the bifurcation diagram(Fig(1.a)). If a is kept beyond the second bifurcation point a_2, and b is tuned, secondary bubbles appear on the arms of the primary bubble. Thus as we shift the map along the a-axis and drift it along the b-axis, the complete bubbling scenario develops in the different slices of the space (X, a, b). This accumulates into what is known as bimodal chaos- chaos restricted or confined to the arms of the primary

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bubble. This can be viewed as a separate scenario to chaos in such systems.

It has been confirmed that the Feigenbaum indices for this scenario with a as control parameter would be the same as the α and δ of the normal period doubling route to chaos[3]. However, detailed RG analysis by Oppo and Politi[4], involving the parameter b also, indicates that if b is kept at a critical value, b_c, where bimodal chaos just disappears, then there is a slowing down in the convergence rate leading to an index which is $(\delta)^{1/2}$. This has been experimentally verified in a CO_2 laser system with modulated losses[5]. The bubbling scenario is seen in the bifurcation diagrams of many nonlinear systems like coupled driven oscillators[6, 7], oscillatory chemical reactions, diode circuits, lasers[8,9], insect populations[10], cardiac cell simulations[11], coupled or modulated maps[12,13], quasi-periodically forced systems[14], DPCM transmission system[15] and traffic flow systems[16] etc. The very fact that this phenomenon appears in such a wide variety of systems makes it highly relevant to investigate and expose the common factor(s) in them i.e., the underlying basic features that make them support bubbles in their bifurcation scenario. The above mentioned continuous systems require maps with at least two parameters of type (1) to model them, the second additive parameter being the coupling strength, secondary forcing amplitude etc. We note that in all the above referred papers no specific mention is made regarding the mechanism of formation of bubbles, probably because the authors were addressing other aspects of the problem. However, there have been a number of isolated attempts to analyse the criteria for bubble formation in a few typical systems. According to Bier and Bountis[3], the two criteria are, the map must possess some symmetry and the first period doubling should occur facing the symmetry line. Later, Stone[17] makes these a little more explicit by stating

that the map should have an extending tail (with a consequent inflection point) and the inflection point should occur to the right of the critical point of the map. It is clear that this applies only to maps with one critical point. The relation of the extending tail to bubbling is briefly discussed in [18] also.

Bistability is an equally interesting and common feature associated with many non-linear systems like a ring laser[19] and a variety of electronic circuits[20] .A recent renewal of interest in such systems arises from the fact that they form ideal candidates for studies related to stochastic resonance phenomena[21]. The bistable behaviour in two parameter maps is shown in the bifurcation diagram in Fig(1.b). To the best of our knowledge, attempts to study any type of conditions for the occurrence of bistability are so far not seen reported in the literature.

Our motivation in the present work is to generalise the criteria reported earlier for bubbling and put them together with more clarity and simplicity. As a byproduct, we succeed in stating the conditions for bistability also along similar lines in systems of type (1), even though these are two mutually exclusive phenomena as far as their occurrence regime is concerned. We provide a detailed graphical analysis, which leads to a simple and comprehensible explanation for the same.

The paper is organized as follows. In section 2, the criteria for bistability and bubbling are stated followed by a brief explanation. The graphical analysis taking two simple cubic maps as examples is included in section 3 and the concluding comments in section 4.

2. The dynamics of bubbling and bistability

For the special class of maps given in (1), the basic criteria for bubbling / bistability can be stated as follows.

The non-linearity in $f(X, a, b)$ must be more than quadratic.

This implies that, $f'(X, a, b)$ (the prime indicating derivative with respect to X), is non-monotonic in X and there exists at least one inflection point X_i, where

$$f''(X_i, a, b) = 0. \tag{2}$$

Then we consider the following two cases.

Case (i)

$$f'''(X_i, a, b) > 0 \tag{3}$$

We define a value of a as a_1, through the relation,

$$f'(X_i, a_1, b) = -1, \tag{4}$$

the first period doubling point of the system. For a value of a close to a_1 but greater than a_1, by adjusting the additive parameter b, the system can be taken through a bubble structure in the bifurcation scenario.

Case (ii)

$$f'''(X_i, a, b) < 0 \tag{5}$$

Here a_1 is defined as the tangent bifurcation point of the system, through the relation

$$f'(X_i, a_1, b) = +1 \tag{6}$$

Then by fixing a greater than a_1, but close to a_1, and tuning b, a bistability region can be produced in the system.

For case (i), the value of a chosen to be greater than a_1 makes $f'(X_i, a, b) < -1$ or

$|f'(X_i, a, b)| > 1$. Moreover, conditions (2) and (3) imply that X_i is a minimum for f', which is concave upwards on both sides of X_i. Hence for a fixed point X_-^* which is to the left of X_i, but in the immediate neighborhood, of X_i, $\left|f'(X_-^*, a, b)\right| < 1$, and hence will be stable.Similarly, fixed point X_+^* to the right of X_i, but near to X_i, $\left|f'(X_+^*, a, b)\right| < 1$ and is stable. Now, the second parameter b is simple additive for the class of maps under consideration and hence f' is independent of b. By adjusting b, the fixed point can be shifted such that $f'(X_-^*, a, b)$ becomes equal to -1, the period doubling point of the map.Then X_-^* will give rise to a 2-cycle with elements X_1^* and X_2^*. Since these are in the neighborhood of X_i, $f'(X_1^*)$ and $f'(X_2^*)$ will be negative so that the product $f'(X_1^*)f'(X_2^*)$ is positive. With further increase of b, a period merging takes place for the 2-cycle, with X_1^* and X_2^* collapsing into X_+^*, which is just stable at the point where $f'(X_+^*) = -1$. Thus in the parameter window (b_1, b_2), a bubble structure is formed.

The situation is exactly reversed for case (ii). Here conditions (2) and (5) makes X_i a maximum of f' and the falls off on both sides of X_i. At a value of $a > a_1$, where a_1 is defined by (6), $f'(X_i, a, b) > +1$. Then in the neighborhood of X_i, a fixed point X_-^*, to the left of X_i, can be stable since $|f'(X_i, a, b)| < 1$. Similarly X_+^* on the right of X_i also will be stable.By adjusting the second parameter b, these will be shifted to their respective tangent bifurcation points, i.e., b_1 where X_+^* is born and b_2 where X_-^* disappears. Then a bistability window is seen in the interval (b_1, b_2).

3. Graphical analysis.

The mechanism of occurrence of bubbling and bistability explained above for maps satisfying the conditions in case (i) and case (ii) respectively can be made more transparent through a detailed graphical analysis. For this we plot the curve C1=$f'(X)$,

the 1-cycle fixed point curve C2=$f(X) - X$ and the 2-cycle curve C3=$f(f(X)) - X$ simultaneously as functions of X, for chosen values of a and b. The zeroes of C2 give the 1-cycle fixed point X^* while those of C3 give the elements of the 2-cycle. Their stability can be checked from the same graph, since the value of the derivative at the fixed points can be read off. We fix the value of a to be greater than a_1, which helps to position the curve C1 in the proper way. By plotting the three curves for different values of b, bistability regions or bubbling sequences can be traced for any given map function of type (1).

For further discussion, we consider two specific forms of maps of the cubic type, which are simple but typical examples for case (i) and (ii). They are,

$$M1 : X_{n+1} = b - aX_n + X_n^3 \qquad (7)$$

$$M2 : X_{n+1} = b + aX_n - X_n^3 \qquad (8)$$

For M1, there are two critical points, $X_{c1} = -\sqrt{a/3}$, which is a maximum and $X_{c2} = \sqrt{a/3}$, which is a minimum. The inflection point in between occurs at $X_i = 0$ and $f''' = 6$. Hence it belongs to case (i) and a_1 as defined by (4) is 1. In Fig(2), the three curves mentioned above are plotted for this map at $a = 1.3$. We start from a value of $b = -1.34$, Fig(2.a) where the fixed point X_-^* is just born via tangent bifurcation since $f'(X_-^*)$ here is $+1$, and the curves C2 and C3 just touches the zero line on the left of X_i at X_-^*. Though C2 has a zero on the right, the slope there is larger than 1 and hence it is unstable, for this value of b. Since b is only additive, increase in the value of b, shifts C2 upwards, resulting in a slow drift of X_-^* from left to right. Thus as b is increased to -0.7(Fig(2.b)), $f'(X_-^*) = -1$ and X_-^* bifurcates into X_1^* and X_2^*. At $b = -0.3$(Fig(2.c)), the 2-cycle is stable with $f'(X_1^*)$ and $f'(X_2^*)$, both negative and their product is positive but less than 1. Note that

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the curve C3 has developed a maximum and a minimum on both sides of X_-^*, which is now unstable, cutting the zero line again at $X_1^* < X_-^*$ and $X_2^* > X_-^*$. As b is further increased, they move apart. Since the value chosen is within the stability window of 2-cycle no further period doubling takes place. As X_-^* crosses X_i, X_1^* & X_2^* move towards each other and merge together at $b = 0.7$(Fig(2.d)) and coincide with the fixed point X_+^*. Further, X_+^* disappears by a reverse tangent bifurcation at $b = 1.34$, when $f'(X_+^*)$ becomes equal to $+1$. Thus the above events lead to the formation of a primary bubble in the window (-0.7,0.7).

By keeping a at a value beyond the second period doubling point a_2, of the map, the merging tendency starts only after the second period doubling and hence secondary bubbles are seen on the arms of the primary bubble. These can be repeated until at $a > a_\infty$, the system is taken to chaos.

Now the above analysis is repeated for map M2, which satisfies the conditions in case (ii) (Fig(3)). Here of the two critical points of the map, $X_{c1} = -\sqrt{a/3}$ is the minimum and $X_{c2} = \sqrt{a/3}$ is the maximum with a positive slope at the point of inflection X_i. a_1 in this case is also 1. Hence in the Fig (3) a is chosen to be 1.4. Fig(3.a) shows the situation for $b = -0.35$, where $f'(X^*) = -1$ and hence the 1-cycle fixed point X_-^* period doubles into a 2-cycle. For lower values of b, we expect the full period doubling scenario since f' is monotonic beyond this point(Fig(3.b)). However, as b is increased to -0.1, $f'(X_+^*) = +1$, where the other 1-cycle, X_+^* to the right of X_i is born by tangent bifurcation(Fig(3.c)). Note that at this point X_-^* is still stable with $\left| f'(X_-^*) \right| < 1$. This continues until $b = +0.1$, where $f'(X_-^*) = +1$ and hence X_-^* disappears(Fig(3.d)). The birth of X_+^* is concurrent with the maximum of C2 touching the zero line $(b = b_1)$ while the disappearance of X_-^* occurs

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as the minimum of C2 touches the zero line ($b = b_2$). As b is increased and C2 is moving up it is clear that the former will take place for a lower b value than the latter as the maximum of C2 occurs for $X > X_i$ and minimum at $X < X_i$ (slope being positive at X_i). Hence $b_1 < b_2$, or there is a window (b_1, b_2), where bistability exists, which in our graph is (-0.1,0.1) for $a = 1.4$. X_+^* is stable beyond this point also and it period doubles as b is increased to $b = +0.35$ where $f'(X_+^*) = -1$. The full Feigenbaum scenario then develops for higher values of b. By keeping a at higher values and tuning b, the bistability can be taken to 2-cycle, 4-cycle and even chaotic regions.

The stability regions of the different types of dynamical behaviour possible for M1 can be marked out in parameter space plot in the (a, b) plane(Fig(4.a)). The cone like region on the left is the stability zone of the 1-cycle fixed point (periodicity, p=1) and it is separated from the escape region by the tangent bifurcation line on both sides. The parabola like curve inside it marks out the 2-cycle (p=2) region, while the smaller parabolas indicate curves along which 4-cycles (p=4) and other higher periodic cycles becomes stable until chaos is reached. The line parallel to the b-axis at a value of $a > a_1$, along which primary bubble is formed, is shown by the dotted line. It is clear that along this line, the system is taken from escape $-¿$ 1-cycle $-¿$2-cycle $-¿$1-cycle $-¿$ escape. Similarly secondary bubbles are formed along a line drawn at $a > a_2$ etc. The parameter space plot for M2 is shown in Fig(4.b). The quadrilateral like region marked as (I) beyond $a > a_1$ is the bistable region for 1-cycle, while quadrilateral (II) is that for 2-cycle etc. The area marked with p=1, is the stability region of 1-cycle while p=2, that for 2-cycle etc. When the system is taken along the dotted line beyond a_1, bistability is seen in the central region,

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followed by period doubling bifurcations to both sides, until chaos is reached.

4. Conclusion.

Although the above discussion is confined to two simple cubic maps, the analysis is repeated for a large number of maps of type (1) chosen from a wide variety of situations covering different functional forms like exponential, trigonometric and polynomial maps. We find that the qualitative behaviour in all cases remain the same and depends only on the criteria (2)-(6). Hence the pattern of scenario detailed in this paper can be taken to be atypical as far as maps of the form (1) are concerned.

The criteria for bistability reported here are certainly novel while those for bubbling are more rigorous and general in nature compared to earlier studies. They can be used as a test to identify maps in which bistability or bubbling is possible and also to isolate the regions in the parameter space (a, b) where they occur. Our main result is that whether it is bistability or bubbling is decided by the sign of the third derivative of the map function at the inflection point. If $f'''(X_i)$ is positive, because of the concave nature of the derivative, tangent bifurcation will precede period doubling as b is increased. Hence bubbling structure is possible. Similarly when $f'''(X_i)$ is negative, curve of f' is convex and hence period doubling precedes tangent bifurcation, leading to bistability. In case $f'''(X_i) = 0$, higher derivatives must be considered for deciding the behaviour.

Bubbling can be looked upon as an extreme case of incomplete period doublings and the latter has been often associated with positive Schwarzian derivative[22]. But for the system under study, it is easy to check that this is always negative (independent of the form of the map function), because of properties (2), (3) and (5). In fact,

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a few such maps have been reported earlier[23] though in a totally different context. The bubbling scenario in maps of the type M1, leads to bimodal chaos that is restricted to the arms of the primary bubble. Such confined chaos or even low periodic behavior prior to that, makes them better models in population dynamics of eco systems than the usual logistic type maps[24]. Attempts to extend the criteria to continuous and higher dimensional systems are underway and will be reported elsewhere.

Acknowledgements

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Figure Captions

Fig.1:-   Bifurcation diagram showing (a) bubble structure and (b) bistable behaviour, for a fixed value of a, with b as the control parameter.

Fig.2:-   The derivative curve C1, the 1-cycle solution curve C2 and the 2-cycle solution curve C3 plotted with the value of a at 1.3 for the map M1. In (a), $b = -1.34$ shows the point where the 1-cycle X_-^* is just born, with $f'(X_-^*) = +1$. (b) With $b = b_1 = -0.7$, $f'(X_-^*) = -1$ hence the X_-^* becomes unstable and the 2-cycle is just born.(c) $b = -0.3$, shows the elements of the stable 2-cycle with X_1^* to the left and X_2^* to the right of the X_-^*, which is unstable now and (d) $b = b_2 = +0.7$, the 1-cycle fixed point X_+^* becomes stable after the merging of X_1^* and X_2^*.

Fig.3:-   Here the curves C1 ,C2 and C3 for the map M2 defined in (8) with $a = 1.4$ plotted. (a) At $b = -0.5$, it is clear from the figure that the 1-cycle solution is unstable and the 2-cycle is stable. (b) $b = -0.35$ gives the first period doubling point i.e., here $f'(X_-^*) = -1$. (c) At $b = b_1 = -0.1$, $f'(X_+^*) = +1$, i.e., the creation of a new fixed point X_+^* by tangent bifurcation. Note that still X_-^* is stable and (d) $b = b_2 = 0.1$, $f'(X_-^*)$ is $+1$. Hence the existing fixed point X_-^* disappears. Thus (b_1, b_2) gives the bistability window.

Fig.4:-   Parameter space plot in (a, b) plane (a) for map M1 and (b) for map M2.

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