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Cohomologies of Affine Jacobi Varieties and Integrable Systems .

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Abstract. We study the affine ring of the affine Jacobi variety of a hyper-elliptic curve. The matrix construction of the affine hyper-elliptic Jacobi varieties due to Mumford is used to calculate the character of the affine ring. By decomposing the character we make several conjectures on the cohomology groups of the affine hyper-elliptic Jacobi varieties. In the integrable system described by the family of these affine hyper-elliptic Jacobi varieties, the affine ring is closely related to the algebra of functions on the phase space, classical observables. We show that the affine ring is generated by the highest cohomology group over the action of the invariant vector fields on the Jacobi variety.

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1 Introduction.

Our initial motivation is the study of integrable systems. Consider an integrable system with $2n$ degrees of freedom, by definition it possesses n integrals in involution. The levels of these integrals are n-dimensional tori. This is a general description, but the particular examples of integrable models that we meet in practice are much more special. Let us explain how they are organized.

The phase space $\mathcal{M}$ is embedded algebraically into the space $\mathbb{R}^N$. The integrals are algebraic functions of coordinates in this space. This situation allows complexification, the complexified phase space $\mathcal{M}^{\mathbb{C}}$ is an algebraic affine variety embedded into $\mathbb{C}^N$. The levels of integrals in the complexified case allow the following beautiful description. The systems that we consider are such that with every one of them one can identify an algebraic curve X of genus n whose moduli are defined by the integrals of motion. On the Jacobian $J(X)$ of this curve there is a particular divisor D (in this paper we consider the case when this divisor coincides with the theta divisor, but more complicated situations are possible). The level of integrals is isomorphic to the affine variety $J(X)-D$. The real space $\mathbb{R}^N \subset \mathbb{C}^N$ intersects with every level of integrals by a compact real sub-torus of $J(X)-D$.

This structure explains why the methods of algebraic geometry are so important in application to integrable models. Closest to the present paper account of these methods is given in the Mumfords's book [1].

Let us describe briefly the results of the present paper. We study the structure of the ring $\mathbf{A}$ of algebraic functions (observables) on the phase space of certain integrable model. The curve X in our case is hyper-elliptic. As is clear from the description given above this ring of algebraic functions is, roughly, a product of the functions of integrals of motion by the affine ring of hyper-elliptic Jacobian. The commuting vector fields defined by taking Poisson brackets with the integrals of motion are acting on $\mathbf{A}$. We shall show that by the action of these vector-fields the ring $\mathbf{A}$ is generated from finite number of functions corresponding to the highest nontrivial cohomology group of the affine Jacobian. We conjecture the form of the cohomology groups in every degree and demonstrate the consistence of our conjectures with the structure of the ring $\mathbf{A}$.

Finally we would like to say that this relation to cohomology groups became clear analyzing the results of papers [2] and [3] which deal with quantum integrable models. Very briefly the reason for that is as follows. The quantum observables are in one-to-one correspondence with the classical ones. Consider a matrix element of some observable between two eigen-functions of Hamiltonians. An eigen-functions written in "coordinate" representation (for "coordinates" we take the angles on the torus) must be considered as proportional to square-root of the volume form on the torus. The matrix element is written as integral with respect to "coordinates", the product of two eigen-functions gives a volume form on the torus, and the operator itself can be considered, at least semi-classically, as a multiplier in front of this volume form, i.e. as coefficient of some differential top form on the torus which is the same as the form of one-half of maximal dimension on the phase space. Further, those operators which correspond to "exact form" have vanishing matrix elements. This is how the relation to the cohomologies appears.

The paper, after the introduction, consists of five sections and six appendices which contain technical details and some proofs.

In section 2 we recall the standard construction of the Jacobi variety which is valid for any Riemann surface.

An algebraic construction of the affine Jacobi variety $J(X) - \Theta$ of a hyper-elliptic curve X is reviewed in section 3 following the book [1]vol.II. This construction is specific to hyper-elliptic curves or more generally spectral curves [5].

In section 4 we study the affine ring of $J(X) - \Theta$ using the description in section 3. The main ingredient here is the character of the affine ring. To be precise we consider the ring $\mathbf{A}_0$ corresponding to the most degenerate curve $y^2 = z^{2g+1}$. The ring $\mathbf{A}$ and the affine ring $\mathbf{A}_f$ of $J(X) - \Theta$ for a non-singular X can be studied using $\mathbf{A}_0$. It is important that $\mathbf{A}_0$ is a graded ring and the character $\mathrm{ch}(\mathbf{A}_0)$ is defined. We calculate it by determining explicitly a $\mathbb{C}$-basis of $\mathbf{A}_0$. The relation between $\mathbf{A}_0$ and $\mathbf{A}_f$ for a non-singular X is given in Appendix E.

A set of commuting vector fields acting on $\mathbf{A}$ is introduced in section 5. This action descends to the quotients $\mathbf{A}_0$ and $\mathbf{A}_f$. The action of the vector fields coincides with the action of invariant vector fields on $J(X)$. With the help of these vector fields we define the de Rham type complexes $(\mathbf{C}^*, d)$, $(\mathbf{C}_0^*, d)$, $(\mathbf{C}_f^*, d)$ with the coefficients in $\mathbf{A}$, $\mathbf{A}_0$, $\mathbf{A}_f$ respectively. The complex $(\mathbf{C}_f^*, d)$ is nothing but the algebraic de Rham complex of $J(X) - \Theta$ whose cohomology groups are known to be isomorphic to the singular cohomology groups of $J(X) - \Theta$. What is interested for us is the cohomology groups of $(\mathbf{C}_0^*, d)$. We calculate the q-Euler characteristic of $(\mathbf{C}_0^*, d)$ and show that it coincides with the quotient of $\mathrm{ch}(\mathbf{A}_0)$ by the character $\mathrm{ch}(\mathcal{D})$ of the space of commuting vector fields. Then, by the Euler-Poincaré principle, $\mathrm{ch}(\mathbf{A}_0)/\mathrm{ch}(\mathcal{D})$ is found to be expressible as the alternating sum of the characters of cohomology groups of $(\mathbf{C}_0^*, d)$. Decomposing independently the explicit formula of $\mathrm{ch}(\mathbf{A}_0)$ into the alternating sum, we make conjectures on the cohomology groups of $(\mathbf{C}_0^*, d)$ which are formulated in the next section.

In section six and in Appendix B we study the singular homology and cohomology groups of $J(X) - \Theta$. The Riemann bilinear relation plays an important role here. We formulate conjectures on the cohomology groups of $(\mathbf{C}^*, d)$, $(\mathbf{C}_0^*, d)$, $(\mathbf{C}_f^*, d)$.

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2 Hyper-elliptic curves and their Jacobians.

Consider the hyper-elliptic curve X of genus g described by the equation:

$$y^2 = f(z),$$

where

$$f(z) = z^{2g+1} + f_1 z^{2g} + \cdots + f_{2g+1}. \tag{1}$$

The hyper-elliptic involution σ is defined by

$$\sigma(z,y) = (z,-y).$$

The Riemann surface X can be realized as two-sheeted covering of the z-sphere with the quadratic branch points which are zeros of the polynomial $f(z)$ and ∞.

A basis of holomorphic differentials is given by:

$$\mu_j = z^{g-j}\frac{dz}{y}, \qquad j = 1, \cdots, g.$$

Choose a canonical homology basis of X: $\alpha_1, \cdots, \alpha_g$, $\beta_1, \cdots, \beta_g$. The basis of normalized differentials is defined as

$$\omega_i = \sum_{j=1}^{g} (M^{-1})_{ij}\mu_j,$$

where the matrix M consists of α-periods of holomorphic differentials μ_i:

$$M_{ij} = \int_{\alpha_j} \mu_i, \qquad i,j = 1\cdots, g. \tag{2}$$

The period matrix

$$B_{ij} = \int_{\beta_i} \omega_j$$

defines a point B in the Siegel upper half space:

$$B_{ij} = B_{ji}, \qquad \mathrm{Im}(B) > 0.$$

The Jacobi variety of X is a g-dimensional complex torus:

$$J(X) = \frac{\mathbb{C}^g}{\mathbb{Z}^g + B\mathbb{Z}^g}.$$

The Riemann theta function associated with $J(X)$ is defined by

$$\theta(\zeta) = \sum_{m\in\mathbb{Z}^g} \exp 2\pi i \left(\tfrac{1}{2}\, {}^t m B m + {}^t m\zeta\right),$$

where $\zeta \in \mathbb{C}^g$. The theta function satisfies

$$\theta(\zeta + m + Bn) = \exp 2\pi i \left(-\tfrac{1}{2}\ {}^t nBn - {}^t n\zeta\right)\ \theta(\zeta),$$

for $m, n \in \mathbb{Z}^g$.

Consider the symmetric product of X, the quotient of the product space by the action of the symmetric group:

$$X(n) = X^n / S_n.$$

The Abel transformation defines the map

$$X(g) \xrightarrow{a} J(X)$$

explicitly given by

$$w_j = \sum_{k=1}^{g} \int_{\infty}^{p_k} \omega_j + \Delta,$$

where $p_1, \cdots, p_g$ are points of X, Δ is the Riemann characteristic corresponding to the choice of ∞ for the reference point. In the present case Δ is a half-period because ∞ is a branch point [1].

The divisor Θ is the $(g-1)$-dimensional subvariety of $J(X)$ defined by

$$\Theta = \{w \mid \theta(w) = 0\}. \tag{3}$$

The main subject of our study is the ring A of meromorphic functions on $J(X)$ with singularities only on Θ. The simple way to describe this ring is provided by theta functions:

$$A = \bigcup_{k=0}^{\infty} \left(\frac{\Theta_k(w)}{\theta(w)^k}\right), \tag{4}$$

where Θ_k is the space of theta functions of order k i.e. the space of regular functions on $\mathbb{C}^g$ satisfying

$$\theta_k(w + m + Bn) = \exp 2k\pi i \left(-\tfrac{1}{2}\ {}^t nBn - {}^t nw\right)\ \theta_k(w).$$

There are k^g linearly independent theta functions of order k.

Let us discuss the geometric meaning of the ring A. It is well known that with the help of theta functions one can embed the complex torus $J(X)$ into the complex projective space as a non singular algebraic subvariety. It can be done, for example, using theta functions of third order:
1. 3^g theta functions of third order define an embedding of $J(X)$ into the complex projective space $\mathbb{P}^{3^g-1}$,
2. a set of homogeneous algebraic equations for these theta functions can be written, which allows to describe this embedding as algebraic one.

Now consider the functions

$$\frac{\Theta_3(w)}{\theta(w)^3}.$$

Obviously, with the help of these functions, we can embed the non-compact variety $J(X) - \Theta$ into the complex affine space $\mathbb{C}^{3^g-1}$. Denote the coordinates in this space by $x_1, \cdots, x_{3^g-1}$, the affine ring of $J(X)-\Theta$ is defined as the ring

$$\mathbb{C}\ [x_1, \cdots, x_{3^g-1}]/(g_\alpha),$$

where (g_α) is the ideal generated by the polynomials $\{g_\alpha\}$ such that $\{g_\alpha = 0\}$ defines the embedding. It is known that the affine ring is the characteristic of the non-compact variety $J(X)-\Theta$ independent of a particular embedding of this variety into affine space. Obviously the ring A defined above is isomorphic to the affine ring. We remark that the above argument on the embedding $J(X)-\Theta$ into an affine space is valid if $(J(X), \Theta)$ is replaced by any principally polarized abelian variety.

Consider $X(g)$ which is mapped to $J(X)$ by the Abel map a. The Riemann theorem says that

$$\theta(w) = 0 \qquad \text{iff} \qquad w = \sum_{j=1}^{g-1} \int_{\infty}^{p_j} \omega + \Delta$$

which allows to describe Θ in terms of the symmetric product. One easily argues that the preimage of Θ under the Abel map is described as

$$D := D_\infty \cup D_0,$$

where

$$\begin{aligned} D_\infty &= \{(p_1, \cdots, p_g) \in X(g) |\ p_i = \infty \text{ for some } i\}, \\ D_0 &= \ \{(p_1, \cdots, p_g) \in X(g) |\ p_i = \sigma(p_j) \text{ for some } i \neq j\ \}. \end{aligned} \tag{5}$$

The Abel map is not one-to-one, and the compact varieties $J(X)$ and $X(g)$ are not isomorphic. However, the affine varieties $J(X) - \Theta$ and $X(g) - D$ are isomorphic since the Abel map

$$X(g) - D \ \xrightarrow{a} J(X) - \Theta$$

is an isomorphism. In what follows we shall study the affine variety $X(g) - D \simeq J(X) - \Theta$.

3 Affine model of hyperelliptic Jacobian.

Consider a traceless 2×2 matrix

$$m(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix},$$

where the matrix elements are polynomials of the form:

$$
\begin{aligned}
a(z) &= a_{\frac{3}{2}} z^{g-1} + a_{\frac{5}{2}} z^{g-2} + \cdots + a_{g+\frac{1}{2}}, \\
b(z) &= z^g + b_1 z^{g-1} + \cdots + b_g, \\
c(z) &= z^{g+1} + c_1 z^g + c_2 z^{g-1} + \cdots + c_{g+1}.
\end{aligned}
\tag{6}
$$

Later we shall set $b_0 = c_0 = 1$. Consider the affine space $\mathbb{C}^{3g+1}$ with coordinates $a_{\frac{3}{2}}, \cdots, a_{\frac{g+1}{2}}$, $b_1, \cdots, b_g$, $c_1, \cdots, c_{g+1}$. Fix the determinant of $m(z)$:

$$
a^2(z) + b(z)c(z) = f(z), \tag{7}
$$

where the polynomial $f(z)$ is the same as used above (1). Comparing each coefficient of z^i $(i = 0, 1, \cdots, 2g)$ of (7) one gets $2g + 1$ different equations. In fact the equations (7) define g-dimensional sub-variety of $\mathbb{C}^{3g+1}$. This algebraic variety is isomorphic to $J(X) - \Theta$ as shown in the book [1]. We shall briefly recall the proof.

Consider a matrix $m(z)$ satisfying (7). Take the zeros of $b(z)$:

$$
b(z) = \prod_{j=1}^{g} (z - z_j)
$$

and set

$$
y_j = a(z_j).
$$

Obviously z_j, y_j satisfy the equation

$$
y_j^2 = f(z_j),
$$

which defines the curve X. So, we have constructed a point of $X(g)$ for every $m(z)$ which satisfies the equations (7). Conversely, for a point $(p_1, \cdots, p_g)$ of $X(g)$, construct the matrix $m(z)$ as

$$
\begin{aligned}
b(z) &= \prod_{j=1}^{g} (z - z_j), \qquad a(z) = \sum_{j=1}^{g} y_j \prod_{k \neq j} \left(\frac{z - z_k}{z_j - z_k} \right), \\
c(z) &= \frac{-a(z)^2 + f(z)}{b(z)},
\end{aligned}
$$

where $z_j = z(p_j)$ is the z-coordinate of p_j. Considering the function $b(z)$ as a function on $X(g)$ one finds that it has singularities when one of z_j equals ∞. The function $a(z)$ is singular at $z_j = \infty$ and also at the points where $z_i = z_j$ but $y_i = -y_j$. This is exactly the description of the variety D. The functions $a(z)$ and $c(z)$ do not add new singularities. Thus we have the embedding of the affine variety $X(g) - D$ into the affine space:

$$
X(g) - D \hookrightarrow \mathbb{C}^{3g+1}.
$$

Therefore we can profit from the wonderful property of the hyper-elliptic Jacobian: it allows an affine embedding into a space of very small dimension equal to $3g + 1$ (compare with $3^g - 1$ which we have for any Abelian variety). Actually, the space $\mathbb{C}^{3g+1}$ occurs foliated with generic leaves isomorphic to the affine Jacobians.

4 Properties of affine ring.

Consider the free polynomial ring $\mathbf{A}$:

$$\mathbf{A} = \mathbb{C}\ [a_{\frac{3}{2}}, \cdots, a_{g+\frac{1}{2}}, b_1, \cdots, b_g, c_1, \cdots, c_{g+1}].$$

On the ring $\mathbf{A}$ one can naturally introduce a grading. Prescribe the degree j to any of generators a_j, b_j, c_j and extend this definition to all monomials in $\mathbf{A}$ by

$$\deg(xy) = \deg(x) + \deg(y).$$

Every monomial of the ring has positive degree (except for 1 whose degree equals 0). Thus, as a linear space, $\mathbf{A}$ splits into

$$\mathbf{A} = \bigoplus_{2p\in\mathbb{Z}_+} \mathbf{A}^{(p)},$$

where $\mathbf{A}^{(j)}$ is the subspace of degree j and $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$. Define the character of $\mathbf{A}$ by

$$\mathrm{ch}(\mathbf{A}) = \sum_{2p\in\mathbb{Z}_+} q^p \dim(\mathbf{A}^{(p)}).$$

Since the ring $\mathbf{A}$ is freely generated by a_p, b_p, c_p one easily finds

$$\mathrm{ch}(\mathbf{A}) = \frac{\left[\frac{1}{2}\right]}{\left[g+\frac{1}{2}\right]!\ [g]!\ [g+1]!}, \tag{8}$$

where, for $k \in \mathbb{Z}_+$,

$$[k] = 1 - q^k, \quad [k]! = [1]\cdots[k], \quad \left[k+\tfrac{1}{2}\right]! = \left[\tfrac{1}{2}\right]\left[\tfrac{3}{2}\right]\cdots\left[k+\tfrac{1}{2}\right].$$

This important formula allows to control the size of the ring $\mathbf{A}$.

The relation of the ring $\mathbf{A}$ to the affine ring A is obvious. The latter is the quotient of $\mathbf{A}$ by the ideal generated by the relations $-\det(m(z)) = f(z)$ where the coefficients of f are considered fixed constants.

From the point of view of integrable models, it is more natural to see $f_1, \cdots, f_{2g+1}$ as variables than complex numbers. If we assign degree j to the variables f_j, all the equations in (7) are homogeneous. Consider the polynomial ring

$$\mathbf{F} = \mathbb{C}\ [f_1, \cdots, f_{2g+1}].$$

The ring $\mathbf{F}$ is graded and its character is

$$\mathrm{ch}(\mathbf{F}) = \frac{1}{[2g+1]!}.$$

The ring $\mathbf{F}$ acts on $\mathbf{A}$, that is, $f(z)$ acts by the multiplication of the left hand side of (7). Consider the space $\mathbf{A}_0$ which consists of $\mathbf{F}$-equivalence classes:

$$\mathbf{A}_0 = \mathbf{A}\ /\ (\mathbf{F}^\times \mathbf{A}), \qquad \mathbf{F}^\times = \sum_{i=1}^{2g+1} \mathbf{F} f_i.$$

Since $\mathbf{F}^\times\mathbf{A}$ is a homogeneous ideal of $\mathbf{A}$, $\mathbf{A}_0$ is a graded vector space:

$$\mathbf{A}_0=\bigoplus_{2p\in\mathbb{Z}_+}\mathbf{A}_0^{(p)}.$$

One can consider the space $\mathbf{A}_0$ as a subspace of $\mathbf{A}$ taking a set of homogeneous representatives of the equivalence classes (being homogeneous they are automatically of smallest possible degree). Consider any homogeneous $x\in\mathbf{A}$. One can write x as

$$x=x^{(0)}+\sum_{i=1}^{2g+1}f_ix_i,$$

where $x^{(0)}\in\mathbf{A}_0$ and x_i is a homogeneous element in $\mathbf{A}$ satisfying $\deg x_i=\deg x-i$. Since the degree of x_i is less than the degree of x, repeating the same procedure for x_i one arrives, by finite number of steps, at

$$x=\sum h_jx_j^{(0)}, \tag{9}$$

where $x_j^{(0)}\in\mathbf{A}_0$, $h_i\in\mathbf{F}$ and the summation is finite. There is an $\mathbf{F}$-linear map:

$$\mathbf{F}\otimes_{\mathbb{C}}\mathbf{A}_0\xrightarrow{m}\mathbf{A},$$

which corresponds to multiplying the elements of $\mathbf{A}_0$ by elements from $\mathbf{F}$ and taking linear combinations. The above reasoning shows that $\mathrm{Im}(m)=\mathbf{A}$. Hence

$$\mathrm{ch}(\mathbf{A}_0)\geq\frac{\mathrm{ch}(\mathbf{A})}{\mathrm{ch}(\mathbf{F})}. \tag{10}$$

The equality takes place iff $\mathrm{Ker}(m)=0$. We shall see that this is indeed the case. Informally the equality $\mathrm{Ker}(m)=0$ is a manifestation of the fact that the space $\mathbb{C}^{3g+1}$ is foliated into g-dimensional sub-varieties, the coordinates f_j describe transverse direction. The pure algebraic proof of this fact is given by the following proposition.

Proposition 1. *The set of elements*

$$\prod_{j=1}^{g}u_{\frac{1+j}{2}}^{i_j}\prod_{k=1}^{g}u_{\frac{g+1+k}{2}}^{l_k},$$

where

$$u_p=\begin{cases}a_p, & p=\text{half-integer}\\ b_p, & p=\text{integer},\end{cases}$$

is a basis of $\mathbf{A}_0$ as a vector space, where $i_1,\cdots,i_g$ are non-negative integers and $l_1,\cdots,l_g$ are 0 or 1.

The proof of Proposition 1 is given in Appendix A. Proposition 1 shows that

$$\mathrm{ch}(\mathbf{A}_0)=\prod_{j=1}^{g}\frac{1}{\left[\frac{1+j}{2}\right]}\prod_{k=1}^{g}\left(1+q^{\frac{g+1+k}{2}}\right)=\frac{\left[\frac{1}{2}\right][2g+1]!}{\left[g+\frac{1}{2}\right]!\ [g]!\ \ [g+1]!}=\frac{\mathrm{ch}(\mathbf{A})}{\mathrm{ch}(\mathbf{F})}, \tag{11}$$

which means that $\mathrm{Ker}(m) = 0$. We summarize this in the following:

Proposition 2. *As an* $\mathbf{F}$ *module,* $\mathbf{A}$ *is a free module,* $\mathbf{A} \simeq \mathbf{F} \otimes_{\mathbb{C}} \mathbf{A}_0$*. In other words every element* $x \in \mathbf{A}$ *can be uniquely presented as a finite sum:*

$$x = \sum h_j x_j^{(0)},$$

where $\{x_j^{(0)}\}$ *is a basis of the* $\mathbb{C}$*-vector space* $\mathbf{A}_0$ *and* $h_j \in \mathbf{F}$*.*

5 Poisson structure and cohomology groups.

The affine model of hyper-elliptic Jacobian is interesting for its application to integrable models. The ring $\mathbf{A}$ that we introduced in the previous section can be supplied with Poisson structure. This fact is also important because introducing the Poisson structure is the first step towards the quantization. The Poisson structure in question is described in r-matrix formalism as follows:

$$\{m(z_1) \otimes I, I \otimes m(z_2)\} = [r(z_1, z_2), m(z_1) \otimes I] - [r(z_2, z_1), I \otimes m(z_2)]. \quad (12)$$

The r-matrix acting in $\mathbb{C}^2 \otimes \mathbb{C}^2$ is

$$r(z_1, z_2) = \frac{z_2}{z_1 - z_2}\left(\tfrac{1}{2}\sigma^3 \otimes \sigma^3 + \sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+\right) + z_2\ \sigma^- \otimes \sigma^-,$$

where σ^3, $\sigma^\pm$ are Pauli matrices.

The variables $z_1, \cdots, z_g$ (zeros of $b(z)$) and $y_j = a(z_j)$ have dynamical meaning of separated variables [4]. The Poisson brackets (12) imply the following Poisson brackets for the separated variables:

$$\{z_i, y_j\} = \delta_{i,j}\ z_i.$$

The determinant $f(z)$ of the matrix $m(z)$ generates Poisson commutative subalgebra:

$$\{f(z_1), f(z_2)\} = 0.$$

It can be shown that the coefficients $f_1, f_2, \cdots, f_g$ and f_{2g+1} belongs to the center of Poisson algebra. The Poisson commutative coefficients $f_{g+1}, \cdots, f_{2g}$ are the integrals of motion. Introduce the commuting vector-fields

$$D_i h = \{f_{g+i}, h\}, \qquad i = 1, \cdots g.$$

For completeness let us describe explicitely the action of these vector-fields on $m(z)$. Define

$$D(z) = \sum_{j=1}^{g} z^{j-1} D_{g+1-j}.$$

Then the Poisson brackets (12) imply:

$$D(z_1)m(z_2) = \frac{1}{z_1 - z_2}[m(z_1), m(z_2)] - [\sigma^- m(z_1)\sigma^-, m(z_2)].$$

One can think of these commuting vector-fields as $D_j = \frac{\partial}{\partial \tau_j}$ where τ_j are "times" corresponding to the integrals of motion f_{g+j}. The "times" τ_j are coordinates on the Jacobi variety, they are related to w as follows

$$\tau = \tfrac{1}{2} M w,$$

where M is the matrix defined in (2). We remark that D_i here coincides with $-2D_i$ in the Mumford's book [1]vol.II. Earlier we have introduced a gradation on the ring $\mathbf{A}$. We can prescribe the degrees to the vector-fields D_j as $\deg(D_j) = j - \frac{1}{2}$ because it can be shown that:

$$D_j \mathbf{A}^{(p)} \subset \mathbf{A}^{(p+j-\frac{1}{2})}.$$

Consider the differential forms

$$f_{i_1 \, \cdots \, i_k} d\tau_{i_1} \, \wedge \cdots \wedge d\tau_{i_k}, \tag{13}$$

with $f_{i_1,\cdots,i_k} \in \mathbf{A}$. These forms span the linear spaces $\mathbf{C}^k$ for $k = 0, \cdots, g$. The differential

$$d = \sum_{j=1}^{g} d\tau_j D_j,$$

acts from $\mathbf{C}^k$ to $\mathbf{C}^{k+1}$. As usual applying d we first apply the vector fields D_j to the coefficients of the differential form and then take exterior product with $d\tau_j$. We have the complex

$$0 \longrightarrow \mathbf{C}^0 \xrightarrow{d} \mathbf{C}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathbf{C}^{g-1} \xrightarrow{d} \mathbf{C}^g \xrightarrow{d} 0.$$

The k-th cohomology group of this complex is denoted by $H^k(\mathbf{C}^*)$. Consider the problem of grading of the spaces $\mathbf{C}^j$. Clearly we have to prescribe the degree to $d\tau_j$ as

$$\deg(d\tau_j) = -j + \tfrac{1}{2}$$

in order that d has degree zero.

Consider the spaces $\mathbf{C}_0^k$ spaned by (13) with $f_{i_1 \, \cdots \, i_k} \in \mathbf{A}_0$. Since the elements of $\mathbf{F}$ are "constants" (commute with D_i), D_i acts on $\mathbf{A}_0$. So, we have the complex $\mathbf{C}_0^*$:

$$0 \longrightarrow \mathbf{C}_0^0 \xrightarrow{d} \mathbf{C}_0^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathbf{C}_0^{g-1} \xrightarrow{d} \mathbf{C}_0^g \xrightarrow{d} 0.$$

This complex is graded. One easily calculates that

$$\mathrm{ch}(\mathbf{C}_0^{g-j}) = q^{\frac{1}{2}(j^2-g^2)} \begin{bmatrix} g \\ j \end{bmatrix} \mathrm{ch}\,(\mathbf{A}_0)\,, \tag{14}$$

where the q-binomial coefficient is defined as

$$\begin{bmatrix} g \\ j \end{bmatrix} = \frac{[g]!}{[j]!\,[g-j]!}.$$

The differential d respects the grading. In this case the q-Euler characteristic can be introduced

$$\chi_q\left(\mathbf{C}_0^*\right) = \mathrm{ch}\left(\mathbf{C}_0^0\right) - \mathrm{ch}\left(\mathbf{C}_0^1\right) + \cdots + (-1)^g \mathrm{ch}\left(\mathbf{C}_0^g\right), \tag{15}$$

which possesses all the essential properties of the usual Euler characteristic. Using the formula (14) one finds

$$\begin{aligned} \chi_q\left(\mathbf{C}_0^*\right) &= (-1)^g q^{-\frac{1}{2}g^2} \ \left[\tfrac{2g-1}{2}\right]! \ \mathrm{ch}\left(\mathbf{A}_0\right) \\ &= (-1)^g q^{-\frac{1}{2}g^2} \ \frac{[2g+1]! \ [\frac{1}{2}]}{[g+\frac{1}{2}] \ [g]! \ [g+1]!}. \end{aligned} \tag{16}$$

Consider the cohomology groups $H^k(\mathbf{C}_0^*)$. The vector spaces $H^k(\mathbf{C}_0^*)$ inherit a grading from $\mathbf{C}_0^j$. Then

$$\chi_q(\mathbf{C}_0^*) = \mathrm{ch}(H^0(\mathbf{C}_0^*)) - \mathrm{ch}(H^1(\mathbf{C}_0^*)) + \cdots + (-1)^g \mathrm{ch}(H^g(\mathbf{C}_0^*)).$$

The q-number in (16) has finite limit for $q \to 1$:

$$\lim_{q\to 1} \chi_q(\mathbf{C}_0^*) = (-1)^g \frac{(2g)!}{(g)! \ (g+1)!} = (-1)^g \left(\binom{2g}{g} - \binom{2g}{g-1}\right). \tag{17}$$

Certainly the fact that the q-Euler characteristic has a finite limit does not mean that cohomology groups are finite-dimensional, but we believe that this is the case. So, we put forward

Conjecture 1. *The spaces* $H^k(\mathbf{C}_0^*)$ *are finite-dimensional.*

More explicitly the cohomology groups will be discussed in the next section.

In the situation under consideration there is an important connection between the algebra $\mathbf{A}$ and the highest cohomology group $H^g(\mathbf{C}_0^*)$.

Proposition 3. *Consider some homogeneous representatives of a basis of the space* $H^g(\mathbf{C}_0^*)$*:*

$$h_\alpha \ d\tau_1 \wedge \cdots \wedge d\tau_g.$$

Arbitrary $x \in \mathbf{A}_0$ *can be presented in the form*

$$x = \sum_\alpha P_\alpha(D_1, \cdots D_g) h_\alpha, \tag{18}$$

where $P_\alpha(D_1, \cdots D_g)$ *are polynomials in* $D_1, \cdots, D_g$ *with* $\mathbb{C}$*-number coefficients.*

Proof. For $x \in \mathbf{A}_0$ construct

$$\Omega = x \ d\tau_1 \wedge \cdots \wedge d\tau_g \in \mathbf{C}_0^g.$$

By the definition of cohomology group we have

$$\Omega = \Omega_0 + d\Omega', \qquad \Omega_0 \in H^g(\mathbf{C}_0^*), \qquad \Omega' \in \mathbf{C}_0^{g-1},$$

which implies that

$$x = h + \sum D_i x_i,$$

with h such that $\Omega_0 = h\ d\tau_1 \wedge \cdots \wedge d\tau_g$, $x_i \in \mathbf{A}_0$. Apply the same procedure to x_i and go on along the same lines. The resulting representation (18) will be achieved in finite number of steps for the reason of grading. QED.

Let us introduce the notation

$$\mathcal{D} = \mathbb{C}\ [D_1, \cdots, D_g].$$

We shall call the expressions of the type (18) the $\mathcal{D}$-descendents of $\{h_\alpha\}$. The interesting question concerning the formula (18) is whether such representation is unique for any x. The answer is that it is not the case, and to understand why it is so we have to return to the formula (16) which can be rewritten as follows:

$$\begin{aligned} q^{-\frac{1}{2}g^2} \mathrm{ch}\,(\mathbf{A}_0) &= \frac{1}{\left[g - \frac{1}{2}\right]!} \mathrm{ch}(H^g(\mathbf{C}_0^*)) - \\ - \frac{1}{\left[g - \frac{1}{2}\right]!} \mathrm{ch}(H^{g-1}&(\mathbf{C}_0^*)) + \frac{1}{\left[g - \frac{1}{2}\right]!} \mathrm{ch}(H^{g-2}(\mathbf{C}_0^*)) - \cdots . \end{aligned} \tag{19}$$

Obviously, the first term in the RHS represents the character of the space of all $\mathcal{D}$-descendents of $\{h_\alpha\}$ (recall that the degree of D_j equals $j - \frac{1}{2}$). This is equivalent to saying that the first term has the same character as the space generated freely over $\mathcal{D}$ by $H^g(\mathbf{C}_0^*)$:

$$\frac{1}{\left[g - \frac{1}{2}\right]!} \mathrm{ch}(H^g(\mathbf{C}_0^*)) = \mathrm{ch}(\mathcal{D})\mathrm{ch}(H^g(\mathbf{C}_0^*)) = \mathrm{ch}\big(\mathcal{D} \otimes_{\mathbf{C}} H^g(\mathbf{C}_0^*)\big).$$

The existence of the second term of the RHS of (19) implies that, in $\mathbf{A}_0$, there are linear relations among $\mathcal{D}$-descendents of $\{h_\alpha\}$ and they are parametrized by the second term. The third term explains that there are relations among linear relations counted by the second term of the RHS of (19) and so on. This is nothing but the usual argument of constructing a resolution of a module. In the present case it is actually possible to construct a free resolution of $\mathbf{A}_0$ as a $\mathcal{D}$ module assuming some conjectures. The construction of the free resolution is given in Appendix F.

Combining Proposition 2 and Proposition 3 one arrives at

Proposition 4. *Let $\{h_\alpha\}$ be the same as in Proposition 3. Then every $x \in \mathbf{A}$ can be presented as*

$$x = \sum_\alpha \mathbf{P}_\alpha\,(D_1, \cdots, D_g)\, h_\alpha, \tag{20}$$

where $\mathbf{P}_\alpha\,(D_1, \cdots, D_g)$ are polynomials in D_1,...,D_g with coefficients from $\mathbf{F}$.

Proof. We shall prove the proposition by the induction on the degree of x. Since $\mathbf{A}^{(p)} = \{0\}$ for $p < 0$, the beginning of induction obviously holds. Suppose that the proposition is true for all elements of degree less than $\deg(x)$. By Proposition 2, there exist x_j such that

$$x = x_0 + \sum_{i=1}^{2g+1} f_i x_i, \quad x_j \in \mathbf{A}_0,$$

where $\deg(x) = \deg(x_0)$ and $\deg(x_i) = \deg(x) - i < \deg(x)$ for $i > 0$. By Proposition 3, there exist polynomials $P_\alpha(D_1, \cdots, D_g)$ with the coefficients in $\mathbb{C}$ such that

$$x_0 = \sum P_\alpha(D_1, \cdots, D_g) h_\alpha + \sum_{i=1}^{2g+1} f_i y_i, \quad y_i \in \mathbf{A},$$

where $\deg y_i = \deg x_0 - i$. Since $\deg(y_i) < \deg(x_0)$, x can be written in the form (20) by the induction hypothesis. QED.

Proposition 4 represents the most important result of this paper. The possibility of presenting every algebraic function on the phase space of the integrable model in the form (20) starting from finite number of functions $\{h_\alpha\}$, which are representatives of the highest cohomology group, is important both in classical and in quantum case. The description of null-vectors follows from the one given above because D_i commute with f_i.

6 Conjectures on cohomology groups.

In the previous section we have seen that the cohomology group $H^g(\mathbf{C}_0^*)$ is important for describing the algebra $\mathbf{A}$. This cohomology group is rather exotic, since the complex $\mathbf{C}_0^*$ corresponds to the case when the algebraic curve X is singular, that is, $y^2 = z^{2g+1}$. In this section we first discuss the relation between $H^k(\mathbf{C}_0^*)$ and the singular cohomology groups of the non-singular affine Jacobi variety $J(X) - \Theta$. For a set of complex numbers $f^0 = (f_1^0, \cdots, f_{2g+1}^0)$ we set

$$\mathbf{A}_{f^0} = \mathbf{A} \otimes_{\mathbf{F}} \mathbb{C}_{f^0}, \quad \mathbb{C}_{f^0} = \mathbf{F} / \sum_{i=1}^{2g+1} \mathbf{F}(f_i - f_i^0),$$

and $f^0(z) = z^{2g+1} + f_1^0 z^{2g} + \cdots + f_{2g+1}^0$. In the case when all $f_i^0 = 0$, $\mathbf{A}_{f^0} = \mathbf{A}_0$. If the curve X: $y^2 = f^0(z)$ is non-singular, $\mathbf{A}_{f^0}$ is isomorphic to the affine ring of $J(X) - \Theta$. Since d commutes with $\mathbf{F}$, the complex $(\mathbf{C}^*, d)$ induces the complex $(\mathbf{C}_{f^0}^*, d)$, where

$$\mathbf{C}_{f^0}^k = \mathbf{C}^k \otimes_{\mathbf{F}} \mathbb{C}_{f^0} = \sum_{i_1 < \cdots < i_k} \mathbf{A}_{f^0} d\tau_{i_1} \wedge \cdots \wedge d\tau_{i_k}.$$

Recall that

$$H^g(\mathbf{C}^*) = \frac{\mathbf{C}^g}{d\mathbf{C}^{g-1}}, \quad H^g(\mathbf{C}_{f^0}^*) = \frac{\mathbf{C}_{f^0}^g}{d\mathbf{C}_{f^0}^{g-1}}.$$

Thus, tensoring $\mathbb{C}_{f^0}$ to the exact sequence

$$\mathbf{C}^{g-1} \xrightarrow{d} \mathbf{C}^g \longrightarrow H^g(\mathbf{C}^*) \longrightarrow 0,$$

we have

$$H^g(\mathbf{C}^*) \otimes_{\mathbf{F}} \mathbb{C}_{f^0} \simeq H^g(\mathbf{C}^*_{f^0}),$$

for any f^0. By Proposition 4 we have

$$H^g(\mathbf{C}^*) = \sum_\alpha \mathbf{F}\Omega_\alpha,$$

where $\{\Omega_\alpha\}$ are representatives of $H^g(\mathbf{C}^*_0)$ in $\mathbf{C}^g$. In other words there is a surjective map of $\mathbf{F}$-modules:

$$\mathbf{F} \otimes_{\mathbb{C}} H^g(\mathbf{C}^*_0) \longrightarrow H^g(\mathbf{C}^*).$$

We conjecture that this map is in fact injective. In general we put forward the following conjecture.

Conjecture 2. (1) $H^k(\mathbf{C}^*)$ *is a free* $\mathbf{F}$*-module for any* k.
(2) $H^k(\mathbf{C}^*) \otimes_{\mathbf{F}} \mathbb{C}_{f^0} \simeq H^k(\mathbf{C}^*_{f^0})$ *for any* k *and* f^0.

Notice that Conjecture 2 implies, in particular, that

$$H^k(\mathbf{C}^*) \simeq \mathbf{F} \otimes_{\mathbb{C}} H^k(\mathbf{C}^*_0).$$

It is known, by the algebraic de Rham theorem (*cf.* [6]), that, if X is non-singular,

$$H^k(\mathbf{C}^*_{f^0}) \simeq H^k(J(X) - \Theta, \mathbb{C}),$$

where the RHS is the singular cohomology group of $J(X) - \Theta$. Thus we have

Corollary of Conjecture 2. *There is an isomorphism:*

$$H^k(\mathbf{C}^*_0) \simeq H^k(\mathbf{C}^*_{f^0}),$$

for any f^0*. In particular, for any non-singular hyper-elliptic curve* X,

$$H^k(\mathbf{C}^*_0) \simeq H^k(X(g) - D, \mathbb{C}).$$

Notice that Conjecture 1 follows from Conjecture 2 because the singular cohomology groups of a non-singular affine variety are finite-dimensional. We shall comment more on Conjecture 2 later, for the moment let us concentrate on the singular cohomology groups of $X(g) - D$ for a non-singular X.

Consider the affine curve $X_{\text{aff}} = X - \{\infty\}$ and its symmetric powers:

$$X_{\text{aff}}(n) = X^n_{\text{aff}}/S_n.$$

Since $X_{\rm aff}$ is affine and connected,

$$H^p(X_{\rm aff},\mathbb{C})=0,\quad p\geq 2,\quad \dim\, H^0(X_{\rm aff},\mathbb{C})=1.$$

The cohomology group $H^1(X_{\rm aff},\mathbb{C})$ is $2g$ dimensional and it is generated by

$$\mu_j=z^{g-j}\frac{dz}{y},\qquad j=-g+1,\cdots,g,$$

in the algebraic de Rham cohomology description of $H^1(X_{\rm aff},\mathbb{C})$. On $H^1(X_{\rm aff},\mathbb{C})$ there is a skew-symmetric bilinear form:

$$\lambda_1\circ\lambda_2={\rm res}_{p=\infty}\left(\lambda_1(p)\int^p\lambda_2\right).$$

Canonical basis ν_j, $j=-g+1,\cdots,\nu_g$, with respect to this form, is defined as one satisfying

$$\nu_i\circ\nu_j=\frac{4}{j-i}\;\delta_{i+j,1}.$$

A particular example of such basis is given in Appendix B.

As in the case of the compact curve X, the cohomology groups of the symmetric products $X_{\rm aff}(n)$ is described as the S_n invariants, $H^*(X_{\rm aff}(n),\mathbb{C})\simeq H^*(X_{\rm aff}^n,\mathbb{C})^{S_n}$ (*cf.* (1.2) in [7]). If we define

$$\begin{aligned}\tilde{\mu}_i&=\mu_i^{(1)}+\cdots+\mu_i^{(n)}, &(21)\\ \mu_i^{(k)}&=1\otimes\cdots\otimes\overset{k}{\breve{\mu}_i}\otimes\cdots\otimes 1\in H^*(X_{\rm aff},\mathbb{C})^{\otimes n},\end{aligned}$$

then they generate the cohomology ring $H^*(X_{\rm aff}(n),\mathbb{C})$. Obviously

$$\begin{aligned}H^1(X_{\rm aff}(n),\mathbb{C})&\simeq\mathbb{C}\tilde{\mu}_{-g+1}\oplus\cdots\oplus\mathbb{C}\tilde{\mu}_g,\\ H^k(X_{\rm aff}(n),\mathbb{C})&\simeq\bigwedge^k H^1(X_{\rm aff}(n),\mathbb{C}). &(22)\end{aligned}$$

Recall that $D=D_0\cup D_\infty$. Obviously, $X_{\rm aff}(g)=X(g)-D_\infty$. Hence there is a map from $H^k(X_{\rm aff}(g),\mathbb{C})$ to $H^k(X(g)-D,\mathbb{C})$. In the Appendix B we prove the following:

Proposition 5. *Consider the natural map:*

$$H^k(X_{\rm aff}(g),\mathbb{C})\;\xrightarrow{i'}H^k(X(g)-D,\mathbb{C}).$$

The kernel of this map is described as follows:

$${\rm Ker}(i')=\omega\wedge H^{k-2}(X_{\rm aff}(g),\mathbb{C}),$$

where

$$\omega=\tfrac{1}{4}\sum_{k=1}^{g}(2k-1)\;\tilde{\nu}_k\wedge\tilde{\nu}_{-k+1}. \tag{23}$$

We remark that ω does not depend on the choice of the canonical basis $\{\nu_j\}$. By Proposition 5 the map i' induces an injective map:

$$W^k := H^k(X_{\text{aff}}(g), \mathbb{C})/\big(\omega \wedge H^{k-2}(X_{\text{aff}}(g), \mathbb{C})\big) \hookrightarrow H^k(X(g) - D, \mathbb{C}). \tag{24}$$

We can make W^k a graded vector space by prescribing the degrees to differential forms as

$$\deg(\tilde{\mu}_j) = -j + \tfrac{1}{2}, \qquad \deg(\omega) = 0.$$

From (22) one easily finds:

$$\begin{aligned} &\mathrm{ch}(W^k) = R_k - R_{k-2}, \\ &R_k = q^{\frac{1}{2}k(k-2g)} \begin{bmatrix} 2g \\ k \end{bmatrix}. \end{aligned}$$

Now we put forward the strong conjecture that the map (24) is in fact surjective and the character of W^k defined here coincides with the character of $H^k(\mathbf{C}_0^*)$:

Conjecture 3. (1) $W^k \simeq H^k(X(g) - D, \mathbb{C})$ for $0 \le k \le g$.
(2) $\mathrm{ch}(W^k) = \mathrm{ch}(H^k(\mathbf{C}_0^*))$.

What does it mean? The divisor D consists of D_∞ and D_0. The forms from W^k describe the part of cohomology groups with singularities on D_∞ only. Our conjecture is that this part exhausts the whole space of the cohomology groups, i.e. that adding exact forms one can move singularities of any form from D_0 to D_∞. This is a strong statement which is rather difficult to prove. The first non-trivial case is $g = 3$ for which we were able to prove Conjecture 3. The details of it will be published elsewhere.

For $k = 1$ we can prove Conjecture 3 (1) for any g. The proof is given in Appendix D.

Let us now present a simple calculation which shows that Conjectures 2, 3 are consistent with the calculation of q-Euler characteristic of $\mathbf{C}_0^*$. Indeed

$$\begin{aligned} &\mathrm{ch}(W^0) - \mathrm{ch}(W^1) + \mathrm{ch}(W^2) - \cdots + (-1)^g \mathrm{ch}(W^g) \\ =&(R_0) - (R_1) + (R_2 - R_0) - (R_3 - R_1) + \cdots + (-1)^g (R_g - R_{g-2}) \\ =&(-1)^g (R_g - R_{g-1}) = \chi_q(\mathbf{C}_0^*). \end{aligned}$$

This calculation was actually the starting point for Conjectures 2, 3. Certainly it does not prove anything, but it shows remarkable consistence between different calculations performed in this paper.

In order to understand the magic of the hyper-elliptic case it is instructive to compare it with the case of an Abelian variety in generic when the divisor Θ is non-singular (which rarely the case for Jacobians of algebraic curves). As explained in the Appendix C in the latter case the following can be proven:

$$W^k \simeq H^k(J-\Theta,\mathbb{C}), \quad k \le g-1,$$
$$W^g \hookrightarrow H^g(J-\Theta,\mathbb{C}).$$

Actually for $g \geq 3$ the space $H^g(J-\Theta,,\mathbb{C})$ is bigger than W^g, the difference of dimensions being

$$\dim\, H^g(J-\Theta,\mathbb{C}) - \dim\, W^g = g! - \frac{(2g)!}{g!(g+1)!}.$$

We would conjecture that the equality $H^g(J-\Theta,\mathbb{C}) \simeq W^g$ specifies hyperelliptic Jacobians.

7 Appendix A. Proof of Proposition 1

We have to determine the basis of $\mathbf{A}_0$. Recall that

$$\mathbf{A}_0 \simeq \mathbf{A}/\sum_{j=1}^{2g+1} f_j \mathbf{A}.$$

Write the equations $f_1 = \cdots = f_{2g+1} = 0$ explicitly:

$$c_k + \sum_{i+j=k, j\neq k} b_i c_j + \sum_{i+j=k-1} a_{i+\frac{1}{2}} a_{j+\frac{1}{2}} = 0, \quad 1 \le k \le g+1, \tag{25}$$

$$\sum_{i+j=k} b_i c_j + \sum_{i+j=k-1} a_{i+\frac{1}{2}} a_{j+\frac{1}{2}} = 0, \quad g+2 \le k \le 2g+1. \tag{26}$$

From (25) c_k $(1 \le k \le g+1)$ can be solved by b_i, a_j. It is sometimes convenient to use the following notation;

$$u_{\frac{g}{2}+1}^{m_1} \cdots u_{g+\frac{1}{2}}^{m_g} = [-m_1, \cdots, -m_g],$$
$$B_0 = \mathbb{C}\,[u_1, u_{\frac{3}{2}}, \cdots, u_{\frac{g+1}{2}}].$$

We use both this bracket notation and the u_j notation to denote a monomial. The ring B_0 is considered as a coefficient in the sequel.

Let us first prove

Proposition A. *In the ring A_0 any monomial $[-m_1, \cdots, -m_g]$ can be written as a linear combination of monomials of the form $[-n_1, \cdots, -n_g]$, $n_1, \cdots, n_g = 0, 1$ with the coefficient in B_0.*

Proof. Define the degree of $[-m_1, \cdots, -m_g]$ as that of the monomial:

$$\deg[-m_1, \cdots, -m_g] = \sum_{k=1}^{g} m_k \frac{g+1+k}{2}.$$

We define a total order on the set of monomials $\{[-m_1, \cdots, -m_g]\}$ by the following rule. Let $P = [-m_1, \cdots, -m_g]$, $P' = [-m_1', \cdots, -m_g']$.

1. If $\deg(P) < \deg(P')$, then $P < P'$.

2. If $\deg(P) = \deg(P')$, compare P and P' by the lexicographical order from the left.

Notice that the product of two elements $P = [-m_1, \cdots, -m_g]$ and $P' = [-m'_1, \cdots, -m'_g]$ is expressed as

$$PP' = [-(m_1 + m'_1), \cdots, -(m_g + m'_g)].$$

The following property obviously holds.

Lemma A 1. *For monomials P_1, P_2, P_3, if $P_1 < P_2$ then $P_1P_3 < P_2P_3$.*

From (25)

$$c_k = -u_k + \cdots, \quad 1 \le k \le g+1,$$

where $\cdots$ part does not contain u_k. Then from (26)

$$u_{\frac{k}{2}}^2 = \cdots, \quad g+2 \le k \le 2g+1. \tag{27}$$

The next lemma describes what kind of monomials appear in the right hand side of (27).

Lemma A2. *The right hand side of (27) is a linear combination of elements of the form u_l, u_lu_m, $u_lu_mu_n$ with the coefficients in B_0.*

Proof. It is sufficient to show that the term like $xu_{i_1}\cdots u_{i_r}$, $r \ge 4$, $x \in B_0$ does not appear in the expression. Since (25), (26) are homogeneous, if $x \ne 0$ and homogeneous, then

$$\deg(xu_{i_1}\cdots u_{i_r}) = k,$$

where we take into account the degree of x. Since $i_1, \cdots, i_r \ge g/2+1$ and $2g+1 \ge k$,

$$\deg(xu_{i_1}\cdots u_{i_r}) \ge \deg(u_{i_1}\cdots u_{i_r}) = i_1 + \cdots + i_r \ge 4\left(\frac{g}{2}+1\right) > k.$$

Thus $x = 0$. QED.

Lemma A3. *In each of the cases in Lemma A2 we have the following statements, where x is a homogeneous element in B_0.*

1. If $u_{k/2}^2 = xu_l + \cdots$, then $u_{k/2}^2 > u_l$.

2. If $u_{k/2}^2 = xu_lu_m + \cdots$, then $u_{k/2}^2 > u_lu_m$.

3. If $u_{k/2}^2 = xu_lu_mu_n + \cdots$, then $u_{k/2}^2 > u_lu_mu_n$.

Proof. 1. Since $\deg(u_{k/2}^2) = k \geq g+2 > g+1/2 \geq l = \deg(u_l)$,the claim follows.

2. If $k > l+m$, there is nothing to be proved. Suppose that $k = l+m$. Then $l < k < m$. Thus comparing by the lexicographical order we have $u_{k/2}^2 > u_l u_m$. The statement of 3 is similarly proved. QED.

Starting from any element $P = [-m_1, \cdots, -m_g]$ we shall show that P can be reduced to the desired form. If some $m_j \geq 2$, then rewrite it using (27). By Lemma A3 every term in the resulting expression is less than P. Repeating this procedure we finally arrive at the linear combinations of $[-n_1, \cdots, -n_g]$, $n_1, \cdots, n_g = 0, 1$ with the coefficients in B_0. Thus Proposition A is proved. QED.

By Proposition A we have

$$\mathrm{ch}(\mathbf{A}_0) \leq \prod_{j=1}^{g} \frac{1}{\left[\frac{1+j}{2}\right]} \prod_{k=1}^{g} \left(1 + q^{\frac{g+1+k}{2}}\right) = \frac{\left[\frac{1}{2}\right][2g+2]!}{\left[g+\frac{1}{2}\right]!\ [g]!\ \ [g+1]!} = \frac{\mathrm{ch}(\mathbf{A})}{\mathrm{ch}(\mathbf{F})}. \quad (28)$$

Thus from (10) we conclude

$$\mathrm{ch}(\mathbf{A}_0) = \frac{\mathrm{ch}(\mathbf{A})}{\mathrm{ch}(\mathbf{F})}$$

which completes the proof of Proposition 1. QED.

8 Appendix B. Proof of Proposition 5.

We define W^k by the LHS of (24):

$$W^k = H^k(X_{\mathrm{aff}}(g), \mathbb{C}) / \big(\omega \wedge H^{k-2}(X_{\mathrm{aff}}(g), \mathbb{C})\big).$$

We first show that the map

$$i' : H^k(X_{\mathrm{aff}}(g), \mathbb{C}) \longrightarrow H^k(X(g) - D, \mathbb{C})$$

satisfies

$$i'\big(\omega \wedge H^{k-2}(X_{\mathrm{aff}}(g), \mathbb{C})\big) = 0$$

and thereby it induces the map

$$i' : W^k \longrightarrow H^k(X(g) - D, \mathbb{C}).$$

Next we shall construct a subspace W_k of the homology group $H_k(X(g) - D, \mathbb{C})$ such that the pairing between W_k and $i'(W^k)$ is non-degenerate. This proves Proposition 5.

Let us study the properties of the differential form ω defined in (23). Consider some differentials λ_j from $H^1(X_{\mathrm{aff}}, \mathbb{C})$, $j = 1, \cdots, k-2$, and construct the g-form:

$$\Omega = d\ \left(\tilde{\kappa} \wedge \tilde{\lambda}_1 \wedge \cdots \wedge \tilde{\lambda}_{k-2}\right) \quad (29)$$

where the one form $\tilde{\kappa}$ is given by

$$\tilde{\kappa} = \sum_{i<j} \kappa^{(ij)}, \qquad \kappa^{(ij)} = \tfrac{1}{4} \tfrac{y_i - y_j}{z_i - z_j} \left(\tfrac{dz_i}{y_i} + \tfrac{dz_j}{y_j} \right)$$

The form under d in RHS of (29) belongs to $\mathbf{C}_f^{k-1}$, that is, it has singularity on the divisor $D = D_0 \cup D_\infty$, but after the differential is applied the singularities on D_0 disappear. Indeed, one easily shows that

$$d\kappa^{(ij)} = \tfrac{1}{4} \sum_{k=1}^{g} (2k-1) \left(\nu_k^{(j)} \nu_{-k+1}^{(i)} - \nu_k^{(i)} \nu_{-k+1}^{(j)} \right), \tag{30}$$

where ν_j are defined as

$$\nu_j = q_j(z) \frac{dz}{y}, \qquad j = -(g-1), \cdots, g-1, g,$$

and q_j is the following polynomial of degree $g - j$:

$$q_j(z) = \mathrm{res}_{p_1 = \infty} \left(\frac{y_1 z_1^{-j}}{z_1 - z} d\sqrt{z_1} \right).$$

The differentials ν_j are normalized at infinity as

$$\nu_j \sim -2 \left(z^{-j} + O(z^{-g-1}) \right) d\sqrt{z} \qquad \text{for} \qquad p \to \infty.$$

The differentials ν_j for $j = 1, \cdots, g$ are holomorphic and ν_j for $j = -g+1, \cdots, 0$ are of the second kind. It is easy to verify that

$$\nu_i \circ \nu_j = \frac{4}{j-i}\ \delta_{i+j,1}.$$

This means, in particular, that, for two cycles γ_1, γ_2 on X

$$\begin{aligned} \int_{\gamma_1 \times \gamma_2} d\kappa^{(12)} &= \tfrac{1}{4} \sum_{k=1}^{g} (2k-1) \left(\int_{\gamma_1} \nu_{-k+1} \int_{\gamma_2} \nu_k - \int_{\gamma_2} \nu_{-k+1} \int_{\gamma_1} \nu_k \right) \\ &= \gamma_1 \circ \gamma_2, \end{aligned} \tag{31}$$

due to Riemann bilinear relation, where $\gamma_1 \circ \gamma_2$ is the intersection number. This is an important property of $d\kappa^{(ij)}$.

The equation (30) means that

$$\Omega = d\ \left(\tilde{\kappa} \wedge \tilde{\lambda}_1 \wedge \cdots \wedge \tilde{\lambda}_{k-2} \right) = -\omega \wedge \tilde{\lambda}_1 \wedge \cdots \wedge \tilde{\lambda}_{k-2}$$

where we have to remind that

$$\omega = \tfrac{1}{4} \sum_{k=1}^{g} (2k-1)\ \tilde{\nu}_k \wedge \tilde{\nu}_{-k+1}.$$

This proves that $i'\Big(\omega \wedge H^{k-2}(X_{\text{aff}}(g), \mathbb{C})\Big) = 0$.

Consider the homology groups of $X_{\text{aff}}(g)$. Taking dual to the relation (22) we obtain a similar relation for the homology groups:

$$H_k(X_{\text{aff}}(g), \mathbb{C}) \simeq \bigwedge^k H_1(X_{\text{aff}}(g), \mathbb{C}).$$

The first homology group $H_1(X_{\text{aff}}(g), \mathbb{C})$ is isomorphic to $H_1(X_{\text{aff}}, \mathbb{C})$. To have an element $\tilde{\delta}$ from $H_1(X_{\text{aff}}(g), \mathbb{C})$ one takes a cycle δ from $H_1(X_{\text{aff}}, \mathbb{C})$ and symmetrizes it over g copies of X_{aff}. In other words, if we fix a point p_0 in X_{aff}, then

$$\begin{aligned}
&\tilde{\delta} = \delta^{(1)} + \cdots + \delta^{(g)}, \\
&\delta^{(i)} = p_0 \otimes \cdots \otimes \delta \otimes \cdots \otimes p_0 \in H_0^{\otimes(i-1)} \otimes H_1 \otimes H_0^{\otimes(g-i)} \hookrightarrow H_1(X_{\text{aff}}^g, \mathbb{C}),
\end{aligned}$$

where $H_j = H_j(X_{\text{aff}}, \mathbb{C})$.

There is an obvious embedding $X(g) - D$ into $X_{\text{aff}}(g)$. It induces a map between the homology groups

$$H_k(X(g) - D, \mathbb{C}) \xrightarrow{\ i\ } H_k(X_{\text{aff}}(g), \mathbb{C}). \tag{32}$$

The meaning of this map is simple: every cycle on $X(g) - D$ is at the same time a cycle on $X_{\text{aff}}(g)$. There are two subtleties:
1. Nontrivial cycle on $X(g) - D$ can be trivial on $X_{\text{aff}}(g)$ i.e. the map (32) can have kernel.
2. On $X_{\text{aff}}(g)$ there are cycles that intersect with D_0 which means that they are not cycles on $X(g) - D$, so, the map (32) can have cokernel.

Let us study the image of the map (32). A k-cycle from $H_k(X_{\text{aff}}(g), \mathbb{Z})$ is a linear combination of elements of the form:

$$\Delta = \tilde{\delta}_1 \wedge \cdots \wedge \tilde{\delta}_k,$$

where $\delta_j \in H_1(X_{\text{aff}}, \mathbb{Z})$. The product

$$\Delta' = \delta_1 \times \cdots \times \delta_k \times p_0 \times \cdots \times p_0$$

defines an element of $H_k(X_{\text{aff}}^g, \mathbb{Z})$. Let π be the projection map $X_{\text{aff}}^g \longrightarrow X_{\text{aff}}(g)$ and π_* the induced map on the homology groups, $\pi_* : H_k(X_{\text{aff}}^g, \mathbb{Z}) \longrightarrow H_k(X_{\text{aff}}(g), \mathbb{Z})$. Then $\Delta = k!\binom{g}{k}\pi_*(\Delta')$ in $H_k(X_{\text{aff}}(g), \mathbb{Z})$.

Thus the cycle Δ belongs to $\text{Im}(i)$ if Δ' does not intersect $\pi^{-1}(D)$. Recall that $D = D_0 \cup D_\infty$. By construction Δ' has no intersection with $\pi^{-1}(D_\infty)$. One easily realizes that Δ' does not intersect with $\pi^{-1}(D_0)$ iff

$$\delta_i \circ \sigma(\delta_j) = 0 \qquad \forall\ i, j.$$

It is rather obvious property of the hyper-elliptic involution that

$$\delta_i \circ \sigma(\delta_j) = -\delta_i \circ \delta_j.$$

Hence we come to the following

Proposition B. *The* $\mathrm{Im}(i)$ *contains linear combination of cycles*

$$\Delta = \tilde{\delta}_1 \wedge \cdots \wedge \tilde{\delta}_k, \quad \delta_1, \cdots, \delta_k \in H_1(X_{\mathrm{aff}}, \mathbb{Z}),$$

such that

$$\delta_i \circ \delta_j = 0 \qquad \forall\ i, j.$$

Let W_k denote the space obtained as $\mathbb{C}$-linear span of cycles in $H_k(X(g)-D, \mathbb{Z})$ corresponding to Δ's in this proposition.

Take a canonical cycles α_i, β_j and set

$$A_i = \tilde{\alpha}_i, \quad A_{i+g} = \tilde{\beta}_i, \quad 1 \leq i \leq g.$$

Define

$$V_{\mathbb{Z}} = H_1(X_{\mathrm{aff}}, \mathbb{Z}) = \oplus_{i=1}^{2g} \mathbb{Z} A_i, \quad V_{\mathbb{C}} = H_1(X_{\mathrm{aff}}, \mathbb{C}) = \oplus_{i=1}^{2g} \mathbb{C} A_i.$$

The symplectic form on $V_{\mathbb{C}}$ is defined by

$$A_i \circ A_j = \pm \delta_{j, i \pm g}.$$

By definition

$$i(W_k) = \mathrm{Span}_{\mathbb{C}}(U_k), \quad U_k = \{\gamma_1 \wedge \cdots \wedge \gamma_k | \gamma_1, \cdots, \gamma_k \in V_{\mathbb{Z}},\ \gamma_i \circ \gamma_j = 0\ \forall i, j\}.$$

Define

$$\tilde{W}_k = \mathrm{Span}_{\mathbb{C}}(\tilde{U}_k), \quad \tilde{U}_k = \{\gamma_1 \wedge \cdots \wedge \gamma_k | \gamma_1, \cdots, \gamma_k \in V_{\mathbb{C}},\ \gamma_i \circ \gamma_j = 0\ \forall i, j\}.$$

Consider the map

$$\begin{aligned} \varphi_k \quad &: \quad \wedge^k V_{\mathbb{C}} \longrightarrow \wedge^{k-2} V_{\mathbb{C}}, \\ \varphi_k \quad &\quad (\gamma_1 \wedge \cdots \wedge \gamma_k) = \sum_{i<j} (-1)^{i+j-1} (\gamma_i \circ \gamma_j) \gamma_{\{ij\}}, \end{aligned} \tag{33}$$

where $\gamma_{\{ij\}}$ is obtained from $\gamma_1 \wedge \cdots \wedge \gamma_k$ removing γ_i and γ_j. It is known that, for $k \leq g$, φ_k is surjective and its kernel $\mathrm{Ker}\varphi_k$ is isomorphic to the k-th fundamental irreducible representation of $\mathrm{Sp}(2g, \mathbb{C})$ (*cf.* Theorem 17.5 [9]). In particular

$$d_k := \dim\ \mathrm{Ker}(\varphi_k) = \binom{2g}{k} - \binom{2g}{k-2}.$$

The following lemma can be easily proved.

Lemma B.*Suppose that* $k \leq g$. *Then*

$$i(W_k) = \tilde{W}_k = \mathrm{Ker}(\varphi_k).$$

As a consequence of the lemma one has in particular

$$\dim W_k \geq \dim\, i(W_k) = \binom{2g}{k} - \binom{2g}{k-2}. \tag{34}$$

Let us show that W_k and $i'(W^k)$ pairs completely. The pairings

$$< \quad , \quad >_1\colon\ H_k(X(g) - D, \mathbb{C}) \otimes H^k(X(g) - D, \mathbb{C}) \longrightarrow \mathbb{C} \tag{35}$$

and

$$< \quad , \quad >_2\colon \wedge^k H_1(X_{\rm aff}(g), \mathbb{C}) \otimes \wedge^k H^1(X_{\rm aff}(g), \mathbb{C}) \longrightarrow \mathbb{C} \tag{36}$$

are related by

$$< \gamma, i'(\eta) >_1 = < i(\gamma), \eta >_2,$$

for $\gamma \in H_k(X(g) - D, \mathbb{C})$, $\eta \in \wedge^k H^1(X_{\rm aff}(g), \mathbb{C})$. The pairing (36) is given by the integral:

$$\begin{aligned} & < \tilde{\gamma}_1 \wedge \cdots \wedge \tilde{\gamma}_k, \tilde{\eta}_1 \wedge \cdots \wedge \tilde{\eta}_k >_2 \\ = \quad & \int_{\tilde{\gamma}_1 \wedge \cdots \wedge \tilde{\gamma}_k} \tilde{\eta}_1 \wedge \cdots \wedge \tilde{\eta}_k = k! \binom{g}{k} \det\Big(\int_{\gamma_i} \eta_j\Big)_{1 \leq i,j \leq k}. \end{aligned}$$

By (31) we have

$$< i(W_k), \omega \wedge^{k-2} H^1(X_{\rm aff}(g), \mathbb{C}) >_2 = 0.$$

Thus the pairing (35), (36) induce pairings

$$< \quad , \quad >_1\colon\ W_k \otimes i'(W^k) \longrightarrow \mathbb{C} \tag{37}$$
$$< \quad , \quad >_2\colon\ i(W_k) \otimes W^k \longrightarrow \mathbb{C}. \tag{38}$$

Since (36) is non-degenerate and $\dim i(W_k) = \dim W^k$, the pairing (38) is non-degenerate. It easily follows from this that the pairing (37) is also non-degenerate. Thus we have proved Proposition 5. QED.

Corollary B. *We have $W_k \simeq i(W_k)$. In particular*

$$\dim W_k = \binom{2g}{k} - \binom{2g}{k-2}.$$

9 Appendix C. The case of generic Abelian variety.

Let (J, Θ) be a principally polarized Abelian variety such that Θ is non-singular. Then

Proposition C. *The dimensions of cohomology groups of $J-\Theta$ are given by*

$$\begin{aligned}\dim H^k(J-\Theta,\mathbb{C}) &= \binom{2g}{k}-\binom{2g}{k-2}, \quad k\le g-1,\\ &= \binom{2g}{g}-\binom{2g}{g-2}+g!-\frac{(2g)!}{g!(g+1)!}, \quad k=g,\\ &= 0, \quad k>g.\end{aligned}$$

Proof. Consider the inclusions $\Theta\subset J\subset (X,J)$ and the induced homology exact sequence:

$$\cdots\longrightarrow H_k(\Theta,\mathbb{C})\longrightarrow H_k(J,\mathbb{C})\longrightarrow H_k(J,\Theta)\longrightarrow H_{k-1}(\Theta,\mathbb{C})\longrightarrow\cdots.$$

Taking the dual sequence of this and using the Poincare-Lefschetz duality we get

$$\begin{aligned}\cdots\longrightarrow H_{k-1}(\Theta,\mathbb{C})\longrightarrow H_k(J-\Theta,\mathbb{C}) &\longrightarrow H_k(J,\mathbb{C})\longrightarrow\\ &\longrightarrow H_{k-2}(\Theta,\mathbb{C})\longrightarrow\cdots.\end{aligned}\tag{39}$$

Since $J-\Theta$ is affine

$$H_k(J-\Theta,\mathbb{C})=0, \quad k>g.$$

Then we have

$$H_k(\Theta,\mathbb{C})\simeq H_{k+2}(J,\mathbb{C}), \quad k\ge g, \quad H_k(\Theta,\mathbb{C})\simeq H_k(J,\mathbb{C}), \quad k\le g-2.$$

It is easy to check that, for $k\le g$, the dual map of

$$H_k(J,\mathbb{C})\longrightarrow H_{k-2}(\Theta,\mathbb{C})$$

is given by wedging the fundamental class $[\Theta]$ of Θ:

$$[\Theta]\wedge : H^{k-2}(\Theta,\mathbb{C})\simeq H^{k-2}(J,\mathbb{C})\longrightarrow H^k(J,\mathbb{C}).\tag{40}$$

Using the representation theory of sl_2 as in the proof of the hard Lefschetz theorem (*c.f.*[6]), the map (40) is injective for $k\le g$. Thus by (39) the following exact sequences hold:

$$\begin{aligned}&0\to H^{k-2}(\Theta,\mathbb{C})\xrightarrow{[\Theta]\wedge} H^k(J,\mathbb{C})\to H^k(J-\Theta,\mathbb{C})\to 0, \quad k<g,\\ &0\to H^{g-2}(\Theta,\mathbb{C})\xrightarrow{[\Theta]\wedge} H^g(J,\mathbb{C})\to H^g(J-\Theta,\mathbb{C})\to\\ &\to H^{g-1}(\Theta,\mathbb{C})\to H^{g+1}(J,\mathbb{C})\to 0.\end{aligned}$$

Proposition C follows from these exact sequences and the fact

$$\chi(\Theta)=(-1)^{g-1}g!.$$

QED.

By Proposition F 3 the fundamental class of Θ coincides with ω in Proposition 5 in the hyper-elliptic case. Thus, if we define W^k in a similar formula to (24), we have

$$W^k \simeq H^k(J-\Theta,\mathbb{C}), \quad k \le g-1,$$
$$W^g \hookrightarrow H^g(J-\Theta,\mathbb{C}).$$

10 Appendix D. The proof of Conjecture 3 for $k=1$

Notice that

$$H^k(X_{\text{aff}}(g),\mathbb{C}) \simeq \wedge^k H^1(X,\mathbb{C}) \simeq \wedge^k H^1(J(X),\mathbb{C}),$$

and $X(g)-D \simeq J(X)-\Theta$. In particular $W^1 \simeq H^1(J(X),\mathbb{C})$.

Proposition D. *For any principally polarized Abelian variety (J,Θ) such that Θ is irreducible we have the isomorphism*

$$H^1(J,\mathbb{C}) \simeq H^1(J-\Theta,\mathbb{C}).$$

Proof. Following [8] we shall use the following notations:

$\mathcal{O}(n\Theta)$: the sheaf of meromorphic functions on J which have poles only on Θ of order at most n,

$\mathcal{O}(*\Theta)$: the sheaf of meromorphic functions on J which have poles only on Θ,

$\Omega^k(n\Theta)$: the sheaf of meromorphic k-forms on J which have poles only on Θ of order at most n,

$\Omega^k(*\Theta)$: the sheaf of meromorphic k-forms on J which have poles only on Θ,

$\Phi^k(n\Theta)$: the sheaf of closed meromorphic k-forms on J which have poles only on Θ of order at most n,

$\Phi^k(*\Theta)$: the sheaf of closed meromorphic k-forms on J which have poles only on Θ,

$$R^k(n\Theta)=\Phi^k(n\Theta)/d\big(\Omega^{k-1}((n-1)\Theta)\big), \quad R^k(*\Theta)=\Phi^k(*\Theta)/d\big(\Omega^{k-1}(*\Theta)\big).$$

In particular

$$\Omega^0(n\Theta)=\mathcal{O}(n\Theta), \quad \Omega^0(*\Theta)=\mathcal{O}(*\Theta).$$

We first recall the description of $H^1(J,\mathbb{C})$ in terms of the differentials of the first and second kinds. Consider the sheaf exact sequence:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}(*\Theta) \xrightarrow{d} d\big(\mathcal{O}(*\Theta)\big) \longrightarrow 0.$$

Since

$$H^k(J, \mathcal{O}(*\Theta)) = 0, \quad k \geq 1,$$

we have

$$H^1(J, \mathbb{C}) \simeq H^0(J, d\mathcal{O}(*\Theta))/dH^0(J, \mathcal{O}(*\Theta)). \tag{41}$$

The numerator in the right hand side of (41) is nothing but the space of differential one forms of the first and the second kinds on J and the denominator is the space of globally exact meromorphic one forms.

On the other hand, by the algebraic de Rham theorem, the first cohomology group of the affine variety $J - \Theta$ is described as

$$H^1(J - \Theta, \mathbb{C}) \simeq H^0(J, \Phi^1(*\Theta))/dH^0(J, \mathcal{O}(*\Theta)). \tag{42}$$

Comparing (41) and (42) what we have to prove is

$$H^0(J, d\mathcal{O}(*\Theta)) \simeq H^0(J, \Phi^1(*\Theta)). \tag{43}$$

Consider the exact sequence

$$0 \longrightarrow d\mathcal{O}(*\Theta) \longrightarrow \Phi^1(*\Theta) \longrightarrow R^1(*\Theta) \longrightarrow 0.$$

The cohomology sequence of this is

$$0 \longrightarrow H^0(J, d\mathcal{O}(*\Theta)) \longrightarrow H^0(\Phi^1(*\Theta)) \longrightarrow H^0(R^1(*\Theta)) \longrightarrow \cdots .$$

From this what should be proved is that the map

$$H^0(\Phi^1(*\Theta)) \longrightarrow H^0(R^1(*\Theta))$$

is a 0-map. To study this map we refer the lemma from [8].

Lemma D.(Lemma 8 [8])

(1) $R^1(*\Theta) \simeq \mathbb{C}_\Theta$, *where* $\mathbb{C}_\Theta$ *is the constant sheaf on* Θ *and the isomorphism is given by*

$$[\frac{d\theta}{\theta}] \longleftarrow [1_\Theta]$$

at any stalk.

(2) $R^1(*\Theta) \simeq R^1(n\Theta)$, $n = 1, 2, \cdots$.

In the proof of Lemma D (1) we use our assumption that Θ is irreducible.

Using this lemma we reduce the problem from "$*\Theta$" to "$n\Theta$" with finite n.

Consider the exact sequence

$$0 \longrightarrow d\mathcal{O}(n\Theta) \longrightarrow \Phi^1((n+1)\Theta) \longrightarrow R^1(*\Theta) \longrightarrow 0, \quad n = 0, 1, \cdots . \tag{44}$$

From the cohomology sequence of it we have the map

$$H^0(J, R^1(*\Theta)) \longrightarrow H^1(J, d\mathcal{O}(n\Theta))$$

which we denote by π_n. Let us prove

$$\mathrm{Ker}\pi_n = 0, \quad n = 0, 1, 2, \cdots.$$

To this end we study $H^1(J, d\mathcal{O}(n\Theta))$. Using the exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}(n\Theta) \xrightarrow{d} d\mathcal{O}(n\Theta) \longrightarrow 0,$$

we easily have

$$H^1(J, d\mathcal{O}(n\Theta)) \simeq H^2(J, \mathbb{C}), \quad n \geq 1, \tag{45}$$

$$H^1(J, d\mathcal{O}) \hookrightarrow H^2(J, \mathbb{C}). \tag{46}$$

The natural maps

$$d\mathcal{O}(n\Theta) \longrightarrow d\mathcal{O}((n+1)\Theta),\ \Phi^1(n\Theta) \longrightarrow \Phi^1((n+1)\Theta),\ R^1(n\Theta) \simeq R^1((n+1)\Theta),$$

and the sequence (44) induce a commutative diagram of cohomology groups. It follows from this commutative diagram and (45), (46) that $\mathrm{Ker}\pi_0 = 0$ implies $\mathrm{Ker}\pi_n = 0$ for $n \geq 1$. Now $\mathrm{Ker}\pi_0 = 0$ follows from

$$H^0(J, d\mathcal{O}) \simeq H^0(J, \Omega^1), \quad H^0(J, \Phi^1(\Theta)) \simeq H^0(J, \Omega^1).$$

The second isomorphism follows from the fact that a meromorphic function on J which has poles only on Θ of order at most one is a constant. Thus the Proposition D is proved. QED.

11 Appendix E.

Recall that, for a set of complex numbers $f^0 = (f_1^0, \cdots, f_{2g+1}^0)$, the ring $\mathbf{A}_{f^0}$ is defined by

$$\mathbf{A}_{f^0} = \mathbf{A} \otimes_{\mathbf{F}} \mathbb{C}_{f^0} = \frac{\mathbf{A}}{\sum_{i=1}^{2g+1} \mathbf{A}(f_i - f_i^0)},$$

where f_i is the coefficient of $f(z)$. If all $f_i^0 = 0$, then $\mathbf{A}_{f^0} = \mathbf{A}_0$. The ring $\mathbf{A}_0$ is graded while $\mathbf{A}_{f^0}$ is not graded unless all $f_i = 0$. Instead $\mathbf{A}_{f^0}$ is a filtered ring for any f^0. Let $\mathbf{A}_{f^0}(n)$ be the set of elements of $\mathbf{A}_{f^0}$ represented by $\oplus_{k \leq n} \mathbf{A}^{(n)}$. Then

$$\mathbf{A}_{f^0} = \cup_{n \geq 0} \mathbf{A}_{f^0}(n), \quad \mathbf{A}_{f^0}(0) = \mathbb{C} \subset \mathbf{A}_{f^0}\left(\tfrac{1}{2}\right) \subset \mathbf{A}_{f^0}(1) \subset \cdots.$$

Consider the graded ring associated with this filtration:

$$\mathrm{gr}\mathbf{A}_{f^0} = \oplus \mathrm{gr}_n \mathbf{A}_{f^0}, \quad \mathrm{gr}_n \mathbf{A}_{f^0} = \frac{\mathbf{A}_{f^0}(n)}{\mathbf{A}_{f^0}(n - \frac{1}{2})}.$$

Since $\deg(f_i^0) = 0$, $\mathrm{gr}\mathbf{A}_{f^0}$ becomes a quotient of $\mathbf{A}_0$. In other words there is a surjective ring homomorphism

$$\mathbf{A}_0 \longrightarrow \mathrm{gr}\mathbf{A}_{f^0}. \tag{47}$$

We shall prove that this map is injective.

Proposition E. *There is an isomorphism of graded rings:*

$$\mathbf{A}_0 \simeq \mathrm{gr}\mathbf{A}_{f^0}.$$

Proof. Notice that the map (47) respects the grading and it is surjective at each grade. By Proposition 2, $\mathbf{A} \simeq \mathbf{F} \otimes_{\mathbb{C}} \mathbf{A}_0$ as a $\mathbb{C}$-vector space. Thus we have $\mathbf{A}_0 \simeq \mathbf{A}_{f^0}$ as a $\mathbb{C}$-vector space for any f^0. In particular the basis of $\mathbf{A}_0$ given in Proposition 1 is also a basis of $\mathbf{A}_{f^0}$. Denote by $\{x_j^{(n)}\}$ the basis of the degree n part $\mathbf{A}_0^{(n)}$. Let us prove that $\{x_j^{(n)}\}$ are linearly independent in $\mathrm{gr}\mathbf{A}_{f^0}$ by the induction on n. For $n = 0$ the statement is obvious. We assume that the statement is true for all m satisfying $m < n$. This means that $\{x_j^{(m)}\}$ is a basis of the degree m part of $\mathrm{gr}\mathbf{A}_{f^0}$ for all $m < n$. In particular $\{x_j^{(m)} | m < n\}$ is a basis of $\mathbf{A}_{f^0}(n - \frac{1}{2})$. Suppose that the relation

$$\sum_j \alpha_j x_j^{(n)} \in \mathbf{A}_{f^0}(n - \tfrac{1}{2}).$$

holds, where some $\alpha_j \neq 0$. Then this means that $\{x_j^{(m)} | m \leq n\}$ are linearly dependent in $\mathbf{A}_{f^0}$. This contradicts the fact that $\{x_j^{(k)}\}$ is a basis of $\mathbf{A}_{f^0}$. Thus $\{x_j^{(n)}\}$ are linearly independent in $\mathrm{gr}\mathbf{A}_{f^0}$. QED.

12 Appendix F. Construction of a free resolution of $\mathbf{A}_0$

Recall that

$$\mathcal{D} = \mathbb{C}\,[D_1, \cdots, D_g]$$

is the polynomial ring generated by the commuting vector fields $D_1, \cdots, D_g$. Then $\mathbf{A}$, $\mathbf{A}_0$ and $\mathbf{A}_{f^0}$ are $\mathcal{D}$ modules. We shall construct a free $\mathcal{D}$-resolution of $\mathbf{A}_0$ assuming Conjecture 2 and 3. To avoid the notational confusion we shall describe the space W^k using the abstract vector space V of dimension $2g$ with a basis v_i, ξ_i $(1 \leq i \leq g)$:

$$V = \oplus_{i=1}^g \mathbb{C} v_i \oplus_{i=1}^g \mathbb{C} \xi_i.$$

Set

$$\omega = \sum_{i=1}^g v_i \wedge \xi_i \in \wedge^2 V$$

and define

$$W^k = \frac{\wedge^k V}{\omega \wedge^{k-2} V}. \tag{48}$$

Assign degrees to the basis elements by

$$\deg(v_i) = -(i - \tfrac{1}{2}), \quad \deg(\xi_i) = i - \tfrac{1}{2}.$$

Then W^k defined by (48) has the same character as W^k defined in Section 6. This justifies the use of the same symbol. Consider the free $\mathcal{D}$-module $\mathcal{D} \otimes_{\mathbb{C}} W^k$ generated by W^k. We shall construct an exact sequence of $\mathcal{D}$-modules of the form

$$0 \leftarrow \mathbf{A}_0 d\tau_1 \wedge \cdots \wedge d\tau_g \leftarrow \mathcal{D} \otimes_{\mathbb{C}} W^g \overset{d}{\leftarrow} \mathcal{D} \otimes_{\mathbb{C}} W^{g-1} \overset{d}{\leftarrow} \cdots \overset{d}{\leftarrow} \mathcal{D} \otimes_{\mathbb{C}} W^0 \leftarrow 0.$$

Define the map

$$\mathcal{D} \otimes_{\mathbb{C}} \wedge^k V \xrightarrow{d} \mathcal{D} \otimes_{\mathbb{C}} \wedge^{k+1} V,$$

by

$$d(P \otimes v_I \wedge \xi_J) = \sum_{i=1}^{g} D_i P \otimes v_i \wedge v_I \wedge \xi_J,$$

where for $I = (i_1, \cdots, i_r)$, $v_I = v_{i_1} \wedge \cdots \wedge v_{i_r}$ etc., $P \in \mathcal{D}$ and $D_i P$ is the product of D_i and P in $\mathcal{D}$. Since the map d commutes with the map taking the wedge with $Q \otimes \omega$ for any $Q \in \mathcal{D}$:

$$d(QP \otimes \omega \wedge v_I \wedge \xi_J) = \sum_{i=1}^{g} QD_i P \otimes \omega \wedge v_i \wedge v_I \wedge \xi_J = (Q \otimes \omega) \wedge d(P \otimes v_I \wedge \xi_J),$$

it induces a map

$$\mathcal{D} \otimes_{\mathbb{C}} W^k \xrightarrow{d} \mathcal{D} \otimes_{\mathbb{C}} W^{k+1}.$$

It is easy to check that the map d satisfies $d^2 = 0$.

Proposition F 1. *The complex*

$$0 \longleftarrow \mathcal{D} \otimes_{\mathbb{C}} W^g \overset{d}{\longleftarrow} \mathcal{D} \otimes_{\mathbb{C}} W^{g-1} \overset{d}{\longleftarrow} \cdots \overset{d}{\longleftarrow} \mathcal{D} \otimes_{\mathbb{C}} W^0 \longleftarrow 0 \tag{49}$$

is exact at $\mathcal{D} \otimes_{\mathbb{C}} W^k$ except $k = g$.

Proof. Notice that the following two facts:
1. The following complex is exact at $\mathcal{D} \otimes \wedge^k V$ except $k \neq g$:

$$0 \longleftarrow \mathcal{D} \otimes_{\mathbb{C}} \wedge^g V \xrightarrow{d} \cdots \overset{d}{\longleftarrow} \mathcal{D} \otimes_{\mathbb{C}} V \overset{d}{\longleftarrow} \mathcal{D} \longleftarrow 0. \tag{50}$$

2. The map $\omega \wedge : \wedge^k V \longrightarrow \wedge^{k+2} V$ is injective for $k \leq g-2$.

The property 1 is the well known property of the Koszul complex. The property 2 is also well known and easily proved using the representation theory of sl_2.

Now suppose that $x \in \mathcal{D} \otimes_{\mathbb{C}} \wedge^k V$, $k < g$ and $dx = \omega \wedge y$ for some $y \in \mathcal{D} \otimes_{\mathbb{C}} \wedge^{k-2} V$. Then $\omega \wedge dy = 0$ and thus $dy = 0$ by the property 2. Then $y = dz$ for some z by the property 1. Thus we have $d(x - \omega \wedge z) = 0$ and $x = dw + \omega \wedge z$ for some w again by the property 1. QED.

Next we shall define a map from $\mathcal{D} \otimes_{\mathbb{C}} W^g$ to $\mathbf{A}_0 d\tau_1 \wedge \cdots \wedge d\tau_g$. To this end we need to identify W^k in this section and that of Section 6, which is defined as the quotient of the cohomology groups of a Jacobi variety. For this purpose we describe the cohomology groups of a Jacobi variety in terms of theta functions.

Define

$$\zeta_i(w) = D_i \log \theta(w) = \frac{\partial}{\partial \tau_i} \log \theta(w).$$

Then, for each i, the differential $d\zeta_i(w)$ defines a meromorphic differential form on $J(X)$ which has double poles on Θ and which is locally exact. This means that $d\zeta_i$ is a second kind differential on $J(X)$. By (41) first and second kinds differential one forms define elements of $H^1(J(X), \mathbb{C})$. The pairing with $H_1(J(X), \mathbb{C})$ is given by integration. The following proposition can be easily proved by calculating periods.

Proposition F 2. *The first and the second kinds differentials $d\tau_1$,...,$d\tau_g$, $d\zeta_1$,...,$d\zeta_g$ give a basis of the cohomology group $H^1(J(X), \mathbb{C})$.*

Thus we can identify V with $H^1(J(X), \mathbb{C}) \simeq H^1(X_{\mathrm{aff}}(g), \mathbb{C})$ by

$$v_i = d\tau_i, \quad \xi_i = d\zeta_i.$$

The next proposition can be easily proved by calculating integrals over two cycles in a similar way to [1](vol.I, p188).

Proposition F 3. *Let ω be defined in Proposition 5. Then we have*

$$\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{g} d\tau_i \wedge d\zeta_i = \omega$$

as elements in $\wedge^2 H^1(J(X), \mathbb{C})$. Moreover ω represents the fundamental class of the theta divisor Θ in $H^2(J(X), \mathbb{C})$.

From this proposition it is possible to identify W^k in this section and that in the previous sections. Define the map

$$\begin{aligned} \wedge^g V &\to \mathbf{A}_{f^0} d\tau_1 \wedge \cdots \wedge d\tau_g, \\ v_I \wedge \xi_J &\mapsto d\tau_I \wedge d\zeta_J, \end{aligned}$$

where for $I=(i_1,\cdots,i_r)$, $d\tau_I=d\tau_{i_1}\wedge\cdots\wedge d\tau_{i_r}$ etc. and f^0 is any set of complex numbers such that $y^2=f^0(z)$ defines a non-singular curve X.

This map extends to the map of $\mathcal{D}$ modules

$$\mathcal{D}\otimes_{\mathbb{C}}\wedge^g V\to \mathbf{A}_{f^0}d\tau_1\wedge\cdots\wedge d\tau_g,$$

in the following manner. Let $P\in\mathcal{D}$ and consider $P\otimes(v_I\wedge\xi_J)$. Write $d\zeta_J=\sum_K F_J^K d\tau_K$, $F_J^K\in\mathbf{A}_{f^0}$. Then we define

$$P\otimes(v_I\wedge\xi_J)\longrightarrow\sum P(F_J^K)d\tau_I\wedge d\tau_K.$$

Since, as a meromorphic differential form on $J(X)$,

$$\sum_{i=1}^g d\tau_i\wedge d\zeta_i=\sum_{i,j=1}^g\frac{\partial^2\log\theta(w)}{\partial\tau_i\partial\tau_j}d\tau_i\wedge d\tau_j=0,$$

the above map induces the map

$$\mathcal{D}\otimes_{\mathbb{C}}W^g\xrightarrow{ev}\mathbf{A}_{f^0}d\tau_1\wedge\cdots\wedge d\tau_g.$$

Denote by $(\mathcal{D}\otimes_{\mathbb{C}}W^g)_n$ the subspace of elements with degree n and by ev_n the restriction of ev to $(\mathcal{D}\otimes_{\mathbb{C}}W^g)_n$. By the definition, for $x\in(\mathcal{D}\otimes_{\mathbb{C}}W^g)_n$, $ev_n(x)\in\mathbf{A}_{f^0}(n+g^2/2)$. By Proposition E, there is a natural isomorphism $\mathbf{A}_0\simeq\mathrm{gr}\mathbf{A}_{f^0}$ (see Appendix E for the filtration of $\mathbf{A}_{f^0}$). Composing the map ev_n with the natural projection map $\mathbf{A}_{f^0}(k)\to\mathrm{gr}_k\mathbf{A}_{f^0}\simeq\mathbf{A}_0^{(k)}$ we obtain the map, which we denote by ev_n too,

$$(\mathcal{D}\otimes_{\mathbb{C}}W^g)_n\xrightarrow{ev_n}(\mathbf{A}_0 d\tau_1\wedge\cdots\wedge d\tau_g)_n,$$

where the RHS means the degree n subspace. Taking the sum of ev_n we finally have the map

$$ev=\oplus_n ev_n:\mathcal{D}\otimes_{\mathbb{C}}W^g\longrightarrow\mathbf{A}_0 d\tau_1\wedge\cdots\wedge d\tau_g,$$

which we also denote by the symbol ev.

If we assume Conjecture 2 and 3, $H^k(\mathbf{C}_0^*)\simeq W^k$. If this holds, then ev is surjective by Proposition 3.

Lemma F. *Suppose that Conjecture 2 and 3 are true. Then the kernel of ev is given by*

$$\mathrm{Ker}(ev)=d\big(\mathcal{D}\otimes W^{g-1}\big).$$

Proof. It is easy to check that

$$\mathrm{Ker}(ev)\supset d\big(\mathcal{D}\otimes W^{g-1}\big).$$

Since ev is surjective and

$$\mathrm{ch}(\mathbf{A}_0)=\mathrm{ch}\Big(\frac{\mathcal{D}\otimes_{\mathbb{C}}W^g}{d\big(\mathcal{D}\otimes_{\mathbb{C}}W^{g-1}\big)}\Big),$$

the claim of the lemma follows. QED.

We summarize the result as

Theorem F. *Suppose that Conjecture 2, 3 are true. Then the following complex gives a resolution of* $\mathbf{A}_0 d\tau_1 \wedge \cdots \wedge d\tau_g$ *as a* $\mathcal{D}$*-module:*

$$0 \longleftarrow \mathbf{A}_0 d\tau_1 \wedge \cdots \wedge d\tau_g \xleftarrow{ev} \mathcal{D} \otimes_{\mathbb{C}} W^g \xleftarrow{d} \cdots \xleftarrow{d} \mathcal{D} \otimes_{\mathbb{C}} W^0 \longleftarrow 0.$$

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