

Symplectic realizations of bihamiltonian structures

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To my wife Larysa

0 Introduction

A smooth manifold M is endowed by a Poisson pair if two linearly independent bivector fields c_1, c_2 are defined on M and moreover $c_\lambda = \lambda_1 c_1 + \lambda_2 c_2$ is a Poisson tensor for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$. A bihamiltonian structure $J = \{c_\lambda\}$ is the whole 2-dimensional family of tensors.

There are two classes of bihamiltonian structures playing important role in the theory of completely integrable systems. The geometries corresponding to these classes are different and so are the ways for appearing of functions in involution.

The first class, called symplectic in this paper, is characterized by the condition that $\text{rank } c_\lambda = \dim M$ (cf. Convention 1.13 concerning the definition of rank) for generic $c_\lambda \in J$. If c_1 is nondegenerate, one can define the so-called recursion operator $c_2 \circ (c_1)^{-1} : TM \rightarrow TM$. Its eigenvalues are in involution with respect to c_1 . Of course, one should impose additional conditions on J in order that these functions are independent and that they form a "complete" set, i.e. the foliation defined by them is lagrangian (not only coisotropic).

One can obtain examples of global symplectic bihamiltonian structures considering holomorphic symplectic manifolds (M, ω) and putting

*Partially supported by the Polish grant KBN 2 PO3A 135 16.

$c_1 = \operatorname{Re} c, c_2 = \operatorname{Im} c$, where $c = (\omega)^{-1}$ is the holomorphic bivector field inverse to ω . Of course, locally they are all the same due to the Darboux theorem and they are quite not interesting from the above point of view since the recursion operator coincides with the complex structure and has only constant eigenvalues $\pm i$. However, these bihamiltonian structures will be important for us and we call them holomorphic symplectic.

The second class consists of degenerate bihamiltonian structures, which are described by the condition $\max_{\lambda} \operatorname{rank} c_{\lambda} < \dim M$. Given such a structure, one can construct the family of functions $F_0 = \sum_{c \in J_0} Z_c$, where $J_0 \subset J$ is a subset of tensors of maximal rank and Z_c stands for the space of local Casimir functions of a Poisson tensor c . It turns out that this family is in involution with respect to any $c_{\lambda} \in J$. Again, keeping in mind the aim of getting the completely integrable system, one should put some restrictions on J .

An elegant and easily checkable condition which guarantees that the family F_0 is locally "complete", i.e. defines a lagrangian foliation on a generic symplectic leaf of J , is given by the Bolsinov-Brailov theorem (see 2.15) and is formulated as follows:

$$(*) \operatorname{rank}(\lambda_1 c_1 + \lambda_2 c_2)(x) = R_0 \text{ for any } (\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \{0\}.$$

Here $R_0 = \max_{\lambda} \operatorname{rank} c_{\lambda}$ and x is a point in a neighbourhood of which we are checking the "completeness" of F_0 . Taking $(*)$ as a starting point we define complete bihamiltonian structures as those satisfying this condition on an open dense subset.

The reader is referred to papers [8]-[12] for the detailed exposition of the geometric and algebraic aspects of bihamiltonian structures based on the classical theory of pencils of operators. Note that according to the terminology of these articles the symplectic and complete bihamiltonian structures mentioned above should be called (micro) Jordan and (micro) Kronecker (cf. Theorem 2.22, below).

The aim of this paper is to study some relations between the above classes. More precisely, we study the reductions of holomorphic symplectic bihamiltonian structures (by means of real foliations) resulting in complete ones. The construction inverse to such a reduction is called a realization; hence the title of the paper. Our main theorem (see 7.1) states the completeness of the reduction J' of the holomorphic symplectic structure J associated with the canonical symplectic form ω on a generic coadjoint orbit $M \subset \mathfrak{g}^*$, where \mathfrak{g} is a complex Lie algebra from a wide class including the semisimple algebras (cf. Convention 5.3) and

the reduction is performed with respect to a real form $G_0 \subset G$ of the (simple, semisimple) complex Lie group adjoint to \mathfrak{g} . Also, the "first integrals", i.e. the elements of the family F_0 corresponding to J' , are calculated (Proposition 7.2). Let us make a few comments on the proof.

Note that generic coadjoint G_0 -orbits in \mathfrak{g}^* satisfy the condition of CR -genericity (see Definition 1.18). The main theorem is a consequence of a relatively simple criterion (Theorem 4.4) of completeness for the reduction J' of a holomorphic symplectic structure J by means of a real CR -generic foliation \mathcal{K} on M . We want to stress that the assumption of the CR -genericity for \mathcal{K} is natural in the context discussed in Section 4. The study of reductions without this assumption seems reasonable, but more complicated.

In order to use the mentioned criterion to the proof of the main theorem, one studies the auxiliary complex Poisson pair c, \tilde{c} on \mathfrak{g}^* , where $c : \mathfrak{g}^* \rightarrow \mathfrak{g}^* \wedge \mathfrak{g}^*$ is the canonical linear Poisson bivector and \tilde{c} is its composition with the real involution corresponding to the real form $\mathfrak{g}_0 \subset \mathfrak{g}$. It turns out that for the reduced Poisson pairs c', \tilde{c}' and c', \tilde{c}' the "first integrals" coincide (see the proof of Theorem 7.1). But it is easy to calculate them for c', \tilde{c}' using the method similar to the classical method of argument translation ([7]). This enable us to control rank of the bivectors from J' in the way required in the criterion.

The last two essential ingredients of the proof are the Gelfand-Zakharevich theorem about the structure of the pair of bivectors in a vector space (see 2.22) and some CR -geometric facts about the G_0 -orbits in \mathfrak{g}^* (Section 6).

Note that the pair c, \tilde{c} is defined canonically and some of the results about it combined with that from Section 6 may be of independent interest (see Proposition 6.5, for example).

The paper is organized as follows. In Section 1 we recall some definitions and facts from the theory of Poisson manifolds and introduce the class of complex Poisson structures, which are generalizations of the standard ones to the case of the complexified tangent bundle. Holomorphic Poisson structures are strictly contained in this class. Also, we recall elementary definitions from the theory of CR -manifolds and adapt some of them to the symplectic context.

Section 2 is devoted to bihamiltonian structures and their relations with the completely integrable systems. We define complete bihamiltonian structures, present some examples and describe their structure from the point of view of the Gelfand-Zakharevich theorem.

In Section 3 we prove that the Poisson reduction (c'_1, c'_2) of a Poisson

pair (c_1, c_2) is again a Poisson pair under the requirement of the linear independence for c'_1, c'_2 . This result follows from the natural behavior of the Schouten bracket with respect to the reduction. We also study the relations between the characteristic distributions of c_λ and c'_λ .

The main theorem of Section 4 (see 4.4) is the criterion mentioned above. It is preceded by the discussion of the linear algebraic aspects of the reductions resulting in complete bihamiltonian structures. In the end of this section a notion of minimal realization is discussed.

The goal of Section 5 is to study the auxiliary Poisson pair c, \tilde{c} from the point of view of Section 2. In particular, it is proved that it is complete and the corresponding "first integrals" are calculated.

In section 6 we show that the coadjoint G_0 -orbits are CR -generic outside some G_0 -invariant real algebraic set in \mathfrak{g}^* and calculate their dimension and CR -dimension. We also show that they are isotropic with respect to the canonical holomorphic symplectic form ω on the corresponding G -orbit.

The concluding section is devoted to the formulation and proof of the main theorem.

Of course, the inspiration for this paper is, besides the mentioned papers of I.M.Gelfand and I.S.Zakharevich, the theory of symplectic realizations for Poisson structures ([18]). Considering realizations of degenerate bihamiltonian structures in symplectic bihamiltonian structures different from holomorphic ones is also meaningful (recently the author was informed by Prof. F.J.Turiel that realizations in a symplectic bihamiltonian structure of different kind, but also with constant coefficients, give an elegant way for reconstructing the bihamiltonian structure from its Veronese web, cf. [9]). However, the author hopes that using of the holomorphic structures opens a new perspective of applying the complex-analytic methods to the theory of real bihamiltonian structures.

We conclude this introduction by the following conjecture: a generic real-analytic complete bihamiltonian structure has a realization in a holomorphic symplectic one; the double complex of differential operators related to the problem of reconstruction the bihamiltonian structure from its Veronese web (see [9]) is a kind of reduction of the d, d^c -bicomplex.

1 Complex Poisson structures and other preliminaries

1.1. Let M be a C^∞ -manifold; write $\mathcal{E}(M)$ ($\mathcal{E}^\mathbb{C}(M)$) for the space of C^∞ -smooth real (complex) valued functions on M . We shall write TM for the tangent bundle and $T^\mathbb{C}M$ for its complexification.

All complex manifolds M will be treated from the C^∞ point of view, so we shall not use special symbols for the underlying real manifolds. The holomorphic tangent bundle will be denoted by $T^{1,0}M$.

For a C^∞ vector bundle $\pi : N \rightarrow M$, let $\Gamma(N)$ denote the space of C^∞ -smooth sections of π . Elements of $\Gamma(\wedge^2 TM)$ ($\Gamma(\wedge^2 T^\mathbb{C}M)$) will be called (complex) bivectors for short.

1.2. Definition A (complex) bivector $c \in \Gamma(\wedge^2 TM)$ ($\Gamma(\wedge^2 T^\mathbb{C}M)$) is called *Poisson* if $[c, c] = 0$.

Here $[,]$ denotes the complex extension of the Schouten bracket which associates a trivector field $[c_1, c_2] \in \Gamma(\wedge^3 T^\mathbb{C}M)$ to two bivectors $c_1, c_2 \in \Gamma(\wedge^2 T^\mathbb{C}M)$. The corresponding local coordinate formula looks as follows:

$$[c_1, c_2]^{ijk}(x) = \frac{1}{2} \sum_{\text{c.p. } ijk} (c_1^{ir}(x) \frac{\partial}{\partial x^r} c_2^{jk}(x) + c_2^{ir}(x) \frac{\partial}{\partial x^r} c_1^{jk}(x)), \quad (1.2.1)$$

where $c_\alpha = c_\alpha^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$, $\alpha = 1, 2$, $\sum_{\text{c.p. } ijk}$ denotes the sum over the cyclic permutations of i, j, k and the summation convention over repeated indices is used (the latter will be used systematically in this paper).

1.3. Definition Let M be a complex manifold. A holomorphic section of the bundle $\wedge^2 T^{1,0}M \subset T^\mathbb{C}M$ will be called a *holomorphic bivector*. In particular, holomorphic bivectors can be considered as complex ones and they will be called *holomorphic Poisson* if, in addition, they are Poisson in the sense of previous definition.

1.4. Definition A hamiltonian vector field $c(f)$ corresponding to a function $f \in \mathcal{E}(M)$ ($\mathcal{E}^\mathbb{C}(M)$) is obtained by the contraction of the differential df and the Poisson bivector c with respect to the first index.

1.5. Proposition A (complex) bivector c is Poisson if and only if an operation $\{, \}_c : \mathcal{E}(M) \times \mathcal{E}(M) \longrightarrow \mathcal{E}(M)$ ($\mathcal{E}^{\mathbb{C}}(M) \times \mathcal{E}^{\mathbb{C}}(M) \longrightarrow \mathcal{E}^{\mathbb{C}}(M)$) given by

$$\{f, g\}_c = c(f)g, \quad f, g \in \mathcal{E}(M) \quad (\mathcal{E}^{\mathbb{C}}(M))$$

is a Lie algebra bracket over \mathbb{R} (\mathbb{C}).

If c is Poisson, then the map $f \mapsto c(f) : \mathcal{E}(M) \rightarrow \text{Vect}(M)$, where $\text{Vect}(M)$ is a Lie algebra of smooth vector fields on M with the commutator bracket, is a Lie algebra homomorphism.

1.6. Definition $\{, \}_c$ is called the Poisson bracket corresponding to a (complex) Poisson bivector c . A family of functions $F \subset \mathcal{E}(M)$ ($\mathcal{E}^{\mathbb{C}}(M)$) is involutive with respect to c if $\{f, g\}_c = 0$ for each two functions $f, g \in F$.

1.7. Definition Consider a (complex) bivector c at $x \in M$ as a map $c_x^{\sharp} : T_x^*M \longrightarrow T_xM$ ($(T_x^{\mathbb{C}}M)^* \longrightarrow T_x^{\mathbb{C}}M$) evaluating the first argument of the bivector on a 1-form. Kernel $\ker c(x)$ and rank $\text{rank } c(x)$ of c at x are defined as that of c_x^{\sharp} . We say that c is nondegenerate if c_x^{\sharp} is an isomorphism. A complex bivector of type $(2, 0)$ on a complex manifold M is called nondegenerate in the holomorphic sense if the restricted sharp map $c^{\sharp} : (T^{1,0}M)^* \longrightarrow T^{1,0}M$ is an isomorphism.

1.8. Definition A characteristic subspace $P_{c,x}$ of a (complex) bivector c at $x \in M$ is defined as $\text{im } c_x^{\sharp}$. A generalized distribution of subspaces $P_c \subset TM$ ($T^{\mathbb{C}}M$) is said to be a characteristic distribution for the bivector c .

Note that a complex bivector of type $(2, 0)$ nondegenerate in holomorphic sense is not nondegenerate since $P_{c,x} = T^{1,0}M \neq T^{\mathbb{C}}M$. We shall usually understand the nondegeneracy of holomorphic bivectors in the holomorphic sense.

1.9. Theorem ([13]) Let c be a real Poisson bivector. The generalized distribution P_c is completely integrable, i.e. there exists a tangent to P_c generalized foliation $\{S_{\alpha}\}_{\alpha \in I}$ on M : $T_x S_{\alpha} = P_{c,x}$ for any $\alpha \in I$ and for any $x \in S_{\alpha}$. The restriction of c to each S_{α} is a nondegenerate Poisson bivector; consequently, S_{α} are symplectic manifolds with the symplectic forms $\omega_{\alpha} = (c|_{S_{\alpha}})^{-1}$.

Here and subsequently the 2-form ω inverse to a nondegenerate bivector c is defined as follows. If $\wedge^2 c^\sharp$ is the extension of the sharp map defined above to the second exterior power of T^*M , then $\omega = (\wedge^2 c^\sharp)^{-1}(c)$. The inverse to a nondegenerate 2-form bivector is defined similarly.

The above theorem is also true in the complex analytic category if we understand P_c as a holomorphic subbundle in $T^{1,0}M$ and the nondegeneracy in the holomorphic sense. The definition of inverse objects in this case is analogous to real one.

1.10. Definition *The submanifolds S_α are called symplectic leaves of a Poisson bivector c .*

1.11. Proposition *Given a complex Poisson bivector $c \in \Gamma(\wedge^2 T^\mathbb{C}M)$, its characteristic distribution $P_c \subset T^\mathbb{C}M$ is involutive, i.e.*

$$[v, w](x) \in P_{c,x}$$

for any complex valued vector fields v, w such that $v(x), w(x) \in P_{c,x}, x \in M$.

In general, one can say nothing about the complete integrability of P_c even if one understands this in spirit of the Newlander-Nirenberg theorem. A nonconstant rank of the subspaces $P_{c,x}$ or $P_{c,x} \cap \overline{P}_{c,x}$ (the overline means the complex conjugation) may be the obstruction here as well as some other reasons (see [17]).

1.12. Example Let $M = \mathbb{C}^3$ with coordinates z_1, z_2, z_3 , $c = \bar{z}_1 \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_3} + \bar{z}_2 \frac{\partial}{\partial z_3} \wedge \frac{\partial}{\partial z_1} + \bar{z}_3 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}$, $f = |z_1|^2 + |z_2|^2 + |z_3|^2$. The bivector c is obviously Poisson. Since $c(f) = 0$, its characteristic subspace $P_{c,x}$ is equal to the $(1,0)$ -tangent space (cf. 1.17) to the 5-dimensional sphere $S \subset M$ centered in 0 and passing through x . Of course, this example is related to the Lie algebra $so(3)$. We shall generalize it in Section 5.

1.13. Convention In the sequel, all Poisson bivectors will be assumed to have maximal rank on an open dense subset in M . For real Poisson bivectors this is equivalent to the following: the union of symplectic leaves of maximal dimension is dense in M .

1.14. Definition Let c be a (complex) Poisson bivector. Define $\text{rank } c$ as $\max_{x \in M} \text{rank } c(x)$.

1.15. Definition A Casimir function $f \in \mathcal{E}(U)$ ($\mathcal{E}^{\mathbb{C}}(U)$) over an open set $U \subset M$ for a (complex) Poisson bivector c is defined by the condition $c(f) = 0$. A space of all Casimir functions over U for c is denoted by $Z_c(U)$ and $Z_{c,x}$ stands for the space of germs of Casimir functions at $x, x \in M$.

Note that if c is real and $\text{rank } c < \dim M$ there exist local nontrivial Casimir functions and their differentials at x span $\ker c(x)$, provided that x is taken from a symplectic leaf of maximal dimension. This is not true concerning the global Casimir functions: it is easy to construct a Poisson bivector c with $\text{rank } c < \dim M$ possessing only trivial ones.

1.16. Definition Let (M, ω) , $\dim M = 2n$, be a symplectic manifold. A submanifold $L \subset M$ is called

- *coisotropic* if $(T_x L)^{\perp \omega(x)} \subset T_x L$ for any $x \in L$;
- *isotropic* if $(T_x L)^{\perp \omega(x)} \supset T_x L$ for any $x \in L$;
- *lagrangian* if $(T_x L)^{\perp \omega(x)} = T_x L$ for any $x \in L$.

A foliation \mathcal{L} on M is *coisotropic* (*isotropic*, *lagrangian*) if so is its every leaf.

Here $\perp \omega(x)$ stands for a skew-orthogonal complement in $T_x M$ with respect to $\omega(x)$. For the third case the following definition is equivalent: $\dim L = n$ and $\omega|_{TL} \equiv 0$.

1.17. We shall need a specific generalization of this definition in the complex case. Let M be a complex manifold with the complex structure $\mathcal{J} : TM \rightarrow TM$. Consider a C^∞ -smooth submanifold $L \subset M$. Write $T_x^{CR} L$ for $T_x L \cap \mathcal{J} T_x L$ and $T_x^{1,0} L$ for $T_x^{\mathbb{C}} L \cap T_x^{1,0} M$, $x \in L$. Another definition for $T_x^{1,0} L$ is the following: $T_x^{1,0} L = \{v - i\mathcal{J}v; v \in T_x^{CR} L\}$.

1.18. Definition ([17],[2]) L is called a *CR-submanifold* in M if $\dim T_x^{1,0} L$ is constant along L ; we say that this number is *CR-dimension* of L ; L is *generic* (*completely real*) if $T_x L + \mathcal{J} T_x L = T_x M$ ($T_x L \oplus \mathcal{J} T_x L = T_x M$) for each $x \in L$.

If a generic CR -submanifold L is given by the equations $\{f_1 = \alpha_1, \dots, f_k = \alpha_k\}$, $f_i \in \mathcal{E}(M)$, such that $df_1 \wedge \dots \wedge df_k \neq 0$ along L , $\alpha_i \in \mathbb{R}$, then $\partial f_1 \wedge \dots \wedge \partial f_k \neq 0$ along L and $T_x^{1,0}L = \{\partial f_1(x), \dots, \partial f_k(x)\}^{\perp 1,0}$, where ∂ is the $(1,0)$ -differential on M , $\perp 1,0$ denotes the annihilator in $T_x^{1,0}M$.

1.19. Definition *A foliation \mathcal{L} on M is a generic (completely real) CR -foliation if its each leaf is a generic (completely real) CR -submanifold.*

1.20. Definition *Let (M, ω) be a holomorphic symplectic manifold. A CR -submanifold $L \subset M$ is*

- *CR -coisotropic if $(T_x^{1,0}L)^{\perp \omega(x)} \subset T_x^{1,0}L$ for any $x \in L$;*
- *CR -isotropic if $(T_x^{1,0}L)^{\perp \omega(x)} \supset T_x^{1,0}L$ for any $x \in L$;*
- *CR -lagrangian if $(T_x^{1,0}L)^{\perp \omega(x)} = T_x^{1,0}L$ for any $x \in L$.*

A CR -foliation \mathcal{L} on M is said to be CR -coisotropic (CR -isotropic, CR -lagrangian) if so is its every leaf.

Here $\perp \omega(x)$ denotes a skew-orthogonal complement in $T_x^{1,0}M$ with respect to the $(2,0)$ -form $\omega(x)$.

Suppose \mathcal{L} is generic and consists of the common level sets of the functions $f_1, \dots, f_k \in \mathcal{E}(M)$, $k \leq n = (1/2) \dim_{\mathbb{C}} M$. Then \mathcal{L} is CR -coisotropic if and only if the family $\{f_1, \dots, f_k\}$ is involutive with respect to the holomorphic Poisson bivector $c = (\omega)^{-1}$. In particular, if $k = n$ one gets CR -lagrangian foliation.

1.21. Definition *Let (M, ω) , $\dim M = 2n$, be a symplectic manifold. A completely integrable system on M is defined as a family of functions $\mathcal{F} \subset \mathcal{E}(M)$ involutive with respect to $c = (\omega)^{-1}$ and containing a subfamily of n functions that are functionally independent almost everywhere on M . In other words, a completely integrable system on M is a lagrangian foliation \mathcal{L} on an open dense subset in M .*

We conclude this section by recalling main definitions concerning hamiltonian actions of Lie groups (see [6] for details).

1.22. Definition Let G be a connected Lie group with the Lie algebra \mathfrak{g} . Assume it is acting on a Poisson manifold M with the Poisson bivector c , i.e. a Lie algebra homomorphism $\rho : \mathfrak{g} \rightarrow \text{Vect}(M)$ is given. The action is called hamiltonian if there exists a Lie algebra homomorphism $\psi : \mathfrak{g} \rightarrow \mathcal{E}(M)$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\psi} & \mathcal{E}(M) \\ \parallel & & \downarrow c(\cdot) \\ \mathfrak{g} & \xrightarrow{\rho} & \text{Vect}(M), \end{array}$$

where $c(\cdot)$ is a Lie algebra homomorphism of taking the hamiltonian vector field (see Proposition 1.5).

2 Bihamiltonian structures and completeness

Let M be a C^∞ -manifold.

2.1. Definition Two linearly independent (complex) Poisson bivectors c_1, c_2 on M form a (complex) Poisson pair if $c_\lambda = \lambda_1 c_1 + \lambda_2 c_2$ is a Poisson bivector for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ (\mathbb{C}^2).

2.2. Proposition A pair of linearly independent (complex) bivectors (c_1, c_2) is Poisson if and only if $[c_1, c_1] = 0, [c_1, c_2] = 0, [c_2, c_2] = 0$.

2.3. Definition Let M be a C^∞ -manifold. A (complex) bihamiltonian structure on M is defined as a two-dimensional linear subspace $J = \{c_\lambda\}_{\lambda \in \mathcal{S}}$ of (complex) Poisson bivectors on M parametrized by a two-dimensional vector space \mathcal{S} over \mathbb{R} (\mathbb{C}). The trivial bihamiltonian structure is the zero-dimensional linear subspace in $\Gamma(\wedge^2 TM)$.

It is clear that every Poisson pair generates a nontrivial bihamiltonian structure and the transition from the latter one to a Poisson pair corresponds to a choice of basis in \mathcal{S} . We shall write (J, c_1, c_2) for a bihamiltonian structure J with a chosen Poisson pair (c_1, c_2) generating J .

2.4. Definition Let (J, c_1, c_2) be a bihamiltonian structure. A complex bihamiltonian structure

$$J^\mathbb{C} = \{\lambda_1 c_1 + \lambda_2 c_2; (\lambda_1, \lambda_2) \in \mathbb{C}^2\}$$

is called the complexification of J .

2.5. Proposition *A complex bihamiltonian structure J is the complexification of some real one if and only if one can choose a generating J Poisson pair (c, \bar{c}) , where $c \in \wedge^2(T^\mathbb{C}M)$, the bar stands for the complex conjugation.*

PROOF. Generate J' by $\operatorname{Re} c, \operatorname{Im} c$. Then $J = (J')^\mathbb{C}$. Conversely, one checks that for (J', c_1, c_2) the complexification $(J')^\mathbb{C}$ is generated by $c_1 \pm ic_2$. q.e.d.

2.6. Definition *Let J be a (complex) bihamiltonian structure and let $J_0 \subset J$ be a subfamily of (complex) Poisson bivectors of maximal rank R_0 (the set $J \setminus J_0$ is at most a finite sum of 1-dimensional subspaces). We say that J is symplectic if $\operatorname{rank} c_\lambda = \dim M$ for any $c_\lambda \in J_0$ and that J is degenerate otherwise.*

2.7. Example Consider a family $J^\mathbb{C}$ generated by a pair (c, \bar{c}) , where $c = (\omega)^{-1}$ is a complex Poisson bivector inverse to a holomorphic symplectic form ω on a complex symplectic manifold M . Since c is holomorphic and \bar{c} is antiholomorphic, we have $[c, \bar{c}] = 0$. Thus $J^\mathbb{C}$ is a bihamiltonian structure. By Proposition 2.5 it is the complexification of the real bihamiltonian structure $(J, \operatorname{Re} c, \operatorname{Im} c)$. This example is fundamental for the paper and we shall need the following fact.

2.8. Proposition *Let M, ω, c and $J^\mathbb{C}$ be as in Example 2.7. Then $J^\mathbb{C}$ is symplectic and the only degenerate bivectors in $J^\mathbb{C}$ are those proportional to c and \bar{c} . Moreover, $\operatorname{rank}_\mathbb{C} c = \operatorname{rank}_\mathbb{C} \bar{c} = \frac{1}{2} \dim_\mathbb{C} M, P_c = T^{1,0}M, P_{\bar{c}} = T^{0,1}M$.*

PROOF. The last assertion is obvious as well as the following equality

$$c \lrcorner \bar{\omega} = 0,$$

where \lrcorner stands for the contraction with respect to the first index. For $\omega_1 = \operatorname{Re} \omega, \omega_2 = \operatorname{Im} \omega, c_1 = \operatorname{Re} c, c_2 = \operatorname{Im} c$ this implies

$$c_1 \lrcorner \omega_1 + c_2 \lrcorner \omega_2 = 0, c_2 \lrcorner \omega_1 - c_1 \lrcorner \omega_2 = 0.$$

We have for $\lambda \in \mathbb{C}$

$$\begin{aligned} (c_1 + \lambda c_2) \lrcorner (\omega_1 - \lambda \omega_2) &= c_1 \lrcorner \omega_1 - \lambda^2 c_2 \lrcorner \omega_2 - \lambda(c_1 \lrcorner \omega_2 - c_2 \lrcorner \omega_1) = \\ &= (1 + \lambda^2) c_1 \lrcorner \omega_1 = (1 + \lambda^2) \frac{1}{4} \operatorname{id}_{TM}. \end{aligned}$$

The last equality is verified directly in the Darboux local coordinates. Thus $(c_1 + \lambda c_2)^{-1} = \frac{4}{1+\lambda^2}(\omega_1 - \lambda\omega_2)$. Since $c_1 \lrcorner \omega_1 = -c_2 \lrcorner \omega_2$, c_2 is also nondegenerate. q.e.d.

2.9. Definition *Let (M, ω) be a complex symplectic manifold. The bihamiltonian structure J and its complexification $J^\mathbb{C}$ from Example 2.7 are called holomorphic symplectic.*

2.10. Given a (complex) bihamiltonian structure J , let F_0 denote the space $\sum_{c \in J_0} Z_c(M) \subset \mathcal{E}(M)$.

The following theorem shows how the degenerate bihamiltonian structures can be applied for constructing the completely integrable systems.

2.11. Theorem *Let J be a degenerate (complex) bihamiltonian structure on M . A family F_0 is involutive with respect to any $c_\lambda \in J$.*

PROOF. Let $c_1, c_2 \in J_0$ be linearly independent, $f_i \in Z_{c_i}, i = 1, 2$. Then

$$\{f_1, f_2\}_{c_\lambda} = (\lambda_1 c_1(f_1) + \lambda_2 c_2(f_1))f_2 = -\lambda_2 c_2(f_2)f_1 = 0. \quad (2.11.1)$$

Now it remains to prove that for any $c \in J_0, f_i \in Z_c, i = 1, 2$, one has $\{f_1, f_2\}_{c_\lambda} = 0$. For that purpose we first rewrite (2.11.1) as

$$c_\lambda(x)(\phi_1, \phi_2) = 0, \quad (2.11.2)$$

where $\phi_i \in \ker c_i(x), i = 1, 2, x \in M$, and the lefthandside denotes a contraction of the bivector with two covectors. Second, we fix x such that $\text{rank } c(x) = R_0$ and approximate $df_2|_x$ by a sequence of elements $\{\phi^i\}_{i=1}^\infty, \phi^i \in \ker c^i(x)$, where $c^i \in J_0, i = 1, 2, \dots$, is linearly independent with c . Finally, by (2.11.2) we get $c_\lambda(x)(df_1|_x, \phi^i) = 0$ and by the continuity $\{f_1, f_2\}_{c_\lambda}(x) = 0$. Since the set of such points x is dense in M , the proof is finished. q.e.d.

In fact this theorem is true for the local Casimir functions (for the germs of Casimir functions).

2.12. Definition *The functions from the family F_0 (see 2.10) are called (global) first integrals of the bihamiltonian structure J . The family of functions $\sum_{c \in J_0} Z_c(U)$ ($\sum_{c \in J_0} Z_{c,x}$) is denoted by $F_0(U)$ ($F_{0,x}$) and its elements are called local first integrals over an open $U \subset M$ (germs of first integrals at $x \in M$).*

In order to obtain a completely integrable system from Casimir functions one should require additional assumptions on the bihamiltonian structure J . Off course, the condition of completeness given below concerns the local Casimir functions (in fact their germs) and may be insufficient for obtaining the completely integrable system. However, it is of use if the local Casimir functions are restrictions of the global ones (see Example 2.20, below).

Given a characteristic distribution $P_c \subset TM$ ($T^{\mathbb{C}}M$) of some (complex) Poisson bivector and a point $x \in M$, let $P_{c,x}^*$ denote a dual space to $P_{c,x}$. Any functional $\phi \in T_x^*M$ ($(T_x^{\mathbb{C}}M)^*$) can be regarded as an element of $P_{c,x}^*$ called the restriction of ϕ to $P_{c,x}$.

2.13. Definition ([3]) *Let J be a (complex) bihamiltonian structure; fix some $c_\lambda \in J$.*

J is called complete at a point $x \in M$ with respect to c_λ if a linear subspace of $P_{c_\lambda,x}^$ generated over \mathbb{R} (\mathbb{C}) by the differentials of the germs $f \in F_{0,x}$ restricted to $P_{c_\lambda,x}$ has dimension $\frac{1}{2} \dim_{\mathbb{R}} P_{c_\lambda,x}$ ($\frac{1}{2} \dim_{\mathbb{C}} P_{c_\lambda,x}$).*

2.14. Proposition *A (complex) bihamiltonian structure J is complete with respect to $c_\lambda \in J$ at a point $x \in M$ if and only if $\dim(\bigcap_{c \in J_0} P_{c,x}) = \frac{1}{2} \dim P_{c_\lambda,x}$.*

PROOF is obvious.

The following theorem is due to A.Brailov (see [3], Theorem 1.1 and Remark after it).

2.15. Theorem *A (complex) bihamiltonian structure J is complete with respect to $c_\lambda \in J_0$ at a point $x \in M$ such that $P_{c_\lambda,x}$ is of maximal dimension if and only if the following condition holds*

$$(*) \text{ rank } c(x) = R_0 \text{ for any } c \in J^{\mathbb{C}} \setminus \{0\} \text{ } (J \setminus \{0\}),$$

where R_0 is as in 2.6.

PROOF of this theorem is a consequence of the following linear algebraic fact.

2.16. Proposition ([3]) *Let V be a vector space over \mathbb{R} (\mathbb{C}) and let J be a two dimensional linear subspace in $\wedge^2 V$. In the real case we let $J^{\mathbb{C}} \subset \wedge^2 V^{\mathbb{C}}$ denote the complexification of the subspace J . We write $J_0 \subset J$ for the subset of bivectors of maximal rank R_0 and $F_0 \subset V^*$*

for the subspace generated by the kernels of bivectors from J_0 . Let $c^\sharp : V^* \rightarrow V$ stand for the corresponding sharp map of $c \in \wedge^2 V$ (cf. 1.7).

Then, given a bivector $c_\lambda \in J_0$, the following two conditions are equivalent:

- (i) $\dim(F_0|_{P_\lambda}) = 1/2 \dim P_\lambda$, where $F_0|_{P_\lambda} = \text{Span}\{\{f|_{P_\lambda}\}_{f \in F_0}\} \subset P_\lambda^*$ and $P_\lambda = c_\lambda^\sharp(V^*)$;
- (ii) $\text{rank } c = R_0$ for any $c \in J^\mathbb{C} \setminus \{0\}$ ($J \setminus \{0\}$).

PROOF. We reproduce the proof from [3] with a small completion.

We perform the proof in the following four steps.

First, we observe that for any two bivectors $a, b \in J \setminus \{0\}$ one has the equality $a^\sharp(F_0) = b^\sharp(F_0)$. Indeed, suppose that a, b are linearly independent. The subspace F_0 is generated by a finite number of kernels $\ker b_1, \dots, \ker b_s, b_i \in J_0$. Without loss of generality, we may assume that $b_i = \alpha_i a + \beta_i b$, where $\alpha_i, \beta_i \neq 0$. Since $(\alpha_i a^\sharp + \beta_i b^\sharp)(\ker b_i) = 0, i = 1, \dots, s$, then $a^\sharp(\ker b_i) = b^\sharp(\ker b_i)$ and, consequently, $a^\sharp(F_0) = b^\sharp(F_0)$.

Second, consider the skew-orthogonal complement $\tilde{F}_0 = (F_0)^{\perp b} = (b^\sharp(F_0))^\perp$ and note that: 1) it does not depend on $b \in J \setminus \{0\}$ (previous step); 2) $F_0 \subset \tilde{F}_0$ (the skew-orthogonal complement of any subspace in V^* with respect to any $b \in J \setminus \{0\}$ contains $\ker b$, in particular $\tilde{F}_0 \supset \ker b, b \in J \setminus \{0\}$); 3) if $a \in J_0$, then $b^\sharp(\tilde{F}_0) \subset a^\sharp(\tilde{F}_0)$ for any $b \in J$ (this is equivalent to $(F_0^{\perp b})^{\perp b} \supset (F_0^{\perp a})^{\perp a}$ or $F_0 + \ker b \supset F_0 + \ker a = F_0$).

Third, given two linearly independent bivectors $a, b \in J$, with $\text{rank } a = R_0$, we define a "recursion" operator $\Phi : \tilde{F}_0/F_0 \rightarrow \tilde{F}_0/F_0$ by the formula $\Phi(\pi(\xi)) = \pi((a^\sharp)^{-1}b^\sharp(\xi))$, where $\xi \in \tilde{F}_0$ and $\pi : \tilde{F}_0 \rightarrow \tilde{F}_0/F_0$ is the natural projection. The operator is correctly defined due to the conditions $a(F_0) = b(F_0)$, $b(\tilde{F}_0) \subset a(\tilde{F}_0)$, and $\ker a \subset F_0$. It is easy to see that the eigenvalues of Φ are precisely those $\lambda \in \mathbb{C}$ for which $\text{rank}(a - \lambda b) < R_0$. In particular (ii) holds if and only if Φ does not have eigenvalues, i.e. $F_0 = \tilde{F}_0$.

Finally, we use the following sequence of subspaces and relations between them

$$\begin{aligned} F_0 &= \pi^{-1}(\pi(F_0)) = \pi^{-1}(F_0|_{P_\lambda}) \subset \pi^{-1}((c^\sharp(F_0|_{P_\lambda}))^\perp) = \\ &= (c_\lambda^\sharp(F_0|_{P_\lambda}))^\perp = (c_\lambda^\sharp(F_0))^\perp = \tilde{F}_0, \end{aligned}$$

where $\pi :: V^* \rightarrow V/P_\lambda^\perp \cong P_\lambda^*$ is the canonical projection and \perp_λ is the annihilator in the sense of the dual pair (P_λ, P_λ^*) . The essential moment here is that $\ker \pi = \ker c_\lambda \subset F_0$; this implies the first equality. The

only inclusion in this sequence is the equality, i.e. $F_0|_{P_\lambda}$ is a lagrangian subspace with respect to $C_\lambda|_{P_\lambda}$, if and only if condition (i) holds. q.e.d.

Theorem 2.15 shows that J is complete with respect to a fixed $c_\lambda \in J_0$ at a point x such that the dimension $P_{c_\lambda, x}$ is maximal if and only if $J = J_0 \cup \{0\}$ and J is complete at x with respect to any nontrivial $c_\lambda \in J$. This motivates the next definition.

2.17. Definition *Let (J, c_1, c_2) be a (complex) bihamiltonian structure. The structure J (the pair (c_1, c_2)) is complete at a point $x \in M$ if condition (*) of Theorem 2.15 holds at x . $J((c_1, c_2))$ is called complete if it is so at any point from some open and dense subset in M . The trivial bihamiltonian structure is complete by definition.*

2.18. Corollary *Let a bihamiltonian structure J be complete at any x from some sufficiently small open set U . Then the functions from $F_0(U)$ (see 2.10) define a foliation \mathcal{L} on U that is lagrangian in any symplectic leaf S_λ of any $c_\lambda|_U, c_\lambda \in J|_{\text{setminus}\{0\}}$. On the overlap of two such sets the corresponding foliations coincide.*

PROOF. The first assertion follows from Theorem 2.15. The second one is a consequence of the uniqueness of the set of local first integrals of a degenerate bihamiltonian structure. q.e.d.

2.19. Definition *The foliation \mathcal{L} described in Proposition 2.18 will be called the bilagrangian foliation of a complete bihamiltonian structure.*

2.20. Example (Method of argument translation, see [7], [3].) Let \mathfrak{g} be a nonabelian Lie algebra, \mathfrak{g}^* its dual space. Fix a basis $\{e_1, \dots, e_n\}$ in \mathfrak{g} with the structure constants $\{c_{ij}^k\}$. The standard linear Poisson bivector on \mathfrak{g}^* is defined as

$$c_1(x) = c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where $\{x_k\}$ are linear coordinates in \mathfrak{g}^* corresponding to $\{e_1, \dots, e_n\}$. In more invariant terms c_1 is described as dual to the Lie-multiplication map $[\cdot, \cdot] : \mathfrak{g} \wedge \mathfrak{g} \longrightarrow \mathfrak{g}$. It is well-known that the symplectic leaves of c_1 are the coadjoint orbits in \mathfrak{g}^* . Now define c_2 as a bivector with constant coefficients $c_2 = c(a)$, where a is a fixed point on any leaf

of maximal dimension. It turns out that c_1, c_2 form a Poisson pair and it is easy to describe the set I of points x for which condition $(*)$ fails. Consider the complexification $(\mathfrak{g}^*)^{\mathbb{C}} \cong (\mathfrak{g}^{\mathbb{C}})^*$ and the sum $\text{Sing}(\mathfrak{g}^{\mathbb{C}})^*$ of symplectic leaves of nonmaximal dimension for the complex linear bivector $c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$, where $z_j = x_j + iy_j$, $j = 1, \dots, n$, are the corresponding complex coordinates in $(\mathfrak{g}^*)^{\mathbb{C}}$. Then I is equal to the intersection of the sets $\mathfrak{g}^* \subset (\mathfrak{g}^*)^{\mathbb{C}}$ and $\overline{a, \text{Sing}(\mathfrak{g}^{\mathbb{C}})^*}$, where $a, \text{Sing}(\mathfrak{g}^{\mathbb{C}})^*$ denotes a cone of complex 2-dimensional subspaces passing through a and $\text{Sing}(\mathfrak{g}^{\mathbb{C}})^*$. In particular, (c_1, c_2) is complete for a semisimple \mathfrak{g} since $\text{Sing} \mathfrak{g}^{\mathbb{C}*}$ has codimension at least 3. Note that this gives rise to completely integrable systems, since the local Casimir functions on \mathfrak{g}^* are restrictions of the global ones, i.e. the invariants of the coadjoint action.

2.21. Example (Bihamiltonian structure of general position on an odd-dimensional manifold, see [9].) Consider a pair of bivectors (a_1, a_2) , $a_i \in \wedge^2 V$, $i = 1, 2$, where V is a $(2m + 1)$ -dimensional vector space; (a_1, a_2) is in general position if and only if it is represented by the Kronecker block of dimension $2m + 1$, i.e.

$$\begin{aligned} a_1 &= p_1 \wedge q_1 + p_2 \wedge q_2 + \dots + p_m \wedge q_m \\ a_2 &= p_1 \wedge q_2 + p_2 \wedge q_3 + \dots + p_m \wedge q_{m+1} \end{aligned} \quad (2.21.1)$$

in an appropriate basis $p_1, \dots, p_m, q_1, \dots, q_{m+1}$ of V . A bihamiltonian structure J on a $(2m + 1)$ -dimensional M is in general position if and only if the pair $(c_1(x), c_2(x))$ is so for any $x \in M$. Such J is complete: it is easy to prove that $J = J_0 \cup \{0\}$, $\dim \bigcap_{c \in J} P_c(x) = n$ and then use Proposition 2.14. In general, a complete Poisson pair at a point is the direct sum of the Kronecker blocks and the zero pair as the corollary of the next theorem shows. This theorem is a reformulation of the classification result for pairs of 2-forms in a vector space ([8], [10]).

2.22. Theorem *Given a finite-dimensional vector space V over \mathbb{C} and a pair of bivectors (c_1, c_2) , $c_i \in \wedge^2 V$, there exists a direct decomposition $V = \oplus_{j=1}^k V_j$, $c_i = \sum_{j=1}^k c_i^{(j)}$, $c_i^{(j)} \in \wedge^2 V_j$, $i = 1, 2$, such that each pair $(c_1^{(j)}, c_2^{(j)})$ is from the following list:*

- (a) the Jordan block: $\dim V_j = 2n_j$ and in an appropriate basis of V_j the matrix of $c_i^{(j)}$ is equal to

$$\begin{pmatrix} 0 & A_i \\ -A_i^T & 0 \end{pmatrix}, i = 1, 2,$$

where $A_1 = I_{n_j}$ (the unity $n_j \times n_j$ -matrix) and $A_2 = J_{n_j}^\lambda$ (the Jordan block with the eigenvalue λ);

- (b) the Kronecker block: $\dim V_j = 2n_j + 1$ and in an appropriate basis of V_j the matrix of $c_i^{(j)}$ is equal to

$$\begin{pmatrix} 0 & B_i \\ -B_i^T & 0 \end{pmatrix}, i = 1, 2,$$

$$\text{where } B_1 = \begin{pmatrix} 100 \dots 00 \\ 010 \dots 00 \\ \dots \\ 000 \dots 10 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 010 \dots 00 \\ 001 \dots 00 \\ \dots \\ 000 \dots 01 \end{pmatrix} \quad ((n_j+1) \times n_j\text{-matrices}).$$

- (c) the trivial Kronecker block: $\dim V_j = 1$, $c_1^{(j)} = c_2^{(j)} = 0$;

2.23. Corollary *Let J be a (complex) bihamiltonian structure. It is complete at a point $x \in M$ if and only if a pair $(c_1(x), c_2(x))$, $c_i(x) \in \wedge^2(T_x^\mathbb{C}M)$, $i = 1, 2$, does not contain the Jordan blocks in its decomposition.*

PROOF follows from the definition of completeness.

The following example of a complete Poisson pair shows that the structure of decomposition to the Kronecker blocks may change from point to point.

2.24. Example ([16]) Let $M = \mathbb{R}^6$ with coordinates $(p_1, p_2, q_1, \dots, q_4)$, $c_1 = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_2}$, $c_2 = \frac{\partial}{\partial p_1} \wedge (\frac{\partial}{\partial q_2} + q_1 \frac{\partial}{\partial q_3}) + \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_4}$. Here we have: two 3-dimensional Kronecker blocks on $\bar{M} \setminus H$, $\bar{H} = \{q_1 = 0\}$; the 5-dimensional Kronecker block and the 1-dimensional zero block on the hyperplane H .

2.25. Remark The decomposition of a pair of bivectors c_1, c_2 in a vector space V to Kronecker blocks is defined noncanonically. For example, let us consider 4-dimensional $V = \text{Span}\{e, p, q_1, q_2\}$, $c_1 = p \wedge q_1$, $c_2 = p \wedge q_2$. Here $V = V_1 \oplus V_2$, where $V_1 = \text{Span}\{e\}$, $V_2 = \text{Span}\{p, q_1, q_2\}$, but instead V_1 one can choose any direct complement to V_2 . However, dimensions of the direct sums for the Kronecker blocks of equal dimension are invariants (see [12],[16]). For instance, dimension of the sum of the trivial Kronecker blocks is equal to $\dim(\ker c_1 \cap \ker c_2)$ (see Proposition 2.26, below).

We conclude the section by a result that will be used later on.

2.26. Proposition *Let V be a vector space over \mathbb{C} and let a pair of bivectors $c_1, c_2 \in \wedge^2 V$ be such that there are no Jordan blocks in the decomposition of Theorem 2.22. Set*

$$\begin{aligned}\mu &= \dim(\ker c_1 \cap \ker c_2) \\ \mu_\lambda &= \dim(\ker c_1 \cap \ker c_\lambda),\end{aligned}$$

where $c_\lambda = \lambda_1 c_1 + \lambda_2 c_2$, $\lambda_1, \lambda_2 \neq 0$. Then $\mu = \mu_\lambda$ and this number is equal to dimension of the sum of the trivial Kronecker blocks.

PROOF. Let (V', a_1, a_2) , $V' \subset V$, $\dim V' = 2m + 1$, $a_i \in \wedge^2 V'$, be a nontrivial Kronecker block. By formula 2.21.1

$$\ker a_\lambda = \lambda_2^m q^{m+1} - \lambda_1^{m-1} \lambda_2 q^m + \dots + (-1)^m \lambda_2^m q^1,$$

where $a_\lambda = \lambda_1 a_1 + \lambda_2 a_2$ and q^1, \dots, q^{m+1} is a part of the basis $p^1, \dots, p^m, q^1, \dots, q^{m+1}$ in V'^* dual to $p_1, \dots, p_m, q_1, \dots, q_{m+1}$ (let us denote these bases by \mathbf{p}, \mathbf{q} . and \mathbf{p}, \mathbf{q} ., correspondingly, and call them adapted to the pair a_1, a_2). The above formula shows that $\ker a_1 \cap \ker a_\lambda = \{0\}$ if $\lambda_2 \neq 0$.

Now, let $V = \oplus_{j=1}^k V_j$, $c_i = \sum_{j=1}^k c_i^{(j)}$, $i = 1, 2$, be the decomposition to the Kronecker blocks and let $V_{k'+1}, \dots, V_k$ be all trivial ones. Consider a basis of V of the following form

$$\mathbf{p}^{(1)}, \mathbf{q}^{(1)}, \dots, \mathbf{p}^{(k')}, \mathbf{q}^{(k')}, r_1, \dots, r_{k-k'},$$

where $\mathbf{p}^{(j)}, \mathbf{q}^{(j)}$ is a basis of V_j adapted to $c_1^{(j)}, c_2^{(j)}$, $j = 1, \dots, k'$, and $r_1, \dots, r_{k-k'}$ generate $V_{k'+1}, \dots, V_k$, respectively. The dual basis will be of the form

$$\mathbf{p}^{(1)}, \mathbf{q}^{(1)}, \dots, \mathbf{p}^{(k')}, \mathbf{q}^{(k')}, r^1, \dots, r^{k-k'}$$

and the above considerations show that $\ker c_1 \cap \ker c_\lambda$ is generated by $r^1, \dots, r^{k-k'}$ if $\lambda_2 \neq 0$; consequently $\dim(\ker c_1 \cap \ker c_\lambda)$ is constant over $\lambda, \lambda_2 \neq 0$ and equal to dimension of the sum of the trivial Kronecker blocks. q.e.d.

3 Reductions and realizations of bihamiltonian structures.

Our next aim is to prove that a Poisson reduction of a bihamiltonian structure is again a bihamiltonian structure. This result follows from the naturality of behavior of the Schouten bracket with respect to the reduction.

3.1. Consider a C^∞ -smooth surjective submersion $p : M \longrightarrow M'$ such that $p^{-1}(x')$ is connected for any $x' \in M'$. The foliation of its leaves will be denoted by \mathcal{K} . Write $p_* : TM \longrightarrow TM'$ for the corresponding tangent bundle morphism, $\bigwedge^k p_* : \bigwedge^k TM \longrightarrow \bigwedge^k TM'$ for its exterior power extension and $\ker \bigwedge^k p_*$ for a subbundle in $\bigwedge^k TM$ that is a kernel of $\bigwedge^k p_*$. Multivector fields on M or M' will be called multivectors for short.

If $(U, \{x^1, \dots, x^l, y^1, \dots, y^{m'}\})$ is a local coordinate system on M such that $m' = \dim M'$ and $y^1, \dots, y^{m'}$ are constant along \mathcal{K} , then the restriction $Z|_U$ of $Z \in \Gamma(\bigwedge^k TM)$ belongs to $\Gamma(\ker \bigwedge^k p_*)(U)$ if and only if each term of its decomposition with respect to $\{\partial_{x^1}, \dots, \partial_{x^l}, \partial_{y^1}, \dots, \partial_{y^{m'}}\}$ contains at least one $\partial_{x^i}, 1 \leq i \leq l$.

3.2. Theorem *Let $Z \in \Gamma(\bigwedge^k TM)$. The following conditions are equivalent:*

- (i) $\mathcal{L}_X Z \in \Gamma(\ker \bigwedge^k p_*) \ \forall X \in \Gamma(\ker p_*)$, where \mathcal{L}_X is a Lie derivation;
- (ii) $\phi_{t,*}^X Z - Z \in \Gamma(\ker \bigwedge^k p_*) \ \forall t \forall X \in \Gamma(\ker p_*)$, where ϕ_t^X denotes the flow of the vector X ;
- (iii) in any local coordinate system $(U, \{x^1, \dots, x^l, y^1, \dots, y^{m'}\})$ on M such that $m' = \dim M'$ and $y^1, \dots, y^{m'}$ are constant on the leaves of p the multivector Z can be written as

$$Z(x, y) = Z'(y) + \tilde{Z}(x, y), \quad (3.2.1)$$

where

$$Z'(y) = Z^{i_1 \dots i_k}(y) \partial_{y^{i_1}} \wedge \dots \wedge \partial_{y^{i_k}} \quad (3.2.2)$$

and $\tilde{Z} \in \Gamma(\ker \wedge^k p_*)(U)$.

If one of these conditions is satisfied for Z , then $Z'(x') = \wedge^k p_*(Z(x))$, $x' \in M'$, $x \in p^{-1}(x')$, is a correctly defined multivector on M' . Moreover, if $(p(U), \{y^1, \dots, y^{m'}\})$ is the induced local coordinate system on M' , then the corresponding local expression for Z' coincides with (3.2.2).

PROOF. In order to prove the last assertion it is sufficient to note that for any two points $x_1, x_2 \in p^{-1}(x)$ there exist $X_1, \dots, X_s \in \Gamma(\ker p_*)$ and $t_1, \dots, t_s \in \mathbb{R}$ such that $\phi_{t_1}^{X_1} \circ \dots \circ \phi_{t_s}^{X_s}(x_1) = x_2$ and then use the second condition.

Obviously, (ii) \Rightarrow (i). To prove the converse we choose a vector bundle direct decomposition $TM = \ker p_* \oplus C$ such that $Z \in \Gamma(C)$ if $Z \notin \Gamma(\ker p_*)$ and C is arbitrary otherwise. Let $\Pi : \Gamma(TM) \rightarrow \Gamma(C)$ be a projection on $\Gamma(C)$ along $\Gamma(\ker p_*)$. Then

$$\frac{d}{dt} \Pi(\phi_{t*}^X Z - Z) = \Pi \frac{d}{dt} (\phi_{t*}^X Z - Z) = \Pi(-\phi_{t*}^X [X, Z]) = 0$$

(we have used the equality $\frac{d}{dt} \phi_{t*}^X Z = -\phi_{t*}^X [X, Z]$ and the fact that $[X, Z] = \mathcal{L}_X Z$, see [15]). Thus $\Pi(\phi_{t*}^X Z - Z)$ is a constant with respect to t multivector and, since $\Pi(\phi_{t*}^X Z - Z)|_{t=0} = \Pi(0) = 0$, we deduce that $\Pi(\phi_{t*}^X Z - Z) \equiv 0$.

The equivalence (i) \Leftrightarrow (iii) follows from the local expression

$$[X, Z]^{i_1 \dots i_k} = \frac{1}{k!} \epsilon_{s_1 \dots s_k}^{i_1 \dots i_k} X^r \partial_r Z^{s_1 \dots s_k} - \frac{1}{(k-1)!} \epsilon_{i s_2 \dots s_k}^{i_1 \dots i_k} Z^{r s_2 \dots s_k} \partial_r X^i \quad (3.2.3)$$

for the Schouten bracket ([15]). Indeed, if one applies (3.2.3) to the local coordinate system from condition (iii) one finds that $\mathcal{L}_X Z \in \Gamma(\ker \wedge^k p_*)$ if and only if (3.2.1) holds. q.e.d.

3.3. Definition We say that a multivector $Z \in \Gamma(\wedge^k TM)$ is projectable or admits the push-forward if one of the conditions of Theorem 3.2 is satisfied. The push-forward, which will be denoted by Z' , is the uniquely defined multivector from $\Gamma(\wedge^k TM')$, see Theorem 3.2.

3.4. Definition A complex multivector $Z \in \Gamma(\wedge^k T^{\mathbb{C}} M)$ admits the push-forward $Z' \in \Gamma(\wedge^k T^{\mathbb{C}} M')$ if the multivectors $\operatorname{Re} Z, \operatorname{Im} Z \in \Gamma(\wedge^k TM)$ do so. We put $Z' = (\operatorname{Re} Z)' + i(\operatorname{Im} Z)'$.

3.5. Corollary *Let c be a (complex) bivector on M admitting the push-forward $c' \in \Gamma(\wedge^2 TM')$ ($c' \in \Gamma(\wedge^2 T^{\mathbb{C}}M')$). Then for any $x' \in M'$ and any $x \in p^{-1}(x')$ the following conditions hold:*

- (i) *the subspace $p_{*,x}(c^{\sharp}((T_x\mathcal{K})^{\perp})) \subset T_{x'}M'$ ($p_{*,x}^{\mathbb{C}}(c^{\sharp}((T_x^{\mathbb{C}}\mathcal{K})^{\perp})) \subset T_{x'}^{\mathbb{C}}M'$), where \perp is the annihilator sign, is independent of x ;*
- (ii) *the kernel of the map $p_{*,x}|_{c^{\sharp}((T_x\mathcal{K})^{\perp})}$ ($p_{*,x}^{\mathbb{C}}|_{c^{\sharp}((T_x^{\mathbb{C}}\mathcal{K})^{\perp})}$) equals $c^{\sharp}((T_x\mathcal{K})^{\perp}) \cap T_x\mathcal{K}$ ($c^{\sharp}((T_x^{\mathbb{C}}\mathcal{K})^{\perp}) \cap T_x^{\mathbb{C}}\mathcal{K}$);*
- (iii) *the characteristic subspace of the push-forward can be described by the following isomorphism*

$$P_{c',x'} \cong c^{\sharp}((T_x\mathcal{K})^{\perp}) / (c^{\sharp}((T_x\mathcal{K})^{\perp}) \cap T_x\mathcal{K})$$

$$(P_{c',x'} \cong c^{\sharp}((T_x^{\mathbb{C}}\mathcal{K})^{\perp}) / (c^{\sharp}((T_x^{\mathbb{C}}\mathcal{K})^{\perp}) \cap T_x^{\mathbb{C}}\mathcal{K})).$$

PROOF. *iii)* follows from *i)* and *ii)*. These last are consequences of Theorem 3.2. q.e.d.

3.6. Remark Although $\dim c^{\sharp}((T_x\mathcal{K})^{\perp}) / c^{\sharp}((T_x\mathcal{K})^{\perp}) \cap T_x\mathcal{K}$ is constant along \mathcal{K} , dimensions of $c^{\sharp}((T_x\mathcal{K})^{\perp})$ and $c^{\sharp}((T_x\mathcal{K})^{\perp}) \cap T_x\mathcal{K}$ may not be so. For example, let $p : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the projection $(x_0, x_1, x_2, x_3) \mapsto (x_1, x_2, x_3)$ and let $c = x_0\partial_{x_0} \wedge \partial_{x_1} + \partial_{x_2} \wedge \partial_{x_3}$. Then c is projectable, but dimension of $c^{\sharp}((T_x\mathcal{K})^{\perp}) = \text{Span}\{\partial_{x_2}, \partial_{x_3}, x_0\partial_{x_0}\}$ jumps at 0.

3.7. Proposition *Let a bivector $Z_i \in \Gamma(\wedge^2 TM)$ admit the push-forward $Z'_i \in \Gamma(\wedge^2 TM')$, $i = 1, 2$. Then a trivector $Z = [Z_1, Z_2] \in \Gamma(\wedge^3 TM)$ admits the push-forward $Z' \in \Gamma(\wedge^3 TM')$ and $Z' = [Z'_1, Z'_2]$.*

PROOF. In any local coordinate system as in condition *(iii)* of Theorem 3.2 Z_i can be written in the form

$$Z_i(x, y) = Z'_i(y) + \tilde{Z}_i(x, y),$$

where $Z'_i(y) = Z'^{ijk}_i(y)\partial_{y^j} \wedge \partial_{y^k}$ and $\tilde{Z}_i \in \Gamma(\ker \wedge^2 p_*)(U)$. By formula (1.2.1)

$$[Z_1, Z_2](x, y) = [Z'_1, Z'_2](y) + \tilde{Z}(x, y),$$

where $\tilde{Z} \in \Gamma(\ker \wedge^3 p_*)(U)$. Thus by Theorem 3.2 Z admits the push-forward Z' and $Z' = [Z'_1, Z'_2]$. q.e.d.

3.8. Corollary *Let (c_1, c_2) be a Poisson pair on M such that c_i admits the push-forward $c'_i \in \Gamma(\wedge^2 TM')$, $i = 1, 2$, and c'_1, c'_2 are linearly independent. Then (c'_1, c'_2) is a Poisson pair on M' .*

PROOF follows immediately from Propositions 2.2 and 3.7. q.e.d.

3.9. Corollary *Let (M, ω) be a complex symplectic manifold and let G be a real Lie group acting on M by biholomorphic symplectomorphisms. Assume that M/G is a manifold. Then $c_1 = \text{Re}(\omega^{-1}), c_2 = \text{Im}(\omega^{-1})$ (see Example 2.7) admit the push-forwards $c'_1, c'_2 \in \Gamma(\wedge^2 T(M/G))$ and (c'_1, c'_2) is a Poisson pair on M/G , provided that c'_1, c'_2 are linearly independent.*

PROOF. It is sufficient to observe that: a) $\mathcal{L}_{X_j} c_i = 0, i = 1, 2$, for generators $X_1, \dots, X_l \in \Gamma TM$ of the G -action; b) an arbitrary vector $X \in \Gamma(\ker p_*)$, where $p : M \rightarrow M/G$ is a natural projection, is expressed as $X = a^j X_j$ for some $a^j \in \mathcal{E}(M)$ and $\mathcal{L}_X c_i = [a^j X_j, c_i] = [a^j, c_i] \wedge X_j \in \Gamma(\ker \wedge^2 p_*), i = 1, 2$ (we have used the standard properties of the Schouten bracket, see [15], p.454). q.e.d.

3.10. Definition *Let $p : M \rightarrow M'$ be as in 3.1. A bihamiltonian structure (J, c_1, c_2) on M is called projectable (via p) if the bivectors c_1, c_2 are so and their push-forwards c'_1, c'_2 are linearly independent or zero. The bihamiltonian structure generated by c'_1, c'_2 on M' will be denoted by J' and will be called the push-forward or reduction of J .*

3.11. Definition *Let $p : M \rightarrow M'$ be as in 3.1 and let J be a projectable bihamiltonian structure. We say that the triple (M, J, \mathcal{K}) is a realization of J' . If, moreover, J is (holomorphic) symplectic (Definitions 2.6, 2.9), we call (M, J, \mathcal{K}) (holomorphic) symplectic realization.*

4 From symplectic to complete

Let $p : M \rightarrow M'$ be as in 3.1 and let J be a projectable symplectic bihamiltonian structure on M with the push-forward J' . In this section we discuss some conditions on the triple (M, J, \mathcal{K}) that guarantee the completeness of J' .

In view of Corollary 3.5, (iii) and the definition of completeness (2.17) our considerations should be linear algebraic in essence.

4.1. So let V be a vector space over \mathbb{C} and let $c_1, c_2 \in \wedge^2 V$ be such that the bihamiltonian structure $J = \{c_\lambda\}_{\lambda \in \mathbb{C}^2}$, $c_\lambda = \lambda_1 c_1 + \lambda_2 c_2$, $\lambda = (\lambda_1, \lambda_2)$, where c_λ is considered as a constant complex bivector field, is symplectic (Definition 2.6). Also, let $K \subset V$ be a subspace such that the push-forwards $c'_1, c'_2 \in \wedge^2(V')$, where $V' = V/K$, (via the canonical projection $p : V \rightarrow V/K$) are linearly independent.

Set $R_0 = \max_{\lambda \in \mathbb{C}^2} \text{rank } c_\lambda$, $R'_0 = \max_{\lambda \in \mathbb{C}^2} \text{rank } c'_\lambda$, where $c'_\lambda = \lambda_1 c'_1 + \lambda_2 c'_2$, and

$$d_\lambda = \dim c_\lambda^\#(K^\perp) / (K \cap c_\lambda^\#(K^\perp)).$$

4.2. Proposition *The condition of completeness*

(**) $\text{rank } c'_\lambda = R'_0$ for any $\lambda \in \mathbb{C}^2 \setminus \{(0, 0)\}$

holds if and only if d_λ is independent of $\lambda \in \mathbb{C}^2 \setminus \{(0, 0)\}$.

PROOF. By Corollary 3.5, (iii) the space $c_\lambda^\#(K^\perp) / (K \cap c_\lambda^\#(K^\perp))$ is isomorphic to the characteristic subspace $P_{c'_\lambda} = (c'_\lambda)^\#(V')$ of the push-forward c'_λ . q.e.d.

Under some additional assumption one can characterize the condition of completeness (**) in terms of the subspace $K \cap c_\lambda^\#(K^\perp)$ itself.

4.3. Proposition *Let $\Lambda_1 = \text{Span}_{\mathbb{C}}\{\hat{\lambda}_1\}, \dots, \Lambda_s = \text{Span}_{\mathbb{C}}\{\hat{\lambda}_s\}$ be the complex lines in \mathbb{C}^2 on which $\text{rank } c_\lambda$ is less than maximal. Set $\Lambda = \bigcup_{i=1}^s \Lambda_i$ and $k_\lambda = \dim K \cap c_\lambda^\#(K^\perp)$. Assume that $K^\perp \cap \ker c_{\hat{\lambda}_i}^\# = \{0\}$, $i = 1, \dots, s$.*

Then $k_\lambda = \text{codim } P_{c'_\lambda}$, where $P_{c'_\lambda}$ is the characteristic subspace of c'_λ .

*Consequently, the condition (**) of Proposition 4.2 holds if and only if*

(i) $k_\lambda = k$ is constant over $\lambda \in \mathbb{C}^2 \setminus \Lambda$;

(ii) $k_{\hat{\lambda}_1} = \dots = k_{\hat{\lambda}_s} = k$.

PROOF. Since $c_\lambda, \lambda \notin \Lambda$ is nondegenerate, $\text{codim}(K + c_\lambda^\#(K^\perp)) = \dim(K \cap c_\lambda^\#(K^\perp))$. On the other hand, $\text{codim}(K + c_\lambda^\#(K^\perp)) = \text{codim } P_{c'_\lambda}$ for any c_λ . Thus $k_\lambda = \text{codim } P_{c'_\lambda}$ for $\lambda \notin \Lambda$.

Now, the condition $K^\perp \cap \ker c_{\hat{\lambda}_i}^\# = \{0\}$ implies the equalities $\dim c_{\hat{\lambda}_i}^\#(K^\perp) = \dim c_\lambda^\#(K^\perp)$, $i = 1, \dots, s$, where $\lambda \notin \Lambda$. Hence $\text{codim}(K + c_{\hat{\lambda}_i}^\#(K^\perp)) = \dim(K \cap c_{\hat{\lambda}_i}^\#(K^\perp))$ and $k_{\hat{\lambda}_i} = \text{codim } P_{\hat{\lambda}_i}$, $i = 1, \dots, s$. q.e.d.

The following theorem gives the necessary and sufficient conditions for the completeness of the reduction J' of a holomorphic symplectic bihamiltonian structure J under an additional assumption corresponding to that in Proposition 4.3. Namely, the foliation \mathcal{K} of the leaves of the projection p is supposed to be a generic CR -foliation.

Let $\hat{\lambda}_1 = (1, i)$, $\hat{\lambda}_2 = (1, -i)$ and let Λ denote the cross $\text{Span}_{\mathbb{C}}\{\hat{\lambda}_1\} \cup \text{Span}_{\mathbb{C}}\{\hat{\lambda}_2\} \subset \mathbb{C}^2$.

4.4. Theorem *Let (M, ω) be a complex symplectic manifold with the corresponding holomorphic symplectic bihamiltonian structure J (see Definition 2.9) and let $p : M \rightarrow M'$ be as in 3.1. Assume that the foliation \mathcal{K} is a generic CR -foliation on M and that $c = \omega^{-1}$ admits the push-forward $c' \in \Gamma(\wedge^2 T^{\mathbb{C}} M')$. For $x' \in M'$, $x \in p^{-1}(x')$, and $\lambda \in \mathbb{C}^2 \setminus \Lambda$ set*

$$\begin{aligned} k_{\lambda}^x &= \dim T_x^{\mathbb{C}} \mathcal{K} \cap (T_x^{\mathbb{C}} \mathcal{K})^{\perp \omega_{\lambda}(x)}, \\ k^x &= \dim T_x^{1,0} \mathcal{K} \cap (T_x^{1,0} \mathcal{K})^{\perp \omega(x)}, \end{aligned}$$

where $\omega_{\lambda} = (c_{\lambda})^{-1} = (\lambda_1 \text{Re } c + \lambda_2 \text{Im } c)^{-1}$. Assume that these numbers are constant along \mathcal{K} (cf. Remark 3.6) and set $k_{\lambda}^{x'} = k_{\lambda}^x$, $k^{x'} = k^x$.

Then $k_{\lambda}^{x'} = \text{codim}_{\mathbb{C}} P_{c', x'}$, $k^{x'} = \text{codim}_{\mathbb{C}} P_{c', x'} = \text{codim}_{\mathbb{C}} P_{\bar{c}', x'}$.

Consequently, the reduction J' of J via p is complete at a point $x' \in M'$ if and only if

(i) $k_{\lambda}^{x'} = k$ is constant in λ ;

(ii) $k^{x'} = k$

(iii) $k = \min_{y' \in M'} k_{\lambda}^{y'}$.

Here $\perp \omega_{\lambda}$, $\perp \omega$ denote the skew-orthogonal complements in $T^{\mathbb{C}} M$, $T^{1,0} M$ with respect to ω_{λ} , ω , correspondingly, i.e. $(T_x^{\mathbb{C}} \mathcal{K})^{\perp \omega_{\lambda}} = c_{\lambda}^{\#}((T_x^{\mathbb{C}} \mathcal{K})^{\perp})$, $(T_x^{1,0} \mathcal{K})^{\perp \omega} = c^{\#}((T_x^{1,0} \mathcal{K})^{\perp})$.

PROOF. If W is a real vector space with a complex structure \mathcal{J} and $Y \subset W$ a subspace, let $W^{1,0}$ denote the space $\{w - i\mathcal{J}w; w \in W\} \subset W^{\mathbb{C}}$ and let $Y^{1,0} = Y^{\mathbb{C}} \cap W^{1,0} = \{y - i\mathcal{J}y; y \in Y \cap \mathcal{J}Y\}$ (cf. 1.17).

Put $V = T_x^{\mathbb{C}} M$, $K = T_x^{\mathbb{C}} \mathcal{K}$. We claim that the assumptions of Proposition 4.3 are satisfied. Indeed, by Proposition 2.8 Λ is appropriate since the only, up to rescaling, degenerate bivectors from family J are c and \bar{c} . On the other hand the condition $T_x \mathcal{K} + \mathcal{J}T_x \mathcal{K} = T_x M$ of CR -genericity for \mathcal{K} implies that $(T_x \mathcal{K})^{\perp} \cap \mathcal{J}^*((T_x \mathcal{K})^{\perp}) = \{0\}$, where

$\mathcal{J}^* : T^*M \rightarrow T^*M$ is adjoint to the complex structure operator \mathcal{J} on TM . This means equalities $((T_x\mathcal{K})^\perp)^{1,0} = \{0\} = ((T_x\mathcal{K})^\perp)^{0,1}$ equivalent to $K^\perp \cap T_x^{1,0}M = 0 = K^\perp \cap T_x^{0,1}M$. Recalling that $T_x^{0,1}M = \ker c(x)$ and $T_x^{1,0}M = \ker \bar{c}(x)$ we get the claim.

Now, put $k_\lambda = k_{\hat{\lambda}}^{x'}$ and $k_{\hat{\lambda}_1} = k_{\hat{\lambda}_2} = k^{x'}$ and apply Proposition 4.3. Conditions (i) and (ii) are equivalent to the constancy of rank for $c_\lambda \in J'(x'), \lambda \neq 0$. Its maximality is guaranteed by (iii). q.e.d.

4.5. Corollary *In the assumptions of Theorem 4.4 suppose that \mathcal{K} is completely real (Definition 1.19). Then if J' is nontrivial it is not complete.*

PROOF. Assume the contrary. By condition (ii) corank of any $c' \in J' \setminus \{0\}$ is 0. This contradicts with the definition of completeness. q.e.d.

Given a complete bihamiltonian structure J' on M' , consider all its realizations with \mathcal{K} being a generic CR -foliation. Then the smallest realizations in this class will be characterized by the smallest difference $T^{1,0}\mathcal{K} \setminus T^{1,0}\mathcal{K} \cap (T^{1,0}\mathcal{K})^{\perp\omega}$.

4.6. Definition *Let J' be a complete bihamiltonian structure on M' . Its realization (M, ω) is called minimal if $T^{1,0}\mathcal{K} \cap (T^{1,0}\mathcal{K})^{\perp\omega} = T^{1,0}\mathcal{K}$, i.e. \mathcal{K} is a CR -isotropic foliation (Definition 1.20).*

We shall give another characterization of the minimal realizations below.

4.7. There is a natural CR -coisotropic foliation $\mathcal{L} \supset \mathcal{K}$ associated with any realization J on (M, ω) of a complete J' . This foliation is built as follows. Consider the "real form" $J'_\mathbb{R}$ of J' , i.e. the following real bihamiltonian structure on M' (cf. Proposition 2.5)

$$\{\lambda_1 \operatorname{Re} c' + \lambda_2 \operatorname{Im} c'\}_{\lambda=(\lambda_1, \lambda_2) \in \mathbb{R}^2},$$

where $c' = p_*c, c = (\omega)^{-1}$. Now take the bilagrangian foliation \mathcal{L}' of J' (see Definition 2.19). The equations for \mathcal{L}' are the functions from the involutive family F'_0 (see 2.10, 2.11). We define \mathcal{L} as $p^{-1}(\mathcal{L}')$. Note that it is CR -coisotropic due to the fact that its equations $f \in p^{-1}(F'_0)$ are in involution with respect to c .

4.8. Proposition *A realization J of a complete bihamiltonian structure J' is minimal if and only if the foliation \mathcal{L} is a CR -lagrangian foliation.*

PROOF. Let $2n, r$ denote rank and corank of the bivector $c' \in J'$, respectively, and let $\dim_{\mathbb{C}} M = 2N$. By Definition 4.6 and Theorem 4.4 J is minimal if and only if $r = d$, where d is CR -dimension of the leaves of \mathcal{K} . On the other hand, since \mathcal{K} is generic, CR -codimension of the leaves is equal to their real codimension, hence $2N - d = 2n + r$. Thus the minimality of J is equivalent to the equality $n + r = N$ that is necessary and sufficient for \mathcal{L} to be CR -lagrangian (see 1.20).

5 Canonical complex Poisson pair associated with complexification of Lie algebra

5.1. Let \mathfrak{g}_0 be a nonabelian Lie algebra over \mathbb{R} and let $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$ be its complexification. All our further results can be formulated and proved using $\mathfrak{g}_0, \mathfrak{g}$ only. But we introduce the corresponding Lie groups for the convenience.

So G_0 will stand for a connected simply connected Lie group with the Lie algebra $\text{Lie}(G_0) = \mathfrak{g}_0$ and G for a connected simply connected complex Lie group with $\text{Lie}(G) = \mathfrak{g}$. One can consider G_0 as a real Lie subgroup in G (see [4], III.6.10).

Write $\mathfrak{g}_0^*, \mathfrak{g}^*$ for the dual spaces. Fix a basis e_1, \dots, e_n in \mathfrak{g}_0 ; let c_{ij}^k be the corresponding structure constants and let $z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n$ be the complex linear coordinates in \mathfrak{g}^* associated to the dual basis in $\mathfrak{g}^* \supset \mathfrak{g}_0^*$. There are the standard linear bivectors $c = c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$ in \mathfrak{g}^* and $c_0 = c_{ij}^k z_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ in \mathfrak{g}_0^* . They can be defined intrinsically for instance as the maps

$$\mathfrak{g}^* \xrightarrow{c} \mathfrak{g}^* \wedge \mathfrak{g}^*, \quad \mathfrak{g}_0^* \xrightarrow{c} \mathfrak{g}_0^* \wedge \mathfrak{g}_0^*$$

dual to the Lie brackets $[\cdot, \cdot] : \mathfrak{g} \wedge \mathfrak{g} \longrightarrow \mathfrak{g}$ and $[\cdot, \cdot] : \mathfrak{g}_0 \wedge \mathfrak{g}_0 \longrightarrow \mathfrak{g}_0$.

It is well-known that the symplectic leaves of c_0 (respectively c) are the coadjoint orbits for G_0 (respectively G). Also, there is a natural (right) action of G_0 on \mathfrak{g}^* :

$$(a + ib)g = \text{Ad}^* g(a) + i \text{Ad}^* g(b), \quad g \in G_0, a, b \in \mathfrak{g}_0.$$

Let $\text{Sing } \mathfrak{g}^*$ be the union of symplectic leaves of nonmaximal dimension for c

5.2. Proposition *The set $\text{Sing } \mathfrak{g}^*$ is algebraic.*

PROOF. The defining polynomials for $\text{Sing } \mathfrak{g}^*$ are minors of m -th order of the $n \times n$ -matrix $\|c_{ij}^k z_k\|$, where $m = \text{rank } c$. q.e.d.

5.3. Convention In the sequel we shall assume that the nonabelian Lie algebra \mathfrak{g} satisfies condition $\text{codim}_{\mathbb{C}} \text{Sing } \mathfrak{g}^* \geq 3$.

5.4. This condition is satisfied by a wide class of Lie algebras including the semisimple ones. Indeed, in the semisimple case we can identify \mathfrak{g}^* and \mathfrak{g} by means of the Killing form. On the other hand, it is well known that the algebraic set of all nonregular (regular means semisimple contained in the unique Cartan subalgebra) elements is at least of codimension three and contains $\text{Sing } \mathfrak{g}^*$.

5.5. Definition *Let us introduce a set*

$$\mathcal{C} = \{z \in \mathfrak{g}^*; \exists(\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}, \lambda_1 z + \lambda_2 \bar{z} \in \text{Sing } \mathfrak{g}^*\},$$

where the bar stands for the complex conjugation corresponding to $\mathfrak{g}_0 \subset \mathfrak{g}$, and call it the incompleteness set (see 5.9 for the explanation of this terminology).

5.6. Proposition *The incompleteness set \mathcal{C} is a real algebraic set of positive codimension.*

PROOF. We use the product $\Pi = \mathfrak{g}^* \times (\mathbb{C}^2 \setminus \{(0, 0)\})$ with the coordinates $z_1, \dots, z_n, \lambda_1, \lambda_2$ and the real algebraic map $\phi : \Pi \longrightarrow \mathfrak{g}^*$ given by the formula

$$(z_1, \dots, z_n, \lambda_1, \lambda_2) \mapsto (\lambda_1 z_1 + \lambda_2 \bar{z}_1, \dots, \lambda_1 z_n + \lambda_2 \bar{z}_n).$$

The set \mathcal{C} can be regarded as $pr_1(\phi^{-1}(\text{Sing } \mathfrak{g}^*))$, where pr_1 is the projection onto \mathfrak{g}^* .

The above construction shows that $\dim_{\mathbb{R}} \mathcal{C} \leq \dim_{\mathbb{R}} \text{Sing } \mathfrak{g}^* + 4$ q.e.d.

5.7. Example Let $\mathfrak{g}_0 = so(3)$, $\mathfrak{g} = sl(2, \mathbb{C}) \cong \mathbb{C}^3$. Then $\text{Sing } \mathfrak{g}^* = \{0\}$, $\mathcal{C} = \{z \in \mathfrak{g}^*; z \text{ linearly independent with } \bar{z}\}$; consequently \mathcal{C} is described by two real equations: $z_1 \bar{z}_2 - z_2 \bar{z}_1 = 0, z_1 \bar{z}_3 - z_3 \bar{z}_1 = 0$. The set $\{z \in \mathfrak{g}^*; z \text{ linearly independent with } \bar{z}\}$ is contained in \mathcal{C} for arbitrary \mathfrak{g} .

5.8. Now, we shall introduce a remarkable pair of complex bivectors on \mathfrak{g}^* playing the crucial role in the sequel of the paper. This pair is (c, \tilde{c}) , where c is as in 5.1 and \tilde{c} is given by $\tilde{c} = c_{ij}^k \bar{z}_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$. One can define \tilde{c} intrinsically by the diagram

$$\begin{array}{ccc} \mathfrak{g}^* & \xrightarrow{c} & \mathfrak{g}^* \wedge \mathfrak{g}^* \\ \uparrow \bar{\cdot} & & \uparrow \tilde{c} \\ \mathfrak{g}^* & = & \mathfrak{g}^*, \end{array}$$

where c is from (5.1.1) and $\bar{\cdot}$ stands for the complex conjugation corresponding to the real form $\mathfrak{g}_0 \subset \mathfrak{g}$.

5.9. Proposition (i) \tilde{c} is G_0 -invariant;

(ii) (c, \tilde{c}) is a complex Poisson pair;

(iii) (c, \tilde{c}) is complete at any point $z \in \mathfrak{g}^* \setminus \mathcal{C}$ (see Definition 2.17).

PROOF. (i) follows from the G -invariance of the bivector c and G_0 -equivariance of $\bar{\cdot}$. (ii) is obtained by direct calculations. The last assertion follows from Proposition 5.6 since the set \mathcal{C} consists precisely of the points of incompleteness for (c, \tilde{c}) . Indeed, $\text{rank}(\lambda_1 c + \lambda_2 \tilde{c})(z) = \text{rank} c_{ij}^k (\lambda_1 z_k + \lambda_2 \bar{z}_k) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$ is less than maximal if and only if $\lambda_1 z + \lambda_2 \bar{z} \in \text{Sing } \mathfrak{g}^*$. q.e.d.

5.10. Definition The bihamiltonian structure generated by c, \tilde{c} will be denoted by \tilde{J} and will be called the canonical bihamiltonian structure.

The end of this section is devoted to the study of the first integrals (Definition 2.12) for the canonical bihamiltonian structure $(\tilde{J}, c, \tilde{c})$.

5.11. Definition Let $r = \text{corank } c$ (codimension of symplectic leaf of maximal dimension). Let us write $\text{rank } \mathfrak{g}$ for r and call this number the rank of \mathfrak{g} .

Note that for the semisimple case this notion of rank coincides with the standard one, i.e. with dimension of a Cartan subalgebra.

5.12. Definition Let $Z_c^{\text{hol}}(U)$ denote the space of holomorphic Casimir functions for c over an open set $U \subset \mathfrak{g}^*$.

An open set $U \subset \mathfrak{g}^* \setminus \text{Sing } \mathfrak{g}^*$ is called admissible if there exist $r = \text{rank } \mathfrak{g}$ functionally independent functions from $Z_c^{\text{hol}}(U)$.

5.13. Proposition *Let a set U be admissible. Given a function $g \in Z_c^{hol}(U)$, define a function $\tilde{g} \in \mathcal{E}^{\mathbb{C}}(U)$ by the formula $\tilde{g} = (\frac{\partial g}{\partial z_i})z_i$. Then the space $Z_{\tilde{c}}(U)$ of (smooth) Casimir functions for \tilde{c} is equal to $\{\tilde{g}; g \in Z_c^{hol}(U)\} \cup \overline{\mathcal{O}}(U)$, where $\overline{\mathcal{O}}(U)$ is the space of antiholomorphic functions over U .*

PROOF. The following calculation shows that $\tilde{g} \in Z_{\tilde{c}}(U)$:

$$\tilde{c}(\tilde{g})_j = c_{ij}^k \bar{z}_k \frac{\partial \tilde{g}}{\partial z_i} = c_{ij}^k \bar{z}_k \overline{\left(\frac{\partial g}{\partial z_i}\right)} = \overline{c(g)}_j = 0$$

(here v_j stands for the j -th component of a vector field $v = v_i \frac{\partial}{\partial z_i}$).

Now, let $g_1, \dots, g_r \in Z_c^{hol}(U)$ be functionally independent. We note that the $(1,0)$ -differentials $\partial \tilde{g}_1, \dots, \partial \tilde{g}_r$ are linearly independent precisely at those points where $\partial g_1, \dots, \partial g_r$ are. Thus by the dimension arguments ($\text{rank } \tilde{c} = \text{rank } c$) the functions $\tilde{g}_1, \dots, \tilde{g}_r$ together with the antiholomorphic functions functionally generate the space $Z_{\tilde{c}}(U)$. q.e.d.

5.14. Definition *Define $\phi_\lambda : \mathfrak{g}^* \rightarrow \mathfrak{g}^*, \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ by $\phi_\lambda(z) = \lambda_1 z + \lambda_2 \bar{z}$. This is a \mathbb{R} -linear isomorphism if $|\lambda_1| \neq |\lambda_2|$ and an epimorphism onto an n -dimensional ($n = \dim_{\mathbb{C}} \mathfrak{g}$) real subspace otherwise.*

An open set $U \subset \mathfrak{g}^ \setminus \mathcal{C}$ is called λ -admissible if the set $\phi_\lambda(U)$ has an admissible neighbourhood.*

An open set $U \subset \mathfrak{g}^ \setminus \mathcal{C}$ is called strongly admissible if it is λ -admissible for any $\lambda \in \mathbb{C}^2 \setminus \{(0,0)\}$.*

5.15. Proposition *Let a set $U \subset \mathfrak{g}^*$ be λ -admissible and let U_λ be an admissible neighbourhood of $\phi_\lambda(U)$.*

Then the space $Z_{c_\lambda}(U)$ of (smooth) Casimir functions for $c_\lambda = \lambda_1 c + \lambda_2 \tilde{c}, \lambda_1 \neq 0$, is equal to $\{g \circ \phi_\lambda|_U; g \in Z_c^{hol}(U_\lambda)\} \cup \overline{\mathcal{O}}(U)$.

PROOF. Again, let $g_1, \dots, g_r \in Z_c^{hol}(U_\lambda)$ be functionally independent. Obviously, the functions $g_{\lambda,1} = g_1 \circ \phi_\lambda, \dots, g_{\lambda,r} = g_r \circ \phi_\lambda$ are Casimir functions for c_λ . They are functionally independent on U since the Jacobi matrices $D = \frac{\partial(g_1, \dots, g_r)}{\partial(z_1, \dots, z_n)}$ and $D_\lambda = \frac{\partial(g_{\lambda,1}, \dots, g_{\lambda,r})}{\partial(z_1, \dots, z_n)}$ are related as follows

$$D_\lambda(z) = \lambda_1 D \circ \phi_\lambda(z).$$

So $g_{\lambda,1}, \dots, g_{\lambda,r}$ and $\overline{\mathcal{O}}(U)$ generate $Z_{c_\lambda}(U)$. q.e.d.

The following proposition shows that strongly admissible sets exist and describes all of them in the semisimple case.

- 5.16. Proposition** (i) *Let $\|\cdot\|$ be a norm in \mathfrak{g}^* . Then any open set $U \subset \mathfrak{g}^* \setminus \text{Sing } \mathfrak{g}^*$ with a sufficiently small diameter $\text{diam } U = \sup_{z, z' \in U} \|z - z'\|$ is strongly admissible.*
- (ii) *Assume that \mathfrak{g} is semisimple. Then any open set $U \subset \mathfrak{g}^* \setminus \mathcal{C}$ is strongly admissible.*

PROOF. (i) We start from the following claim: if a set U is admissible, then the set $\lambda U = \{\lambda u; u \in U\}$ is so for any $\lambda \in \mathbb{C} \setminus \{0\}$. Indeed, the bivector c is homogeneous with homogeneity degree 1: $h_{\lambda,*}c = \lambda c$, where $h_{\lambda}(z) = \lambda z$. Hence, if g_1, \dots, g_r are independent Casimir functions for c over U , then $(h_{\lambda}^*)^{-1}g_1, \dots, (h_{\lambda}^*)^{-1}g_r$ are so over $h_{\lambda}(U) = \lambda U$.

Now, assume that the norm is so chosen that $\|z\| = \|\bar{z}\|$. Then the inequality

$$\|\lambda_1 z + \lambda_2 \bar{z} - \lambda_1 z' - \lambda_2 \bar{z}'\| \leq |\lambda_1| \|z - z'\| + |\lambda_2| \|\bar{z} - \bar{z}'\|$$

shows that

$$\text{diam } \phi_{\lambda}(U) \leq (|\lambda_1| + |\lambda_2|) \text{diam } U, \quad (5.16.1)$$

where ϕ_{λ} is from Definition 5.14.

Next, choose a point $z \in U$ (note that z is linearly independent with \bar{z} , see Example 5.7) and consider the map $\mathbb{C}^2 \ni \lambda \mapsto \phi_{\lambda}(z) \in \mathfrak{g}^*$. The image of the unit sphere $S^1 = \{|\lambda_1|^2 + |\lambda_2|^2 = 1\}$ under this map can be covered by a finite number of admissible balls B_1, \dots, B_m . Inequality 5.16.1 shows that shrinking U if needed one can get the following

$$\phi_{\lambda}(U) \subset \cup_{i=1}^m B_i \quad \forall \lambda \in S^1.$$

Hence, for sufficiently small $U \ni z$ the set $\phi_{\lambda}(U)$, where $\lambda \in S^1$, possesses an admissible neighbourhood and by the above proved claim the same is true for $\phi_{\lambda}(U)$, $\lambda \neq 0$. Since all norms on \mathfrak{g}^* are equivalent this completes the proof.

(ii) It is enough to note that there exists a set $g_1(z), \dots, g_r(z)$, $r = \text{rank } \mathfrak{g}$ of global holomorphic Casimir functions for c that are functionally independent on $\mathfrak{g}^* \setminus \text{Sing } \mathfrak{g}^*$. One can identify \mathfrak{g} and \mathfrak{g}^* by means of the Killing form and take for g_1, \dots, g_r an algebraic basis of the ring of G -invariant polynomials on \mathfrak{g} . The functional independence of these functions on $\mathfrak{g}^* \setminus \text{Sing } \mathfrak{g}^*$ is established in Theorem 0.1 of [14]. q.e.d.

We summarize the above results in the following Proposition.

5.17. Proposition *Let $U \subset \mathfrak{g}^*$ be a strongly admissible set and let U_λ be an admissible neighbourhood of $\phi_\lambda(U)$. The set of first integrals (see Definition 2.12) $F_0(U)$ of $(\tilde{J}, c, \tilde{c})$ over U is (functionally) generated by the functions from the sets $F_1(U)$ and $\overline{\mathcal{O}}(U)$, where the last one is the set of antiholomorphic functions on U and $F_1(U)$ is in turn generated by $Z_c^{hol}(U)$, $\{\tilde{g}; g \in Z_c^{hol}(U)\}$, $\{g \circ \phi_\lambda|_U; g \in Z_c^{hol}(U_\lambda)\}$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, $\lambda_1, \lambda_2 \neq 0$.*

PROOF follows from Propositions 5.13, 5.15 and from the definition of the set $F_0(U)$. q.e.d.

The following proposition will be crucial in the proof of our main result (Theorem 7.1). As usual, given a subspace $V \subset (T_z^{\mathbb{C}} \mathfrak{g}^*)^*$, we set $V^{1,0} = V \cap (T_z^{1,0} \mathfrak{g}^*)^*$.

5.18. Proposition *Let*

$$\mu(z) = \dim(\ker c(z))^{1,0} \cap (\ker \tilde{c}(z))^{1,0}$$

and let

$$\mu_\lambda(z) = \dim(\ker c(z))^{1,0} \cap (\ker c_\lambda(z))^{1,0}$$

for $c_\lambda = \lambda_1 c + \lambda_2 \tilde{c}$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, $\lambda_1, \lambda_2 \neq 0$.

Then

- (i) $\mu(z) = \mu_\lambda(z)$ for any $z \in \mathfrak{g}^* \setminus \mathcal{C}$;
- (ii) *there exists a real algebraic set $\mathcal{R}, \mathcal{C} \subset \mathcal{R} \subset \mathfrak{g}^*$, where \mathcal{C} is the incompleteness set (see Definition 5.5), such that $\mu(z) = \mu$ is constant and minimal on $\mathfrak{g}^* \setminus \mathcal{R}$ and any set with these properties contains \mathcal{R} ;*
- (iii) *if \mathfrak{g} is semisimple the set $\mathcal{R} \setminus \mathcal{C}$ is empty and $\mu(z) \equiv 0$ on $\mathfrak{g}^* \setminus \mathcal{C}$.*

PROOF. (i) We shall use the completeness of the bihamiltonian structure (J, c, \tilde{c}) at any $z \in \mathfrak{g}^* \setminus \mathcal{C}$ (see proof of Proposition 5.9). By Theorem 2.22 and Corollary 2.23 the pair $c(z), \tilde{c}(z) \in \bigwedge^2 T_z^{1,0} \mathfrak{g}^*$ does not have the Jordan blocks in its decomposition. Thus we can use Proposition 2.26 to deduce that $\mu(z) = \mu_\lambda(z)$.

(ii) To prove this condition it is sufficient to note that the subspace $(\ker c(z))^{1,0} \cap (\ker \tilde{c}(z))^{1,0}$ annihilates the sum of characteristic subspaces $P_{c,z} + P_{\tilde{c},z}$. Put $\mathcal{R} = \{z \in \mathfrak{g}^*; \dim(P_{c,z} + P_{\tilde{c},z}) < m\}$, where $m =$

$\max_z \dim(P_{c,z} + P_{\tilde{c},z})$. The defining polynomials for \mathcal{R} are the minors of m -th order of the $2n \times n$ -matrix

$$\begin{vmatrix} c_{ij}^k z_k \\ c_{ij}^k \bar{z}_k \end{vmatrix}.$$

If the set \mathcal{R} defined above lies in \mathcal{C} , let us change the definition and put $\mathcal{R} = \mathcal{C}$.

It remains to prove the inclusion $\mathcal{C} \subset \mathcal{R}$ in the case $\mathcal{C} \not\subset \mathcal{R}, \mathcal{C} \neq \mathcal{R}$. Introduce a set $\mathcal{R}_\lambda = \{z \in \mathfrak{g}^*; \dim(P_{c,z} + P_{c_\lambda,z}) < m_\lambda\}$, where $m_\lambda = \max_z \dim(P_{c,z} + P_{c_\lambda,z}), \lambda_2 \neq 0$. Then by (i) $\mathcal{R}_\lambda, \lambda_2 \neq 0$ coincides with \mathcal{R} outside \mathcal{C} . Since $\mathcal{R} = Cl(\mathcal{R} \setminus \mathcal{C})$ and $\mathcal{R}_\lambda = Cl(\mathcal{R}_\lambda \setminus \mathcal{C})$ (Zariski closures), one gets $\mathcal{R} = \mathcal{R}_\lambda$.

Now, let $z \in \mathcal{C}$ and let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2, \lambda_2 \neq 0$ be such that $\text{rank } c_\lambda(z) < R_0$ (cf. the definition of completeness, 2.17). Then $\dim(P_{c,z} + P_{c_\lambda,z}) < m_\lambda$. Consequently, $\mathcal{C} \subset \mathcal{R}_\lambda = \mathcal{R}$. If the only (up to the proportionality) bivector of nonmaximal rank in the family $\{c_\lambda(z)\}$ is c , then $\dim(P_{c,z} + P_{\tilde{c},z}) < m$ and again $\mathcal{C} \subset \mathcal{R}$.

(iii) It follows from Proposition 2.26 that $\dim(\ker c(z))^{1,0} \cap (\ker \tilde{c}(z))^{1,0}$ is equal to dimension of the sum of the trivial Kronecker blocks for the pair $c(z), \tilde{c}(z)$. If we consider c, \tilde{c} as elements of $\Gamma(\wedge^2 T^{1,0} \mathfrak{g}^*)$, the same arguments as for the method of argument translation ([16], Theorem 4.1) show that in the semisimple case the trivial Kronecker blocks are absent for any $z \in \mathfrak{g}^* \setminus \mathcal{C}$. q.e.d.

5.19. Definition We call the set \mathcal{R} from Proposition 5.18 the *Kronecker irregularity set* and the number μ the *trivial Kronecker dimension* of the Lie algebra \mathfrak{g} .

The following example shows that for nonsemisimple Lie algebras the set $\mathcal{R} \setminus \mathcal{C}$ may be nonempty and the trivial Kronecker dimension may be nonzero.

5.20. Example Let $\mathfrak{g} = \text{Span}\{p_1, p_2, q_1, \dots, q_4, f_1, \dots, f_4, g_1, \dots, g_4\}$ be a fourteen-dimensional Lie algebra with the standard linear Poisson bivector $c = \frac{\partial}{\partial p_1} \wedge (f_1 \frac{\partial}{\partial q_1} + \dots + f_4 \frac{\partial}{\partial q_4}) + \frac{\partial}{\partial p_2} \wedge (g_1 \frac{\partial}{\partial q_1} + \dots + g_4 \frac{\partial}{\partial q_4})$. Then \mathcal{R} is given by one real equation

$$\begin{vmatrix} f_1 & f_2 & f_3 & f_4 \\ g_1 & g_2 & g_3 & g_4 \\ \bar{f}_1 & \bar{f}_2 & \bar{f}_3 & \bar{f}_4 \\ \bar{g}_1 & \bar{g}_2 & \bar{g}_3 & \bar{g}_4 \end{vmatrix} = 0.$$

The set $\text{Sing } \mathfrak{g}^*$ consists of the points where the vectors $f_1 \frac{\partial}{\partial q_1} + \dots + f_4 \frac{\partial}{\partial q_4}, g_1 \frac{\partial}{\partial q_1} + \dots + g_4 \frac{\partial}{\partial q_4}$ are linearly dependent, i.e. the defining equations for $\text{Sing } \mathfrak{g}^*$ are $f_1 g_2 - f_2 g_1 = 0, f_1 g_3 - f_3 g_1 = 0, f_1 g_4 - f_4 g_1 = 0$. However, the proof of Proposition 5.6 shows that $\text{codim}_{\mathbb{R}} \mathcal{C} \geq \text{codim}_{\mathbb{R}} \text{Sing } \mathfrak{g}^* - 4$; consequently, in our example $\text{codim}_{\mathbb{R}} \mathcal{C} \geq 6 - 4 = 2$ and $\mathcal{C} \neq \mathcal{R}$.

Here $\mu = 8$ since $f_1, \dots, f_4, g_1, \dots, g_4$ are the common Casimir functions for $c\tilde{c}$.

Also, μ will be nonzero for all reductive nonsemisimple Lie algebras.

Note that the above examples agree with our Convention 5.3.

6 CR -geometry of real coadjoint orbits

We retain the notations and conventions from the previous section. The reader is referred to Section 1 for the CR -geometric concepts used below.

6.1. Proposition (i) *The bivectors $c_1 = \text{Re } c, c_2 = \text{Im } c$ are Poisson.*

(ii) *The coadjoint action of $G_{\mathbb{R}}$, where $G_{\mathbb{R}}$ is G considered as a real Lie group, is hamiltonian with respect to c_1, c_2 (see Definition 1.22).*

(iii) *The generalized distribution of subspaces tangent to the G_0 -orbits is generated by the vector fields $c(z_i) + c(\bar{z}_i), i = 1, \dots, n$.*

PROOF. (i) The more general statement that the pair c_1, c_2 is Poisson is proved by the same arguments as in Example 2.7.

(ii) First, we shall prove that the holomorphic coadjoint action of G on \mathfrak{g}^* is hamiltonian in holomorphic sense with respect to c . Consider the antirepresentation $Ad^* : G \rightarrow \mathfrak{g}^*$. The corresponding Lie algebra action $\rho : \mathfrak{g} \rightarrow \text{Vect}^{hol}(\mathfrak{g}^*)$, where $\text{Vect}^{hol}(\mathfrak{g}^*)$ is the Lie algebra of holomorphic vector fields on \mathfrak{g}^* , can be described as follows. The vector field $\rho(v), v \in \mathfrak{g}$, is equal to

$$z \mapsto ad^*(v)z : \mathfrak{g}^* \rightarrow \mathfrak{g}^* \cong T_z^{1,0} \mathfrak{g}^*.$$

On the other hand, if $e_1, \dots, e_n, c_{ij}^k$ are as in 5.1, then

$$\langle ad^*(e_i)z, e_j \rangle = \langle z, ad(e_i)e_j \rangle = \langle z, [e_i, e_j] \rangle = \langle z, c_{ij}^k e_k \rangle = c_{ij}^k z_k,$$

where we put $z = z_1 e^1 + \dots + z_n e^n$. Hence, $\rho(e_i) = c_{ij}^k z_k \frac{\partial}{\partial z_j} = c(z_i)$ and the corresponding antihomomorphism $\psi : \mathfrak{g} \rightarrow \mathcal{O}(\mathfrak{g}^*)$ (cf. Definition 1.22) is defined by $e_i \mapsto z_i, i = 1, \dots, n$.

Now, the Lie algebra action $\rho_{\mathbb{R}} : \mathfrak{g}_{\mathbb{R}} \cong \mathfrak{g}_0 \oplus i\mathfrak{g}_0 \rightarrow \text{Vect}(\mathfrak{g}^*)$, where \cong is over \mathbb{R} , corresponding to the antirepresentation $Ad^* : G_{\mathbb{R}} \rightarrow \mathfrak{g}^*$ is described by the formulas

$$\begin{aligned} \langle ad^*(e_i)z, e_j \rangle &= c_{ij}^k x_k, \langle ad^*(ie_i)z, e_j \rangle = c_{ij}^k y_k \\ \langle ad^*(e_i)z, ie_j \rangle &= c_{ij}^k y_k, \langle ad^*(ie_i)z, ie_j \rangle = -c_{ij}^k x_k. \end{aligned}$$

Consequently,

$$\rho_{\mathbb{R}}(e_i) = c_{ij}^k x_k \frac{\partial}{\partial x_j} + c_{ij}^k y_k \frac{\partial}{\partial y_j} = (c + \bar{c})(z_i + \bar{z}_i) = (c - \bar{c})(z_i - \bar{z}_i)$$

and

$$\rho_{\mathbb{R}}(ie_i) = c_{ij}^k y_k \frac{\partial}{\partial x_j} - c_{ij}^k x_k \frac{\partial}{\partial y_j} = (c + \bar{c})(z_i - \bar{z}_i) = (c - \bar{c})(z_i + \bar{z}_i).$$

(iii) This condition follows from the proof of (ii) and from the obvious equality $(c + \bar{c})(z_i + \bar{z}_i) = c(z_i) + \overline{c(\bar{z}_i)}$. q.e.d.

6.2. Proposition *Let \mathcal{O} be a G_0 -orbit through $z_0 \in \mathfrak{g}^*$. Then \mathcal{O} is a generic CR-manifold in the G -orbit $G(z_0)$;*

PROOF. The G_0 -invariance of the complex structure \mathcal{J} on \mathfrak{g}^* implies the constancy of $\dim T_z \mathcal{O} \cap \mathcal{J}T_z \mathcal{O}$, $z \in \mathcal{O}$. To prove the genericity we note that the tangent bundle $T\mathcal{O}$ is generated by the vector fields $c(z_j) + \overline{c(z_j)}$, $j = 1, \dots, n$ (Proposition 6.1.(iii)), and that \mathcal{J} acts on them as follows

$$\mathcal{J}(c(z_j) + \overline{c(z_j)}) = i(c(z_j) - \overline{c(z_j)}). \quad (6.2.1)$$

Thus $T\mathcal{O} + \mathcal{J}T\mathcal{O}$ is generated by the real and imaginary parts of the vector fields $c(z_j)$, $j = 1, \dots, n$, spanning $T^{1,0}G(z_0)$. Hence $T_z \mathcal{O} + \mathcal{J}T_z \mathcal{O} = T_z^{\mathbb{C}}G(z_0)$. q.e.d.

The next proposition gives some characterization (another one can be found in 6.5) of CR-dimension of a G_0 -orbit.

6.3. Proposition *Let \mathcal{O} be a G_0 -orbit through $z_0 \in \mathfrak{g}^*$. Write G^z (respectively G_0^z) for the stabilizer of $z \in \mathfrak{g}^*$ in G (respectively G_0) and \mathfrak{g}^z (\mathfrak{g}_0^z) for the corresponding Lie algebra. Then for any $z \in \mathcal{O}$*

$$T_z^{1,0} \mathcal{O} \cong \mathfrak{g}^z / (\mathfrak{g}_0^z)^{\mathbb{C}}.$$

PROOF. We study the embedding of $T_z\mathcal{O}$ in the tangent space $T_zG(z_0)$ to a G -orbit $G(z_0)$. At the Lie algebra level it is equal to a map

$$\iota : \mathfrak{g}_0/\mathfrak{g}_0^z \longrightarrow \mathfrak{g}/\mathfrak{g}^z$$

induced by the inclusions $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}$, $\mathfrak{g}_0^z \hookrightarrow \mathfrak{g}^z$. The intersection $\iota(\mathfrak{g}_0/\mathfrak{g}_0^z) \cap \mathcal{J}\iota(\mathfrak{g}_0/\mathfrak{g}_0^z)$, where \mathcal{J} is the complex structure on $\mathfrak{g}/\mathfrak{g}^z$ induced by the multiplication by i , is equal to

$$\{s + \mathfrak{g}_0^z; s \in \mathfrak{g}_0, \exists t \in \mathfrak{g}_0, s - it \in \mathfrak{g}^z\}.$$

We define a map

$$\phi : \mathfrak{g}^z \longrightarrow \iota(\mathfrak{g}_0/\mathfrak{g}_0^z) \cap \mathcal{J}\iota(\mathfrak{g}_0/\mathfrak{g}_0^z) \cong T_z\mathcal{O} \bigcap \mathcal{J}T_z\mathcal{O}$$

by the formula

$$\mathfrak{g}^z \ni s - it \mapsto s + \mathfrak{g}^z,$$

where $s, t \in \mathfrak{g}_0$. The kernel of ϕ is equal to $\{s - it \in \mathfrak{g}^z; s \in \mathfrak{g}^z\} = \{s - it \in \mathfrak{g}^z; s \in \mathfrak{g}_0^z = \mathfrak{g}^z \cap \mathfrak{g}_0\}$. Since $ad^*s(x) = ad^*s(y) = 0$ for $s \in \mathfrak{g}_0^z, x = \operatorname{Re} z, y = \operatorname{Im} z$, one gets

$$\begin{aligned} 0 &= ad^*(s - it)(x + iy) = -iad^*(t)(x + iy) \Rightarrow \\ &ad^*(t)x = ad^*(t)y = 0 \Rightarrow t \in \mathfrak{g}_0^z. \end{aligned}$$

Thus $\ker \phi = (\mathfrak{g}_0^z)^\mathbb{C}$. The surjectivity of ϕ is obvious. q.e.d.

6.4. Proposition *Let \mathcal{O} be a G_0 -orbit through $z_0 \in \mathfrak{g}^* \setminus \operatorname{Sing} \mathfrak{g}^*$ and let $g_1, \dots, g_r \in Z_c^{hol}(U), r = \operatorname{rank} \mathfrak{g}$, be independent holomorphic Casimir functions for c in some neighbourhood $U \subset \mathfrak{g}^* \setminus \operatorname{Sing} \mathfrak{g}^*$ of z_0 . Then $T^{1,0}\mathcal{O}$ is generated over U by the vector fields $c(\tilde{g}_1), \dots, c(\tilde{g}_r)$ (see Proposition 5.13 for the notations).*

PROOF. By formula (6.2.1) $T\mathcal{O} \cap \mathcal{J}T\mathcal{O}$ is generated by linear combinations $\alpha^j(c(z_j) + \overline{c(z_j)}), \alpha^j \in \mathcal{E}(\mathfrak{g}^*)$, such that there exist $\beta^j \in \mathcal{E}(\mathfrak{g}^*)$ satisfying the equality

$$\alpha^j(c(z_j) + \overline{c(z_j)}) = \beta^j i(c(z_j) - \overline{c(z_j)}).$$

This implies

$$(\alpha^j + i\beta^j)c(z_j) + (\alpha^j - i\beta^j)\overline{c(z_j)} = 0$$

and

$$\gamma^j c(z_j) = 0, \tag{6.4.1}$$

where we put $\gamma^j = \alpha^j + i\beta^j$.

In order to calculate all vector functions $\gamma = (\gamma^1, \dots, \gamma^n)$ satisfying (6.4.1) one observes two facts. First, that the vector functions $\gamma_{(m)} = (\frac{\partial g_m}{\partial z_1}, \dots, \frac{\partial g_m}{\partial z_n})$, $m = 1, \dots, r$, satisfy (6.4.1). Second, the dimension arguments show that any $\gamma(z)$ for which (6.4.1) holds is a linear combination of $\gamma_{(1)}(z), \dots, \gamma_{(r)}(z)$ if $z \in U$.

In other words, $T^{CR}\mathcal{O}$ is generated by $(\frac{\partial g_m}{\partial z_j} + \frac{\partial \bar{g}_m}{\partial \bar{z}_j})(c(z_j) + \overline{c(z_j)})$, $i = 1, \dots, r$, and $T^{1,0}\mathcal{O}$ by

$$\begin{aligned} (\frac{\partial g_m}{\partial z_j} + \overline{(\frac{\partial g_m}{\partial z_j})})(c(z_j) + \overline{c(z_j)}) - i\mathcal{J}(\frac{\partial g_m}{\partial z_j} + \overline{(\frac{\partial g_m}{\partial z_j})})(c(z_j) + \overline{c(z_j)}) = \\ (\frac{\partial g_m}{\partial z_j} + \overline{(\frac{\partial g_m}{\partial z_j})})(c(z_j) + \overline{c(z_j)} + c(z_j) - \overline{c(z_j)}) = \\ 2c(g_m) + 2\overline{(\frac{\partial g_m}{\partial z_j})}c(z_j) = 2c(\tilde{g}_m). \end{aligned}$$

q.e.d.

In the next result we describe dimension and once more CR -dimension of generic G_0 -orbits.

6.5. Corollary *Let \mathcal{O} be any G_0 -orbit lying in the complement to the Kronecker irregularity set \mathcal{R} (see Definition 5.19), which is G_0 -invariant by Proposition 5.9. Then*

- (i) $\dim_{\mathbb{C}} T_z^{1,0}\mathcal{O}$, $z \in \mathcal{O}$, equals $r - \mu$, where $r = \text{rank } \mathfrak{g}$ and μ is the trivial Kronecker dimension; in particular, if \mathfrak{g} is semisimple, $\dim_{\mathbb{C}} T_z^{1,0}\mathcal{O} = r$;
- (ii) $\dim \mathcal{O} = n - \mu$, where $n = \dim \mathfrak{g}$.

PROOF. (i) follows immediately from Propositions 6.4, 5.13, and 5.18. (ii) is a consequence of (i) and Proposition 6.3 (since $\dim_{\mathbb{C}} \mathfrak{g}^z = r$). q.e.d.

The following corollary characterize generic G_0 -orbits in \mathfrak{g}^* from the symplectic point of view.

6.6. Corollary *Let \mathcal{O} be a G_0 -orbit through $z_0 \in \mathfrak{g}^* \setminus \text{Sing } \mathfrak{g}^*$. Then \mathcal{O} is a CR -isotropic submanifold in M (Definition 1.20).*

PROOF. First we shall show the G_0 -invariance of the functions $\tilde{g}_1, \dots, \tilde{g}_r$. We use the equality

$$c_{ij}^l \frac{\partial g_m}{\partial z_i} + c_{ij}^k z_k \frac{\partial^2 g_m}{\partial z_i \partial z_l} = 0 \quad (6.6.1)$$

obtained by the differentiation of the equality $c_{ij}^k z_k \frac{\partial g_m}{\partial z_i} = 0$ with respect to z_l . Conjugating (6.6.1) and multiplying by z_l one gets

$$\begin{aligned} 0 &= c_{ij}^l z_l \overline{\left(\frac{\partial g_m}{\partial z_i} \right)} + c_{ij}^k \bar{z}_k \overline{\left(\frac{\partial^2 g_m}{\partial z_i \partial z_l} \right)} z_l = \\ &= (c_{ij}^k z_k \frac{\partial}{\partial z_i} + c_{ij}^k \bar{z}_k \frac{\partial}{\partial \bar{z}_i}) \left[\overline{\left(\frac{\partial g_m}{\partial z_l} \right)} z_l \right] = \\ &= -(c(z_j) + \overline{c(z_j)}) \tilde{g}_m. \end{aligned}$$

This proves the claim.

Now, recall that $(T^{1,0}\mathcal{O})^{\perp\omega}$ is generated by the vector fields $c(f)$, where f runs through all G_0 -invariant functions. Thus by Proposition 6.4 $T^{1,0}\mathcal{O} \subset (T^{1,0}\mathcal{O})^{\perp\omega}$. q.e.d.

7 Main theorem: completeness of reductions for generic coadjoint orbits

We retain notations and conventions from two preceding sections.

7.1. Theorem *Let U be an open set in a G -orbit $M \subset \mathfrak{g}^*$ such that $U' = U/G_0$ is a smooth manifold and let $p : U \rightarrow U'$ be the canonical projection. Write J for the holomorphic symplectic bihamiltonian structure on M associated with the restriction of the standard holomorphic symplectic form $\omega = (c|_M)^{-1}$.*

Then the reduction J' (cf. Corollary 3.9 and Definition 3.10) of $J|_U$ via p is a complete bihamiltonian structure at any $z' \notin p(U \cap \mathcal{R})$, where \mathcal{R} is the Kronecker irregularity set (see Definition 5.19). In particular, J' is complete on U' if U is not contained in \mathcal{R} .

The realization J of J' is minimal (see Definition 4.6).

PROOF. We are going to use Theorem 4.4 and the notations from it.

The foliation \mathcal{K} of leaves of p is a generic CR -foliation due to Proposition 6.2. The numbers k_λ^z, k^z are constant in z if $z' = p(z)$ is fixed due

to the G_0 -invariance of all ingredients. Thus the assumptions of Theorem 4.4 are satisfied.

By Corollaries 6.5 and 6.6 the number $k^{z'} = \dim T_z^{1,0} \mathcal{K} \cap (T_z^{1,0} \mathcal{K})^{\perp \omega(z)}$, $z \in p^{-1}(z')$, equals $r - \mu$, where $r = \text{rank } \mathfrak{g}$ and μ is the trivial Kronecker dimension (see Definition 5.19), for any $z' \in U'$. We now shall prove that the number $k_\lambda^{z'} = \dim T^\mathbb{C} \mathcal{K} \cap (T^\mathbb{C} \mathcal{K})^{\perp \omega_\lambda(z)}$, $z \in p^{-1}(z')$ satisfies the inequality

$$k_\lambda^{z'} \geq k^{z'} \quad (7.1.1)$$

for any $z' \in M'$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \Lambda$.

For that purpose we shall put $c_\lambda = \lambda_1 \text{Re } c + \lambda_2 \text{Im } c$, $\omega_\lambda = (c_\lambda|_M)^{-1}$ and use the fact that $(T^\mathbb{C} \mathcal{K})^{\perp \omega_\lambda}$, is generated by the vector fields $c_\lambda(f)$, where f varies through the functions constant along \mathcal{K} . Given a point $z \in U \setminus \mathcal{R}$, we shall define r functions $g_{\lambda,1}, \dots, g_{\lambda,r}$ in a neighbourhood U_z of z such that the vector fields $c_\lambda(g_{\lambda,1}), \dots, c_\lambda(g_{\lambda,r})$ are tangent to \mathcal{K} and $r - \mu$ of them are independent on U_z .

Let us choose U_z to be strongly admissible (see Definition 5.14 and Proposition 5.16) and put

$$g_{\lambda,1} = g_1(\tilde{\lambda}_2 z + \tilde{\lambda}_1 \bar{z}), \dots, g_{\lambda,r} = g_r(\tilde{\lambda}_2 z + \tilde{\lambda}_1 \bar{z}),$$

where $(\tilde{\lambda}_1, \tilde{\lambda}_2) = (\frac{1}{2}(\lambda_1 - i\lambda_2), \frac{1}{2}(\lambda_1 + i\lambda_2)) \in \mathbb{C}^2$ is such that $c_\lambda = \tilde{\lambda}_1 c + \tilde{\lambda}_2 \bar{c}$ and g_1, \dots, g_r are functionally independent Casimir functions for c . These functions are functionally independent over U_z and their ∂ -differentials generate $(\ker(\tilde{\lambda}_2 c + \tilde{\lambda}_1 \bar{c}))^{1,0}$ (see Proposition 5.15). On the other hand, $\ker c_\lambda = \ker c$ and Proposition 5.18 implies that over U_z there are exactly $r - \mu$ independent among the vector fields $c_\lambda(g_{\lambda,1}), \dots, c_\lambda(g_{\lambda,r})$.

The nondegeneracy of c_λ implies the independence of the vector fields $c_\lambda(g_{\lambda,1}), \dots, c_\lambda(g_{\lambda,r})$ at $z \in M \setminus \mathcal{C}$. The following equalities show that these vector fields are tangent to \mathcal{K} ($T\mathcal{K}$ is spanned by $c(z_i) + \overline{c(z_i)}$, $i = 1, \dots, n$, see Proposition 6.1)

$$\begin{aligned} c_\lambda(g_{\lambda,m}) &= \tilde{\lambda}_1 c(g_{\lambda,m}) + \tilde{\lambda}_2 \bar{c}(g_{\lambda,m}) = \\ \tilde{\lambda}_1 \frac{\partial g_{\lambda,m}}{\partial z_i} c(z_i) + \tilde{\lambda}_2 \frac{\partial g_{\lambda,m}}{\partial \bar{z}_i} \bar{c}(\bar{z}_i) &= \tilde{\lambda}_1 \tilde{\lambda}_2 \frac{\partial g_m}{\partial z_i} |_{\tilde{\lambda}_2 z + \tilde{\lambda}_1 \bar{z}} (c(z_i) + \overline{c(z_i)}). \end{aligned}$$

Here we used the obvious identities

$$\frac{\partial g_{\lambda,m}}{\partial z_i} = \tilde{\lambda}_2 \frac{\partial g_m}{\partial z_i} |_{\tilde{\lambda}_2 z + \tilde{\lambda}_1 \bar{z}},$$

$$\frac{\partial g_{\lambda,m}}{\partial \bar{z}_i} = \tilde{\lambda}_1 \frac{\partial g_m}{\partial z_i} |_{\tilde{\lambda}_2 z + \tilde{\lambda}_1 \bar{z}}.$$

Thus we have proved (7.1.1) that is equivalent in view of Theorem 4.4 to the following

$$\text{rank } c'_\lambda(z') \leq \text{rank } c'(z'), z' \in U' \setminus p(\mathcal{R}).$$

By the lower semi-continuity of the function $f(\lambda) = \text{rank } c'_\lambda, \lambda \in \mathbb{C}^2$, this gives

$$\text{rank } c'_\lambda(z') = \text{rank } c'(z'), z' \in U' \setminus p(\mathcal{R}).$$

Thus we have obtained the constancy of $k_\lambda^{z'}$ in λ and the equality $k_\lambda^{z'} = k^{z'}$ for $z' \in U' \setminus p(\mathcal{R})$. Since this number is also independent of z' , condition (iii) of Theorem 4.4 is satisfied.

The minimality of the realization J for J' follows from Corollary 6.6. q.e.d.

We finish the paper by a characterization of the first integrals (see Definition 2.12) of the reduction J' . It turns out that they are intimately related with the first integrals of the canonical bihamiltonian structure \tilde{J} (see Section 5).

7.2. Proposition (i) *Let $U \subset \mathfrak{g}^*$ be a strongly admissible set (see Definition 5.14) and let $F_1(U)$ be as in Proposition 5.17. Then the functions from $F_1(U)$ are G_0 -invariant.*

(ii) *Retaining the hypotheses of Theorem 7.1 assume that U is strongly admissible. Then the set of first integrals $F'_0(U')$ of the bihamiltonian structure J' over U' is equal to the set $(F_1(U))'$ consisting of the elements of the set $F_1(U)$ regarded as functions on $U' = U/G_0$.*

PROOF. (i) The elements of $Z_c^{hol}(U)$ are G_0 -invariant by the definition, the G_0 -invariance of the functions from $\{\tilde{g}; g \in Z_c^{hol}(U)\}$ is established in the proof of Proposition 6.6. To prove it for the elements from $\{g \circ \phi_\lambda|_U; g \in Z_c^{hol}(U_\lambda)\}$, where U_λ is an admissible neighbourhood of $\phi_\lambda(U)$, it is sufficient to notice the G_0 -equivariance of the map ϕ_λ .

(ii) It follows from the proof of Theorem 7.1 that the mentioned there functions $g_{\lambda,1}, \dots, g_{\lambda,r} \in \mathcal{F}_1(U)$ (now one can put $U_z = U$) being reduced generate the space of the Casimir functions $Z_{c'_\lambda}$, where c'_λ is the reduction of the bivector $c_\lambda, \lambda \in \mathbb{C}^2 \setminus \Lambda$. Moreover, by the proof of Proposition 6.6 the functions $\tilde{g}_1, \dots, \tilde{g}_r$ generate the space $Z_{c'}$ after the reduction. q.e.d.

7.3. Corollary *Let $U \subset \mathfrak{g}^*$ be a strongly admissible set. Then the common level sets of functions from the family $\{\operatorname{Re} f, \operatorname{Im} f; f \in F_1(U)\}$ form a foliation on U that is a CR-lagrangian foliation (see Definition 1.20).*

PROOF. this foliation is the CR-lagrangian foliation associated with the minimal realization $(U, J|_U, \mathcal{K})$ of the complete bihamiltonian structure J' (see 4.7, 4.8). q.e.d.

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