Elements of a Global Operator Approach to Wess-Zumino-Novikov-Witten Models

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Abstract

Elements of a global operator approach to the WZNW models for compact Riemann surfaces of arbitrary genus g with N marked points were given by Schlichenmaier and Sheinman. This contribution reports on the results. The approach is based on the multi-point Krichever-Novikov algebras of global meromorphic functions and vector fields, and the global algebras of affine type and their representations. Using the global Sugawara construction and the identification of a certain subspace of the vector field algebra with the tangent space to the moduli space of the geometric data, Knizhnik-Zamalodchikov equations are defined. Some steps of the approach of Tsuchia, Ueno and Yamada to WZNW models are presented to compare it with our approach.

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1 Introduction

Wess-Zumino-Novikov-Witten (WZNW) models provide important examples of a two-dimensional conformal field theory. They can roughly be described as follows. The gauge algebra of the theory is the affine algebra associated to a finite-dimensional gauge algebra (i.e. a simple finite-dimensional Lie algebra). The geometric data consists of a compact Riemann surface (with complex structure) of genus g and a finite number of marked points on this surface. Starting from representations of the gauge algebra the space of conformal blocks can be defined. It depends on the geometric data. Varying the geometric data should yield a bundle over the moduli space of the

geometric data. In [8] Knizhnik and Zamolodchikov considered the case of genus 0 (i.e. the Riemann sphere). There, changing the geometric data consists in moving the marked points on the sphere. The space of conformal blocks could completely be found inside the part of the representation associated to the finite-dimensional gauge algebra. On this space an important set of equations, the Knizhnik-Zamolodchikov (KZ) equations, was introduced. In a general geometric setting, solutions are the flat sections of the bundle of conformal blocks over the moduli space with respect to the KZ connection.

For higher genus it is not possible to realize the space of conformal blocks inside the representation space associated to the finite-dimensional algebra. There exists different attacks to the generalization. Some of them add additional structure on these representation spaces (e.g. twists, representations of the fundamental group,..). Here I do not have the place to pay proper reference to all these approaches. Let me only give a few names: Bernard [1],[2], Felder and Wieczerkowski [3], [4], Hitchin [5], and Ivanov [6].

An important approach very much in the spirit of the original Knizhnik-Zamolodchikov approach was given by Tsuchia, Ueno and Yamada [21]. In Section 2 I will present a very short outline of their theory. The main point in their approach is that at the marked points, after choosing local coordinates, local constructions are done. In this setting the well-developed theory of representations of the traditional affine Lie algebras (Kac-Moody algebras of affine type) can be used. It appears a mixture between local and global objects and considerable effort is necessary to extend the local constructions to global ones.

Oleg Sheinman and myself propose a different approach to the WZNW models which uses consequently only global objects. These objects are the Krichever-Novikov (KN) algebras [9] and their representations, respectively their multi-point generalizations given by me [11]. An outline of this approach is presented in Section 3. In the remaining sections more details on the construction are given. Here I only want to point out that a subspace of the KN algebra of vector fields is identified with tangent directions on the moduli space of the geometric data. Conformal blocks can be defined. We are able to incorporate a richer theory because in our set-up we are able to deal with more general representations of the global algebras.

Finally, with the help of the global Sugawara construction for the higher genus and multi-point situation proven in [14], it is possible to define the higher genus multi-point Knizhnik-Zamolodchikov equations (see Definition 7.3). The complete proofs of the results appeared in [15]. A detailed study of the KZ equations, resp. of the connection is work in progress [16].

2   Outline of the Tsuchya-Ueno-Yamada approach

Let me first recall a few steps of the approach of Tsuchia, Ueno and Yamada to the WZNW models. I will concentrate on the steps which are of relevance in our approach.

More details can be found in [21], resp. in the more pedagogical introduction [22].

(1) A finite-dimensional complex simple Lie algebra $\mathfrak{g}$ with $(.|.)$ a symmetric, nondegenerate, invariant bilinear form (e.g. the Killing form) is fixed. This Lie algebra is the finite-dimensional gauge algebra of the theory. Some authors prefer to call this algebra the horizontal algebra.

(2) Fix a natural number N and a compact Riemann surface M of arbitrary genus g (resp. a smooth projective curve over $\mathbb{C}$). Take a tuple of N distinct points $(P_1, P_2, \ldots, P_N)$ on M and around every such point P_i a coordinate ξ_i. This defines the geometric data

$$\Upsilon' = (M, (P_1, P_2, \ldots, P_N), (\xi_1, \xi_2, \ldots, \xi_N)) . \tag{1}$$

(3) The affine algebra (or Kac-Moody algebra of affine type) associated to $\mathfrak{g}$ is given as

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}\left[t^{-1}, t\right] \oplus \mathbb{C}\, c \tag{2}$$

with Lie structure

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + (x|y)\, n \cdot \delta_m^{-n}\, c, \qquad \forall n, m \in \mathbb{Z}\,, \tag{3}$$

$$[x \otimes t^n, c] = 0, \qquad \forall n \in \mathbb{Z}\,. \tag{4}$$

Note that in this approach one has to consider Laurent series instead of the usual Laurent polynomials. In our approach we will return to Laurent polynomials. By ignoring the central element c one sees that $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}\left[t^{-1}, t\right]$ carries also a Lie algebra structure. This algebra is called the loop algebra (associated to $\mathfrak{g}$). There is a well-defined theory of highest weight representations H_λ of the affine algebra $\widehat{\mathfrak{g}}$ associated to a level and certain weights λ of the finite dimensional Lie algebra $\mathfrak{g}$, see [7]. Recall that the central element c operates as level$\times id$ on H_λ.

Another Lie algebra appearing in this context is the Virasoro algebra V which is the Lie algebra with basis $\{l_n, n \in \mathbb{Z}\} \cup \{c_1\}$ and Lie structure

$$[l_n, l_m] = (m - n)\, l_{n+m} + \delta_m^{-n} \frac{n^3 - n}{12}\, c_1, \qquad \forall n, m \in \mathbb{Z}\,, \tag{5}$$

$$[l_n, c_1] = 0, \qquad \forall n \in \mathbb{Z}\,. \tag{6}$$

Starting from a highest weight representation of the affine algebra the Sugawara construction defines also a representation of the Virasoro algebra on the representation space. Further down I will describe the Sugawara construction in a general geometric setting. This setting will incorporate also the classical Sugawara construction. Note that in the theory a graded structure is implicitly given and is employed. For example the Virasoro algebra becomes a graded affine Lie algebra by defining $\deg(l_n) := n$ and $\deg(c_1) := 0$.

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(4) After fixing the level globally, one assigns to every marked point P_i a heighest weight representation H_{λ_i} of the gauge algebra. Set $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_N)$ and

$$H_\lambda := H_{\lambda_1} \otimes H_{\lambda_2} \cdots \otimes H_{\lambda_N} . \tag{7}$$

The space H_λ is a representation space for the Lie algebra $\widehat{\mathfrak{g}_{(N)}}$, where this algebra is defined as the one-dimensional central extension of N copies of the loop algebra. Here the i-th copy of the loop algebra operates on the i-th factor in the tensor product and is associated to the i-th point P_i. To distinguish the different copies we use t_i for the affine parameter t in the loop algebra.

(5) Up to now the geometry was not really involved. This changes in this step. The affine parameter t_i corresponding formally to P_i via the assignment of the representation H_{λ_i} to this point, will be identified with the coordinate ξ_i at this point. One considers the algebra $A(\Upsilon')$ of meromorphic functions on M which have poles at most at the points $\{P_1, P_2, \ldots, P_N\}$, and sets $\mathfrak{g}(\Upsilon') := \mathfrak{g} \otimes A(\Upsilon')$. This algebra is also called the block algebra. By taking the Laurent expansion of $f \in A(\Upsilon')$ at the point P_i with respect to the coordinate ξ_i there, we get an embedding $\mathfrak{g}(\Upsilon') \to \overline{\mathfrak{g}}_N$ by assigning to f the corresponding Laurent series in the affine parameter t_i in the i-th copy of $\overline{\mathfrak{g}}$. The cocycle defining the central extension vanishes on $\mathfrak{g}(\Upsilon')$. Hence it can be considered as a subalgebra of $\widehat{\mathfrak{g}_{(N)}}$.

(6) The space of conformal blocks (also called chiral blocks) are defined as the coinvariants

$$V_\lambda := H_\lambda / \mathfrak{g}(\Upsilon') H_\lambda . \tag{8}$$

In some context is is better to work with the dual objects $V_\lambda^* = \mathrm{Hom}_{\mathbb{C}}(V_\lambda, \mathbb{C})$. This space can be described as the space of linear forms on H_λ vanishing on $\mathfrak{g}(\Upsilon') H_\lambda$. The vector spaces V_λ turn out to be finite-dimensional. Their dimension is given by the Verlinde formula.

(7) One of the motivations of [21] was to supply a proof of the Verlinde formula. For this the authors pass to the moduli space $\mathcal{M}_{g,N}^\infty$ of the data Υ' (with the obvious identifications under isomorphisms). Clearly, not only the goal to proof the Verlinde formula forces one to consider moduli spaces but also the concept of quantization requires to consider all possible configurations. Now everything has to be sheavified. One obtains the sheaf of conformal blocks over moduli space. On this sheaf the Knizhnik-Zamolodchikov connection is constructed. The sheaf is indeed a vector bundle, hence the dimension of the conformal blocks will be constant along the moduli space, only depending on the genus g, the number N of marked points, and the associated weights λ (of course the Lie algebra $\mathfrak{g}$ and the level will be fixed). The construction of the connection involves the Sugawara construction which is done locally on the Riemann surface. By introducing a projective connection on the Riemann surface it is possible to show that it globalizes.

(8) It is shown that the essential data can be given in terms of the moduli space $\mathcal{M}_{g,N}$ of smooth projective curves of genus g with N marked points, by forgetting the coordinate systems at the marked points.

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(9) Finally, by passing to the Deligne-Mumford-Knudsen boundary of $\mathcal{M}_{g,N}$ corresponding to stable singular curves and by proving factorization rules (e.g. behaviour of the conformal blocks under normalization of the singular curves) it is possible to express the dimension of the space of conformal blocks for the (g, N, λ) situation in dimensions of spaces of conformal blocks for lower genera.

3 Outline of the global operator approach

The approach presented in Section 2 is very successful and a beautiful piece of mathematics. Nevertheless it has some problems.

(a) It is necessary to choose coordinates around the points P_i. The authors have to work over infinite-dimensional moduli spaces. Finally, a complicated reduction process is needed to reduce the relevant data to the moduli space of curves with marked points. Note that by Tsuchimoto [20] some simplifications with respect to this problem has been given.

(b) The proof that the local Sugawara construction globalizes (with the help of a projective connection) on the Riemann surface is difficult.

(c) Some objects of the theory are only defined locally, other objects globally.

(d) The representation H_λ of the affine algebra $\widehat{\mathfrak{g}_{(N)}}$ does not see the geometry.

We (Oleg Sheinman and myself) propose a different approach to the WZNW models which still stays in the algebraic-geometric setup. In this section I will outline what has been done so far. Some more details will be given in the following sections. First note that the geometric data we consider consist of a compact Riemann surface M (or a smooth projective curve over $\mathbb{C}$ – I will use the terms interchangeable) and N (distinct) marked points:

$$\Upsilon = (M, (P_1, P_2, \ldots, P_N)) . \tag{9}$$

In contrast to the data (1) it does not contain coordinates. Again, we fix a simple finite-dimensional complex Lie algebra $\mathfrak{g}$.

(1) We replace all local algebras by algebras of Krichever-Novikov(KN)-type and their multi-point generalizations. They consists of meromorphic objects which might have poles only at the points $\{P_1, P_2, \ldots, P_N\}$ and a fixed reference point P_∞. This can be done for the function algebra, the affine algebra, the vector field algebra, the differential operator algebra. Central extensions can be defined. They carry an almost-graded structure (see the definition below) induced by the vanishing order at $\{P_1, P_2, \ldots, P_N\}$, resp. at P_∞. In particular, all these algebras can be decomposed into subalgebras of elements of positive degree corresponding roughly to elements holomorphic at $\{P_1, P_2, \ldots, P_N\}$ and subalgebras of elements of negative degree corresponding roughly to elements holomorphic at P_∞ and a "critical" finite-dimensional subspace spanned by other elements.

(2) In the next step we consider highest weight representations of the higher genus affine algebras. Examples are given again by representations $\widehat{W}_\lambda$ which are constructed from weights assigned to the marked points. For highest weight representations the Sugawara construction works and gives a representation of the centrally extended vector field algebra (see [14], [13]).

(3) The objects make direct sense over a dense subset of $\mathcal{M}_{g,N}$. We obtain sheaf versions of our objects.

(4) Note that the block algebra $\mathfrak{g}(\Upsilon)$ considered above is naturally a subalgebra of our higher genus multi-point affine algebra. There is no need to choose coordinates. It is possible to define again

$$\widehat{V}_\lambda := \widehat{W}_\lambda / \mathfrak{g}(\Upsilon)\widehat{W}_\lambda \ . \tag{10}$$

as the space of conformal blocks.

(5) At a fixed point b of the moduli space $\mathcal{M}_{g,N}$ the basis elements l_k of the "critical subspace" of the vector field algebra can be identified with the tangent directions X_k to the moduli space at b. Additionally the elements operate as Sugawara operators $T[l_k]$ on the representations $\widehat{W}_\lambda$ defined above b. This allows us to define the formal Knizhnik-Zamolodchikov(KZ) equations to be

$$(\partial_k + T[l_k])\, \Phi = 0, \qquad k = 1, \ldots, 3g - 3 + N \ . \tag{11}$$

Here Φ is assumed to be a section of $\widehat{W}_\lambda$, resp. of $\widehat{V}_\lambda$, and ∂_k is an action of X_k on $\widehat{W}_\lambda$ which fulfills suitable conditions.

In the following sections I will give additional information on these steps. Details appeared in [15]. Further work is in progress [16]. In particular what has to be done is to construct a projectively flat connection on the sheaf of conformal blocks and to extend the construction to the whole moduli space. Especially it should be extended its boundary to obtain again a proof of the Verlinde formula.

4 The Krichever-Novikov objects

Let M be a compact Riemann surface of genus g. Let A be a fixed set of finitely many points on M which is splitted into two non-empty subsets I and O. This is the general situation dealt with in [11]. Here it is enough to consider

$$I := \{P_1, P_2, \ldots, P_N\}, \qquad O := \{P_\infty\} \ . \tag{12}$$

The classical (Virasoro) situation is: $M = \mathbb{P}^1$ with $I = \{z = 0\}$ and $O = \{z = \infty\}$. The Krichever-Novikov situation [9] is: M an arbitrary Riemann surface with $I = \{P_+\}$ and $O = \{P_-\}$.

Let $\mathcal{K}$ be the canonical bundle, i.e. the bundle whose local sections are the holomorphic 1-differentials, and $\mathcal{K}^\lambda$ (for $\lambda \in \mathbb{Z}$) its tensor powers, resp. for λ negative,

the tensor power of its dual. Denote by $F^\lambda(A)$ the space of meromorphic sections of $\mathcal{K}^\lambda$ consisting of those elements which are holomorphic outside of A. We set $\mathcal{A}(A) := F^0(A)$ for the associative algebra of functions and $\mathcal{L}(A) := F^{-1}(A)$ for the Lie algebra of vector fields (with the usual Lie bracket as Lie structure). If the set A is clear from the context, we will drop it in the notation. By multiplication with the elements of $\mathcal{A}$ the space F^λ becomes a module over $\mathcal{A}$. By taking the Lie derivative with respect to the vector fields it becomes also a Lie module over $\mathcal{L}$.

Theorem 4.1. *([11], [10]) There exists a decomposition*

$$F^\lambda = \bigoplus_{n \in \mathbb{Z}} F_n^\lambda \ , \tag{13}$$

where the F_n^λ are subspaces of dimension N. By defining the elements of F_n^λ to be the homogeneous elements of degree n, the algebras $\mathcal{A}$ and $\mathcal{L}$ are almost-graded (Lie) algebras and the F_n^λ are almost-graded (Lie) modules over $\mathcal{A}$, respectively over $\mathcal{L}$.

For the convenience of the reader let me recall the definition of an almost-grading at the example of the Lie algebra $\mathcal{L}$. A Lie algebra $\mathcal{L}$ is called almost-graded if it can be decomposed (as vector space)

$$\mathcal{L} = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n, \quad \dim \mathcal{L}_n < \infty \ , \tag{14}$$

such that there exists $L, K \in \mathbb{Z}$ with

$$[\mathcal{L}_n, \mathcal{L}_m] \quad \subseteq \quad \bigoplus_{h=n+m-L}^{n+m+K} \mathcal{L}_h, \quad \forall n, m \in \mathbb{Z} \ . \tag{15}$$

This definition (suitable modified) works for the associative algebra $\mathcal{A}$ and the modules F^λ. In our situation we always can do with lower shift $L = 0$.

The almost-grading is important to obtain a decomposition of the algebras into positive and negative parts and to define the concept of highest weight representations (see below). It is also necessary to obtain an embedding of the algebras $\mathcal{A}$ and $\mathcal{L}$ (and more generally also of $\mathcal{D}$, the Lie algebra of differential operators, see [12]) into $\overline{gl}(\infty)$ via their action on the modules F^λ. Here $\overline{gl}(\infty)$ denotes the algebra of (both-sided) infinite matrices with finitely many diagonals.

The degree is introduced by the order of $f \in F^\lambda$ at I. More precisely, F_n^λ is given by exhibiting a basis $f_{n,p}^\lambda$, $p = 1, \ldots, N$. For the generic situation ($\lambda \neq 0, 1$, $g \neq 1$, and a generic choice of A) the element is fixed up to multiplication with a scalar by

$$\operatorname{ord}_{P_i}(f_{n,p}^\lambda) = \begin{cases} n - \lambda, & i = p \\ n - \lambda + 1, & i \neq p, i = 1, \ldots, N \\ -N(n+1-\lambda) + (2\lambda - 1)(g-1), & i = \infty \ . \end{cases} \tag{16}$$

For the non-generic situation for finitely many n the prescription at $i = \infty$ has to be adjusted. A detailed prescription is given in [11] and [10]. To fix the element $f_{n,p}^\lambda$ uniquely it is necessary to fix a coordinate ξ_i centered at P_i, for $i = 1, \ldots, N$ and to require $f_{n,p}^\lambda(\xi_p)_| = \xi_p^{n-\lambda}(1 + O(\xi_p))d\xi_p^\lambda$. Note that the fixing does not really depend on the full information in the coordinate. It depends only on the first infinitesimal neighbourhood. In detail: Let $\xi_p' = a_1 \xi_p + \sum_{j \geq 2} a_j \xi_p^j$ (clearly with $a_1 \neq 0$) be another coordinate centered at P_p then the normalization will only depend on the value a_1.

We have the important duality

$$F^\lambda \times F^{1-\lambda} \to \mathbb{C}, \quad \langle f, g \rangle = \frac{1}{2\pi\,\mathrm{i}} \int_C f \cdot g = -\operatorname{res}_{P_\infty}(f \cdot g), \tag{17}$$

where C is any separating cycle for $(P_1, P_2, \ldots, P_N)$ and P_∞ which is cohomologous to (-1) $\times$ the circle around P_∞. For the scaled basis elements we obtain

$$\langle f_{n,p}^\lambda, f_{m,r}^{1-\lambda} \rangle = \delta_n^{-m}\delta_p^r . \tag{18}$$

Again note that the duality relation (18) will not depend on the coordinates chosen. For special values of λ we set

$$A_{n,p} := f_{n,p}^0, \quad e_{n,p} := f_{n,p}^{-1}, \quad \omega^{n,p} := f_{-n,p}^1, \quad \Omega^{n,p} := f_{-n,p}^2. \tag{19}$$

In particular $(A_{n,p}, \omega^{n,p})$ and $(e_{n,p}, \Omega^{n,p})$ are dual systems of basis elements.

Next we decompose our algebras. First we consider $\mathcal{A}$, the algebra of functions:

$$\mathcal{A} = \mathcal{A}_-' \oplus \mathcal{A}_{(0)}' \oplus \mathcal{A}_+,$$

$$\mathcal{A}_-' = \bigoplus_{n \leq -K-1} \mathcal{A}_n, \quad \mathcal{A}_+ = \bigoplus_{n \geq 1} \mathcal{A}_n, \quad \mathcal{A}_{(0)}' = \langle A_{n,p} \mid 1 \leq p \leq N, \; -K \leq n \leq 0 \rangle .$$

Here $\mathcal{A}_n$ is the subspace of $\mathcal{A}$ consisting of the elements of degree n and K is the constant appearing in the definition of the almost-grading (15) for the algebra $\mathcal{A}$. The constant depends on the genus g and the number of points N. From the almost-grading it follows that $\mathcal{A}_+$ and $\mathcal{A}_-'$ are subalgebras. In general $\mathcal{A}_{(0)}'$ is only a subspace. For our purpose here it is more convenient to take as $\mathcal{A}_- \supseteq \mathcal{A}_-'$ the subalgebra of meromorphic functions vanishing at P_∞, make a change of basis in the lowest degree part $\mathcal{A}_{-K}$ in $\mathcal{A}_{(0)}$, and take $\mathcal{A}_{(0)} \subseteq \mathcal{A}_{(0)}'$ correspondingly smaller. This yields the decomposition

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_{(0)} \oplus \mathcal{A}_+ . \tag{20}$$

A completely analogous decomposition exists for the vector field algebra $\mathcal{L}$

$$\mathcal{L} = \mathcal{L}_- \oplus \mathcal{L}_{(0)} \oplus \mathcal{L}_+ . \tag{21}$$

The vector fields in $\mathcal{L}_+$ are vanishing of order 2 at the points $\{P_1, P_2, \ldots, P_N\}$, and the vector fields in $\mathcal{L}_-$ are vanishing of order 2 at the point P_∞. The space $\mathcal{L}_{(0)}$ is $(3g - 3 + 2N + 2)$-dimensional. We call it the "critical strip".

For further reference let me identify the basis elements of the critical strip. First we have N elements $e_{0,p}$, $p = 1, \ldots, N$ vanishing at all P_i, $i = 1, \ldots, N$, but not vanishing of 2nd order at every point. Next we have N elements $e_{-1,p}$, $p = 1, \ldots, N$ regular at all P_i, $i = 1, \ldots, N$, but not vanishing at every point. Later we will see that they will be responsible for moving the points P_p, $p = 1, \ldots, N$. In the middle we have $3g - 3$ elements which are neither regular at $\{P_1, P_2, \ldots, P_N\}$ nor at P_∞. They will correspond to deformations of the complex structure of M. Finally we have two elements regular of order 1, resp. of order 0 at P_∞.

For the algebras we can construct central extensions $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{L}}_R$ defined via the following geometric cocycles

$$\mathcal{A}: \qquad \gamma(g, h) := \frac{1}{2\pi \, \mathrm{i}} \int_C g \, dh, \tag{22}$$

$$\mathcal{L}: \qquad \chi_R(E, f) := \frac{1}{24\pi \, \mathrm{i}} \int_C \left(\frac{1}{2} \left(e''' f - e f''' \right) - R \left(e' f - e f' \right) \right) dz . \tag{23}$$

Here R is a projective connection on M. The cocycles are local in the sense that there exists T and S such that for all $n, m \in \mathbb{Z}$: $\gamma(\mathcal{A}_n, \mathcal{A}_m) \neq 0 \implies T \leq m + n \leq 0$ and $\chi_R(\mathcal{L}_n, \mathcal{L}_m) \neq 0 \implies S \leq m + n \leq 0$. Again the T and S can be explicitly calculated [10]. Restricted to the $(+)$ and $(-)$ subalgebras in (20) and (21) the cocycles are vanishing. In particular, the subalgebras can be considered in a natural way as subalgebras of the central extensions $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{L}}_R$. By setting $\deg(t) := 0$ for the central elements the almost-grading can be extended to the central extensions.

Different connections R yield cohomologous cocycles χ_R, hence equivalent central extensions (see [10] for details). In the classical situation of $\mathbb{P}^1$ with two marked points one obtains the Virasoro algebra.

5 The affine multi-point Krichever-Novikov algebras

Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra (e.g. semi-simple or abelian) and $(.|.)$ a symmetric, nondegenerate invariant bilinear form on $\mathfrak{g}$. The (higher genus multi-point) loop algebra is defined to be

$$\overline{\mathfrak{g}} := \mathfrak{g} \otimes \mathcal{A} \qquad \text{with} \quad [x \otimes g, y \otimes h] := [x, y] \otimes (g \cdot h) . \tag{24}$$

Its elements can be considered as $\mathfrak{g}$-valued meromorphic functions on M which are holomorphic outside of $\{P_1, P_2, \ldots, P_N, P_\infty\}$. A central extension $\widehat{\mathfrak{g}}$ is given as vector space $\widehat{\mathfrak{g}} = \mathbb{C} \oplus \overline{\mathfrak{g}}$ with Lie structure $(\widehat{x} := (0, x), \ t := (1, 0))$

$$[\widehat{x \otimes f}, \widehat{y \otimes g}] := \widehat{[x, y] \otimes (f \cdot g)} - (x, |y) \, \gamma(f, g) \cdot t , \qquad [t, \widehat{\mathfrak{g}}] = 0 , \tag{25}$$

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and γ from (22). We call this algebra higher genus multi-point affine algebra. For $g = 0$ and 2 points the classical affine algebras are obtained (with respect to Laurent polynomials). For higher genus and two points it was introduced by Krichever-Novikov [9] and extensively studied by Sheinman [17],[18],[19]. Its generalization to higher genus and an arbitrary number of marked points was given in [12].

By the decomposition (20) of $\mathcal{A}$ we obtain a decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_- \oplus \widehat{\mathfrak{g}}_{(0)} \oplus \widehat{\mathfrak{g}}_+ . \tag{26}$$

In particular, $\widehat{\mathfrak{g}}_{(0)} = \mathcal{A}_{(0)} \otimes \mathfrak{g} \oplus \mathbb{C}\, t$. Using the duality (18) we see that $1 = \sum_{p=1}^{N} A_{0,p}$ and hence that via $\mathfrak{g} \cong \mathfrak{g} \otimes 1 \hookrightarrow \widehat{\mathfrak{g}}$ the finite dimensional Lie algebra $\mathfrak{g}$ can be naturally considered as subalgebra of $\widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_+$, of $\overline{\mathfrak{g}}$, and of $\widehat{\mathfrak{g}}$.

6   Verma modules and Sugawara construction

In this section we assume $\mathfrak{g}$ to be a simple complex Lie algebra. Choose $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra, $\mathfrak{b}$ a corresponding Borel subalgebra, and $\mathfrak{n}$ a corresponding upper nilpotent subalgebra. We choose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N), \lambda_p \in \mathfrak{h}^*$, take N copies of everything, and label them by $p = 1, \ldots, N$. We set

$$\mathfrak{g}_{(N)} := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N, \quad \mathfrak{b}_{(N)} := \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_N . \tag{27}$$

Let W_p be a one-dimensional vector space over $\mathbb{C}$ with basis w_p for $p = 1, \ldots, N$. On W_p a one-dimensional representation of $\mathfrak{b}_p$ is defined via

$$h_p \omega_w := \lambda_p(h_p) w_p, \quad n_p w_p := 0, \qquad h_p \in \mathfrak{h}_p, \ n_p \in \mathfrak{n}_p . \tag{28}$$

This yields a representation of $\mathfrak{b}_{(N)}$ on $W = \bigotimes_{p=1}^{N} W_p$. As usual the Lie algebra acts via "Leibniz rule" on the tensor product, and on the p-th tensor factor only the p-th summand operates non-trivially. We call this representation τ_λ.

Remark. It is possible to allow twists to get a richer theory. They can be obtained via automorphisms $\varphi_p : \mathfrak{g} \to \mathfrak{g} = \mathfrak{g}_p$ given by $\varphi_p = Ad\, \gamma_p,\ \gamma_p \in G$, where G is the Lie group associated to $\mathfrak{g}$.

Lemma 6.1. *The subspace* $\overline{\mathfrak{g}}_0 \oplus \mathbb{C}\, t \oplus \widehat{\mathfrak{g}}_+$ *is a Lie subalgebra of* $\widehat{\mathfrak{g}}$ *and* $\psi : \overline{\mathfrak{g}}_0 \oplus \mathbb{C}\, t \oplus \widehat{\mathfrak{g}}_+ \to \widehat{\mathfrak{g}_{(N)}}$ *defined by*

$$\psi(\sum_{p=1}^{N} x_p \otimes A_{0,p}) := (x_1, \ldots, x_N), \quad \psi(t) := 0, \quad \psi(\widehat{\mathfrak{g}}_+) := 0 \tag{29}$$

is a Lie homomorphism.

Proof. We calculate

$$[x_p \otimes A_{0,p}, y_r \otimes A_{0,r}] = [x_p, y_r] \otimes (A_{0,p} A_{0,r}) = [x_p, y_r] \otimes (A_{0,r} \delta_p^r + \sum_{h>0} \sum_{s=1}^N \alpha_{(0,p),(0,r)}^{(h,s)} A_{h,s}) \;,$$

with $\alpha_{(0,p),(0,r)}^{(h,s)} \in \mathbb{C}$. First, this shows that the subspace is indeed closed under the Lie bracket. Note that the defining cocycle for the central extension vanishes for the subalgebra. Next, we see that all mixing terms are of higher degree. They will be annulated under ψ. Only the term with $p = r$ will survive. This shows that ψ is a Lie homomorphism. $\qquad\square$

Denote by $\widehat{\mathfrak{b}}$ the subalgebra $\overline{\mathfrak{b}}_0 \oplus \mathbb{C} \, t \oplus \widehat{\mathfrak{g}}_+$ (recall $\overline{\mathfrak{b}}_0 = \mathfrak{b} \otimes \mathcal{A}_0$). By restricting ψ to $\widehat{\mathfrak{b}}$ we obtain from the lemma that $\psi : \widehat{\mathfrak{b}} \to \mathfrak{b}_{(N)}$ is a Lie homomorphism. Now we can define for $\delta \in \mathbb{C}$ a representation of $\widehat{\mathfrak{b}}$ on W by

$$\tau_{\lambda,\delta}(x \oplus \alpha t \oplus x_+) := \tau_\lambda(\psi(x)) + \alpha \delta \cdot id \;. \tag{30}$$

Definition 6.2. The linear space

$$\widehat{W}_{\lambda,\delta} := U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{b}})} W \tag{31}$$

with its natural structure of a $\widehat{\mathfrak{g}}$-module is called the Verma module of the Lie algebra $\widehat{\mathfrak{g}}$ corresponding to (λ, δ) where $\lambda \in (\mathfrak{h}^*)^N$ is the weight of the Verma module and $\delta \in \mathbb{C}$ is the level of the Verma module.

As usual $U(\widehat{\mathfrak{g}})$ denotes the universal enveloping algebra.

Now the question arises can the finite-dimensional representations of $\mathfrak{g}$ be recovered. The answer is yes. It is given by

Proposition 6.3. *The $\widehat{\mathfrak{g}}$-module $\widehat{W}_{\lambda,\delta}$ is under the natural embedding of $\mathfrak{g}$ into $\widehat{\mathfrak{g}}$ also a $\mathfrak{g}$-module and contains the $\mathfrak{g}$-module*

$$\widetilde{W}_\lambda = \widetilde{W}_{\lambda_1} \otimes \cdots \widetilde{W}_{\lambda_N} \;, \tag{32}$$

where the $\widetilde{W}_{\lambda_i}$ are the heighest weight modules of $\mathfrak{g}$ of weight λ_i.

The proof is straightforward. The module $\widetilde{W}_\lambda$ lies in the degree zero part of $\widehat{W}_{\lambda,\delta}$, but there are other elements of degree zero not corresponding to $\widetilde{W}_\lambda$.

Denote by $\widehat{\mathfrak{g}}_-^* = \overline{\mathfrak{g}}_-^* \subseteq \widehat{\mathfrak{g}}$ the subalgebra of $\mathfrak{g}$-valued meromorphic functions which are regular at P_∞, then we can define the space of conformal blocks as the space of coinvariants

$$\widehat{V}_{\lambda,\delta} := \widehat{W}_{\lambda,\delta} / \widehat{\mathfrak{g}}_-^* \widehat{W}_{\lambda,\delta} \;. \tag{33}$$

Verma modules are examples of admissible modules $\widehat{W}$ of $\widehat{\mathfrak{g}}$ in the following sense:

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1. the central element t operates as $c \cdot id$ with $c \in \mathbb{C}$, (c is called the level),

2. for all $w \in \widehat{W}$ we have $\widehat{\mathfrak{g}}_n.w = 0$, for all $n \gg 0$.

For admissible modules we showed [14] (see also [13]) that there exists a Sugawara construction which yields a representation of the centrally extended vector field algebra. Here I can only give the main ideas. Choose $(u_i)_{i=1,\dots,\dim \mathfrak{g}}$ and $(u^i)_{i=1,\dots,\dim \mathfrak{g}}$ a system of dual basis of $\mathfrak{g}$ with respect to the bilinear form $(.|.)$. The current associated to the basis element u_i is given as

$$J_i(Q) = \sum_{n \in \mathbb{Z}} \sum_{p=1}^{N} u_i(n, p) \omega^{n,p}(Q) , \quad Q \in M. \tag{34}$$

Here we used $u_i(n, p)$ to denote the operator corresponding to $u_i \otimes A_{n,p}$ on $\widehat{W}$. The current is a formal operator-valued 1-differential on M. The similar expression is used for the current $J^i(Q)$ associated to u^i, resp. for the current associated to an arbitrary element $x \in \mathfrak{g}$. The energy-momentum tensor is defined as the formal sum

$$T(Q) := \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}} :J_i(Q) J^i(Q): , \quad Q \in M . \tag{35}$$

In this expression $:...:$ denotes some normal ordering, which moves the positive degree elements to the right. Using the admissibility and the normal ordering we can conclude that the energy momentum tensor is indeed a well-defined formal operator-valued 2-differential on M written as

$$T(Q) = \sum_{k \in \mathbb{Z}} \sum_{r=1}^{N} L(k, r) \Omega^{k,r}(Q) , \quad Q \in M. \tag{36}$$

Let $k^\vee$ be the dual Coxeter number (i.e. $2k^\vee$ is the eigenvalue of the Casimir operator in the adjoint representation), then

Theorem 6.4. *[14] Assume $\mathfrak{g}$ to be simple or abelian and $k^\vee$ the dual Coxeter number, resp. $k^\vee = 0$ for the abelian case. Assume $\widehat{W}$ to be an admissible module of $\widehat{\mathfrak{g}}$ with level c, then the $L_{k,r}$ are well-defined operators on $\widehat{W}$. If $c + k^\vee \neq 0$ then the rescaled operators*

$$L^*_{k,r} = -\frac{1}{c + k^\vee} L_{k,r} \tag{37}$$

define a representation of a centrally extended vector field algebra $\widehat{\mathcal{L}}$.

It is shown in [14] that a different normal ordering yields an equivalent central extension.

7 Moduli spaces and formal Knizhnik-Zamolodchikov equations

The construction done in the previous sections globalizes over a dense subset of the moduli space $\mathcal{M}_{g,N}$ of smooth projective curves of genus g with N marked points. Recall that two configurations $m = (M, (P_1, \ldots, P_N))$ and $m' = (M', (P'_1, \ldots, P'_N))$ describe the same point in moduli space if there is an algebraic isomorphism $\varphi : M \to M'$ which maps $P_i \to P'_i$ for $i = 1, \ldots, N$. "Bad" points in moduli space correspond to configurations which admit nontrivial automorphisms. At such points singularities can occur. For $N \geq 1$ and $g \geq 2$ the generic moduli point does not admit any nontrivial automorphism. Over the locus Y of these points there exists a universal family of curves with marked points (a versal family would suffice). The situation for $g = 0$ and $g = 1$ is a little bit different. Here one should better work with the configuration space. This situation will not be covered here, see [14].

Beside $\mathcal{M}_{g,N}$ we have also to consider $\mathcal{M}_{g,N+1}$. By forgetting the last marked point we obtain a morphism $f : \mathcal{M}_{g,N+1} \to \mathcal{M}_{g,N}$. Denote by Y the dense open subset of $\mathcal{M}_{g,N}$, where the universal family exists and set $\widetilde{Y} = f^{-1}(Y)$ then there exists also a universal family over $\widetilde{Y}$. Recall that a universal family over $\widetilde{Y}$ consists of a suitable well-behaved morphism (i.e. proper and flat) $\pi : \mathcal{U} \to \widetilde{Y}$ for $\tilde{b} = [(M, P_1, P_2, \ldots, P_N, P_\infty)] \in \widetilde{Y}$ with $\pi^{-1}(\tilde{b}) \cong M$, and $N + 1$ sections

$$\sigma_i : \widetilde{Y} \to \mathcal{U}, \quad \text{with} \quad \sigma_i(\tilde{b}) = P_i \in \pi^{-1}(\tilde{b}), \qquad i = 1, \ldots, N, \infty . \tag{38}$$

Note that for $g \geq 2$ the family $\mathcal{U}$ can be obtained by pulling back the universal curve defined over an open dense subset of $\mathcal{M}_{g,0}$ via the morphism corresponding to forgetting the points. For $g = 2$ we have to start with $\mathcal{M}_{2,1}$.

Now we fix a reference section $\widehat{\sigma}_\infty$. This corresponds to choosing a reference point for every M depending algebraically on varying M. Inside of $\widetilde{Y}$ we have the subset

$$Y' = \{[(M, P_1, P_2, \ldots, P_N, P_\infty)] \in \widetilde{Y} \mid P_\infty = \widehat{\sigma}_\infty(M)\} . \tag{39}$$

By genericity the forget morphism restricted to Y' is 1:1 onto Y. We will identify in the following Y' with Y. But note that this identification will not be canonical. It depends on the reference section chosen.

Let $\pi : \mathcal{U} \to Y$ be the projection. For every open subset U of Y set

$$\widetilde{S}_U := \sum_{i=1}^{N} \sigma_i(U) + \widehat{\sigma}_\infty(U) \tag{40}$$

for the divisor of sections on $\mathcal{U}_{|\pi^{-1}(U)}$. The sheaf $\mathcal{A}_Y$ is defined as the sheaf over Y given by defining $\mathcal{A}_Y(U)$ to be the space of functions on $\pi^{-1}(U)$ with possible poles along $\widetilde{S}_U$. The same works for the central extension $\widehat{\mathcal{A}}_Y$, the loop algebra $\overline{\mathfrak{g}}_Y = \mathcal{A}_Y \otimes \mathfrak{g}$ and the affine algebra $\widehat{\mathfrak{g}}_Y = \overline{\mathfrak{g}}_Y \oplus \mathcal{O}_Y \cdot t$. Here $\mathcal{O}_Y$ denotes the structure sheaf of Y. The central extensions are given in a natural globalization of (22), resp. (25).

Definition 7.1. A sheaf $\mathcal{W}$ of $\mathcal{O}_Y$-modules is called a sheaf of representations for the affine algebra sheaf $\widehat{\mathfrak{g}}_Y$ if the $\mathcal{W}(U)$ are modules over $\widehat{\mathfrak{g}}_Y(U)$ for every U fulfilling the obvious compatibility conditions on the sheaf level.

Admissible representation sheaves are defined in a similar way. The Verma module construction can be made on the sheaf level. This yields admissible representation sheaves $\widehat{W}_{(\lambda,\delta),Y}$. The sheaf of conformal blocks is the quotient sheaf

$$\widehat{V}_{(\lambda,\delta),Y} := \widehat{W}_{(\lambda,\delta),Y} / \widehat{\mathfrak{g}}^*_{-,Y} \widehat{W}_{(\lambda,\delta),Y} \ . \tag{41}$$

Next we want to describe the tangent space at a moduli point.
Let $b = [(M, P_1, P_2, \dots, P_N)]$ be the moduli point and set $S = \sum_{i=1}^N P_i$, then the Kodaira-Spencer map gives an isomorphism of the tangent space to the moduli space with certain cohomology spaces

$$\mathrm{T}_{[M]}\mathcal{M}_{g,0} \cong \mathrm{H}^1(M, T_M), \qquad \mathrm{T}_b\mathcal{M}_{g,N} \cong \mathrm{H}^1(M, T_M(-S)) \ . \tag{42}$$

Here T_M is the holomorphic tangent line bundle on M, i.e. $T_M = \mathcal{K}^{-1}$. The following theorem proven in [15] gives an identification of the tangent space at the moduli point b with a certain subspace of the critical strip of the vector field algebra associated to the geometric data $(M, (P_1, \dots, P_N, \widehat{\sigma}_\infty(M))$.

Theorem 7.2. *There exists a natural isomorphism*

$$(\mathcal{L}_{k-2} \oplus \mathcal{L}_{k-3} \cdots \oplus \mathcal{L}_{-1}) \oplus \mathcal{L}^*_{(0)} \cong \mathrm{H}^1(M, T_M(-kS)) \ , \quad k \geq 0 \ . \tag{43}$$

Here $\mathcal{L}^*_{(0)}$ are the elements of the reduced critical strip generated by the basis elements with poles at $\{P_1, P_2, \dots, P_N, P_\infty\}$. For $g \geq 2$ its dimension is $3g - 3$ and it starts with $\mathcal{L}_{-2}$ from above.

Let me only indicate the construction of these isomorphisms. It is based on the calculation of the Cech cohomology of $T_M(-kS)$ with respect to the affine (resp. Stein) covering of M given as follows. Let U_∞ be a disk around P_∞ and set $U_1 := M \setminus S$. Note that U_1 is affine. Then $U_1 \cap U_\infty = U^*_\infty$ is the disc with P_∞ removed. Hence the Cech 1-cocycles can be given as sections on U^*_∞. In this way $f \in \mathcal{L} \to f_{|U^*_\infty}$ gives a linear map from the vector field algebra to the Cech 1-cocycles, and further to the cohomology group $\mathrm{H}^1(M, T_M(-kS))$. The restrictions of elements coming from outside of the strip given in the formulation of the theorem can by their very definition be extended either to U_∞ or to U_1 with the required zero-order at S. Hence they are coboundaries. A closer examination shows that the basis of the rest stays linearly independent in cohomology. By dimension count it follows that the map is an isomorphism.

To give an example: for the element $e_{-1,p}$ (with $p = 1, \dots, N$) the lowest order term of its expansion at the point P_p has the form $\partial/\partial\xi_p$. At all other marked points it has a zero. Under the isomorphism it corresponds to moving the marked point P_p on M.

Now take $X_k \in T_b\mathcal{M}_{g,N} \cong \mathcal{L}^*_{(0)} \oplus \mathcal{L}_{-1}$ a tangent vector corresponding to an element l_k of the critical strip. We assume that X_k operates linearly as operator ∂_k on the space of sections of a representation sheaf $\mathcal{W}$. For Φ a section we set

$$\nabla_k\Phi := (\partial_k + T[l_k])\Phi, \quad \text{with} \quad T[l] := \frac{-1}{(c+k^\vee)\,2\pi\,\mathrm{i}} \int_C T(Q)l(Q) \qquad (44)$$

the Sugawara operator, which corresponds to l.

Definition 7.3. The formal Knizhnik-Zamolodchikov equations are defined to be the set of equations

$$\nabla_k\Phi = 0, \qquad k = 1, \dots, 3g - 3 + N . \qquad (45)$$

Note that these equations can be expressed in terms of the geometric data of the curve and the points which can be moved. In [15] the equations have been explicitely calculated for genus 0 and 1.

Acknowledgments

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