Quantum Diffusions and Appell Systems *

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Abstract

Within the algebraic framework of Hopf algebras, random walks and associated diffusion equations (master equations) are constructed and studied for two basic operator algebras of Quantum Mechanics i.e the Heisenberg-Weyl algebra (hw) and its q-deformed version hw_q. This is done by means of functionals determined by the associated coherent state density operators. The ensuing master equations admit solutions given by hw and hw_q-valued Appell systems.

1. Introduction. We work in the general framework of the so called quantum probability theory[17] and more specifically along the research line relating random walks, diffusions and Markov transition operators to Lie-Hopf algebras[14, 15, 16]. Our aim is to construct algebraic random walks and their diffusion limit in terms of master equations[19]. We work with two basic operator algebras of Quantum Mechanics[8] i.e the Heisenberg-Weyl algebra (hw) and its q-deformed version hw_q[13], and use their Hopf algebra like structures for our construction (Chapt. 2). The density of the two functionals needed are constructed by the associated to those algebras coherent states vectors[11]. As the random walks take place on the manifold of these coherent states vectors it is important to investigate the geometrical features of them (Chapt. 3). Then a limiting procedure leads to the master (diffusion) equations for the case of hw random walk (Chapt. 5) and the case of hw_q random walk (Chapt. 6), correspondingly. The solutions of the resulting master equations of motion for certain general elements of the respective operator algebras are obtained in terms of the associated operator valued Appell systems[7]. Certain generalities of classical Appell systems are discussed in Chapt. 4[18]. Finally, some technicalities such as ordering formulae for generators of the two hw algebras[10], as well as some Baker-Campbell-Hausdorff decompositions formulae for the $SU(1,1)$ group elements[21] are summarized in Appendices A and B.

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2. Hopf Algebras. A Hopf algebra [1] $\mathcal{A} = \mathcal{A}(\mu, \eta, \Delta, \epsilon, S)$ over a field k is a vector space equipped with an algebra structure with homomorphic associative product map $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, and a homomorphic unit map $\eta : k \to \mathcal{A}$, that are related by $\mu \circ (\eta \otimes id) = id = \mu \circ (id \otimes \eta)$, together with a coalgebra structure with a homomorphic coassociative coproduct map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and a homomorphic counit map $\epsilon : \mathcal{A} \to k$, that are related between them by $(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta$. Both products satisfy the compatibility condition of bialgebra i.e $(\mu \otimes \mu) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta) = \Delta \circ \mu$, where $\tau(x \otimes y) = y \otimes x$ stands for the twist map. If η or ϵ is not defined in $\mathcal{A}$ we speak about non unital or non counital Hopf algebra.

Suppose we have a functional $\phi : \mathcal{A} \to \mathbf{C}$, defined on $\mathcal{A}$, let us define the operator $T_\phi : \mathcal{A} \to \mathcal{A}$ as $T_\phi = (\phi \otimes id) \circ \Delta$, then $\epsilon \circ T_\phi = \phi$, namely the counit aids to pass from the operator to its associated functional. From this relation we can define the convolution product $\psi * \phi$, between functionals as follows [15]:

$$\begin{aligned}\epsilon \circ T_\psi T_\phi &= \epsilon \circ (\psi \otimes id) \circ \Delta \circ (\phi \otimes id) \circ \Delta = (\phi \otimes \psi) \circ (id \otimes id \otimes \epsilon) \circ (id \otimes \Delta) \circ \Delta \\ &= (\phi \otimes \psi) \circ \Delta = \phi * \psi \, ,\end{aligned} \tag{1}$$

and in general $\epsilon \circ T^n_\phi = \epsilon \circ T_{\phi^{*n}} = \phi^{*n}$. These last relations imply that the transition operators form a discrete semigroup wrt their composition with identity element $T_\epsilon \equiv id$ (due to the axioms of Hopf algebra) and generator T_ϕ, while the functionals form a dual semigroup wrt the convolution with identity element e and generator ϕ, and that these two semigroups are homomorphic to each other. and the

We recall now two algebras and their structural maps that concerns us here:
i) *Heisenberg-Weyl algebra hw*: this is the algebra of the quantum mechanical oscillator and is generated by the creation, annihilation and the unit operator $\{a^\dagger, a, \mathbf{1}\}$ respectively which satisfy the commutation relation (Lie bracket) $[a, a^\dagger] = \mathbf{1}$, while $\mathbf{1}$ commutes with the other elements. This algebra possesses a natural non counital Hopf algebra structure (or bialagebra-like cf. [14], Chap. 3), with comultiplication defined as

$$\begin{aligned}\Delta^{(n-1)} a &= n^{-\frac{1}{2}} (a \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes a \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes \cdots \otimes a) \, , \\ \Delta^{(n-1)} a^\dagger &= n^{-\frac{1}{2}} (a^\dagger \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes a \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes \cdots \otimes a^\dagger) \, , \\ \Delta \mathbf{1} &= \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \, .\end{aligned} \tag{2}$$

Let us also define the so called number operator $N = a^\dagger a$ with the following commutation relations with the generators of hw:

$$[a, a^\dagger] = \mathbf{1} \quad , \; [N, a^\dagger] = a^\dagger \quad , \quad [N, a] = -a \, . \tag{3}$$

The module which carries the unique irreducible and infinite dimensional representation of the oscillator algebra is the Hilbert-Fock space $\mathcal{H}_F$ which is generated by a starting (or "vacuum") state vector $|0\rangle \in \mathcal{H}$ and is given as $\mathcal{H} = \{|n\rangle = \frac{(a^\dagger)^n}{n!} |0\rangle \, , n \in \mathbf{Z}_+$.

ii) *The q-deformed Heisenberg-Weyl algebra* hw_q: The q-deform Heisenberg-Weyl algebra is generated by the elements $hw_q =< b, b^\dagger, q^N, q^{-N}, \mathbf{1} >$ that satisfy the relations

$$\begin{aligned} bb^\dagger - q^{-1} b^\dagger b = q^N \quad , \quad q^N q^{-N} = \mathbf{1} \, , \\ q^N b q^{-N} = q^{-1} b \quad , \quad q^N b^\dagger q^{-N} = q b^\dagger \, .\end{aligned} \tag{4}$$

For real q the Fock representation space is spanned by the vectors $\{|n\rangle = \frac{(b^\dagger)^n}{\sqrt{[n]_q!}}|0\rangle, n \in \mathbf{Z}_+\}$, where $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ and $[n]_q = [1]_q[2]_q \cdots [n]_q$. In the Fock space representation of this algebra we have the additional relations $b^\dagger b = [N]_q$, $bb^\dagger = [N+1]_q$. This algebras has no satisfactory Hopf structure but still as will be seen below we can define algebraic random walks on it and study their diffusion limit. To this end let us make the transformations[13] $a_q = q^{N/2}b$ and $a_q^\dagger = b^\dagger q^{N/2}$, and obtain the resulting algebra

$$a_q a_q^\dagger - q^2 a_q^\dagger a_q = \mathbf{1} \tag{5}$$

which is the new form of the hw_q algebra[20]. Although not an algebra homomorphisms we will use below the coassociative maps

$$\Delta a_q \quad = \quad a_q \otimes \mathbf{1} + \mathbf{1} \otimes a_q \quad , \quad \Delta a_q^\dagger = a_q^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes a_q^\dagger \ . \tag{6}$$

3. Coherent States . For our needs here a brief introduction to the concept of coherent states (CS) on Lie groups goes as follows: consider a Lie group $\mathcal{G}$, with a unitary irreducible representation $T(g)$, $g \in \mathcal{G}$, in a Hilbert space $\mathcal{H}$. We select a reference vector $|\Psi_0\rangle \in \mathcal{H}$, to be called the "vacuum" state vector, and let $\mathcal{G}_0 \subset \mathcal{G}$ be its isotropy subgroup, i.e for $h \in \mathcal{G}_0$, $T(h)|\Psi_0\rangle = e^{i\varphi(h)}|\Psi_0\rangle$. The map from the factor group $\mathcal{M} = \mathcal{G}/\mathcal{G}_0$ to the Hilbert space $\mathcal{H}$, introduced in the form of an orbit of the vacuum state under a factor group element, defines a CSV $|x\rangle = T(\mathcal{G}/\mathcal{G}_0)|\Psi_0\rangle$ labelled by points $x \in \mathcal{M}$ of the coherent state manifold. Coherent states form an (over)complete set of states, since by means of the Haar invariant measure of the group $\mathcal{G}$ *viz.* $d\mu(x)$, $x \in \mathcal{M}$, they provide a resolution of unity, $\mathbf{1} = \int_\mathcal{M} d\mu(x)\,|x\rangle\langle x|$. As a consequence, any vector $|\Psi\rangle \in \mathcal{H}$ is analyzed in the CS basis, $|\Psi\rangle = \int_\mathcal{M} d\mu(x)\Psi(x)\,|x\rangle$, with coefficients $\Psi(x) = \langle x|\Psi\rangle$. We should note here that the square integrability of the vectors Ψ will impose some limits on the growth parameters of the functions $\Psi(x)$ (cf. [11] and references therein).

What concerns us here is mostly the geometry of the CS manifold $\mathcal{M}$. This is due to the fact that the random walks and their diffusion limits that will be study below will be given in terms of functionals associated with coherent states so that the random walks will be induced on the functions defined on $\mathcal{M}$ (passive description) or on the operators acting on the functions defined on $\mathcal{M}$ (active description). Although only the latter description will be studied here in terms of the quantum master equations, it should be obvious that the geometry of the background manifold $\mathcal{M}$ namely both the Riemannian and the symplectic geometry (the symplectic geometry especially in the case of non stationary random walks), will manifest itself in the associated diffusion equations. Specifically below it will be shown that the hw random walk takes place on the flat complex plane $\mathbf{C}$ with canonical symplectic structure, while the deformed hw_q random walk takes place on a q-deformed surface of revolution with modified, due to q-deformation, Riemannian and symplectic geometry. This fact provides a further motivation for studying random walks and diffusions within the present algebraic framework since in this way we are able to study these phenomena taking place on non trivial spaces. Details constructions and studies can be found elsewhere[2], here we summarize some relevant information:

Let us first specialize to the HW group: The hw-CS is defined by the realation

$$|\alpha\rangle = e^{\alpha a^\dagger - \bar{\alpha} a}|0\rangle = \mathcal{N} e^{\alpha a^\dagger}|0\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \ . \tag{7}$$

It is an (over)complete set of states with respect to the measure $d\mu(\alpha) = \frac{1}{\pi} e^{-|\alpha|^2} d^2\alpha$ for the non-normalized CS, and $\alpha \in \mathcal{M} = HW/U(1) \approx \mathbf{C}$ is the CS manifold. Since $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$, $\mathcal{M}$ is the flat canonical phase plane with the standard line element $ds^2 = d\alpha d\bar{\alpha}$. Also the symplectic 2-form $\omega = i d\alpha \wedge d\bar{\alpha}$ is associated to the canonical Poisson bracket $\{f, g\} = i(\partial^f_\alpha \partial^g_{\bar{\alpha}} - \partial^f_{\bar{\alpha}} \partial^g_\alpha)$.

Next we turn to the hw_q case: The definition of the hw_q-CS reads[2]

$$|\,|\alpha\rangle_q = e_q^{\alpha a_q^\dagger}\,|0\rangle = e^{\alpha A_q^\dagger}\,|0\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}}\,|n\rangle \ , \tag{8}$$

where $[n] = \frac{q^{2n}-1}{q^2-1}$. The states are first defined in terms of the q-deformed exponential function $e_q^x = \sum_{n\geq o} \frac{x^n}{[n]!}$ and the q-creation operator and then equivalently by exponentiation of the operator $A_q^\dagger = \frac{N}{[N]} a_q^\dagger$, that satisfies with the hw_q elements the hw algebra relations[2]

$$[a_q, A_q^\dagger] = \mathbf{1} \quad , \quad [A_q, a_q^\dagger] = \mathbf{1} \ . \tag{9}$$

The q-CS is an (over)complete set of states[9] with respect to the measure $d\mu(\alpha)_q = \frac{1}{\pi}(e_q^{|\alpha|^2})^{-1} d_q^2\alpha$, and wrt the Jackson q-integral[6]. If $q = e^\lambda$, then since $a_q\,|\alpha\rangle_q = \alpha\,|\alpha\rangle_q$, the q-CS manifold $\mathcal{M}$ is a non flat surface of revolution with q-deformed induced curvature with curvature scalar $R = \lambda^2 12(1 + 2\,|\alpha|^2 + \mathcal{O}(\lambda^3))$. Also the symplectic 2-form ω is modified by the q-deformation as $\omega = \{i - \frac{\lambda^2}{2}\,|\alpha|^2\,(|\alpha|^2 + 2) + \mathcal{O}(\lambda^3)\} d\alpha \wedge d\bar{\alpha}$ [2].

The density operator (state) ρ will be used below to determine functionals of some Hopf operator algebras $\mathcal{A}$, so here we introduce the general concept and give its construction in terms of convex combinations of projectors of coherent states. Let a Hilbert vector space $\mathcal{H}$ that carries a unitary irreducible representation of $\mathcal{A}$ of finite or infinite dimension. The set

$$\mathcal{S} = \{\rho \in \mathrm{End}(\mathcal{H}) : \rho \geq 0, \rho^\dagger = \rho, tr\rho = 1\} \ , \tag{10}$$

namely the set of non-negative, Hermitian, trace-one operators acting on $\mathcal{H}$ form a convex subspace of $\mathrm{End}(\mathcal{H})$, which is the convex hull of the set

$$\mathcal{S}_P = \{\rho \in \mathcal{S} \quad , \quad \rho^2 = \rho\} \equiv \mathcal{H}/U(1) \ , \tag{11}$$

namely of the set of pure density operators (states), that are in one-to-one correspondance with the state vectors of $\mathcal{H}$. Two kinds of ρ density operators that will be used in the sequel are constructed by hw-CS and hw_q-CS. Explicitly from the pure density operators $|\pm\alpha\,\rangle\langle\,\pm\alpha| \in \mathcal{S}_P$ and the q-deformed ones $|\alpha\rangle_{qq}\langle\alpha| \equiv |\alpha\,\rangle\langle\,\alpha|_q \in \mathcal{S}_P$, we form convex combination belonging to the convex hull of $\mathcal{S}_P$ i.e

$$\begin{aligned} \rho &= p\,|\alpha\,\rangle\langle\,\alpha| + (1-p)\,|{-\alpha}\,\rangle\langle\,-\alpha| \ , \\ \rho_q &= p\,|\alpha\,\rangle\langle\,\alpha|_q + (1-p)\,|{-\alpha}\,\rangle\langle\,-\alpha|_q \ . \end{aligned} \tag{12}$$

4. Appell Systems. Classical Appell polynomials [18] on the real line are polynomials $\{h_n(x); n \in \mathbf{N}\}$ of degree n that satisfy the condition $(d/dx)h_n(x) = nh_n(x)$. A class of such systems is the shifted moment sequences $h_n(x) = \int_{-\infty}^{\infty}(x+y)^n p(dy)$, for some positive real measure p with finite moments. The class of Appell polynomials includes cases such as the divided sequences, the Bernoulli polynomials and the Hermite polynomials, which correspond to the Gaussian measure $p = p(dy) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$. Some important properties of the Appell polynomial sets that have been investigated are the following: Hermite polynomials are the only Appell polynomials associated to the ordinary derivative operator that are also orthogonal [18](e, b, c), similarly Charlier polynomials are the only Appell systems associated to the difference operator that are also orthogonal[18](d), while the Rogergs q-Hermite polynomials are the only Appell systems associated to Askey-Wilson q-derivative operator that are orthogonal too[18](e).

The following Hopf algebraic reformulation of the real line Appell systems (i.e non polynomials necesserily) motivates their generalization to more general spaces. Let $\mathcal{A} = \mathbf{R}[[X]]$ the algebra of the real formal power series generated by pointwise multiplication $fg(x) = f(x)g(x), f, g \in \mathcal{A}$. Then $\mathcal{A}$ becomes a Hopf algebra with comultiplication $(\Delta f)(x, y) = f(x + y)$ and counit $\epsilon(id) = 1$, $\epsilon(X) = 0$, where id is the identity function and $X(x) = x$ stands for the coordinate function. For a given functional $\phi : \mathcal{A} \to \mathbf{C}$ and a chosen basis $(x^n), n \in \mathbf{Z}_+$ in $\mathcal{A}$, it is easy to verify that the relation $h_n(x) = (\phi \otimes id) \circ \Delta x^n = T_\phi x^n$ defines an Appell systems and is equivalent to the preceding definition. Specifically for $\phi = \int_{-\infty}^{\infty} p(dy)$ with p the Gaussian measure we obtain the Hermite polynomials if we make the identifications $x \otimes 1 \equiv x$ and $1 \otimes x \equiv iy$. This algebraic definition has been used extensively to introduce Appell systems in non commuting algebras[7]. Here we will utilize it to define below Appell systems on two important operator algebras of Quantum Mechanics i.e the Heisenberg algebra and the q-deformed Heisenberg algebra and to show that the resulting operator valued Appell systems are solutions of quantum master equations that are constructed respectively as limits of random walks defined on these algebras.

5. Diffusion on C. Let $\phi(\cdot) = \mathrm{Tr}\rho(\cdot) \equiv < \rho, \cdot >$, a functional defined on the enveloping Heisenberg-Weyl algebra $\mathcal{U}(hw)$, where $\rho = p\,|\alpha\rangle\langle\alpha| + (1-p)\,|-\alpha\rangle\langle -\alpha|$, i.e the ρ density operator is given as a convex sum of pure state density operators. The action of the transition operator $T_\phi = (\phi \otimes id) \circ \Delta$ on the generating monomials of $\mathcal{U}(hw)$ (where we ignore the numerical factors in the comultiplication of eq.(2)) reads,

$$\begin{aligned} T_\phi((a^\dagger)^m a^n) &= (\phi \otimes id) \circ \Delta((a^\dagger)^m a^n) \\ &= \sum_{i=0}^{m} \sum_{j=0}^{n} \begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} [p\alpha^{*i}\alpha^j + (1-p)(-\alpha)^i(-\alpha)^j](a^\dagger)^{m-i} a^{n-j} \\ &= p(a^\dagger + \alpha^*)^m (a + \alpha)^n + (1-p)(a^\dagger - \alpha^*)^m (a - \alpha)^n \ . \end{aligned} \tag{13}$$

For a general element $f(a, a^\dagger) \in \mathcal{U}(hw)$ that is normally ordered, namely the annihilation operator a is placed to the right of the creation operator $a^\dagger$, denoted by $\hat{f}(a, a^\dagger) = \sum_{m,n \geq 0} c_{mn} (a^\dagger)^m a^n$, the action of the linear operator T_ϕ becomes

$$T_\phi(\hat{f}(a, a^\dagger)) = p\hat{f}(a + \alpha, a^\dagger + \alpha^*) + (1-p)\hat{f}(a - \alpha, a^\dagger - \alpha^*) \tag{14}$$

By means of the CS eigenvector property and the normal ordering of the f element we also compute the value of functional viz.

$$\phi(\hat{f}(a, a^\dagger)) = p\hat{f}(\alpha, \alpha^*) + (1-p)\hat{f}(-\alpha, -\alpha^*) \ . \tag{15}$$

Let us consider the displacement operator $D_\alpha = e^{\alpha a^\dagger - \alpha^* a}$ which acts with the group adjoint action on any element f of the $\mathcal{U}(hw)$ algebra viz.[11]

$$AdD_a(f) = Ade^{\alpha a^\dagger - \alpha^* a}(f) = Ade^{ad(\alpha a^\dagger - \alpha^* a)}(f) = D_\alpha f D_\alpha^\dagger, \tag{16}$$

where $ad(X)f = [X, f]$ and $ad(X)ad(X)f = [X, [X, f]]$ and similarly for higher powers, stands for the Lie algebra adjoint action that is defined in terms of the Lie commutator. Explicitly the action of the displacement operator on the generators of $\mathcal{U}(hw)$ reads $AdD_{\pm\alpha}(a) = a \mp \alpha$ and $AdD_{\pm\alpha}(a^\dagger) = a^\dagger \mp \alpha^*$. By means of these expressions we rewrite the action of the preceding transition operator as

$$T_\phi(\hat{f}(a, a^\dagger)) = [pAdD_{-\alpha} + (1-p)AdD_\alpha]\hat{f} \tag{17}$$

Next we want to compute the limiting transition operator

$$\begin{aligned} T_t &\equiv T_{\phi_t} \equiv \lim_{n\to\infty} T_\phi^n \\ &= \lim_{n\to\infty} [p(1 + ad(-\alpha a^\dagger + \alpha^* a) + \frac{1}{2} adad(-\alpha a^\dagger + \alpha^* a) + \cdots) \\ &+ (1-p)(1 + ad(\alpha a^\dagger - \alpha^* a) + \frac{1}{2} adad(\alpha a^\dagger - \alpha^* a) + \cdots]^n . \end{aligned} \tag{18}$$

If we introduce the parameters $t \in \mathbf{R}$ and $c, \gamma \in \mathbf{C}$ by means of the relations,

$$2\alpha(p - \frac{1}{2}) = \frac{tc}{n} \quad , \quad \frac{\alpha^2}{2} = \frac{t\gamma}{n} , \tag{19}$$

and then take $\alpha \to 0, n \to \infty$, with t, c, γ fixed, we use the limit $(1 + \frac{Z}{n})^n \to e^Z$, to arrive at the limiting Markov operator $T_t = e^{tad\mathcal{L}}$, where

$$\mathcal{L} = -ca^\dagger + c^* a + \gamma(a^\dagger)^2 - \gamma^* a^2 - |\gamma| (a^\dagger a + aa^\dagger). \tag{20}$$

By construction T_t is the time evolution operator for any element f of $\mathcal{U}(hw)$ i.e $f_t = T_t(f)$ and forms a continous semigroup $T_t T_{t'} = T_{t+t'}$ under composition. This yields the diffusion equation obeyed by f_t, which will be taken to be normally ordered hereafter. By time derivation of the equation

$$\phi_t(\hat{f}) =< \rho, \hat{f}_t >=< \rho, e^{tad\mathcal{L}} \hat{f} >=< e^{-tad\mathcal{L}^\dagger} \rho, \hat{f} >=< \rho_t, \hat{f} > , \tag{21}$$

we obtain the diffusion equation $\frac{d}{dt}\hat{f}_t = \mathcal{L}\hat{f}_t$, as well as the dual one satisfied by the ρ density operator viz. $\frac{d}{dt}\rho_t = \mathcal{L}^\dagger \rho_t$. To simplify and eventually solve the ensuing equations we will assume here that the parameter γ introduced above is a complex variable with random argument of zero average and constant non zero magnitude. Then if we average over random γ the equations of motion only the term proportional to the amplitude of γ will be retained. If in addition we consider the case of an symmetric random walk i.e $p = 1/2, c = 0$ the equation of motion becomes

$$\frac{d}{dt}\hat{f}_t = -2|\gamma| \left[a^\dagger \hat{f}_t a + a\hat{f}_t a^\dagger - N\hat{f} - \hat{f}(N+1)\right] . \tag{22}$$

This is a quantum master equation of the Lindblad type[12] which will be shown to admit a solution in terms of a operator valued Appell system associated with the generator of that equation. We may introduce the following operators[3]

$$K_+ f = a^\dagger f a \quad K_- f = afa^\dagger \quad K_0 f = \frac{1}{2}(a^\dagger af + faa^\dagger) , \tag{23}$$

and $K_c f = [a^\dagger a, f]$. These operators acting on the elements f of the enveloping algebra $\mathcal{U}(hw)$, generate the $su(1,1)$ Lie algebra defined by the commutation relations

$$[K_-, K_+] = 2K_0 \quad , \quad [K_0, K_\pm] = \pm K_\pm, \tag{24}$$

where K_c is the central element (Casimir operator) of the algebra. In terms of these operators the quantum master equation (22) is cast in the form

$$\frac{d}{dt}\hat{f}_t = -2\,|\gamma|\,(-2K_0 + K_+ + K_-)\hat{f}_t \; . \tag{25}$$

Use of the disentangling theorem (Baker-Campbell-Hausdorff formula) of a general $SU(1,1)$ group element (c.f Appendix A), allows to express the solution of the quantum master equation in the form

$$\hat{f}_t = \exp(A_+K_+)\exp(\ln A_0 K_0)\exp(A_-K_-)(\hat{f}) = \exp(B_-K_-)\exp(\ln B_0 K_0)\exp(B_+K_+)(\hat{f}) \; , \tag{26}$$

if the normally or respectively antinormally ordered BCH decomposition is used. Above $\hat{f} = \sum_{n\geq 0} c_{mn}(a^\dagger)^s a^t$, stands for the initial time operator which can be a general element of the enveloping algebra $\mathcal{U}(hw)$. Specifically in the case of normally ordered decomposition with initial operator taken as $\hat{f} = (a^\dagger)^m a^n$ the solution of the quantum master equation is obtained by means of the actions issued in eq.(23) and by the antinormal-to-normal reordering relations among the generators of the $\mathcal{U}(hw)$ algebra (c.f Appemdix B). An arduous but straightforward calculation yields the normal ordered solution:

$$\hat{f}_t = \exp(A_+K_+)\exp(\ln A_0 K_0)\exp(A_-K_-)((a^\dagger)^s a^t) =$$

$$\sum_{k\geq 0}\sum_{l\geq 0}\sum_{m\geq 0}\sum_{i=0}^{min(k,s)}\sum_{j=0}^{min(k+t-i,k)}\sum_{u=0}^{l}\sum_{v=0}^{u}\sum_{q=0}^{l-u}\sum_{w=0}^{v}\sum_{f=0}^{min(q,x)}\sum_{h=0}^{min(y+q-f,w)} \times$$

$$\frac{A_-^k}{k!}\frac{\overline{A}_0^l}{l!}\frac{A_+^m}{m!} d^i_{k,s} d^j_{k+t-i,k} d^f_{q,x} d^h_{y+q-f,w} \overline{d}_{l-u,q}\overline{d}_{v,w}\frac{1}{2^l}\begin{pmatrix} l \\ u \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}(a^\dagger)^{x+w+m-f-h}a^{y+q+m-f-h} \tag{27}$$

where $x = s+k+q-i-j$, $y = t+k+w-i-j$ and $\overline{A}_0 = \ln A_0$, with $A_0 = \frac{1}{1-4|\gamma|t}$ and $A_\pm = \frac{-2|\gamma|t}{1-2|\gamma|t}$. A similar solution can be obtained for the antinormal BCH decomposition. We can therefore state the results in the following

Proposition 1. *The solution of the quantum master equation* $\frac{d}{dt}\hat{f}_t = \mathcal{L}\hat{f}_t$ *where the generator* $\mathcal{L}(\hat{f}_t) = -2\,|\gamma|\left[a^\dagger\hat{f}_t a + a\hat{f}_t a^\dagger - N\hat{f} - \hat{f}(N+1)\right]$ *of Lindblad type generates the semigroup of Markov transition operators* $T_t = e^{t\mathcal{L}}$ *acting on the enveloping algebra* $\mathcal{U}(hw)$*, is given by the associated* $\mathcal{U}(hw)$*-valued Appell system which in its normally ordered form is given by equation (27).*

We note also that the dual master equation satisfied by the density operator can easily be solved along the above lines in terms of the associated Appell system.

6. ***q*-Diffusion.** Let $\phi_\phi(\cdot) = \mathrm{Tr}\rho_q(\cdot) \equiv <\rho_q, \cdot>$, a functional defined on the enveloping q-Heisenberg-Weyl algebra $\mathcal{U}_q(hw)$, where $\rho_q = p\,|\alpha\rangle\langle\alpha|_q + (1-p)\,|-\alpha\rangle\langle-\alpha|_q$ is the ρ

density operator given as a convex sum of pure state q-density operators. The action of transition operator $T^q_\phi = (\phi_q \otimes id) \circ \Delta$ on the monomials of $\mathcal{U}_q(hw)$, with Δ map given is eq.(6) reads,

$$\begin{aligned} T_{\phi_q}(a^\dagger)^m_q a^n_q) &= (\phi_q \otimes id) \circ \Delta((a^\dagger)^m_q a^n_q) \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} \begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} [p\alpha^{*i}\alpha^j + (1-p)(-\alpha)^i(-\alpha)^j](a^\dagger)^{m-i}_q a^{n-j}_q \\ &= p(a^\dagger_q + \alpha^*)^m (a_q + \alpha)^n + (1-p)(a^\dagger_q - \alpha^*)^m (a_q - \alpha)^n \; . \end{aligned} \tag{28}$$

On an element $f(a_q, a^\dagger_q)$ of the enveloping algebra $\mathcal{U}_q(hw)$ that is normally ordered, namely the annihilation operator a_q is placed to the right of the creation operator $a^\dagger_q$, that is expressed as $\hat{f}(a_q, a^\dagger_q) = \sum_{m,n\geq 0} c_{mn}(a^\dagger)^m_q a^n_q$, the action of the linear operator T_{ϕ_q} becomes

$$T_{\phi_q}(\hat{f}(a_q, a^\dagger_q)) = p\hat{f}(a_q + \alpha, a^\dagger_q + \alpha^*) + (1-p)\hat{f}(a_q - \alpha, a^\dagger_q - \alpha^*) \; . \tag{29}$$

By means of the q-CS eigenvector property and the normal ordering of the element f we also compute the value of functional viz.

$$\phi_q \hat{f}(a_q, a^\dagger_q) = p\hat{f}(\alpha_q, \alpha^*_q) + (1-p)\hat{f}(-\alpha, -\alpha^*) \; . \tag{30}$$

Let us now consider the displacement operator $D^q_\alpha = e^{\alpha A^\dagger_q - \alpha^* a_q}$, which acts with the following adjoint action on any element f of the $\mathcal{U}_q(hw)$ algebra, $AdD^q_a(f) = Ade^{\alpha A^\dagger_q - \alpha^* a_q}(f) = e^{ad(\alpha A^\dagger_q - \alpha^* a_q)}(f) = D^q_\alpha f D^q_{-\alpha}$. We should emphasize at this point that $D^{q\dagger}_\alpha \neq D^q_{-\alpha}$. This is an important difference from the preceding undeformed case with $q = 1$, which stems from the fact the though eq.(5) is valid the two involved operators are not Hermitian conjugate to each other. This fact would not permit us to proceed for the construction of quantum diffusion equation in a manner analogous to the $q = 1$ case. Instead here we will restrict the space of solutions of the resulting q-master equation from the whole algebra $\mathcal{U}_q(hw)$ to the commuting subalgebra generated either by monomials of the creation operator $\{(a^\dagger)^m_q, m \in \mathbf{Z}_+\}$ or of the annihilation operator $\{a^m_q, m \in \mathbf{Z}_+\}$ alone. Notice however that such a choice would be undesirable from the physical point of view since it would not allow us to study Hermitian solutions of the ensuing master equation.
Then the explicit action of the q-displacement operator on the generators of $\mathcal{U}_q(hw)$ reads $AdD^q_{\pm\alpha}(a_q) = a_q \mp \alpha$ and $AdD^{q\dagger}_{\mp\alpha}(a^\dagger_q) = a^\dagger_q \mp \alpha^*$. By means of these expressions we rewrite the action of the preceding q-transition operator on an analytic formal power series $f(a_q)$ as

$$T_{\phi_q}(f(a_q)) = [pAdD^q_{-\alpha} + (1-p)AdD^q_\alpha](f(a_q)) \; . \tag{31}$$

We wish to compute the limiting transition operator

$$\begin{aligned} T^q_t &\equiv T_{\phi^q_t} \equiv \lim_{n\to\infty}(T_{\phi_q})^n \\ &= \lim_{n\to\infty}[p(1 + ad(-\alpha A^\dagger_q + \alpha^* a_q) + \frac{1}{2}adad(-\alpha A^\dagger_q + \alpha^* a_q) + \cdots) \\ &+ (1-p)(1 + ad(\alpha A^\dagger_q - \alpha^* a_q) + \frac{1}{2}adad(\alpha A^\dagger_q - \alpha^* a_q) + \cdots]^n \; . \end{aligned} \tag{32}$$

If we introduce the parameters $t \in \mathbf{R}$ and $c, \gamma \in \mathbf{C}$ by means of the same relations (19) as in the $q = 1$ case, then we will obtain the limiting q-transition operator $T^q_t = e^{tad\mathcal{L}_q}$, where $\mathcal{L}_q = -cA^\dagger_q + c^* a_q + \gamma(A^\dagger_q)^2 - \gamma^* a^2_q - |\gamma|\,(A^\dagger_q a_q + a_q A^\dagger_q)$.

To simplify this q-master equation we will assume as in the undeformed case that the parameter γ is a complex variable with random argument of zero average and constant non zero magnitude. Then if we average over random γ the equation of motion then only terms proportional to the amplitude of γ will be retained. If in addition we consider the case of an symmetric random walk i.e $p = 1/2, c = 0$ the equation of motion becomes

$$\frac{d}{dt} f_t = -2\,|\gamma| \left[A_q^\dagger f_t a + a f_t A_q^\dagger - N f_t - f_t (N+1) \right] . \tag{33}$$

This is a q-quantum master equation of the Lindblad type[12] which will be shown to admit a solution in terms of a operator valued Appell system associated with the generator of that equation. We may introduce as in the preceding undeformed case the following operators

$$K_+ f = A_q^\dagger f a_q \quad K_- f = a_q f A_q^\dagger \quad K_0 f = \frac{1}{2}(A_q^\dagger a_q f + f a_q A_q^\dagger) , \tag{34}$$

and $K_c f = [A_q^\dagger a_q, f]$. These operators acting on the elements f of the enveloping algebra $\mathcal{U}_q(hw)$, generate the $su(1,1)$ Lie algebra defined as in eq. (24). In terms of these operators the q-quantum master equation (33) is cast in the form

$$\frac{d}{dt} f_t = -2\,|\gamma| \,(-2K_0 + K_+ + K_-) f_t . \tag{35}$$

Use of the disentangling theorem (Baker-Campbell-Hausdorff formula) of a general $SU(1,1)$ group element (c.f Appendix A), allows to express the solution of the quantum q-master equation in the form

$$\hat{f}_t = \exp(A_+K_+)\exp(\ln A_0 K_0)\exp(A_-K_-)(\hat{f}) = \exp(B_-K_-)\exp(\ln B_0 K_0)\exp(B_+K_+)(\hat{f}) , \tag{36}$$

if the normally or respectively the antinormally ordered BCH decomposition is used. Above we choose $f = \sum_{n \geq 0} c_n a_q^n$, to stand for the initial time operator which can be a general element of the subalgebra of $\mathcal{U}_q(hw)$ that is generated by the q-annihilation operator. Specifically in the case of normally ordered decomposition with initial operator taken as $f = a_q^t$ the solution of the quantum q-master equation is obtained by means of the actions issued in eq.(34). A straightforward calculation yields the solution:

$$f_t = \exp(A_+K_+)\exp(\ln A_0 K_0)\exp(A_-K_-)(a^t) = \sum_{k\geq 0}\sum_{l \geq 0}\sum_{m \geq 0}\sum_{r=0}^{l} \times$$

$$\frac{A_-^k}{k!}\frac{\overline{A}_0^l}{l!}\frac{A_+^m}{m!}\frac{1}{2^m}\begin{pmatrix} l \\ r \end{pmatrix}(a^\dagger)_q^{k+m}(N+k)^{l-r}(N+k+t+1)^r a_q^{k+t+m} , \tag{37}$$

where the A's have the same values as before. A similar solution can be obtained for the antinormal BCH decomposition. We can therefore state the results in the following

Proposition 2. *The solution of the quantum q-master equation $\frac{d}{dt} f_t = \mathcal{L}_q f_t$ where the operator $\mathcal{L}_q(f_t) = -2\,|\gamma| \left[A_q^\dagger f_t a_q + a_q f_t A_q^\dagger - N f_t - f_t(N+1)\right]$ of Lindblad type generates the semigroup of q-Markov transition operators $T_t^q = e^{t\mathcal{L}_q}$ acting on the enveloping algebra $\mathcal{U}_q(hw)$, is given by the associated $a_q^\dagger a_q$-valued Appell system which is given by equation (37).*

We note also that the dual q-master equation satisfied by the density operator can easily be solved along the above lines in terms of the associated Appell system.

7. Discussion. A novel way for constructing quantum master equations has been provided with solutions given by certain sets of operator valued functions that constitute a generalization of the concept of classical Appell polynomial. This entire approach is algebraic and utilizes concepts and tools from the powerfully structure of Hopf algebra. The choice of the dual partner of that algebra structure, namely the coherent states and their adjoint density operators, offers a chance to investigate random walks on non trivial geometries.

The prospect of such a framework is rich enough to allow for random walks constructed on e.g non commuting spaces with braided/smash structure[4] or on Lie groups, quantum groups and quantum modules and comodules. The kinds of Appell systems resulting in those cases might provide new challences to the theory of Special Functions. Some of these issues will be taken up in a forthcoming communication[5].

Appendix A. The disentangling theorem (Baker-Campbell-Hausdorff formula)[8] of a general $SU(1,1)$ group element[21] $g(a_+, a_0, a_-)$ in the normal $\{K_+^a K_0^b K_-^c : a, b, c \in \mathbf{Z}_+\}$, and antinormal $\{K_-^a K_0^b K_+^c : a, b, c \in \mathbf{Z}_+\}$ ordering of the generators of the enveloping algebra $\mathcal{U}(su(1,1))$ reads respectively:

$$
\begin{aligned}
g(a_+, a_0, a_-) &= \exp(\alpha_+ K_+ + \alpha_0 K_0 + \alpha_- K_-) \\
&= \exp(A_+ K_+) \exp(\ln A_0 K_0) \exp(A_- K_-) \,, \\
&= \exp(B_- K_-) \exp(\ln B_0 K_0) \exp(B_+ K_+) \,,
\end{aligned} \tag{38}
$$

where $A_\pm(a_0) = \frac{(a_\pm/\phi)\sinh\phi}{\cosh\phi - (a_0/2\phi)\sinh\phi}$, $A_0 = (\cosh\phi - (a_0/2\phi)\sinh\phi)^{-2}$ and $B_\pm(a_0) = -A_\pm(-a_0)$, $B_0 = (\cosh\phi + (a_0/2\phi)\sinh\phi)^2$, with $\phi^2 = ((\alpha_0/2)^2 - a_+ a_-)$. The relations between the two types of ordered decompositions is based on the formulae $A_\pm = \frac{B_0 B_\pm}{1 - B_0 B_+ B_-}$, $A_0 = \frac{B_0}{(1 - B_0 B_+ B_-)^2}$, and $B_\pm = \frac{A_\pm}{A_0 - A_+ A_-}$, $B_0 = 1/A_0(A_0 - A_+ A_-)^2$.

Appendix B. Relations among ordered basic monomials of the enveloping algebra $\mathcal{U}(hw)$[10]. From antinormal to normal ordering:

$$
a^i (a^\dagger)^j = \sum_{l=0}^{min(i,j)} d_{i,j}^l (a^\dagger)^{j-l} a^{j-l} = \sum_{l=0}^{min(i,j)} l! \begin{pmatrix} i \\ l \end{pmatrix} \begin{pmatrix} j \\ l \end{pmatrix} (a^\dagger)^{j-l} a^{j-l} \,, \tag{39}
$$

From number operator to normal ordering:

$$
N^k = \sum_{l=1}^{k} c_{k,l} (a^\dagger)^l a^l \,, \tag{40}
$$

where $\overline{c}_{k+1,l} = \overline{c}_{k,l-1} + l\overline{c}_{k,l}$, and these coefficients are recognized as the Stirling numbers of second kind.
From number operator to antinormal ordering:

$$
N^k = \sum_{l=1}^{k} \overline{d}_{k,l} a^l (a^\dagger)^l \,, \tag{41}
$$

where $\overline{d}_{k+1,l} = \overline{d}_{k,l-1} - (l+1)\overline{d}_{k,l}$, with $\overline{d}_{0,0} = 1$.

Relations among ordered basic monomials of the enveloping algebra $\mathcal{U}_q(hw)$[10]. From antinormal to normal ordering:

$$a_q^i (a^\dagger)_q^j = \sum_{l=0}^{min(i,j)} \overline{b}_{i,j}^l (a^\dagger)_q^{j-l} a_q^{j-l} = \sum_{l=0}^{min(i,j)} q^{l(l-i-j)+ij} [l]! \begin{bmatrix} i \\ l \end{bmatrix}_q \begin{bmatrix} j \\ l \end{bmatrix}_q (a^\dagger)_q^{j-l} a_q^{j-l} \ . \quad (42)$$

We note that for $q \to 1$ the $\overline{b}_{i,j}^l \to d_{i,j}^l$. From normal to antinormal ordering:

$$(a^\dagger)^i a^j = \sum_{l=0}^{min(i,j)} b_{i,j}^l a^{j-l} (a^\dagger)^{i-l} = \sum_{l=0}^{min(i,j)} (-)^l q^{l(l-i-j)-ij)} [l]! \begin{bmatrix} i \\ l \end{bmatrix}_q \begin{bmatrix} j \\ l \end{bmatrix}_q a^{j-l} (a^\dagger)^{i-l} \ . \quad (43)$$

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