Normal ordering for deformed boson operators and operator-valued deformed Stirling numbers

Jacob Katriel* and Maurice Kibler†

Institut de Physique Nucléaire de Lyon

IN2P3-CNRS et Université Claude Bernard

43 Boulevard du 11 Novembre 1918

F-69622 Villeurbanne Cedex, France

Abstract

The normal ordering formulae for powers of the boson number operator $\hat{n}$ are extended to deformed bosons. It is found that for the "M-type" deformed bosons, which satisfy $aa^\dagger - qa^\dagger a = 1$, the extension involves a set of deformed Stirling numbers which replace the Stirling numbers occurring in the conventional case. On the other hand, the deformed Stirling numbers which have to be introduced in the case of the "P-type" deformed bosons, which satisfy $aa^\dagger - qa^\dagger a = q^{-\hat{n}}$, are found to depend on the operator $\hat{n}$. This distinction between the two types of deformed bosons is in harmony with earlier observations made in the context of a study of the extended Campbell-Baker-Hausdorff formula.

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*Permanent address: Department of Chemistry, Technion - Israel Institute of Technology, Haifa 32000, Israel. email: chr09kt@technion.

†email: Kibler@ipnl.in2p3.fr.

1 Introduction

The transformation of a second-quantized operator into a normally ordered form, in which each term is written with the creation operators preceding the annihilation operators, has been found to simplify quantum mechanical calculations in a large and varied range of situations. Techniques for the accomplishment of this ordering have been developed and are widely utilized [1,2]. A particular subclass of problems and techniques involves situations in which the operators of interest commute with the number operator. More specifically, one is interested in transforming an operator which is a function of the number operator into a normally ordered form, or transforming an operator each of whose terms has an equal number of creation and annihilation operators corresponding to each degree of freedom, into an equivalent operator expressed in terms of the number operator only.

In the present article we consider the corresponding problem for the deformed bosons which have been investigated very extensively in the last three years [3,4] in connection with the recent interest in the properties and applications of quantum groups.

2 Stirling and deformed Stirling numbers

The Stirling numbers of the first (s) and second (S) kinds were introduced in connection with the expression for a descending product of a variable x as a linear combination of integral and positive powers of that variable, and the inverse relation, respectively [5]

$$x\,(x-1)\ \cdots\ (x-k+1) = \sum_{m=1}^{k}\ s(k,m)\ x^m \tag{1}$$

$$x^m = \sum_{k=1}^{m} S(m,k)\ x\,(x-1)\ \cdots\ (x-k+1)\ . \tag{2}$$

Using these defining relations it is easy to show that the Stirling numbers satisfy the recurrence relations

$$s(k+1,m) = s(k,m-1) - k\ s(k,m) \tag{3}$$

and

$$S(m+1,k) = S(m,k-1) + k\, S(m,k)\ , \tag{4}$$

with the initial values $s(1,1) = S(1,1) = 1$ and the "boundary conditions" $s(i,j) = S(i,j) = 0$ for $i < 1$, $j < 1$ and for $i < j$. The combinatorial significance of the Stirling numbers has been amply discussed [6].

Several generalisations of the Stirling numbers appeared in the mathematical literature [7-12]. In anticipation of further development we shall refer to them generically as deformed Stirling numbers. In this context we wish to distinguish between the two widely used forms of "deformed numbers" $[x]_M = \frac{q^x-1}{q-1}$, the usual choice in the mathematical literature on q-analysis [13], and $[x]_P = \frac{q^x-q^{-x}}{q-q^{-1}}$, which is common to the recent physical literature and to the literature on quantum groups. A generalisation was recently proposed by Wachs and White [12], which can be written in the form $[x]_G = \frac{q^x-p^x}{q-p}$. This form contains $[x]_M$ and $[x]_P$ as special cases, corresponding to the choices $p = 1$ and $p = q^{-1}$, respectively. We shall write $[x]_{M(q)}$, $[x]_{P(q)}$ and $[x]_{G(p,q)}$ instead of the symbols introduced above whenever the choice of the parameters q and/or p will have to be explicated. The identities

$$[x]_{P(q)} = q^{-x+1}\ [x]_{M(q^2)} \qquad [x]_{G(p,q)} = p^{x-1}\ [x]_{M(q/p)} \qquad [x]_{G(p,q)} = (\sqrt{pq})^{x-1}\ [x]_{P(\sqrt{q/p})} \tag{5}$$

illustrate the notation and exhibit some of the elementary properties of these deformed numbers.

One of the generalisations of the Stirling numbers [10] involves a descending product of M-type deformed numbers expressed in terms of the powers of the M-type deformed number $[x]_M$

$$[x]_M\ [x-1]_M\ \cdots\ [x-k+1]_M = \sum_{m=1}^{k}\ s_q(k,m)\ [x]_M^m \tag{6}$$

and the corresponding inverse relation

$$[x]_M^m = \sum_{k=1}^{m}\ S_q(m,k)\ [x]_M\ [x-1]_M\ \cdots\ [x-k+1]_M\ . \tag{7}$$

Using the defining relations it is easy to show that the deformed Stirling numbers $s_q(k,m)$ and $S_q(m,k)$, which are referred to in the mathematical literature as q-Stirling numbers of the first and second kind, respectively, satisfy the recurrence relations

$$s_q(k+1,m) = q^{-k} \Big(s_q(k,m-1) - [k]_M \; s_q(k,m) \Big) \tag{8}$$

and

$$S_q(m+1,k) = q^{k-1} \; S_q(m,k-1) + [k]_M \; S_q(m,k) \; , \tag{9}$$

with "boundary conditions" and initial values identical with those specified above for the conventional Stirling numbers.

A slight modification in the form of the descending product, replacing the factors $[x-i]_M$ by $[x]_M - [i]_M$, results in the relations [8-10]

$$[x]_M \; ([x]_M - [1]_M) \; \cdots \; ([x]_M - [k-1]_M) = \sum_{m=1}^{k} \; \tilde{s}_q(k,m) \; [x]_M^m \tag{10}$$

and

$$[x]_M^m = \sum_{k=1}^{m} \; \tilde{S}_q(m,k) \; [x]_M \; ([x]_M - [1]_M) \; \cdots \; ([x]_M - [k-1]_M) \; . \tag{11}$$

Starting with these defining relations and using the identity [9]

$$[a]_M - [b]_M = q^b [a-b]_M \tag{12}$$

we obtain the recurrence relations

$$\tilde{s}_q(k+1,m) = \tilde{s}_q(k,m-1) - [k]_M \; \tilde{s}_q(k,m) \tag{13}$$

and

$$\tilde{S}_q(m+1,k) = \tilde{S}_q(m,k-1) + [k]_M \; \tilde{S}_q(m,k) \; , \tag{14}$$

where the "boundary conditions" and initial values are, again, as above. Note that

$$\tilde{s}_q(k,m) = q^{k(k-1)/2} \; s_q(k,m) \qquad\qquad \tilde{S}_q(m,k) = q^{-k(k-1)/2} \; S_q(m,k) \; . \tag{15}$$

The two sets of deformed Stirling numbers of the first and second kinds, as well as the conventional Stirling numbers to which they reduce in the limit $q \to 1$, satisfy the following dual relations

$$\sum_{m=1}^{k} s_q(k,m) \; S_q(m,k') = \delta(k,k') \tag{16}$$

and

$$\sum_{k=1}^{m} S_q(m,k)\; s_q(k,m') = \delta(m,m')\ . \tag{17}$$

An additional set of deformed Stirling numbers of the second kind was recently introduced by Wachs and White [12]. Their definition is motivated by combinatorial considerations and has no algebraic origin. Their recurrence relation reads

$$S_{p,q}(m+1,k) = p^{k-1}\; S_{p,q}(m,k-1) + [k]_G\; S_{p,q}(m,k) \tag{18}$$

and it reduces to (14) for $p = 1$.

3 Some algebraic properties of deformed boson operators

In the context of recent interest in quantum groups and their realization, three types of deformed boson operators have been introduced [3,4,14]. The most straightforward definition starts by postulating a Fock space on which creation (a), annihilation ($a^\dagger$) and number ($\hat{n}$) operators are defined in analogy with the conventional boson operators. The general form postulated is

$$a\,|l>= \sqrt{[l]}\;|l-1> \qquad a^\dagger\,|l>= \sqrt{[l+1]}\;|l+1> \qquad \hat{n}\,|l>= l\,|l>\ . \tag{19}$$

It follows immediately that $a^\dagger a = [\hat{n}]$ and $aa^\dagger = [\hat{n}+1]$. The two widely used forms of the deformed bosons are obtained by choosing either $[l] = [l]_M = \frac{q^l-1}{q-1}$ or $[l] = [l]_P = \frac{q^l-q^{-l}}{q-q^{-1}}$. A generalisation was recently proposed by Chakrabarti and Jagannathan [14]. We shall adhere to the notation introduced by Wachs and White [12] and write this generalisation in the form $[l] = [l]_G = \frac{q^l-p^l}{q-p}$, which is trivially modified relative to that introduced in Ref. [14]. As a consequence of a remark made in the previous section, this third type of deformed boson contains the first two as special cases.

The deformed bosons as defined by Eq. (19) are not associated with any a priori specification of a (possibly deformed) commutation relation. Choosing a parameter Q, which does not have to be related to the two parameters p and q so far introduced, the deformed bosons are found to satisfy the deformed commutation relation

$$[a, a^\dagger]_Q = aa^\dagger - Qa^\dagger a = \phi(\hat{n}) = \frac{1}{q-p}\Big(q^{\hat{n}}(q-Q) + p^{\hat{n}}(Q-p)\Big) \; . \tag{20}$$

Since the choice of Q is arbitrary we can opt to be guided by the requirement that the form of $\phi(\hat{n})$ be as simple as possible or by some other relevant criterion. The conventional choice $Q = q$, to which we will eventually adhere, results in

$$a\, a^\dagger - q\, a^\dagger\, a = \phi_M(\hat{n}) = 1 \; , \tag{21}$$

$$a\, a^\dagger - q\, a^\dagger\, a = \phi_P(\hat{n}) = q^{-\hat{n}} \tag{22}$$

and

$$a\, a^\dagger - q\, a^\dagger\, a = \phi_G(\hat{n}) = p^{\hat{n}} \tag{23}$$

for the M-type, P-type and G-type bosons, respectively. We do not label the creation and annihilation operators by indices such as M, P or G because the nature of these operators is always obvious from the context. The choice $Q = p$ results in $\phi(\hat{n}) = q^{\hat{n}}$ for all the three cases. For the M-type bosons ($p = 1$) this choice implies $Q = 1$, i.e., the deformed commutation relation becomes $a\, a^\dagger - a^\dagger\, a = q^{\hat{n}}$. For the P-type bosons ($p = q^{-1}$) this choice is the familiar alternative to Eq. (22), namely $a\, a^\dagger - q^{-1} a^\dagger\, a = q^{\hat{n}}$. In a recent study of the extension of the Campbell-Baker-Hausdorff formula to deformed bosons [16], it was noted that the choice $Q = q$ is the most suitable one for the M-type bosons, but that $Q = q + q^{-1} - 1$ seems to have some advantages for the P-type bosons. From the same point of view, one would choose $Q = q + p - 1$ for the G-type bosons.

We shall also need the relation

$$[a^k, a^\dagger]_{Q^k} = \Phi(k, \hat{n})\; a^{k-1} \tag{24}$$

which can be viewed as an extension of Eq. (20) in the sense that $\Phi(1, \hat{n}) = \phi(\hat{n})$. One easily finds that

$$\Phi(k, \hat{n}) = \frac{1}{q-p}\Big(q^{\hat{n}}\; (q-Q)\; [k]_{G(Q,q)} + p^{\hat{n}}\; (Q-p)\; [k]_{G(Q,p)}\Big) \; . \tag{25}$$

We shall retain the conventional choice $Q = q$ for the three cases specified above. With this choice we get

$$\Phi_M(k, \hat{n}) = [k]_M \qquad \Phi_P(k, \hat{n}) = [k]_P \; q^{-\hat{n}} \qquad \Phi_G(k, \hat{n}) = [k]_G \; p^{\hat{n}} \; . \tag{26}$$

4 Normal ordering of powers of the deformed number operator

The relevance of the ordinary Stirling numbers to the normal ordering of powers of the boson number operator was demonstrated in Ref. [15]. In the present section we consider some normal ordering properties of the deformed bosons specified by the parameter choice $p = 1$ and $Q = q$, which corresponds to the M-type boson operators and to the deformed commutation relation (21). Up to a trivial interchange of p and q this is the only combination of parameters for which the deformed commutator does not depend on $\hat{n}$. The other types of deformed boson operators are considered in the following section where it is found that they differ in a significant respect from the case presently considered.

In order to express an integral power of $[\hat{n}]_M$ in a normally ordered form we can either formally write such an expansion and obtain a recurrence relation for the coefficients by applying Eq. (21) or use the deformed Stirling numbers of the second kind directly. We shall present both approaches because of the intrinsic interest of each one of them.

In the direct approach, we start from the expansion

$$[\hat{n}]_M^m = (a^\dagger a)^m = \sum_{k=1}^{m} \; c(m,k) \; (a^\dagger)^k a^k \; . \tag{27}$$

Expressing $(a^\dagger a)^{m+1}$ by means of Eq. (27) and using Eq. (21), we obtain a recurrence relation which is identical with the one satisfied by $S_q(m,k)$, Eq. (9). Moreover, it is obvious from the defining equation (27) that $c(1,1) = S_q(1,1) = 1$. Thus, $c(m,k) = S_q(m,k)$.

A different derivation can be obtained by using the identity

$$\prod_{i=0}^{k-1} \; [\hat{n} - i]_M = (a^\dagger)^k a^k \; . \tag{28}$$

This identity follows by noting that application of both sides of Eq. (28) on any member of the complete set $\{|l>; l = 0, 1, \cdots\}$ of eigenstates of the number operator results in $\prod_{i=0}^{k-1} [l-i]_M$. Using Eq. (7) we obtain

$$[\hat{n}]_M^m = \sum_{k=1}^{m} S_q(m,k) \prod_{i=0}^{k-1} [\hat{n}-i]_M \tag{29}$$

and substituting Eq. (28) we get the desired normally ordered expansion

$$[\hat{n}]_M^m = \sum_{k=1}^{m} \; S_q(m,k) \; (a^\dagger)^k a^k \; . \tag{30}$$

We note in passing that an equivalent expansion could have been obtained starting from the identity

$$\prod_{i=0}^{k-1} ([\hat{n}]_M - [i]_M) = q^{k(k-1)/2} \; (a^\dagger)^k a^k \; . \tag{31}$$

This identity can be proved either by induction or by considering the effect of both sides on the complete set of eigenstates of the number operator. Using (11) and (31), we obtain the normally ordered expansion of $[\hat{n}]_M^m$ in the form

$$[\hat{n}]_M^m = \sum_{k=1}^{m} \; \tilde{S}_q(m,k) \; q^{k(k-1)/2} \; (a^\dagger)^k \; a^k \tag{32}$$

which is related to (30) by Eq. (15).

In order to obtain the inverse relation, expressing a normally ordered product as a function of the number operator, we note that Eqs. (6) and (28) lead to

$$(a^\dagger)^k a^k = [\hat{n}]_M \; [\hat{n}-1]_M \cdots [\hat{n}-k+1]_M = \sum_{m=1}^{k} \; s_q(k,m) \; [\hat{n}]_M^m \; . \tag{33}$$

5 Operator-valued deformed Stirling numbers

In the present section, we attempt to derive the normally ordered expansion of a power of the number operator for arbitrarily deformed bosons. Allowing p, q and Q to be arbitrary, we demand

$$[\hat{n}]_G^m = \sum_{k=1}^{m} (a^\dagger)^k \; \hat{S}(m,k,\hat{n}) \; a^k \; . \tag{34}$$

Using the general relation (24), we derive the recurrence relation

$$\hat{S}(m+1,k,\hat{n}) = Q^{k-1}\ \hat{S}(m,k-1,\hat{n}+1) + \hat{S}(m,k,\hat{n})\ \Phi(k,\hat{n})\ . \qquad (35)$$

The "boundary conditions" and initial values, for all values of $\hat{n}$, are the same as those following Eq. (4).

The M-type bosons ($p=1$), with the choice $Q=q$ which yields $\Phi_M(k,\hat{n}) = [k]_M$, were studied in section 4. For this case, $\hat{S}(m,k,\hat{n})$ does not depend on $\hat{n}$. More specifically, Eq. (35) then reduces to Eq. (9). For the G-type bosons, we found in section 3 that by choosing $Q=q$ we obtain $\Phi_G(k,\hat{n}) = [k]_G\ p^{\hat{n}}$; consequently, we have

$$\hat{S}_G(m+1,k,\hat{n}) = q^{k-1}\ \hat{S}_G(m,k-1,\hat{n}+1) + \hat{S}_G(m,k,\hat{n})\ [k]_G\ p^{\hat{n}}\ . \qquad (36)$$

Note that in the general case $\hat{S}_G(m,k,\hat{n})$ depends on the operator $\hat{n}$. The special cases $p=1$ and $p=q^{-1}$ are contained in Eq. (36). The dependence of $\hat{S}_G(m,k,\hat{n})$ on $\hat{n}$ for all cases except $p=1$ can be taken to imply that we have actually failed to obtain a normally ordered expansion for $[\hat{n}]_G^m$ in terms of a finite sum in $(a^\dagger)^k a^k$ with $k=1,\ 2,\ \cdots,\ m$.

The structure of the recurrence relation (36) indicates that the dependence on $\hat{n}$ of the deformed Stirling numbers $\hat{S}_G(m,k,\hat{n})$ can be expressed in terms of the factor $p^{(m-k)\hat{n}}$. Defining the ($\hat{n}$-independent) reduced Stirling numbers of the second kind $\Xi(m,k)$ through

$$\hat{S}_G(m,k,\hat{n}) = q^{k(k-1)/2}\ p^{(m-k)\hat{n}}\ \Xi(m,k) \qquad (37)$$

we obtain the recurrence relation

$$\Xi(m+1,k) = p^{m-k+1}\ \Xi(m,k-1) + [k]_G\ \Xi(m,k) \qquad (38)$$

with the initial condition $\Xi(1,1)=1$.

To obtain the "inverse relation" to (34), expressing a normally ordered term $(a^\dagger)^k a^k$ by means of a polynomial in $[\hat{n}]_G$, we need the "G-arithmetic" identity

$$[a-b]_G = q^{-b}([a]_G - p^{a-b}[b]_G)\ , \qquad (39)$$

which follows from the two identities

$$[a+b]_G = q^b [a]_G + p^a [b]_G \tag{40}$$

and

$$[-b]_G = -(pq)^{-b}\ [b]_G\ . \tag{41}$$

We now proceed to obtain the desired relation

$$(a^\dagger)^k a^k = \sum_{m=1}^{k} \hat{s}_G(k,m,\hat{n})\ [\hat{n}]_G^m\ . \tag{42}$$

Since $(a^\dagger)^{k+1}a^{k+1} = (a^\dagger)^k\ [\hat{n}]_G\ a^k = (a^\dagger)^k\ a^k\ [\hat{n}-k]_G$, we can use Eqs. (39) and (42) to obtain the recurrence relation

$$\hat{s}_G(k+1,m,\hat{n}) = q^{-k}\Big(\hat{s}_G(k,m-1,\hat{n}) - p^{\hat{n}-k}\ [k]_G\ \hat{s}_G(k,m,\hat{n})\Big)\ . \tag{43}$$

Note that for $p=1$ this recurrence relation reduces to Eq. (8).

Introducing the ($\hat{n}$-independent) reduced Stirling numbers of the first kind $\xi(k,m)$ such that

$$\hat{s}_G(k,m,\hat{n}) = q^{-k(k-1)/2}\ p^{(k-m)\hat{n}}\ \xi(k,m) \tag{44}$$

in Eq. (43), we obtain the recurrence relation

$$\xi(k+1,m) = \xi(k,m-1) - p^{-k}\ [k]_G\ \xi(k,m)\ . \tag{45}$$

The exponential dependence on $\hat{n}$ of the deformed Stirling numbers of the first kind, $\hat{s}_G(k,m,\hat{n})$, means that we have not been able to express $(a^\dagger)^k a^k$ as a *polynomial* in $\hat{n}$ but we did express it as a *function* of $\hat{n}$.

In order to derive the bi-orthogonality relations between the deformed Stirling numbers of the first and second kinds, we first rewrite Eq. (34) in the form

$$[\hat{n}]_G^m = \sum_{k=1}^{m} (a^\dagger)^k\ a^k\ \hat{S}_G(m,k,\hat{n}-k)\ . \tag{46}$$

Using Eq. (37) we obtain

$$\hat{S}_G(m,k,\hat{n}-k) = p^{k(k-m)}\hat{S}_G(m,k,\hat{n})\ . \tag{47}$$

Defining $\Xi'(m,k) = p^{k(k-m)}\Xi(m,k)$, we obtain relations of the form of Eqs. (16) and (17) with $\Xi'(m,k)$ replacing $S_q(m,k)$ and $\xi(k,m)$ replacing $s_q(k,m)$.

6 A generating function for the deformed Stirling numbers of the first kind

We start by transforming the q-binomial theorem [13] into a G-binomial theorem. By introducing the symbol

$$(\lambda; x)^{(l)} = (\lambda + x)\ (p\lambda + qx)\ (p^2\lambda + q^2 x) \cdots (p^{l-1}\lambda + q^{l-1}x) \tag{48}$$

we have

$$(\lambda; x)^{(l)} = \sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix}_G p^{i(i-1)/2}\ q^{(l-i)(l-i-1)/2}\ \lambda^i\ x^{l-i}\ , \tag{49}$$

where

$$\begin{bmatrix} l \\ i \end{bmatrix}_G = \frac{[l]_G!}{[i]_G!\ [l-i]_G!} \tag{50}$$

is a G-binomial coefficient and $[k]_G! = [1]_G[2]_G \cdots [k]_G$. Equation (49) can be proved by induction, using the G-binomial coefficient recurrence relation

$$\begin{bmatrix} l + 1 \\ i \end{bmatrix}_G = p^{\,l+1-i} \begin{bmatrix} l \\ i - 1 \end{bmatrix}_G + q^i \begin{bmatrix} l \\ i \end{bmatrix}_G\ , \tag{51}$$

which follows from the definition of the G-binomial coefficient on using the G-arithmetic relation (40).

Now, from the identity

$$\frac{(a^\dagger)^k a^k}{[k]_G!}\ |l> = \begin{bmatrix} l \\ k \end{bmatrix}_G |l> \tag{52}$$

we obtain

$$\sum_{k=0}^{m} p^{k(k-1)/2}\ q^{(l-k)(l-k-1)/2}\ \lambda^k\ \frac{(a^\dagger)^k a^k}{[k]_G!}\ |l> = (\lambda; 1)^{(l)}\ |l> \tag{53}$$

which can be written as an operator identity

$$\sum_{k=0}^{\infty} p^{k(k-1)/2}\ q^{(\hat{n}-k)(\hat{n}-k-1)/2}\ \lambda^k\ \frac{(a^\dagger)^k a^k}{[k]_G!} = (\lambda; 1)^{(\hat{n})}\ . \tag{54}$$

To obtain an expression for $(a^\dagger)^k a^k$ as a function of the number operator $\hat{n}$, we have to expand the right-hand side of Eq. (54) in powers of λ. The coefficient of λ^k can be extracted by writing

$$(a^\dagger)^k a^k = \frac{[k]_G!}{k!}\, p^{-k(k-1)/2}\, q^{-(\hat{n}-k)(\hat{n}-k-1)/2}\, \frac{\partial^k}{\partial\lambda^k}(\lambda;1)^{(\hat{n})}\Big|_{\lambda=0} . \tag{55}$$

The identities

$$[m]_{G(p^k,q^k)} = \frac{[km]_{G(p,q)}}{[k]_{G(p,q)}} \tag{56}$$

and

$$[km]_{G(p,q)} = \sum_{i=1}^{k} \begin{pmatrix} k \\ i \end{pmatrix} (q-p)^{i-1}\, [m]^i_{G(p,q)}\, p^{m(k-i)} \tag{57}$$

are found to be useful when implementing Eq. (55). (To avoid possible confusion we point out that the symbol appearing in Eq. (57) is the conventional binomial coefficient.) Note that for the conventional bosons, for which $p = q = 1$, Eq. (55) reduces to an expression [15] which can be related to the well-known generating function for the conventional Stirling numbers of the first kind [5].

7 Discussion

In the present article we found that the normal ordering formulae for powers of the boson number operator can be extended in a simple and natural way to the M-type bosons, which satisfy $[a, a^\dagger]_q = 1$. However, for the P-type bosons, which satisfy $[a, a^\dagger]_q = q^{-\hat{n}}$, as well as for the more general G-type bosons, we found that the extension of the conventional boson analysis results in "normal-ordering" expressions with $\hat{n}$-dependent coefficients.

The marked difference between the M-type bosons and all the others has already been noted before, in the context of the extension of the Campbell-Baker-Hausdorff formula for products of exponential operators [16]. While the observations pointed out above set apart the M-type bosons, the following may be taken to set apart the P-type bosons disfavourably, within the general set of G-type bosons: Taking the Hamiltonian of the deformed harmonic oscillator to be $\mathcal{H} = \frac{\hbar\omega_0}{2}(a^\dagger a + a a^\dagger)$ and expanding in powers of $s = \ln q$ and $t = \ln p$ (which we assume to be sufficiently small), we find that

$$\mathcal{H} = \hbar\omega_0[\frac{s+t}{8} \; + \; (1 - \frac{s+t}{2})(\hat{n} + \frac{1}{2}) \; + \; \frac{s+t}{2}(\hat{n} + \frac{1}{2})^2 + \cdots] . \tag{58}$$

Apart from an irrelevant shift of the energy zero and a renormalization of the frequency into $\omega = \omega_0(1 - \frac{s+t}{2})$ this Hamiltonian contains a quadratic anharmonicity unless $s = -t$, i.e., unless $p = q^{-1}$. It is true that a quadratic anharmonicity will emerge even for the P-type oscillator ($p = q^{-1}$) as a residue of the fourth order term, but it will be associated with a fourth order anharmonicity which may well be inconsistent with the experimental spectrum of some system of interest, such as a diatomic molecule.

We finally point out that a coordinate and a conjugate momentum can be defined for the deformed oscillator by means of the relations $\hat{x} = (a^\dagger + a)/\sqrt{2}$ and $\hat{p} = i(a^\dagger - a)/\sqrt{2}$. Application of Eq. (20) with the choice $Q = 1$ results in (for $\hbar\omega_0 = 1$)

$$[\hat{x}, \hat{p}] = i[a, a^\dagger] = i[1 + \Big(s + t - \frac{(s+t)^2}{2}\Big)\hat{n} + \frac{s^2 + st + t^2}{2}\hat{n}(\hat{n} + 1) + \cdots] \ , \tag{59}$$

from which follows the deformed uncertainty relation.

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