Integrality of L^2-Betti numbers

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Last compiled February 1, 2008; last edited October 15, 2001 or later

Abstract

The Atiyah conjecture predicts that the L^2-Betti numbers of a finite CW-complex with torsion-free fundamental group are integers. We show that the Atiyah conjecture holds (with an additional technical condition) for direct and inverse limits of directed systems of groups for which it is true. As a corollary it holds for residually torsion-free solvable groups, e.g. for pure braid groups or for positive 1-relator groups with torsion free abelianization.

Putting everything together we establish a new class of groups for which the Atiyah conjecture holds, which contains all free groups and in particular is closed under taking subgroups, direct sums, free products, extensions with elementary amenable quotient, and under direct and inverse limits of directed systems.

MSC: 55N25 (homology with local coefficients), 16S34 (group rings, Laurent rings), 46L50 (non-commutative measure theory)

1 Introduction

1.1. Remark. This is a corrected version of an older paper with the same title. The proof of one of the basic results of the earlier version contains a gap, as was kindly pointed out to me by Pere Ara. This gap could not be fixed. Consequently, in this new version everything based on this result had to be removed.

In [1] Atiyah introduced L^2-Betti numbers of closed manifolds in terms of the kernel of the Laplacian on the universal covering. There, he asked what the possible values of these numbers are, in particular whether they are always integers if the fundamental group is torsion-free. We call this the Atiyah conjecture. More precisely, we consider the following algebraic question:

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1.2. Definition. Let $\mathbb{Z} \subset \Lambda \subset \mathbb{Q}$ be an additive subgroup of the rationals, and K a subring of $\mathbb{C}$. We say a discrete group G fulfills the *Atiyah conjecture of order Λ over KG* if

$$\dim_G(\ker A) \in \Lambda \qquad \forall A \in M(m \times n, KG),$$

where $\ker A$ is the kernel of the induced map $A\colon l^2(G)^n \to l^2(G)^m$. Let $\mathrm{pr}_{\ker A}$ be the projection onto $\ker A$. Then

$$\dim_G(\ker A) := \mathrm{tr}_G(\mathrm{pr}_{\ker A}) := \sum_{i=1}^{n} \langle \mathrm{pr}_{\ker A} e_i, e_i \rangle_{l^2(G)^n},$$

where $e_i \in l^2(G)^n$ is the vector with the trivial element of $G \subset l^2(G)$ at the i^{th}-position and zeros elsewhere. tr_G is the canonical finite trace on the von Neumann algebra of operators on $l^2(G)^n$ commuting with the right G-action, the so called *Hilbert $\mathcal{N}G$-module maps.*

The group G is said to fulfill the *strong Atiyah conjecture over KG* if Λ is generated by $\{|F|^{-1} \mid F < G \text{ finite}\}$.

1.3. Remark. In [15] it is shown that the so called Lamplighter group G does not satisfy the Atiyah conjecture over $\mathbb{Z}G$. However, there is no bound on the orders of finite subgroups of G, in contrast to all examples of groups for which the strong Atiyah conjecture is known.

In Definition 1.2, we can replace the kernels by the closures of the images or by the cokernels because these numbers by additivity of the G-dimension [20, 1.4] pairwise sum up to the dimension of the domain or range, which is an integer. Replacing A by A^*A we may also assume that $n = m$. Because we can multiply A with a unit in Q without changing its kernel, the Atiyah conjecture over KG implies the Atiyah conjecture over QG, where Q is the quotient field of K in $\mathbb{C}$. Usually therefore we will assume that K is a field.

The Atiyah conjecture of order Λ over $\mathbb{Z}G$ is equivalent to the statement that all L^2-Betti numbers of finite CW-complexes with fundamental group G are elements of Λ [20, Lemma 2.2], i.e. to the original question of Atiyah.

Fix a coefficient field $K \subset \mathbb{C}$. If not stated otherwise, the term “Atiyah conjecture” means “Atiyah conjecture over KG”.

We introduce a new method of proof for the Atiyah conjecture which is based on the approximation results in [26]. In [26] it is proved that in many cases if a group G is the direct or inverse limit of a directed system of groups G_i, then L^2-Betti numbers over G are limits of L^2-Betti numbers defined over the groups G_i. More precisely:

1.4. Definition. (compare [26, 1.11]) Let $\mathcal{G}$ be the smallest class of groups which contains the trivial group, is subgroup closed and is closed under direct or inverse limits of directed systems, and under extensions with amenable quotient.

The amenable quotient is not necessarily a quotient group, one relaxes this to a suitable notion of an amenable (discrete) quotient space (by a subgroup which is not normal) (compare [26, 4.1]).

This class of groups is very large. It contains most groups which naturally occur in geometry, e.g. all residually finite groups. It would be interesting to find an example of a group which does not belong to $\mathcal{G}$.

1.5. Proposition. *Let $G_i \in \mathcal{G}$ (with $\mathcal{G}$ as in Definition 1.4) be a directed system of torsion-free groups which fulfill the strong Atiyah conjecture. Then their (direct or inverse) limit G fulfills the strong Atiyah conjecture over $\mathbb{Q}G$.*

1.6. Corollary. *The strong Atiyah conjecture over $\mathbb{Q}G$ is true if G is a pure braid group, or a positive 1-relator group (in the sense of [5]) with torsion free abelianization.*

The corollary will be explained in Example 2.8.

There is one significant difference to all other results about the Atiyah conjecture obtained so far: the approximation result is proved only over $\mathbb{Q}G$ instead of $\mathbb{C}G$. This does not effect the original question of Atiyah. But it is relevant for the zero divisor conjecture (by Lemma 1.7). In a forthcoming paper [10] we will show how to enlarge $\mathbb{Q}$ at least to the field of algebraic numbers in $\mathbb{C}$.

Proposition 1.5 can be used to give an alternative proof of the Atiyah conjecture for free groups, compare Example 2.5.

Linnell obtained the most general positive results about the Atiyah conjecture so far [17, 18]. His interest stems from the zero divisor conjecture and the following observation (compare e.g. [16]):

1.7. Lemma. *If for a torsion-free group the strong Atiyah conjecture is true over KG then the ring KG has no non-trivial zero divisors.*

Linnell uses essentially two lines of argument for the Atiyah conjecture: algebraic considerations using K-theoretic information (in particular Moody's induction theorem [23, Theorem 1]) to obtain the Atiyah conjecture for elementary amenable groups or for extensions with elementary amenable quotient.

For a free group F, Linnell devised a completely different argument, making use of Fredholm module techniques which were used before to prove that there are no non-trivial projectors in C_r^*F.

Linnell defines [17, 18]:

1.8. Definition. The class $\mathcal{C}$ is the smallest class of groups which contains all free groups and is closed under directed union and extensions with virtually abelian quotients. The class $\mathcal{C}'$ consists of extensions of direct sums of free groups with elementary amenable quotient.

Putting his two approaches together, Linnell proves in [17, 1.5]

1.9. Theorem. *The strong Atiyah conjecture over $\mathbb{C}G$ is true for groups in the classes $\mathcal{C}$ which have a bound on the orders of finite subgroups.*

The proof of the Atiyah conjecture for the class $\mathcal{C}'$ given in [18] contains a gap (which parallels the gap mentioned in Remark 1.1). However, we will shown in Corollary 4.4 how to fill this gap and conclude that the strong Atiyah conjecture over $\mathbb{Q}G$ is true if $G \in \mathcal{C}'$.

We use the new results to define the following class of groups for which the Atiyah conjecture is true:

1.10. Definition. Let $\mathcal{D}$ be the smallest non-empty class of groups such that:

(1) If G is torsion-free and A is elementary amenable, and we have a projection $p\colon G \to A$ such that $p^{-1}(E) \in \mathcal{D}$ for every finite subgroup E of A, then $G \in \mathcal{D}$.

(2) $\mathcal{D}$ is subgroup closed.

(3) Let $G_i \in \mathcal{D}$ be a directed system of groups and G its (direct or inverse) limit. Then $G \in \mathcal{D}$.

Observe that $\mathcal{D}$ contains only torsion-free groups.

1.11. Theorem. *If $G \in \mathcal{D}$ then the strong Atiyah conjecture over $\mathbb{Q}G$ is true for G.*

1.12. Corollary. *If $G \in \mathcal{D}$ then the ring $\mathbb{Q}G$ does not contain non-trivial zero divisors.*

1.13. Proposition. *All torsion-free groups in $\mathcal{C}$ and $\mathcal{C}'$ are also contained in $\mathcal{D}$. Moreover, $\mathcal{D}$ is closed under direct sums, direct products and free products.*

$\mathcal{C}$ is not closed under direct sums. $\mathcal{C}'$ has this property, but it is not closed under free product. $\mathcal{D}$ is an enlargement which repairs these deficits.

1.14. Example. It follows that $G := \mathbb{Z} * (\mathbb{Z} * \mathbb{Z} \times \mathbb{Z} * \mathbb{Z})$ fulfills the Atiyah conjecture over $\mathbb{Q}G$. The L^2-Betti numbers of G are computed as $b_1^{(2)}(G) = 1 = b_2^{(2)}(G)$ and $b_k^{(2)}(G) = 0$ for $k \neq 1, 2$. By [24, 9.3] $G \notin \mathcal{C}$. Using ideas of Reich, one can also show $G \notin \mathcal{C}'$, so that indeed we cover additional groups.

Organization of the paper

Section 2 gives an overview over elementary results about the Atiyah conjecture. We also prove Proposition 1.5 and discuss applications of this proposition. In Sections 3 and 4 we prove Theorem 1.11 and Proposition 1.13. Section 5 describes possible generalizations. In Section 6 we extend Linnell's proof of the Atiyah conjecture for free groups to another class of groups, the so called treelike groups, and characterize the treelike groups.

2 The Atiyah conjecture for limits of groups

Here we collect a few statements where the Atiyah conjecture for some groups implies its validity for other groups.

The following propositions are well known.

2.1. Proposition. *([18, 8.6])*
If H is a subgroup of index n in G and H fulfills the Atiyah conjecture of order $\Lambda \subset \mathbb{Q}$ over KH, then G fulfills the Atiyah conjecture of order $\frac{1}{n}\Lambda$ over KG.

2.2. Proposition. *If G fulfills the Atiyah conjecture of order $\Lambda \subset \mathbb{Q}$ over KG, and if U is a subgroup of G, then U fulfills the Atiyah conjecture of order Λ over KH.*

Proof. This follows from the fact that the U-dimension of the kernel of a matrix over KU acting on $l^2(U)^n$ coincides with the G-dimension of the same matrix, considered as an operator on $l^2(G)^n$ [26, 3.1]. □

2.3. Proposition. *Let G be the directed union of groups $\{G_i\}_{i\in I}$ and assume that each G_i fulfills the Atiyah conjecture of order $\Lambda_i \subset \mathbb{Q}$ over KG_i. Then G fulfills the Atiyah conjecture of order Λ over KG, where Λ is the additive subgroup of $\mathbb{Q}$ generated by $\{\Lambda_i\}_{i\in I}$.*

Proof. A matrix over KG, having only finitely many non-trivial coefficients, already is a matrix over KG_i for some i. The G_i-dimension and the G-dimension of the kernel of the matrix coincide by [26, 3.1]. □

For groups in $\mathcal{G}$ we have good approximation results for L^2-Betti numbers [26, 6.9]. We use these to give a proof of the Atiyah conjecture for suitable groups. For this it is necessary to work over the rational group ring instead of the complex group ring.

2.4. Proposition. *Let G_i be a directed system of groups which fulfill the Atiyah conjecture over $\mathbb{Q}G_i$ of common order $\frac{1}{L}\mathbb{Z}$. Suppose all the groups G_i lie in the class $\mathcal{G}$ (defined in Definition 1.4). Then their direct or inverse limit G fulfills the Atiyah conjecture of order $\frac{1}{L}\mathbb{Z}$ over $\mathbb{Q}G$.*

This is a direct consequence of the assumption and the approximation result [26, 6.9], since there it is shown that each L^2-Betti number over the limit group is the limit of L^2-Betti numbers over the groups in the sequence. By assumption, all of these lie in $\frac{1}{L}\mathbb{Z}$, which is a closed subset of $\mathbb{R}$. Therefore the same is true for the limit L^2-Betti number.

Note that this implies the strong Atiyah conjecture if we know that L is the least common multiple of the orders of finite subgroups of G (provided this number exists), in particular if all G_i are torsion-free and fulfill the strong Atiyah conjecture. Hence Proposition 1.5 is a special case of Proposition 2.4.

2.5. Example. We use Proposition 2.4 to give a different proof that a free group F fulfills the strong Atiyah conjecture over $\mathbb{Q}F$. By [22] free groups are residually torsion-free nilpotent. More precisely, if we consider the descending central series $F \supset F_1 \supset F_2 \supset \dots$ then $\bigcap F_k = \{1\}$ and F/F_k is torsion-free. It follows that the groups F/F_k are nilpotent and poly-$\mathbb{Z}$. Using the methods of the ring-theoretic parts of [17] it is easy to prove the strong Atiyah conjecture (over the complex group ring) for poly-$\mathbb{Z}$ groups. We will repeat the proof in Section 3. Alternatively, one can rely on a different type of ring theory and in particular avoid the use of the rings DG and $\mathcal{U}G$ of Definition 3.3: If G is poly-$\mathbb{Z}$, then $\mathbb{C}G$ is Noetherian [25, 8.2.2]. Using Moody's induction theorem [23, Theorem 1] we see that the G-theory of $\mathbb{C}G$ is trivial, which by [28, 4.6] coincides with the K-theory. Therefore a finite projective resolution of a $\mathbb{C}G$-module can be replaced by a finite free resolution. By [25, 8.2.18] $\mathbb{C}G$ has finite cohomological dimension. Together this implies that every finitely generated module over $\mathbb{C}G$ has a finite free resolution. If A is a matrix over $\mathbb{C}G$, its kernel in $(\mathbb{C}G)^n$ is finitely generated and we define its G-dimension as the alternating sum of the ranks of the modules in a finite free resolution. One can prove that this number, which is an integer, coincides with the G-dimension of $\ker A \subset l^2(G)^n$ (this is e.g. done in the thesis [7, 5.4.1]).

We therefore arrive at the inverse system $\cdots \to F/F_2 \to F/F_1$ where each of the quotients F/F_k is torsion-free and fulfills the strong Atiyah conjecture and lies in $\mathcal{G}$. F is a subgroup of the inverse limit of this system. Application of Proposition 1.5 and Lemma 2.2 gives the result.

There is of course one drawback: the approximation method proves the Atiyah conjecture only for matrices over $\mathbb{Q}F$ instead of $\mathbb{C}F$.

In view of Proposition 2.4 the following lemma is of importance.

2.6. Lemma. *The class $\mathcal{D}$ is contained in $\mathcal{G}$.*

Proof. By definition, $\mathcal{G}$ is subgroup closed and closed under direct and inverse limits of directed systems, and under extensions with elementary amenable quotients. It follows that $\mathcal{D}$ is contained in $\mathcal{G}$.

$\square$

2.7. Corollary. *The strong Atiyah conjecture over $\mathbb{Q}G$ is true if G is residually torsion free of class $\mathcal{C}$, in particular if G is residually torsion-free solvable (or more generally residually torsion-free elementary amenable).*

Proof. Let $\mathcal{X}$ be a class of groups. We call a group G residually of class $\mathcal{X}$ if G has a sequence of normal subgroups $G \supset H_1 \supset H_2 \supset \dots$ with $\bigcap_{k\in\mathbb{N}} H_k = \{1\}$ and such that the quotients $G/H_k \in \mathcal{X}$ $\forall k \in \mathbb{N}$. Another (more common) definition of residuality is to require for every $g \in G$ a homomorphism $\phi_g : G \to X_g$ with $\phi_g(g) \neq 1$ where $X_g \in \mathcal{X}$. If $\mathcal{X}$ is closed under direct sums then for countable groups G the two definitions are equivalent. This is the case for the class of torsion-free elementary amenable groups (but not for the class $\mathcal{C}$).

If G has a sequence of normal subgroups $H_1 \supset H_2 \supset \dots$ with trivial intersection such that $G/H_k \in \mathcal{C}$ and G/H_k are torsion-free, then G is a subgroup of the inverse limit of $(G/H_k)_{k\in\mathbb{N}}$. Combining Lemma 2.6 and Proposition 1.13, $\mathcal{C} \subset \mathcal{G}$. Because of Theorem 1.9 we can apply Proposition 1.5 and Lemma 2.2 to conclude that G fulfills the strong Atiyah conjecture over $\mathbb{Q}G$. □

2.8. Example. Let F be a free group on the free generators $\{a_i\}_{i\in\mathbb{N}}$ and assume $c \in F$ generates its own centralizer. Choose $V \subset \mathbb{N}$. Let A be a free abelian countably generated group A and r one of the free generators.

The following groups are residually torsion free of class $\mathcal{C}$ and hence by Corollary 2.7 fulfill the strong Atiyah conjecture over the rational group ring:

(1) Pure (also called colored) braid groups.

(2) Positive 1-relator groups G with $H_1(G,\mathbb{Z})$ torsion free. A group is a 1-relator group if it has a presentation

$$G =< g_1, \dots, g_n | R := \prod_{k=1}^{N} g_{i_k}^{n_k} = 1 >$$

with one relation R. It is called positive if one finds such a presentation with $n_k \geq 0$ for all k. By [21, IV.5.2.] G is torsion free if and only if R is not a proper power in the free group generated by $g_1, \dots, g_n$. It follows from the presentation that $H_1(G) = G/[G,G]$ is torsion free if and only if the greatest common divisor of $s_1, \dots, s_n$ is 1, where $s_r = \sum_{i_k=r} n_k$ is the exponent sum of g_r in R. Observe that therefore $H_1(G,\mathbb{Z})$ torsion free for a positive 1-relator group implies that G is torsion free.

(3) $E_1 :=< r, a_i; i \in V \mid c = r^n >$;

(4) $E_2 :=< a_i; i = 1, \dots, k \mid a_1^{n_1} a_2^{n_2} \dots a_k^{n_k} = 1 >$, with $k > 3$ and $n_i \in \mathbb{Z}$;

(5) $E_3 := F *_{c=r} A$.

Proof. First, we deal with the positive 1-relator groups. If $R = 1$, G is free and we have nothing to prove. By [5, Theorem 1] positive 1-relator groups are residually solvable. We want to prove that all the derived series quotients $G/G^{(n)}$ are torsion free. We can assume that $R \neq 1$, and since all n_k are non-negative, the second differential in the presentation 2-complex $X^{(2)}$ of G is non-trivial (i.e. at least one of the s_k is not zero), and therefore $H_2(X^{(2)}, \mathbb{Z}) = 0$. By assumption $H_1(G, \mathbb{Z})$ is torsion free. Now [27, Theorem A] implies that $G/G^{(n)}$ is torsion free for all n.

By [14, Theorem 2.6], the pure braid groups are residually torsion-free nilpotent. The group E_1 is residually torsion-free nilpotent by [4, Theorem 1]. E_2 is residually free by [3, p. 414]. E_3 is residually free [3, Theorem 8]. Free groups and therefore E_1, E_2, E_3 are residually torsion-free solvable. Every nilpotent and every solvable group is contained in $\mathcal{C}$. □

2.9. Remark. In [19] a condition is given when the Atiyah conjecture for a group H implies the Atiyah conjecture for finite extensions of H. The pure braid groups satisfy this condition, so that in particular the strong Atiyah conjecture for the full braid groups follows.
The condition is also satisfied for many positive one-relator groups.

3 Extensions with elementary amenable quotient

The following proposition is implicit in Linnell's work. He only deals with the complex group ring, therefore we give a complete proof here.

3.1. Proposition. *Let $1 \to H \to G \to A \to 1$ be an exact sequence of groups. Assume that G is torsion free and A is elementary amenable. For every finite subgroup $E < A$ let H_E be the inverse image of E in G. Assume that $K = \bar{K} \subset \mathbb{C}$ and for all finite subgroups $E < G$ that H_E fulfills the strong Atiyah conjecture over KH_E. Then G fulfills the strong Atiyah conjecture over KG.*

3.2. Corollary. *Suppose H is torsion-free and fulfills the strong Atiyah conjecture over KH with $K = \bar{K}$. If G is an extension of H with elementary amenable torsion-free quotient then G fulfills the strong Atiyah conjecture.*

Proof. By assumption, the only finite subgroup of G/H is the trivial group and the Atiyah conjecture is true for its inverse image H. □

We essentially follow Linnell's lines in [17] to prove Proposition 3.1. First we repeat a few lemmas.

3.3. Definition. Given the group von Neumann algebra $\mathcal{N}G$ we define $\mathcal{U}G$ to be the algebra of all unbounded operators affiliated to $\mathcal{N}G$ (compare e.g. [18, Section 8]), i.e. such that all their spectral projections belong to $\mathcal{N}G$.

We have to consider $KG \subset \mathcal{N}G \subset \mathcal{U}G$. Let DG be the division closure of KG in $\mathcal{U}G$, i.e. the smallest subring of $\mathcal{U}G$ which contains KG and which has the property that whenever $x \in DG$ is invertible in $\mathcal{U}G$ then $x^{-1} \in DG$.

The Atiyah conjecture is strongly connected to ring theoretic properties of DG:

3.4. Lemma. *Let G be a torsion-free group. G fulfills the strong Atiyah conjecture over KG if and only if the division closure DG of KG in $\mathcal{U}G$ is a skew field.*

Proof. The "only if" part is proved in [18]. Linnell states the result only for $K = \mathbb{C}$ but the proof carries over verbatim to the more general case. For the converse suppose DG is a skew field. We have to show that for $A \in M(m \times n, KG)$ the G-dimension of $\operatorname{coker}(A \otimes_{KG} \mathrm{id}_{l^2G})$ is an integer, or equivalently that the G-dimension of $\operatorname{coker}(A \otimes_{KG} \mathrm{id}_{\mathcal{U}G})$ is an integer. Since tensor products are right exact, this amounts to the fact that the G-dimension of $(\operatorname{coker}(A) \otimes_{KG} DG) \otimes_{DG} \mathcal{U}G$ is integral. However, $\operatorname{coker}(A) \otimes_{KG} DG$ is a (finitely generated, since A is a finite matrix) module over the skew field DG, therefore isomorphic to $(DG)^N$ for a suitable integer N. Then $\operatorname{coker}(A) \otimes_{KG} \mathcal{U}G \cong (\mathcal{U}G)^N$ which has G-dimension $N \in \mathbb{N}$. □

3.5. Lemma. *Suppose H is a normal subgroup of G with infinite cyclic quotient. Let $(\mathcal{U}H)G$ be the subring of $\mathcal{U}G$ generated by G and $\mathcal{U}H$. If $t \in G - H$ and $x = 1 + q_1 t + \cdots + q_k t^k \in (\mathcal{U}H)G$ with $q_i \in \mathcal{U}H$ then x is invertible in $\mathcal{U}G$ and in particular is not a zero divisor.*

Proof. If $q \in \mathcal{U}H$ then one finds $\alpha, \beta \in \mathcal{N}H$ with $q = \alpha\beta^{-1}$ by [6, Theorem 1 and proof of Theorem 10] or [24, 2.7]. Applying the same to q^* we may as well achieve $q = \beta^{-1}\alpha$. Apply this now inductively to $q_1 = \beta_1^{-1}\alpha_1$, $\beta_1 q_2 = \beta_2^{-1}\alpha_2, \dots$ to get the non-zero divisor $\beta = \beta_1 \dots \beta_k \in \mathcal{N}H$ (which is invertible in $\mathcal{U}H$) such that $\beta q_i \in \mathcal{N}H$ for $i = 1, \dots, k$. It follows from [16, Theorem 4] that βx is not a zero divisor in $\mathcal{N}G$, therefore it becomes invertible in $\mathcal{U}G$. □

3.6. Lemma. *Let DG be the division closure of KG in $\mathcal{U}G$, where $K = \bar{K}$. If $\alpha \in \operatorname{Aut}(G)$ then*

$$\alpha(DG) = DG^* = DG.$$

Proof. Of course $\alpha(DG)$ is the division closure of $\alpha(KG)$ in $\alpha(\mathcal{U}G)$. Since $\alpha(KG) = KG$ and $\alpha(\mathcal{U}G) = \mathcal{U}G$ we conclude $\alpha(DG) = DG$. The proof for the anti-automorphism $*$ is identical. Here $KG = KG^*$ since K is closed under complex conjugation. □

3.7. Definition. Mainly for proofs by induction we will use the following constructions of classes of groups in addition to Definition 1.10 and Definition 1.4:

- For a class $\mathcal{X}$ of groups, $L\mathcal{X}$ denotes the class of all groups which are locally of class $\mathcal{X}$, i.e. every finitely generated subgroup belongs to $\mathcal{X}$. Such groups are directed unions of groups in $\mathcal{X}$.
- For two classes of groups $\mathcal{X}$ and $\mathcal{Y}$, the class $\mathcal{X}\mathcal{Y}$ consists of all groups G with normal subgroup $H \in \mathcal{X}$ and quotient $G/H \in \mathcal{Y}$. Let $\mathcal{X}^\times$ consist of all finite direct sums of groups in $\mathcal{X}$.
- $(\mathcal{FREE})$ is the class of free groups, $(\mathcal{FIN})$ is the class of finite groups, and $\mathcal{B}$ is the class of finitely generated virtually abelian groups.
- The elementary amenable groups are denoted by $\mathcal{A}$. This is the smallest class of groups which contains all abelian and all finite groups and is closed under extensions and directed unions.

Proof of Proposition 3.1.
Case 1: G/H finitely generated free abelian.
By induction we can immediately reduced to the case where G/H is infinite cyclic. Since DH is a skew field an arbitrary non-zero element of $DH * G/H$ has after multiplication with units in $DH * G/H$ the shape $1 + q_1 t + q_2 t^2 + \dots q_k t^k$ where t is a generator of G/H. Lemma 3.5 implies therefore that each element of $(DH)G - \{0\}$ becomes invertible in $\mathcal{U}G$, in particular $(DH)G$ has no zero divisors. Therefore its Ore completion is a skew field which embeds into $\mathcal{U}G$. It embeds into DG and is division closed, therefore both coincide, DG is a skew field and by Lemma 3.4 the strong Atiyah conjecture holds for G.

If, as a next step, G/H is finitely generated free abelian, induction immediately implies that the strong Atiyah conjecture holds for G.
Case 2: G/H finitely generated virtually abelian.
It has a finitely generated free abelian normal subgroup A_0. Its inverse image G_0 in G fulfills the strong Atiyah conjecture by the previous step. Since H fulfills the strong Atiyah conjecture, by Lemma 3.4 the division closure DH of KH in $\mathcal{U}H$ is a skew field. Conjugation with an element of G gives an automorphism of H. Using Lemma 3.6, by [17, 2.1] the subring $(DH)G$ generated by DH and G is a crossed product: $(DH)G = DH * (G/H)$. The same applies to every finite extension $H \lhd H_E$, therefore $(DH)H_E =$

$DH * (H_E/H)$. By [17, 4.2] for these finite extensions $(DH)H_E = DH_E$, since DH is a skew field and hence Artinian.

By assumption all the groups H_E are torsion-free and fulfill the strong Atiyah conjecture. By Lemma 3.4 $DH_E = DH * E$ then is a skew field for every finite $E < G/H$. Because of [18, 4.5], $DH * G/H = (DH)G$ is an Ore domain. The proof shows that its Ore completion (a skew field!) is obtained by inverting $DH * (G_0/H) - \{0\}$. Since $DH * (G_0/H) \subset DG_0$, and the latter is a skew field contained in $\mathcal{U}G$, all inverses $(DH * (G_0/H) - \{0\})^{-1}$ are contained in $\mathcal{U}G$. Therefore the same is true for the Ore completion of $DH * G/H$, which is a skew field and as a result coincides with the division closure DG. By Lemma 3.4 the strong Atiyah conjecture holds for G.
Case 3: A elementary amenable.
This is done by transfinite induction, where we use roughly the description of elementary amenable groups given in [13, Section 3]: Let $\mathcal{A}_0$ be the class of finite groups, and for an ordinal α set $\mathcal{A}_{\alpha+1} := (L\mathcal{A}_\alpha)\mathcal{B}$. If α is a limit ordinal set $\mathcal{A}_\alpha = \bigcup_{\beta<\alpha} \mathcal{A}_\beta$. Then $\mathcal{A} = \bigcup_{\alpha \geq 0} \mathcal{A}_\alpha$. One can easily show, as in [17, 4.9], that each $\mathcal{A}_\alpha$ and $L\mathcal{A}_\alpha$ is subgroup closed and closed under extensions with finite quotient.

The case $A \in \mathcal{A}_0$ is trivial, because then the conclusion is part of the assumptions.

Let α be the least ordinal with $A \in \mathcal{A}_\alpha$. Then $\alpha = \beta + 1$. Assume by induction that the statement is true for $\mathcal{A}_\beta$. If $A = \bigcup_{i \in I} A_i$ with $A_i \in \mathcal{A}_\beta$, then $G = \bigcup G_i$ where G_i is the inverse image of A_i in G, i.e. an extension of H by G_i. By induction and Lemma 2.3 the Atiyah conjecture is true for G, therefore it is true if the quotients belong to $L\mathcal{A}_\beta$.

Let now A be an extension with kernel $B \in L\mathcal{A}_\beta$ and quotient Q which is finitely generated virtually abelian. We want to apply the induction hypothesis and Case 2 to $1 \to G_1 \to G \to Q \to 1$, where for a finite subgroup E of Q we let G_E be the inverse image of E in G (in particular G_1 is the inverse image of B). To apply the results of Case 2 we have to establish the Atiyah conjecture for all the groups G_E. We obtain exact sequences $1 \to H \to G_E \to B_E \to 1$ where B_E is a finite extension of B. Hence $B_E \in L\mathcal{A}_\beta$. Note that B_E is a subgroup of A, therefore each finite subgroup of B_E is a finite subgroup of A, too. This implies that the assumptions of the proposition are fulfilled for $1 \to H \to G_E \to B_E \to 1$ and therefore G_E fulfills the strong Atiyah conjecture by induction.

Hence the assumptions of Case 2 are fulfilled for $1 \to G_1 \to G \to Q \to 1$ and the result follows. □

In the course of the proof we also established:

3.8. Lemma. *If, in the situation of Proposition 3.1, G/H is finitely generated virtually abelian then the ring DG is an Ore localization of $DH * G/H$.*

3.9. Example. Let $G \in \mathcal{G}$ (as defined in Definition 1.4) be a torsion-free group which fulfills the strong Atiyah conjecture with $K = \bar{K} \subset \mathbb{C}$, and $f : G \to G$ a group homomorphism. Then the mapping torus group

$$T_f := < t, G| \; tgt^{-1} = f(g), gh = (gh) \quad g, h \in G >$$

fulfills the strong Atiyah conjecture over $\mathbb{Q}T_f$. If f is injective, then T_f fulfills the strong Atiyah conjecture even over KT_f and we don't need the assumption $G \in \mathcal{G}$.

Proof. Let E_f be the direct limit of $G \xrightarrow{f} G \xrightarrow{f} G \to \dots$. Then E_f fulfills the strong Atiyah conjecture by Proposition 2.4 or, if f is injective, by Proposition 2.3. The claim follows from Proposition 3.1 because T_f is an extension of E_f with quotient $\mathbb{Z}$. □

4 Proofs of the Atiyah conjecture

In this section we prove Theorem 1.11 and Proposition 1.13 and give additional information about $\mathcal{D}$.

4.1. Lemma. *Let $H = F_1 \times F_2 \times \cdots$ be a direct sum of non-abelian free groups. Let $H \lhd G$ be a torsion-free finite extension of H. Then every finitely generated subgroup of G is contained in a group V which fits into an extension $1 \to U \to V \to A \to 1$ where U is a subgroup of a direct sum of free groups and A is torsion-free elementary amenable.*

Proof. By [18, 13.2] every finitely generated subgroup of G is contained in a finite extension of a finitely generated product of free groups. So we may assume that G (and H) are finitely generated. We now use the ideas of [18, 13.4] where Linnell proves a related result. For a group V let S_nV be the intersection of normal subgroups of index $\le n$. Note that this is a characteristic subgroup, and if V is finitely generated, S_nV has finite index in S by [25, 8.2.7]. Set $H_n := S_nF_1 \times S_nF_2 \dots S_nF_k$. Let H_n' be the commutator subgroup of H_n. Then H/H_n' is virtually abelian and torsion-free: it is the direct sum of $F_i/(S_nF_i)'$ containing the abelian subgroup $S_nF_i/(S_nF_i)'$ of finite index, and whenever F is free and $N \lhd F$ is a normal subgroup then F/N' is torsion free (compare e.g. the last paragraph in the proof of 3.4.8 of [29]).

Since G is torsion-free, by [18, 13.4] for every prime number p there is a subgroup $K_p < H$ which is normal in G such that G/K_p is virtually abelian and does not contain a finite subgroup of index p. For n_p sufficiently large $H'_{n_p} < K_p$. We do this for all prime factors of $|G/H|$ and let n be the maximum of the n_p (which exists since G/H is finite). Then $1 \to H_n' \to G \to A \to 1$ is an extension where A is virtually abelian. We also have an extension $1 \to H/H_n' \to A \to G/H \to 1$. Since H/H_n' is torsion-free, every torsion element of A is mapped to an element of the same order in G/H. Therefore it suffices to check for a prime factor p of $|G/H|$ and $x \in A$ with $x^p = 1$ that $x = 1$. But we also have the extension $1 \to K_p/H_n' \to A \to G/K_p \to 1$ where $K_p/H_n' < H/H_n'$ is torsion-free and G/K_p does not contain p-torsion. This concludes the proof. □

Proof of Theorem 1.11 and Corollary 1.12.
Remember that every group in $\mathcal{D}$ is torsion-free and belongs by Lemma 2.6 to $\mathcal{G}$. The result follows immediately from Proposition 2.2, Proposition 2.4, and Proposition 3.1.

The corollary is an immediate consequence of the theorem (because of Lemma 3.4). □

Proof of Proposition 1.13.
By [22], every finitely generated free group is residually torsion-free nilpotent. Consequently, the same is true for every direct sum of free groups. Therefore, these groups are subgroups of inverse limits of torsion-free nilpotent (and therefore elementary amenable) groups. It follows that direct sums of arbitrary free groups (as directed unions of sums of finitely generated ones) and subgroups of these belong to $\mathcal{D}$. By Lemma 4.1, every torsion-free finite extension of a direct sum of free groups then also belongs to $\mathcal{D}$. It now follows immediately from the definition of $\mathcal{C}'$ that every torsion-free element of $\mathcal{C}'$ belongs to $\mathcal{D}$.

Consider (for every ordinal α) the subclasses $\mathcal{C}_\alpha$ of $\mathcal{C}$ with

$$\mathcal{C}_0 := (\mathcal{FREE})(\mathcal{FIN}) \quad \text{and} \quad \mathcal{C}_{\alpha+1} := (L\mathcal{C})\mathcal{B}.$$

If α is a limit ordinal, set $\mathcal{C}_\alpha := \bigcup_{\beta<\alpha} \mathcal{C}_\beta$. By induction, we proof that every torsion-free element of $\mathcal{C}_\alpha$ belongs to $\mathcal{D}$. We have just shown this for $\alpha = 0$. If α is a limit ordinal, nothing is to proof. By [17, Lemma 4.9], each $L\mathcal{C}_\alpha$ is closed under extensions with finite quotients. If $G \in \mathcal{C}_{\alpha+1}$ is torsion-free, we have an extension $1 \to H \to G \xrightarrow{p} A \to 1$ with A elementary amenable (since A is abelian by finite) and $H \in L\mathcal{C}_\alpha$. If $E < A$ is finite, therefore $p^{-1}(E) \in L\mathcal{C}_\alpha$ (as a finite extension of H). Moreover, $p^{-1}(E)$ is torsion-free, since it is a subgroup of the torsion-free group G. By induction, $p^{-1}(E) \in \mathcal{D}$. It follows that $G \in \mathcal{D}$. This finishes the induction step. Since $\mathcal{C} = \bigcup_\alpha \mathcal{C}_\alpha$ (by [17, Lemma 4.9]), the statement about $\mathcal{C}$ follows.

Next, we prove that $\mathcal{D}$ is closed under direct sums. Since every direct sum is a union of finite direct sums, by induction it suffices to consider direct sums with two summands. Fix $G \in \mathcal{D}$.

The start of the induction is the trivial observation that $\{1\} \times G \in \mathcal{D}$. For the induction step, we have to check that $H \times G \in \mathcal{D}$ in the following three cases:

(1) There is an exact sequence $1 \to U \to H \xrightarrow{p} A \to 1$ with A elementary amenable and such that $p^{-1}(E) \times G \in \mathcal{D}$ for every finite subgroup E of A. But then we have an exact sequence

$$1 \to U \times G \to H \times G \xrightarrow{q} A \to 1$$

and $q^{-1}(E) = p^{-1}(E) \times G$ for every finite subgroup E of A, and it follows that $H \times G \in \mathcal{D}$.

(2) H is a direct or inverse limit of a directed system of groups H_i, and $H_i \times G \in \mathcal{D}$ for every i. Then $H \times G$ is the direct or inverse limit of the directed system $H_i \times G$ by Lemma 4.2, and therefore $H \times G \in \mathcal{D}$.

(3) If $H \subset U$ and $U \times G \in \mathcal{D}$, then $H \times G \subset U \times G$ which implies $H \times G \in \mathcal{D}$.

The detailed construction of families $\mathcal{D}_\alpha$ such that this proof applies is standard (compare the similar construction of $\mathcal{C}_\alpha$ used above) and is left to the reader.

Every direct product is the inverse limit of finite direct sums. It follows that $\mathcal{D}$ is also closed under taking direct products.

We now proof by induction that $\mathcal{D}$ is closed under free products with (arbitrary) free groups. The induction starts with the observation that $\{1\} * F \in \mathcal{D}$ if F is free, what we have already checked.

For the induction step, we again have to check three cases.

(1) Assume $H \subset U$ and $U * F \in \mathcal{D}$ for every free group F. Then $H * F \subset U * F$ and consequently $H * F \in \mathcal{D}$.

(2) Assume we have an exact sequence $1 \to U \to H \to A \xrightarrow{p} 1$ with A elementary amenable and $p^{-1}(E) * F \in \mathcal{D}$ for every free group F and every finite subgroup E of A. Consider the projection $q := p * 1 \colon H * F \to A$. Then for every $E < A$ Lemma 4.3 implies $q^{-1}(E) \cong p^{-1}(E) * F'$ with some possibly different free group F'. Since the induction hypothesis applies to $p^{-1}(E) * F'$ if E is finite, it follows that $H * F \in \mathcal{D}$.

(3) Assume H is the direct or inverse limit of a directed system of groups H_i, and $H_i * F \in \mathcal{D}$ for every free group F. By [26, Lemma 2.9 and 2.10] $H * F$ is contained in the direct or inverse limit of the directed system $H_i * F$. It follows that $H * F \in \mathcal{D}$.

For arbitrary free products observe that $G * H$ is a subgroup of $(G \times H) * (G \times H)$ which is a subgroup of $(G \times H) * \mathbb{Z}$ (it is contained in the kernel of the projection onto $\mathbb{Z}$ by [11, Lemma 5.5]). Since $\mathcal{D}$ is subgroup closed, $G * H \in \mathcal{D}$ if $G, H \in \mathcal{D}$ follows from what we have proved about direct sums and free products with free groups. The class $\mathcal{D}$ is closed under arbitrary free products by induction and considering directed unions (an arbitrary free product is the directed union of finite free products). This finishes the proof. $\square$

The following lemmas were needed in the proof of Proposition 1.13.

4.2. Lemma. *If π is the direct or inverse limit of a system of groups π_i and if G is any group, then $\pi \times G$ is the direct or inverse limit of $\pi_i \times G$.*

Proof. Compare e.g. in [26, 2.8 and 2.9]. $\square$

4.3. Lemma. *Assume $p\colon H \to A$ a homomorphism of groups. Consider $q := p * 1\colon H * G \to A$. Then, for every subgroup E of A, $q^{-1}(E) = p^{-1}(E) * (*_{t\in T}(t^{-1}Gt))$, where T is a transversal for $p^{-1}(E)$ in H, i.e. a system of representatives for the cosets $H/p^{-1}(E)$.*

Proof. Obviously, q maps $p^{-1}(E)$ and $t^{-1}Gt$ to E. On the other hand, a normal form argument implies that each element in $q^{-1}(E)$ lies in the subgroup of $H * G$ generated by $p^{-1}(E)$ and $t^{-1}Gt$ ($t \in T$). The normal form argument also implies that this subgroup is isomorphic to the free product $p^{-1}(E) * (*_{t\in T}(t^{-1}Gt))$. $\square$

We now explain how to fill the gap (pointed out in Section 1) of Linnell's result about the Atiyah conjecture for the class $\mathcal{C}'$, using our methods (and obtaining a slightly weaker conclusion).

4.4. Corollary. *Assume $G \in \mathcal{C}'$ and there is a bound on the orders of the finite subgroups of G. Then the strong Atiyah conjecture is true over $\mathbb{Q}G$.*

Proof. Linnell's proof in [18, Section 13] applies as soon as the strong Atiyah conjecture over $\mathbb{Q}G$ is established for every direct sum of free groups G. However, we have seen that these groups belong to $\mathcal{D}$ and therefore satisfy the strong Atiyah conjecture over $\mathbb{Q}G$. $\square$

5 Generalizations

In the second part of the paper we concentrated on torsion-free groups. By Linnell's methods, many of the results hold if instead of this one requires a bound on the orders of finite subgroups. In particular, a version of Proposition 3.1 remain true if Q is not assumed to be torsion-free but only has a bound on the orders of its finite subgroups. One proves this by transfinite induction, using a little bit more of the machinery of Linnell [17, 18]. Correspondingly, one can then (similarly to $\mathcal{D}$) define a larger class of groups (not necessarily torsion-free) for which the strong Atiyah conjecture is true.

Another possible generalization is given by only considering the 1-dimensional Atiyah conjecture. This is defined as in Definition 1.2, but one does consider only 1×1-matrices A, i.e. $A \in KG$. This is interesting because already the strong 1-dimensional Atiyah conjecture for torsion-free groups implies the zero divisor conjecture. Moreover, Linnell [16] proves that the 1-dimensional strong Atiyah conjecture for torsion-free groups is stable under extensions with right-orderable groups.

One immediately checks that there is a 1-dimensional versions of Proposition 2.4. Hence we can define a class of groups similar to $\mathcal{D}$ but which is closed under extensions with right-orderable quotient (instead of extensions with elementary amenable quotient), and the 1-dimensional strong Atiyah conjecture over $\mathbb{Q}G$ is true if G is contained in this class. In particular, $\mathbb{Q}G$ is zero divisor free.

6 Treelike groups

6.1. Proposition. *Let H be a finite group. It follows that H fulfills the Atiyah conjecture of order $\frac{1}{L}\mathbb{Z}$ over KH, with $L = |H|$. Let G be a group and Ω and Δ sets with commuting G-action from the left and H-action from the right such that Ω and Δ are free H-sets. Let $\Omega = \Omega' \amalg X$ and assume that Δ and Ω' are free G-sets and X consists of $1 \le r < \infty$ H-orbits. Assume that there is a bijective map $\phi : \Delta \to \Omega$ which is an H-almost G-map and a right H-blockwise map (compare Definition 6.2).*

Then G fulfills the Atiyah conjecture of order $\frac{1}{rL}\mathbb{Z}$ over KG.

We used the following notation:

6.2. Definition. The map $\phi : \Delta \to \Omega$ is an *H-almost G-map* if for each $g \in G$ the set $\{x \in \Delta \| \; \phi(gx) \neq g\phi(x)\}$ is contained in only finitely many H-orbits.
It is called a *right H-blockwise* map if each H-orbit of Δ is mapped bijectively to an H-orbit of Ω.

6.3. Remark. Whenever the assumptions of Proposition 6.1 are fulfilled, we can replace H by $\{1\}$ (and therefore get $L = 1$). However, X then consists of $r \times |H| = rL$ $\{1\}$-orbits (i.e. points), and the conclusion is unchanged.

Proof of Proposition 6.1. Our proof generalizes Linnell's approach in [18].

Assume that we have the G-H sets Ω and Δ as above. Given $A \in M_n(KG)$, the G-operation induces bounded operators A_Δ on $l^2(\Delta)^n$ and $A_{\Omega'}$ on $l^2(\Omega')^n$ in the obvious way. Set $A_\Omega := A_{\Omega'} \oplus 0$ on $l^2(\Omega)^n = l^2(\Omega')^n \oplus l^2(X)^n$.

Let $P \in M_n(\mathcal{N}G)$ be the projection onto the image of A. Since Δ and Ω' are free G-sets one can extend the above construction to P. It follows immediately that P_Δ is the projection onto the image of A_Δ and P_Ω the projection onto the image of A_Ω. Since the H-actions commute with the G-actions, all operators constructed in this way are H-equivariant.

There are only finitely many $g \in G$ such that $A =: (a_{ij}) \in M_n(\mathbb{C}G)$ has a non-trivial coefficient of g in at least one of the a_{ij}. For each of these, there are only finitely many H-orbits in Δ on which $\phi(gx) \neq g\phi(x)$. Let Δ_0 be the union of this finite number of H-orbits in Δ, and set $\Delta_c = \Delta - \Delta_0$ and $\Omega_c := \phi(\Delta_c)$.

Restricted to $l^2(\Delta_c)^n$, $\phi^* A_\Omega \phi$ and A_Δ coincide:

$$A_\Delta|_{l^2(\Delta_c)^n} = \phi^* A_\Omega \phi|_{l^2(\Delta_c)^n}. \tag{6.4}$$

Choose an exhaustion $\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \dots \Delta$ of Δ (i.e. $\Delta = \bigcup_{k\in\mathbb{N}} \Delta_k$) where each Δ_k consists of finitely many (entire) H-orbits. Then (since ϕ is an H-blockwise map) $\Omega_k := \phi(\Delta_k)$ also consists of finitely many H-orbits.

If Δ consists of uncountably many H-orbits, we have to use an exhaustion index by an uncountable directed index set. The proof remains essentially unchanged. Since our applications only deal with the countable setting for simplicity we stick to the index set $\mathbb{N}$.

As $\Delta = \bigcup_{k\in\mathbb{N}} \Delta_k$ we have $A_\Delta(l^2(\Delta)^n) = \overline{\bigcup_k A_\Delta(l^2(\Delta_k)^n)}$.

Let $P_{\Delta,k}$ be the projection onto $\overline{A_\Delta(l^2(\Delta_k)^n)}$, and $P_{\Omega,k}$ the projection onto $\overline{A_\Omega(l^2(\Omega_k)^n)}$.

Because in the matrix $A = (a_{ij})$ only finitely many elements of G have non-trivial coefficient in at least one of the a_{ij}, A has finite propagation, i.e. for every $k \in \mathbb{N}$ we find $n(k) \in \mathbb{N}$ such that $A_\Delta(l^2(\Delta_k)^n) \subset l^2(\Delta_{n(k)})^n$. Similarly (if $n(k)$ is chosen big enough) $A_\Omega(l^2(\Omega_k)^n) \subset l^2(\Omega_{n(k)})^n$. Particularly, $A_\Delta(l^2(\Delta_k)^n)$ and $A_\Omega(l^2(\Omega_k)^n)$ are the ranges of finite matrices over KH to which the Atiyah conjecture applies and by assumption on H

$$\begin{aligned} \operatorname{tr}_H(P_{\Delta,k}) &= \dim_H(A_\Delta(l^2\Delta_k)^n) \in \frac{1}{L}\mathbb{Z}, \\ \operatorname{tr}_H(P_{\Omega,k}) &= \dim_H(A_\Omega(l^2\Omega_k)^n) \in \frac{1}{L}\mathbb{Z}. \end{aligned} \tag{6.5}$$

Let $P_{\Delta,k,c}$ be the projection onto $A_\Delta(l^2(\Delta_k \cap \Delta_c)^n)$ and $P_{\Delta,c}$ the projection onto $A_\Delta(l^2(\Delta_c)^n)$. Define $P_{\Omega,k,c}$ and $P_{\Omega,c}$ correspondingly. By Equation (6.4) $P_{\Delta,k,c} = \phi^* P_{\Omega,k,c}\phi$ and $P_{\Delta,c} = \phi^* P_{\Omega,c}\phi$. In particular, if $x \in \Delta^n \subset l^2(\Delta)^n$ we have (with ϕ diagonally extended to Δ^n)

$$\begin{aligned} \langle P_{\Delta,k,c}x, x\rangle_{l^2(\Delta)^n} &= \langle P_{\Omega,k,c}\phi(x), \phi(x)\rangle_{l^2(\Omega)^n}, \\ \langle P_{\Delta,c}x, x\rangle_{l^2(\Delta)^n} &= \langle P_{\Omega,c}\phi(x), \phi(x)\rangle_{l^2(\Omega)^n}. \end{aligned} \tag{6.6}$$

Set $Q_{\Delta,k} := P_{\Delta,k} - P_{\Delta,k,c}$, $Q_\Delta := P_\Delta - P_{\Delta,c}$, and define $Q_{\Omega,k}$ and Q_Ω correspondingly. These are the projections onto the complement of the smaller sets in the larger ones.

Note that the images of the Q's are already obtained from $l^2(\Delta_0)^n$ and $l^2(\Omega_0)^n$, respectively. More precisely, let A_0 be the restriction of A_Δ to

$l^2(\Delta_0)^n$. Then $Q_{\Delta,k}A_0$ induces a weakly exact sequence (i.e. the images are dense in the kernels)

$$0 \to \ker(Q_{\Delta,k}A_0) \to l^2(\Delta_0)^n \xrightarrow{Q_{\Delta,k}A_0} \mathrm{im}(Q_{\Delta,k}) \to 0. \tag{6.7}$$

We only have to check weak exactness at $\mathrm{im}(Q_{\Delta,k})$. $Q_{\Delta,k}(A(l^2(\Delta)^k))$ is dense in $\mathrm{im}(Q_{\Delta,k})$. But $A(l^2(\Delta_k)^n) = A(l^2(\Delta_0)^n) + A(l^2(\Delta_k \cap \Delta_c)^n)$ and $Q_{\Delta,k}(A(l^2(\Delta_k \cap \Delta_c)^n)) = 0$. For the kernel we have the weakly exact sequence

$$0 \to \ker(A_0) \to \ker(Q_{\Delta,k}A_0) \stackrel{A_0}{\to} \overline{A_\Delta(l^2(\Delta_0)^n) \cap A_\Delta(l^2(\Delta_k \cap \Delta_c)^n)} \to 0, \tag{6.8}$$

which would be clear with $A_\Delta(l^2(\Delta_k \cap \Delta_c)^n)$ replaced by $\ker(Q_{\Delta,k})$ —but if $f \in A_\Delta(l^2(\Delta_k)^n)$ and in particular if $f \in A_\Delta(l^2(\Delta_0)^n)$, then $f \in \ker(Q_{\Delta,k})$ if and only if $f \in A_\Delta(l^2(\Delta_k \cap \Delta_c)^n)$. This is true since all the vector spaces we are considering here are finite dimensional (a consequence of the assumption that H is finite).

In the same way one gets the weakly exact sequences

$$0 \to \ker(Q_\Delta A_0) \to l^2(\Delta_0) \xrightarrow{Q_\Delta A_0} \mathrm{im}(Q_\Delta) \to 0, \tag{6.9}$$

$$0 \to \ker(A_0) \to \ker(Q_\Delta A_0) \stackrel{A_0}{\to} \overline{A_\Delta(l^2(\Delta_0)^n) \cap A_\Delta(l^2(\Delta_c)^n)} \to 0. \tag{6.10}$$

Observe that

$$\overline{A_\Delta(l^2(\Delta_0)^n) \cap A_\Delta(l^2(\Delta_c)^n)} = \overline{\bigcup_{k\in\mathbb{N}} A_\Delta(l^2(\Delta_0)^n) \cap A_\Delta(l^2(\Delta_k \cap \Delta_c)^n)},$$

hence because of continuity of $\dim_H$ [20, 1.4]

$$\dim_H(\overline{A_\Delta(l^2(\Delta_0)^n) \cap A_\Delta(l^2(\Delta_c)^n)}) = \lim_{k\to\infty} \dim_H(\overline{A_\Delta(l^2(\Delta_0)^n) \cap A_\Delta(l^2(\Delta_k \cap \Delta_c)^n)}). \tag{6.11}$$

Using the additivity of $\dim_H$ under short weakly exact sequences [20, 1.4], Equation (6.11) implies together with Equation (6.7), (6.8), (6.9), and (6.10) (where all numbers are finite and bounded by $\dim_H(l^2(\Delta_0)^n)$)

$$\mathrm{tr}_H(Q_\Delta) = \dim_H(\mathrm{im}(Q_\Delta)) = \lim_{k\to\infty} \dim_H(\mathrm{im}(Q_{\Delta,k})) = \lim_{k\to\infty} \mathrm{tr}_H(Q_{\Delta,k}). \tag{6.12}$$

Exactly in the same way one obtains

$$\mathrm{tr}_H(Q_\Omega) = \lim_{k\to\infty} \mathrm{tr}_H(Q_{\Omega,k}). \tag{6.13}$$

Choose a set of representatives T for the H-orbits of Δ. Because ϕ is an H-blockwise bijective map, $\phi(T)$ will be a set of representatives for the H-orbits of Ω. For $t \in T$ set $t_i := (0, \dots, 0, t, 0, \dots, 0) \in \Delta^n$ (t at the i-th position). Then (by the definition of the H-trace)

$$\operatorname{tr}_H(Q_\Delta) = \sum_{t\in T}\sum_{i=1}^{n} \langle Q_\Delta t_i, t_i\rangle_{l^2(\Delta)},$$
$$\operatorname{tr}_H(Q_\Omega) = \sum_{t\in T}\sum_{i=1}^{n} \langle Q_\Omega \phi(t_i), \phi(t_i)\rangle_{l^2(\Delta)}.$$

Because Δ is a free G-set, $G\cdot t \subset \Delta$ can for each $t \in T$ be identified with G and therefore

$$\sum_{i=1}^{n} \langle P_\Delta t_i, t_i\rangle_{l^2(\Delta)^n} = \operatorname{tr}_G(P) = \dim_G(\operatorname{im}(A)).$$

Similarly, since Ω' is a free G-set, if $\phi(t) \in \Omega'$ then

$$\sum_{i=1}^{n} \langle P_\Omega \phi(t_i), \phi(t_i)\rangle_{l^2(\Omega)^n} = \sum_{i=1}^{n} \langle P_\Omega \phi(t_i), \phi(t_i)\rangle_{l^2(\Omega')^n} = \operatorname{tr}_G(P) = \dim_G(\operatorname{im}(A)).$$

However, if $\phi(t) \in X$ then by the construction of P_Ω

$$\langle P_\Omega \phi(t_i), \phi(t_i)\rangle_{l^2(\Omega)} = 0.$$

Therefore

$$\sum_{t\in T}\sum_{i=1}^{n} (\langle P_\Delta t_i, t_i\rangle_{l^2(\Delta)^n} - \langle P_\Omega \phi(t_i), \phi(t_i)\rangle_{l^2(\Omega)^n}) = r \dim_G(\operatorname{im}(A)) \qquad (6.14)$$

(where $r = |\phi(T) \cap X|$ is the number of H-orbits in X). Note that the sum is finite so that convergence is not an issue here. We want to use this equation to determine the value of $\dim_G(\operatorname{im}(A))$.

From our splitting of P_Δ and P_Ω we see that for each $t \in T$ and $i \in \{1, \dots, n\}$

$$\begin{aligned}&\langle P_\Delta t_i, t_i\rangle_{l^2(\Delta)^n} - \langle P_\Omega \phi(t_i), \phi(t_i)\rangle_{l^2(\Omega)^n}\\ &= \underbrace{\langle P_{\Delta,c} t_i, t_i\rangle - \langle P_{\Omega,c} \phi(t_i), \phi(t_i)\rangle}_{\overset{(6.6)}{=}0} + \langle Q_\Delta t_i, t_i\rangle - \langle Q_\Omega \phi(t_i), \phi(t_i)\rangle. \end{aligned} \qquad (6.15)$$

In the same way

$$\begin{aligned}&\langle P_{\Delta,k} t_i, t_i\rangle_{l^2(\Delta)^n} - \langle P_{\Omega,k} \phi(t_i), \phi(t_i)\rangle_{l^2(\Omega)^n}\\ &= \underbrace{\langle P_{\Delta,k,c} t_i, t_i\rangle - \langle P_{\Omega,k,c} \phi(t_i), \phi(t_i)\rangle}_{\overset{(6.6)}{=}0} + \langle Q_{\Delta,k} t_i, t_i\rangle - \langle Q_{\Omega,k} \phi(t_i), \phi(t_i)\rangle. \end{aligned} \qquad (6.16)$$

Note that by Equation (6.5) for each finite k, all the operators in Equation (6.16) are of finite H-rank individually and therefore we can write

$$\begin{aligned} \operatorname{tr}_H(Q_{\Delta,k}) - \operatorname{tr}_H(Q_{\Omega,k}) &= \sum_{t\in T}\sum_{i=1}^{n} \langle Q_{\Delta,k}t_i, t_i\rangle - \langle Q_{\Omega,k}\phi(t_i), \phi(t_i)\rangle \\ &\overset{(6.16)}{=} \operatorname{tr}_H(P_{\Delta,k}) - \operatorname{tr}_H(P_{\Omega,k}) \overset{(6.5)}{\in} \frac{1}{L}\mathbb{Z}. \end{aligned} \tag{6.17}$$

This implies, since $\frac{1}{L}\mathbb{Z}$ is discrete,

$$\operatorname{tr}_H(Q_\Delta) - \operatorname{tr}_H(Q_\Omega) \overset{(6.12),\,(6.13)}{=} \lim_{k\to\infty} (\operatorname{tr}_H(Q_{\Delta,k}) - \operatorname{tr}_H(Q_{\Omega,k})) \overset{(6.17)}{\in} \frac{1}{L}\mathbb{Z}. \tag{6.18}$$

Putting everything together we see

$$\begin{aligned} r\dim_H(\operatorname{im}(A)) &\overset{(6.14)}{=} \sum_{t\in T}\sum_{i=1}^{n} (\langle P_\Delta t_i, t_i\rangle_{l^2(\Delta)^n} - \langle P_\Omega\phi(t_i), \phi(t_i)\rangle_{l^2(\Omega)^n}) \\ &\overset{(6.15)}{=} \sum_{t\in T}\sum_{i=1}^{n} (\langle Q_\Delta t_i, t_i\rangle - \langle Q_\Omega\phi(t_i), \phi(t_i)\rangle) \\ &= \operatorname{tr}_H(Q_\Delta) - \operatorname{tr}_H(Q_\Omega) \overset{(6.18)}{\in} \frac{1}{L}\mathbb{Z}. \end{aligned}$$

Since $A \in M_n(\mathbb{C}G)$ was arbitrary, Proposition 6.1 follows. □

6.19. Definition. A group G is called *treelike* (of *finity* r), if there are free G-sets Δ and Ω', a non-empty trivial G-set X with $0 \neq r \in \mathbb{N}$ elements and an almost G-map

$$\alpha : \Delta \to \Omega' \amalg X.$$

Julg and Valette [12] show that free groups are treelike of finity 1. On the other hand, Dicks and Kropholler [9] prove that if G is treelike of finity 1 then G is free.

As an immediate consequence of Proposition 6.1 with $H = \{1\}$ we get:

6.20. Theorem. *Let Q be a treelike group of finity r. Then Q fulfills the Atiyah conjecture of order $\frac{1}{r}\mathbb{Z}$ over $\mathbb{C}Q$.*

We generalize in this section the main result in [9] and show that the class of treelike groups of finity r coincides with the class of groups which contain a free group of index r. So, in view of Lemma 2.2, Theorem 6.20 does not provide new cases of the Atiyah conjecture.

6.21. Theorem. *The following statements are equivalent for a group G:*

(1) G is treelike of finity which divides r.

(2) G acts on a tree and all vertex stabilizers have order which divides r.

(3) G is the fundamental group of a graph of groups where the order of each vertex group divides r.

(4) G contains a free subgroup of finite index d which divides r.

(5) G acts freely on a union of d trees where d divides r.

(6) G contains a free subgroup of finite index and the order of every finite subgroup of G divides r.

All statements which don't involve treelikeness are well known. Those proofs for which no reference in a standard textbook could be found are included here.

Proof. (1) $\implies$ (2): Suppose G is treelike of finity r. That means we have two free G-sets Δ and Ω' and a bijective almost G-map $\theta : \Delta \to \Omega := \Omega' \amalg (\amalg_{i=1}^{r}\{*\})$. Let V be the set of all maps with same domain and range which coincide with θ outside a finite subset of Δ. By [9, 2.4] V is the vertex set of a G-tree with finite edge stabilizers. We want to show that the vertex stabilizers are finite, and their order is divisible by r.

That means we want to show that there is no way to change θ at only finitely many points in the domain to obtain an f fixed by $H < G$ unless H is finite and its order divides r. Remember that the action of G on the maps from Δ to Ω is given by conjugation: $f^g(x) = g^{-1}f(gx)$. Therefore, f being an H-fixed point is equivalent to f being an H-map.

Each map $f \in V$ has an index $i(f) := \sum_{x\in\Omega}(\left|f^{-1}(x)\right| - 1)$. This is well defined if the inverse image of almost every point of Ω consists of one point. In particular, it is well defined for the bijective map θ and $i(\theta) = 0$. If we change a map at one point (or iteratively at finitely many points), this index remains well defined and unchanged (the inverse image of one point will be smaller by one, whereas the inverse image of another point will be larger by one). In particular, $i(f) = 0$ for every $f \in V$.

Assume that $f \in V$ is an H-map for $H < G$. Then f will miss, say, k of the r H-fixed points of Ω, and the inverse image of the remaining $r - k$ of those will consist of unions of free H-orbits. In addition, the inverse image of each point in a given free H-orbit of Ω will be of equal size. If H is infinite then $k = r$ because no point can have an infinite inverse image (else $i(f)$ is not defined). But then f is not injective because $i(f) = 0$. Therefore the inverse image of at least one free H-orbit in Ω consists of more than one orbit, which is impossible because $i(f)$ is defined. This contradiction shows that no infinite subgroup H of G fixes a point $f \in V$.

If H is finite and f is an H-map then the fact that Δ and Ω' are free H-sets implies as above $0 = i(f) = -r + |H| \cdot N$, i.e. r is divisible by the order of H.

(2) $\implies$ (3): This is the structure theorem for groups acting on a tree, compare e.g. [8, I.4.1.].

(3) $\implies$ (4) This is proved as [2, Theorem 8.3]. The statement follows also easily from the arguments of [8, IV.1.6], where is is shown that G contains a free normal subgroup of finite index.

(4) $\implies$ (5): Let F be a free subgroup of index d in G. Choose a free basis $\{f_i\}$ of F. Construct the following graph: the elements of G are the vertices, and two vertices g and h are joined by an edge if and only if $gf_i = h$ for a suitable f_i in the basis of F.

The graph is a free left G-set in the obvious way (with transitive action on the vertices). The component of the trivial element is the Cayley graph of F with respect to the generators $\{f_i\}$ and therefore is a tree [8, I.8.2]. The vertex sets of the other components are the translates of F. Consequently, the graph consists of d trees.

(5) $\implies$ (1): A construction of Julg and Valette gives an almost G-map from the set of edges to the set of vertices united with d additional points which are fixed under the G-action. It is done as follows: choose base points for each of the trees. These base points are mapped to the d additional points. Each other vertex is mapped to the first edge of the geodesic starting at this vertex and ending at the basepoint of its component. This geodesic and therefore the edge is unique because we are dealing with trees. Since the G-action was free, we get a treelike structure for G with finity d.

(6) $\implies$ (2): A group which contains a free subgroup of finite index acts on a tree with finite vertex stabilizers [8, IV.1.6]. The stabilizers are subgroups of G, therefore their order divides r.

(3) $\implies$ (6): By [8, IV.1.6], if G is the fundamental group of a graph of groups with finite vertex groups where the order is bounded by r, then it contains a free subgroup of finite index. Moreover, every finite subgroup of G is contained in a conjugate of a vertex group [8, I.7.11], i.e. the order of every finite subgroup divides r. □

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