TOPOLOGICAL REPRESENTATIONS OF POSETS

R.BRESLAV[1], A.STAVROVA[1], AND R.R.ZAPATRIN[2]

Abstract. In [10] an arbitrary poset P was proved to be isomorphic to the collection of subsets of a space $\mathcal{M}$ with two closures $\mathbf{C}_1$ and $\mathbf{C}_2$, which are closed in the first closure and open in other – $\mathrm{C}_1\mathrm{O}_2(\mathcal{M},\mathbf{C}_1,\mathbf{C}_2)$. As a space for this representation an algebraic dual space P^* was used. Here we extend the theory of algabraic duality for posets generalizing the notion of an ideal. This approach yields a sufficient condition for the collection of $\mathrm{C}_1\mathrm{O}_2$-subsets of $A \subset P^*$ (with respect to induced closures) to be isomorphic to P. Applying this result to certain classes of posets we prove some representation theorems and get a topological characterization of orthocomplementations.

1. Introduction

Since Stone introduced the topological representation of Boolean algebras [4] there was a lot of attempts to generalize this result: the Stone-like representations of orthopsets by Mayet and Tkadlec [5, 8], different topological representations of distributive [6, 7] and arbitrary (by Hartonas, Dunn and Urquhart) [3, 9] lattices. We follow the construction introduced in [10] where algebraic dual space P^* is endowed with two closures $\mathbf{C}_1$ and $\mathbf{C}_2$ in such a way that the collection of all subsets of P^* which are closed in $\mathbf{C}_1$ and open in $\mathbf{C}_2$ ordered by set inclusion (we denote this collection by $\mathrm{C}_1\mathrm{O}_2(P^*,\mathbf{C}_1,\mathbf{C}_2)$) is isomorphic to the initial poset P:

$$\mathrm{C}_1\mathrm{O}_2(A,\mathbf{C}_1,\mathbf{C}_2) \approx P \tag{1}$$

The representation (1) of P works for arbitrary poset P. However, for particular classes of posets the 'universal set' P^* can be contracted to a smaller one $A \subseteq P^*$ with the closures $\mathbf{C}_1$, $\mathbf{C}_2$ induced from P^*. In this paper we show that the representations of specific classes of posets mentioned above all have the form

[1]The Centre of Mathematics, Nevsky pr., 39, 191011, St.Petersburg, Russia
[2]Department of Mathematics, SPb UEF, Griboyedova 30–32, 191023, St.Petersburg, Russia

$$\mathrm{C_1O_2}(A, \mathbf{C}_1, \mathbf{C}_2) \approx P$$

and differ only by the choice of $A \subseteq P^*$.

1.1. **Spaces with two closures.** Mapping $\mathbf{C} : \exp(\mathcal{M}) \to \exp(\mathcal{M})$ we call *closure* if

1. $A \subset \mathbf{C}(A)$;
2. $\mathbf{C}(\mathbf{C}(A)) = \mathbf{C}(A)$;
3. if $A \subset B$ then $\mathbf{C}(A) \subset \mathbf{C}(B)$.

A set $A \subset \mathcal{M}$ is *closed* (or $\mathbf{C}$-*closed*) if $A = \mathbf{C}(A)$, A is *open* if $\overline{A} = \mathcal{M} \setminus A$ is closed and *clopen* if it is both closed and open. Note, that any intersection of closed sets is closed, and $\mathbf{C}(A)$ is the intersection of all closed sets which contain A. $\mathcal{K} \subset \exp(\mathcal{M})$ is called the *base* of closure $\mathbf{C}$ ($\mathbf{C} = \mathrm{clos}\,(\mathcal{K})$) if any closed set is an intersection of elements of $\mathcal{K}$.

The closure $\mathbf{C}$ is *exact* if $\mathbf{C}(\emptyset) = \emptyset$, and *topological* if $\mathbf{C}(A \cup B) = \mathbf{C}(A) \cup \mathbf{C}(B)$. Note, that exact topological closure defines topology on $\mathcal{M}$. For a closure $\mathbf{C}$ on $\mathcal{M}$ define $\mathrm{CO}(\mathcal{M}, \mathbf{C})$ to be the collection of all clopen subsets of $\mathcal{M}$. Obviously $\mathrm{CO}(\mathcal{M}, \mathbf{C})$ ordered by set inclusion is a bounded orthoposet. It was shown by Mayet and Tkadlec [5, 8], that for an arbitrary bounded orthoposet P there is a space $\mathcal{M}$ with closure $\mathbf{C}$ such that $P \approx \mathrm{CO}(\mathcal{M}, \mathbf{C})$.

If we define two closures on $\mathbf{C}_1$ and $\mathbf{C}_2$ $\mathcal{M}$, then by $\mathrm{C_1O_2}(\mathcal{M}, \mathbf{C}_1, \mathbf{C}_2)$ we denote the collection of all subsets of $\mathcal{M}$ which are both $\mathbf{C}_1$-closed and $\mathbf{C}_2$-open, ordered by set inclusion. We can say nothing about the structure of $\mathrm{C_1O_2}(\mathcal{M}, \mathbf{C}_1, \mathbf{C}_2)$ except it is a poset, moreover, as it was shown in [10] for an arbitrary poset P one can build a space with two closures such that $P \approx \mathrm{C_1O_2}(\mathcal{M}, \mathbf{C}_1, \mathbf{C}_2)$.

1.2. **Algebraic duality for posets.** For a poset P its *algebraic dual space* P^* is the set of all isotone mappings from P to poset $\mathbf{2} = \{0, 1\}$ with $0 < 1$. Here we develop the techniques needed to build the representation.

Consider $A \subset P^*$. A set I we call an *ideal* (with respect to A or A-*ideal*) if I is an intersection of kernels of some mappings $x \in A$ (i.e. $I = \bigcap x^{-1}(0)$). Dually, the intersection of co-kernels F we call a *filter* ($F = \bigcap x^{-1}(1)$).

For $B \subset P^*$ we define an ideal $I(B)$ (filter $F(B)$) to be the intersection of kernels (co-kernels, respectively) of $x \in B$.

For $Q \subset P$ define an ideal $[Q)_A$ (resp., filter $(Q]_A$) – the intersection of ideals (resp., filters) containing Q.

Note that ideals with respect to P^* coincide with order ideals (I is an order ideal if $q \in I$ and $p \leq q$ implies $p \in I$). In general A-ideals are always order ideals, but the converse is not always true.

We say that $A \subset P^*$ is *full* if for all $p \not\leq q$ there exists $x \in A$ such that $x(p) = 1$, $x(q) = 0$.

$A \subset P^*$ is called *separating* if for any disjoint ideal I and filter F there exists $x \in A$ such that $x\big|_I = 0$ and $x\big|_F = 1$.

In some cases discussed in section 3 $\boldsymbol{(p]}_A$ and $\boldsymbol{(p]}_A$ coincide with lower and upper cones of p respectively. Due to the following obvious lemma the separating set is full in this case.

Lemma 1. *Let A be a separating subset of P^* and $\boldsymbol{[p)}_A \cap \boldsymbol{(q]}_A = \emptyset$ for all $q \not\leq p \in P$. Then A is full.*

2. Topological representation: the general case

Define two closures on P^*. For $p \in P$ consider two subsets of P^*:

$$\mathcal{UP}(p) = \{x \mid x(p) = 1\} \qquad \mathcal{LO}(p) = \{x \mid x(p) = 0\}.$$

Then define closures $\mathbf{C}_1$, $\mathbf{C}_2$ in the following way:

$$\mathbf{C}_1 = \operatorname{clos}\{\mathcal{UP}(p)\}_{p\in P} \quad \text{and} \quad \mathbf{C}_2 = \operatorname{clos}\{\mathcal{LO}(p)\}_{p\in P}.$$

Note, that since $\overline{\mathcal{UP}(p)} = \mathcal{LO}(p)$ all $\mathcal{UP}(p)$ are $\mathrm{C_1O_2}$-sets.

On $A \subset P^*$ consider closures $\mathbf{C}_{1A}, \mathbf{C}_{2A}$ induced by $\mathbf{C}_1$ and $\mathbf{C}_2$ (i.e. $\mathbf{C}_{iA}(X) = \mathbf{C}(X) \cap A$). Let $\mathcal{UP}_A(p) = \mathcal{UP}(p) \cap A$ and $\mathcal{LO}_A(p) = \mathcal{LO}(p) \cap A$, then

$$\mathbf{C}_{1A} = \operatorname{clos}\{\mathcal{UP}_A(p)\}_{p\in P} \quad \text{and} \quad \mathbf{C}_{2A} = \operatorname{clos}\{\mathcal{LO}_A(p)\}_{p\in P}.$$

We omit the index A in $\mathbf{C}_{iA}, \mathcal{UP}_A$ etc. when it is clear which subspace is meant.

The following equations show the relation between the closures introduced on A and A-ideals:

$$\mathbf{C}_1(X) = \bigcap_{p\in F(X)} \mathcal{UP}(p) \quad \text{and} \quad \mathbf{C}_2(X) = \bigcap_{p\in I(X)} \mathcal{LO}(p).$$

Theorem 2. *Let $A \subset P^*$. Consider $\sigma : P \to \mathrm{C_1O_2}(A, \mathbf{C}_{1A}, \mathbf{C}_{2A})$ which maps p to $\mathcal{UP}(p)$, then*

1. *σ is isotone;*
2. *if A is full then σ is injective;*
3. *if A is separating then σ is surjective.*

Proof. (1) Since $p \leq q$ implies $x(p) \leq x(q)$ for all $x \in P^*$ then $p \leq q$ implies $\mathcal{UP}(p) \subset \mathcal{UP}(q)$, so σ is isotone.
(2) For $p \neq q$ either $p \not\leq q$ or $q \not\leq p$, so there exists $x \in A : x(p) \neq x(q)$, then exactly one of $\mathcal{UP}(p)$, $\mathcal{UP}(q)$ contains x and $\mathcal{UP}(p) \neq \mathcal{UP}(q)$.
(3) Let $B \in \mathrm{C_1O_2}(A, \mathbf{C}_{1A}, \mathbf{C}_{2A})$, then $B = \mathbf{C}_{1A}(B)$ and $\overline{B} = \mathbf{C}_{2A}(\overline{B})$. Consider $Q = I(\overline{B}) \cap F(B) = I \cap F$. If $Q = \emptyset$ there exists $x \in A$: $x\big|_I = 0$ and $x\big|_F = 1$, so $x \in \mathcal{UP}(p)$ for all $p \in F$ and $x \in \mathcal{LO}(q)$ for all $q \in I$. Thus $x \in B$ and $x \in \overline{B}$ simultaniously, so $Q \neq \emptyset$. For $p \in Q$ we have $B \subset \mathcal{UP}(p)$, $\overline{B} \subset \mathcal{LO}(p) = \overline{\mathcal{UP}(p)}$ and $B = \mathcal{UP}(p)$. □

Corollary 2.1. *Let A be a full and separating subspace of P^*, then* $P \approx \mathrm{C_1O_2}(A, \mathbf{C}_{1A}, \mathbf{C}_{2A})$.

To get the topological representation of an arbitrary poset we prove

Lemma 3. *P^* is full and separating.*

Proof. For disjoint ideal I and filter F, which are in this case order ideal and filter, consider $x : x(p) = 0$ for $p \in I$ and $x(p) = 1$ otherwise. Obviously $x \in P^*$ and separates I and F. Applying lemma 1 we see that P^* is full. □

This leads us to the following theorem:

Theorem 4. *Let P be an arbitrary poset, then* $P \approx \mathrm{C_1O_2}(P^*, \mathbf{C}_1, \mathbf{C}_2)$.

Due to the following lemma in the case of bounded poset P subspaces A of P^* can be reduced:

Lemma 5. *Let P be a bounded poset, $A \subset P^*$ be full and separating, then $A \setminus \{\mathbf{0}, \mathbf{1}\}$, where $\mathbf{0}, \mathbf{1} \in P^*$ are constant mappings, is also full and separating.*

Proof. Note that the ideals (filters) with respect to $A \setminus \{\mathbf{0}, \mathbf{1}\}$ coincide with the proper A-ideals (A-filters) and for disjoint nonempty I and F the separating mapping $x \in A$ is not constant. □

3. Topological representations: special cases

We apply the results of previous section to some special classes of posets.

3.1. **Orthoposets.** The bounded poset P is called an *orthoposet* if there exists an anti-isotone mapping $(\cdot)' : P \to P$ (*orthocomplementation*) such that $p = (p')'$, $p \vee p' = 1$ and $p \wedge p' = 0$. For an orthoposet define its *orthodual* space $P^{*\prime}$ to be the set of all $x \in P^*$ such that $x(p') = (x(p))'$.

Lemma 6. *$P^{*\prime}$ is full and separating.*

Proof. For disjoint ideal I and filter F consider $x : x(p) = 0$ for $p \in I \cup F'$, $x(p) = 1$ for $p \in I' \cup F$, otherwise $x(p) = y(p)$ for some $y \in P^{*\prime}$. Obviously $x \in P^{*\prime}$ and separates I and F, so $P^{*\prime}$ is separating. As $[p)_{P^{*\prime}}$ is the lower cone of p for all $p \in P$ $P^{*\prime}$ is full according to lemma 1. $\square$

Since $\mathcal{UP}_{P^{*\prime}}(p') = \mathcal{LO}_{P^{*\prime}}(p)$, the bases of closures $\mathbf{C}_1$ and $\mathbf{C}_2$ coincide and $\mathbf{C}_1 = \mathbf{C}_2$. Denote

$$\mathbf{C} = \mathbf{C}_1 = \mathbf{C}_2$$

Then $\mathrm{C_1O_2}$-sets are $\mathbf{C}$-clopen. Applying theorem 2 we have

Theorem 7. *Let P be an orthoposet, then there exists a closure space $(\mathcal{M}, \mathbf{C})$ such that $P \approx \mathrm{CO}(\mathcal{M}, \mathbf{C})$.*

The representation obtained in previous theorem coicides with that described by Mayet [5] and Tkadlec [8].

Now we use the notion of full separating subspace to characterize all orthocomplementations which can be introduced on a bounded poset P. Any orthocomplementation $(\cdot)'$ defines a full separating subspace of P^* on which the closures $\mathbf{C}_1$ and $\mathbf{C}_2$ coincide. Let $\mathcal{S}$ be the collection of full separating subspaces of P^* where $\mathbf{C}_1 = \mathbf{C}_2$. Consider $A \in \mathcal{S}$ then the set complementation on $\mathrm{C_1O_2}(A, \mathbf{C}_1, \mathbf{C}_2) \approx P$ is an orthocomplementation, so with every $A \in \mathcal{S}$ we can associate an orthocomplementation $(\cdot)'^A$ on P.

Theorem 8. *All orthocomplementations on P are in one-to-one correspondence with maximal (with respect to set inclusion) elements of $\mathcal{S}$.*

Proof. For $A \in \mathcal{S}$ all $x \in A$ preserves $(\cdot)'^A$ because $x(p) = x(p'^A) = 1$ implies $x \in \mathcal{UP}(p)$ and $x \in \mathcal{UP}(p'^A) = \overline{\mathcal{UP}(p)}$ (the similar contradiction holds for $x(p) = 0$). It means that $A \subset P^{*\prime A}$, so all maximal elements of $\mathcal{S}$ are of the form $P^{*\prime A}$. Thus any orthodual space $P^{*\prime}$ is a subspace of $P^{*\prime A}$ for some A. Obviously, orthocomplementation associated with $P^{*\prime A}$ is $(\cdot)'^A$ and the one associated with $P^{*\prime}$ is $(\cdot)'$. Since $P^{*\prime} \subset P^{*\prime A}$ and orthocomplementations are induced by set complementation we get that $(\cdot)' = (\cdot)'^A$ and $P^{*\prime} = P^{*\prime A}$, so all orthodual spaces, defined by different orthocomplementations on P, are maximal in $\mathcal{S}$. $\square$

3.2. **Distributive lattices.** According to the Stone representation theorem any Boolean algebra is isomorphic to the collection of all clopen sets in some topological space. Since Boolean algebra is an orthocomplemented distributive lattice one can expect distributive lattice to be represented as the collection of $\mathrm{C}_1\mathrm{O}_2$-sets of some space with two topological closures. We are going to construct such a representation which follows from theorem 2 and is different from Priestley [6] and Rieger [7].

For a lattice L let $L^{*\vee\wedge} \subset L^*$ be the set of all lattice morphisms (isotone mappings preserving lattice operations) from L to $\mathbf{2}$. Note that $L^{*\vee\wedge}$-ideal is always lattice ideal (an order ideal I is called lattice ideal if $a, b \in I$ implies $a \vee b \in I$).

Lemma 9. *For any distributive lattice L the ideals (filters) with respect to $L^{*\vee\wedge}$ coincide with the lattice ideals (filters). Besides that, $L^{*\vee\wedge}$ is full and separating.*

Proof. First we prove that for disjoint lattice ideal I and filter F there exists $x \in L^{*\vee\wedge}$ such that $x|_I = 0; x|_F = 1$ (it means that $L^{*\vee\wedge}$ separates lattice ideals). Suppose I_0 to be the maximal lattice ideal containing I which is disjoint with F. The set-complement of I_0 is a filter [2], thus the mapping x: $x|_{I_0} = 0$, $x|_{L\setminus I_0} = 1$ preserves $\vee$ and $\wedge$. For an arbitrary $p \in L$ the upper cone of p is a lattice filter. Then we get every lattice ideal I to be the intersection of kernels of all x_p, which separates I and the upper cone of p, over all $p \notin I$, so I is an ideal with respect to $L^{*\vee\wedge}$ (recall the definition of ideal in section 1.2). Hence, the separating property for $L^{*\vee\wedge}$ is equivalent to the fact that $L^{*\vee\wedge}$ separates lattice ideals, which was proved above. $L^{*\vee\wedge}$ is full by lemma 1. □

Theorem 10. *For any distributive lattice L there exists a space with two topological closures* $(\mathcal{M}, \mathbf{C}_1, \mathbf{C}_2)$ *such that* $L \approx \mathrm{C}_1\mathrm{O}_2(\mathcal{M}, \mathbf{C}_1, \mathbf{C}_2)$.

Proof. The only thing we need to prove is that the closures $\mathbf{C}_1$, $\mathbf{C}_2$ induced on $L^{*\vee\wedge}$ are topological. Since elements of $L^{*\vee\wedge}$ preserve both $\vee$ and $\wedge$ we have $\mathcal{UP}(p \vee q) = \mathcal{UP}(p) \cup \mathcal{UP}(q)$ and $\mathcal{LO}(p \wedge q) = \mathcal{LO}(p) \cup \mathcal{LO}(q)$, so the bases of $\mathbf{C}_1$ and $\mathbf{C}_2$ are closed under finite set union, therefore the closures themselves are topological. □

Corollary 10.1. *A lattice L is distributive iff $L^{*\vee\wedge}$ is a full separating subspace of L^*.*

Proof. This follows from lemma 9, the fact that for any lattice L the closures induced on $L^{*\vee\wedge}$ are topological, and that for any space $\mathcal{M}$ with two topological closures $\mathrm{C}_1\mathrm{O}_2(\mathcal{M}, \mathbf{C}_1, \mathbf{C}_2)$ is a distributive lattice. □

3.3. **Boolean algebras.** Here we present a proof of the Stone representation theorem:

Theorem 11 (Stone)**.** *Any Boolean algebra B is isomorphic to the collection of all clopen subsets of a topological space.*

Proof. Since B is a bounded distributive lattice, $B^{*\vee\wedge} \setminus \{\mathbf{0}, \mathbf{1}\}$ is full and separating. Every lattice morphism of Boolean algebras preserves orthocomplementation and, as in the case of orthoposets, the topological closures $\mathbf{C}_1$ and $\mathbf{C}_2$ do coincide.

Associating with every element of $B^{*\vee\wedge} \setminus \{\mathbf{0}, \mathbf{1}\}$ its kernel (that is a maximal lattice ideal) one get the Stone space of Boolean algebra originally described in [4]. □

Corollary 11.1. *Let L be a distributive lattice, then L is a Boolean algebra iff closures $\mathbf{C}_1$ and $\mathbf{C}_2$ coincide on $L^{*\vee\wedge} \setminus \{\mathbf{0}, \mathbf{1}\}$.*

References

[1] G. Birkhoff, *Lattice theory*, American Mathematical Society, Providence, Rhode Island, 3rd ed., 1967.

[2] G. Grätzer, *General lattice theory*, Academic Press, 1977.

[3] J.M. Dunn, C. Hartonas, Stone Duality for Lattices, *Algebra Universalis* **37** (1997) 391.

[4] M.H. Stone, The Representation of Boolean Algebras, *Bull. Amer. Math. Soc.* **44** (1938) 807.

[5] R. Mayet, Une dualité pour les ensembles ordonnés complémentés, *Comptes Rendus de l'Academie des Sciences de Paris, sér. I Math.* **294** (1982) 63.

[6] H.A. Priestley, Representation of Distributive Lattices by means of Ordered Stone Spaces, *Bull. London Math. Soc.* **2** (1970) 186.

[7] L. Rieger, A Note on Topological Representation of Distributive Lattices, *Časopsis pro Pěstovani Matematiky a Fysiky* **74** (1949) 55.

[8] J. Tkadlec, Partially additive measures and set representation of orthoposets, *Journal of Pure and Applied Algebra* **86** (1993) 79.

[9] A. Urquhart, A topological representation theory for lattices, *Algebra Universalis* **8** (1978) 45.

[10] R.R. Zapatrin, Algebraic duality in the theory of partially ordered sets, *Pure Mathematics and Applications* **9**, 3/4 (1998) 485.

[11] R.R. Zapatrine, *Les espaces duaux pour les ensembles ordonnés arbitraires*, e-print http://xxx.lanl.gov/abs/math.GN/0001080