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WEIGHTED TRACES ON ALGEBRAS OF PSEUDO-DIFFERENTIAL OPERATORS AND GEOMETRY ON LOOP GROUPS

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Abstract

Using *weighted traces* which are linear functionals of the type

$$A \to tr^Q(A) := \left(tr(AQ^{-z}) - z^{-1}tr(AQ^{-z})\right)_{z=0}$$

defined on the whole algebra of (classical) pseudo-differential operators (P.D.O.s) and where Q is some positive invertible elliptic operator, we investigate the geometry of loop groups in the light of the cohomology of pseudo-differential operators. We set up a geometric framework to study a class of infinite dimensional manifolds in which we recover some results on the geometry of loop groups, using again weighted traces. Along the way, we investigate properties of extensions of the Radul and Schwinger cocycles defined with the help of weighted traces.

Résumé

A l'aide de *traces pondérées* qui sont des fonctionnelles linéaires du type:

$$A \to tr^Q(A) := \left(tr(AQ^{-z}) - z^{-1}tr(AQ^{-z})\right)_{z=0}$$

définies sur toute l'algèbre des opérateurs pseudo-différentiels classiques, Q étant un opérateur elliptique inversible, on étudie la géométrie de l'espace des lacets à la lumière de la cohomologie des opérateurs pseudo-différentiels. On met en place un cadre géométrique afin d'étudier une classe de variétés de dimension infinie, cadre dans lequel on retrouve, toujours à l'aide des traces pondérées, des résultats concernant la géométrie des lacets. Ces traces pondérées nous permettent aussi d'étendre la notion de cocycle de Radul et de Schwinger et d'en étudier certaines propriétés.

Introduction

The paper is built up from two parts: the first one presents algebraic tools which are used in the second part to extend some geometric concepts to the infinite dimensional context. The approach to the geometry of loop groups we present in the second part uses weighted traces and it is new to our knowledge. Weighted traces offer a useful tool to prove both algebraic and geometric results, some of which had been proved elsewhere by other methods.

Let us describe the contents of the first part of the paper (sections 1-4). The Lie algebra of interest in our context is the infinite dimensional Lie algebra of pseudo-differential operators (P.D.Os) acting on sections of some finite rank vector bundle E based on a closed manifold M (its dimension does not yet play a role at this stage). The Lie bracket is given by the operator bracket. It is well known that when the manifold is connected and of dimension strictly larger than 1, the only trace on this algebra, i.e the only $\mathbb{C}$ valued linear functional which satisfies the *tracial property* namely $tr[A, B] = 0$, is the Wodzicki residue trace [W], see also [K] for a review.

Here we shall consider linear functionals on this algebra which arise as *zeta function regularized traces* . They involve a *weight* given by a positive self-adjoint elliptic operator and in general depend on the choice of the weight hence the terminology *weighted traces* which we shall use here. One can find such "traces" (sometimes implicitely) in the literature on determinant bundles [BF] (in particular in the connection and the curvature) and more generally when investigating geometry in infinite dimensions (in particular for the notion of minimality of submanifolds [MRT],[AP]). Here we will be using them to define the Ricci curvature on current groups and first Chern form on loop groups.

These weighted traces extend the so called canonical trace of Kontsevich and Vishik [KV1] defined on the subalgebra of P.D.Os with integer order that lie in the odd class (defined below) provided the dimension of the underlying manifold is odd. On this subalgebra (which contains ordinary differential operators) the weighted traces actually obey the tracial property $tr[A, B] = 0$ which is not the case on the whole algebra of P.D.Os.

Here, rather than searching for subalgebras on which the "weighted traces" actually are traces, we focus on the obstruction that prevents them from being traces on a bigger subalgebra of P.D.Os. It is measured by the coboundary of the weighted trace and yields a generalization of the well known Radul cocycle (see [KK], [R], [M2]). It is of infinite dimensional essence since it can be expressed in terms of a Wodzicki residue which is a purely infinite dimensional trace.

With this idea in mind of studying infinite dimensional obstructions rather than restricting ourselves to subalgebras on which they vanish, using weighted traces we build up bilinear functionals on *the whole* algebra of classical P.D.Os. These restrict to the twisted Radul [M2] and Schwinger [S], [M2], [CFNW] cocycles on the (rather small) subalgebra of P.D.Os that lie in g_{res}, the algebra of bounded operators A such that $[A, \epsilon]$ is of order no larger than $-\frac{dim M}{2}$ where ϵ is the "sign" of some self-adjoint elliptic operator.

Finally, we investigate the link between our Schwinger functional and a generalization of another cocycle arising in the context of a central extension of the group G_{res} of invertible operators in g_{res}. These relations boil down to well-known [PS] identifications of these cocycles on the restriction to g_{res} for the obstruction to such identifications which is given in terms of Radul cocycles vanishing on this subalgebra.

Let us now turn to a more geometric point of view which leads us to the second part of the paper (sections 5-9). We now consider families of pseudo-differential operators, extending the notion of weighted trace to such families. On top of the algebraic obstructions mentioned above, due to the dependence on the weight, there are obstructions of geometric type which arise when trying to generalize properties of classical geometric concepts to infinite dimensions.

Current groups $Map(M, G)$ where M is a manifold and G is a Lie group, offer a first tractable example of infinite dimensional manifold. As spaces of classical paths in Wess-Zumino Witten models, they play an important role in quantum field theories. They have been the topic of many an investigation and particularly from a geometric point of view (e.g. [DL], [F1,2], [P], [PS], [SP], [Wu]). In a pioneering article [F1], Freed suggested a way to geeneralize to these groups some of the methods available to study finite dimensional Lie groups. In particular, he defined a Ricci curvature on current groups and a first Chern form on the $H^{\frac{1}{2}}$ based loop group which is Kähler. Other methods [SW] have since then been suggested, leading to the same expressions. In [F1] (see also [F2], [SW]) the author uses a "conditioned" trace which involves taking a "two step trace", namely first the trace on the Lie algebra of the finite dimensional Lie group G and then, when the operator obtained this way is trace-class, taking its trace. He shows that the (conditioned) first Chern form on the based loop group is proportional to the symplectic form, from which follows on one hand that it is closed and hence defines the first Chern class, and on the other hand that it is Kähler-Einstein.

Here we suggest a more general approach to defining Ricci curvature and first Chern forms on a class of manifolds which includes the current groups mentioned above. Carried out (in a left invariant way) to current Lie groups equipped with a weight given by a left invariant field of elliptic operators, the notion of weighted trace enables us to define the *weighted Ricci curvature* as a weighted pseudo-trace of $R : Z \to \Omega(Z, \cdot)\cdot$ where Ω is the curvature tensor and the *weighted first Chern form* as a weighted pseudo-trace of the curvature, provided the operators involved are classical P.D.Os.

We express the weighted first Chern form in terms of a pull-back by the adjoint representation of a cocycle on g_{res}, thus relating the closed two form given by the Chern form with a closed two cochain.

A well known result by Kuiper [Ku] shows that a Hilbert manifold is parallelisable since $GL(H)$ is contractible for a Hilbert space H (the model space of the manifold). This might seem in conflict with the fact that the first Chern class does not vanish. But in our approach, the model space H being a space of sections of some (finite rank) vector bundle E based on a closed manifold M, the structure group $GL(H)$ reduces to a non contractible group, namely the group $Ell_0^*(M, E)$ of zero order invertible elliptic operators acting on these sections.

This opens a road to many questions, such as finding a criteria for the weighted first Chern form and higher order forms to be closed, investigating the holonomy group which we expect to be non trivial because of the reasons mentioned above. This article yields a geometric setting in which such questions make sense for a class of Hilbert manifolds beyong the example of loop groups. It confronts this approach with other approaches in the specific case of loop groups.

1. Weighted traces on the algebra of (classical) P.D.Os

In [KV 1], the authors introduced a new trace type functional TR on operators in $PDO(M,E)$ with order in $\alpha_0 + \mathbb{Z}$ where α_0 is some non integer complex number called the canonical trace. Avoiding integer orders has to do with the fact that a positive homogeneous distribution on $\mathbb{R}^n/\{0\}$ of non integer order can be extended in a unique way to a positive homogeneous distribution on $\mathbb{R}^n$ [H]. In this section we first briefly recall the construction of this trace TR and some of its properties. We use this trace as a tool to build linear functionals of pseudo-differential operators which we call "weighted traces" and to study some of their properties. Such functionals were also considered in [MN] to prove a pseudo-differential generalization of the Atiyah-Patodi-Singer index theorem. For the sake of self-containedness, we recall the proof of this fact. We refer the reader to Appendix A for notations and basic facts concerning pseudo-differential operators.

- *The Kontsevich and Vishik canonical trace*

Let us first describe the general lines of the construction in a heuristic way. For a (classical) pseudo-differential operator locally given by:

$$Au(x) := \int_{\mathbb{R}^n} a(x,\xi)\hat{u}(\xi)d\xi$$

where $\sigma_A(x,\xi)$ is the (locally defined) total symbol (see Appendix A), we would like to define

$$\text{"}TR(A) := \int_M \int_{\mathbb{R}^n} tr_x \sigma_A(x,\xi) d\xi dvol(x)\text{"}$$

which in general does not make sense since $a(x,\xi)$ typically has components of degree $\geq -dimM$. It however does make sense for an operator of order $< -dimM$ and yields the ordinary trace. We should therefore find a way of only picking up the finite term in such an expression.

Let us consider a classical P.D.O A of order α with symbol $\sigma = \sum_{j=0}^N \Psi\sigma_{\alpha-j} + \sigma_{(N)}$ (see [Sh] for a discussion about such assumptions on the symbol) where $\sigma_{\alpha-j} \in S^{\alpha-j}(\mathbb{R}^n)$, $\sigma_{(N)} \in S^{\alpha-N-1}(\mathbb{R}^n)$ and where Ψ is a smooth function on $\mathbb{R}^n$ which is zero in $B(0,\frac{1}{4})$ and equal to 1 on $\mathbb{R}^n - B(0,\frac{1}{2})$. Since $\sigma_{(N)}(\xi) = O(|\xi|^{\alpha-N})$ we have $\int_{B(0,R)} \sigma_{(N)}(\xi)d\xi = \int_{\mathbb{R}^n} \sigma_{(N)}(\xi)d\xi + O(R^{\alpha+n-N})$. On the other hand, splitting the integrals $\int_{B(0,R)} \Psi\sigma_{\alpha-j}(\xi)d\xi = \int_{B(0,1)} \Psi\sigma_{\alpha-j}(x,\xi)d\xi + \int_{B(0,R)/B(0,1)} \Psi\sigma_{\alpha-j}(x,\xi)d\xi$, we use the fact that $\sigma_{\alpha-j}$ is homogeneous of order $\alpha - j$ to express the last integral. If α is an integer then there is an integer j_0 such that $\alpha - j_0 + n = 0$ and we have for $N > j_0$:

$$\begin{aligned}
&\sum_{j=0}^N \int_{B(0,R)/B(0,1)} \Psi\sigma_{\alpha-j}(x,\xi)d\xi = \sum_{j=0}^N \int_1^R \int_{|\xi|=1} r^{\alpha-j+n-1}\sigma_{\alpha-j}(x,\xi)d\xi dr \\
&\sim_{R\to\infty} \sum_{j=0, n+\alpha-j\neq 0}^N \frac{1}{\alpha+n-j} R^{\alpha-j+n} \int_{|\xi|=1} a_{\alpha-j}(x,\xi)d\xi + logR \int_{|\xi|=1} a_{-n}(x,\xi)d\xi + \text{constant term.}
\end{aligned}$$

Finally this yields the existence of an asymptotic expansion in $R \to +\infty$ and of a constant (w.r.to R) $C(a(x,\cdot))$ such that

$$\begin{aligned}
&\int_{B(0,R)} a(x,\xi)d\xi \\
&\sim_{R\to\infty} \sum_{j=0, n+\alpha-j\neq 0}^N \frac{1}{\alpha+n-j} R^{\alpha-j+n} \int_{|\xi|=1} \sigma_{\alpha-j}(x,\xi)d\xi + logR \int_{|\xi|=1} a_{-n}(x,\xi)d\xi + C(a(x,\cdot)).
\end{aligned}$$

Because of the logarithmic term in R, one does not expect the finite part

$$f.p.\left(\int_{\mathbb{R}^n} a(x,\xi)d\xi\right) = LIM_{R\to\infty} \int_{B(0,R)} \sigma(x,\xi)d\xi$$

to be invariant under a change of variable of $\mathbb{R}^n$. However if the order α is not an integer, there is no logarithmic divergence and $f.p. \int_{\mathbb{R}^n} \sigma(x,\xi) d\xi$ is independent of the local representation $a(x,\xi)$ of A:

Proposition [KV1, 2] (see also [Le] in Prop. 5.2): *Provided $A \in PDO(M,E)$ has non integer order:*

$$TR(A) := \int_M tr_{E_x} \left(f.p.(\int_{\mathbb{R}^n} \sigma_A(x,\xi) d\xi) \right) dvol(x) \tag{1.1}$$

where σ_A is the symbol of A, is well-defined and satisfies the tracial property:

$$TR([A,B]) = 0 \quad \forall A \in PDO(M,E), B \in PDO(M,E), \textit{such that} \quad ord(A) + ord(B) \notin \mathbb{Z}.$$

Here tr_{E_x} denotes the trace on the fibre E_x of the bundle E above the point x.

The linear functional TR coincides with the usual trace for P.D.Os of order with real part strictly smaller than minus the dimension of the underlying manifold the operators are acting on. TR is in fact a trace functional i.e $TR[A,B] = 0$ for any $A, B \in PDO(M,E)$ such that $ordA + ordB \in \mathbb{C} - \mathbb{Z}$ (see Proposition 3.2 in [KV2]).

- *A fundamental property of the canonical trace*

Following Kontsevich and Vishik, we shall call a local family $A_z \in PDO(M,E)$ with distribution kernels (locally) *weakly holomorphic* if the following conditons are satisfied:

(i) The order α_z of A_z is (locally) holomorphic in z,

(ii)The kernel $A_z(x,y)$ of A_z is (locally) holomorphic in z for x, y in disjoint local charts,

(iii)Given any local chart U on M, the homogeneous components $\sigma_{A_z,\alpha_z-j}(x, \frac{\xi}{|\xi|})$ of the symbol $\sigma_{A_z}(x,\xi)$ are (locally) holomorphic functions in z on the restriction to U of the cotangent sphere bundle S^*M.

(iv) When x and y belong to a common local chart U, the difference between $A_z(x,y)$ and the truncated kernel $\sum_{j=0}^N \int \rho(|\xi|) \sigma_{A_z,\alpha_z-j}(x,\xi) exp(i\langle x-y, \xi\rangle d\xi$ of class $C^{k(N)}$ for some $k(N)$ increasing with N tends to a (locally) holomorphic kernel on $U \times U$ when $N \to \infty$.

In (iii) the topology on the space of symbols is given by the supremium norm of the symbol and its derivatives and in (iv) the convergence is to be understood in the sense of weak convergence of distributions [KV1].

The following property of the canonical trace plays a fundamental part in these notes:

Fundamental property(see [KV2] Proposition 3.4 and [KV1] Th.3.13)

For any (local weak) holomorphic family $A(z)$ of classical P.D.Os on M, $z \in U \subset \mathbb{C}$, $ordA_z = \alpha(z)$ where α is holomorphic and α' does not vanish, the function $TR(A_z)$ is meromorphic with no more than simple poles at $z = m \in U \cap \mathbb{Z}$.

- *Wodzicki residue*

Applying the fundamental property to the family $A_z^Q := AQ^{-z}$ where $Q \in Ell^{*,+}_{ord>0}(M,E)$, $A \in PDO(M,E)$ leads to the notion of *Wodzicki residue* [W] (see also [K] for a review):

$$res(A) := ordQ \cdot Res_{z=0} TR(A_z^Q) \tag{1.2}$$

which is in fact independent of Q. This can be carried out for any operator $Q \in Ell^+_{ord>0}(M,E)$ which might not be injective, replacing it by the operator $Q + P_Q$ in the above formulas so that $A_z^Q = A(Q + P_Q)^{-z}$ where P_Q denotes the orthogonal projection of Q onto its kernel which is finite dimensional since Q is elliptic and the manifold closed. The projection is orthogonal for the inner product on the space

$C^\infty(M,E)$ of smooth sections of E induced by the hermitian structure $\langle \cdot, \cdot \rangle_x$ on the fibre over $x \in M$ and the Riemannian volume measure μ on M:

$$\langle \sigma, \rho \rangle := \int_M d\mu(x) \langle \sigma(x), \rho(x) \rangle_x \quad \forall \sigma, \rho \in C^\infty(M,E)$$

The Wodzicki residue defines a trace functional on the algebra $PDO(M,E)$ of classical P.D.Os and vanishes for any classical P.D.O with non integer order. Since res also vanishes on a smoothing operator, it induces a trace functional on the symbol algebra of $PDO(M,E)$ [W]. It is in fact the unique trace functional on $PDO(M,E)$ provided M is connected and has dimension > 1.

An important feature of the Wodzicki residue we need to keep in mind here is that it vanishes on any trace-class operator as can easily be checked from the above definition.

Kontsevich and Vishik (see [KV2] see (3.16)) furthermore show that, given a local weak holomorphic family A_z of operators of order $\alpha(z)$, the poles of $TR(A_z)$ at entire points are expressed in terms of Wodzicki residues:

$$Res_{z=m} TR(A_z) = -\frac{1}{\alpha'(\alpha^{-1}(m))} res \left(\sigma(A_{\alpha^{-1}(m)}) \right). \tag{1.3}$$

• *The weighted trace*

The fundamental property leads yet to another linear functional on the algebra $PDO(M,E)$ which is not a trace but interesting all the same because it does not vanish on trace class operators for which it coincides with the ordinary trace.

Given $Q \in Ell^{*,+}_{ord>0}(M,E)$ of order $ordQ$, we call the *Q-weighted trace* of an operator $A \in PDO(M,E)$:

$$tr^Q(A) := \left[TR(AQ^{-z}) - \frac{1}{ordQ \cdot z} res(A) \right]_{z=0}. \tag{1.4}$$

Here again, this extends to the case when Q is not injective setting:

$$tr^Q(A) := \left[TR(A(Q+P_Q)^{-z}) - \frac{1}{ordQ \cdot z} res(A) \right]_{z=0} \tag{1.5}$$

where as before P_Q is the orthogonal projection onto the kernel of Q.

Warning! To simplify notations, we shall often assume that Q is injective subintending that when it is not, one should replace Q by $Q+P_Q$.

As we shall soon see, although refered to as weighted traces here, these functionals do not satisfy the tracial property $tr^Q[A,B] = 0$ and hence do not deserve the name "trace". We shall all the same keep to this abusive terminology which turns out to be very convenient.

• *Dependence on the weight*

Unlike the Wodzicki residue, it generally depends on the choice of Q. The dependence is intrinsically infinite dimensional since it is measured in terms of a Wodzicki residue. Indeed, let Q_1, Q_2 be two operators in $Ell^{*,+}_{ord>0}(M,E)$ with same order q, applying the fundamental property and $(*)$ to the (locally around zero) holomorphic family $A(\frac{Q_1^{-z} - Q_2^{-z}}{z})$ of order $\alpha(z) := ordA - zq$, we find (see Prop.2.2 in [KV2]):

$$\begin{aligned} tr^{Q_1}(A) - tr^{Q_2}(A) &= \lim_{z \to 0} TR(A(\frac{Q_1^{-z} - Q_2^{-z}}{z})) \\ &= -q^{-1} \cdot res(A(logQ_1 - logQ_2)) \end{aligned} \tag{1.6}$$

which is well defined since $logQ_1 - logQ_2$ lies in $PDO(M,E)$.

However when the underlying manifold is odd dimensional and for any odd-class classical P.D.O A with integer order, the TR-generalized zeta function $z \to TR(AQ^{-z})$ is regular at 0 and $tr^Q(A)$ obtained as the limit when $z \to 0$ of these expressions is independent of the choice of Q (see [KV1] Prop. 4.1 where it is denoted by $Tr_{(-1)}$). This limit can be seen as an extension of the canonical trace to operators with integer orders.

Although weighted traces are not tracial, they have a useful covariance property:

Lemma 1

Let $C \in PDO(M, E)$ be injective and bounded. With the same notations as above, we have:

$$tr^{C^{-1}QC}(A) = tr^Q(CAC^{-1}). \tag{1.7}$$

Proof: For $z \in \mathbb{C}$ with real part large enough, we have:

$$\begin{aligned} TR(A(C^{-1}QC)^{-z}) &= TR(AC^{-1}Q^{-z}C) \\ &= TR(CAC^{-1}Q^{-z}) \end{aligned}$$

where we have used that TR is tracial. Taking the renormalized limit then yields the result. •

2. The Radul cocycle as a coboundary of the weighted trace

In this section we investigate the coboundary of the weighted traces introduced above. We show how the cocycles obtained in this way relate to the Radul cocycle which arises in geometric quantization [M2]. In the context of regularized determinants, it is related to the multiplicative anomaly [D]. This cocycle was already investigated in [MN] where the authors express the coboundary of weighted traces in terms of a Wodzicki residue (see [MN] Lemma 13).

We shall need some definitions of Lie algebra cohomology (see e.g. [M1]).

• *Lie algebra cohomology:* Let L be a Lie algebra and V an L-module (the action of L on V is denoted by a "dot"). A *cochain of degree n* (or *n-cochain*) with values in V is an antisymmetric multilinear map $c : L \times L \times \cdots \times L$ (n times) $\to V$. Let $C^n(L, V)$ denote the space of all n-cochains and let us define the coboundary operator:

$$\delta : C^n(L, V) \to C^{n+1}(L, V)$$

$$\begin{aligned} (\delta c^n)(x_1, x_2, \cdots, x_{n+1}) &= \sum_{i<j} (-1)^{i+j} c^n([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{n+1}) \\ &+ \sum_{i=1}^{n+1} (-1)^{i+1} x_i \cdot c^n(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1}) \end{aligned}$$

Taking $V = \mathbb{C}$ with the trivial zero action of L on $\mathbb{C}$, $x \cdot z := 0$ for any $x \in L$, $z \in \mathbb{C}$ we have :

$$(\delta c^n)(x_1, x_2, \cdots, x_{n+1}) = \sum_{i<j} (-1)^{i+j} c^n([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{n+1}).$$

In particular $\delta^2 = 0$. An *n-cocycle* is a cochain of degree n with vanishing coboundary. Let us denote by $B^n(L, V)$ the set of n-coboundaries, by $Z^n(L, V)$ the set of n cocycles and let us call $H^n(L, V) := Z^n(L, V)/B^n(L, V)$ the n-th cohomology space.

• *The weighted Radul cocycle on $PDO(M, E)$*

We now apply this construction to $L := PDO(M, E)$ and the 1-cochain given by a weighted trace.

Let $Q \in Ell^{*,+}_{ord>0}(M, E)$. The Q-weighted traces tr^Q do not satisfy the cyclicity property thus leading to a cocycle given by its coboundary:

$$c^Q_R(A, B) := \delta tr^Q(A, B) = tr^Q[A, B] \quad \forall A, B \in PDO(M, E) \tag{2.1}$$

which we call the Q-weighted Radul cocycle. This terminology is justified on the grounds of the following proposition which relates c_R^Q to a more familiar expression of the classical Radul cocycle.

Proposition 1

(i) For any $A \in PDO(M,E)$ the operator $[logQ, A]$ lies in $PDO(M,E)$.

(ii) For any $A, B \in PDO(M,E)$ we have:

$$c_R^Q(A,B) = -\frac{1}{ordQ} res([logQ, A]B) = \lim_{z\to 0} TR(Q^{-z}[A,B]). \tag{2.2}$$

Remark

This relation expresses the algebraic obstruction preventing a weighted trace from being tracial in terms of a Wodzicki residue, which is a trace of purely infinite dimensional type.

Proof:

(i) We first check that $[logQ, A]$ lies in $PDO(M,E)$ if the order $ordA$ of A is strictly positive.

$$\begin{aligned}[A, logQ] &= (A \cdot logQ - logQ \cdot A)\\ &= A\left(logQ - \frac{ordQ}{ordA} log(|A|+1)\right) + \left(\frac{ordQ}{ordA} log(|A|+1) - logQ\right) A.\end{aligned}$$

Since the difference of two logarithms of PDOs of same order is a P.D.O, this proves that if the order of A is strictly positive then $[logQ, A]$ is a P.D.O. Now if A has negative order, let $k = \frac{|ordA|+1}{ordQ}$. Then setting $P = Q^k$ we have that P is of order equal to $|ordA| + 1$ and we have:

$$\begin{aligned}P(AlogQ - (logQ)A) &= Q^k A\ logQ - logQ\ Q^k A\\ &= [Q^k A, logQ],\end{aligned}$$

which by the previous results applied to $Q^k A$ which has strictly positive order, shows it is a P.D.O.

(ii) For Rez large enough we have:

$$\begin{aligned}TR(Q^{-z}[A,B]) &= TR([Q^{-z}, A]B) + TR([A, Q^{-z}B])\\ &= TR([Q^{-z}, A]B) + TR([AQ^{\frac{-z}{2}}, Q^{\frac{-z}{2}}B])\\ &= TR([Q^{-z}, A]B).\end{aligned}$$

In the same way we have:

$$\begin{aligned}TR(Q^{-z}[A,B]) &= TR([A,B]Q^{-z})\\ &= TR(A[B,Q^{-z}]) + TR(AQ^{-z}B - BAQ^{-z})\\ &= TR(A[B,Q^{-z}]) + TR([AQ^{\frac{-z}{2}}, Q^{\frac{-z}{2}}B])\\ &= TR(A[B,Q^{-z}]).\end{aligned}$$

Appplying the fundamental property and $(*)$ to the family $z^{-1}[Q^{-z}, A]B$ of order $\alpha(z) = -z \cdot ordQ + ordA + ordB$ we find:

$$\begin{aligned}\lim_{z\to 0} TR([Q^{-z}, A]B) &= Res_{z=0} TR(z^{-1}[Q^{-z}, A]B)\\ &= \frac{1}{ordQ} res\left(\frac{d}{dz}_{/z=0}([Q^{-z}, A]B)\right)\\ &= -\frac{1}{ordQ} res([logQ, A]B).\end{aligned}$$

In a similar way we have:

$$\lim_{z\to 0} TR(A[B, Q^{-z}]) = -\frac{1}{ordQ} res(A[B, logQ]).$$

This proposition shows that the Q-weighted Radul cocycle generalizes the usual Radul cocycle obtained for $\sigma(Q) = |\xi|$ [CFNW] (see formula (24)),[M2], [R] (see formula (41)), [KK] (see formula (1)). In this particular case, the Radul cocycle was also considered in [KV2] in relation to multiplicative anomalies for determinants of elliptic operators (see also [D]) but it was later considered for more general operators Q in [MN] and proved (see Lemma 13) to be a coboundary in the Hochschild cohomology of pseudo-differential operators. As a consequence of the above proposition, the weighted Radul cocycle vanishes on the algebra of odd-class P.D.Os with integer order whenever the underlying manifold is odd dimensional.

To each $\mathbb{C}$-valued cocycle c_R^Q corresponds a central extension (see e.g [M1], [Ki], [Rog]) which we shall denote by $PDO(M,E)^Q := \{(A,\lambda), A \in PDO(M,E)\}$ with Lie bracket:

$$[(A,\lambda),(B,\mu)]^Q := \Big([A,B], c_R^Q(A,B)\Big).$$

In other words we have the exact sequence of Lie algebras:

$$0 \to \mathbb{C} \to PDO(M,E)^Q \to PDO(M,E) \to 0$$

Two such extensions $PDO(M,E)^{Q_1}$ and $PDO(M,E)^{Q_2}$ are equivalent since for $Q_1, Q_2 \in Ell_{>0}^{+*}(M,V)$ the cocycles $c_R^{Q_1}$ and $c_R^{Q_2}$ are cohomologous. Indeed their difference

$$c_R^{Q_1} - c_R^{Q_2} = res\left((logQ_1 - logQ_2)[\cdot,\cdot]\right)$$

is the coboundary of the 1 cochain $A \to res\left((logQ_1 - logQ_2) \quad \cdot\right)$.

Since the difference of two logarithms is a zero order P.D.O, this latter 1 cochain is a particular example of a family of cochains parametrized by the algebra $PDO^0(M,E)$ of operators in $PDO(M,E)$ of order *zero*:

$$res^P := res(P \quad \cdot) \tag{2.3}$$

where P is a zero order P.D.O. The coboundary $(A,B) \to res(P[A,B])$ does not vanish in general. The case when $P = \epsilon(D)$ is the sign of a self-adjoint operator D gives rise to Mickelsson's res' linear functional.

3. A weighted Schwinger functional

In this section D denotes a self-adjoint elliptic operator acting on smooth sections of a vector bundle E with strictly positive order. An example of such a bundle is given by the spinor bundle over an odd dimensional spin compact manifold M and the operator D by the Dirac operator on M.

Adopting notations which are frequently used in the context of geometric quantization, let $\epsilon(D) := D + P_D(|D| + P_D)^{-1}$ denote the sign of D which defines a classical P.D.O. of order 0. The operator $Q := |D|$ lies in $Ell^+_{ord>0}(M,E)$.

- *A Schwinger functional*

We define the Schwinger functional:

$$\begin{aligned} &PDO(M,E) \to \mathbb{C} \\ &\qquad (A,B) \to c_S^D(A,B) := \frac{1}{2} tr^{|D|}(\epsilon(D)[\epsilon(D),A][\epsilon(D),B]). \end{aligned} \tag{3.1}$$

The terminology "Schwinger functional" is motivated by the fact (as we shall see later) that on some subalgebras of $PDO(M,E)$ it coincides with the usual Schwinger cocycle.

A straightforward computation yields:

$$c_S^D(A,B) = tr^{|D|}([A,\epsilon(D)]B) = tr^{|D|}(A[\epsilon(D),B]).$$

Let us introduce the linear functional

$$\begin{aligned} PDO(M,E) &\to \mathbb{C} \\ A &\to tr_\epsilon^D(A) := tr^{|D|}(\epsilon(D)A) \end{aligned}$$

which we call the *signed weighted trace* of A. As before the terminology "trace" is not appropriate here since this functional does not in general have vanishing coboundary.

Proposition 3

Let $\mathcal{S}$ be a subalgebra of $PDO(M,E)$. The following conditions are equivalent (all the cocycles are Lie algebra cocycles):

1) c_S^D is a 2-cocycle on $\mathcal{S}$,

2) c_S^D is an antisymmetric bilinear form on $\mathcal{S}$,

3) The bilinear map

$$(A,B) \to c_{TR}^D(A,B) := tr^{|D|}[\epsilon(D)A,B]$$

defines a 2-cocycle on $\mathcal{S}$,

4) c_{TR}^D is an antisymmetric bilinear form on S,

5) For any $A,B \in \mathcal{S}$ we have:

$$res^{\epsilon(D)}[A,[log|D|,B]] = 0, \tag{3.2}$$

Provided one of these conditions is fufilled, the following relation holds:

$$c_{TR}^D - c_S^D = \delta tr_\epsilon^D$$

so that the cocycles c_{TR}^D and c_S^D are cohomologous.

Remarks

1) $c_{TR}^D(A,B)$ coincides with $-2c'(A,B)$ defined in (2.11) of [M2].

2) Part 5) of this proposition shows that the obstruction to the cocycle property of the various functionals involved arises as a residue $res^{\epsilon(D)}[A,[log|D|,B]]$. Here again the obstruction is therefore purely infinite dimensional.

The following definitions and the lemma below will be used in the proof.

We shall set:

$$\tilde{c}_{TR}(A,B) := tr^{|D|}[A\epsilon(D),B]$$

and

$$\bar{c}_{TR} := \frac{c_{TR}^D + \tilde{c}_{TR}^D}{2}.$$

Lemma 2

$$c_{TR}^D(B,A) = -\tilde{c}_{TR}^D(A,B) \quad \forall A,B \in PDO(M,E)$$

so that $\bar{c}_{TR}^D$ is antisymmetric. Moreover

$$\begin{aligned} \bar{c}_{TR}^D(A,B) &= \delta tr_\epsilon^D(A,B) + \frac{1}{2}\Big(tr^{|D|}([\epsilon(D),B]A) - Tr^{|D|}([A,\epsilon(D)]B)\Big) \\ &= \frac{1}{2}\delta tr_\epsilon^D(A,B) + \frac{1}{2}tr^{|D|}\left(A\epsilon(D)B - B\epsilon(D)A\right) \end{aligned}$$

where δtr_ϵ^D denotes the coboundary of the signed weighted trace in the Lie algebra cohomology.

Proof: Let us first see how c_{TR}^D transforms when exchanging A and B.

$$\begin{aligned}
c_{TR}^D(A,B) &= tr^{|D|}[\epsilon(D)B, A] \\
&= tr^{|D|}(\epsilon(D)BA - A\epsilon(D)B) \\
&= -tr^{|D|}(A\epsilon(D)B - BA\epsilon(D)) \\
&= -tr^{|D|}[A\epsilon(D), B] \\
&= -\tilde{c}_{TR}^D(A,B)
\end{aligned}$$

From this it follows that $\bar{c}_{TR}^D$ is antisymmetric. The second statement then easily follows from Lemma 2 and the definition of $\bar{c}_{TR}^D$:

$$\begin{aligned}
\bar{c}_{TR}^D(A,B) &= \frac{1}{2}\left(c_{TR}^D(A,B) - c_{TR}^D(B,A)\right) \\
&= \frac{1}{2}\left(c_\epsilon(A,B) - c_\epsilon(B,A)\right) + \frac{1}{2}\left(tr^{|D|}([\epsilon(D),B]A - tr^{|D|}([\epsilon(D),A]B)\right) \\
&= c_\epsilon^D(A,B) - \frac{1}{2}\left(tr^{|D|}([\epsilon(D),B]A + tr^{|D|}([A,\epsilon(D)]B)\right) \\
&= \frac{1}{2}c_\epsilon^D(A,B) + \frac{1}{2}\left(tr^{|D|}(A\epsilon(D)B - B\epsilon(D)A)\right).
\end{aligned}$$

Proof of proposition 3:

2) $\Leftrightarrow$ 4) : A straightforward computation yields:

$$\begin{aligned}
c_S^D(A,B) &= -c_S^D(B,A) \\
&\Leftrightarrow tr^{|D|}(A[\epsilon(D),B]) = -tr^{|D|}(B[\epsilon(D),A]) \\
&\Leftrightarrow tr^{|D|}[A\epsilon(D),B] = tr^{|D|}[A,B\epsilon(D)] \\
&\Leftrightarrow \tilde{c}_{TR}^D(A,B) = c_{TR}(A,B) \\
&\Leftrightarrow -c_{TR}^D(B,A) = c_{TR}(A,B)
\end{aligned}$$

where we have used the result of Lemma 2.

3) $\Leftrightarrow$ 4) : The implication from left to right is clear since a 2-cocycle is antisymmetric. Let us prove the other implication 4) $\Rightarrow$ 3) which amounts to showing that when c_{TR} is antisymmetric it defines a cocycle. Since c_{TR}^D is antisymmetric, we have $c_{TR}^D = \tilde{c}_{TR}^D = \bar{c}_{TR}^D$. By Lemma 2, it is sufficient to show that ω defined by $\omega(A,B) \equiv Tr^{|D|}(A\epsilon(D)B - B\epsilon(D)A)$ has vanishing coboundary $\delta\omega$. A direct computation yields:

$$\delta\omega(A,B,C) = c_{TR}^D(A,[B\epsilon(D),C\epsilon(D)]) + c_{TR}^D(B,[C\epsilon(D),A\epsilon(D)]) + c_{TR}^D(C,[A\epsilon(D),B\epsilon(D)])$$

from which follows that:

$$\begin{aligned}
\delta\omega(A\epsilon(D),B\epsilon(D),C\epsilon(D)) &= c_{TR}^D(A\epsilon(D),[B,C]) + c_{TR}^D(B\epsilon(D),[C,A]) + c_{TR}^D(C\epsilon(D),[A,B]) \\
&= \tilde{c}_{TR}^D(A\epsilon(D),[B,C]) + \tilde{c}_{TR}^D(B\epsilon(D),[C,A]) + \tilde{c}_{TR}^D(C\epsilon(D),[A,B]) \\
&= tr^{|D|}[A,[B,C]] + Tr^{|D|}[B,[C,A]] + Tr^{|D|}[C,[A,B]] \\
&= 0
\end{aligned}$$

Since any P.D.O A can be written $A = A_1\epsilon(D)$ where $A_1 \equiv A\epsilon(D)$ is a P.D.O, the result follows.

4) $\Leftrightarrow$ 5) : Since this computation is similar to the one used in the proof of Lemma 2 we shall skip some intermediate steps here. Let d denote the order of D and let us set $l := log|D|$.

$$\begin{aligned}
-d \cdot c_{TR}^D(B,A) &= -d \cdot c_R^{|D|}(\epsilon(D)B, A) \\
&= res(\epsilon(D)[l,B]A) \\
&= res(\epsilon(D)A[l,B]) - res(\epsilon(D)[A,[l,B]]) \\
&= -d \cdot \tilde{c}_{TR}^D(B,A) - res(\epsilon(D)[A,[l,B]]) \\
&= d \cdot c_{TR}^D(A,B) - res(\epsilon(D)[A,[l,B]])
\end{aligned}$$

hence

$$c_{TR}^D(B,A) + c_{TR}^D(A,B) = -\frac{1}{d} \cdot res(\sigma(\epsilon(D)[A,[l,B]]))$$

and c_{TR} is antisymmetric if and only if condition (3.2) is satisfied.

so that c_{TR}^D is antisymmetric if and only if $res(\epsilon(D)[A,[l,B]]) = 0$ for any A, B in the algebra under consideration.

1) $\Leftrightarrow$ 2) : Only the implication from right to left is non trivial. To prove it, we use Lemma 2 once again by which we have:

$$\bar{c}_{TR}^D(A,B) = \frac{1}{2}\delta tr_\epsilon^D(A,B) + \frac{1}{2}Tr^{|D|}\left(A\epsilon(D)B - B\epsilon(D)A\right).$$

On the other hand, since c_S^D is antisymmetric, we have:

$$c_S^D(A,B) = \frac{1}{4}tr^{|D|}(\epsilon(D)\,[[\epsilon(D),A],[\epsilon(D),B]]\,.$$

Then a direct computation yields:

$$c_S^D(A,B) = -\frac{1}{2}(\delta tr_\epsilon^D)(A,B) + \frac{1}{2}tr^{|D|}\left(A\epsilon(D)B - B\epsilon(D)A\right).$$

Hence

$$\bar{c}_{TR}^D - c_S^D = \delta tr_\epsilon^D$$

From this identity follows that, provided c_S^D and c_{TR}^D are cocycles, then they are cohomologous.

- *The algebra* $PDO(M,E)_{res}^D$

Let us introduce a subalgebra $PDO(M,E)_{res}^D$of $PDO(M,E)$ the definition of which is close to the algebra g_{res} (they coincide up to the fact that we require the operators to be P.D.Os) which plays a substancial part in geometric quantization techniques (see e.g. [PS]):

$$\begin{aligned}
PDO(M,E)_{res}^D &= \{A \in PDO(M,E), \quad ordA \leq 0 \quad ord([A,\epsilon(D)]) < -\frac{dimM}{2}\} \\
&= g_{res}^D \cap PDO(M,E)
\end{aligned} \tag{3.3}$$

where as before $dimM$ is the dimension of the underlying manifold and where $ordA$ denotes the order of the operator A. Here $g_{res}^D := \{A \in \mathcal{B}(L^2(M,E)), \quad [A,\epsilon(D)] \quad \text{is Hilbert Schmidt}\}$ where $L^2(M,E)$ is the closure of the space $C^\infty(M,E)$ of smooth sections of E for the L^2 inner product induced by the hermitian structure on E and the Riemannian volume measure on M as above, $\mathcal{B}(L^2(M,E))$ denoting the algebra of bounded operators on this Hilbert space.

An immediate consequence of Proposition 3 is the following Corollary:

Corollary 1

Let $\mathcal{S}$ be a subalgebra of $PDO(M,E)$. If $\mathcal{S}$ is stable under the map $A \to [log|D|, A]$ i.e if

$$A \in \mathcal{S} \Rightarrow [log|D|, A] \in \mathcal{S} \tag{3.4}$$

and if moreover

$$res(\epsilon(D)[A,B]) = 0 \quad \forall A, B \in \mathcal{S} \tag{3.5}$$

hold on the subalgebra S, then so does relation (R) and c_{TR}^D and c_S^D define cohomologous cocycles on $\mathcal{S}$.

Corollary 2

On the algebra $PDO(M,E)_{res}^D$ the Schwinger functional coincides with the usual Schwinger cocycle and c_{TR}^D with the usual twisted Radul cocycle and we have:

$$c_S^D \equiv c_{TR}^D.$$

Proof: We need to check that $PDO(M,E)_{res}^D$ fulfills assumptions (3.4) and (3.5) of Corollary 1. As before we set $l := log|D|$. Since $[\epsilon(D), [l, A]] = [l, [\epsilon(D), A]]$, the order of $[\epsilon(D), [l, A]]$ is the same as that of $[\epsilon(D), A]$ so that if A lies in $PDO(M,E)_{res}^D$, so does $[l, A]$. Hence assumption (3.4) is satisfied on $PDO(M,E)_{res}^D$. An easy computation yields:

$$res(\epsilon(D)[A,B]) = -\frac{1}{2} res(\epsilon(D)[[\epsilon(D), A], [\epsilon(D), B]])$$

which vanishes on $PDO(M,E)_{res}^D$ since it is the trace of an operator of degree strictly smaller than $-dimM$, the operator $\epsilon(D)$ being of order 0 and the operators $[\epsilon(D), A]$, $[\epsilon(D), B]$ being both of order strictly smaller than $-\frac{dimM}{2}$. Hence assumption (ii) is satisfied on $PDO(M,E)_{res}^D$. Corollary 1 implies that both c_S^D and c_{TR}^D are cocycles on $PDO(M,E)_{res}^D$ and that they are cohomologous.

On $PDO(M,E)_{res}^D$ the cocycle c_S^D reads:

$$c_S^D(A,B) = \frac{1}{2} tr(\epsilon(D)[\epsilon(D), A][\epsilon(D), B])$$

where now tr is an ordinary trace and it therefore coincides with the usual Schwinger cocycle see e.g (according to the author, the definition might change by a constant factor) [CFNW], [PS], [M2], [S].

The expression of c_{TR}^D in terms of a residue obtained in section 1 (Proposition 1) yields

$$c_{TR}^D(A,B) = -\frac{1}{ordD} res([log|D|, \epsilon(D)A]B)$$

thus relating c_{TR}^D with Mickelssons [M2] twisted Radul cocycle (they coincide up to a factor -2).

4. Cocycles and group representations

Given $D \in Ell_{ord>0}^{s.a}(M,E)$, we have a natural polarization of the space $H \equiv L^2(M,E)$ given by:

$$H = H_+(D) \oplus H_-(D)$$

where $H_+(D) = \pi_+(D)(H)$, $H_-(D) = \pi_-(D)(H)$, $\pi_+(D) = \frac{1+\epsilon(D)}{2}$, $\pi_-(D) = \frac{-\epsilon(D)+Id}{2}$. Notice that with this choice for $\epsilon(D)$, $\pi_+(D)$ is 1 on $KerD$ and $\pi_-(D)$ vanishes on $KerD$. Of course we could have chosen the other convention namely $\pi_-(D)$ to be -1 on $KerD$ and $\pi_+(D)$ to vanish on $KerD$. In this polarization, let us write an operator $A \in PDO(M,E)$ as a matrix:

$$A \equiv \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix}.$$

Since $\epsilon(D) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we have

$$[\epsilon(D), A] = 2 \begin{bmatrix} 0 & A_{+-} \\ -A_{-+} & 0 \end{bmatrix}.$$

The operator $\epsilon(D)$ being a P.D.O, the operator $\begin{bmatrix} 0 & A_{+-} \\ -A_{-+} & 0 \end{bmatrix}$ is also P.D.O.

For $D \in Ell^{s.a}_{ord>0}(M, E)$ we introduce the following bilinear functional:

$$\begin{aligned} \lambda^D : PDO(M, E) &\to \mathbb{C} \\ (A, B) &\to tr^{|D|}([A_{++}, B_{++}] - [A, B]_{++}). \end{aligned} \tag{4.1}$$

On $PDO(M, E)^D_{res}$ it coincides with the Lie algebra cocycle corresponding to a central extension of the group:

$$G^D_{res} \equiv \{A \in GL(H), A_{+-} \quad \text{and} \quad A_{-+} \quad \text{are Hilbert-Schmidt}\} \tag{4.2}$$

as described in [PS] (6.6.5).

The Schwinger functional c^D_S not being antisymmetric in general (since according to proposition 3 it then becomes a cocycle), it is natural to introduce a *mean Schwinger functional* in a similar way to what we we did for the twisted Radul cocycle:

$$\bar{c}^D_S(A, B) := \frac{c^D_S(A, B) - c^D_S(B, A)}{2} = \frac{1}{4} tr^{|D|} \left[[\epsilon(D), A], [\epsilon(D), B]\right].$$

Lemma 3

Let $D \in Ell^{s.a}_{ord>0}(M, E)$, let $A \in PDO(M, E)$, $B \in PDO(M, E)$. Then

$$\begin{aligned} 2\lambda^D(A, B) &= 2tr^{|D|}(B_{+-}A_{-+} - A_{+-}B_{-+}) \\ &= \bar{c}^D_S(A, B) - c^{|D|}_R(A_{+-}, B_{-+}) - c^{|D|}_R(A_{-+}, B_{+-}) \end{aligned}$$

Furthermore λ^D is a cocycle on $PDO(M, E)^D_{res}$ and the following relation holds on $PDO(M, E)^D_{res}$:

$$\begin{aligned} \frac{1}{2} c^D_S(A, B) &= \lambda^D(A, B) \\ &= TR(A_{-+}B_{+-} - A_{+-}B_{-+}) \\ &= \frac{1}{4} TR\left(\epsilon(D)[\epsilon(D), A][\epsilon(D), B]\right) \end{aligned}$$

confirming formula (6.6.6) of [PS].

Remark : Although the individual components $A_{++}, A_{+-}, A_{-+}, A_{--}$ of the two by two matrix of A in the polar decomposition might not be P.D.Os, $c^{|D|}_R(A_{+-}, B_{-+}) = tr^{|D|}[A_{+-}, B_{-+}]$ and $c^{|D|}_R(A_{-+}, B_{+-}) = tr^{|D|}[A_{-+}, B_{+-}]$ are well defined since these are weighted traces of operator brackets which are P.D.Os as projections of operator brackets of P.D.Os of the type $[[\epsilon(D), A], [\epsilon(D), B]]$.

Proof: Let us start with the general case. The first equality follows from a direct computation and the definition of the weighted Radul cocycle. The second equality follows from the fact that

$$\epsilon(D)[[\epsilon(D), A], [\epsilon(D), B]] = -4 \begin{bmatrix} A_{+-}B_{-+} - B_{+-}A_{-+} & 0 \\ 0 & B_{-+}A_{+-} - A_{-+}B_{+-} \end{bmatrix}$$

since this yields:

$$\bar{c}^D_S(A, B) = 2tr^{|D|}(A_{+-}B_{-+} - B_{+-}A_{-+}) + c^{|D|}_R(B_{+-}, A_{-+}) + c^{|D|}_R(B_{-+}, A_{+-}).$$

When considering the case of $PDO(M,E)^D_{res}$, the Radul cocycles $c_R^{|D|}(B_{+-},A_{-+})$ and $c_R^{|D|}(B_{-+},A_{+-})$ vanish since $B_{+-},A_{-+},B_{-+},A_{+-}$ are Hilbert-Schmidt and the weighted traces $tr^{|D|}$ become ordinary traces TR. This completes the proof.

Noting that $[\epsilon(D),A]=-[\epsilon(D),A]^*$ is equivalent to $A_{+-}=-A^*_{-+}$ for an operator $C:H_+(D)\to H_-(D)$ we shall set:

$$j(C)\equiv\begin{bmatrix}0 & -C^*\\ C & 0\end{bmatrix}.$$

Lemma 4

Let $D\in Ell^{s.a}_{ord>0}(M,E)$, let $A:H_+(D)\to H_-(D)$, $B:H_+(D)\to H_-(D)$ such that $j(A)\in PDO(M,E)$, and $j(B)\in PDO(M,E)$. Then

$$\begin{aligned}\bar c_S^D(j(A),j(B)) &= \delta(tr_\epsilon^D)(j(A),j(B))\\ &= 2tr^{|D|}(B^*A-A^*B)+c_R^{|D|}(A,B^*)+c_R^{|D|}(A^*,B)\end{aligned}$$

Proof: Since $[\epsilon(D),j(A)]=-2j(A)$ we have:

$$\begin{aligned}\bar c_S^D(j(A),j(B)) &= \frac{1}{4}tr_\epsilon^D([[\epsilon(D),j(A)],[\epsilon(D),j(B)]]\\ &= tr_\epsilon^D([j(A),j(B)])\\ &= c_\epsilon^D(j(A),j(B))\\ &= tr_\epsilon^D\begin{bmatrix}-A^*B+B^*A & 0\\ 0 & -AB^*+BA^*\end{bmatrix}\\ &= 2tr^D(B^*A-A^*B)+c_R^{|D|}(A,B^*)-c_R^{|D|}(B,A^*)\\ &= 2tr^D(B^*A-A^*B)+c_R^{|D|}(A,B^*)+c_R^{|D|}(A^*,B)\end{aligned}$$

On the grounds of this proposition we set for two operators A,B such that $A^*B\in PDO(M,E)$:

$$\omega^D(A,B)\equiv -itr^{|D|}(A^*B-B^*A).$$

ω^D relates to the mean Schwinger functional $\bar c_S$ as follows:

Corollary 3

Let $A:H_+(D)\to H_-(D)$ and $B:H_+(D)\to H_-(D)$ be operators such that $j(A)$ and $j(B)$ lie in $PDO(M,E)$. Then

$$\bar c_S^D(j(A),j(B))=2i\omega^D(A,B)+c_R^{|D|}(A,B^*)+c_R^{|D|}(A^*,B).$$

Whenever A and B are Hilbert-Schmidt, then

$$\bar c_S^D(j(A),j(B))=2i\omega^D(A,B)=2tr(A^*B-B^*A)$$

where tr is now an ordinary trace.

5. Geometry on weighted bundles

• *Weighted vector bundles*

We shall say that a Hilbert space H lies in the class $\mathcal{CH}$ whenever there is a compact boundaryless compact smooth Riemannian manifold M, a finite rank smooth vector bundle E based on M and $s>\frac{dimM}{2}$ such that $H = H^s(M, E)$. Typically, letting G be a Lie group and $Lie(G)$ be its Lie algebra, the Lie algebra $H^s(M, Lie(G))$ of the Hilbert current group $H^s(M, G)$ lies in $\mathcal{CH}$.

Let $\mathcal{CE}$ be the class of Hilbert vector bundles $\mathcal{E} \to X$ based on a (possibly infinite dimensional) manifold X with fibres modelled on a separable Hilbert space H in the class $\mathcal{CH}$ defined previously and with *transition maps in* $PDO(M, E)$ when H is the space of sections in some Sobolev class of a vector bundle E based on M.

$\mathcal{CX}$ denotes the class of infinite dimensional manifolds X with tangent bundle TX in $\mathcal{CE}$. Since the transition maps are bounded, they correspond to operators of order 0 and since they are invertible, they are in fact elliptic operators of order zero so that they lie in $Ell(M, E)$.

To illustrate this setting let us give examples of manifolds, resp. vector bundles in the class $\mathcal{CX}$, resp. $\mathcal{CE}$.

• *Examples*

i) Finite rank vector bundles lie in the class $\mathcal{CE}$. To see this we take a manifold $M = \{*\}$ reduced to a point $*$, the bundle E to be trivial of the type $\{*\} \times \mathbb{R}^d$ (or $\{*\} \times \mathbb{C}^d$ if the bundle is complex). The transition functions belong to $Ell(\{*\}, E) = Gl_d(\mathbb{R})$ (or $Gl_d(\mathbb{C})$ if the bundle is complex).

ii) If G denotes a Lie group and $s > \frac{dim M}{2}$, then the current group $H^s(M, G)$ is a Hilbert Lie group and can be equipped with a left invariant atlas $\phi_\gamma(u)(x) := exp_{\gamma(x)}(u(x)) \quad \forall x \in M, \quad \forall \gamma \in H^s(M, G)$ where $exp_{\gamma(x)}$ is the exponential coordinate chart at point $\gamma(x)$ induced by a left invariant Riemannian metric on G. The transition functions are given by multiplication operators which indeed are P.D.Os.

iii) More generally, let N be a Riemannian manifold, then the space $H^s(M, N)$ is a Hilbert manifold with tangent space at a point γ given by $H^s(M, \gamma^* TN)$. This manifold, which is modelled on $H^s(M, \mathbb{R}^n)$ where n is the dimension of N, can be equipped with an atlas induced by the exponential map exp^N on N in a similar way to the above description, a local chart being of the type $\phi_\gamma(u)(x) = exp^N_{\gamma(x)}(u(x))$. The transition functions are locally given by multiplication operators and hence define PDO's.

• *Bundles of operators*

Let $\mathcal{E}$ be a C^∞ vector bundle in the class $\mathcal{CE}$ based on a manifold B and let it be modelled on a separable Hilbert space H. For any $b \in B$ we shall denote by $PDO_b(\mathcal{E})$ the class of operators A_b acting densely on the fibre $\mathcal{E}_b$ above b such that for any local trivialisation $\phi : \mathcal{E}|_{U_b} \to U_b \times H$ around b the operator $\phi^\sharp A(b) := \phi(b) A_b \phi(b)^{-1}$ (where $\phi(b) : \mathcal{E}_b \to H$ (obtained after localizing it using smooth cut-off functions) is the isomorphism induced by the trivialization) lies in $PDO(M, E)$. This definition involves local charts but it is in fact independent of the choice of local chart. Indeed, for another local chart (U, ψ) we have $\psi^\sharp A(b) = \psi(b) \circ \phi(b)^{-1} \phi^\sharp A_b \phi(b) \circ \psi(b)^{-1}$, and since the transition functions $\psi(b) \circ \phi(b)^{-1}$ are given by classical P.D.Os, the condition $\phi^\sharp A \in PDO(M, E)$ is independent of the choice of ϕ. In a similar way, the notion of order defined by the order of the P.D.O in the local chart, does not depend on the choice of local chart.

We shall also denote by $Ell_b(\mathcal{E})$ the class of operators A_b acting densely on $\mathcal{E}_b$ such that for any local trivialisation $\phi : \mathcal{E}/U_p \to U_p \times H$ around b the operator $\phi : \mathcal{E}/U_b \to U_b \times H$ around b the operator, $(\phi^\sharp A)(b)$ lies in $Ell(M, E)$. Here again, the definition involves a choice of local chart but is in fact independent of that choice.Indeed, the principal symbol being multiplicative and using the characterization of ellipticity in terms of invertibility of the principal symbol, one easily checks that the condition $\phi^\sharp A \in Ell(M, E)$ is independent of the choice of ϕ.

This gives rise to two bundles $PDO(\mathcal{E}) := \bigcup_{b \in B} PDO_b(\mathcal{E})$ and $Ell(\mathcal{E}) := \bigcup_{b \in B} Ell_b(\mathcal{E})$ with fibre at point b given respectively by $PDO_b(\mathcal{E})$ and $Ell_b(\mathcal{E})$.

Notice that when $\mathcal{E}$ is a bundle of finite rank, we have $PDO(\mathcal{E}) = Hom(\mathcal{E})$ and $Ell(\mathcal{E}) = GL(\mathcal{E})$ since the underlying manifold M reduces to a point and the vector bundle to a vector space, namely the model space of $\mathcal{E}$.

• *Weighted traces*

A weight on a smooth finite rank vector bundle in the class $\mathcal{CE}$ is a smooth section of $Ell(\mathcal{E})$ of operators with constant order which is locally a positive self-adjoint (elliptic) operator. If $\mathcal{E}$ has finite rank, it is simply given by a section of $GL(\mathcal{E})$ which locally is a positive self-adjoint operator and hence corresponds to the choice of a Riemannian metric.

If the base manifold is either locally compact and a countable union of compacts or if it is a paracompact Hilbert manifold modelled on a separable Hilbert space, then it has a smooth partition of unity [L] and such a global section can be built up patching up local sections of $Ell(\mathcal{E})$, i.e maps from an open neighborhood in the base manifold B to $Ell(M, E)$ (this construction is similar to the one that gives the existence of a Riemannian metric under the same conditions).

Let $\mathcal{E} \in \mathcal{CE}$ be modelled on H. We define the weighted pseudo-trace of a field of operators:

$$\begin{aligned} \Gamma(PDO(\mathcal{E})) &\to \Gamma(X, \mathbb{R})(\text{or} \quad \Gamma(X, \mathbb{C})) \\ A &\to tr^Q(A) \end{aligned} \tag{5.1}$$

locally; in a local chart (U, ϕ) it is defined by the $\phi^\sharp$-weighted pseudo-trace $tr^{\phi^\sharp Q}(\phi^\sharp A)$ of $\phi^\sharp A$. Although the definition involves a choice of local chart, because of the covariance property it is in fact independent of this choice. Indeed if ϕ and ψ are two local charts around x and if we set $C := \psi \circ \phi^{-1}$ in formula (1.7), we have:

$$tr^{\psi^\sharp Q}(\psi^\sharp A) = tr^{C\phi^\sharp Q C^{-1}}(C\phi^\sharp A C^{-1}) = tr^{\phi^\sharp Q}(\phi^\sharp A).$$

Remark

Looking back at definition (1.3), strictly speaking, in order to make sure the field P_Q of orthogonal projections is smooth, we should assume that $KerQ$ has constant dimension. This is an artificial difficulty which comes from the specific construction of the zeta function renormalization since it involves taking complex powers of an invertible operator $(Q + P_Q)^{-z}$. However, bearing in mind that the heat-kernel renormalization only involves the exponential e^{-tQ} and the fact that $Lim_{z\to 0} tr(A(Q + P_Q)^{-z}) = Lim_{\epsilon\to 0} tr(Ae^{-\epsilon Q})$ (which can be shown via the Mellin transform), we see that the jumps in the dimension of the kernel of Q do not affect the renormalized trace. From now on, we shall assume that Q is invertible, otherwise one just replaces Q by $Q' := Q + P_Q$ where P_Q is the orthogonal projection onto the kernel of Q.

- *Variations of weighted traces*

Proposition 4

Let $(\mathcal{E}, Q)$ be a weighted vector bundle equipped with a connection ∇ and let α, β be two $PDO(\mathcal{E})$ valued forms on the base manifold of $\mathcal{E}$.

1)

$$tr^Q([\alpha, \beta]) = -\frac{1}{ordQ} res([logQ, \alpha]\beta) \tag{5.2}$$

2) Provided both $[\nabla, logQ]$ and $[\nabla, \alpha]$ are sections of $PDO(\mathcal{E})$, we have

$$[\nabla, tr^Q](\alpha) := dtr^Q(\alpha) - = tr^Q([\nabla, \alpha]) = -\frac{1}{ordQ} res(\alpha \cdot [\nabla, logQ]). \tag{5.3}$$

Here $ordQ$ denotes the (constant) order of the field of elliptic operators Q .

Proof

(5.2) follows from Proposition 1 (compare with (2.2))

As in the proof of Proposition 1, to prove (5.3) we use the fundamental property of the canonical trace TR. Let us first consider a smooth (for the natural topology on classical P.D.Os induced by the topology of uniform convergence in all derivatives on the classical symbols) one parameter family of operators $A_t \in PDO(M, E)$ with constant order a and a smooth one parameter family $Q_t \in Ell^+_{ord>0}(M, E)$ (for the natural Fréchet topology on $PDO(M,E)$) with constant order q, t varying in $]0,1[$. Applying formula (1.3) to $A_z := \frac{A_0(Q_t^{-z}-Q_0^{-z})}{z}$ (in which case $\alpha(z) = a - qz$) and then going to the limit when $t \to 0$ yields:

$$\frac{d}{dt}_{/t=0} tr^{Q_t}(A_0) = -\frac{1}{q} res(A_0 \frac{d}{dt}_{/t=0} log Q_t).$$

let us now also consider a 1-parameter smooth family (A_t) of P.D.Os of constant order, then:

$$\frac{d}{dt}_{/t=0} tr^{Q_t}(A_t) = tr^{Q_0}(\frac{d}{dt}_{/t=0} A_t) - \frac{1}{ordQ_0} res(A_0 \frac{d}{dt}_{/t=0} log Q_t).$$

Similarly, given any local trivialization around a point b_0 in the base manifold B of $\mathcal{E}$, we have:

$$dtr^Q(\alpha) = tr^Q(d\alpha) - \frac{1}{ordQ} res(\alpha \cdot dlog Q). \tag{5.4}$$

Let us write $\nabla = d + \theta$ in this local trivialization. Since by assumption $[\nabla, \alpha]$ is a section of $PDO(\mathcal{E})$, we have that locally $d\alpha + [\theta, \alpha] \in PDO(M,E)$ and hence, since $d\alpha \in PDO(M,E)$ as the differential of a P.D.O, we conclude that $[\theta, \alpha]$ also lies in $PDO(M,E)$. Applying the first part of the lemma yields:

$$tr^Q([\theta, \alpha]) = -\frac{1}{ordQ} res([log Q, \theta]\alpha). \tag{5.5}$$

Combining (5.4) and (5.5) we find:

$$\begin{aligned}
dtr^Q(\alpha) &= tr^Q(d\alpha) - \frac{1}{ordQ} res(\alpha \cdot dlog Q) \\
&= tr^Q([\nabla, \alpha]) - tr^Q([\theta, \alpha]) - \frac{1}{ordQ} res(\alpha \cdot dlog Q) \\
&= tr^Q([\nabla, \alpha]) + \frac{1}{ordQ} res([log Q, \theta]\alpha) - \frac{1}{ordQ} res(\alpha \cdot dlog Q) \\
&= tr^Q([\nabla, \alpha]) - \frac{1}{ordQ} res(\alpha[\nabla, log Q]).
\end{aligned}$$

$\bullet$

- *Ricci curvature on weighted manifolds*

Let $X \in \mathcal{CX}$ be equipped with a Levi-Civita connection ∇^X such that for any tangent vector fields U, V the map:

$$R^X(U,V) : W \to \Omega^s(W,U)V \quad \forall W \in \Gamma(TX)$$

is a section of $PDO(TX)$ where Ω^X denotes the curvature of ∇^X. Then, given a weight Q on X we can define the *Q-weighted Ricci curvature*:

$$r_1^{X,Q} := tr^Q(R^X). \tag{5.6}$$

Since tr^Q coincides with the ordinary trace on trace-class operators if X is finite dimensional, it then coincides with the ordinary Ricci curvature.

- *Weighted first Chern form on Kähler manifolds*

Let us now assume $X \in \mathcal{CX}$ is Kähler and let ∇^X be the a Kähler connection, let Q be a weight on X. In a similar way, provided that for any holomorphic tangent vector field U and for any antiholomorphic tangent vector field $\bar{V}$ the map $\Omega^X(U, \bar{V})$ is a section of $PDO(TX)$, we define the *Q-weighted first Chern form*:

$$r_1^{X,Q}(U, \bar{V}) := tr^Q(\Omega^X(U, \bar{V})) \tag{5.7}$$

where Ω^X denotes the curvature of ∇^X. If X is finite dimensional, it coincides with the ordinary first Chern form.

Unlike the first Chern form on a finite dimensional manifold, it is not closed in general. This follows from the following proposition:

Corollary 4

Let (X, ∇^X) be a Kähler manifold eqXipped with a weight Q and let $ordQ$ be the (constant) order of Q. Then, provided $[\nabla^X, logQ]$ is a $PDO(TX)$ valued one form and provided the curvature Ω^X is a $PDO(TX)$ valued two form, we have

$$dr_1^Q = -\frac{1}{ordQ} res([\nabla^X, logQ]\Omega^X). \tag{5.8}$$

Proof: Applying Proposition 4 to $\alpha := \Omega^X$, we find:

$$\begin{aligned} dtr^Q(\Omega^X) &= tr^Q([\nabla^X, \Omega^X]) - \frac{1}{ordQ} res(\Omega^X[\nabla^X, logQ]) \\ &= -\frac{1}{ordQ} res(\Omega^X[\nabla^X, logQ]) \end{aligned}$$

where we have used the Bianchi identity $[\nabla^X, \Omega^X] = 0$. •

• *Remark*

From Kuiper's results [Ku] on the contractibility of the unitary group of a separable Hilbert space, we know that the orthonormal frame bundle $O(\mathcal{E})$ of a hermitian vector bundle with fibres modelled on a Hilbert space, is topologically trivial. This is the case for the class of manifolds we are investigating, and from this topological triviality one might expect that whenever the first Chern class is closed, teh corresponding characteristic class should vanish. This is not the case as we shall see shortly in the case of current groups, but in our setting the non triviality seems to come from the fact that we restict ourselves to P.D.Os. Indeed, the holonomy bundle is a reduction of the frame bundle and for a manifold X in the class $\mathcal{CX}$ (recall that we only allowed transition maps which were P.D.Os) modelled on the Hilbert space $H := H^s(M, E)$ for some $s > \frac{dimM}{2}$, the structure group of the frame bundle is

$$GL^{Ell}(H) := GL(H) \cap PDO(M, E) = GL(H) \cap Ell(M, E) = Ell_0^*(M, E) \tag{5.9}$$

where $Ell_0^*(M, E)$ denotes the group of invertible zero order elliptic operators acting on sections of E. But it is a well-known fact that the pathwise connected component of identity of $Ell_0^*(M, E)$ has a non trivial fundamental group isomorphic to $K_0(S^*M)$ where S^*M is the unit sphere in the cotangent bundle of M (see Appendix B).

6. Weighted Lie groups

In this section, we apply the constructions and results of the previous section to the case of weighted Lie groups, thus preparing for the next section where we will specialize to current groups. Here $\mathcal{G}$ is an infinite dimensional Hilbert Lie group in the class $\mathcal{CX}$ with Lie algebra $Lie(\mathcal{G}) = H^s(M, E)$ (for some $s > \frac{dimM}{2}$ and some hermitian vector bundle E based on some manifold M).

• *Left invariant weights*

A natural weight on $\mathcal{G}$ is given by a *left invariant field* of operators

$$Q(\gamma) := L_{\gamma_*} Q_0 L_{\gamma_*}^{-1} \quad \forall \gamma \in \mathcal{G} \tag{6.1}$$

where $Q_0 \in Ell^+_{ord>0}(M, E)$ is any weight on the Lie algebra $Lie(\mathcal{G})$ and where L_γ denotes left multiplication.

As a consequence of Proposition 4 we have:

Corollary 5

Let ∇ be a left invariant connection on $\mathcal{G}$ induced by a left invariant one form $\theta_0 \in Lie(\mathcal{G}) \otimes Hom(Lie(\mathcal{G}))$. Then, given a left invariant p-form ω on $\mathcal{G}$ which we identify with $\omega_0 \in \Lambda^p(Lie\mathcal{G}) \otimes PDO(M, E)$, we have:

$$[\nabla, tr^Q](\omega) := dtr^Q(\omega) - tr^Q([\nabla, \omega]) = -tr^{Q_0}([\theta_0, \omega_0]) = \frac{1}{ordQ_0} res([logQ_0, \theta_0]\omega_0)$$

where the bracket is an operator bracket so that $[\theta_0, \omega_0](X, X_1, \cdots, X_p) = [\theta_0(X), \omega_0(X_1, \cdots, X_p)]$.

Proof: Recall that for a left invariant p-form α on $\mathcal{G}$ we have:

$$d\alpha(X_1, \cdots, X_{p+1}) = \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{p+1})$$

whereby the $X_i's$ are left invariant vector fields. Applying this to $\alpha = tr^Q(\omega)$ and $\alpha = \omega$ and then taking the Q-weighted trace yields $dtr^Q(\omega) = tr^Q(d\omega)$. Hence

$$\begin{aligned}
[\nabla, tr^Q](\omega) &:= dtr^Q(\omega) - tr^Q([\nabla, \omega]) \\
&= -tr^Q([\theta, \omega]) \\
&= -tr^{Q_0}([\theta_0, \omega_0]) \\
&= \frac{1}{ordQ_0} res([logQ_0, \theta_0]\omega_0).
\end{aligned}$$

• *Diagonal weights*

Let us now consider the particular case when E is a trivial bundle $E = M \times V$ where V is a finite dimensional hermitian vector space. We can restrict ourselves to (left invariant) *diagonal weights*, meaning by this that:

$$Q_0 := \bar{Q}_0 \otimes 1_V \tag{6.2}$$

where 1_V is the identity opertor on V so that (6.1) reads:

$$Q(\gamma) = L_{\gamma_*} \left(\bar{Q}_0 \otimes 1_V\right) L_{\gamma_*}^{-1}. \tag{6.3}$$

In that case, for a left invariant p-form induced by ω_0 as in the above corollary we have:

$$\begin{aligned}
tr^Q(\omega) &= tr^{Q_0}(\omega_0) \\
&= Lim_{z\to 0} tr(\omega_0(Q_0 + P_{Q_0})^{-z}) \\
&= Lim_{z\to 0} tr(\omega_0(\bar{Q}_0 + P_{\bar{Q}_0})^{-z} \otimes 1) \\
&= Lim_{z\to 0} tr((\bar{Q}_0 + P_{\bar{Q}_0})^{-z} tr_V(\omega_0)) \\
&= tr^{\bar{Q}_0}(tr_V(\omega_0))
\end{aligned} \tag{6.4}$$

where tr_V denotes the finite dimensional trace on the Lie algebra V. Hence, provided $tr_V(\omega_0)$ is itself a trace-class operator (i.e a P.D.O of order $< -\frac{dimM}{2}$) then

$$tr^Q(\omega) = tr(tr_V(\omega_0)) \tag{6.5}$$

(where tr is an ordinary trace) is independent of $\bar{Q}_0$. This is the type of situation we shall be working with in these notes. Since there is no weight dependence in that case, we can choose any operator $\bar{Q}_0$ and in particular the Laplace operator Δ on the Riemannian manifold M.

7. The case of current groups

We now specialize to the current group $H^s(M,G)$ ($s > \frac{dim M}{2}$) of H^s maps from the compact Riemannian manifold M to a semi-simple Lie group G of compact type (which ensures that the Killing form is non degenerate and that the adjoint representation ad on the Lie algebra is antisymmetric for this bilinear form). $H^s(M,G)$ is a Hilbert Lie group with Lie algebra $H^s(M, Lie(G))$ where $Lie(G)$ denotes the Lie algebra of G so that the current group $H^s(M,G)$ is a manifold in the class $\mathcal{CX}$, the underlying finite rank vector bundle E being trivial since $E = M \times Lie(G)$ and we equip $H^s(M,G)$ with the *left invariant and diagonal weight* Q (see (6.2) with $V = Lie(G)$):

$$Q(\gamma) := L_\gamma(\bar{Q} \otimes 1_{Lie(G)})L_\gamma^{-1} = L_\gamma \left(\Delta \otimes 1_{Lie(G)}\right) L_\gamma^{-1} \quad \forall \gamma \in H^s(M,G). \tag{7.1}$$

• *A left invariant metric*

The operator $Q_0 := \Delta \otimes 1_{Lie(G)}$ is an elliptic operator of order 2 acting densely on $H^s(M, Lie(G))$. It is positive for the scalar product

$$\langle \cdot, \cdot \rangle_0 := \int_M dvol(x)(\cdot, \cdot)_{Lie(G)} \tag{7.2}$$

where $(\cdot,\cdot)_{Lie(G)}$ is a scalar product on $Lie(G)$ given by minus the Killing form. $H^s(M,G)$ is equipped with a left invariant metric defined in terms of the following scalar product on $H^s(M, Lie(G))$:

$$\langle \cdot, \cdot \rangle_0^s := \langle (Q_0 + P_{Q_0})^{\frac{s}{2}} \cdot, (Q_0 + P_{Q_0})^{\frac{s}{2}} \cdot \rangle_0. \tag{7.3}$$

• *The Levi-Civita connection*

The corresponding the Levi-Civita connection is a left invariant connection $\nabla^s = d + \theta^s$ where θ^s is a left invariant $Hom(H^s(M, Lie(G))$ valued one form on $H^s(M,G)$ given by a map $\theta_0^s : Lie(\mathcal{G}) \to Hom(Lie\mathcal{G})$ (see [F1] formula (1.9) up to a sign mistake):

$$\theta_0^s(U) = \frac{1}{2}\Big(ad_U + (Q_0 + P_{Q_0})^{-s} ad_U (Q_0 + P_{Q_0})^s - (Q_0 + P_{Q_0})^{-s} ad_{(Q_0 + P_{Q_0})^s U}\Big) \tag{7.4}$$

where U is an element of the Lie algebra $H^s(M, Lie(G))$.

• *The curvature*

The curvature Ω^s is a left invariant two form given by an element $\Omega_0^s \in \Lambda^2(H^s(M, Lie(G)) \otimes Hom(H^s(M, Lie(G))$:

$$\Omega_0^s(U,V) = \theta_0^s \wedge \theta_0^s(U,V) = [\theta^s(U), \theta^s(V)] - \theta_0^s([U,V]) \quad U, V \in H^s(M, Lie(G)). \tag{7.5}$$

Warning

We shall henceforth identify left invariant forms with forms on the Lie algebra $H^s(M, Lie(G))$ so that for left invariant fields X, Y induced respectively by elements $X_0, Y_0 \in H^s(M, Lie(G))$ we have $\theta^s(X) = \theta_0^s(X_0)$ and $\Omega^s(X,Y) = \Omega_0^s(X_0, Y_0)$.

D.Freed shows ([F1] prop. 1.14) that for smooth $X, Y \in H^s(M, Lie(G))$ the map

$$R^s(X,Y) : Z \to \Omega^s(Z,X)Y$$

is a pseudo-differential operator (with Sobolev coefficients) of order $max(-1,-2s)$. Freed in fact proves the result for smooth X and Y but his proof easily extends to the case when $X, Y \in H^s(M, Lie(G))$ since it involves a counting of the order of the operator $R^s(X,Y)$ which is independent of the degree of regularity of its coefficients. Notice that for $s \in \frac{1}{2}\mathbb{N} - \{0\}$, we have $max(-1,-2s) = -1$.

We further quote from [F1]:

Lemma 5

For $X, Y \in H^s(M, Lie(G))$ the operator $tr_{Lie(G)} R^s(X,Y)$ is a classical pseudo-differential operator of order $-2q$ where $q = min(1, 2s)$ and where as before $tr_{Lie(G)}$ denotes the finite dimensional trace on $Lie(G)$.

Proof: We refer the reader to [F1] Proposition 1.18 where this result was proven in the case when $X, Y \in C^\infty(M, Lie(G))$. •

• *The Ricci curvature*

Following the general procedure described in section 5, let us define a weighted Ricci curvature on the Lie group $H^s(M, G)$ and compare it to that of [F1].

For any diagonal weight Q, as a consequence of (6.5) we have:

$$Ricc^{s,Q}(X,Y) := tr^Q(R^s(X,Y)) = tr^{\bar{Q}_0}\left(tr_{Lie(G)}(R^s(X,Y))\right). \tag{7.6}$$

Hence provided $min(1, 2s) > \frac{dimM}{2}$, using Lemma 5 we can write:

$$Ricc^{s,Q}(X,Y) = tr(tr_{Lie(G)}(R^s(X,Y))) \tag{7.7}$$

wher tr is now an ordinary trace, since in that case, $tr_{Lie(G)}(R^s(X,Y))$ is trace-class. This holds in particular for any *loop group* $H^s(S^1, G)$ with $s > \frac{1}{2}$.

It is easy to check that in that case and for any left invariant weight Q the Wodzicki residue $res(R^s(X,Y))$ of the operator $R^s(X,Y)$ vanishes and that the Q-weighted Ricci curvature is given by an ordinary limit:

$$Ricc^{s,Q}(X,Y) = lim_{z\to 0}(TR(R^s(X,Y)Q^{-z})) \tag{7.8}$$

however this limit depends on the choice of Q.

• *The case when s is an integer and M is odd dimensional*

There are some cases for which the Q-dependence vanishes; we shall need a preliminary lemma to single them out.

Lemma 6

For two left invariant vector fields X, Y on $H^s(M, G)$ and $s \in \mathbb{N}$ the operator $R^s(X,Y)$ lies in the odd-class.

Proof

From the expression of the Levi-Civita curvature (7.5) one sees that the operator $R^s(X,Y)$ is built up from compositions and linear combinations of operators in the odd class provided Q_0 is in the odd class, since ad_X and adX both lie in the odd class since they are built up from multiplication operators. Since for any integer s, the operator Q_0^s does lie in the odd class and since the product of odd-class operators lies in the odd class, the result follows.

Proposition 5

When the underlying manifold is odd dimensional and s is an integer, the (left invariant) weighted Ricci curvature on $H^s(M, G)$ equipped with the left invariant connection (7.4) is independent of the choice of the weight Q among operators in the odd class and we have for any two left invariant vector fields X and Y on $H^s(M, G)$:

$$Ricc^{s,Q}(X,Y) = TR_{odd}(R^s(X,Y)). \tag{7.9}$$

Proof :

As we saw in section 1 (formula (1.6)), the dependence on the choice of Q is measured in terms of a residue since for two weights Q_1 and Q_2 we have:

$$tr^{Q_1}(R^s(X,Y)) - tr^{Q_2}(R^s(X,Y)) = -q^{-1}\left(res(R^s(X,Y)(logQ_1 - logQ_2))\right)$$

where q is the common order of Q_1 and Q_2. If both Q_1 and Q_2 lie in the odd-class so does $logQ_1 - logQ_2$ (see Proof of Proposition 4.1 in [KV1]), and since $R^s(X,Y)$ also lies in the odd class, the operator $R^s(X,Y)(logQ_1 - logQ_2))$ lies in the odd-class. Thus in odd dimensions its Wodzicki residue vanishes. •

There is a priori no reason for a similar property to hold for $s = \frac{1}{2}$, which is the case we shall be focusing on in the following. The Q-weighted Ricci curvature will in general depend on the choice of Q.

8. The case of the based loop group $H_e^{\frac{1}{2}}(S^1, G)$

We now specialize to the case when $M = S^1$ and $s = \frac{1}{2}$, applying the results of the previous section to a based loop group. Notice that for $s = \frac{1}{2}$, H^s maps are not even continuous in that case, and in practice, one works with $H^{\frac{1}{2}+\epsilon}, \epsilon > 0$ in order to have continuous objects.

The space $H_e^s(S^1, G) \subset H^s(S^1, G)$ of G valued loops with value e_G at a given point, where e_G is the identity element in G. It is a Hilbert manifold with Lie algebra the based loop algebra $H_0^s(S^1, Lie(G))$ of maps in the loop algebra which all coincide at point 0. $H_e^s(S^1, G)$ is equipped with an almost complex structure, for which the metric is in fact Kähler when $s = \frac{1}{2}$ [F1]. We equip it as before with a left invariant diagonal weight Q, the operator Q being the same one used to define the connection θ^s.

- *An almost complex structure on* $H_e^{\frac{1}{2}}(S^1, G)$

We introduce a Dirac operator $\bar{D}_0 = z\frac{d}{dz} = -i\frac{d}{dt}$ (with $z = e^{it}$) acting on $C^\infty(S^1, \mathbb{C})$ and we set $D_0 := \bar{D}_0 \otimes 1_{Lie(G)} = z\frac{d}{dz} \otimes 1_{Lie(G)}$, D denoting the left invariant field of operators generated by D_0. We shall choose $Q_0 := D_0^2 = \Delta \otimes 1_{Lie(G)}$ where Δ is the Laplacian on functions, to define the left invariant diagonal weight Q. D_0 is injective when restricted to based loops and the sign $\epsilon(D_0) := D_0|D_0|^{-1}$ of the Dirac operator is a pseudo-differential operator of order 0 which yields a conjugation on $H_0^{\frac{1}{2}}(S^1, Lie(G))$ since $\epsilon(D_0)^2 = 1$. We have the splitting:

$$H := H_0^{\frac{1}{2}}(S^1, Lie(G)) = H_+ \oplus H_- \tag{8.1}$$

where

$$H_+ := Ker(\epsilon(D_0) - 1) = \pi_+ \left(H_0^{\frac{1}{2}}(S^1, Lie(G)) \right)$$

and

$$H_- := Ker(\epsilon(D_0) + 1) = \pi_- \left(H_0^{\frac{1}{2}}(S^1, Lie(G)) \right).$$

Here π_+ and π_+ are the orthogonal projections w.r. to the scalar product $\langle \cdot, \cdot \rangle_0^{\frac{1}{2}}$ defined in (7.3) $\pi_+ := \frac{\epsilon(D_0)+1}{2}, \pi_- := \frac{-\epsilon(D_0)+1}{2}$.

The set $\{u_n(t) := e^{int}, n \in \mathbb{Z}\}$ yields a C.O.N.S of eigenvectors of D_0 corresponding to the set of eigenvalues $\{\lambda_n := n, n \in \mathbb{Z}\}$. Then the set $\{u_n^+(t) := e^{int}, n \in \mathbb{N}\}$ spans H_+ and the set $\{u_n^-(t) := e^{-int}, n \in \mathbb{N}\}$ spans H_-. The map J_0 defined by $J_0 u_n^+ := -iu_n^+$, $J_0 u_n^- := iu_n^-$ or equivalently by $J_0 := i\epsilon(D_0)$ obeys the relation $J_0^2 = -1$ and yields a natural almost complex structure on H for which the $(1,0)$ part $H^{1,0} := Ker(J + i)$ coincides with H_+, the $(0,1)$ part $H^{1,0} := Ker(J - i) = H_+$ with H_-. By left invariance of the metric on $H_0^{\frac{1}{2}}(S^1, Lie(G))$, this gives rise to a left invariant almost complex structure $J(\gamma) := L_\gamma J_0 L_\gamma^{-1}$ on the Lie group $H_e^{\frac{1}{2}}(S^1, G)$.

- *The Kähler connection on* $H_e^{\frac{1}{2}}(S^1, G)$

$H_e^{\frac{1}{2}}(S^1, G)$ is equipped with a left invariant symplectic (hence closed see [Pr]) form:

$$\omega(X,Y) := \int_0^1 \langle X_0', Y_0 \rangle dt$$

where X, Y are two left invariant vector fields generated by X_0, Y_0 and where the "prime" denotes derivation with respect to t. One can check that the invariant bilinear form given by $B(X,Y) := \Omega(X, JY)$ yields back the $H^{\frac{1}{2}}$ Sobolev metric given by (7.3). The associated left invariant hermitian form reads:

$$\langle\langle \cdot, \cdot \rangle\rangle_0^{\frac{1}{2}} := \langle \cdot, \epsilon(D_0) \quad \cdot \rangle_0^{\frac{1}{2}} = \langle D_0 \cdot, \cdot \rangle_{L^2}. \tag{8.2}$$

where we have used the fact that $\epsilon(D_0)_0 Q_0^{\frac{1}{2}} = \epsilon(D_0)|D_0| = D_0$ and where $\langle \cdot, \cdot, \rangle_{L^2}$ denotes the L^2 hermitian product. Hence S is a Kähler form on the manifold $H^{\frac{1}{2}}(S^1, G)$ equipped with the $H^{\frac{1}{2}}$ left invariant Riemannian metric given by (7.3) and the left invariant complex structure J defined above. As a consequence, the Levi-Civita connection $\nabla^{\frac{1}{2}}$ on $H_e^{\frac{1}{2}}(S^1, G)$ is Kählerian, meaning by this that $[\nabla^{\frac{1}{2}}, J] = 0$.

Since $\nabla^{\frac{1}{2}}$ commutes with J, $\theta^{\frac{1}{2}}$ defined in (7.4) (with $s = \frac{1}{2}$) stablizes each of the spaces H_+ and H_-. Its restriction to H_+:

$$\phi_0 := \left[\theta_0^{\frac{1}{2}}\right]_{++} \tag{8.3}$$

defines a left invariant the (complex) Kähler connection ϕ. In the next lemma we use the splitting $H = H_+ \oplus H_-$ to write any operator $A \in Hom(H_0^{\frac{1}{2}}(S^1, Lie(G))$ as a matrix:

$$A = \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix}.$$

Following [F1] we introduce the Töplitz operators $T_X := (ad_X)_{++}, \quad X \in H$.

Lemma 7

For $U \in H_+$,

$$\phi(U) = D^{-1} T_U D.$$

For $\bar{V} \in H_-$,

$$\phi(\bar{V}) = T_{\bar{V}}.$$

Proof:

1) For $U \in H_+$, we have $|D_0|U = D_0 U$. Hence, setting $U = \alpha \otimes a$, $V := \beta \otimes b$, $\alpha, \beta \in H^{\frac{1}{2}}(S^1, \mathbb{C})$, $a, b \in Lie(G)$, we have:

$$\begin{aligned} |D_0|^{-1} ad_{|D_0|U} V &= |D_0|^{-1}(D_0\alpha)\beta \otimes ad_a b \\ &= |D_0|^{-1} D_0(\alpha\beta) \otimes ad_a b - |D_0|^{-1}\alpha D_0 \beta \otimes ad_a b \\ &= |D_0|^{-1} D_0 ad_U - |D_0|^{-1} ad_U D_0 \\ &= \epsilon(D_0) ad_U - |D_0|^{-1} ad_U D_0. \end{aligned}$$

Inserting this into (7.4) yields:

$$\theta^{\frac{1}{2}}(U) = \frac{1}{2}(ad_U + |D_0|^{-1} ad_U |D_0| + |D_0|^{-1} ad_U D_0 - \epsilon(D_0) ad_U)$$

which, when restricted to H_+ and using (8.3) leads to:

$$\begin{aligned} \phi_0(U) &= \pi_- ad_{U|H_+} + |D_0|^{-1} ad_U D_{0|H_+} \\ &= D_0^{-1} T_U D_0 \end{aligned}$$

since ad_U stablizes H_+ for $U \in H_+$.

2) For $\bar{V} \in H_-$, $|D_0|\bar{V} = -D_0\bar{V}$ and hence in a similar manner as above we have:

$$|D_0|^{-1}ad_{|D_0|\bar{V}} = |D_0|^{-1}ad_{\bar{V}}D_0 - \epsilon(D_0)ad_{\bar{V}}$$

which yields in turn:

$$\theta^{\frac{1}{2}}(\bar{V}) = \frac{1}{2}(ad_{\bar{V}} + \epsilon(D)ad_{\bar{V}} - |D|^{-1}ad_{\bar{V}}D + |D|^{-1}ad_{\bar{V}}|D|).$$

When restricted to H_+ and using (8.3), this reads:

$$\phi(\bar{V}) = \pi_+ ad_{\bar{V}|_{H_+}} = T_{\bar{V}}.$$

9. The first Chern form on $H_e^{\frac{1}{2}}(S^1, G)$ and the cohomology of $PDO(S^1, Lie(G))$

- *The first Chern form on* $H_e^{\frac{1}{2}}(S^1, G)$

The curvature (which also stablizes H_+) is given as in (7.5) by:

$$\Omega(X, \bar{Y}) := [\phi(X), \phi(\bar{Y})] - \phi([X, \bar{Y}]) = \left(\Omega^{\frac{1}{2}}(X, \bar{Y})\right)^{1,0}. \tag{9.1}$$

From [F1] (see Remarks above Theorem 2.20), we also know that $\Omega(X, \bar{Y})$ is a classical pseudo-differential operator of order -1 and hence that it is not trace-class. However $tr_{Lie(G)}\Omega(X, \bar{Y})$ being of order -2 (see [F1] Remarks above Theorem 2.20), is trace-class. We now define the weighted first Chern form on $H_e^{\frac{1}{2}}(S^1, G)$ according to (5.7) where the weighted Kähler manifold here is $(H_e^{\frac{1}{2}}(S^1, G), Q)$:

$$r_1^Q := tr^Q(\Omega)$$

Here the complex curvature is given by (9.1) and the trace is taken w.r. to the hermitian metric. If the weight is diagonal then

$$r_1^Q = tr(tr_{LieG}\Omega).$$

This follows from (6.5) and shows our definition of weighted first Chern form coincides (with a good choice of the weight) with the "two step trace" used by Freed [F1] to make sense of the trace of the curvature (see Theorem 2.20 in [F1]).

In fact only the $(1, 0)$ part $Q^{1,0} := Q_{++}$ of Q comes into play in the expression $tr^Q(\Omega)$ since Ω is the $(1, 0)$ part of $\Omega^{\frac{1}{2}}$. Before we give an expression of the first Chern form, we need wome preliminary results.

Lemma 8

For the diagonal weight Q on $H^{\frac{1}{2}}(S^1, G)$ chosen as in (7.1), we have

$$tr^Q(\phi(Z)) = 0 \quad \forall Z \in H^{\frac{1}{2}}(S^1, Lie(G)). \tag{9.2}$$

Proof: It is sufficient to establish the formula for Z of the type $\gamma \otimes c \in H^s(M, \mathbb{R}) \otimes Lie(G)$. If $Z = \gamma \otimes c \in H_-$, then $\phi(Z) = (ad_Z)_{++}$ and hence (compare with (6.5))

$$\begin{aligned}
tr^Q(\phi(Z)) &= tr^Q((ad_Z)_{++}) \\
&= Lim_{z\to 0} tr\left((ad_Z)_{++}\bar{Q}_0^{-z} \otimes 1_{Lie(G)})\right) \\
&= Lim_{z\to 0} tr\left((M_\gamma \bar{Q}_0^{-z} \otimes ad_c)_{++}\right) \\
&= tr^{\bar{Q}_0}(M_\gamma)_{++} tr_{LieG}(ad_c) \\
&= 0
\end{aligned}$$

since ad_c is antisymmetric. M_γ denotes the multiplication operator by γ and Lim the renormalized limit. Here we have used the fact that Q_0 stablizes H_+.

If $Z \in H_+$, then $\phi(Z) = D^{-1} T_Z D = D^{-1} (ad_Z)_{++} D$ and

$$tr^Q(D^{-1}(ad_Z)_{++}D) = tr^Q((ad_Z)_{++}) = 0.$$

Proposition 6

Let $H_e^{\frac{1}{2}}(S^1, G)$ be equipped with a left invariant diagonal weight Q. The Q-weighted first Chern form is the pull-back by ϕ of the Q_0 Radul cocycle $c_R^{Q_0}$ on $PDO(S^1, Lie(G))$ and coïncides with Freed's conditioned first Chern form [F1]. For any $X, Y \in H_0^{\frac{1}{2}}(S^1, Lie(G))$ induced by $X_0 \in H_+$ and $\bar{Y}_0 \in H_-$ we have:

$$r_1^Q(X, \bar{Y}) = c_R^{Q_0}(\phi(X), \phi(\bar{Y})) \tag{9.3}$$

where tr denotes the ordinary trace and λ^D was defined in formula (4.1).

Proof: The first term in the expression (9.1) of the curvature is a bracket of P.D.Os so that taking the weighted pseudo-trace yields $tr^{Q_0}[\phi(X), \phi(\bar{Y})] = c_R^{Q_0}(\phi(X), \phi(\bar{Y}))$. The trace of the second term in (9.1) vanishes by lemma 7 and we find

$$r_1^Q(X, \bar{Y}) = tr^Q(\Omega(X, \bar{Y})) = tr^{Q_0}([\phi(X), \phi(\bar{Y})] = c_R^{Q_0}(\phi(X), \phi(\bar{Y}))$$

as announced.

•

Before we go on to the next proposition, let us recall various ways of expressing the Killing form on the Lie algebra $Lie(G)$:

$$\begin{aligned}
\langle a, b \rangle_{Lie(G)} &= tr_{Lie(G)}(ad_b^* ad_a) \\
&= \sum_{i=1}^{d} \langle [a, c_i], [b, c_i] \rangle_{Lie(G)} \\
&= -tr_{Lie(G)}[a, [b, \cdot]] \\
&= -tr_{Lie(G)}[b, [a, \cdot]].
\end{aligned}$$

where $c_i, i = 1, \cdots, d$ varies in an O.N.B of $Lie(G)$.

Proposition 7

The map:

$$\begin{aligned}
ad : C^\infty(S^1, Lie(G)) &\to C^\infty(S^1, Hom(Lie(G))) \\
X := \sum_{n \in \mathbb{Z}} a_n z^n &\to ad_X = \sum_{n \in \mathbb{Z}} ad_{a_n} z^n
\end{aligned}$$

extends to a map:

$$ad : H^{\frac{1}{2}}(S^1, Lie(G)) \to g_{res}^D$$

with the notations of (3.3) replacing M by S^1 and E by the trivial bundle $S^1 \times Lie(G)$. In particular, $[ad_X]_{+-}$ and $[ad_X]_{-+}$ are Hilbert-Schmidt operators on $L^2(S^1, Lie(G))$.

Proof: (this proof is close to that of proposition 6.3.1 in chapter 6 of [PS])

For $b \in L$, $q \in \mathbb{Z}$ and $X = \sum_{n \in \mathbb{Z}} a_n z^n \in C^\infty(S^1, Lie(G))$ we have:

$$\begin{aligned}
ad_X(bz^q) &= \sum_{n \in \mathbb{Z}} ad_{a_n}(b) z^{n+q} \\
&= \sum_{p \in \mathbb{Z}} ad_{a_{p-q}}(b) z^p
\end{aligned}$$

so that ad_X is represented by an $\mathbb{Z}\times\mathbb{Z}$ matrix with entries $ad_{X\,p,q} = ad_{a_{p-q}} \in Hom(Lie(G))$. We compute the Hilbert-Schmidt norm of $(ad_{X_{+-}}$ denoted by $\| \cdot \|_{HS}$. We shall use the fact that for $a \in Lie(G)$ we have $\|ad_a\|_{HS}^2 = \|a\|^2$ where this last norm is the one corresponding to the Killing form.

$$
\begin{aligned}
\|ad_{X+-}\|_{HS}^2 &= \sum_{q<0} \sum_{i=1}^{dimLie(G)} \langle (ad_X)_{+-} b_i z^q, (ad_X)_{+-} b_i z^q \rangle \\
&= \sum_{p\geq 0, q<0} \sum_{i=1}^{dimLie(G)} \langle ad_{a_{p-q}}(b_i) z^p, ad_{a_{p-q}}(b_i) z^p \rangle \\
&= \sum_{p\geq 0, q<0} \sum_{i=1}^{dimLie(G)} \langle ad_{a_{p-q}}(b_i), ad_{a_{p-q}}(b_i) \rangle_{Lie(G)} \\
&= \sum_{p\geq 0, q<0} \|ad_{X\,p,q}\|_{HS}^2 \\
&= \sum_{k>0} k \|ad_{a_k}\|_{HS}^2 \\
&= \sum_{k>0} k \|a_k\|^2
\end{aligned}
$$

where $(b_i)_{i=1,\cdots,dimLie(G))}$ is an orthonormal basis of $Lie(G)$. In the same way, we have:

$$
\|ad_{X-+}\|_{HS}^2 = \sum_{k<0} \|a_k\|^2 |k|.
$$

From the above expressions of the Hilbert-Schmidt norms of ad_{X+-} and ad_{X-+}, it follows that these are Hilbert-Schmidt whenever $\sum_{k\in\mathbb{Z}} |k| \|a_k\|^2$ is finite. But this is the condition for X to lie in $H^{\frac{1}{2}}(S^1, L)$.

- *The first Chern form in terms of the Kähler form*

Recall that the exterior derivative of a form can be expressed in terms of a Wodzicki residue (see (5.8)). Unlike in the finite dimensional setting, here the weighted first Chern form r_1^Q might therefore not be closed. The following proposition shows that it relates to two closed forms, the Kähler form ω on the one hand (this fact had already been shown by Freed) and the pull-back by the adjoint representation of the cocycle λ^D on the other hand.

Lemma 9

The pull-back $ad^*\lambda^D$ of the cochain λ^D defined by (4.1) is closed on $H^{\frac{1}{2}}(S^1, Lie(G))$.

Proof: By Proposition 7, we know that ad takes its values in g_{res}^D. On the other hand λ_D is indeed a cocycle on g_{res}^D. Combining these two facts yields, for three left invariant vector fields X, Y, Z:

$$
\begin{aligned}
\delta(ad^*\lambda^D)(X,Y,Z) &= ad^*\lambda^D([X,Y],Z) - ad^*\lambda^D([Y,Z],X) + ad^*\lambda^D([Z,X],Y) \\
&= \delta\lambda^D(ad_X, ad_Y, ad_Z) \\
&= 0
\end{aligned}
$$

where $\delta\lambda^D$ denotes the coboundary of λ^D.

Proposition 8

The first Chern form is given by:

$$
r_1^Q(U,\bar{V}) = ad^*\lambda^D(U,\bar{V}) = -i\omega(U,\bar{V})
$$

for $U, V \in H^{\frac{1}{2}}(S^1, Lie(G))$ and it is a closed form. Here ω is the symplectic form on $H_e^{\frac{1}{2}}(S^1, G)$ defined in section 8.

Proof: From the results of Proposition 7 combined with the description of λ^D given in Lemma 3, we know that $\lambda^D(ad_U, ad_{\bar{Y}}) = tr(ad_{\bar{V}+-}ad_{U-+} - ad_{U+-}ad_{\bar{V}-+})$ is an ordinary trace and hence it reads:

$$\lambda^D(ad_U, ad_{\bar{V}}) = \sum_{q\in\mathbb{N}, i=1,\cdots,d} \left(\langle ad_{\bar{V}+-}ad_{U-+}z^q c_i, z^q c_i\rangle - \langle ad_{U+-}ad_{\bar{V}-+}z^q c_i, z^q c_i\rangle\right)$$

where $c_i, i = 1, \cdots, d$ varies in an O.N.B of $Lie(G)$. Since $L^2(S^1, Lie(G))$ is spanned by elements of the type $z^n a, n \in \mathbb{Z}, a \in Lie(G)$, it is sufficient to compute this last sum for $U = z^n a$, $\bar{V} = z^{-p}b$, $a, b \in Lie(G)$, $n \in \mathbb{Z}, p \in \mathbb{Z}$. For $q \in \mathbb{N}, c \in Lie(G)$ we have:

$$(ad_U)_{-+}(z^q c) = \begin{matrix} 0 & \text{if} \quad n+q>0 \\ [a,c]z^{n+q} & \text{if} \quad n+q \leq 0 \end{matrix}$$

and

$$(ad_{\bar{V}})_{+-}(ad_U)_{-+}(z^q c) = \begin{matrix} 0 & \text{if} \quad n+q>0 \quad \text{or} \quad n-p+q<0 \\ [b,[a,c]]z^{n-p+q} & \text{if} \quad n+q\leq 0 \quad \text{and} \quad n-p+q \geq 0. \end{matrix}$$

In the same way

$$(ad_{\bar{V}})_{-+}(z^q c) = \begin{matrix} 0 & \text{if} \quad -p+q>0 \\ [b,c]z^{n+q} & \text{if } p+q \leq 0 \end{matrix}$$

and

$$(ad_U)_{+-}(ad_V)_{-+}(z^q c) = \begin{matrix} 0 & \text{if} \quad -p+q>0 \quad \text{or} \quad n-p+q<0 \\ [a,[b,c]]z^{n-p+q} & \text{if} \quad -p+q\leq 0 \quad \text{and} \quad n-p+q \geq 0. \end{matrix}$$

Finally this yields (since the only non vanishing terms correspond to $n - p = 0$):

$$\begin{aligned} \lambda^D(ad_U, ad_{\bar{V}}) &= -\sum_{0<q\leq n, i=1,\cdots,d} \langle [a,[b,c_i]]z^q, c_i z^q\rangle \\ &= -ntr_{Lie(G)}(c \to [a,[b,c]]) \\ &= n\langle a, b\rangle_{Lie(G)} \\ &= -i\omega(z^n a, z^{-n}b) \\ &= -i\omega(U, \bar{V}) \end{aligned}$$

which yields the result.

A similar computation [F1] (see Theorem 2.20) shows that the weighted first Chern form can be expressed in terms of the symplectic form ω and yields the result.

Appendix A. Classical elliptic pseudo-differential operator

This Appendix gives a brief presentation of the basic tools in our framework, namely classical pseudo-differential operators and particularly elliptic ones, their logarithms and their complex powers. Classical references are [G], [Se], [Sh].

• *The symbol set*

Let U be an open subset of $\mathbb{R}^d$. Given $\alpha \in \mathbb{C}$, let us denote by $S^\alpha(U)$ the set of complex valued smooth function

$$U \times \mathbb{R}^d \to \mathbb{R}$$
$$(U, \xi) \to \sigma(U, \xi)$$

satisfiying the following property. Given any compact subset K of U and any two multiindices $\gamma = (\gamma_1, \cdots, \gamma_d) \in \mathbb{N}^d$, $\delta = (\delta_1, \cdots, \delta_d) \in \mathbb{N}^d$, there exists a constant $C_{\alpha,\beta}(K)$ such that

$$|D_x^\gamma D_\xi^\delta \sigma(x,\xi)| \leq C_{\gamma,\delta}(K)(1+|\xi|)^{Re\alpha-|\delta|} \quad \forall x \in K, \forall \xi \in \mathbb{R}^d.$$

An element of $S^\alpha(x)$ is called a *symbol of order* α. Let $S^{\leq m}(x)$ denote the set of symbols of order $\leq m$. The *principal part* of the symbol $\sigma \in S^\alpha(x)$ (*or principal symbol*) is defined as follows:

$$\sigma_\alpha(x,\xi) = \lim_{t \to +\infty} \frac{\sigma(U, t\xi)}{t^\alpha}.$$

A *smoothing symbol* is a symbol in

$$S^{-\infty}(x) \equiv \bigcap_{k \in \mathbb{N}} S^{\leq -k}(U)$$

and the relation

$$\sigma \simeq \tilde{\sigma} \Leftrightarrow \sigma - \tilde{\sigma} \in S^{-\infty}(U)$$

defines an equivalence relation on $S(x)$.

A symbol of order α is called a *classical symbol* if there exist $\sigma_{\alpha-j} \in S^{\alpha-j}(x)$, $j \in \mathbb{N}$ such that:

$$\sigma(x,\xi) \simeq \sum_{j=0}^{\infty} \sigma_{\alpha-j}(x,\xi)$$

which are positively homogeneous, i.e

$$\sigma_{\alpha-j}(x, t\xi) = t^{\alpha-j}\sigma_{\alpha-j}(x,\xi) \quad \forall t \in \mathbb{R}^+.$$

Following Kontsevich and Vishik [KV 1], we shall say that a classical symbol lies in the *odd-class* if the positively homogeneous components $\sigma_{\alpha-j}$ are moreover homogeneous i.e:

$$\sigma_{\alpha-j}(x, t\xi) = t^{\alpha-j}\sigma_{\alpha-j}(x,\xi) \quad \forall t \in \mathbb{R}.$$

• *From symbols to pseudo-differential operators*

To a symbol $\sigma \in S^\alpha(U)$ we can associate the *pseudo-differential operator (P.D.O.) with symbol* σ defined by:

$$A : C_c^\infty(x) \to \mathbb{C}^\infty(\mathbb{R}^d)$$
$$u \to \left(U \to Au(x) = \int_{\mathbb{R}^d} e^{i\xi\cdot xU}\sigma(U,\xi)\hat{u}(\xi)d\xi\right).$$

where $C_c^\infty(U)$ denotes the space of complex valued smooth functions with compact support in U. The *principal symbol* of A is given by the principal part $\sigma_P(A)$ of its symbol $\sigma(A)$. If the symbol is classical, we shall call the corresponding P.D.O *classical*. The set of classical P.D.Os of order α (resp. $\leq m$) is denoted by $PDO^\alpha(U)$ (resp. $PDO^{\leq m}(U)$).

A *smoothing P.D.O.* is an operator in

$$PDO^{-\infty}(U) \equiv \bigcap_{k \in \mathbb{N}} PDO^{\leq -k}(U)$$

and there is an exact sequence:

$$0 \to PDO^{-\infty}(U) \to PDO^{\leq m}(U) \to S^{\leq m}(U) \to 0.$$

An *ordinary differential operator* of order $m \in \mathbb{N}$ is defined by a polynomial symbol (the polynomial being of order m) in ξ:

$$\sigma(x, \xi) = \sum_{j=0}^{m} a_k(x) \xi^k.$$

A differential operator is *local* i.e $u = 0 \Rightarrow Au = 0$. But a pseudo-differential operator, because of the smearing produced by the Fourier transform, is not local. It is only pseudo-local i.e if u is smooth on an open set U then Pu is also smooth on any open subset $V \subset U$.

The various classes of symbols introduced previously induce corresponding classes of pseudo-differential operators. A *classical pseudo-differential* is a P.D.O such that its symbol has components given by classical symbols and an *odd-class classical P.D.O* is a classical P.D.O such that its symbol has components given by symbols in the odd class. Notice that ordinary differential operators with integer order provide examples of P.D.Os in the odd class.

• *pseudo-differential operators acting on sections of vector bundles*

The notion of pseudo-differential operator can be carried out to operators acting on sections of vector bundles. Let $\pi_E : E \to M$, $\pi_F : F \to M$ be two smooth vector bundles with rank r_E and r_F respectively based on a smooth manifold M with dimension d. An operator

$$P : \Gamma(M, E) \to \Gamma(M, F)$$

acting from the space $\Gamma(M, E)$ of smooth sections of E to the space $\Gamma(M, F)$ of smooth sections of F is called a *pseudo-differential operator of order α* if given a neighborhood of any point $m \in M$, there is a local trivialization i.e a morphism:

$$\phi : (U, U \times \mathbb{C}^{r_E}, U \times \mathbb{C}^{r_F}) \to (M, E, F)$$

where U is an open subset of $\mathbb{R}^d$, the induced linear map $\phi^\sharp A : C_c^\infty(U, U \times \mathbb{C}^{r_E}) \to C_c^\infty(U, U \times \mathbb{C}^{r_F})$ has a symbol $\sigma(\phi^\sharp A) \in C^\infty(U \times \mathbb{R}^d) \otimes \mathcal{M}_{r_E, r_F}(\mathbb{C})$ with matrix components in $S^\alpha(U)$. Here $\mathcal{M}_{k,l}$ denotes the space of $k \times l$ matrices with coefficients in $\mathbb{C}$. It is a *classical P.D.O* if $\sigma(\phi^\sharp A)$ is a classical symbol. These definitions involve a choice of trivialization but can be shown to be independent of this choice.

$\sigma(\phi^\sharp A)$ is called the *(formal) symbol* of A and is only defined locally. However its principal part $\sigma_P(\phi^\sharp A)$ is independent of the choice of coordinate charts and is therefore defined globally. We shall denote it by $\sigma_P(A)$. It is called the *principal symbol* of the P.D.O.

Let us denote by $PDO^\alpha(M, E, F)$ the space of all classical P.D.Os of order α and by $PDO^{\leq m}(M, E, F)$ the space of all classical P.D.Os of order $\leq m$. When $E = F$, we shall denote these spaces by $PDO^\alpha(M, E)$ (resp. $PDO^{\leq m}(M, E)$) and when E is the trivial bundle $M \times \mathbb{C}$ by $PDO^\alpha(M)$ (resp.$PDO^{\leq m}(M)$). The *symbol set* $S^{\leq m}(M, E, F)$ is defined by the exact sequence:

$$0 \to PDO^{-\infty}(M, E, F) \to PDO^{\leq m}(M, E, F) \to S^{\leq m}(M, E, F) \to 0$$

where $PDO^{-\infty}(M,E,F) := \bigcap_{k\geq 0} PDO^{\leq -k}(M,E,F)$.

When M is compact, their is a notion of product of two pseudo-differential operators and

$$PDO(M,E) := \bigcup_{m\in\mathbb{Z}} PDO^{\leq m}(M,E)$$

defines an associative algebra.

From now on we shall assume M is a smooth compact manifold without boundary.

•*Admissible elliptic pseudo-differential operators*

Let E and F be as before two finite rank vector bundles based on a smooth manifold M. When $r_E = r_F$ a pseudo-differential operator $P : \Gamma(E) \to \Gamma(F)$ of order m is called *elliptic* if its principal symbol $\sigma_P^m(x,\xi)$ is an invertible matrix for $\xi \neq 0$. Let $Ell(M,E)$ denote the set of elliptic classical P.D.Os.

Let us denote by $Ell^*_{ord>0}(M,E)$ the class of invertible elliptic operators of strictly positive order. Since M is compact the spectrum $spec(A)$ of such an operator consists of isolated eigenvalues with finite multiplicity [Sh]. There is therefore a disc D_R of radius $R>0$ around the origin which does not contain any point of the spectrum. We shall say that A has *spectral cut* L_θ if there is a ray $L_\theta = \{\lambda \in \mathbb{C}, arg\lambda = \theta\}$ in the complex plane which does not intersect the spectrum of A. Such an operator will be called *admissible* and we shall denote by $Ell^{*,adm}_{ord>0}(M,E)$ the set of such admissible operators. Any invertible elliptic operator with strictly positive order such that the matrix given by its principal symbol has no eigenvalues in some non empty conical neighborhood Λ of a ray in the spectral plane is admissible since in that case at most a finite number of eigenvalues of the operator are contained in Λ [Sh].

Let us introduce some notations. $Ell^{s.a}_{ord>0}(M,E)$, resp. $Ell^{+}_{ord>0}(M,E)$ denotes the set of self-adjoint, resp. positive self-adjoint elliptic operators with strictly positive order. Adding an upper index $*$ restricts to injective operators so that $Ell^{*s.a}_{ord>0}(M,E)$, resp. $Ell^{*+}_{ord>0}(M,E)$ denotes the set of self-adjoint injective, resp. positive self-adjoint injective elliptic operators with strictly positive order and we have following inclusions:

$$Ell^{*+}_{ord>0}(M,E) \subset Ell^{*s.a}_{ord>0}(M,E) \subset Ell^{*,adm}_{ord>0}(M,E).$$

• *Complex powers and logarithms of elliptic operators*

Let $A \in Ell^{*,adm}_{>0}(M,E)$ with spectral cut L_θ. For $Rez<0$, the complex power A_θ^z of A is a bounded operator on any space $H^s(M,E)$ of sections of E of Sobolev class H^s defined by the contour integral:

$$A_\theta^z = \frac{i}{2\pi}\int_{\Gamma_\theta} \lambda^z (A-\lambda I)^{-1} d\lambda$$

where $\Gamma_\theta = \Gamma_{1,\theta}\cup\Gamma_{2,\theta}\cup\Gamma_{3,\theta}$ $\Gamma_{1,\theta} = \{\lambda = re^{i\theta}, r\geq R\}$,$\Gamma_{2,\theta} = \{\lambda = Re^{i\phi}, \theta\geq\phi\geq-\theta\}$, $\Gamma_{3,\theta} = \{\lambda = re^{i(\theta-2\pi)}, r\geq R\}$. Here $\lambda^z = exp(zlog\lambda)$ where $log\lambda = log|\lambda| + i\theta$ on $\Gamma_{1,\theta}$ and $log\lambda = log|\lambda| + i(\theta-2\pi)$ on $\Gamma_{3,\theta}$.

This definition is independent of the choice of R but depends on the choice of θ and yields for any $z\in\mathbb{C}$ an elliptic operator A_θ^z of order $z\cdot ord(A)$. When $z=-k, k\in\mathbb{N}$, then A^z coincides with A^{-k} of order $-k\cdot ord(A)$. When M is Riemannian, E is hermitian and A is essentially self-adjoint, then A_θ^z is independent of the choice of θ and coincides with the complex powers defined using spectral representation.

In the following we shall focus on operators in $Ell^{+}_{ord>0}(M,E)$ in which case we shall use the principal branch of the logarithm, taking $\theta=\pi$ and simply drop the mention θ.

For arbitrary $k\in\mathbb{Z}$, the map $z\to A_\theta^z$ defines a holomorphic function from $\{z\in\mathbb{C}, Rez<k\}$ to the space of bounded linear maps $\mathcal{L}(H^s(M,E)\to H^{s-kordA}(M,E))$ for any $s\in\mathbb{R}$ and we can set:

$$log_\theta A \equiv \left[\frac{\partial}{\partial z}A_\theta^z\right]_{z=0}$$

which defines a (non classical) P.D.O operator of zero order and hence a bounded operator from $H^s(M,E)$ to $H^{s-\epsilon}(M,E)$ for any $\epsilon > 0$ and any $s \in \mathbb{R}$. In local coordinates (x,ξ) on T^*M, the symbol of $log_\theta A$ reads:

$$\sigma_{log_\theta A}(x,\xi) = ord(A) log|\xi| Id + \text{ a classical P.D.O symbol of order 0.}$$

Hence, although the logarithm of an injective elliptic classical pseudo-differential operator with admissible cut L_θ is not itself a classical pseudo-differential operator, for two operators $A \in Ell^{*,adm}_{ord>0}(M,E)$, $B \in Ell^{*,adm}_{ord>0}(M,E)$ admitting spectral cuts L_θ and L_ϕ:

$$\frac{log_\theta A}{ordA} - \frac{log_\phi B}{ordB} \in PDO^0(M,E).$$

Appendix B

In this appendix, we recall why the first fundamental group of $Ell_0^{*,0}(M,E)$ is non trivial, where $Ell_0^{*,0}(M,E)$ denotes the pathwise connected component of identity in the group of invertible zero order elliptic P.D.Os.

E denotes a finite rank vector bundle based on a compact boundaryless Riemannian manifold M equipped with a connection ∇. We keep the notations of Appendix A. To begin with, let us describe the topology on $Ell_0^*(M,E)$.

•*A Fréchet structure on the algebra of P.D.Os and their symbols*

The space $SPDO(M,E)$ of symbols of classical pseudo-differential operators is a Fréchet space when equipped with the following family of semi-norms labelled by multiindices $\gamma \in \mathbb{N}^d$, $\delta \in \mathbb{N}^d$ and $i \in \{1,\cdots,N\}$, $k \in \mathbb{N}$:

$$\|\sigma\|_{\gamma,\delta,k} := max_i \left(sup_{y\in\overline{V_i},\xi\in\mathbb{R}^d,\|\xi\|=1} \|D_y^\gamma D_\xi^\delta \sigma_k(y,\xi)\| \right)$$

where σ_k is the homogeneous component of order k, $\{V_i\}_{i=1,\cdots,N}$ is a finite open cover (with $\bar{V_i}$ compact) of M associated to a partition of unity $\{V_i,\xi_i\}_{i=1,\cdots,N}$ on M subordinated to some finite open covering $\{U_i\}_{i=1,\cdots,N}$.

This topology combined with natural semi-norms on kernels of compact operators, induces a Fréchet structure on the space of classical pseudo-differential operators $PDO(M,E)$ via the identification of operators with their symbol up to a smoothing operator (see [KV1] section 3)

•*From elliptic operators to their principal symbols*

Let $S_P Ell_0^*(M,E)$ denote the group of principal symbols of operators in $Ell_0^*(M,E)$ and let $Ell_0^{*,0}(M,E)$, resp. $S_P Ell_0^{*,0}(M,E)$ be the pathwise connected component of identity of these topological spaces.

Lemma

$$\Pi_1(S_P Ell_0^{*,0}(M,E)) = \Pi_1(Ell_0^{*,0}(M,E)).$$

Proof: We show that $Ell_0^{*,0}(M,E)$ and $S_P Ell_0^{*,0}(M,E)$ have same homotopy type. The result then follows.

Let

$$f : Ell_0^{*,0}(M,E) \to S_P Ell_0^{*,0}(M,E)$$
$$A \to \sigma_P(A).$$

The connection on E and the metric on M give a canonical way of assigning to a symbol σ, an operator $Op(\sigma(A))(u)(x)$ with same principal symbol (see e.g. [BML] page 188 and references therein):

$$Op(\sigma(A))(u)(x) := \int_{T_x^* M} e^{ix\cdot\xi}\sigma(\xi)\hat{u}(\xi)d\xi$$

where the Fourier transform is defined using the exponential map on M and parallel transport on E. This gives rise to a map:

$$g : S_P Ell_0^{*,0}(M,E) \to Ell_0^{*,0}(M,E)$$
$$\sigma \to Op\,(\sigma)\,.$$

Then $f \circ g = Id$. One does not expect that $g \circ f = Id$ because $A = Op(\sigma_P(A))(1+K)$ for some (uniquely defined) compact operator K. However we have $g \circ f \sim Id$ where $\sim$ denotes homotopy of maps. Indeed, we can build a map $(A,s) \to F(A,s)$ on $Ell_0^*(M,E) \times [0,1]$ such that $F(A,0) = g \circ f(A) = Op\,(\sigma_P(A))$ and $F(A,1) = A$ for any $A \in Ell_0^*(M,E)$. Set

$$F(A,s) := (1-s)Op\,(\sigma_P(A)) + sA = Op\,(\sigma_P(A))\,(1+sK),$$

then $F(A,s) \in Ell_0(M,E)$ since $\sigma_P(F(A,s)) = \sigma_P\left(Op\left(\sigma_P(A)\right)\right) = \sigma_P(A)$ and A is elliptic of order zero. The operator $1+sK$ might not be invertible so let us show how we can modify it continuously into a family $\tilde{F}(A,s) \in Ell_0^*(M,E)$. Let $\gamma(s) := Id + K_s$. Let us denote by S_K the spectrum of K. Since K can be extended into a compact operator of $L^2(M,E)$, S_K is compact and consists of isolated points, up to 0 which may be an accumulation point. The operator $Id+sK$ is not invertible whenever $-\frac{1}{s}$ is an eigenvalue of K. Since $s \in [0,1]$, we have $-\frac{1}{s} \leq -1$. The set $]-\infty,-1] \cap S_K$ is finite, so $\gamma(s)$ is invertible, up to a finite number of $s \in [0,1]$. We can therefore modify γ continously into some continuous path $\tilde{\gamma}(s) := 1+\phi(s)K$ with values in the set $\{Id + K \; invertible : K \in PDO^{\leq -1}(M,E)\}$, where $PDO^{\leq \alpha}(M,E)$ denotes the subspace of pseudodifferential opertors with order $\leq \alpha$. Setting $\tilde{F}(A,s) := Op\left(\sigma_P(A)\right)(1+\phi(s)K)$ yields the result.

- *The fundamental group* $\Pi_1(Ell_0^*(M,E))$

We shall discuss the case when E is trivial, but the results extend to the non trivial case using the fact that M being compact, for large enough N, there is a vector bundle F based on M such that $E \oplus F \simeq 1_N$ where 1_N is the trivial bundle of rank N on M.

Let $\mathcal{A}$ be a complex unitary Banach algebra and let N be a positive integer large enough. Then ([Ka], chIII, Th1.25) the fundamental group $\Pi_1(GL_N(\mathcal{A}))$ is isomorphic to $K(\mathcal{A})$, where $GL_N(\mathcal{A})$ is the group of invertible maps of $\mathcal{A}^N$ and $K(\mathcal{A})$ is the Grothendieck group of finitely generated projective $\mathcal{A}$-modules. Applying that result to the Banach space $C(S^*M)$ of continuous functions on S^*M, we find that:

$$\Pi_1(GL_N(C(S^*M))) \simeq K_0(S^*M) := K(C(S^*M)).$$

Using the classical result (see e.g. [Di], vol. 3) that on a compact manifold a continuous function is homotopic to a smooth function and that two homotopic smooth functions are smoothly homotopic, we have:

$$\Pi_1(GL_N(C^\infty(S^*M))) = \Pi_1(GL_N(C(S^*M))).$$

It follows that for the trivial bundle 1_N based on M of rank N large enough:

$$\Pi_1\left(S_P Ell_0^{*,0}(M,1_N)\right) \simeq \Pi_1(GL_N(C(S^*M))) \simeq K_0(S^*M).$$

On the other hand, $\Pi_1(GL_N(C(S^*M)))$ being generated by one parameter subgroups $exp(2\pi itp)$, $p : (C(S^*M))^N \to (C(S^*M))^N$, such that $p^2 = p$ [Ka], is non trivial so that $\Pi_1(Ell_0^*(M,1_N))$ is non trivial as announced.

Remark: In fact, since the arcwise connected component of identity $Ell^{*,0}(M,E)$ of the group $Ell^*(M,E)$ of invertible elliptic operators has the same fundamental group as $Ell_0^{*,0}(M,E)$ consisting of those that have zero order, this shows that $\Pi_1(Ell^{*,0}(M,E)) = \Pi_1(Ell_0^{*,0}(M,E)) \simeq K_0(S^*M)$ is generated by one parameter subgroups $exp(2\pi itP)$ of the above type and hence non trivial.

REFERENCES

[AP] M.Arnaudon, S.Paycha, *Regularisable and minimal orbits for group actions in infinite dimensions*, Comm.Math.Phys. **191** , 641–662 (1998)

[BF] J.M.Bismut, D.Freed,*The Analysis of elliptic families I*, Comm.Math.Phys. **106**, 159–176 (1986)

[BGV] N. Berline, E. Getzler, M. Vergne, *Heat-kernels and Dirac operators*, Springer Verlag 1991

[CFNW] M.Cederwall, G.Ferretti, B. Nilsson, A.Westerberg, *Schwinger terms and Cohomology of pseudodifferential operators*, Comm.Math.Phys. **175** , 203–220 (1996)

[CE] H. Cartan, S. Eilenberg, *Homological algebra*, Princeton University Press (1956)

[C1] A. Connes,*Non commutative differential geometry*, Publications I.H.E.S , n.62 (1985) p. 257-360

[C2] A. Connes,*Non commutative geometry*, Academic Press, San Diego, 1994

[D] C.Ducourtioux,*Ph.D thesis in preparation*

[Di] J. Dieudonné, *Eléments d'analyse 3*, Gauthier-Villars (1970)

[DL] B. Driver, T. Lohrenz, *Log Sobolev Inequalities on loop groups* , Journal of Functional Analysis **140**, 381-438 (1996)

[FLPTT] N. Fagella, A. Lesne, S. Paycha, L. Tedeschini-Lalli, S.T. Tsou, *Renormalization, Proceedings of a workshop*, Paris 1997

[F1] D. Freed, *The geometry of loop groups*, Journ. Diff. Geom. **28**, 223–276 (1988)

[F2] D. Freed, *Flag manifolds and infinite dimensional Kähler geometry*, in "Infinite dimensional groups with applications, Ed. V. Kac, Math. Sci. Res. Inst. Publ., Vol 4, Springer, New York, 1985

[G] P. Gilkey, *Invariance theory. The heat equation and the Atiyah-Singer index theorem*, Publish or Perish (1984), second edition, CRC Press (1995)

[H] L.Hörmander, *The analysis of linear partial differential operators*, I. Grundlagen der mathematischen Wiss. **256**, Springer Verlag (1983)

[Ka] M.Karoubi, *K-theory (An introduction)*, Springer Verlag 1978

[K] Ch. Kassel, *Le résidu non commutatif [d'après Wodzicki]*, Séminaire Bourbaki **708** (1989)

[Ki] A. Kirillov, *Elements of theory of representation*, Berlin, Springer Verlag, 1976

[KOS] H.H Kuo, N. Obata, K. Saito, *Lévy Laplacians of generalized functions on a nuclear space*,

[KK] O.S. Kravenchko, B.A. Khesin, *A central extension of the algebra of pseudodifferential symbols*, Funct.Anal.Appl. **25**, 152–154 (1991)

[KV1] M.Kontsevich, S. Vishik, *Determinants of elliptic pseudodifferential operators*, Max Planck Preprint (1994)

[KV2] M.Kontsevich, S. Vishik, *Geometry of determinants of elliptic operators* in Functional Analysis on the Eve of the 21st Century **Vol. I**
(ed. S.Gindikin, J.Lepowski,R.L.Wilson) Progress in Mathematics (1994)

[Ku] N.H.Kuiper, *The homotopy type of the unitary group of Hilbert spaces*, Topology 3 (1965)

[L] S. Lang, *An introduction to differentiable manifolds*, Interscience, John Wiley (1967)

[LM] B.Lawson, M.L Michelsohn, *Spin Geometry, Princeton Mathematical Series*, 1989

[Le] M.Lesch, *On the non commutative residue for pseudo-differential operators with log-polyhomogeneous symbols*, Annals of global analysis and geometry **17** (1998)

[MRT] Y. Maeda, S. Rosenberg, P.Tondeur*The curvature of gauge orbits* in " Global Analysis and Modern Mathematics" edited by K. Uhlenbeck, Publich or Perish (1994)

[M] J. Marsden, *Applications of Global Analysis in Mathematical Physics*, Publish or Perish, 1974

[M1] J.Mickelsson, *Current algebras and groups*,New York, Plenum Press, 1989

[M2] J.Mickelsson, *Second Quantization, anomalies and group extensions*, Lecture notes given at the "Colloque sur les Méthodes Géométriques en physique, C.I.R.M, Luminy, June 1997

[MN] R. Melrose and V. Nestor, *Homology of pseudo-differential operators I. manifolds with boundary*, Preprint: funct-an/9606005, Oct. 98

[Pa] S. Paycha, *Renormalized traces as a looking glass into infinite dimensional geometry; a proposal to extend concepts of Riemannian geometry to a class of Hilbert manifolds*, Preprint 1999

[Pr] A. Pressley, *The Energy flow on, the loop space of a compact Lie group*, Journ. of the LOndon Math. Soc. Vol 26, 1982

[PS] A. Pressley, G. Segal, *Loop groups*, Oxford Univ.Press (1988)

[R] A.O.Radul, *Lie algebras of differential operators, their central extensions, and W-algebras*, Funct.Anal.Appl. **25**, 25–39(1991)

[Rog] C. Roger, *extensions centrales d'algebres et de groupes de Lie de dimension infinie, algebres de Virasoro et généralisations*, Prépublications de l'institut Girard Desargues (1995)

[S] J. Schwinger, *Field theory commutators*, Phys. Rev. Lett. **3**, 296-297 (1959)

[Se] R.T. Seeley, *Complex powers of an elliptic operator, Singular integrals*, Proc. Symp. Pure Math., Chicago, Amer.Math. Soc., Providence, 288-307 (1966)

[Sh] M.A. Shubin, *pseudodifferential Operators and Spectral Theory*, Springer Verlag 1987

[SW] M. Spera, T. Wurzbacher, *Differential geometry of Grassmannian embedding of based loop groups*, Preprint 1998

[W] M. Wodzicki, *Non commutative residue* in Lecture Notes in Mathematics **1289** Springer Verlag (1987)

[Wu] T. Wurzbacher, *Symplectic geometry of the loop space of a Riemannian manifold*, Journal of Geometry and Physics **16**, 345-384 (1995)