IRREDUCIBLE DEGENERATIONS OF PRIMARY KODAIRA SURFACES

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ABSTRACT. We classify irreducible d-semistable degenerations of primary Kodaira surfaces. As an application we construct a canonical partial completion for the moduli space of primary Kodaira surfaces.

INTRODUCTION

A smooth compact complex surface with trivial canonical bundle is a K3 surface, a 2-dimensional complex torus, or a primary Kodaira surface. Normal crossing degenerations of such surfaces have attracted much attention. For example, Kulikov analyzed projective degenerations of K3 surfaces [13]. His results were generalized by Persson and Pinkham to degenerations of surfaces with trivial canonical bundle whose central fiber has algebraic components [21]. Conversely, Friedman character-ized the singular K3 surfaces from Kulikov's list that deform to smooth K3 surfaces by his notion of d-semistability [6].

For non-Kähler degenerations no general results seem to be known. We chose primary Kodaira surfaces as our object of study because they lack some complica-tions that the more interesting classes of tori and K3 surfaces have. On the other hand, Kodaira surfaces have a lot in common with tori and, via the Kummer con-struction, with K3 surfaces. The phenomena one sees in degenerations of Kodaira surfaces should therefore also be observable in the other classes of surfaces with trivial canonical bundle.

By a result of Borcea the moduli space $\mathfrak{K}$ of smooth Kodaira surfaces is isomor-phic to a countable union of copies of $\mathbb{C} \times \Delta^*$ [4]. The parameters correspond to the j-invariant of the elliptic base and a refined j-invariant of the fiber. As Δ^* can not be completed to a closed Riemann surface by adding finitely many points, it is clear that by studying degenerations we can at most hope for a partial completion of $\mathfrak{K}$. The explicit form of the moduli space also suggests the existence of families leading to a degeneration of the base or the fiber to a nodal elliptic curve. We will see that this is indeed the case. This leads to a partial completion $\mathbb{C} \times \Delta^* \subset \mathbb{P}^1 \times \Delta \setminus (\infty, 0)$ of each component of $\mathfrak{K}$. However, as elliptic curves can degenerate to any k-cycle of rational curves, this is not the full story. Rather we obtain a whole hierarchy of such completions, that are linked by non-Hausdorff phenomena at the boundary. Moreover, we will see that at the most interesting point $(\infty, 0)$ the picture becomes complicated. We were not able to fully clarify what happens there. If one restricts to normal crossing surfaces one should certainly blow up this point. On the other hand, we will also make the presumably not so surprising observation that, as in

the case of abelian varieties [1], it does not suffice to restrict to normal crossing varieties. We did however find families of generalized Kodaira surfaces mapping properly to $\Delta \times \mathbb{P}^1$, whose singularities are at most products of normal crossing singularities.

The bulk of the paper is concerned with a classification of irreducible, d-semi-stable, locally normal crossing surfaces X with $K_X = 0$. Three different types occur. Our main result is a description of the resulting completion $\mathfrak{K} \subset \overline{\mathfrak{K}}$, derived by deformation theory. The three different types correspond to three different parts in the boundary $\mathfrak{B} = \overline{\mathfrak{K}} \setminus \mathfrak{K}$. Each part is a countable union of copies of Δ^* or $\mathbb{C}$. Locally along the boundary divisor the completion $\mathfrak{K} \subset \overline{\mathfrak{K}}$ looks like the blowing-up of $(\infty, 0) \in \mathbb{P}^1 \times \Delta$, with two points on the exceptional divisor removed.

Some examples for degenerations of Kodaira surfaces have previously been given by Friedman and Shepherd-Barron in [7].

This article is divided into six sections. The first section contains general facts about smooth Kodaira surfaces. In the second section we describe their potential degenerations and show that three types are possible. Sections 3–5 contain an analysis of each type. In the final section we assemble our results in terms of moduli spaces, complemented by some examples of smoothable surfaces with singularities that are products of normal crossings.

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This paper is dedicated to Hans Grauert on the occasion of his 70th birthday. The second author wants to take this opportunity to express his gratitude for the support and mathematical stimulus he received from him as one of his last students. It was a great pleasure to learn from him.

1. Smooth Kodaira surfaces

In this section we collect some facts on primary Kodaira surfaces. Suppose B, E are two elliptic curves, and endow E with a group structure. Let $f : X \to B$ be a holomorphic principal E-bundle. The canonical bundle formula gives $K_X = 0$, so the Kodaira dimension is $\kappa(X) = 0$. As a topological space, X is the product of the 1-sphere with a 1-sphere-bundle $g : M \to B$. Let $e(g) \in H^2(B, \mathbb{Z})$ be its Euler class. The homological Gysin sequence

$$H_2(B, \mathbb{Z}) \xrightarrow{e(g)} H_0(B, \mathbb{Z}) \longrightarrow H_1(M, \mathbb{Z}) \longrightarrow H_1(B, \mathbb{Z}) \longrightarrow 0$$

and the Künneth formula yield $H_1(X, \mathbb{Z}) = \mathbb{Z}^3 \oplus \mathbb{Z}/d\mathbb{Z}$ with $d = e(g) \cap [B]$. We call the integer $d \geq 0$ the *degree* of X. Bundles of degree $d = 0$ are 2-dimensional complex tori. A smooth compact complex surface X with an elliptic bundle structure of degree $d > 0$ is called a *primary Kodaira surface*. For simplicity, we refer to such surfaces as Kodaira surfaces.

Kodaira surfaces have three invariants. The first invariant is the degree $d > 0$. It determines the underlying topological space. The Universal Coefficient Theorem gives

$$H^1(X, \mathbb{Z}) = \mathbb{Z}^3 \quad \text{and} \quad H^2(X, \mathbb{Z}) = \mathbb{Z}^4 \oplus \mathbb{Z}/d\mathbb{Z}.$$

Hence the degree d is also the order of the torsion subgroup of the Néron-Severi group $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$.

The second invariant is the j-invariant of B. It depends only on X: Since $b_1(X) = 3$ is odd, X is nonalgebraic. Moreover, since $f : X \to B$ has connected fibers it is the algebraic reduction and so the fibration structure does not depend on choices.

The third invariant is an element $\alpha \in \Delta^*$. Here $\Delta^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ is the punctured unit disk. The Jacobian Pic^0_X has dimension $h^1(\mathcal{O}_X) = 2$, and the quotient $\mathrm{Pic}^0_X / \mathrm{Pic}^0_B$ is isomorphic to $\mathbb{C}^*$. We have $h^2(\mathcal{O}_X(-X_b)) = h^0(\mathcal{O}_X(X_b)) = 1$ for each $b \in B$. So the map on the left in the exact sequence

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X_b, \mathcal{O}_{X_b}) \longrightarrow H^2(X, \mathcal{O}_X(-X_b)) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow 0$$

is surjective. Thus $\mathrm{Pic}^0_X / \mathrm{Pic}^0_B \to \mathrm{Pic}^0_{X_b}$ is an epimorphism. By semicontinuity, the kernel equals $\mathrm{NS}(B) = \mathbb{Z}$ and is generated by a well-defined element $\alpha \in \Delta^*$.

According to [3], p. 145, the principal E-bundle X is an associated fibre bundle $X = (L \backslash 0) \times_{\mathbb{C}^*} \mathbb{C}^*/\langle \alpha \rangle$ for some line bundle $L \to B$ of degree d. One can check that this α agrees with the invariant α above. It follows that the isomorphism classes of Kodaira surfaces correspond bijectively to the triples $(d, j, \alpha) \in \mathbb{Z}_{>0} \times \mathbb{C} \times \Delta^*$. Moreover, each Kodaira surface with invariant (d, j, α) is the quotient of a properly discontinuous free $\mathbb{Z}^2$-action on $\mathbb{C}^* \times \mathbb{C}^*$ defined by

$$\Phi(z_1, z_2) = (\beta z_1, z_1^d z_2) \quad \text{and} \quad \Psi(z_1, z_2) = (z_1, \alpha z_2).$$

Here $\beta \in \Delta^*$ is defined as follows: Consider the j-invariant as a function $j : H \to \mathbb{C}$ on the upper half plane. It factors over $\exp : H \to \Delta^*$, $\tau \mapsto \exp(2\pi\sqrt{-1}\tau)$. Now $\beta = \exp(\tau)$ with $j(\tau) = j$. This uniformization illustrates Borcea's result on the moduli space of Kodaira surfaces [4]. Moreover, it suggests the application of toric geometry for the construction of degenerations.

The following observation will be useful in the sequel:

Lemma 1.1. *Suppose $X \to B$ is a principal E-bundle of degree $d \geq 0$. Let G be a finite group of order w acting on X. If the induced action on B is free, then w divides d, and the quotient X/G is a principal E-bundle of degree d/w.*

Proof. Set $X' = X/G$ and $B' = B/G$. Then B' is an elliptic curve, $B \to B'$ is Galois and the induced fibration $X' \to B'$ is a principal E-bundle. To calculate its degree d', we use a characterization of d in terms of $\pi_1(X)$. Since the universal covering of E is contractible the higher homotopy groups of E vanish. The beginning of the homotopy sequence of the fibrations $X \to B$, $X' \to B'$ thus leads to the following diagram of central extensions:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1(E) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(B) & \longrightarrow & 0 \\
 & & \mathrm{id}\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_1(E) & \longrightarrow & \pi_1(X') & \longrightarrow & \pi_1(B') & \longrightarrow & 0 \,.
\end{array}
$$

Let $h_1, h_2 \in \pi_1(X')$ map to generators of $\pi_1(B')$. Since $[\pi_1(B') : \pi_1(B)] = w$ there exists an integral matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with determinant w such that

$$g_1 = ah_1 + bh_2, \quad g_2 = ch_1 + dh_2$$

map to generators of $\pi_1(B)$. Then $g_1, g_2 \in \pi_1(X) \subset \pi_1(X')$ and $[g_1, g_2]$, $[h_1, h_2]$ are generators for the commutator subgroups $[\pi_1(X), \pi_1(X)]$ and $[\pi_1(X'), \pi_1(X')]$ respectively. Now the degrees d, d' being the orders of the torsion subgroups of $H_1(X, \mathbb{Z}) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ and of $H_1(X', \mathbb{Z}) = \pi_1(X')/[\pi_1(X'), \pi_1(X')]$

they can be expressed as divisibility of a generator of the commutator subgroup. Write $[h_1, h_2] = d' \cdot x$ for some primitive $x \in \pi_1(E)$. Then

$$[g_1, g_2] = [ah_1 + bh_2, ch_1 + dh_2] = (ad - bc)[h_1, h_2] = wd' \cdot x$$

shows $d = wd'$ as claimed.   □

2. D-SEMISTABLE SURFACES WITH TRIVIAL CANONICAL CLASS

Our objective is the study of degenerations of smooth Kodaira surfaces. We consider the following class of singular surfaces:

Definition 2.1. A reduced compact complex surface X is called *admissible* if it is irreducible, has locally normal crossing singularities, is d-semistable, and satisfies $K_X = 0$.

The sheaf of first order deformations $\mathcal{T}_X^1 = \mathcal{E}xt^1(\Omega_X^1, \mathcal{O}_X)$ is supported on $D = \mathrm{Sing}(X)$. Following Friedman [6], we call a locally normal crossing surface X d-*semistable* if $\mathcal{T}_X^1 \simeq \mathcal{O}_D$. This is a necessary condition for the existence of a global smoothing with smooth total space.

Smooth admissible surfaces are either K3 surfaces, 2-dimensional complex tori, or Kodaira surfaces. Throughout this section, X will be a *singular* admissible surface. We will see that three types of such surfaces are possible. A finer classification is deferred to subsequent sections.

Let $\nu : S \to X$ be the normalization, $C \subset S$ its reduced ramification locus, $D \subset X$ the reduced singular locus, and $\varphi : C \to D$ the induced morphism. Then S is smooth. The surface X can be recovered from the commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & S \\ \varphi \downarrow & & \downarrow \nu \\ D & \longrightarrow & X, \end{array}$$

which is cartesian and cocartesian. It gives rise to a long exact Mayer-Vietoris sequence

$$\ldots \longrightarrow H^p(X, \mathcal{O}_X) \longrightarrow H^p(S, \mathcal{O}_S) \oplus H^p(D, \mathcal{O}_D) \longrightarrow H^p(C, \mathcal{O}_C) \longrightarrow \ldots .$$

The ideals $\mathcal{O}_S(-C) \subset \mathcal{O}_S$ and $\mathcal{I}_D = \nu_*(\mathcal{O}_S(-C)) \subset \mathcal{O}_X$ are the conductor ideals of the inclusion $\mathcal{O}_X \subset \nu_*(\mathcal{O}_S)$. They coincide with the relative dualizing sheaf

$$\nu_*(\omega_{S/X}) = \mathrm{Hom}(\nu_*(\mathcal{O}_S), \mathcal{O}_X).$$

Hence $K_S = -C$; in particular, the Kodaira dimension is $\kappa(S) = -\infty$. The Enriques-Kodaira classification of surfaces ([3], Chap. VI) tells us that S is either ruled or has $b_1(S) = 1$. In the latter case, one says that S is a surface of *class VII*.

Following Deligne and Rapoport [5] we call the seminormal curve obtained from $\mathbb{P}^1 \times \mathbb{Z}/n\mathbb{Z}$ by the relations $(0, i) \sim (\infty, i + 1)$, a *Néron polygon*.

Lemma 2.2. *Each connected component of $C \subset S$ is an elliptic curve or a Néron polygon. For each singular connected component $D' \subset D$, the number of irreducible components in $\nu^{-1}(D') \subset C$ is a multiple of 6.*

Proof. The adjunction formula gives $\omega_C = \mathcal{O}_C$. Obviously, C has ordinary nodes. By [5], Lemma 1.3, each component of C is an elliptic curve or a Néron polygon. Let s be the number of singularities and c the number of irreducible components in D'. The preimage $\nu^{-1}(D') \subset S$ is a disjoint union of Néron polygons. It has $2c$ irreducible components and $3s$ singularities. For Néron polygons, these numbers coincide. □

We will use below the following *triple point formula* for d-semistable normal crossing surfaces:

Lemma 2.3. *([20], Cor.2.4.2) Let $D' \subset D$ be an irreducible component and $C_1' \cup C_2' \subset C$ its preimage in S. Then $-(C_1')^2 - (C_2')^2$ is the number of triple points of D'.*

Let $g : S \to S'$ be a minimal model and $C' = g(C)$. Then $K_{S'} = -C'$, so $C' \subset S'$ has ordinary nodes, and $g : S \to S'$ is a sequence of blowing-ups with centers over $\text{Sing}(C')$. Recall that S is a *Hopf surface* if its universal covering space is isomorphic to $\mathbb{C}^2 \setminus 0$.

Proposition 2.4. *Suppose S is nonalgebraic. Then S is a Hopf surface. The curve $C = C_1 \cup C_2$ is a disjoint union of elliptic curves, and X is obtained by gluing them together.*

Proof. Since S has algebraic dimension $a(S) < 2$, no curve on S has positive selfintersection. Suppose C contains a Néron polygon $C' = C_1 \cup \ldots \cup C_m$. As X is d-semistable, the triple point formula Lemma 2.3 implies $C_i^2 = -1$. So $(C')^2 = \sum C_i^2 + \sum_{i \neq j} C_i \cdot C_j = -m + 2m > 0$, contradiction. Hence C is the disjoint union of elliptic curves. Consider the exact sequence

$$H^0(S, \mathcal{O}_S) \longrightarrow H^0(C, \mathcal{O}_C) \longrightarrow H^1(S, \omega_S).$$

By the Kodaira classification, $b_1(S) = 1$, consequently $h^1(\mathcal{O}_S) = 1$, thus C has at most two components. Suppose $\varphi : C \to D$ induces a bijection of irreducible components. Then each component of C double-covers its image in D. So the map on the left in the exact sequence

$$H^1(S, \mathcal{O}_S) \oplus H^1(D, \mathcal{O}_D) \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow 0$$

is surjective, contradicting $K_X = 0$. Consequently $C = C_1 \cup C_2$ consists of two elliptic curves which are glued together in X. Let $n \geq 0$ be the number of blowing-ups in $S \to S'$. The normal bundle of C has degree $K_S^2 = K_{S'}^2 - n = -n$. Again by the triple point formula it follows $n = 0$. Hence S is minimal. By [18], Prop. 3.1, S is a Hopf or Inoue surface. The latter case is impossible here because Inoue surfaces contain at most one elliptic curve ([18], Prop. 1.5). □

Proposition 2.5. *Suppose S is nonrational algebraic. Then S is a $\mathbb{P}^1$-bundle over an elliptic curve B. The curve $C = C_1 \cup C_2$ is the disjoint union of two sections, and X is obtained by gluing them together.*

Proof. By the Enriques classification, the minimal model S' is a $\mathbb{P}^1$-bundle over a nonrational curve B. By Lüroth's Theorem, C does not contain Néron polygons, hence is the disjoint union of elliptic curves. The Hurwitz formula implies that B is also an elliptic curve. Since K_S has degree -2 on the ruling, we infer $C = -K_S$ has at most two irreducible components. As in the preceding proof, we conclude that C consists of two components, which are identified in X. Let $n \geq 0$ be the number of

blowing-ups in $S \to S'$. The normal bundle of C has degree $K_S^2 = K_{S'}^2 - n = -n$. Since X is d-semistable, $n = 0$ follows. Hence S is a $\mathbb{P}^1$-bundle. $\square$

Proposition 2.6. *Suppose S is a rational surface. Then it is the blowing-up of a Hirzebruch surface in two points P_1, P_2 in disjoint fibers F_1, F_2. The curve C is a Néron 6-gon, consisting of the two exceptional divisors, the strict transforms of F_1, F_2, and the strict transforms of two disjoint sections whose union contains P_1 and P_2. The surface X is obtained by identifying pairs of irreducible components in C.*

Proof. The minimal model S' is either $\mathbb{P}^2$ or a Hirzebruch surface of degree $e \neq 1$. Since each representative of $-K_{S'}$ is connected the curve C must be connected. Suppose C is an elliptic curve. Then the map on the left in the exact sequence

$$H^1(S, \mathcal{O}_S) \oplus H^1(D, \mathcal{O}_D) \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow 0$$

is surjective, contradicting $K_X = 0$. Consequently, $C = C_1 \cup \ldots \cup C_m$ is a Néron polygon. The triple point formula gives $\sum C_i^2 = -m$, hence

$$K_S^2 = \sum C_i^2 + \sum_{i \neq j} C_i \cdot C_j = -m + 2m = m.$$

Let $n \geq 0$ be the number of blowing-ups in $S \to S'$, so $K_S^2 = K_{S'}^2 - n$. For $S' = \mathbb{P}^2$ this gives $m = 9 - n$. If S' is a Hirzebruch surface, $m = 8 - n$ holds instead. According to Lemma 2.2, the natural number m is a multiple of 6. The only possibilities left are $S' = \mathbb{P}^2$ and C' a Néron 3-gon, or S' a Hirzebruch surface and C' a Néron 4-gon. From this it easily follows that X is obtained from a blowing-up of a Hirzebruch surface as stated. $\square$

Remark 2.7. For the results of this section one can weaken the hypothesis that X is d-semistable. It suffices to assume that the invertible $\mathcal{O}_D$-module $\mathcal{T}_X^1$ is numerically trivial. One might call such surfaces *numerically d-semistable*. They will occur as locally trivial deformations of d-semistable surfaces.

3. Hopf surfaces

In this section we analyze the geometry and the deformations of admissible surfaces X whose normalization S is *nonalgebraic*. By Proposition 2.4, such S is a Hopf surface containing a union of two disjoint isomorphic elliptic curves $C = C_1 \cup C_2$.

As a topological space, Hopf surfaces are certain fibre bundles over the 1-sphere, whose fibres are quotients of the 3-sphere (see [10], Thm. 9, and [11]). By Hartog's Theorem, the action of $\pi_1(S)$ on the universal covering space $\mathbb{C}^2 \setminus 0$ extends to $\mathbb{C}^2$, fixing the origin. We call a Hopf surface *diagonizable* if $\pi_1(S)$ is contained in the maximal torus $\mathbb{C}^* \times \mathbb{C}^* \subset \mathrm{GL}_2(\mathbb{C})$ up to conjugacy inside the group of all biholomorphic automorphisms of $\mathbb{C}^2$ fixing the origin. This is precisely the class of Hopf surfaces we are interested in:

Proposition 3.1. *A Hopf surface S is diagonizable if and only if it contains two disjoint elliptic curves $C_1, C_2 \subset S$ with $-K_S = C_1 + C_2$. Moreover, a diagonizable Hopf surface has fundamental group $\pi_1(S) = \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$.*

Proof. Suppose $\pi_1(S) \subset \mathrm{GL}_2(\mathbb{C})$ consists of diagonal matrices. Let $C_1, C_2 \subset S$ be the images of $\mathbb{C}^* \times 0$ and $0 \times \mathbb{C}^*$, respectively. The invariant meromorphic 2-form $dz_1/z_1 \wedge dz_2/z_2$ on $\mathbb{C}^2$ yields $-K_S = C_1 + C_2$.

Conversely, assume the condition $-K_S = C_1 + C_2$. An element $\alpha \in \pi_1(S)$ is called a *contraction* if $\lim_{n\to\infty} \alpha^n(U) = \{0\}$ holds for the unit ball U in $\mathbb{C}^2$. According to a classical result [22], equation 44, in suitable coordinates a contraction takes the form

$$\alpha(z_1, z_2) = (\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2)$$

with $0 < |\alpha_1| \le |\alpha_2| < 1$ and $\lambda(\alpha_1 - \alpha_2^m) = 0$. We claim that each contraction $\alpha \in \pi_1(S)$ has $\lambda = 0$. Suppose not. The quotient of $\mathbb{C}^2 \setminus 0$ by $\langle \alpha \rangle$ is a primary Hopf surface and defines a finite étale covering $f : S' \to S$. Let $C' \subset S'$ be the image of $\mathbb{C}^* \times 0$. Then [12], p. 696, gives $-K_{S'} = (m+1)C'$. On the other hand,

$$-K_{S'} = -f^*(K_S) = f^{-1}(C_1) + f^{-1}(C_2)$$

holds. Consequently, a nonempty subcurve of $(m+1)C'$ is base point free, so S' is elliptic. Now according to [12], Theorem 31, ellipticity of $S' = (\mathbb{C}^2 \setminus 0)/(\alpha)$ implies $\lambda = 0$, contradiction.

Next we claim that $\pi_1(S)$ is contained in $\mathrm{GL}_2(\mathbb{C})$, at least up to conjugacy. Suppose not. A nonlinear fundamental group $\pi_1(S)$ is necessarily abelian ([12], Thm. 47). By [10], p. 231, there is a contraction $\alpha \in \pi_1(S)$ with $\lambda \ne 0$, contradiction.

Now we can assume $\pi_1(S) \subset \mathrm{GL}_2(\mathbb{C})$. Write $\pi_1(S) = G \cdot Z$ as a semidirect product, where $G = \{\gamma \in \pi_1(S) \mid |\det(\gamma)| = 1\}$, and $Z \simeq \mathbb{Z}$ is generated by a contraction $\alpha = (\alpha_1, \alpha_2)$, acting diagonally. If $\alpha_1 \ne \alpha_2$ then [23], p. 24, ensures that S is diagonalizable. It remains to treat the case $\alpha_1 = \alpha_2$. Then Z is central and we only have to show that G is abelian. Let S' be the quotient of $\mathbb{C}^2 \setminus 0$ by $\langle \alpha \rangle$. The canonical projection $\mathbb{C}^2 \setminus 0 \to \mathbb{P}^1$ induces an elliptic bundle $S' \to \mathbb{P}^1$. This defines an elliptic structure $g : S = S'/G \longrightarrow \mathbb{P}^1/G$. Suppose $\mathbb{P}^1 \to \mathbb{P}^1/G$ has more than 2 ramification points. Let F_b be the reduced fiber over $g \in \mathbb{P}^1/G$ with multiplicity m_b and $C_i = F_{b_i}$; the canonical bundle formula yields

$$
\begin{aligned}
0 &= C_1 + C_2 + K_S = F_{b_1} + F_{b_2} - 2F + \sum_{b \in \mathbb{P}^1/G} (m_b - 1)F_b \\
&= (1 - m_{b_1})F_{b_1} + (1 - m_{b_2})F_{b_2} + \sum_{b \in \mathbb{P}^1/G} (m_b - 1)F_b > 0,
\end{aligned}
$$

contradiction. Consequently there are only 2 ramification points. Therefore G is a subgroup of $\mathbb{C}^*$, hence abelian. This finishes the proof of the equivalence.

As for the fundamental group, it must have rank 1 for $b_1 = 1$. If there were more than one generator needed for the torsion part the action could not be free on $(\mathbb{C}^* \times 0) \cup (0 \times \mathbb{C}^*)$. □

Suppose that S is a diagonalizable Hopf surface with $\pi_1(S) \simeq \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \subset \mathrm{GL}_2(\mathbb{C})$ consisting of diagonal matrices. The free part of $\pi_1(S)$ is generated by a contraction $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 \in \Delta^*$. The torsion part is generated by a pair $\zeta = (\zeta_1, \zeta_2)$ of primitive n-th roots of unity. Note that they must be primitive, again since the action is free on the coordinate axes minus the origin. Choose $\tau_i \in \mathbb{C}$ and $n_i \in \mathbb{Z}$ with $\alpha_i = \exp(2\pi\sqrt{-1}\tau_i)$ and $\zeta_i = \exp(2\pi\sqrt{-1}n_i/n)$. Consider the lattice

$$\Lambda_i = \langle \tau_i, n_i/n, 1 \rangle = \mathbb{Z} \cdot \tau_i + \mathbb{Z} \cdot 1/n \subset \mathbb{C}.$$

The images $C_1, C_2 \subset S$ of $\mathbb{C}^* \times 0$ and $0 \times \mathbb{C}^*$ take the form

$$C_i = \mathbb{C}^* / \langle \alpha_i, \zeta_i \rangle = \mathbb{C}/\Lambda_i.$$

Without loss of generality we can assume $C = C_1 \cup C_2$. In fact, if S is not elliptic then C_1, C_2 are the only curves on S at all. Otherwise the elliptic fibration comes from a rational function of the form z_1^a/z_2^b on $\mathbb{C}^2 \setminus 0$. Then $|ab| \neq 1$ iff the elliptic fibration $S \to \mathbb{P}^1$ has multiple fibers. In this case the canonical bundle formula implies $C = C_1 \cup C_2$. Otherwise we can apply a linear coordinate change on $\mathbb{C}^2$ to achieve that the preimages of the components of C are $\mathbb{C}^* \times 0$ and $0 \times \mathbb{C}^*$.

Now assume there is an isomorphism $\psi : C_2 \to C_1$. Set $D = C_1$, $C = C_1 \cup C_2$, and $\varphi = \mathrm{id} \cup \psi : C \to D$. The cocartesian diagram

$$
\begin{array}{ccc}
C & \longrightarrow & S \\
\varphi \downarrow & & \downarrow \nu \\
D & \longrightarrow & X
\end{array}
$$

defines a normal crossing surface X. Since each translation of the elliptic curve C_2 extends to an automorphism of S fixing C_1, we can assume that $\psi : C_2 \to C_1$ respects the origin. Hence ψ is defined by a homothety $\mu : \mathbb{C} \to \mathbb{C}$ with $\mu \Lambda_2 = \Lambda_1$. In other words, there is a matrix

$$
(3.1) \qquad \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \qquad \text{with} \qquad \begin{array}{l} \mu \tau_2 = a\tau_1 + b/n \\ \mu/n = c\tau_1 + d/n. \end{array}
$$

The following two propositions describe the properties of the surface X in terms of $\pi_1(S) \subset \mathrm{GL}_2(\mathbb{C})$ and $\mu \in \mathbb{C}$.

Proposition 3.2. *The condition $K_X = 0$ holds if and only if $a = d = 1$ and $c = 0$.*

Proof. As $\nu^*(K_X) = 0$ in any case K_X is numerically trivial. The condition $K_X = 0$ is thus equivalent to $H^2(X, \mathcal{O}_X) \neq 0$. Assume $K_X = 0$. Then the map on the left in the exact sequence

$$
H^1(S, \mathcal{O}_S) \oplus H^1(D, \mathcal{O}_D) \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow 0
$$

is not surjective. Hence $H^1(\mathcal{O}_D)$ and $H^1(\mathcal{O}_S)$ have the same image in $H^1(\mathcal{O}_C)$. According to the commutative diagram

$$
\begin{array}{ccccc}
H^1(S, \mathcal{O}_S) & \longrightarrow & H^1(C, \mathcal{O}_C) & \xleftarrow{\varphi^*} & H^1(D, \mathcal{O}_D) \\
\mathrm{bij} \downarrow & & \downarrow \mathrm{inj} & & \downarrow \mathrm{inj} \\
H^1(S, \mathbb{C}) & \longrightarrow & H^1(C, \mathbb{C}) & \xleftarrow{\varphi^*} & H^1(D, \mathbb{C})
\end{array}
$$

from Hodge theory, the image of $H^1(D, \mathbb{C})$ in $H^1(C, \mathbb{C})$ contains the image of $H^1(S, \mathbb{C})$. So the composition

$$
H^1(S, \mathbb{C}) \longrightarrow H^1(C, \mathbb{C}) \xrightarrow{(\mathrm{id}, \psi^*)^{-1}} H^1(D, \mathbb{C}) \oplus H^1(D, \mathbb{C})
$$

factors over the diagonal $H^1(D, \mathbb{C}) \subset H^1(D, \mathbb{C})^{\oplus 2}$. Dually, the composition

$$
H_1(D, \mathbb{C}) \oplus H_1(D, \mathbb{C}) \xrightarrow{(\mathrm{id}, \psi_*)^{-1}} H_1(C, \mathbb{C}) \longrightarrow H_1(S, \mathbb{C})
$$

factors over the addition map $H_1(D, \mathbb{C})^{\oplus 2} \to H_1(D, \mathbb{C})$. In other words, the composition

$$
\Lambda_1 \xrightarrow{(\mathrm{id}, -\mathrm{id})} \Lambda_1 \oplus \Lambda_1 \xrightarrow{\mathrm{id} \oplus \left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix} \right)} \Lambda_1 \oplus \Lambda_2 \xrightarrow{(1,0,1,0)} \mathbb{Z}
$$

is zero. It is described by $(1 - d, c)$; thus $d = 1$, $c = 0$, and in turn $a = 1$. Hence the condition is necessary. The converse is shown in a similar way. $\square$

Remark 3.3. Inserting the above conditions into equation 3.1, one sees that the homothety $\mu : \mathbb{C} \to \mathbb{C}$ must be the identity, and then $\tau_2 = \tau_1 + b/n$. In particular S is elliptically fibered.

Proposition 3.4. *Suppose $K_X = 0$. Then X is d-semistable if and only if the congruences $(n_1 - n_2)^2 \equiv 0$ and $b(n_1 - n_2) \equiv 0$ modulo n hold.*

Proof. Let N_i be the normal bundle of $C_i \subset S$. We can identify $\mathcal{T}_X^1$ with $N_2 \otimes \psi^*(N_1)$. The bundle N_1 can be obtained as quotient of the normal bundle of $\mathbb{C}^* \times 0$ in $\mathbb{C}^* \times \mathbb{C}^*$, which is trivial, by the induced action of $\pi_1(S) = \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$: $N_1 = \mathbb{C}^* \times \mathbb{C}^*/(\alpha, \zeta)$ with

$$\alpha \cdot (z_1, v) = (\alpha_1 z_1, \alpha_2 v) \quad \text{and} \quad \zeta \cdot (z_1, v) = (\zeta_1 z_1, \zeta_2 v).$$

After pull back along $\mathbb{C} \to \mathbb{C}^*, z \mapsto \exp(2\pi\sqrt{-1}z)$, the bundle N_1 can also be seen as the quotient of the trivial bundle $\mathbb{C} \times \mathbb{C}$ by $\pi_1(C_1) = \Lambda_1$ acting via

$$\tau_1 \cdot (w, v) = (w + \tau_1, \alpha_2 v), \quad n_1/n \cdot (w, v) = (w + n_1/n, \zeta_2 v), \quad 1 \cdot (w, v) = (w + 1, v).$$

Choose m_1 with $m_1 n_1 \equiv 1$ modulo n. Then $1/n \in \Lambda_1$ acts via $1/n \cdot (w, v) = (w + 1/n, \zeta_2^{m_1} v)$. An analogous situation holds for N_2. Inserting $\Psi = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ we obtain that $\mathcal{T}_X^1 = N_2 \otimes \psi^*(N_1)$ is the quotient of $\mathbb{C} \times \mathbb{C}$ by the action

$$\tau_2 \cdot (w, v) = (\tau_2 + w, \alpha_1 \alpha_2 \zeta_2^{bm_1} v) \quad \text{and} \quad 1/n \cdot (w, v) = (1/n + w, \zeta_1^{m_2} \zeta_2^{m_1} v).$$

The isomorphism class of $\mathcal{T}_X^1 = N_2 \otimes \psi^*(N_1)$ depends only on the element

$$\tau_2 \longmapsto \alpha_1 \alpha_2 \zeta_2^{bm_1}, \quad 1/n \longmapsto \zeta_1^{m_2} \zeta_2^{m_1}$$

in $\mathrm{Hom}_{\mathbb{Z}}(\Lambda_2, \mathbb{C}^*)$. To check for triviality of $\mathcal{T}_X^1$, consider the commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Hom}_{\mathbb{Z}}(\Lambda_2, \mathbb{C}) & \xrightarrow{\;\mathrm{pr}\;} & \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \overline{\mathbb{C}}) & \xrightarrow{\;\simeq\;} & H^1(C_2, \mathcal{O}_{C_2}) \\
\mathrm{exp} \downarrow & & & & \downarrow \mathrm{exp} \\
\mathrm{Hom}_{\mathbb{Z}}(\Lambda_2, \mathbb{C}^*) & \longrightarrow & \mathrm{Pic}(C_2) & \underset{\simeq}{\longleftarrow} & H^1(C_2, \mathcal{O}_{C_2}^*),
\end{array}
$$

explained in [15], Sect. I.2. Here pr is the projection of $\mathrm{Hom}_{\mathbb{Z}}(\Lambda_2, \mathbb{C}) \simeq \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ onto the $\mathbb{C}$-antilinear homomorphisms $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \overline{\mathbb{C}})$. Lifting the homomorphism defining $N_2 \otimes \varphi^*(N_1)$ from $\mathrm{Hom}_{\mathbb{Z}}(\Lambda_2, \mathbb{C}^*)$ to $\mathrm{Hom}_{\mathbb{Z}}(\Lambda_2, \mathbb{C})$, we obtain

$$\tau_2 \longmapsto \tau_1 + \tau_2 + \frac{bm_1 n_2}{n}, \quad 1/n \longmapsto \frac{m_2 n_1 + m_1 n_2}{n}.$$

Using the diagram, we infer that X is d-semistable if and only if this homomorphism is $\mathbb{C}$-linear up to an integral homomorphism. The latter condition is equivalent to the equation

$$(\tau_1 + \tau_2 + \frac{bm_1 n_2}{n} + e) - \tau_2 n(\frac{m_2 n_1 + m_1 n_2}{n} + f) = 0$$

for certain integers e, f. Now proceed as follows: Substitute $\tau_1 = \tau_2 - b/n$; the coefficients of τ_1 and 1 give two equations; since e and f are variable this leads to two congruences modulo n; finally use $n_1 m_1 \equiv 1$, $n_2 m_2 \equiv 1$ modulo n to deduce the stated congruences. $\square$

We seek a coordinate free description of X. We call an automorphism of a Néron 1-gon a *rotation* if the induced action on the normalization $\mathbb{P}^1$ fixes the branch points (compare [5], Sect. 3.6).

Proposition 3.5. *Let X be a complex surface. The following are equivalent:*

- (i) *X is admissible with nonalgebraic normalization S.*
- (ii) *There is an elliptic principal bundle $X' \to B'$ of degree $e > 0$ over the Néron 1-gon B' and a (cyclic) Galois covering $g : X' \to X$ such that the Galois group G acts effectively on B' via rotations.*

Proof. Suppose (ii) holds. The normalization S' of X' is an elliptic principal bundle of degree $e > 0$ over $\mathbb{P}^1$, hence nonalgebraic. Since $g : X' \to X$ induces a finite morphism $g' : S' \to S$, the surface S is nonalgebraic as well. It is easy to check $K_X = 0$ using the assumption that G acts via rotations.

Conversely, assume (i). As we have seen, S is a diagonizable Hopf surface. The exact sequence

$$0 \longrightarrow \pi_1(S') \longrightarrow \pi_1(S) \longrightarrow \mathrm{PGL}_2(\mathbb{C})$$

determines another diagonizable Hopf surface S' and a Galois covering $S' \to S$. The Galois group $G = \pi_1(S)/\pi_1(S')$ is cyclic of order $m = n/\gcd(n, n_1 - n_2, b)$, and the order of the torsion subgroup in $\pi_1(S')$ is $n' = \gcd(n, n_1 - n_2)$, as simple computations show.

The projection $\mathbb{C}^2 \setminus 0 \to \mathbb{P}^1$ induces an elliptic principal bundle $f' : S' \to \mathbb{P}^1$. The Galois group G acts effectively on $\mathbb{P}^1$ and fixes $0, \infty$. Let $C_1', C_2' \subset S'$ be the fibres over the fixed points. The quotient $\mathbb{P}^1/G$ is a projective line, and we obtain an induced elliptic structure $f : S \to \mathbb{P}^1/G$. Note that f has two multiple fibres of order m, whose reductions are C_1, C_2. Next, the canonical isomorphism $\psi' : C_2' \to C_1'$ gives a normal crossing surface X'. Clearly, it is an elliptic principal bundle of degree $e = n' > 0$ over the Néron 1-gon B' obtained from $\mathbb{P}^1$ by the relation $0 \sim \infty$. The congruences $(n_1 - n_2)^2 \equiv 0 \equiv b(n_1 - n_2)$ modulo n are equivalent to $n'(n_1 - n_2) \equiv 0 \equiv n'b$. A straightforward calculation shows that this ensures that the free G-action on S' descends to a (free) action on X'. The quotient is $X = X'/G$. □

Remark 3.6. At this point we are in position to classify the admissible surfaces with nonalgebraic normalization. The elliptic principal bundles $X' \to B'$ are in one-to-one correspondence with pairs $(\alpha, e) \in \Delta^* \times \mathbb{N}$, while up to isomorphism the action of G on B' only depends on $|G|$. A further discrete invariant belongs to possibly non-isomorphic lifts of the G-action to X'.

We call the invariants e and $w = |G|$ of X the *degree* and the *warp* of X respectively. The congruences in Proposition 3.4 imply that w divides e. This is important for the construction of smoothings:

Theorem 3.7. *Suppose X is an admissible surface (Definition 2.1) with nonalgebraic normalization, of degree e and warp w. Then X deforms to a smooth Kodaira surface of degree e/w.*

Proof. Let $X' \to X$ be the Galois covering from Proposition 3.5. It suffices to construct a G-equivariant deformation of X'.

The deformation of X' will be obtained as infinite quotient of a toric variety belonging to an infinite fan. Our construction fibers over a similar construction of a smoothing of the Néron 1-gon due to Mumford ([2], Ch.I,4) that we now recall for the reader's convenience. Mumford considered the fan generated by the cones

$$\bar{\sigma}_m = \Big\langle (m, 1), (m + 1, 1) \Big\rangle, \quad m \in \mathbb{Z}.$$

There is a $\mathbb{Z}$-action on the associated toric variety W belonging to the linear (right-) action by $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ on the fan. Let $q : W \to \mathbb{C}$ be the morphism coming from the projection $\mathrm{pr}_2 : \mathbb{Z}^2 \to \mathbb{Z}$. Note that the central fiber is an infinite chain of rational curves (a Néron ∞-gon), while the general fiber is isomorphic to $\mathbb{C}^*$. Now $\mathbb{Z}$ acts fiberwise and the action is proper and free over the preimage of $\Delta \subset \mathbb{C}$. Moreover, $\mathbb{Z}$ acts transitively on the set of irreducible components of the central fiber. The quotient is the desired deformation of the Néron 1-gon.

For our purpose, consider the infinite fan in $\mathbb{Z}^3$ generated by the two-dimensional cones

$$\sigma_m = \left\langle (m, e \cdot \tbinom{m}{2}, 1), (m+1, e \cdot \tbinom{m+1}{2}, 1) \right\rangle, \quad m \in \mathbb{Z}.$$

Let V be the corresponding 3-dimensional smooth toric variety and $V \to \mathbb{C}$ the toric morphism belonging to the projection $\mathrm{pr}_3 : \mathbb{Z}^3 \to \mathbb{Z}$. Now the special fibre V_0 is isomorphic to a $\mathbb{C}^*$-bundle over the Néron ∞-gon. As neighbouring cones σ_m, σ_{m+1} are not coplanar, its degree over each irreducible component of the Néron ∞-gon is nonzero. A simple computation shows that its degree is e. This time the $\mathbb{Z}$-action on the fan is given by the automorphism

$$\Phi = \begin{pmatrix} 1 & e & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in \mathrm{SL}_3(\mathbb{Z}).$$

Again the induced $\mathbb{Z}$-action on the preimage $U \subset V$ of $\Delta \subset \mathbb{C}$ is proper and free. Choose $\alpha \in \Delta^*$ such that the fibres of $X' \to B'$ are isomorphic to $\mathbb{C}^*/\langle \alpha^w \rangle$. Let $\Psi : V \to V$ be the automorphism extending the action of $(1, \alpha, 1) \in (\mathbb{C}^*)^3$. Then $\mathfrak{X}' = U/\langle \Phi, \Psi^w \rangle$ is a smooth 3-fold, endowed with a projection $\mathfrak{X}' \to \mathbb{C}$. The general fibres $\mathfrak{X}_t$, $t \in \Delta^*$, are smooth Kodaira surfaces of degree e with fibres $\mathbb{C}^*/\langle \alpha^w \rangle$ and basis $\mathbb{C}^*/\langle t \rangle$. The special fibre $\mathfrak{X}_0'$ is isomorphic to X'.

Finally we extend the G-action on X' to $\mathfrak{X}'$. The automorphism $\Psi : V \to V$ descends to a free G-action on $\mathfrak{X}'$. Replacing α by another primitive w-th root of α^w if necessary, the induced action on $\mathfrak{X}_0'$ coincides with the given action on X'. Consequently, $\mathfrak{X} = \mathfrak{X}'/G$ is the desired smoothing of $X = \mathfrak{X}_0$. The action of Ψ on B' is free. Therefore, according to Lemma 1.1, the general fibres $\mathfrak{X}_t$ are smooth Kodaira surfaces of degree e/w. $\qquad\square$

The next task is to determine the versal deformation of X. We have to calculate the relevant cohomology groups:

Lemma 3.8. *For a locally normal crossing surface X with $K_X = 0$ and non-algebraic normalization it holds $h^0(\Theta_X) = 1$, $h^1(\Theta_X) = 1$ and $h^2(\Theta_X) = 0$.*

Proof. The Ext spectral sequence together with a local computation gives

$$H^p(X, \Theta_X) = \mathrm{Ext}^p(\Omega_X^1/\tau_X, \mathcal{O}_X),$$

see [6], page 88ff. Here $\tau_X \subset \Omega_X^1$ denotes the torsion subsheaf. In view of Serre duality

$$\mathrm{Ext}^p(\Omega_X^1/\tau_X, \mathcal{O}_X)^\vee \simeq H^{2-p}(X, \Omega_X^1/\tau_X)$$

it thus suffices to compute the cohomology of Ω_X^1/τ_X. Consider the exact sequence

$$0 \longrightarrow \Omega_X^1/\tau_X \longrightarrow \Omega_S^1 \longrightarrow \Omega_D^1 \longrightarrow 0$$

of $\mathcal{O}_X$-modules ([6], Prop. 1.5). Note that the map on the right is an *alternating sum*. The inclusion $H^0(X, \Omega_X^1/\tau_X) \subset H^0(S, \Omega_S^1) = 0$ yields $h^2(\Theta_X) = 0$. Moreover,

$$0 \longrightarrow H^0(D, \Omega_D^1) \longrightarrow H^1(X, \Omega_X^1/\tau_X) \longrightarrow H^1(S, \Omega_S^1)$$

and $h^{1,1}(S) = 0$ gives $h^1(\Theta_X) = 1$. With $b_3(S) = 1$, $h^1(\omega_S) = h^1(\mathcal{O}_S) = 1$ and degeneracy of the Fröhlicher spectral sequence for smooth compact surfaces ([3], IV, Thm. 2.7) we have $h^{1,2}(S) = 0$. Now the exact sequence

$$0 \longrightarrow H^1(D, \Omega_D^1) \longrightarrow H^2(X, \Omega_X^1/\tau_X) \longrightarrow H^2(S, \Omega_S^1)$$

gives $h^0(\Theta_X) = 1$.   □

Proposition 3.9. *Let X be as in Theorem 3.7. Then $\dim T_X^0 = 1$, $\dim T_X^1 = 2$ and $\dim T_X^2 = 1$.*

Proof. Since X is locally a complete intersection the E_2 term of the spectral sequence $E_2^{p,q} = H^p(\mathcal{T}_X^q) \Rightarrow T_X^{p+q}$ has at most one non-trivial differential, which is $H^0(\mathcal{T}_X^1) \to H^2(\Theta_X)$. The previous proposition shows first degeneracy at E_2 level and in turn gives the stated values for $\dim T_X^i$.   □

Theorem 3.10. *X is an admissible surface with nonalgebraic normalization, of degree e and warp w. Then the semiuniversal deformation $p : \mathfrak{X} \to V$ of X has a smooth, 2-dimensional base V. The locally trivial deformations are parameterized by a smooth curve $V' \subset V$, and $V \setminus V'$ corresponds to smooth Kodaira surfaces of degree e/w.*

Proof. Since $h^1(\Theta_X) = 1$ and $h^2(\Theta_X) = 0$, the locally trivial deformations are unobstructed, and V' is a smooth curve. Since $\dim T_X^1 = 2$ and since X deforms to smooth Kodaira surfaces of degree e/w, which move in a 2-dimensional family, the base V is a smooth surface. We saw in the proof of the previous proposition that the restriction map $T^1 \to H^0(X, \mathcal{T}_X^1)$ is surjective. According to [6], Proposition 2.5, the total space $\mathfrak{X}$ is smooth. Now Sard's Lemma implies that the projection $\mathfrak{X} \to V$ is smooth over $V \setminus V'$, at least after shrinking V.   □

Remark 3.11. The referee pointed out that V' should have an interpretation as versal deformation space of the elliptic curve $D \subset X$. This is indeed the case: Since the restriction of the Kodaira-Spencer map to $T_{V'}$ generates $H^1(\Theta_X)$ it suffices to show that the composition

$$(3.2) \qquad\qquad H^1(\Theta_X) \longrightarrow H^1(\Theta_X \otimes \mathcal{O}_D) \longrightarrow H^1(\Theta_D)$$

is surjective. This statement is stable under étale covers, so we may assume that there is an elliptic fibration $p : X \to B$ over the Néron 1-gon (Proposition 3.5). The double curve D is the fiber over the node of B, and by relative duality $R^1 p_*(\Theta_{X/B}) = \mathcal{O}_B$. The base change theorem implies that the restriction map $R^1 p_*(\Theta_{X/B}) \to H^1(\Theta_D)$ is surjective. On global sections this map is nothing but the composition of

$$\mathbb{C} = H^0(B, R^1 p_*(\Theta_{X/B})) \longrightarrow H^1(X, \Theta_X)$$

with (3.2). Therefore the latter map is surjective too.

4. Ruled surfaces over elliptic curves

In this section we study admissible surfaces X whose normalization S is *algebraic and nonrational*, as in Proposition 2.5.

Let B' be an elliptic curve and $f : S \to B'$ a $\mathbb{P}^1$-bundle with two disjoint sections $C_1, C_2 \subset S$. Put $C = C_1 \cup C_2$ and $D = C_1$. Let $\psi : C_2 \to C_1$ be an isomorphism and $\varphi = \mathrm{id} \cup \psi : C \to D$ the induced double covering. The cocartesian diagram

$$
\begin{array}{ccc}
C & \longrightarrow & S \\
{\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \nu} \\
D & \longrightarrow & X
\end{array}
$$

defines a normal crossing surface X. By abuse of notation we write ψ also for the induced automorphism $(f|_{C_1}) \circ \psi \circ (f|_{C_2})^{-1}$ of B'.

Proposition 4.1. *We have $K_X = 0$ if and only if $\psi : B' \to B'$ is a translation.*

Proof. As in the proof of Proposition 3.2, K_X is trivial iff the images of $H^1(S, \mathcal{O}_S)$ and $H^1(D, \mathcal{O}_D)$ in $H^1(C, \mathcal{O}_C)$ coincide. Notice that the map $\phi^* : H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_C) \simeq H^1(\mathcal{O}_D) \oplus H^1(\mathcal{O}_D)$ is the diagonal embedding. It does not depend on ϕ.

Suppose ψ is a translation. Then the images of $H^1(\mathcal{O}_S)$ and $H^1(\mathcal{O}_D)$ both agree with $(f|_C)^* H^1(\mathcal{O}_{B'})$.

Conversely, assume that ψ is not a translation. Then it acts on $H^1(\mathcal{O}_E)$ as a nontrivial root of unity. Consequently, $H^1(\mathcal{O}_S)$ and $H^1(\mathcal{O}_C)$ have different images in $H^1(\mathcal{O}_D)$. $\square$

The next task is to calculate the sheaf of first order deformations $\mathcal{T}_X^1$. We call the degree $e = -\min \left\{ A^2 \mid A \subset S \text{ a section} \right\}$ of S also the *degree* of X. In our case $e \geq 0$, since $S \to B'$ has two disjoint sections. Let w be the order, possibly 0, of the automorphism $\psi : B' \to B'$. Call it the *warp* of X.

Proposition 4.2. *Suppose $K_X = 0$. Then X is d-semistable if and only if the warp w divides the degree e.*

Proof. We identify C_1, C_2 with B' via $f : S \to B'$. Set $\mathcal{L}_i = f_*(\mathcal{I}_{C_i}/\mathcal{I}_{C_i}^2)$. Then $\deg(\mathcal{L}_i) = -C_i^2$ and $\mathcal{L}_1 \otimes \mathcal{L}_2 = \mathcal{O}_{B'}$. On the other hand, $\mathcal{T}_X^1$ is the dual of $\mathcal{L}_1 \otimes \psi^*(\mathcal{L}_2)$. The kernel of the homomorphism

$$
\phi_{\mathcal{L}} : B' \longrightarrow \mathrm{Pic}^0(B'), \quad b \longmapsto T_b^*(\mathcal{L}) \otimes \mathcal{L}^{-1}
$$

is the subgroup $B'_e \subset B'$ of e-torsion points ([14], Lem. 4.7). Choose an origin $0 \in B'$ and some $b \in B'$ such that ψ is the translation $T_b : B' \to B'$. So $\mathcal{T}_X^1$ is trivial if and only if $b \in B'_e$, which means $w | e$. $\square$

Remark 4.3. Here we see that a complete set of invariants of admissible X with nonrational algebraic normalization and positive degree is: The j-invariant of B', the degree $e > 0$, and a translation ψ of B' of finite order. In fact, $\mathbb{P}^1$ bundles over B' of degree $e > 0$ with two disjoint sections are of the form $\mathbb{P}(\mathcal{O}_{B'} \oplus L^\vee)$ with L a line bundle of degree e. Up to pull-back by translations the latter are all isomorphic.

We come to the construction of smoothings:

Theorem 4.4. *Suppose X is an admissible surface (Definition 2.1) with algebraic, nonrational normalization, of degree e (possibly 0) and warp w. Then X deforms to an elliptic principal bundle of degree e/w over an elliptic curve.*

Proof. In order to construct the desired smoothing, we first pass to a Galois covering of X. The ruling $f : S \to B'$ yields a bundle $X \to B$ over the isogenous elliptic curve $B = B'/\langle\psi\rangle$ whose fibres are Néron w-gons. Let $G \subset \mathrm{Pic}^0(B') \subset \mathrm{Aut}(B')$ be the group of order w generated by ψ. Consider the surface $S' = S \times G$ and the isomorphisms

$$\psi_j : C_2 \times \left\{\psi^j\right\} \longrightarrow C_1 \times \left\{\psi^{j+1}\right\}, \quad (s, \psi^j) \longmapsto (\psi(s), \psi^{j+1})$$

for $j \in \mathbb{Z}/w\mathbb{Z}$. The corresponding relation defines a normal crossing surface X' with w irreducible components. Clearly, X' is d-semistable with $K_{X'} = 0$. The surface S' is endowed with the free G-action $\psi : (s, \psi^j) \mapsto (s, \psi^{j+1})$. It descends to a free G-action on X' with quotient $X = X'/G$.

We proceed similarly as in Theorem 3.7. Let V be the smooth toric variety corresponding to the infinite fan in $\mathbb{Z}^3$ generated by the cones

$$\sigma_n = \Big\langle (0, n, 1), (0, n+1, 1) \Big\rangle, \quad n \in \mathbb{Z}.$$

The projection $\mathrm{pr}_3 : \mathbb{Z}^3 \to \mathbb{Z}$ defines a toric morphism $V \to \mathbb{C}$. The special fibre V_0 is the product of a Néron ∞-gon with $\mathbb{C}^*$. The automorphism

$$\Psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in \mathrm{SL}_3(\mathbb{Z}).$$

acting from the right on row vectors maps σ_n to σ_{n+1}. By abuse of notation we also write Ψ for the induced automorphism of V. Choose $\alpha \in \Delta^*$ with $B' = \mathbb{C}^*/\langle\alpha^w\rangle$ and $B = \mathbb{C}^*/\langle\alpha\rangle$. The automorphism $(t_1, t_2, t_3) \mapsto (\alpha t_1, t_1^{e/w} t_2, t_3)$ of $(\mathbb{C}^*)^3$ extends to another automorphism Φ of V. The action of $\langle\Phi, \Psi\rangle$ on the preimage $U \subset V$ of $\Delta \subset \mathbb{C}$ is proper and free. Hence $\mathfrak{X}' = U/\langle\Phi^w, \Psi^w\rangle$ is a smooth 3-fold, endowed with a projection $\mathfrak{X}' \to \Delta$. The general fibres $\mathfrak{X}'_t$, $t \in \Delta^*$, are smooth Kodaira surfaces of degree e. The special fibre is $\mathfrak{X}'_0 = X'$. The action of $(\Phi\Psi)^w$ descends to a free G-action on $\mathfrak{X}'$, which coincides with the given action on $\mathfrak{X}'_0 = X'$. Hence $\mathfrak{X} = \mathfrak{X}'/G$ is the desired smoothing. $\qquad\square$

We head for the calculation of the versal deformation of X.

Proposition 4.5. *Suppose $K_X = 0$. Then the following holds:*
  (i) *If $e > 0$, then $h^0(\Theta_X) = 1$, $h^1(\Theta_X) = 2$, and $h^2(\Theta_X) = 1$.*
  (ii) *If $e = 0$, then $h^0(\Theta_X) = 2$, $h^1(\Theta_X) = 3$, and $h^2(\Theta_X) = 1$.*

Proof. As in the proof of Lemma 3.8 we use $H^p(\Theta_X) \simeq H^{2-p}(\Omega^1_X/\tau_X)$ and the exact sequence

$$0 \longrightarrow \Omega^1_X/\tau_X \longrightarrow \Omega^1_S \longrightarrow \Omega^1_D \longrightarrow 0.$$

Recall $h^{1,0}(S) = h^{1,0}(D) = 1$. The map on the right in

$$0 \longrightarrow H^0(X, \Omega^1_X/\tau_X) \longrightarrow H^0(S, \Omega^1_S) \longrightarrow H^0(D, \Omega^1_D)$$

is zero, since it is an alternating sum, so $h^2(\Theta_X) = 1$. Next we consider

$$0 \longrightarrow H^0(D, \Omega^1_D) \longrightarrow H^1(X, \Omega^1_X/\tau_X) \longrightarrow H^1(S, \Omega^1_S) \longrightarrow H^1(D, \Omega^1_D).$$

We have $h^{1,1}(S) = 2$ and $h^{1,1}(D) = 1$. The class of a fibre $F \subset S$ in $H^{1,1}(S)$ maps to zero in $H^{1,1}(D)$. For $e = 0$, this also holds for the class of the section $C_1 \subset S$,

and $h^1(\Theta_X) = 3$ follows. For $e > 0$, the image of the section does not vanish, and $h^1(\Theta_X) = 2$ holds instead. Finally, the sequence

$$H^1(S, \Omega_S^1) \longrightarrow H^1(D, \Omega_D^1) \longrightarrow H^2(X, \Omega_X^1/\tau_X) \longrightarrow H^2(S, \Omega_S^1) \longrightarrow 0$$

is exact. Now $h^{1,2}(S) = 1$ yields $h^0(\Theta_X) = 2$ for $e = 0$, and $h^0(\Theta_X) = 1$ for $e > 0$. $\square$

Corollary 4.6. *Suppose $K_X = 0$ and $e = 0$. Then X does not deform to a smooth Kodaira surface.*

Proof. Write $S = \mathbb{P}(\mathcal{O}_{B'} \oplus \mathcal{L})$ for some invertible $\mathcal{O}_{B'}$-module $\mathcal{L}$ of degree 0. Moving the gluing parameter ψ and the isomorphism class of B' and $\mathcal{L}$ gives a 3-dimensional locally trivial deformation of X. Since $h^1(\Theta_X) = 3$, the space parameterizing the locally trivial deformations in the semiuniversal deformation $p : \mathfrak{X} \to V$ of X is a smooth 3-fold. Now each fibre $\mathfrak{X}_t$ deforms to an elliptic principal bundle of degree 0 (Thm. 4.4), and the embedding dimension of V is $\dim T_X^1 = 4$. This shows that V is a smooth 4-fold with an open dense set parameterizing complex tori. Hence no fibre $\mathfrak{X}_t$ is isomorphic to a smooth Kodaira surface. $\square$

Proposition 4.7. *Let X be as in Theorem 4.4 with $e > 0$. Then $\dim T_X^0 = 1$, $\dim T_X^1 = 3$ and $\dim T_X^2 = 2$.*

Proof. In view of Proposition 4.5 we only have to show degeneracy of the spectral sequence of tangent cohomology $E_2^{p,q} = H^p(\mathcal{T}_X^q) \Rightarrow T_X^{p+q}$ at E_2 level, cf. Proposition 3.9. This is the case iff $T_X^1 \to H^0(\mathcal{T}_X^1)$ is surjective. Now by d-semistability $H^0(\mathcal{T}_X^1)$ is one-dimensional and any generator has no zeros. Geometrically degeneracy of the spectral sequence therefore means the existence of a deformation of X over $\Delta_\varepsilon := \operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2$ that is not locally trivial. This is what we established by explicit construction in Theorem 4.4. More precisely, let $\mathfrak{X} \to \Delta$ be a deformation of X. For $P \in D$ the image of the Kodaira-Spencer class $\kappa \in T_X^1$ of this deformation in $\mathcal{T}_{X,P}^1 \simeq \mathcal{E}xt^1(\Omega_{X,P}^1, \mathcal{O}_{X,P})$ is the Kodaira-Spencer class of the induced deformation of the germ of X along D. In appropriate local coordinates such deformations have the form $xy - \varepsilon f(z) = 0$ with f inducing the section of $\mathcal{E}xt^1(\Omega_{X,P}^1, \mathcal{O}_{X,P})$. Therefore $f \not\equiv 0$ iff the total space $\mathfrak{X}$ is smooth at P. $\square$

Theorem 4.8. *Suppose X is admissible with algebraic, nonrational normalization, of degree $e > 0$ and warp w. Let $p : \mathcal{X} \to V$ be the semiuniversal deformation of X. Then $V = V_1 \cup V_2$ has two irreducible components, and the following holds:*

(i) *V_2 is a smooth surface and parameterizes the locally trivial deformations.*

(ii) *$V_1 \cap V_2$ is a smooth curve and parameterizes the d-semistable locally trivial deformations.*

(iii) *V_1 is a smooth surface, and $V_1 \setminus V_1 \cap V_2$ parameterizes smooth Kodaira surfaces of degree e/w.*

Proof. A similar situation has been found by Friedman in his study of deformations of d-semistable K3 surfaces. We follow the proof of [6], Theorem 5.10 closely.

Deformation theory provides a holomorphic map $h : T_X^1 \to T_X^2$ with $V = h^{-1}(0)$, whose linear term is zero, and whose quadratic term is given by the Lie bracket $1/2[v, v]$ (cf. e.g. [19]). Moving the two parameters $j(E) \in \mathbb{C}$, $\psi \in B'$ and using $h^1(\Theta_X) = 2$, we see that the locally trivial deformations are parameterized by a smooth surface $V_2 \subset T_X^1$. Let $h_2 : T_X^1 \to \mathbb{C}$ be a holomorphic map with $V_2 = h_2^{-1}(0)$,

and $h_1 : T_X^1 \to T_X^2$ a holomorphic map with $h = h_1 h_2$. Let $L_1 : T_X^1 \to T_X^2$ and $L_2 : T_X^1 \to \mathbb{C}$ be the corresponding tangential maps and set $V_1 = h_1^{-1}(0)$.

We proceed by showing that $V_1 \cap V_2$ is a smooth curve, or equivalently that the intersection of $\mathrm{kern}(L_1)$ with $H^1(\Theta_X) = \mathrm{kern}(L_2)$ is 1-dimensional. The smoothing of X constructed in Theorem 4.4 obviously has a smooth total space. As in the proof of Proposition 3.9 we see that its Kodaira-Spencer class $k \in T_X^1$ generates $\mathcal{T}_X^1$.

Let $L : H^1(\Theta_X) \to H^1(\mathcal{T}_X^1)$ be the linear map $v \mapsto [v, k]$. By [6], Proposition 4.5, its kernel are the first order deformations which remain d-semistable. Since we can destroy d-semistability by moving the gluing $\psi \in \mathrm{Pic}^0(E)$, this kernel is 1-dimensional. As in [6], page 109, one shows that for $v \in H^1(\Theta_X)$:

$$L(v) = [v, k] = 1/2[v + k, v + k] = L_1(v + k) \cdot L_2(v + k) = L_1(v) \cdot L_2(k).$$

Using $L_2(k) \neq 0$ we infer $\mathrm{kern}\, L_1 \cap H^1(\Theta_X) = \mathrm{kern}\, L \simeq \mathbb{C}$. It follows that $V_1 \cap V_2$ is a smooth curve, which parameterizes the d-semistable locally trivial deformations, and V_1 must be a smooth surface. Our local computation together with the interpretation of $j(E) \in \mathbb{C}$ as coordinate on $V_1 \cap V_2$ shows that V_1 is nothing but the base of the smoothing of X constructed in Theorem 4.4. In particular, $V_1 \setminus V_1 \cap V_2$ parameterizes smooth Kodaira surfaces. $\square$

## 5. RATIONAL SURFACES AND HONEYCOMB DEGENERATIONS

The goal of this section is to analyze rational admissible surfaces. According to Proposition 2.6, the ramification curve $C \subset S$ of the normalization $\nu : S \to X$ is a Néron 6-gon $C = C_0 \cup \ldots \cup C_5$ with $K_S = -C$. We suppose $C_i \cap C_{i+1} \neq \emptyset$, regarding the indices as elements in $\mathbb{Z}/6\mathbb{Z}$. The singular locus $D \subset X$ is the seminormal curve with normalization $\mathbb{P}^1 \times \mathbb{Z}/3\mathbb{Z}$ and imposed relations $(0, i) \sim (0, j)$ and $(\infty, i) \sim (\infty, j)$ for $i, j \in \mathbb{Z}/3\mathbb{Z}$. The gluing $\varphi : C \to D$ identifies pairs of irreducible components in C. Since the resulting surface should be normal crossing the fibers of $S \to X$ have cardinality at most 3. There are two possibilities left. The first alternative is $\varphi(C_i) = \varphi(C_{i+3})$ and $\varphi(C_i \cap C_{i+1}) = \varphi(C_{i+2} \cap C_{i+3})$ for all $i \in \mathbb{Z}/6\mathbb{Z}$. We call it *untwisted* gluing. The second alternative is $\varphi(C_i) = \varphi(C_{i+3})$, $\varphi(C_{i+1}) = \varphi(C_{i-1})$, $\varphi(C_{i+2}) = \varphi(C_{i-2})$ and $\varphi(C_i \cap C_{i+1}) = \varphi(C_{i-1} \cap C_{i-2})$ for some $i \in \mathbb{Z}/6\mathbb{Z}$. Call it *twisted* gluing. The distinction is important for the canonical class:

Proposition 5.1. *The condition $K_X = 0$ holds if and only if the gluing $\varphi : C \to D$ is untwisted.*

Proof. As in Propositions 3.2 and 4.1 the condition $K_X = 0$ is equivalent to $H^2(\mathcal{O}_X) \neq 0$. By the exact sequence

$$H^1(S, \mathcal{O}_S) \oplus H^1(D, \mathcal{O}_D) \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow 0$$

and with $h^{0,1}(S) = 0$ this holds precisely if $\varphi^* : H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_C)$ vanishes. The bicoloured graphs attached to the curves C, D ([5], section 3.5) are

$$\Gamma(C) \qquad\qquad \Gamma(D)$$
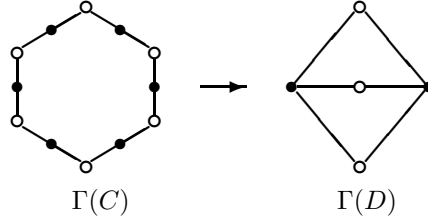
Here the white vertices are the irreducible components, the black vertices are the singularities, and the edges are the ramification points on the normalization. Since the curves in question are seminormal, the map $\varphi^* : H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_C)$ coincides with $\varphi^* : H^1(\Gamma(D), \mathbb{C}) \to H^1(\Gamma(C), \mathbb{C})$. A direct calculation left to the reader shows that it vanishes if and only if φ is untwisted. $\square$

From now on we assume that the gluing $\varphi : C \to D$ is untwisted. According to Proposition 2.6, the surface S is obtained from a Hirzebruch surface $f' : S' \to \mathbb{P}^1$ by blowing-up twice. Our conventions are that $C_0, C_3 \subset S$ are sections of the induced fibration $f : S \to \mathbb{P}^1$, that $C_1 \cup C_2$, $C_4 \cup C_5$ are the fibres over $0, \infty \in \mathbb{P}^1$, and that C_1, C_4 are exceptional for the contraction $r : S \to S'$. Let us also assume that the image of C_0 in S' is a minimal section. If $e \geq 0$ is the degree of the Hirzebruch surface S', this means $C_0^2 = -e - 1$ and $C_3^2 = e - 1$.

Set $D_i = \varphi(C_i)$ for $i \in \mathbb{Z}/3\mathbb{Z}$. The space of all untwisted gluings $\varphi : C \to D$ is a torsor under

$$(5.1) \qquad\qquad \mathrm{Aut}^0(D) = \prod \mathrm{Aut}^0(D_i) \simeq (\mathbb{C}^*)^3.$$

The action of $\phi \in \mathrm{Aut}^0(D)$ is given by composing $\varphi \mid C_0 \cup C_1 \cup C_2$ with ϕ. We call $\varphi \mid C_0 \cup C_3$ the *vertical* gluing. The ruling $f : S \to \mathbb{P}^1$ yields a preferred vertical gluing, which identifies points of the same fibre. Every other vertical gluing differs from the preferred one by a *vertical gluing parameter* $\zeta \in \mathbb{C}^*$. We call its order $w \geq 0$ the *warp* of X. The warp is important for the calculation of $\mathcal{T}_X^1$:

Proposition 5.2. *The surface X is d-semistable if and only if the warp w divides the degree e.*

Proof. The inclusions $D_i \cup D_j \subset D$ give an injection $\mathrm{Pic}(D) \subset \prod_{i \neq j} \mathrm{Pic}(D_i \cup D_j)$. We proceed by calculating the class of $\mathcal{T}_X^1 \mid D_0 \cup D_1$. Consider the normal crossing surface $\bar{S}$ defined by the cocartesian diagram

$$\begin{array}{ccc} C_2 \cup C_5 & \longrightarrow & S \\ \downarrow & & \downarrow \\ D_2 & \longrightarrow & \bar{S} \end{array}$$

Let $\bar{C}_i \subset \bar{S}$ be the images of $C_i \subset S$. The ideals $\mathcal{I}_{01}, \mathcal{I}_{34} \subset \mathcal{O}_{\bar{S}}$ of the Weil divisors $\bar{C}_{01} = \bar{C}_0 \cup \bar{C}_1$ and $\bar{C}_{34} = \bar{C}_3 \cup \bar{C}_4$ are invertible. A local computation shows that

$$(\mathcal{T}_X^1 \mid D_0 \cup D_1)^\vee \simeq \mathcal{I}_{01}/\mathcal{I}_{01}^2 \otimes \psi^*(\mathcal{I}_{34}/\mathcal{I}_{34}^2),$$

where $\psi : \bar{C}_{01} \longrightarrow \bar{C}_{34}$ is the isomorphism obtained from the induced gluing $\bar{\varphi} : \bar{C}_{01} \cup \bar{C}_{34} \to D_0 \cup D_1$. Another local computation gives

$$\mathcal{I}_{01}/\mathcal{I}_{01}^2 \mid \bar{C}_0 \simeq \mathcal{I}_{C_0}/\mathcal{I}_{C_0}^2(-C_0 \cap C_{-1}), \quad \mathcal{I}_{01}/\mathcal{I}_{01}^2 \mid \bar{C}_1 \simeq \mathcal{I}_{C_1}/\mathcal{I}_{C_1}^2(-C_0 \cap C_1).$$

It follows that $\mathcal{T}_X^1 \mid D_0 \cup D_1$ is trivial if and only if $\zeta^e = 1$. A similar argument applies to $D_0 \cup D_2$ and $D_1 \cup D_2$.   □

Remark 5.3. The previous two propositions yield the following classification of admissible X with rational normalization and positive degree e: According to Equation 5.1, the isomorphism class of X is determined by $e > 0$ and 3 gluing parameters. By Proposition 5.2 the vertical one is an e-th root of unity ζ. For (e, ζ) fixed the automorphisms of S act on the remaining two horizontal gluing parameters with quotient isomorphic to $\mathbb{C}^*$.

We come to the existence of smoothings:

Theorem 5.4. *Suppose X is an admissible surface (Definition 2.1) with rational normalization, of degree e (possibly 0) and warp w. Then X deforms to an elliptic principal bundle of degree e/w over an elliptic curve.*

Proof. It should not be too surprising that the construction is a modification of both the constructions in Theorems 3.7 and 4.4.

First, we simplify matters by passing to a Galois covering. Let $G \subset \mathbb{C}^*$ be the group of order w generated by the vertical gluing parameter $\zeta \in \mathbb{C}^*$. Since $w|e$, the G-action $z \mapsto \zeta z$ on $\mathbb{P}^1$ lifts to a G-action on the Hirzebruch surface S', and hence to the blowing-up S. The diagonal action on $\tilde{S} = S \times G$ is free. Let $\psi_i : C_i \to C_{i+3}$ be the isomorphisms induced by the gluing $\varphi : C \to D$. This gives G-equivariant isomorphisms

$$\psi_{ij} : C_i \times \{\zeta^j\} \longrightarrow C_{i+3} \times \{\zeta^{j+1}\}, \quad (s, \zeta^j) \longmapsto (\psi_i(s), \zeta^{j+1}).$$

Let X' be the normal crossing surface obtained from $\tilde{S}$ modulo the relation imposed by the ψ_{ij}. Then G acts freely on X' with quotient $X = X'/G$.

The next task is to construct a G-equivariant smoothing of X'. Again toric geometry enters the scene. Consider the infinite fan in $\mathbb{Z}^4$ generated by the cones

$$\sigma_{m,n} = \left\langle (m, e \cdot \binom{m}{2}, 1, 0), (m+1, e \cdot \binom{m+1}{2}, 1, 0), (0, n, 0, 1), (0, n+1, 0, 1) \right\rangle$$

for $m, n \in \mathbb{Z}$. Let V be the corresponding 4-dimensional smooth toric variety. The projection $\mathrm{pr}_{34} : \mathbb{Z}^4 \to \mathbb{Z}^2$ defines a toric morphism $\mathrm{pr}_{34} : V \to \mathbb{C}^2$. The special fibre V_0 is isomorphic to a bundle of Néron ∞-gons over the Néron ∞-gon. Let $f : \hat{\mathbb{C}}^2 \to \mathbb{C}^2$ be the blowing-up of the origin $0 \in \mathbb{C}^2$ with the closed toric orbits removed and $E \subset \hat{\mathbb{C}}^2$ the exceptional set. It is isomorphic to $\mathbb{C}^*$. The cartesian diagram

$$\begin{array}{ccc} \hat{V} & \longrightarrow & V \\ \downarrow & & \downarrow \\ \hat{\mathbb{C}}^2 & \longrightarrow & \mathbb{C}^2 \end{array}$$

defines a smooth toric 4-fold $\hat{V}$, which is an open subset of the blowing-up of V with center $V_0 \subset V$. The exceptional divisor $\hat{V}_E = \hat{V}_0$ of $\hat{V} \to V$ is isomorphic to $E \times V_0$. The exceptional divisor E will be a parameter space of the whole construction via the $\mathbb{C}^*$-bundle $\hat{\mathbb{C}} \to E$ induced by the canonical map $\mathbb{C}^2 \setminus 0 \to \mathbb{P}^1$.

Let $Z \subset \hat{V}$ be the set of 1-dimensional toric orbits. Each fibre Z_t, $t \in E$, consists of the discrete set of non-normal-crossing singularities in $\hat{V}_t \simeq V_0$, as illustrated by Figure 1. Let $\bar{V} \to \hat{V}$ be the blowing-up with center $Z \subset \hat{V}$. The exceptional divisor $\bar{Z} \subset \bar{V}$ is easy to determine. Each fibre $\hat{V}_t$, $t \in E$, is smooth, except for the points

in Z_t. At such points a local equation for $\hat{V}_t$ is $T_1 T_2 = t T_3 T_4$, which is an affine cone over a smooth quadric in $\mathbb{P}^2$. Hence $\bar{Z}$ is a disjoint union of smooth quadrics in $\mathbb{P}^3$, each isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The whole exceptional divisor is $\bar{Z} \simeq E \times Z \times \mathbb{P}^1 \times \mathbb{P}^1$. The picture over a fixed $t \in E$ is depicted in Figure 2.
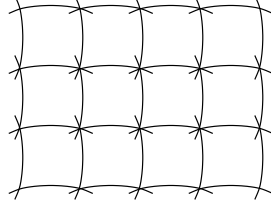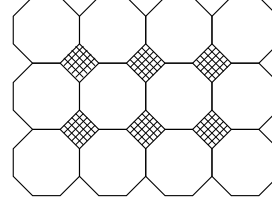

Fig. 1   Fig. 2

Here the octagons lie on the strict transform of V_t, and the squares are contained in the exceptional divisor $\bar{Z}_t$.

We seek to contract $\bar{Z} \subset \bar{V}$ along one of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $F \subset \bar{Z}$ be a $\mathbb{P}^1$-fibre. A local calculation shows that $\mathcal{O}_F(\bar{Z})$ has degree -1, and that the Cartier divisors $K_{\bar{V}}$ and $\mathcal{O}_{\bar{V}}(\bar{Z})$ coincide in a neighborhood of $F \subset \bar{V}$. So the Nakano contraction criterion [17] applies, and there exists a contraction $\bar{V} \to \tilde{V}$ which restricts to the projection

$$\mathrm{pr}_{123} : \bar{Z} = E \times Z \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow E \times Z \times \mathbb{P}^1.$$

The special fibre $\tilde{V}_0$ resembles an infinite system of honeycombs:
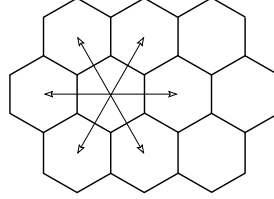

Fig. 3

Finally, we want to arrive at compact surfaces, so we seek to divide out a cocompact group action. Consider the two commuting automorphisms

$$\Phi = \begin{pmatrix} 1 & e & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

of $\mathbb{Z}^5$ acting from the right on row vectors. We regard our fan in $\mathbb{Z}^4$ as a fan in $\mathbb{Z}^5 = \mathbb{Z}^4 \oplus \mathbb{Z}$. Torically, the trivial $\mathbb{Z}$-factor amounts to going over to $V \times \mathbb{C}^*$, which will give a horizontal gluing parameter. Since $\Phi(\sigma_{m,n}) = \sigma_{m+1,n}$ and $\Psi(\sigma_{m,n}) = \sigma_{m,n+1}$, we get an induced action of the discrete group $\mathbb{Z}^2 = \langle \Phi, \Psi \rangle$ on $V \times \mathbb{C}^*$. Let $U \subset V$ be the preimage of $\Delta^2 \subset \mathbb{C}^2$. The action is proper and free on $U \times \mathbb{C}^*$. Let $\tilde{U} \subset \tilde{V}$ be the corresponding preimage. It is easy to see that the action on $V \times \mathbb{C}$ induces an action on $\tilde{V} \times \mathbb{C}^*$ which is proper and free on $\tilde{U} \times \mathbb{C}^*$. So

the quotient $\mathfrak{X}' = (\tilde{U} \times \mathbb{C}^*)/\langle \Phi, \Psi^w \rangle$ is a smooth complex 5-fold. The action on the special fibre is indicated in Figure 3 by the arrows. The general fibres $\mathfrak{X}'_t$, $(t, \lambda) \in f^{-1}(\Delta^* \times \Delta^*) \times \mathbb{C}^*$ are elliptic bundles of degree e over the elliptic curve $\mathbb{C}^*/\langle t_1 \rangle$ with fibre $\mathbb{C}^*/\langle t_2 \rangle$, where $f(t) = (t_1, t_2)$. Some special fibre $\mathfrak{X}'_t$, $t \in E \times \mathbb{C}^*$, is isomorphic to X', since t_5 moves through all horizontal gluings.

It remains to extend the G-action on X' to a G-action on $\mathfrak{X}'$. The automorphism

$$(\mathbb{C}^*)^5 \longrightarrow (\mathbb{C}^*)^5, \quad (t_1, t_2, t_3, t_4, t_5) \longmapsto (\zeta t_1, t_2 t_4, t_3, t_4, t_5)$$

of the torus extends to an automorphism of the torus embedding V. It induces a free G-action on $\mathfrak{X}'$, which is the desired extension. Set $\mathfrak{X} = \mathfrak{X}'/G$. General fibres $\mathfrak{X}_t$, $t \in (\mathbb{C}^2 \setminus 0) \times \mathbb{C}^*$ are elliptic bundles of degree e/w over an elliptic curve. Some special fibre $\mathfrak{X}_t$, $t \in E \times \mathbb{C}^*$ is isomorphic to X. $\qquad \square$

We turn our attention to the versal deformation of X and calculate the relevant cohomology groups:

Proposition 5.5. *Suppose $K_X = 0$. Then the following holds:*

  (i) *If $e > 0$, then $h^0(\Theta_X) = 1$, $h^1(\Theta_X) = 2$, and $h^2(\Theta_X) = 0$.*
  (ii) *If $e = 0$, then $h^0(\Theta_X) = 2$, $h^1(\Theta_X) = 3$, and $h^2(\Theta_X) = 0$.*

Proof. As in Lemmas 3.8 and 4.5 we use $H^p(X, \Theta_X) \simeq H^{2-p}(X, \Omega^1_X/\tau_X)$ and the exact sequence

$$0 \longrightarrow \Omega^1_X/\tau_X \longrightarrow \Omega^1_S \longrightarrow \Omega^1_{\tilde{D}} \longrightarrow 0.$$

Here $\tilde{D}$ is the normalization of D. The inclusion $H^0(\Omega^1_X/\tau_X) \subset H^0(\Omega^1_S)$ and $b_1(S) = 0$ gives $h^2(\Theta_X) = 0$. Next consider the exact sequence

$$0 \longrightarrow H^1(X, \Omega^1_X/\tau_X) \longrightarrow H^1(S, \Omega^1_S) \longrightarrow H^1(\tilde{D}, \Omega^1_{\tilde{D}})$$

The task is to determine the rank of the map on the right. This is done as in [6], p. 91: One has to compute the images in $H^{1,1}(\tilde{D})$ of the classes of $C_i \subset S$ using intersection numbers on S. For $e > 0$ this gives $h^1(\Theta_X) = 2$, whereas for $e = 0$ the result is $h^1(\Theta_X) = 3$. We leave the actual computation to the reader. Finally, the exact sequence

$$H^1(S, \Omega^1_S) \longrightarrow H^1(\tilde{D}, \Omega^1_{\tilde{D}}) \longrightarrow H^2(X, \Omega^1_X/\tau_X) \longrightarrow H^2(S, \Omega^1_S),$$

together with the preceding observations and $b_3(S) = 0$ gives the stated values for $h^0(\Theta_X)$. $\qquad \square$

Corollary 5.6. *Suppose $K_X = 0$ and $e = 0$. Then X does not deform to a smooth Kodaira surface.*

Proof. The locally trivial deformations of X are unobstructed since $h^2(\Theta_X) = 0$. They define a smooth 3-fold $V' \subset V$ in the base of the semiuniversal deformation $\mathfrak{X} \to V$. It is easy to see that it is given by the three gluing parameters in $\varphi : C \to D$. Since each fibre $\mathfrak{X}_t$ deforms to a complex torus (Thm. 5.4), V is smooth of dimension 4 and no fibre can be isomorphic to a smooth Kodaira surface. $\qquad \square$

Proposition 5.7. *Let X be as in Theorem 5.4 with $e > 0$. Then $\dim T^0_X = 1$, $\dim T^1_X = 3$ and $\dim T^2_X = 2$.*

Proof. This is shown as the similar statement of Proposition 4.7. $\qquad \square$

Theorem 5.8. *Suppose X is admissible with rational normalization, of degree $e > 0$ and warp w. Let $p : \mathfrak{X} \to V$ be the semiuniversal deformation of X. Then $V = V_1 \cup V_2$ consists of two irreducible components, and the following holds:*

(i) V_2 is a smooth surface parameterizing the locally trivial deformations.
(ii) $V_1 \cap V_2$ is a smooth curve parameterizing the locally trivial deformations which remain d-semistable.
(iii) $V_1 \setminus V_1 \cap V_2$ parameterizes smooth Kodaira surfaces of degree e/w.

Proof. The argument is as in the proof of Theorem 4.8. $\square$

6. THE COMPLETED MODULI SPACE AND ITS BOUNDARY

In this final section, we take up a global point of view and analyze degenerations of smooth Kodaira surfaces in terms of moduli spaces. Here we use the word "moduli space" in the broadest sense, namely as a topological space whose points correspond to Kodaira surfaces. We do not discuss wether it underlies a coarse moduli space or even an analytic stack or analytic orbispace.

Let $\mathfrak{K}_d = \mathbb{C} \times \Delta^*$ be the moduli space of smooth Kodaira surfaces of degree $d > 0$, and $\mathfrak{K} = \cup_{d>0} \mathfrak{K}_d$ their union. The points of $\mathfrak{K}$ correspond to the isomorphism classes $[X]$ of smooth Kodaira surfaces. The topology on $\mathfrak{K}$ is induced by what we suggest to call the *versal topology* on the set $\mathfrak{M}$ of isomorphism classes of all compact complex spaces. The versal topology is the finest topology on $\mathfrak{M}$ rendering continuous all maps $V \to \mathfrak{M}$ defined by flat families $\mathfrak{X} \to V$ which are versal for all fibres. The complex structure on $\mathfrak{K}$ comes from Hodge theory: One can view $\mathfrak{K}_d$ as the period domain of polarized pure Hodge structures of weight 2 on $H^2(X, \mathbb{Z})$/torsion divided by the automorphism group of this lattice. According to [4], the induced structure of a ringed space on $\mathfrak{K}_d$ is the usual complex structure on $\mathbb{C} \times \Delta^*$.

Let $\mathfrak{K} \subset \overline{\mathfrak{K}}$ be the space obtained by adding all admissible surfaces in the closure of $\mathfrak{K} \subset \mathfrak{M}$. The surfaces parameterized by the boundary $\mathfrak{B} = \overline{\mathfrak{K}} \setminus \mathfrak{K}$ are called *d-semistable* Kodaira surfaces. These are nothing but the admissible surfaces deforming to smooth Kodaira surfaces. The boundary decomposes into three parts

$$\mathfrak{B} = \mathfrak{B}^h \cup \mathfrak{B}^r \cup \mathfrak{B}^e$$

according to the three types of admissible surfaces. Here $\mathfrak{B}^h$ refers to the surfaces whose normalization are Hopf surfaces, $\mathfrak{B}^r$ to the surfaces with rational normalization, and $\mathfrak{B}^e$ to surfaces whose normalization is ruled over an elliptic base. We refer to these parts of $\overline{\mathfrak{K}}$ as Hopf, rational and elliptic ruled stratum respectively.

Proposition 6.1. *The irreducible components of the boundary $\mathfrak{B}$ are smooth complex curves. The components in $\mathfrak{B}^h$ are isomorphic to $\Delta^* \simeq \{\infty\} \times \Delta^*$, the components in $\mathfrak{B}^r$ are isomorphic to $\mathbb{C}^*$, and the components in $\mathfrak{B}^e$ are isomorphic to $\mathbb{C} \simeq \mathbb{C} \times \{0\}$.*

Proof. This follows from Remarks 3.6, 4.3 and 5.3. $\square$

Locally, the completion $\mathfrak{K} \subset \overline{\mathfrak{K}}$ is isomorphic to the blowing-up of $(\infty, 0) \in \mathbb{P}^1 \times \Delta$, but with the points $0, \infty \in \mathbb{P}^1$ on the exceptional divisor removed. This follows in particular from the construction in Theorem 5.4 of a family over the blow up of $\mathbb{C}^2$ with 2 points removed, with fiber over $(t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*$ a smooth Kodaira surface with invariants $(j, \alpha) = (\exp(t_1), t_2)$. To complete further we need to enlarge the class of generalized Kodaira surfaces. The least one would hope for is that any family $\mathfrak{X}^* \to \Delta^*$ of generalized Kodaira surfaces, that can be completed to a proper family over Δ, can be completed by a generalized Kodaira surface.

We first show that this is impossible if we consider only normal crossing surfaces.

Theorem 6.2. *Let $\mathfrak{X} \to \Delta$ be a degeneration of d-semistable Kodaira surfaces with elliptically ruled normalization. Assume that $\mathfrak{X}$ is bimeromorphic to a Kähler manifold, and that the j-invariant of the base of $\mathfrak{X}_t$ tends to 0 with $t \in \Delta$. Then $\mathfrak{X}_0$ is not of normal crossing type.*

Proof. Let $\tilde{\mathfrak{X}} \to \mathfrak{X}$ be the normalization. Let $\mathfrak{B}$ be the component of the Douady space of holomorphic curves in $\tilde{\mathfrak{X}}$ that contain a fiber of $\tilde{\mathfrak{X}}_t \to B_t$ for general $t \in \Delta$. Since these curves are contained in fibers of $\tilde{\mathfrak{X}} \to \Delta$ there is a map $\mathfrak{B} \to \Delta$. By the Kähler assumption this map is proper [8]. Let $\tilde{\mathfrak{X}}' \to \mathfrak{B}$ be the universal family. The universal map $\tilde{\mathfrak{X}}' \to \tilde{\mathfrak{X}}$ is an isomorphism over Δ^*, hence bimeromorphic. By desingularisation we may dominate this map by successively blowing up $\tilde{\mathfrak{X}}$ in smooth points and curves. This can be arranged to keep the property "normal crossing". In other words, we can assume that $\tilde{\mathfrak{X}}' = \tilde{\mathfrak{X}}$ is a fibration by rational curves over a degenerating family $\mathfrak{B} \to \Delta$ of elliptic curves.

Let $\mathfrak{C}', \mathfrak{C}'' \subset \tilde{\mathfrak{X}}$ be the two components of the preimage of the singular locus of $\mathfrak{X}$ mapping onto Δ. For any $b \in \mathfrak{B}$ the corresponding rational curve F_b intersects $\mathfrak{C}', \mathfrak{C}''$ in one point each. Let $b_0 \in \mathfrak{B}_0$ be a node of the nodal elliptic curve over $0 \in \Delta$. Then each intersection $\mathfrak{C}' \cap F_{b_0}$ and $\mathfrak{C}'' \cap F_{b_0}$ gives a point of multiplicity at least 2 on $\tilde{\mathfrak{X}}_0$. As $\mathfrak{C}', \mathfrak{C}''$ are identified under $\tilde{\mathfrak{X}} \to \mathfrak{X}$ this leads to a point of multiplicity at least 4 on $\mathfrak{X}_0$. □

There are however various completions if we admit *products of normal crossing singularities*. In dimension 2 the only such singularity that is not normal crossing is a point (X, x) of multiplicity 4. It has as completed local ring

$$\mathcal{O}_{X,x}^\wedge = \mathbb{C}[[T_1, \dots, T_4]]/(T_1 T_2, T_3 T_4).$$

In particular, it is a complete intersection and hence still Gorenstein. We will call (X, x) a *quadrupel point*. The singular locus $C \subset X$ consists of the four coordinate lines $C_1, \dots, C_4$, and C_i, C_j lie on the same irreducible component iff $i - j \not\equiv 2$ modulo 4. The embedding dimension of (X, x) is 4. It is therefore not embeddable into a smooth 3-fold. Its appearance here is perhaps not so surprising, as it occurs also in certain compactifications of moduli of polarized abelian surfaces [16].

We were nevertheless unable to select a natural class of generalized Kodaira surfaces that would satisfy the mentioned completeness property. Therefore we content ourselves to define two surfaces with quadrupel points, connecting the elliptic ruled stratum to the Hopf stratum and to the rational stratum respectively. A surface connecting the Hopf stratum with the rational stratum could not be found, although such surfaces should probably exist.

Fix an integer $e > 0$. Let S be the Hirzebruch surface of degree e. Choose two sections $C_0, C_2 \subset S$ with $C_0^2 = -e$, $C_2^2 = e$, and let C_1, C_3 be the fibers over $0, \infty \in \mathbb{P}^1$. We define the surface X_1 by gluing C_1, C_3 by any isomorphism (all choices give isomorphic results), and C_0, C_2 by an isomorphism of finite order w over $\mathbb{P}^1$. Note that X_1 is normal crossing except at one quadrupel point. As in the proof of Proposition 5.1 one checks $K_{X_1} = 0$.

Proposition 6.3. *Suppose w divides e. There is a family $\mathfrak{X} \to \mathbb{C}^2$ with the following properties:*

1. *$\mathfrak{X}_0 \simeq X_1$*
2. *the fibers over $\mathbb{C}^* \times \{0\}$ are d-semistable Kodaira surfaces of warp w and degree e with elliptic ruled normalization*

    3. *the fibers over $\{0\} \times \mathbb{C}^*$ are d-semistable Kodaira surfaces of warp w and degree e with nonalgebraic normalization*

Proof. In the proof of Theorem 5.4 we constructed a toric morphism $V \to \mathbb{C}^2$ with central fiber a bundle of Néron ∞-gons over the Néron ∞-gon. The restriction of $\Phi, \Psi \in \mathrm{GL}(\mathbb{Z}^5)$ defined there to $\mathbb{Z}^4 \simeq \mathbb{Z}^4 \oplus \{0\}$ defines an action of $\mathbb{Z}^2$ on the fan defining V. The induced action on V is proper and free, and commutes with the map to $\mathbb{C}^2$. We obtain a proper family $\mathfrak{X}' = V/\mathbb{Z}^2 \to \mathbb{C}^2$. By construction of V there is also an action of $G \simeq \mathbb{Z}/w\mathbb{Z}$ on $\mathfrak{X}'/\mathbb{C}^2$. The family $\mathfrak{X}'/G \to \mathbb{C}^2$ has the desired properties. □

Finally we study a surface connecting the rational and the elliptic stratum, under the expense of changing the warp. Again we fix an integer $e > 0$. Let $S = S' \cup S''$ be the disjoint union of two Hirzebruch surfaces of degrees $e+1$ and $e-1$ respectively. Choose two sections $C_0', C_2' \subset S'$ with $(C_0')^2 = -(e+1)$ and $(C_2')^2 = e+1$, and let C_1', C_3' be the fibres over $0, \infty \in \mathbb{P}^1$. Make the analogous choices C_0'', C_2'' etc. for S''. We glue C_i' to C_{i+2}'' in a way that preserves the intersection points indicated in the following figure by circles and crosses.
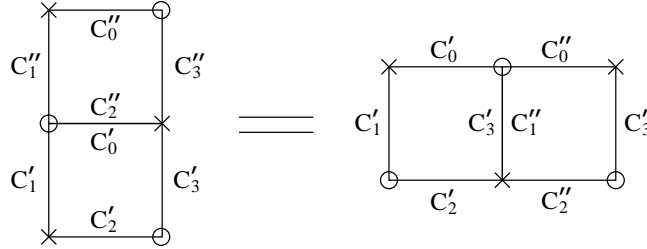

Fig. 4

For odd i the isomorphism $C_i' \to C_i''$ can be chosen arbitrarily; for even i we glue compatibly with the projections $S' \to \mathbb{P}^1$, $S'' \to \mathbb{P}^1$. The result is a reduced surface X_2 that is normal crossing except at two points corresponding to the circles and the crosses in the figure. It is a bundle of Néron 1-gons over $\mathbb{P}^1$. Again as in the proof of Proposition 5.1, one verifies that $K_{X_2} = 0$ holds.

Proposition 6.4.   1. *X_2 deforms to d-semistable Kodaira surfaces with rational normalization of degree e and warp 1.*
    2. *X_2 deforms to d-semistable Kodaira surfaces with elliptic ruled normalization of degree $2e$ and warp 2.*

Proof. (i) The idea is to modify a fiberwise degeneration of Hirzebruch surfaces. Let $\widetilde{S}', \widetilde{S}''$ be Hirzebruch surfaces of degree e. Define curves $\widetilde{C}_i' \subset \widetilde{S}'$ and $\widetilde{C}_i'' \subset \widetilde{S}''$ as above.

Let $p : \widetilde{\mathfrak{Z}} \to \Delta$ be a flat family whose general fibres $\widetilde{\mathfrak{Z}}_t$, $t \neq 0$, are Hirzebruch surfaces of degree e; the closed fiber $\widetilde{\mathfrak{Z}}_0$ is the union of $\widetilde{S}'$ and $\widetilde{S}''$ with $\widetilde{C}_2'$ and $\widetilde{C}_0''$ identified, analogously to the left-half of Figure 4. Such a family can be constructed either by toric methods or as follows. Let $L \to \mathbb{P}^1$ be the $\mathbb{C}^*$-bundle of degree e. Let $\mathfrak{C} \to \Delta$ be a versal deformation of the nodal rational curve $xy = 0$ in $\mathbb{P}^2$. The action $t \cdot (z, w) = (tz, t^{-1}w)$ of $\mathbb{C}^*$ on $\mathfrak{C}_0$ extends to an action on $\mathfrak{C}$ over Δ. We may take $\widetilde{\mathfrak{Z}} = (L \times \mathfrak{C})/\mathbb{C}^*$ with $\mathbb{C}^*$ acting diagonally.

The points $\widetilde{C}'_0 \cap \widetilde{C}'_1$ and $\widetilde{C}''_2 \cap \widetilde{C}''_3$ on $\widetilde{\mathfrak{Z}}_0$ can be lifted to sections of $\widetilde{\mathfrak{Z}} \to \Delta$. Let $\mathfrak{Z} \to \widetilde{\mathfrak{Z}}$ be the blowing-up of these sections. The strict transform of $\widetilde{C}'_0 \cup \widetilde{C}''_2$ is a Cartier divisor on $\mathfrak{Z}_0$. It can be extended to an effective Cartier divisor on $\mathfrak{Z}$, whose associated line bundle we denote by $\mathcal{L}$. Consider the factorization $\mathfrak{Z} \to \Delta \times \mathbb{P}^1$ of $\mathfrak{Z} \to \Delta$. The relative base locus of $\mathcal{L}$ over $\Delta \times \mathbb{P}^1$ is obviously finite. According to the Fujita-Zariski Theorem ([9], Thm. 1.10) the corresponding invertible $\mathcal{O}_{\mathfrak{Z}}$-module $\mathcal{L}$ is relatively semiample over $\mathbb{P}^1 \times \Delta$. Moreover, $\mathcal{L}$ is relatively ample over $\mathbb{P}^1 \times \Delta^*$. Let $\mathfrak{Z} \to \mathfrak{S}$ be the corresponding contraction. The exceptional locus of this contraction is the strict transform of $\widetilde{C}'_1 \cup \widetilde{C}''_3$.

Now $\mathfrak{S} \to \Delta$ has as general fiber the blowing up of a Hirzebruch surface of degree e in two points as needed for the construction of d-semistable Kodaira surfaces with rational normalization. The central fiber consists of a union of two Hirzebruch surfaces of degrees $e+1$ and $e-1$ respectively, with a section of degree $e+1$ glued to a section of degree $-(e-1)$. So this is a partial normalization of X_2. The gluing morphism $\varphi : C \to D$ on the special fibre $\mathfrak{S}_0 = S$ extends to a gluing morphism over Δ. The corresponding cocartesian diagram gives the desired flat family $\mathfrak{X} \to \Delta$ with $\mathfrak{X}_0 = X_2$.

(ii) For the elliptic case consider the partial normalization Y of X_2 obtained from S by identifying C'_1 with C''_3 and C'_3 with C''_1, see the right half of Figure 4. It is the projective closure of a line bundle over the Néron 2-gon. In this picture the two disjoint sections $C_0 = C'_0 \cup C''_0$, $C_2 = C'_2 \cup C''_2$ are the zero section and the section at infinity. Let $\mathfrak{B} \to \Delta$ be a smoothing of B. Since the line bundle defining Y extends to $\mathfrak{B}$ there exists an extension $\mathfrak{Y} \to \mathfrak{B}$ of $Y \to B$ with disjoint sections $\mathfrak{C}_0, \mathfrak{C}_2 \subset \mathfrak{Y}$. The gluing $C_0 \to C_2$ is given by an automorphism of B of order 2. This automorphism can be extended to an automorphism of $\mathfrak{B}/\Delta$ of the same order. An appropriate identification of the sections now yields the desired deformation of X_2.   □

References

[1] V. Alexeev: Complete moduli in the presence of semiabelian group action. preprint 1999

[2] A. Ash, D. Mumford, M. Rapoport, Y. Tai: Smooth compactifications of locally symmetric varieties. Math Sci Press, Brookline, 1975.

[3] W. Barth, C. Peters, A. Van de Ven: Compact complex surfaces. Ergeb. Math. Grenzgebiete (3) 4, Springer, Berlin etc., 1984.

[4] C. Borcea: Moduli for Kodaira surfaces. Compos. Math. 52, 373–380 (1984).

[5] P. Deligne, M. Rapoport: Les schémas de modules de courbes elliptiques. In: P. Deligne, W. Kuyk (eds.), Modular Functions of one Variable II, pp. 143–316. Lect. Notes Math. 349. Springer, Berlin etc., 1973.

[6] R. Friedman: Global smoothings of varieties with normal crossings. Ann. Math. (2) 118, 75–114 (1983).

[7] R. Friedman, N. Shepherd-Barron: Degenerations of Kodaira surfaces. In: R. Friedman, D. Morrison (eds.), The birational geometry of degenerations, pp. 261–275. Prog. Math. 29. Birkhäuser, Boston etc, 1983.

[8] A. Fujiki: On the Douady space of a compact complex space in the category $\mathcal{C}$. II. Publ. Res. Inst. Math. Sci. 20, 461–489 (1984).

[9] T. Fujita: Semipositive line bundles. J. Fac. Sci. Univ. Tokyo 30, 353–378 (1983).

[10] M. Kato: Topology of Hopf surfaces. J. Math. Soc. Japan 27, 222–238 (1975).

[11] M. Kato: Erratum to "Topology of Hopf surfaces". J. Math. Soc. Japan 41, 173–174 (1989).

[12] K. Kodaira: On the structure of compact complex analytic surfaces II, III. Am. J. Math. 88, 682–721 (1966); 90, 55–83 (1968).

[13] V. Kulikov: Degenerations of K_3 surfaces and Enriques surfaces. Math. USSR, Izv. 11, 957–989 (1977).

[14] H. Lange, C. Birkenhake: Complex abelian varieties. Grundlehren Math. Wiss. 302. Springer, Berlin etc., 1992.

[15] D. Mumford: Abelian varieties. Tata Institute of Fundamental Research Studies in Mathematics 5. Oxford University Press, London, 1970.

[16] I. Nakamura: On moduli of stable quasi abelian varieties. Nagoya Math. J. 58, 149–214 (1975).

[17] S. Nakano: On the inverse of monoidal transformation. Publ. Res. Inst. Math. Sci., Kyoto Univ. 6, 483–502 (1971).

[18] K. Nishiguchi: Canonical bundles of analytic surfaces of class VII_0. In: H. Hijikata, H. Hironaka et al. (eds.), Algebraic geometry and commutative algebra II, pp. 433–452. Academic Press, Tokyo, 1988.

[19] V. Palamodov: Deformations of complex spaces. Russ. Math. Surveys 31:3, 129–197 (1976).

[20] U. Persson: On degenerations of algebraic surfaces. Mem. Am. Math. Soc. 189 (1977).

[21] U. Persson, H. Pinkham: Degeneration of surfaces with trivial canonical bundle. Ann. Math. (2) 113, 45–66 (1981).

[22] S. Sternberg: Local contractions and a theorem of Poincaré. Amer. J. Math. 79, 809–824 (1957).

[23] J. Wehler: Versal deformation of Hopf surfaces. J. Reine Angew. Math. 328, 22–32 (1981).

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