INTERCHANGING HOMOTOPY LIMITS AND COLIMITS IN CAT

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ABSTRACT. Let I,J be small categories and $C : I \times J \longrightarrow \mathrm{CAT}$ a functor to the category of small categories. We show that if I has a final object then the canonical map $\mathrm{hocolim}_J \,\mathrm{holim}_I\, C \longrightarrow \mathrm{holim}_I \,\mathrm{hocolim}_J\, C$ is a strong homotopy equivalence.

Let $C : I \longrightarrow \mathbf{CAT}$ be a functor going from a small category I to the large category of all small categories. By the *homotopy colimit* of C we mean the Grothendieck construction ([T]):

$$\operatorname*{hocolim}_I C := \int_I C$$

and by the *homotopy limit* we mean the pullback:

$$\begin{array}{ccc} \mathrm{holim}_I\, C & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow{\scriptstyle id} \\ \mathrm{HOM}(I, \mathrm{hocolim}_I\, C) & \xrightarrow[\pi_*]{} & \mathrm{HOM}(I, I) \end{array}$$

Here HOM is the functor category, π_* is induced by the natural projection $\pi : \mathrm{hocolim}_I\, C \longrightarrow I$, 0 is the category with only one map and *id* maps the only object of 0 to the identity functor. A reason for using the above definitions is that taking nerves one recovers the usual homotopy (co)limits for simplicial sets, up to homotopy in the case of hocolim ([T]) and up to isomorphism in the case of holim ([L]). Before we state the main result of this paper, we need a definition.

Definition. A *pseudo final* object in a category I is a an object $e \in I$ together with a natural map $\epsilon : 1 \longrightarrow e$ going from the identity to the constant functor.

Theorem 1. *Let I and J be small categories and let $C : I \times J \longrightarrow \mathrm{CAT}$ be a functor. Then there is a* faithful *functor:*

$$\iota : \operatorname*{hocolim}_J \operatorname*{holim}_I C \to \operatorname*{holim}_I \operatorname*{hocolim}_J C$$

which is natural in all variables involved. If in addition the category I has a pseudo-final object, then there also exists a functor:

$$p : \operatorname*{holim}_I \operatorname*{hocolim}_J C \to \operatorname*{hocolim}_J \operatorname*{holim}_I C$$

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such that $p\iota = 1$ *and a natural map* $1 \to \iota p$. *The functor* p *is natural in* J *and* C *and for functors* $I \to I'$ *which preserve* e *and* ϵ. *In particular, under this assumption, the functor* ι *is a homotopy equivalence.*

Before we prove the theorem we explain our choice of notation and recall a description of $\operatorname{holim}_I C$ in terms of objects and arrows. We use a somewhat non-standard notation for functors $I \to \mathrm{CAT}$. On objects we use subscripts instead of parenthesis; thus C_i (and not $C(i)$) means the value of C at the object $i \in I$. Also, we omit the letter C on arrows, so we use the same letter (α, β, γ,...) for a map in I and for its image through the functor C. As we use i, j, k, ... for objects in I and x_i, y_i, z_i, ... for objects of C_i, this should arise no confusion. For example $\alpha : i \to j$ is a map in I while αx_i is the object of C_j obtained by applying the image of α through C to the object $x_i \in C_i$; i.e. αx_i really means $C(\alpha)(x_i)$. If $\rho : x_i \to y_i$ is a map in C_i and $\alpha : i \to j$ is in I, we write $\alpha(\rho)$ for $C(\alpha)(\rho)$.

Here is a description of both $\operatorname{hocolim}_I C$ and $\operatorname{holim}_I C$ in terms of objects and arrows. An object of $\operatorname{hocolim}_I C$ is a pair $x_i := (i, x)$ where $i \in I$ and $x \in C_i$. A map $x_i \to y_j$ is a pair (α, ρ) with $\alpha : i \to j \in I$ and $\rho : \alpha x \to y \in C_j$. Composition is defined as in a semidirect product: $(\alpha, \rho)(\beta, \mu) = (\alpha\beta, \alpha(\rho)\mu)$. On the other hand, $\operatorname{holim}_I C$ is the category of all pairs of families $(x, \rho) := (\{x_i\}_{i\in I}, \{\rho_\alpha\}_{\alpha\in I})$, indexed respectively by the objects and the maps of I, where $x_i \in C_i$, and if $\alpha : i \to j$ is a map in I, then $\rho_\alpha : \alpha x_i \to x_j$ is a map in C_j. The family $\{\rho_\alpha\}_{\alpha\in I}$ is subject to the conditions:

$$\rho_1 = 1 \qquad\qquad \rho_{\alpha\beta} = \rho_\alpha \alpha(\rho_\beta) \tag{1}$$

In the first equality, the 1 on the left is an identity map $1 : i \to i$ and the 1 on the right is the identity of x_i in C_i; in the second equality, $i_0 \xleftarrow{\alpha} i_1 \xleftarrow{\beta} i_2$ are composable maps in I and it should be interpreted as an equality in the set of maps $\alpha\beta x_{i_2} \to x_{i_0}$ in C_{i_0}. A map $f : (x, \rho^x) \to (y, \rho^y)$ in $\operatorname{holim}_I C$ is a family of maps $f_i : x_i \to y_i \in C_i$ indexed by the objects of I such that the following diagram commutes for every map $\alpha : i \to j \in I$:

$$\begin{tikzcd} \alpha x_i \arrow[r, "f_i"] \arrow[d, "\rho^x"'] & \alpha y_i \arrow[d, "\rho^y"] \\ x_j \arrow[r, "f_j"'] & y_j \end{tikzcd} \tag{2}$$

Proof of Theorem 1. To begin with, we write down what each of the categories involved is. An object of $\operatorname{hocolim}_I \operatorname{holim}_J C$ is a pair (j, x) where $j \in J$ is an object and $x = \{(x_{ij}, {\rho_\alpha}^x) : i, \alpha \in I\}$ is a family of objects $x_{ij} \in C_{ij}$, one for each $i \in I$, together with a family of maps $\rho_\alpha : \alpha x_{ij} \to x_{i',j} \in C_{i'j}$, one for each map $\alpha : i \to i' \in I$. (Hereafter we shall write $x = \{(x_{ij}, {\rho_\alpha}^x\})$, omitting the specification of where the indexes lie.) The ρ_α satisfy condition (1) for maps in I. A map between an object (j, x) and an object (k, y) is a pair (β, f) where $\beta : j \to k \in J$ and $f = \{f_i\}$ is a family of maps $f_i \beta x_{ij} \to y_{ik}$ such that diagram (2) commutes. Maps are composed by $(\gamma, g)(\beta, f) = (\gamma\beta, \{g_i\gamma f_i : i \in I\})$. An object of the category $\operatorname{holim}_I \operatorname{hocolim}_J C$ is a family of pairs $(j(i), x_{ij(i)})$ $(i \in I)$ where $j(i) \in J$ and $x_{ij(i)} \in C_{ij(i)}$, together with a family of pairs $(\beta_\alpha, \rho_\alpha) = (\beta^x_\alpha, \rho^x_\alpha)$, $(\alpha : i \to i' \in I)$ where $\beta_\alpha : j(i) \to j(i') \in J$ and $\rho_\alpha : \alpha\beta_\alpha x_{ij(i)} \to x_{i',j(i')} \in C_{i',j(i')}$ are maps, and ρ_1 and β_1 are identity maps. The β satisfy $\beta_{\alpha'\alpha} = \beta_{\alpha'}\beta_\alpha$ and the ρ satisfy (1). A map between the object $x = ((j(i), x_{ij(i)}), (\beta_\alpha, \rho^x_\alpha))$ and the object $y = ((k(i), y_{i,k(i)}), (\beta, \rho^y_\alpha))$ is a family of pairs (β_i, f_i) $(i \in I)$, where $\beta_i : j(i) \to k(i) \in J$ and $f_i : \beta_i x_{ij(i)} \to y_{ik(i)} \in C_{ik(i)}$ are maps. The (β_i, f_i) satisfy ${\beta_\alpha}^y\beta_i = \beta_{i'}{\beta_\alpha}^x$ and $\rho^y_\alpha{\beta_\alpha}^y\alpha(f_i) = f_{i'}\beta_{i'}(\rho^x_\alpha)$ for each map $\alpha : i \to i' \in I$. Composition is defined as $(\gamma, g)(\beta, f) = \{(\gamma_i\beta_i, g_i\gamma_i(f_i))\}$. We define ι as sending the object (j, x) to the object $(\{(j, x_{ij})\}, \{(1_j, \rho^x_\alpha)\})$ and the map (β, f) to the map $(\beta, \{f_i\})$. It is clear from the definition that ι is a functor that maps $\operatorname{hocolim}_J \operatorname{holim}_I C$ faithfully into $\operatorname{holim}_I \operatorname{hocolim}_J C$. Moreover the functor ι identifies $\operatorname{hocolim}_J \operatorname{holim}_I C$ with the subcategory of $\operatorname{holim}_I \operatorname{hocolim}_J C$ of those objects x as above which have constant $j = j(i)$ and constant $\beta_\alpha = 1_j$, and with arrows the families $\{(\beta_i, f_i)\}$ with $\beta_i = \beta$, a constant map. Now assume I has a pseudo-final object $\epsilon : 1 \to e$. Define p as $p(\{(j(i), x_{ij(i)})\}, \{(\beta_\alpha, \rho_\alpha)\}) = (\{(j(e), \beta_{\epsilon_i}x_{ij(i)})\}, \{(1_{j(e)}, \beta_{\epsilon_i}(\rho_\alpha))\})$ on objects and by $p\{(\beta_i, f_i)\} = (\beta_e, \{\beta_{\epsilon_i}(f_i)\})$ on arrows. It is tedious but straightforward to verify that p is a functor from $\operatorname{holim}_I \operatorname{hocolim}_J C$ to $\operatorname{hocolim}_J \operatorname{holim}_I C$. Once this is verified, it is clear that $p\iota = 1$. The natural map $\theta : 1 \to \iota p$ is defined as follows. Given an object $x \in \operatorname{holim}_I \operatorname{hocolim}_J C$, define $\theta(x)_i = \{(\beta^x_{\epsilon_i}, 1_{\beta_{\epsilon_i}x_{ij(i)}})\} : \{(j(i), x_{ij(i)})\} \to \{(j(e), {\beta_{\epsilon_i}}^x x_{ij(i)})\}$. Another tedious but straightforward verification shows that θ is a natural map. $\square$

Warning. Write NC for the nerve of C. Note that the theorem does not imply that $\operatorname{hocolim}_J \operatorname{holim}_I NC \approx \operatorname{holim}_I \operatorname{hocolim}_J NC$. This is because, unlike holim, hocolim commutes with nerves only up to weak equivalence, not isomorphism, and holim, unlike hocolim, does not preserve all weak equivalences, only those between fibrant simplicial sets. Theorem 1.2 may still be true for simplicial sets but it certainly cannot be derived in this way. Also note that, even when C is fibrant, $\operatorname{hocolim}_J C$ need not be so, and thus the spectral sequence for holim may converge to the wrong homotopy type. The following example illustrates these pathologies.

Tricky Example. Fix a prime number p. Let $C : \mathbb{N}^{op} \times \mathbb{N} \to \mathrm{CAT}$ be given by the following diagram:

$$
\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
(\mathbb{Z}/p^3)_\delta & \xrightarrow{\ p\ } & (\mathbb{Z}/p^3)_\delta & \xrightarrow{\ p\ } & (\mathbb{Z}/p^3)_\delta & \xrightarrow{\ p\ } & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
(\mathbb{Z}/p^2)_\delta & \xrightarrow{\ p\ } & (\mathbb{Z}/p^2)_\delta & \xrightarrow{\ p\ } & (\mathbb{Z}/p^2)_\delta & \xrightarrow{\ p\ } & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
(\mathbb{Z}/p)_\delta & \xrightarrow{\ p\ } & (\mathbb{Z}/p)_\delta & \xrightarrow{\ p\ } & (\mathbb{Z}/p)_\delta & \xrightarrow{\ p\ } & \dots
\end{array}
$$

Here $(\mathbb{Z}/p^m)_\delta$ is the discrete category, the vertical maps are the natural projections, and p means 'multiply by p'. One checks that $\operatorname{hocolim}_{\mathbb{N}} \operatorname{holim}_{\mathbb{N}^{op}} C\delta \approx \operatorname{holim}_{\mathbb{N}^{op}} \operatorname{hocolim}_{\mathbb{N}}(C)_\delta \approx (\mathbb{Q}_p)_\delta$, the discrete category of the p-adic field. However the hocolim of each of the rows has the weak homotopy type of a point, by [BK] and [T]. Note this is not in contradiction with the fact that holim preserves weak equivalences of fibrant simplicial sets ([BK]) nor the fact that it preserves adjoint functors ([L]). Indeed the category $L_m := \operatorname{hocolim}_{\mathbb{N}} C_{m,-}$ has $\mathbb{N} \times \mathbb{Z}/p^m$ as set of objects, and $\hom((n_0, a_m), (n_1, b_m)) = \{*\}$ if both $n_0 \leq n_1$ and $p^{n_1 - n_0} a_m = b_m$ and the empty set otherwise. Thus NL_m is not fibrant, because not every map in L_m is an isomorphism, and it does not have initial or final object, i.e., the map $L_m \to 0$ does not have an adjoint.

References

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