Non-linear connections on phase space and the Lorentz force law

John H. van Drie
Meridien Research
Kalamazoo, MI USA
www.johnvandrie.com

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Abstract

The equations of parallel transport for a non-linear connection on phase space are examined. It is shown that, for a free-particle Lagrangian, the connection term first-order in momentum reproduces the geodesic equation of General Relativity and the term zeroth-order in the momentum reproduces the Lorentz electromagnetic force law. Hence from one mathematical expression, a non-linear parallel transport equation, one can derive the interaction laws for both the gravitational and electromagnetic forces. These equations are free of the difficulties associated with formalism of Weyl, which forms the basis for the theory of Yang and Mills.

1 Introduction

After Einstein's description of the gravitational force as a manifestations of the curvature of space, numerous methods were proposed for a similar, geometric explanation of the Lorentz force law [1-4]. Weyl's proposal [1] is especially significant, as it is the basis for the field equations of Yang-Mills [5], the basis of modern field theory. Wu and Yang [6] have noted, however, that the Weyl/Yang-Mills approach leads to certain mathematical difficulties, when viewed in its original differential-geometric context. Presented here is another differential-geometric method for deriving the Lorentz electromagnetic force law which is free of these difficulties. The method described here develops a differential-geometric formalism using relativistic phase space as the underlying structure (unlike General Relativity and Weyl's theory, which use spacetime and its tangent space as the underlying structure). This method provides a conceptually clearer understanding of the relationship between the physical quantity of Newtonian **force**, and the mathematical quantity of **connection**. The connection under consideration is a more general type of connection than that of Riemannian geometry, the mathematical basis of General Relativity. Rather than considering *geodesics*, i.e. paths of minimum distance (the distance determined by the metric), the method here uses the equations of *parallel transport*, which under appropriate constraints reduce to the geodesic equation. The relaxation of these constraints gives rise to new mathematical terms, one of which it is shown may be interpreted physically as the electromagnetic field.

2 The geometric structure of phase space

Phase space is the natural setting for studying the dynamics of an N-particle system. The purpose of this section is to describe the *geometric* structure of phase space, and establish in a general way the relation between the mathematical, geometric quantity "connection" with the physical quantity "force". Given 4-dimensional spacetime and a particle whose position at a point in proper time τ may be characterized by the coordinates $\{x^\mu(\tau)\}_{\mu=0}^3$, and a Lagrangian $L(x^\mu, \frac{dx^\mu}{d\tau}, \tau)$ describing this physical system, one defines momenta as

$$\begin{aligned} p_i &= \frac{\partial L}{\partial \dot{x}^i} \\ p_0 &= \sum_i p_i \dot{x}^i - L \end{aligned}$$

where i ranges over the 3 spatial coordinates, 0 denotes the time coordinate, and $\dot{x}^i = \frac{dx^i}{dx^0}$. One constructs the phase space Π , such that each point in Π is described by coordinates (x^μ, p_μ). We will consider two infinitesimally displaced points in spacetime, $\mathbf{x}$ and $\mathbf{x}'$, and will define Π_x as the subspace of Π representing all possible momenta the particle may take at a point $\mathbf{x}$. If, at a time τ, the particle is at point $\mathbf{x}$, we may associate with it some momentum, i.e. some element $p \in \Pi_x$. Similarly, if at a time $\tau + d\tau$, the particle is at $\mathbf{x}'$, we may associate it with some $p' \in \Pi_{x'}$. In the classical Newtonian view, the change in momentum over this interval of time $d\tau$ represents the forces acting on the particle. However, viewing this same situation geometrically, we see that a subtle point has been overlooked in this classical Newtonian view:

> there is no way *a priori* to associate some element $p \in \Pi_x$ with some $p' \in \Pi_{x'}$, since $\mathbf{x}$ and **x'** are infinitesimally displaced.

We must first parallel transport p' *back to* Π_x, *and then compute the difference between it and* $p \in \Pi_x$. Thus, *we must define a* **connection** *before we can construct the rate of change of momentum representing the Newtonian force.* How does one choose a connection? Newtonian mechanics is usually based on a Euclidean connection, with forces described by terms in the Lagrangian. Instead, we will consider only free-particle Lagrangians, and construct connections such that the parallel-transported change in momentum is zero along a particle's path. In other words, rather than ascribing differences between p and p' to an external force (i.e. forces cause deviations from Euclidean straightness), we will ascribe the difference between p and p' to the parallel transport of $p \in \Pi_x$ to $p' \in \Pi_{x'}$. Loosely speaking, *we shall describe "forces" in terms of connections*. Conceptually, this is identical to Einstein's programme in the General Theory of Relativity. This identification of force with connection allows one to identify stress-energy with curvature, where mathematical identities on the curvature automatically produce the necessary conservation theorems on stress-energy [7]. However, the approach described here generalizes Einstein's approach, by going beyond the affine connections derived from a metric in Riemannian geometry, to the most general expression for a non-linear connection on phase space. Whereas in Riemannian geometry, the metric is the fundamental quantity, from which quantities such as the connection and curvature are derived, these connections we consider are the most fundamental quantity from which other geometric quantities such as curvature are derived [8]. Such connections may not in general be derived from a metric. By adopting this more general mathematical framework, we can "geometrize" a wider variety of forces, notably the electromagnetic force.

3 Linear connections on phase space

An arbitrary connection on phase space may be defined by a Pfaffian ideal of 1-forms θ_μ [9],

$$\theta_\mu = dp_\mu - f_{\mu\nu}(x,p)dx^\nu \tag{1}$$

where the $f_{\mu\nu}$ are arbitrary functions of position $\mathbf{x}$ and momentum p. (We use here a pseudo-Euclidean metric $g_{\mu\nu} = \eta_{\mu\nu}, \eta_{00} = -1, \eta_{11} = \eta_{22} = \eta_{33} = 1$, 0 otherwise). We shall first consider the homogeneous-linear connection $f_{\mu\nu}(x,p) = h_{\mu\nu\alpha}(x)p^\alpha$, where $h_{\mu\nu\alpha}$ are arbitrary functions of position. Defining the connection 1-forms as

$$\omega_{\mu\alpha} = h_{\mu\nu\alpha}(x)dx^\nu$$

(equivalent to Cartan's connection 1-forms), equation 1 becomes

$$\theta_\mu = dp_\mu - \omega_{\mu\alpha}p^\alpha \tag{2}$$

for this homogeneous-linear connection. We can now calculate explicitly the equations for the curves defined by this Pfaffian ideal. Denoting by $\mathbf{u}$ the unit tangent vector to these curves, we know that $\mathbf{u}\rfloor\theta_\mu = 0$ for all μ, where $\rfloor$ denotes contraction of a vector with a 1-form. Equation 2 then becomes

$$\mathbf{u}\rfloor\theta_\mu = 0 = \mathbf{u}\rfloor dp_\mu - \mathbf{u}\rfloor(\omega_{\mu\alpha}p^\alpha)$$

.

$$0 = u^\nu \frac{\partial p_\mu}{\partial x^\nu} - u^\nu(h_{\mu\nu\alpha}p^\alpha)$$

.

$$0 = \frac{dp_\mu}{d\tau} - h_{\mu\nu\alpha}p^\alpha u^\nu \tag{3}$$

where τ is a parametrization of the curve, and where $\frac{dp_\mu}{d\tau}$ denotes the derivative along the curve, and where we have taken

$$\frac{dp_\mu}{d\tau} = \mathbf{u} \rfloor dp_\mu = u^\alpha \frac{\partial p_\mu}{\partial x^\alpha}$$

Assuming a free point-particle Lagrangian, with mass m (a scalar constant), for which [10],

$$p_\mu = m u_\mu$$

equation 3 becomes

$$0 = m \frac{du_\mu}{d\tau} - m h_{\mu\nu\alpha} u^\alpha u^\nu,$$

or,

$$\frac{du_\mu}{d\tau} - h_{\mu\nu\alpha} u^\alpha u^\nu = 0. \tag{4}$$

If one associates the coefficient $h_{\mu\nu\alpha}$ for our homogeneous- linear connection with $\Gamma_{\mu\nu\alpha}$, the affine connection of Riemannian geometry,

$$h_{\mu\nu\alpha} = -\Gamma_{\mu\nu\alpha}$$

equation 4 becomes the "geodesic equation" for a Riemannian space [11],

$$\frac{du_\mu}{d\tau} + \Gamma_{\mu\nu\alpha} u^\alpha u^\nu = 0.$$

Einstein demonstrated that this equation describes the forces of gravitation, once the curvature of space is related to stress energy, $T_{\mu\nu}$[12]:

$$T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

where

$$R = g^{\mu\nu} R_{\mu\nu},$$

and

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}.$$

where $,\alpha$ denotes derivation, i.e.

$$\Gamma^\alpha_{\mu\alpha,\nu} = \frac{\partial \Gamma^\alpha_{\mu\alpha}}{\partial x^\nu}.$$

4 Non-linear connections on phase space

4.1 Derivation of parallel transport equation on phase space

We now return to equation 1, solutions to which describe the paths of parallel transport in phase space, to relax the assumption of homogeneous-linearity which led to equation 2. Now we shall consider connections of the form

$$f_{\mu\nu}(p,x) = \overset{(0)}{h}_{\mu\nu}(x) + \overset{(1)}{h}_{\mu\nu\alpha}(x) p^\alpha$$

where the $\overset{(k)}{h}_{\mu\nu\alpha_1\ldots\alpha_k}(x)$ are functions only of x. Defining the set of connection 1-forms

$$\overset{(0)}{\omega}_{\mu} = \overset{(0)}{h}_{\mu\nu}\, dx^{\nu}$$

.

$$\overset{(1)}{\omega}_{\mu\alpha} = \overset{(1)}{h}_{\mu\nu\alpha}\, dx^{\nu}$$

equation 1 becomes

$$\theta_{\mu} = dp_{\mu} - \overset{(0)}{\omega}_{\mu} - \overset{(1)}{\omega}_{\mu\alpha}\, p^{\alpha}$$

As before, we explicitly calculate the equations for the curves which annul this Pfaffian ideal, by contracting the unit tangent vector **u** with all of the 1-forms θ_{μ}:

$$\mathbf{u}\rfloor\theta_{\mu} = 0$$

.

$$0 = \mathbf{u}\rfloor dp_{\mu} - \mathbf{u}\rfloor \overset{(0)}{\omega}_{\mu} - \mathbf{u}\rfloor \overset{(1)}{\omega}_{\mu\alpha}\, p^{\alpha}$$

Recalling our earlier use of the derivative along the curve, and explicitly writing the contraction, this becomes

$$0 = \frac{dp_{\mu}}{d\tau} - u^{\gamma}\, \overset{(0)}{h}_{\mu\gamma} - u^{\gamma}\, \overset{(1)}{h}_{\mu\gamma\alpha}\, p^{\alpha}$$

Assuming again a point-particle Lagrangian, $p_{\mu} = mu_{\mu}$, this becomes

$$0 = m\frac{du_{\mu}}{d\tau} - u^{\gamma}\, \overset{(0)}{h}_{\mu\gamma} - mu^{\gamma}\, \overset{(1)}{h}_{\mu\gamma\alpha}\, u^{\alpha}$$

or, in a form where the mass-dependence of the different terms is explicit,

$$0 = \frac{du_{\mu}}{d\tau} - \frac{1}{m}\, \overset{(0)}{h}_{\mu\gamma}\, u^{\gamma} - \overset{(1)}{h}_{\mu\gamma\alpha}\, u^{\gamma}u^{\alpha} \tag{5}$$

Notice that, expressed in this form, *the equivalence principle is manifest.* Only one term may be without a mass term, the first-order term. Mathematically, this must describe paths which particles will follow independent of their mass, which physically corresponds to gravitation, as Einstein's General theory shows. Other geometrically-derived forces, the zeroth-order term, and 2nd-order terms and higher, correspond to forces whose effects will be dependent on the mass of the particle.

4.2 Physical interpretation of the lowest order term of this parallel transport equation

In order to interpret equation 5, we will associate the connection term $\overset{(0)}{h}_{\mu\gamma}$, which is independent of momentum but a function of position, with the Faraday tensor of electromagnetism [13],

$$F_{\mu\gamma} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_x \\ -E_y & -B_z & 0 & B_y \\ -E_z & B_x & -B_y & 0 \end{pmatrix}$$

where $\mathbf{E} = (E_x, E_y, E_z)$ is the electric-field spatial 3-vector, and $\mathbf{B} = (B_x, B_y, B_z)$ is the magnetic-field spatial 3-vector. Combining

$$\overset{(0)}{h}_{\mu\gamma} = eF_{\mu\gamma}$$

where e is the electric charge, with the definition of $\mathbf{F}$ and equation 5, and setting the first and second order terms, $\overset{(1)}{h}_{\mu\gamma\alpha}$ and $\overset{(2)}{h}_{\mu\gamma\alpha\beta}$ to zero, we are led to

$$\frac{d}{dt}(m\mathbf{v}) = e(\mathbf{E} - \mathbf{v} \times \mathbf{B}) \tag{6}$$

where $\mathbf{v}$ is the velocity spatial 3-vector, and $\frac{d}{dt}$ denotes the derivative with respect to coordinate time. This equation expresses the time-rate-of-change of the Newtonian momentum, $m\mathbf{v}$, to a term which has precisely the form of the Lorentz force law. Hence by starting with equation 5, which was derived mathematically from the expression for a non-linear connection on phase space (equation 1), we are led to a physical equation, the Lorentz force law, whose validity is derived empirically. Therefore, this development may be interpreted as a geometric derivation of the Lorentz force law. If we repeat this derivation, one sees that equation 5 describes the path of a particle in a Riemannian connection $\Gamma_{\mu\nu\alpha} = -\overset{(1)}{h}_{\mu\gamma\alpha}$ in the presence of an electromagnetic field [14].

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = \frac{e}{m} F^\alpha_\beta u^\beta$$

4.3 Other aspects of the physical interpretation of the non-linear parallel transport law

We have shown how the classical electrodynamic interaction law may be derived from the mathematics of non-linear connections on phase space. A proper understanding of the additional terms present in this expression would require a quantum-mechanical formulation of this classical law. Recall that the quantum-mechanically, the electromagnetic interaction is introduced by applying the "minimal substitution" $\mathbf{p} \to \mathbf{p} - e\mathbf{A}$ to the path equations in the absence of an electromagnetic field[15], where $\mathbf{A}$ is the vector potential, whose exterior derivative is the Faraday tensor $\mathbf{F} = \mathbf{dA}$[16]. However, in our derivation via the non-linear parallel transport law, we have lost the simple understanding of the vector potential. Hence, a straightforward application of equation 5 to quantum mechanics cannot be made. Possibilities for making the application to quantum mechanics, by looking for the equations of parallel transport of $|\xi\rangle$ which provide the correct equations of parallel transport for $p_\mu = -i\hbar\langle\xi|\frac{\partial}{\partial x^\mu}|\xi\rangle$, where ξ is the 4-component spinor of the Dirac equation, will be examined separately. One important qualitative feature of a quantum-mechanical theory of the electromagnetic interaction based on this approach is already apparent: we make the following associations between mathematical and physical quantities

$$\begin{array}{cc}
\textbf{connection} & \textbf{force} \\
\downarrow d & \downarrow d \\
\textbf{curvature} & \textbf{sources} \\
\downarrow d & \downarrow d \\
0 & 0 \\
(Bianchi\ identity) & (conservation\ of\ energy)
\end{array}$$

exactly as can be made in Einstein's General theory[7]. This is in contrast to the geometric derivation of the electromagnetic interaction law devised by Weyl [1], as has been noted by Wu and Yang [6] (this table is taken from their Table I):

$$\begin{array}{cc}
\textbf{connection} & \textbf{gauge potential} \\
\downarrow d & \downarrow d \\
\textbf{curvature} & \textbf{field strength} \\
\downarrow & \downarrow d \\
? & \textbf{sources}
\end{array}$$

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6 References

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[10] Ref. 7, p. 201.
[11] *ibid.*, p. 224.
[12] *ibid.*
[13] *ibid.*, p. 73.
[14] *ibid.*, eq. 20.41.
[15] R. P. Feynman, *Quantum Electrodynamics*, Reading, MA: W. A. Benjamin, 1962, p. 4.
[16] Ref. 7, p. 569. Note that the minimal substitution works classically as well: another way to "derive" equation (20.41) in [7] is to take the usual geodesic equation, equation 4, and apply the "minimal substitution", $\mathbf{p} \rightarrow \mathbf{p} - e\mathbf{A}$, or equivalently $\mathbf{dp} \rightarrow \mathbf{dp} - e\mathbf{F}$.