

n -Dimensional Bateman Equation

and

Painlevé Analysis of Wave Equations

by

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Abstract: In the Painlevé analysis of nonintegrable partial differential equations one obtains differential constraints describing the movable singularity manifold. We show that, for a class of n -dimensional wave equations, these constraints have a general structure which is related to the n -dimensional Bateman equation. In particular, we derive the expressions of the singularity manifold constraint for the n -dimensional sine-Gordon -, Liouville -, Mikhailov -, and double sine-Gordon equation, as well as two 2-dimensional polynomial field theory equations, and prove that their singularity manifold conditions are satisfied by the n -dimensional Bateman equation. Finally we give some examples.

1 Introduction

The Painlevé analysis, as a test for integrability of partial differential equations (PDEs), was proposed by Weiss, Tabor and Carnevale in 1983 [26]. It is a generalization of the singular point analysis for ordinary differential equations (ODEs), which dates back to the work of Sofia Kovalevskaya of 1889 [11]. She studied the Euler-Poisson equations in the complex domain and found conditions under which the only movable singularities exhibited by the solutions were ordinary poles, leading to her discovery of a new first integral. In the late nineteenth century Paul Painlevé completely classified first order ODEs [17], as well as a large class of second order ODEs [18, 19], on the basis that the only movable singularities their solutions exhibit, are ordinary poles. This special property is today known as the Painlevé property (see, for example [4, 12, 20]). We also say that an ODE is of *Painlevé type*, by which we mean that it belongs to the class of equations in Painlevé's classification, or that it can be transformed to one of the equations in that class; therefore an ODE which has the Painlevé property. The list of ODEs, classified by Painlevé, is given in the book of Davis [5].

We consider a PDE to be integrable if it can be solved by an inverse scattering transform (we refer to the book [1], and references therein). A PDE which is integrable possess the Painlevé property, which means that its solutions are single-valued in the neighbourhood of non-characteristic movable singularity manifolds [1, 15, 21]. In this sense the method described by Weiss, Tabor and Carnevale [26] proposes a necessary condition of integrability, also known as the Painlevé test, which is analogous to the algorithm for ODEs described by Ablowitz, Ramani and Segur [2] which determines whether a given ODE has the Painlevé property. One seeks a solution of a given PDE (in rational form) in the form of a Laurent series (also known as the Painlevé expansion)

$$u(\mathbf{x}) = \phi^{-m}(\mathbf{x}) \sum_{j=0}^{\infty} u_j(\mathbf{x}) \phi^j(\mathbf{x}), \quad (1.1)$$

where $u_j(\mathbf{x})$ are analytic functions of the complex variables $\mathbf{x} = (x_0, x_1, \dots, x_{n-1})$ (we do not change notation for the complex domain), with $u_0 \neq 0$, in the neighbourhood of a non-characteristic movable singularity manifold defined by $\phi(\mathbf{x}) = 0$ (the pole manifold),

where ϕ is an analytic function of \mathbf{x} . The PDE is said to pass the Painlevé test if, on substituting (1.1) in the PDE, one obtains the correct number of arbitrary functions as required by the Cauchy-Kovalevsky theorem, given by the expansion coefficients in (1.1), whereby ϕ should be one of the arbitrary functions. The coefficient in the Painlevé expansion, where the arbitrary functions are to appear, are known as the resonances. If a PDE satisfies the Painlevé test, it is usually [16] possible to construct Bäcklund transformations and Lax pairs [6, 20, 24], which then proves the sufficient condition of integrability.

Recently some attention was given to the construction of exact solutions of nonintegrable PDEs by the use of a truncated Painlevé series [3, 7, 22, 23]. On applying the Painlevé expansion to nonintegrable PDEs one obtains conditions on ϕ at the resonances; the singular manifold conditions. By truncating the series one usually obtains additional constraints on the singularity manifolds, leading to compatibility problems for the solution of ϕ [7, 23, 25]. It has been known for some time that the 2-dimensional Bateman equation

$$\phi_{x_0x_0}\phi_{x_1}^2 + \phi_{x_1x_1}\phi_{x_0}^2 - 2\phi_{x_0}\phi_{x_1}\phi_{x_0x_1} = 0, \quad (1.2)$$

plays an important role in the Painlevé analysis of 2-dimensional nonintegrable PDEs [25].

In the present paper we show that the general solution of the n -dimensional Bateman equation, as generalized by Fairlie [9], solves the singularity manifold condition at the resonance for a class of wave equations. In the present paper we consider the n -dimensional ($n \geq 3$) sine-Gordon -, Liouville -, Mikhailov equation, and double sine-Gordon equation. The Painlevé test of the 2-dimensional double sine-Gordon equation was analyzed by Weiss [25], and resulted in the singularity constrained (1.2). Weiss pointed out that the 2-dimensional Bateman equation (1.2) can be linearized by a Legendre transformation. Moreover, it is invariant under the Moebius group and admits the general implicit solution

$$x_0f_0(\phi) + x_1f_1(\phi) = c, \quad (1.3)$$

where f_0 and f_1 are arbitrary smooth functions and c is an arbitrary real constant. In the following section we derive the explicit relation between the singularity manifold and the 2-dimensional Bateman equation for two 2-dimensional polynomial wave equations. Finally we give some examples which demonstrate the use of our Propositions.

2 Propositions

Fairlie [9] proposed the following n -dimensions Bateman equation:

$$\det \begin{pmatrix} 0 & \phi_{x_0} & \phi_{x_1} & \cdots & \phi_{x_{n-1}} \\ \phi_{x_0} & \phi_{x_0 x_0} & \phi_{x_0 x_1} & \cdots & \phi_{x_0 x_{n-1}} \\ \phi_{x_1} & \phi_{x_0 x_1} & \phi_{x_1 x_1} & \cdots & \phi_{x_1 x_{n-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{x_{n-1}} & \phi_{x_0 x_{n-1}} & \phi_{x_1 x_{n-1}} & \cdots & \phi_{x_{n-1} x_{n-1}} \end{pmatrix} = 0. \quad (2.1)$$

Equation (2.1) generalizes the 2-dimensional Bateman equations (1.2) in n dimensions. It admits the following general implicit solution [9]

$$\sum_{j=0}^{n-1} x_j f_j(\phi) = c, \quad (2.2)$$

where f_j are n arbitrary smooth functions.

We consider the n -dimensional generalization of the well known 2-dimensional sine-Gordon -, Liouville -, and Mikhailov equations, given respectively by

$$\begin{aligned} \square_n u + \sin u &= 0 \\ \square_n u + \exp(u) &= 0 \\ \square_n u + \exp(u) + \exp(-2u) &= 0, \end{aligned} \quad (2.3)$$

as well as the double sine-Gordon equation in n dimensions:

$$\square_n u + \sin \frac{u}{2} + \sin u = 0. \quad (2.4)$$

Here \square_n denotes the d'Alembert operator in n -dimensional Minkowski space, and is defined by

$$\square_n := \frac{\partial^2}{\partial x_0^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}.$$

It is well known that the wave equations (2.3) are integrable for $n = 2$ (see, for example, [1]).

Before we state our Proposition for the singularity manifolds of those equations, we introduce some notations and a Lemma. We call the $(n+1) \times (n+1)$ -matrix, the determinant of which defines the n -dimensional Bateman equation (2.1), the n -dimensional Bateman matrix and denote this matrix by B_{n+1}^n . The subscript of B shows the size of the matrix while the superscript gives the dimension (the number of variables of ϕ), i.e., for the n -dimensional Bateman matrix (2.1), the associated Bateman matrix is

$$B_{n+1}^n = \begin{pmatrix} 0 & \phi_{x_0} & \phi_{x_1} & \cdots & \phi_{x_{n-1}} \\ \phi_{x_0} & \phi_{x_0 x_0} & \phi_{x_0 x_1} & \cdots & \phi_{x_0 x_{n-1}} \\ \phi_{x_1} & \phi_{x_0 x_1} & \phi_{x_1 x_1} & \cdots & \phi_{x_1 x_{n-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{x_{n-1}} & \phi_{x_0 x_{n-1}} & \phi_{x_1 x_{n-1}} & \cdots & \phi_{x_{n-1} x_{n-1}} \end{pmatrix}. \quad (2.5)$$

In particular the submatrices of the above n -dimensional Bateman matrix are of importance, i.e., the submatrices B_p^n , where $3 \leq p \leq n+1$. These submatrices, which we call n -dimensional Bateman submatrices, are obtained by deleting rows and corresponding columns of B_{n+1}^n . We give the following

DEFINITION. *Let*

$$M_{x_{j_1} x_{j_2} \dots x_{j_r}}$$

denote the determinant of a Bateman submatrix, that remains after the rows and columns containing the derivatives $\phi_{x_{j_1}}, \phi_{x_{j_2}}, \dots, \phi_{x_{j_r}}$ have been deleted from the n -dimensional

Bateman matrix (2.5). Let

$$j_1, \dots, j_r \in \{0, 1, \dots, n-1\}, \quad j_1 < j_2 < \dots < j_r, \quad r \leq n-2, \quad \text{for } n \geq 3.$$

Then $M_{x_{j_1} x_{j_2} \dots x_{j_r}}$ are the determinants of the Bateman matrices B_{n+1-r}^n . We call the equations

$$M_{x_{j_1} x_{j_2} \dots x_{j_r}} = 0 \tag{2.6}$$

the minor n -dimensional Bateman equations.

Note that the n -dimensional Bateman equation (2.1) has $n!/[r!(n-r)!]$ minor n -dimensional Bateman equations. Consider an example: If $n = 5$ and $r = 2$, then there exist 10 minor Bateman equations, one of which is given by $M_{x_2 x_3}$, i.e.,

$$\det \begin{pmatrix} 0 & \phi_{x_0} & \phi_{x_1} & \phi_{x_4} \\ \phi_{x_0} & \phi_{x_0 x_0} & \phi_{x_0 x_1} & \phi_{x_0 x_4} \\ \phi_{x_1} & \phi_{x_0 x_1} & \phi_{x_1 x_1} & \phi_{x_1 x_4} \\ \phi_{x_4} & \phi_{x_0 x_4} & \phi_{x_1 x_4} & \phi_{x_4 x_4} \end{pmatrix} = 0. \tag{2.7}$$

We can now state the following

LEMMA. If ϕ satisfies the n -dimensional Bateman equation (2.1), then it satisfies any minor Bateman equation

$$M_{x_{j_1} x_{j_2} \dots x_{j_r}} = 0$$

with

$$j_1, \dots, j_r \in \{0, 1, \dots, n-1\}, \quad j_1 < j_2 < \dots < j_r, \quad r \leq n-2, \quad \text{for } n \geq 3.$$

Proof: By implicitly differentiating the general solution (2.2) of the n -dimensional Bateman equation (2.1), it is easily shown that any minor n -dimensional Bateman equation is satisfied by this solution. Since (2.2) is the general solution of the n -dimensional Bateman equation, the proof is concluded. \square

We now prove

PROPOSITION 1. *For $n \geq 3$, the singularity manifold conditions of the n -dimensional sine-Gordon -, Liouville -, and Mikhailov equations (2.3), are satisfied by the solution of the n -dimensional Bateman equation (2.1).*

Proof: We do the proof for the sine-Gordon equation. For the Liouville - and Mikhailov equation, the proofs are similar. By the substitution

$$v(\mathbf{x}) = \exp[iu(\mathbf{x})]$$

the n -dimensional sine-Gordon equation takes the following form:

$$v\Box_nv - (\nabla_nv)^2 + \frac{1}{2}(v^3 - v) = 0, \quad (2.8)$$

where

$$(\nabla_nv)^2 := \left(\frac{\partial v}{\partial x_0}\right)^2 - \sum_{j=1}^{n-1} \left(\frac{\partial v}{\partial x_j}\right)^2.$$

The dominant behaviour of (2.8) is 2, so that the Painlevé expansion is

$$v(\mathbf{x}) = \sum_{j=0}^{\infty} v_j(\mathbf{x})\phi^{j-2}(\mathbf{x}).$$

The resonance is at 2 and the first two coefficients in the Painlevé expansion have the following form:

$$v_0 = -4(\nabla_n\phi)^2, \quad v_1 = 4\Box_n\phi.$$

We first consider $n = 3$. The singularity manifold condition at the resonance is then given by

$$\det \begin{pmatrix} 0 & \phi_{x_0} & \phi_{x_1} & \phi_{x_2} \\ \phi_{x_0} & \phi_{x_0 x_0} & \phi_{x_0 x_1} & \phi_{x_0 x_2} \\ \phi_{x_1} & \phi_{x_0 x_1} & \phi_{x_1 x_1} & \phi_{x_1 x_2} \\ \phi_{x_2} & \phi_{x_0 x_2} & \phi_{x_1 x_2} & \phi_{x_2 x_2} \end{pmatrix} = 0,$$

which is the 3-dimensional Bateman equation $\det B_4^3 = 0$, as defined by (2.1).

Consider now $n \geq 4$. The condition at the resonance can be written as follows:

$$\sum_{j_1, j_2, \dots, j_{n-3}=1}^{n-1} M_{x_{j_1} x_{j_2} \dots x_{j_{n-3}}} - \sum_{j_1, j_2, \dots, j_{n-4}=1}^{n-1} M_{x_0 x_{j_1} x_{j_2} \dots x_{j_{n-4}}} = 0, \quad (2.9)$$

where

$$j_1 < j_2 < \dots < j_{n-3},$$

and $M_{x_{j_1} x_{j_2} \dots x_{j_{n-3}}}$, $M_{x_0 x_{j_1} x_{j_2} \dots x_{j_{n-4}}}$ are minor n -dimensional Bateman equations, i.e., the determinants of 4×4 Bateman matrices B_4^n . By the Lemma give above, equation (2.9) is satisfied by the solution of the n -dimensional Bateman equation (2.1). \square

We now consider the double sine-Gordon equation in n dimensions (2.4):

$$\square_n u + \sin \frac{u}{2} + \sin u = 0.$$

It was shown by Weiss [25], that for $n = 2$ this equation does not pass the Painlevé test, and that the singularity manifold condition is given by the Bateman equation (1.2).

For n dimensions we prove the following

PROPOSITION 2. *For $n \geq 2$, the singularity manifold condition of the double sine-Gordon equation (2.4) is satisfied by the solution of the n -dimensional Bateman equation (2.1).*

Proof: By the substitution

$$v(\mathbf{x}) = \exp \left[\frac{i}{2} u(\mathbf{x}) \right]$$

the rational form of the double sine-Gordon equation (2.4) is obtained as

$$v \square v + (\nabla v_n)^2 + \frac{1}{4}(v^3 - v) + \frac{1}{4}(v^4 - 1) = 0.$$

The Painlevé expansion takes the form

$$v(\mathbf{x}) = \sum_{j=0}^{\infty} v_j(\mathbf{x}) \phi^{j-1}(\mathbf{x})$$

and the resonance is 2. The first two expansion coefficients are

$$v_0 = -4(\nabla_n v)^2, \quad v_1 = \frac{2}{v_0} \square_n \phi - \frac{1}{2}$$

For the singularity manifold condition we have to consider four cases:

Case $n = 2$: At the resonance we obtain (1.2), i.e.,

$$\det B_3^2 = 0.$$

Case $n = 3$: The condition now takes the following form:

$$8 \det B_4^3 + (M_{x_1} + M_{x_2} - M_{x_0}) v_0 = 0.$$

Case $n \geq 4$: The condition at the resonance can be written as follows:

$$\begin{aligned} & 8 \left(\sum_{j_1, j_2, \dots, j_{n-3}=1}^{n-1} M_{x_{j_1} x_{j_2} \dots x_{j_{n-3}}} \right) \\ & + \left(\sum_{j_1, j_2, \dots, j_{n-2}=1}^{n-1} M_{x_{j_1} x_{j_2} \dots x_{j_{n-2}}} - \sum_{j_1, j_2, \dots, j_{n-3}=1}^{n-1} M_{x_0 x_{j_1} x_{j_2} \dots x_{j_{n-3}}} \right) v_0 = 0, \end{aligned}$$

where

$$j_1 < j_2 < \cdots < j_{n-3} < j_{n-2}.$$

By the above Lemma the proof is concluded. \square

We now consider two well known nonlinear polynomial field theory equations, the so-called nonlinear Klein-Gordon equations:

$$\square_2 u + u^k = 0 \tag{2.10}$$

with $k = 2, 3$. In light-cone coordinates, i.e.,

$$x_0 \longrightarrow \frac{1}{2}(x_0 - x_1), \quad x_1 \longrightarrow \frac{1}{2}(x_0 + x_1),$$

(2.10) takes the form

$$\frac{\partial^2 u}{\partial x_0 \partial x_1} + u^k = 0. \tag{2.11}$$

It should be noted that the 2-dimensional Bateman equation remains invariant under the light-cone coordinates. Therefore, for our purpose we can work with (2.11) instead of (2.10). In [8] it was shown that the nonlinear Klein-Gordon equation (2.11), with $k = 3$, does not pass the Painlevé test. We are now interested in the relation between the 2-dimensional Bateman equation (1.2) and the singularity manifold condition of (2.11) for the case $k = 2$ as well as $k = 3$.

We prove the following

PROPOSITION 3. *The solution of the 2-dimensional Bateman equation (1.2) satisfies the singularity manifold condition of the nonlinear Klein-Gordon equation (2.11) for $k = 2$ and $k = 3$.*

Proof: First we consider equation (2.11) with $k = 3$, i.e.,

$$\frac{\partial^2 u}{\partial x_0 \partial x_1} + u^3 = 0. \quad (2.12)$$

For the Painlevé expansion

$$u(x_0, x_1) = \phi^{-m}(x_0, x_1) \sum_{j=0}^{\infty} u_j(x_0, x_1) \phi^j(x_0, x_1), \quad (2.13)$$

we find that the dominant behaviour is -1, the resonance is 4, and the first three expansion coefficients in expansion (2.13) are

$$\begin{aligned} u_0^2 &= 2\phi_{x_0}\phi_{x_1}, \\ u_1 &= -\frac{1}{3u_0^2} (u_0\phi_{x_0x_1} + u_{0x_1}\phi_{x_0} + u_{0x_0}\phi_{x_1}), \\ u_2 &= \frac{1}{3u_0^2} (u_{0x_0x_1} - 3u_0u_1^2), \\ u_3 &= \frac{1}{u_0^2} (u_2\phi_{x_0x_1} + u_{2x_1}\phi_{x_0} + u_{2x_0}\phi_{x_1} + u_{1x_0x_1} - 6u_0u_1u_2). \end{aligned}$$

At the resonance we obtain the following singularity manifold condition:

$$\Phi\sigma - (\phi_{x_0}\Phi_{x_1} - \phi_{x_1}\Phi_{x_0})^2 = 0, \quad (2.14)$$

where Φ is the 2-dimensional Bateman equation given by (1.2), i.e.,

$$\Phi = \phi_{x_0x_0}\phi_{x_1}^2 + \phi_{x_1x_1}\phi_{x_0}^2 - 2\phi_{x_0}\phi_{x_1}\phi_{x_0x_1},$$

and σ contains derivatives of ϕ with respect to x_0 and x_1 . The explicit form of σ is not interesting for our proof. The explicit appearance of Φ (2.14) concludes the proof for the nonlinearity $k = 3$.

For the equation

$$\frac{\partial^2 u}{\partial x_0 \partial x_1} + u^2 = 0 \quad (2.15)$$

the singularity manifold condition is somewhat more complicated. The dominant behaviour of (2.15) is -2 and the resonance is 6. The first five expansion coefficients in the Painlevé expansion take the following form:

$$\begin{aligned} u_0 &= -6\phi_{x_0}\phi_{x_1}, \\ u_1 &= \frac{1}{\phi_{x_0}\phi_{x_1} + u_0} (u_{0x_1}\phi_{x_0} + u_{0x_0}\phi_{x_1} + u_0\phi_{x_0x_1}), \\ u_2 &= -\frac{1}{2u_0} (u_{0x_0x_1} + u_1^2 - u_{1x_1}\phi_{x_0} - u_{1x_0}\phi_{x_1} - u_1\phi_{x_0x_1}), \\ u_3 &= -\frac{1}{2u_0} (u_{1x_0x_1} + 2u_1u_2), \\ u_4 &= -\frac{1}{\phi_{x_1}\phi_{x_0} + u_0} (u_3\phi_{x_0x_1} + u_{2x_0x_1} + 2u_1u_3 + u_{3x_1}\phi_{x_0} + u_{3x_0}\phi_{x_1} + u_2^2), \\ u_5 &= -\frac{1}{6\phi_{x_0}\phi_{x_1} + 2u_0} (2u_1u_4 + 2u_4\phi_{x_0x_1} + 2u_{4x_0}\phi_{x_1} + 2u_{4x_1}\phi_{x_0} + 2u_2u_3 + u_{3x_0x_1}). \end{aligned}$$

At the resonance the singularity manifold condition is a PDE of order six, which consists of 372 terms (!) all of which are derivatives of ϕ with respect to x_0 and x_1 . This condition may be written in the following form:

$$\sigma_1\Phi + \sigma_2\Psi + (\phi_{x_0}\Psi_{x_1} - \phi_{x_1}\Psi_{x_0} - \sigma_3\Psi - \sigma_4\Phi)^2 = 0, \quad (2.16)$$

where Φ is the 2-dimensional Bateman equation (1.2), and

$$\Psi = \phi_{x_0}\Phi_{x_1} - \phi_{x_1}\Phi_{x_0}, \quad \Phi = \phi_{x_0x_0}\phi_{x_1}^2 + \phi_{x_1x_1}\phi_{x_0}^2 - 2\phi_{x_0}\phi_{x_1}\phi_{x_0x_1}.$$

Here $\sigma_1, \dots, \sigma_4$ consist of derivatives of ϕ with respect to x_0 and x_1 . Their explicit form is not interesting. By (2.16) it is clear that the general solution of the Bateman equation satisfies the singularity manifold condition for (2.15). \square

Due to its enormous complexity in higher dimensions, we were not able to find the explicit relations between the singularity manifold for higher dimensional equations of the form

$$\square_n u + u^k = 0 \tag{2.17}$$

and the n -dimensional Bateman equation (or minor Bateman equations). We

CONJECTURE. *In n -dimensions, the solution of the n -dimensional Bateman equation (2.1) satisfies the singularity manifold condition of (2.17) for $k = 2, 3$.*

Some examples of (2.17) are also given below, and these are consistent with this view.

3 Application

According to a conjecture by Ablowitz, Ramani and Segur [2], every ODE that can be obtained by a Lie symmetry reduction (similarity reduction) of a PDE, which is solvable by the inverse scattering transform method, has the Painlevé property. Some weak form of this conjecture was proved in [13]. On the other hand, if we would consider a nonintegrable 2-dimensional PDE, then it is possible that some of the ODEs resulting by some reduction Ansatz of the PDE, may also be of Painlevé type. In particular, the reduced ODE would fulfil the necessary condition to be of Painlevé type (pass the Painlevé test for ODEs) for those Ansätze for which the new independent variable satisfies the condition on the singularity manifold of the given PDE. By the Propositions stated in the previous section, we know that the condition on the singularity manifold is satisfied by the n -dimensional Bateman equation for our class of equations. Thus, the Propositions, lead to the following

COROLLARY. *The nonlinear wave equations (2.3), (2.4), (2.12) (2.15) can be reduced to ODEs which satisfy the necessary condition to be of Painlevé type, if and only if the*

new independent variables of the reduced ODEs satisfy the corresponding n -dimensional Bateman equation (1.2).

This means that if we were to reduce one of the nonintegrable n -dimensional PDEs discussed in our paper into an ODE with independent variable ω by, for example, an Ansatz of the form

$$u(x_0, x_1, \dots, x_{n-1}) = f_1(x_0, x_1, \dots, x_{n-1})\varphi(\omega) + f_2(x_0, x_1, \dots, x_{n-1}), \quad (3.1)$$

then we can easily test the necessary condition of integrability of the resulting ODE by checking whether ω satisfies the n -dimensional Bateman equation (2.1). This would be the same as to perform the Painlevé test on the resulting ODE. By Lie symmetry analysis of PDEs one is able to systematically construct Ansätze which reduce the PDEs to ODEs according to their Lie transformation group properties (see for example [10]). By the above Corollary one is now able to classify the group invariants (that are independent of u) for the given PDEs, and determine which group invariants may result in ODE's of Painlevé type, without performing the Painlevé analysis on the actual reduced ODEs, but by merely checking whether the invariants satisfy the n -dimensional Bateman equation (2.1). One must note that the reduction Ansatz is not necessarily related to a classical Lie symmetry invariant. One can obtain very interesting reduction Ansätze by the use of the so-called conditional symmetries, or Q -symmetries (see [10] for some interesting examples).

Below we give some examples of the stated Corollary. A more systematic analysis and classification of the the equations treated here, will be presented in a future paper.

EXAMPLE 1. Consider the 3-dimensional Liouville equation [10] , i.e.,

$$\square_3 u + \lambda \exp(u) = 0, \quad (3.2)$$

with the Ansatz

$$\begin{aligned} u(x_0, x_1, x_2) &= \varphi(\omega) - 2 \ln(\alpha_0 y_0 - \alpha_1 y_1 - \alpha_2 y_2) \\ \omega(x_0, x_1, x_2) &= (\alpha_0 y_0 - \alpha_1 y_1 - \alpha_2 y_2)(\beta_0 y_0 - \beta_1 y_1 - \beta_2 y_2)^a \end{aligned} \quad (3.3)$$

where $a \in \mathcal{Q} \setminus \{0\}$ and

$$\begin{aligned}\alpha_0^2 - \alpha_1^2 - \alpha_2^2 &= \alpha_0\beta_0 - \alpha_1\beta_1 - \alpha_2\beta_2 = 0, \\ \beta_0\beta_0 - \beta_1\beta_1 - \beta_2\beta_2 &< 0, \\ y_\mu &= x_\mu + a_\mu, \quad \mu = 0, 1, 2.\end{aligned}$$

Here ω , given by (3.3), satisfies the 3-dimensional Bateman equation $\det B_4^3 = 0$, so that by the Corollary we are ensured that the reduced ODE, resulting from Ansatz (3.3), satisfies the necessary condition to be of Painlevé type. Ansatz (3.3) leads to the following ODE:

$$a^2\omega^2 \frac{d^2\varphi}{d\omega^2} + a(a-1)\omega \frac{d\varphi}{d\omega} + \lambda \exp(\varphi) = 0. \quad (3.4)$$

Equation (3.4) is of Painlevé type and admits the general solution

$$\varphi(\omega) = -2 \ln \left[\frac{\sqrt{-\lambda}}{\sqrt{2}c_1} \omega^{-1/a} \cos(c_1\omega^{1/a} + c_2) \right]; \quad \lambda < 0 \quad (3.5)$$

$$\varphi(\omega) = -2 \ln \left[\frac{\sqrt{\lambda}}{\sqrt{2}c_1} \omega^{-1/a} \cosh(c_1\omega^{1/a} + c_2) \right]; \quad \lambda > 0. \quad (3.6)$$

By (3.5) and the Ansatz (3.3) an exact solution of the Liouville equation (3.2) follows:

$$u(x_0, x_1, x_2) = -2 \ln \left[\frac{\sqrt{-\lambda}}{\sqrt{2}c_1} \omega^{-1/a} \cos(c_1\omega^{1/a} + c_2) \right] - 2 \ln(\alpha_0 y_0 - \alpha_1 y_1 - \alpha_2 y_2); \quad \lambda < 0$$

$$u(x_0, x_1, x_2) = -2 \ln \left[\frac{\sqrt{\lambda}}{\sqrt{2}c_1} \omega^{-1/a} \cosh(c_1\omega^{1/a} + c_2) \right] - 2 \ln(\alpha_0 y_0 - \alpha_1 y_1 - \alpha_2 y_2); \quad \lambda > 0$$

$$\omega(x_0, x_1, x_2) = (\alpha_0 y_0 - \alpha_1 y_1 - \alpha_2 y_2)(\beta_0 y_0 - \beta_1 y_1 - \beta_2 y_2)^a, \quad y_\mu = x_\mu + a_\mu, \quad \mu = 0, 1, 2.$$

This example can easily be extended to n dimensions.

EXAMPLE 2. Consider the 4-dimensional sine-Gordon equation [10], i.e,

$$\square_4 u + \sin(u) = 0. \quad (3.7)$$

By the Ansatz

$$\begin{aligned} u(x_0, x_1, x_2, x_3) &= \varphi(\omega) \\ \omega(x_0, x_1, x_2, x_3) &= \frac{x_2 - x_3(x_0 + x_1)}{\sqrt{1 + (x_0 + x_1)^2}} + f(x_0 + x_1), \end{aligned} \quad (3.8)$$

where f is an arbitrary smooth function of its argument, (3.7) reduces to the following integrable ODE:

$$\frac{d^2 \varphi}{d\omega^2} - \sin \varphi = 0. \quad (3.9)$$

It easy to show that ω , given by (3.8), satisfies the 4-dimensional Bateman equation $\det B_5^4 = 0$. Equation (3.9) can be integrated in terms of Jacobi elliptic functions to obtain exact solutions of the 4-dimensional sine-Gordon equation (3.7).

EXAMPLE 3. Consider the 2-dimensional nonlinear Klein-Gordon equation

$$u_{x_0 x_1} + \lambda u^3 = 0. \quad (3.10)$$

We demonstrate that by the given Corollary and the Ansatz

$$u(x_0, x_1) = h(x_0, x_1)\varphi(\omega), \quad (3.11)$$

where ω satisfies the 2-dimensional Bateman equation (1.2) i.e.,

$$x_0 f_0(\omega) + x_1 f_1(\omega) = c,$$

we are able to construct ODEs which pass the Painlevé test. Ansatz (3.11) leads to

$$\left(\frac{fgh}{(x_0 f_0 + x_1 f_1)^2} \right) \frac{d^2 \varphi}{d\omega^2}$$

$$\begin{aligned}
& + \left(\frac{h(\dot{f}_0 f_1 + f_0 \dot{f}_1)}{(x_0 \dot{f}_0 + x_1 \dot{f}_1)^2} - \frac{f g h(x_0 \ddot{f}_0 + x_1 \ddot{f}_1)}{(x_0 \dot{f}_0 + x_1 \dot{f}_1)^3} - \frac{h_{x_1} f_0 + h_{x_0} f_1}{(x_0 \dot{f}_0 + x_1 \dot{f}_1)} \right) \frac{d\varphi}{d\omega} \\
& + h_{x_0 x_1} \varphi + \lambda h^3 \varphi^3 = 0.
\end{aligned} \tag{3.12}$$

Here $h = h(x_0, x_1)$, $f_i = f_i(\omega)$, and $\dot{f}_i \equiv df_i/d\omega$ ($i = 0, 1$). For example, let

$$h(x_0, x_1) = \frac{1}{x_0}, \quad f_1(\omega) = -1,$$

then (3.12) reduces to

$$\ddot{\varphi} + \left(2 \frac{\dot{f}}{f} - \frac{\ddot{f}}{\dot{f}} \right) \dot{\varphi} - \left(\frac{\lambda \dot{f}^2}{f} \right) \varphi^3 = 0. \tag{3.13}$$

Equation (3.13) satisfies the necessary condition to be of Painlevé type (it passes the Painlevé test for ODEs), which is in agreement with the above Corollary, as we are using the general solution of the 2-dimensional Bateman equation (1.2). Note that for $f_0(\omega) = \omega$ we obtain the same ODE which was obtained with a Lie symmetry analysis in [8]. We remark that the use of the general solution (1.3) of the Bateman equation (1.2), in the construction of exact solutions of (3.10), is clearly limited. A more effective approach, to obtain exact solutions, would be to linearize the 2-dimensional Bateman equation by the Legendre transformation, as outlined by Webb and Zank [23]. However, this is not the purpose of the present paper.

EXAMPLE 4. Consider the 4-dimensional nonlinear Klein-Gordon equation

$$\square_4 u + \lambda u^3 = 0, \tag{3.14}$$

where $\lambda \in \mathcal{R}$. Asymptotic solutions of (3.14) were constructed in [14] by the use of the Poincaré group $P(1, 3)$ and its invariants. By composing the group invariants, we obtain the following Ansatz for (3.14):

$$u(x_0, x_1, x_2, x_3) = \varphi(\omega)$$

$$\omega(x_0, x_1, x_2, x_3) = \beta_1(<\tilde{\mathbf{p}}, \mathbf{x}> + a_1) - \beta_2(<\tilde{\boldsymbol{\alpha}}, \mathbf{x}> + a_2) - \beta_3(<\tilde{\boldsymbol{\beta}}, \mathbf{x}> + a_3) \quad (3.15)$$

$$+ a \ln \left\{ \alpha_1(<\tilde{\mathbf{p}}, \mathbf{x}> + a_1) - \alpha_2(<\tilde{\boldsymbol{\alpha}}, \mathbf{x}> + a_2) - \alpha_3(<\tilde{\boldsymbol{\beta}}, \mathbf{x}> + a_3) \right\}.$$

Here $<\tilde{\mathbf{p}}, \mathbf{x}> \equiv p_0x_0 - p_1x_1 - p_2x_2 - p_3x_3$, $<\tilde{\boldsymbol{\alpha}}, \mathbf{x}> \equiv \alpha_0x_0 - \alpha_1x_1 - \alpha_2x_2 - \alpha_3x_3$, $<\tilde{\boldsymbol{\beta}}, \mathbf{x}> \equiv \beta_0x_0 - \beta_1x_1 - \beta_2x_2 - \beta_3x_3$ and a_j ($j = 0, 1, 2, 3$) are arbitrary real constants, whereas $\alpha_j, \beta_j, \tilde{\alpha}_\mu, \tilde{\beta}_\mu, \tilde{p}_\mu$ ($j = 1, 2, 3$; $\mu = 0, 1, 2, 3$) are real constants which must satisfy the following conditions:

$$\beta_1^2 - \beta_2^2 - \beta_3^2 = -1, \quad \alpha_1^2 - \alpha_2^2 - \alpha_3^2 = \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3 = 0 \quad (3.16)$$

$$<\tilde{\mathbf{p}}, \tilde{\mathbf{p}}> = 1, \quad <\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\alpha}}> = <\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\beta}}> = -1,$$

$$<\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}> = <\tilde{\boldsymbol{\alpha}}, \tilde{\mathbf{p}}> = <\tilde{\boldsymbol{\beta}}, \tilde{\mathbf{p}}> = 0. \quad (3.17)$$

Here ω , given by (3.15), satisfies the 4-dimensional Bateman equation $\det B_5^4 = 0$, and the reduced equation

$$\frac{d^2\varphi}{d\omega^2} + \lambda\varphi^3 = 0 \quad (3.18)$$

is of Painlevé type. The general solution of (3.18) is given in terms of Jacobi elliptic functions [5].

EXAMPLE 5. Consider the 4-dimensional nonlinear Klein-Gordon equation

$$\square_4 u + \lambda_1 u + \lambda_2 u^3 = 0, \quad (3.19)$$

where $\lambda_1, \lambda_2 \in \mathcal{R}$. By the invariants of the Poincaré group, and its Lie subalgebras, the following two Ansätze are, for example, possible:

$$u(x_0, x_1, x_2, x_3) = \varphi(\omega_1)$$

$$\omega_1 = \frac{c}{2} \left\{ <\tilde{\boldsymbol{\gamma}}, \mathbf{x}>^2 + \left[<\tilde{\boldsymbol{\beta}}, \mathbf{x}> + \frac{1}{4} (<\tilde{\mathbf{p}}, \mathbf{x}> + <\tilde{\boldsymbol{\alpha}}, \mathbf{x}>)^2 \right]^{1/2} \right\}$$

$$+ q_1 <\tilde{\boldsymbol{\gamma}}, \mathbf{x}> - q_2 \left[<\tilde{\boldsymbol{\beta}}, \mathbf{x}> + \frac{1}{4} (<\tilde{\mathbf{p}}, \mathbf{x}> + <\tilde{\boldsymbol{\alpha}}, \mathbf{x}>)^2 \right], \quad (3.20)$$

and

$$u(x_0, x_1, x_2, x_3) = \varphi(\omega_2)$$

$$\omega_2(x_0, x_1, x_2, x_3) = -q_3 \left[\langle \tilde{\mathbf{p}}, \mathbf{x} \rangle^2 - \langle \tilde{\boldsymbol{\alpha}}, \mathbf{x} \rangle^2 - \langle \tilde{\boldsymbol{\beta}}, \mathbf{x} \rangle^2 \right]^{1/2}, \quad (3.21)$$

where $\langle \tilde{\mathbf{p}}, \mathbf{x} \rangle \equiv \tilde{p}_0 x_0 - \tilde{p}_1 x_1 - \tilde{p}_2 x_2 - \tilde{p}_3 x_3$, $\langle \tilde{\boldsymbol{\alpha}}, \mathbf{x} \rangle \equiv \tilde{\alpha}_0 x_0 - \tilde{\alpha}_1 x_1 - \tilde{\alpha}_2 x_2 - \tilde{\alpha}_3 x_3$, and $\langle \tilde{\boldsymbol{\beta}}, \mathbf{x} \rangle \equiv \tilde{\beta}_0 x_0 - \tilde{\beta}_1 x_1 - \tilde{\beta}_2 x_2 - \tilde{\beta}_3 x_3$. Here c and q_3 are arbitrary nonzero real constants, whereas the rest of the real parameters have to satisfy condition (3.17) and

$$\langle \tilde{\gamma}, \tilde{\gamma} \rangle = -1, \quad \langle \tilde{\gamma}, \tilde{\mathbf{p}} \rangle = \langle \tilde{\boldsymbol{\beta}}, \tilde{\gamma} \rangle = \langle \tilde{\boldsymbol{\alpha}}, \tilde{\gamma} \rangle = 0, \quad q_1^2 + q_2^2 = q \neq 0.$$

By the above Ansätze the following ODEs are respectively obtained:

$$(2c\omega_1 + q) \frac{d^2\varphi}{d\omega_1^2} + 2c \frac{d\varphi}{d\omega_1} - \lambda_1 \varphi + \lambda_2 \varphi^3 = 0, \quad (3.22)$$

$$q_3\omega_2 \frac{d^2\varphi}{d\omega_2^2} + 2q_3 \frac{d\varphi}{d\omega_2} + \lambda_1\omega_2\varphi - \lambda_2\omega_2\varphi^3 = 0. \quad (3.23)$$

Equations (3.22) and (3.23) are not of Painlevé type, which is in agreement with the fact that ω_1 and ω_2 do not satisfy the 4-dimensional Bateman equation $\det B_5^4 = 0$.

A systematic classification of integrable reductions of the above given multidimensional wave equations, by the use of the Propositions and Corollary stated here, will be the subject of a future paper.

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