Buckley-Leverett flow on limited-scale fractal curves

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Abstract

The 1D two-phase Buckley-Leverett flow along a limited-scale fractal shape streamline is considered. A power-law model (crude model) and an F^α-calculus model (accurate model) are proposed for such process. Analytical solutions of the corresponding initial value problems are obtained. The proposed models are compared for the case when a fractal streamline is the von Koch curve. It is shown that both obtained models demonstrate the similar time evolution of saturation profiles in a qualitative sense. Also, if the flood front position (position of saturation discontinuity) is known, then a quantitative agreement between the models can be vastly improved.

Keywords: Two-phase flow in porous media, Fractal curve, Power law, Fractal calculus
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1. Introduction

In the last decades, experimental researches show that inhomogeneous porous media may possess self-similarity properties [1, 2, 3, 4, 5]. Therefore, the concept of fractals is applicable to more precise simulation of some transport phenomena in such media (e.g., viscous fingers [2], percolation effects [6], and some other [7, 8]).

One way to model transport processes in a fractal medium is to use the fractional calculus approach [9]. In particular, fractional differential equations can be exploited to model processes with a residual memory arising due to the fractality of medium (see, e.g., [10], [11]). However, in most cases fractional analogues of classical models are more complicated due to unusual properties of fractional derivatives such as the generalized Leibniz rule and nonlocality [12]. Jumarie [13] suggested the fractional-order derivative with the classical Leibniz rule. But afterward, Tarasov [14] proved that the differentiation operators satisfying this rule must have an integer order. A similar statement holds for the local derivatives [15]. In addition, Uchaikin [16] showed that the interpretation of a fractional diffusion equation as the equation describing the diffusion in a fractal medium can be incorrect. Thus, modeling physical processes in a fractal medium by fractional derivatives raises some questions and difficulties.

The main problem of a mathematically rigorous description of processes associated with fractal curves is that such curves are not ordinary differentiable. This is due to the infinity of a classical length of regular fractal curves between any two different points. The Hausdorff measure allows one to evaluate this "length" (see, e.g., [17]). But finding exact values of this measure for fractals is difficult. Different estimates are only known for this measure for some fractals (see, e.g., [18] for the Koch curve). Therefore, the calculus for fractal subsets of a real line (F^α-calculus) with respect to the "truncated" Hausdorff measure are developed in [19, 20]. This modified measure is more suitable for numerical calculations. In [21], the F^α-calculus is derived for fractal curves embedded in space $\mathbf{R^n}, n > 1$. The theorems of conjugacy of F^α-calculus with ordinary calculus were proved in [20, 21]. The F^α-derivative have a simple geometric interpretation (identical to integer-order derivative), unlike the fractional derivatives (cf. [22, 23]). This fact allows one to construct more rigorous modified models of processes associated with fractals. Some relations of classical mechanics on fractal subsets of real line are derived in [24]. Modified Fourier and Laplace transforms are introduced in [25]. The F^α-Fokker-Planck equation and the discussion its fundamental solution in the case of a null drift coefficient are presented in [26]. An F^α-generalization of the Henstock-Kurzweil integral for singular and unbounded fractal function is proposed in [27]. Results of properties investigation of the F^α-fractional derivatives are presented in [28, 29]. The generalisation of F^α-calculus to a Cartesian product of fractal subsets of $\mathbf{R}$ are given in [30]. The extensive literature on the F^α-calculus is reviewed in [31].

It is well-known that the self-similarity is closely connected with power laws [17]. Using this fact, a continuity equation and Darcy's law were generalized to the processes in fractal media by substituting power laws for geometric coefficients (a fluid "geometric" mass and a permeability) [32, 33, 34]. In [33], a practical evaluation technique for corresponding power laws parameters was developed. Application of power laws for taking into account the fractal structure of porous media in the transport processes is extensively analysed in the survey papers [35, 36].

The objectives of this work are: (i) to propose a new two-phase F^α-Buckley-Leverett model which can describe the fluid

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flow along an approximate fractal shape streamline; (ii) to identify a power-law model application area for this process. The structure of the paper is as follows. Section 2 is devoted to deriving and investigating two modified Buckley-Leverett models. Subsections 2.1 is concerned with a power-law generalization of one-phase filtration model. In Subsection 2.2, we present the necessary definitions and some necessary facts about the F^α-calculus [21]. Subsections 2.3 and 2.4 are devoted to derive two modifications of the Buckley-Leverett model for fractal curves and to obtain exact solutions for corresponding Cauchy problems. Finally, the results of comparison of the obtained modified models are discussed in Section 3.

2. Description of Buckley-Leverett models on fractal curves

2.1. Power law in one-phase flow

A typical property of objects with a fractal structure is an unusual mass distribution which can be described by the Mandelbrot ratio [17] $M \sim L^D$, where M is the mass, L is the characteristic size of the spatial domain, D is the Hausdorff dimension. By using this relation, the groundwater flow equation (see, e.g., [37])

$$\frac{\partial\,(mp)}{\partial t} - \frac{\partial}{\partial x}\left(\frac{k}{\mu}\frac{\partial p}{\partial x}\right) = 0 \tag{1}$$

can be generalized (see, e.g., [2, 33]) to fractal media case as

$$\frac{\partial}{\partial t}\left(m\sigma x^{D-1}p\right) - \frac{\partial}{\partial x}\left(\frac{k}{\mu}Kx^{D-1-\omega}\frac{\partial p}{\partial x}\right) = 0. \tag{2}$$

Here t is the time variable, x is the spatial variable, m is the porosity, p is the fluid pressure, k is the permeability of porous media, μ is the fluid viscosity, σ is the constant depending only on fractal media properties (σx^{D-1} is the fluid "geometric density"), K and ω are the constants describing anomalous conductivity properties of the fractal media.

2.2. Power law Buckley-Leverett model

We assume that a porous medium possess a fractal system of capillaries. Let us consider the initial-value problem for a system of equations describing a 1D two-phase flow of immiscible fluids in a such medium within the framework of Buckley-Leverett theory (see, e.g., [38]).

Using Eq. (2) for each phases, we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(m\sigma x^{D-1}s_1\right) + \frac{\partial}{\partial x}\left(\frac{kKx^{D-1-\omega}}{\mu_1}f_1\left(s_1\right)\frac{\partial p}{\partial x}\right) = 0,\\
&\frac{\partial}{\partial t}\left(m\sigma x^{D-1}s_2\right) - \frac{\partial}{\partial x}\left(\frac{kKx^{D-1-\omega}}{\mu_2}f_2\left(s_2\right)\frac{\partial p}{\partial x}\right) = 0,
\end{aligned} \tag{3}$$

for $t_0 < t \leq T,\ l \leq x \leq L$. Here $s_i,\ \mu_i,\ f_i\left(s_i\right)\ (i = 1, 2)$ are the saturation, the viscosity, and the relative permeability of phase i, respectively. The wetting phase is corresponded to $i = 1$. The initial conditions

$$s_1(t_0, x) = r(x),\ \ s_2(t_0, x) = 1 - r(x)$$

will be used for (3). Here the function $r(x) : [l, L] \to [0, 1]$ is assume to be invertible.

The saturations satisfy the condition $s_1 + s_2 = 1$. Using this condition Eq. (3), can be rewritten as

$$\begin{aligned}
&(f_1(s) + \mu_0 f_2(s))\frac{\partial p}{\partial x} = \frac{C(t)}{Kx^{D-1-\omega}},\\
&\frac{\partial s}{\partial t} + \frac{q(t)}{m\sigma x^{D-1}}\frac{\partial G(s)}{\partial x} = 0,
\end{aligned} \tag{4}$$

where

$$\mu_0 = \frac{\mu_1}{\mu_2},\ \ s = s_1,\ \ q(t) = -\frac{k}{\mu_1}\,C\left(t\right),\ \ G(s) = \frac{f_1(s)}{f_1(s) + \mu_0 f_2(s)},$$

and $C(t)$ is an arbitrary function. Eq. (4) is a power law generalization of Buckley-Leverett model.

The solution of the Cauchy problem for Eq. (4) with the initial condition $s(t_0, x) = r(x)$ can be obtained in an explicit form. The substitution

$$Q = \int_{t_0}^{t} q(t)d\tau,\ \ \ W = \frac{m\sigma x^D}{D}$$

leads Eq. (4) to

$$\frac{\partial s}{\partial Q} + G'(s)\frac{\partial s}{\partial W} = 0.$$

Using the method of characteristics (see, e.g., [39]), we get

$$W = x(s, t_0) + G'(s)Q,\ \ x(s, t_0) = r^{-1}(s).$$

In terms of t, x this solution takes the form

$$\frac{m\sigma x^D}{D} = x(s, t_0) + G'(s)\int_{t_0}^{t} q(\tau)d\tau. \tag{5}$$

2.3. Brief introduction to F^α-calculus

In this section, we recall necessary definitions and preliminary results of F^α-calculus (see [21] for more details).

A fractal curve $F \subset \mathbf{R^n}$ is continuously parametrizable if there exists a function $w : [a, b] \to F \subset \mathbf{R^n}, a < b$ which is continuous, one-to-one, and onto F.

Let us consider a finite subdivizion of the interval $[a, b]$

$$P_{[a,b]} = \{x_0 = a, x_1, \ldots, x_N = b\}\,,\ \ x_{i-1} < x_i,\ \ i = 1, \ldots, N < \infty\,.$$

Then, with $\|\cdot\|$ denoting the Euclidean norm, the coarse grained mass $\gamma^\alpha_\delta(F, a, b)$ is given by

$$\gamma^\alpha_\delta(F, a, b) = \inf_{P_{[a,b]}:|P|\leq\delta}\sum_{i=0}^{n-1}\frac{\|w(x_{i+1}) - w(x_i)\|^\alpha}{\Gamma(\alpha+1)},\ \ n < \infty, \tag{6}$$

where $|P| = \max\limits_{0\leq i\leq n-1}\left(x_{i+1} - x_i\right), \alpha > 0$.

The mass function of the continuously parametrizable curve F is given by

$$\gamma^\alpha(F, a, b) = \lim_{\delta\to 0}\gamma^\alpha_\delta(F, a, b).$$

In [21] the Monte Carlo algorithm is proposed for computing this function.

The γ-dimension of F is given by

$$dim_\gamma(F) = inf\{\alpha : \gamma^\alpha(F, a, b) = 0\} = sup\{\alpha : \gamma^\alpha(F, a, b) = \infty\}.$$

The γ-dimension of a self-similar curve is equal to the Hausdorff dimension.

Hereafter, we will assume that $\alpha = dim_\gamma(F)$.

Let $F \subset \mathbf{R}^\mathbf{n}$ be a parametrizable fractal curve. A number l is called F-limit, as $\theta' \to \theta$, if for any $\varepsilon > 0$ exists $\delta > 0 : \theta' \in F$, $|\theta' - \theta| < \delta \Rightarrow |f(\theta') - l| < \varepsilon$. If such a number exists it is denoted by

$$l = \lim_{\theta' \to \theta}{}^F f(\theta').$$

Consider the function $J(\theta) = \gamma^\alpha\left(F, a, w^{-1}(\theta)\right)$. The F^α-derivative of function $f : F \to \mathbf{R}$ at $\theta \in F$ is defined as

$$D_F^\alpha f(\theta) = \lim_{\theta' \to \theta}{}^F \frac{f(\theta') - f(\theta)}{J(\theta') - J(\theta)},$$

if the limit exists.

By $B(F)$ denote the class of bounded functions $h : F \to \mathbf{R}$. By φ denote the map $B(F) \to B([S_F^\alpha(a), S_F^\alpha(b)])$ such that for any $x \in [a, b]$

$$\varphi[f](S_F^\alpha(x)) = f(w(x)). \tag{7}$$

Let $h \in B(F)$ be an ordinarily differentiable function on $[S_F^\alpha(a_0), S_F^\alpha(b_0)]$. Then $D_F^\alpha h(\theta)$ exists for all $\theta \in F$ (see [21]) and

$$D_F^\alpha h(\theta) = \left.\frac{dg(v)}{dv}\right|_{v=J(\theta)}. \tag{8}$$

2.4. F^α-Buckley-Leverett model

The F^α-derivative give one the possibility to naturally modify the continuity equation and the Darcy law to describe a fluid flow along a fractal curve. We have

$$\frac{\partial m\overline{p}}{\partial t} - D_{F,\theta}^\alpha \overline{u} = 0, \quad \overline{u} = \frac{k}{\mu} D_{F,\theta}^\alpha \overline{p}.$$

Then the F^α-analogue of Eq. (1) takes the form

$$\frac{\partial(m\overline{p})}{\partial t} - D_{F,\theta}^\alpha\left(\frac{k}{\mu} D_{F,\theta}^\alpha \overline{p}\right) = 0. \tag{9}$$

In [26, 32] the second absolute moments of Eq. (9) and (2) are investigated, respectively. It is shown that they are proportional to each other, if $\omega = 0$.

As previously, consider a two-phase flow along a fractal curve. The saturations satisfy the conditions $\bar{s}_1 + \bar{s}_2 = 1$, $\bar{s}_1(t_0, \theta) = \bar{r}(\theta)$, $\bar{s}_2(t_0, \theta) = 1 - \bar{r}(\theta)$. Similarly to Subsection 2.2, using Eq. (9) for each phase, we get for the wetting phase

$$\begin{aligned} &(f_1(\bar{s}) + \mu_0 f_2(\bar{s})) D_{F,\theta}^\alpha \bar{p} = \bar{C}(t), \\ &\frac{\partial \bar{s}}{\partial t} + \frac{q(t)}{m} D_{F,\theta}^\alpha G(\bar{s}) = 0, \end{aligned} \tag{10}$$

where

$$\bar{s} = \bar{s}_1(t, \theta),\ \mu_0 = \mu_1/\mu_2,\ \bar{p} = \bar{p}(t, \theta),\ \bar{q}(t) = -\frac{k}{\mu_1}\bar{C}(t),$$

$$G(\bar{s}) = \frac{f_1(\bar{s})}{f_1(\bar{s}) + \mu_0 f_2(\bar{s})}\ t_0 < t \leq T,\ \theta(x) \in F,\ a \leq x \leq b,$$

and $\bar{C}(t)$ is an arbitrary function.

Eq. (10) will be called the Buckley-Leverett flow F^α-model.

The solution of Cauchy type problem for Eq. (10) with the initial condition $\bar{s}(t_0, x) = \bar{r}(x)$ can be obtained in an explicit form.

Let $\bar{s}_\varphi = \varphi[\bar{s}]$. Then, taking into account (7) and (8), we obtain

$$\frac{\partial \bar{s}_\varphi}{\partial t} + \frac{q(t)}{m}\frac{\partial \mathrm{G}(\bar{s}_\varphi)}{\partial y} = 0,\ \bar{s}_\varphi = s(y, t),\ y = J(\theta).$$

This equation has the solution (see, e.g., [39])

$$my = y(\bar{s}_\varphi, t_0) + G'(\bar{s}_\varphi)\int_{t_0}^t q(\tau)d\tau,\ y(\bar{s}_\varphi, t_0) = J(r^{-1}(\bar{s}_\varphi)). \tag{11}$$

The image of (11) under the map φ^{-1} is the solution of Eq. (10):

$$mJ(\theta) = y(\bar{s}, t_0) + G'(\bar{s})\int_{t_0}^t q(\tau)\, d\tau$$

or, in the parametric form,

$$m\gamma^\alpha(F, a, x) = y(\bar{s}, t_0) + G'(\bar{s})\int_{t_0}^t q(\tau)d\tau, \tag{12}$$

where $\bar{s} = \bar{s}(w(x), t),\ a \leq x \leq b$.

3. Comparison of Buckley-Leverett models on fractal curves

In this section, we will conduct a comparative analysis of both models obtained in the previous section by using an example of initial-value problem with known analytical solution.

We introduce the modification of coarse grained mass function (6) as

$$\gamma_{\delta,\Delta}^\alpha(F, a, b) = \inf_{P_{[a,b]}:|P|_{max}\leq\delta, |P|_{min}\geq\Delta} \sum_{i=0}^{n-1} \frac{\|w(x_{i+1}) - w(x_i)\|^\alpha}{\Gamma(\alpha + 1)},\ n < \infty,$$

where $|P|_{\min} = \min_{0\leq i\leq n-1}(x_{i+1} - x_i)$, $\alpha > 0$. The additional parameter Δ is necessary for investigation the behaviour of fractal functions during the refinement of the subdivision. The parameters δ and Δ can be interpreted as the lower and upper boundaries of the scales whereby the self-similarity is preserved.

Obviously, the solutions (5) and (12) of both considered models (like a classical model) have multivalued regions. As usual, the physically impossible multivalued solution is replaced by a discontinuity. The position of the discontinuity $x = x_{fr}$ (or $x = \bar{x}_{fr}$ in F^α-model) is determined by Maxwell rule: the area equality condition for the regions 1 and 2 in Fig. 1 (see [40] for more details).

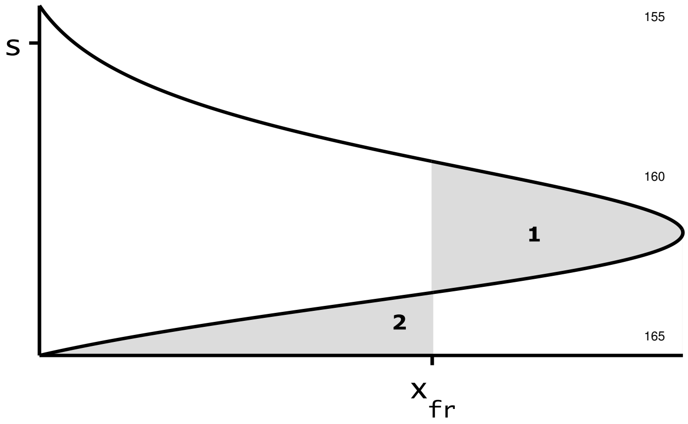

Figure 1: Illustration of Maxwell rule.

We consider two different ways for coupling of F^α-model and power-law model. In the first one, the "natural" condition of the fractals mass equality

$$\gamma_{\delta_1,\Delta_1}(F, 0, L) = M(L; \sigma) \tag{13}$$

is used. In the second one, the power law coefficient $\sigma = \sigma'$ for fixed $t = t' > 0$ is calculated from the mass functions equality condition at a moving interface of F^α-model:

$$\gamma_{\delta',\Delta'}(F, a, \bar{x}_{fr}(t')) = M(\bar{x}_{fr}(t'); \sigma'), \tag{14}$$

where

$$M(x; \sigma) = \int_0^x \sigma y^{D-1} dy = \frac{\sigma x^D}{D}.$$

Then, we get

$$\sigma' = D\gamma_{\delta',\Delta'}(F, 0, \bar{x}_{fr}(t'))(\bar{x}_{fr}(t'))^{-D}.$$

As a test example for comparing two consider models, we take the von Koch curve as a fractal curve F. A convenient parametrization $\theta = \theta(x) \in F,\ x \in [0, 1]$ of this curve is given in [41].

Then we have $a = l = 0,\ b = L = 1,\ \alpha = D = \ln(4)/\ln(3)$. Other parameters are taken as follows: $t_0 = 0$, $s(x, 0) = \bar{s}(\theta, 0) = 0$, $q(t) = \bar{q}(t) = 1$, $f_1(s) = s^2$, $f_2(s) = (1-s)^2$, $\mu_0 = 0.4$, $m = 0.7$.

For F^α-model, the wetting phase saturation $\bar{s}(x, t)$ can be found from Eq. (12) if and only if the mass function $\gamma(F, 0, x)$ is known for a fractal curve. For a limited-scale fractal curve the coarse grained mass $\gamma_{\delta,\Delta}(F, 0, x)$ should be used instead of $\gamma(F, 0, x)$. This function can be calculated by the Monte Carlo algorithm proposed in [21]. Results of computation $\gamma_{\delta,\Delta}(F, 0, x)$ for two pairs δ, Δ are shown in Fig. 2. Using these results, corresponding saturations have been calculated by Eq. (12) (see Fig. 3). It can be seen from Figs. 2, 3 that for chosen parameters of models the refinement of subdivision $P_{[a,b]}$ leads to small perturbation of saturation.

In Fig. 4, the graphs of saturations $\bar{s}(x, t; \delta, \Delta)$ and $s(x, t; \sigma)$ calculated by (12) and (5) respectively are plotted for the case when the "natural" condition (13) holds and identical initial conditions are used. It can be seen that there is a significant difference between results predicted by F^α-model and power-law model. Thus, in this case the power-law model can not be considered as the adequate approximations for F^α-model.

Now, let us consider the case when the models are coupled by the condition (14). The results of its comparison for two pairs δ, Δ and identical initial conditions are shown in Figs. 5, 6. It is follows from (14) that for power-law model $\sigma' = \sigma'(t')$ since $\bar{x}_{fr} = \bar{x}_{fr}(t')$. Therefore, the mass function $M(x; \sigma')$ also depends on t'. In Figs. 5a, 6a this function is plotted for four different values of t'. The corresponding saturation $s(x, t; \sigma)$ has been found from Eq. (5). It is plotted in Figs. 5b, 6b. To illustrate the Maxwell rule, the multivalued solution is shown in Fig. 5b for $t = 0.05$. It can be seen from Figs. 5b, 6b that modified models have the same structure of solutions: multivalence, counts of inflection points, monotonely decreasing with respect to spatial variable, and time evolution. Also, both models give closed values for discontinuity positions. Thus, power-law model and F^α-model coupled by the condition (14) are more close to each other than that coupled by "natural" condition (13).

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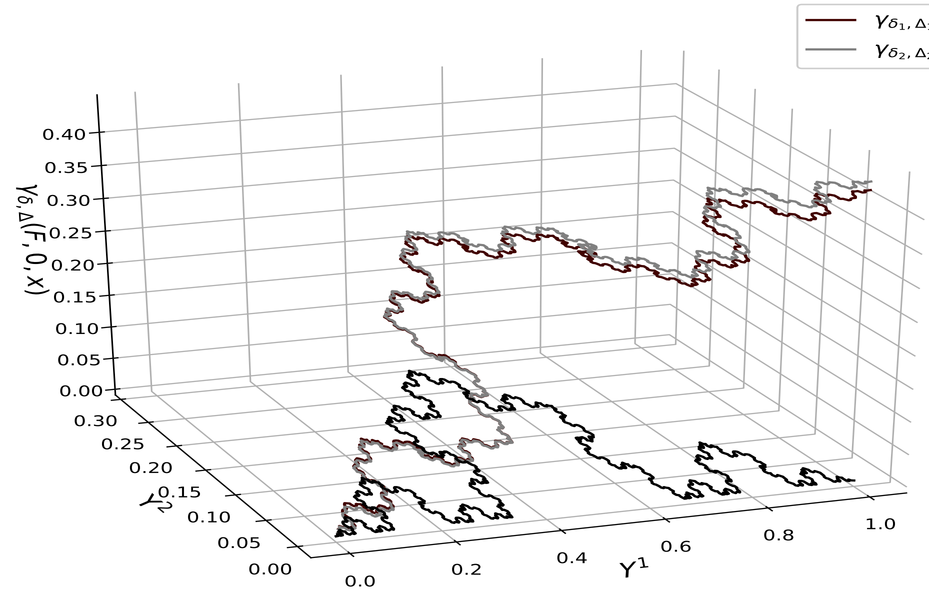

Figure 2: Graphs of coarse grained mass function $\gamma_{\delta,\Delta}(F, 0, x)$ for $\delta = \delta_1 = 0.033, \Delta = \Delta_1 = 0.000411$ (dark grey line) and $\delta = \delta_2 = 0.018, \Delta = \Delta_2 = 0.000148$ (light grey line).

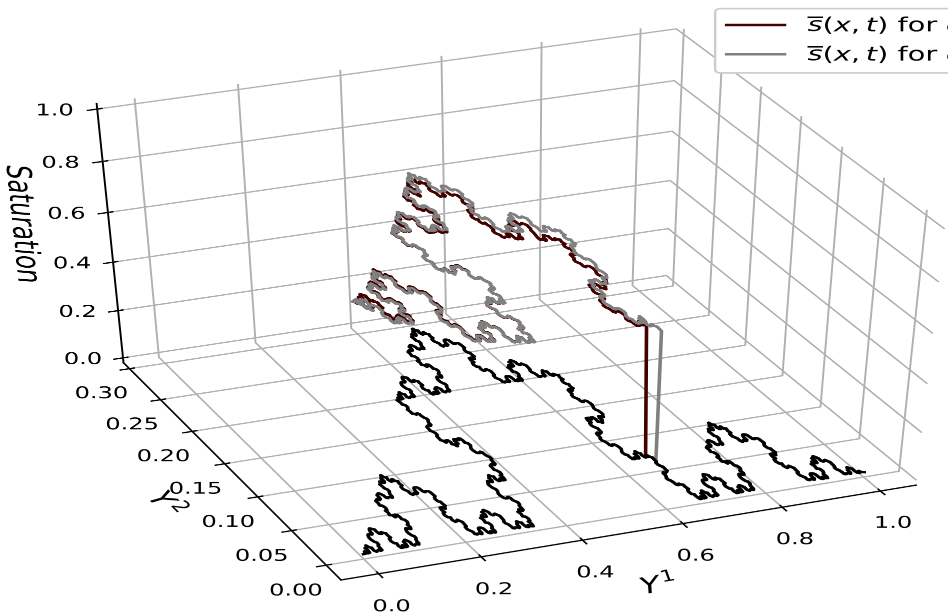

Figure 3: Graphs of saturation $\bar{s}(x, t; \delta, \Delta)$ for $\delta = \delta_1 = 0.033, \Delta = \Delta_1 = 0.000411$ (dark grey line) and $\delta = \delta_2 = 0.018, \Delta = \Delta_2 = 0.000148$ (light grey line).

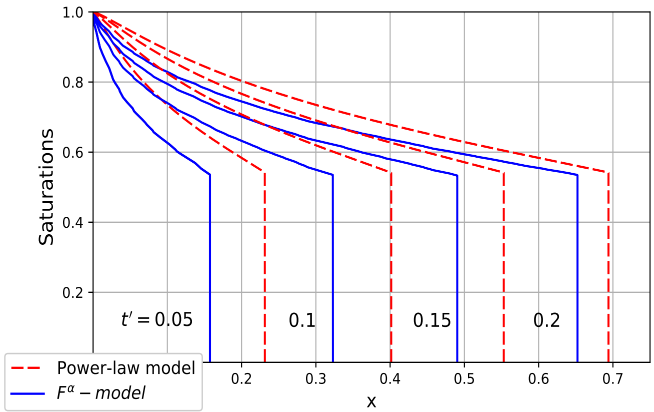

Figure 4: Graphs of saturations $\bar{s}(x,t;\delta,\Delta)$, $s(x,t;\sigma)$ with the power law coefficient from "natural" condition $\gamma_{\delta,\Delta}(F,a,b) = M(L;\sigma)$ for $\delta = \delta_1 = 0.033, \Delta = \Delta_1 = 0.000411$.

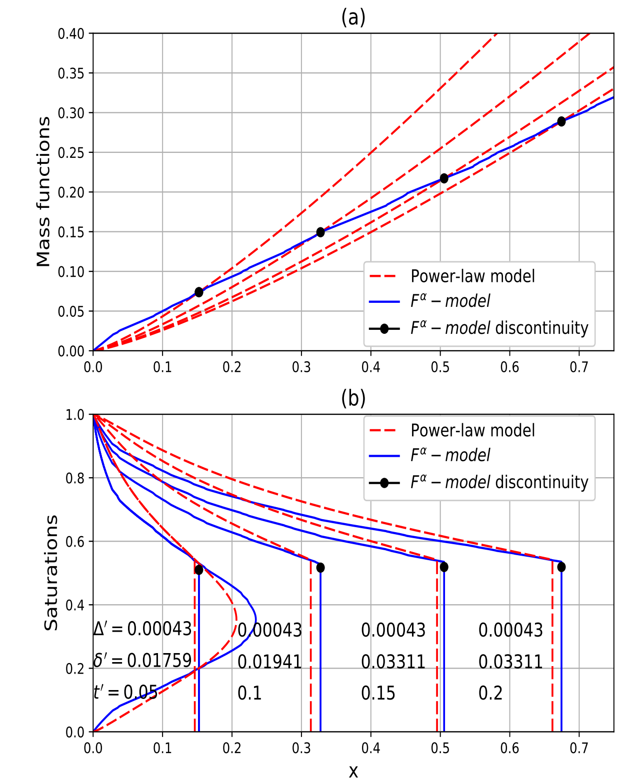

Figure 5: Graphs of coarse grained mass function $\gamma_{\delta_1,\Delta_1}(F,0,x)$ for $\delta_1 = 0.033, \Delta_1 = 0.000411$, saturations $\bar{s}(x,t;\delta',\Delta')$ and corresponding mass functions $M(x;\sigma')$, saturations $s(x,t;\sigma')$ (δ',Δ',σ' depend on $\bar{x}_{fr}(t')$).

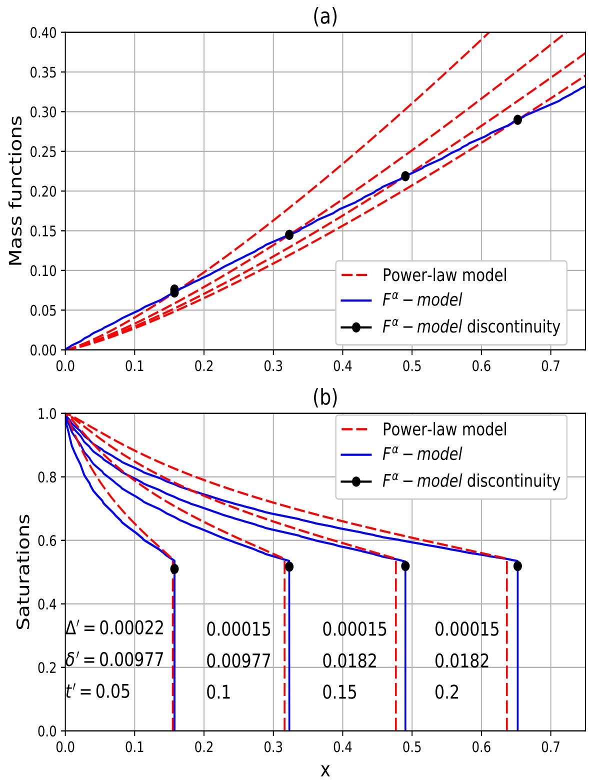

Figure 6: Graphs of saturations $\bar{s}(x,t;\delta',\Delta')$, $s(x,t;\sigma')$ (δ',Δ' and corresponding σ' depend on $\bar{x}_{fr}$) and mass functions $\gamma_{\delta_2,\Delta_2}(F,0,x)$, $M(x;\sigma')$ for $\delta_2 = 0.018, \Delta_2 = 0.000148$.

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