# Numerical Investigation of Radial Steady-State Fluid Flow Model with Riesz Potential

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# Contents

1. Fractional integrals and modification approach

2. Modified model and investigation of steady-state case

3. Investigation of non-steady-state case

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1. Fractional integrals and modification approach

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The left-sided and the right-sided Riemann–Liouville fractional integrals of order $\alpha$ (one-dimensional)

$$({}_{\mathrm{a}}I_x^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^x \frac{f(y)dy}{(x-t)^\alpha}, \qquad ({}_{\mathrm{x}}I_b^\alpha f)(x) = \frac{-1}{\Gamma(1-\alpha)}\int_x^b \frac{f(y)dy}{(x-t)^\alpha}, \qquad 0<\alpha<1.$$

**Riesz potential operator** of order $\alpha$ for radial function (multidimensional Riemann–Liouville integral)

$$(K_r^\alpha f)(r) = \frac{1}{\left(2^\alpha \pi \Gamma(\alpha/2)\right)}\int_{B^R} \frac{f(y)dy}{|r-y|^{2-\alpha}},$$

where $B^R -$ is the ball with radius $R$.

S. G. Samko, A. A. Kilbas, O. I. Marichev, et al., Fractional integrals and derivatives, 1993.

There is a class of functions for which the **one-dimensional representation**[1] of Riesz operator is valid. This formula in $\mathbb{R}^2$-case has the form.

$$(K_r^\alpha f)(r) = 2^{-\alpha}({}_0I_{\rho^2}^{\alpha/2}[s^{-\alpha/2}{}_sI_{R^2-}^{\alpha/2} f(\sqrt{\tau})])(\rho)|_{\rho=r^2},$$

$$({}_0I_{\rho^2}^{\alpha/2} g)(\rho) = \frac{1}{\Gamma(\alpha/2)}\int_0^{\rho^2} g(s)(\rho^2 - s)^{\alpha/2-1} ds,$$

$$({}_sI_{R^2}^{\alpha/2} h)(s) = \frac{1}{\Gamma(\alpha/2)}\int_s^{R^2} h(\tau)(\tau - s)^{\alpha/2-1} d\tau,$$

where ${}_0I_{\rho^2}^{\alpha/2} g$, ${}_sI_{R^2}^{\alpha/2}$ – are left-sided and right-sided Riemann–Liouville integral.

**Further**, we assume that all functions satisfy hypotheses of the one-dimensional representation theorem.

[1] B. Rubin, “One-dimensional representation, inversion, and certain properties of the riesz potentials of radial functions,” Mathematical Notes 34,751–757 (1983).

## Classical linear model

$$\frac{\partial m\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0, u = -k/\mu \frac{\partial p}{\partial x}, \beta = \frac{\partial \rho}{\partial P} \equiv const$$ (fluid is incompressible),

where $\mu$ – viscosity, $u$ – filtration speed, $m$ – porosity, $p$ – pressure, $k$ – permeability, $\rho$ – density.

## Classical radial model

$$\frac{\partial m\rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho u)}{\partial r} = 0, u = -\frac{k}{\mu}\frac{\partial p}{\partial r},$$

## Fractional linear model[1]

$$u = -k\mu\, {}_{\mathrm{a}}I_x^\alpha \frac{\partial p}{\partial x}$$

## Fractional radial model

$$\cancel{u = -k/\mu\, {}_{\mathrm{a}}I_r^\alpha \frac{\partial p}{\partial r}}$$

$$\cancel{u = \frac{\partial}{\partial r}(-k/\mu\, {}_{\mathrm{a}}I_r^\alpha p)}$$

$$u = -k/\mu K_r^\alpha \frac{\partial p}{\partial r}$$

(qualitative difference with practice: Constant solution/non-invariant with respect to the choice of point reference)

[1]A. Chang, H. Sun, Y. Zhang, C. Zheng, and F. Min, “Spatial fractional darcy’s law to quantify fluid flow in natural reservoirs,” Physica A: Statistical Mechanics and its Applications 519, 119–126 (2019).

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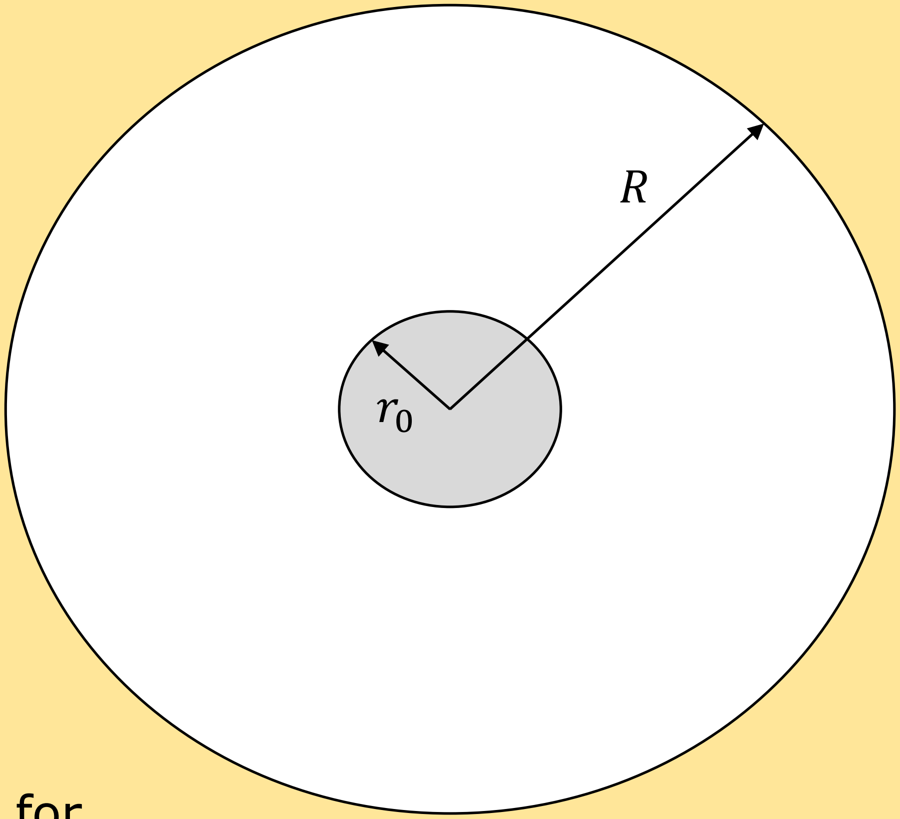
$$\frac{\partial p}{\partial t} = \gamma \, \frac{\partial}{\partial r}\left( r K_r^\alpha \frac{\partial p}{\partial r}\right),$$

$$\gamma = \frac{k}{\mu m \beta}, \beta = \frac{\partial \rho}{\partial P}, 0 < r < R, 0 < \alpha < 1,$$

$$\boxed{p(r) = p_w, 0 \le r \le r_0,}$$

$$p(R) = p_c, \qquad p_w, p_c, \gamma \equiv const,$$

$$(K_r^\alpha f)(r) = (2^\alpha \pi \Gamma(\alpha/2))^{-1} \int_{B^R} \frac{f(y)dy}{|r-y|^{2-\alpha}},$$




where $r_0$ –is the well radius,
$R$ – is the reservoir radius,
$p(r)$ – is the pressure,
$K_r^\alpha$ – Riesz potential operator of order $\alpha$ for radial function $\mathbb{R}^2$,
$B^R$ – is the ball with radius $R$,
$p_w$ – bottomhole pressure,
$p_c$– boundary pressure.

Substituting $x = r^2$ in (1), we get

$$2\sqrt{x}\frac{d}{dx}\left(\sqrt{x}kK_x^\alpha\left[2\sqrt{x}\frac{dp}{dx}\right]\right) = 0, \qquad p = p(x), r_0^2 < x < R^2,$$

$$p(x) = p_w, 0 \le x \le r_0^2, p(R^2) = p_c,$$

$$(K_x^\alpha f)(x) = 2^{-\alpha}({}_0I_x^{\alpha/2}[x^{-\alpha/2}({}_xI_{R^2}^{\alpha/2} f)(x)])(x).$$

Let us consider a finite-difference mesh for $x$:

$$\omega = \{x_i, i = 0, \dots, N, x_0 = r_m^2, x_N = R^2, x_{n_1} = r_0^2, 0 < n_1 < N.\}$$

**Method of weight functions**[1].

The left-side fractional trapezoidal formula in this case has the form

$$({}_0I_x^{\alpha/2} g)(x)|_{x=x_j} \approx \sum_{k=0}^{j} a_{k,j} g_k, \; g_k = g(x_k),$$

$$a_{k,j} = \frac{1}{\Gamma(\alpha/2)}\begin{cases} \int_{x_0}^{x_1}(x_j - x)^{\alpha/2-1}\frac{x_1 - x}{x_1 - x_0}dx, & \mathrm{k} = 0, \\ \int_{x_k}^{x_{k+1}}(x_j - x)^{\alpha/2-1}\frac{x_{k+1} - x}{x_{k+1} - x_k}dx + & \\ + \int_{x_{k-1}}^{x_k}(x_j - x)^{\alpha/2-1}\frac{x - x_{k-1}}{x_k - x_{k-1}}dx, & \mathrm{k} = 1, \dots, \mathrm{j} - 1, \\ \int_{x_{j-1}}^{x_j}(x_j - x)^{\alpha/2-1}\frac{x - x_{k-1}}{x_j - x_{j-1}}dx, & \mathrm{k} = \mathrm{j}. \end{cases}$$

[1] N. S. Bakhvalov, N. Zhidkov, and G. Kobel'kov, Numerical Methods [in Russian] (Moscow: Nauka, 1973).

For the right-side integral, we get

$$({}_xI_{R^2}^{\alpha/2}h)(x)|_{x=x_j} \approx \sum_{k=j}^{N} b_{k,n}h_k,$$

$$b_{k,j} = \frac{1}{\Gamma(\alpha/2)}\begin{cases} \int_{x_j}^{x_{j+1}} (x-x_j)^{\alpha/2-1}\frac{x_{j+1}-x}{x_{j+1}-x_j}dx, & \mathrm{k = j,} \\ \int_{x_k}^{x_{k+1}} (x-x_j)^{\alpha/2-1}\frac{x_{k+1}-x}{x_{k+1}-x_k}dx + & \\ + \int_{x_{k-1}}^{x_k} (x-x_j)^{\alpha/2-1}\frac{x-x_{k-1}}{x_k-x_{k-1}}dx, & \mathrm{k = j+1,\dots,N-1,} \\ \int_{x_{N-1}}^{x_N} (x-x_j)^{\alpha/2-1}\frac{x-x_{N-1}}{x_N-x_{N-1}}dx, & \mathrm{k = N.} \end{cases}$$

Combining one-dimensional representation formula, (5) and (6), we obtain

$$(K_x^\alpha f)(x)|_{x=x_n} \approx 2^{-\alpha}\sum_{i=0}^{n}\left[a_{i,n}x_i^{-\alpha/2}\left(\sum_{j=i}^{N} b_{j,i}f_j\right)\right] = \sum_{i=0}^{N} A_{i,n}f_i,$$

$$A_{i,n} = 2^{-\alpha}x_i^{-\alpha/2}\begin{cases} a_{i,n}\sum_{j=i}^{nX} b_{j,i}, & \mathrm{i = 0,\dots,n,} \\ 0, & \mathrm{i = n+1,\dots,N.} \end{cases}$$

$$2\sqrt{x}\frac{d}{dx}\left(\sqrt{x}kK_x^{\alpha}\left[2\sqrt{x}\frac{dp}{dx}\right]\right) = 0, p = p(x), r_0^2 \le x < R^2,$$

$$p(x) = p_w, 0 \le x \le r_0^2, p(R^2) = p_c.$$

$$(K_x^{\alpha} f)(x) = 2^{-\alpha}({}_0I_x^{\alpha/2}[x^{-\alpha/2}({}_xI_{R^2}^{\alpha/2} f)(x)])(x).$$

Integrating (4) in $x$, we get

$$K_x^{\alpha}\left[2\sqrt{x}\frac{dp}{dx}\right] = \frac{c}{\sqrt{x}}, c \equiv const.$$

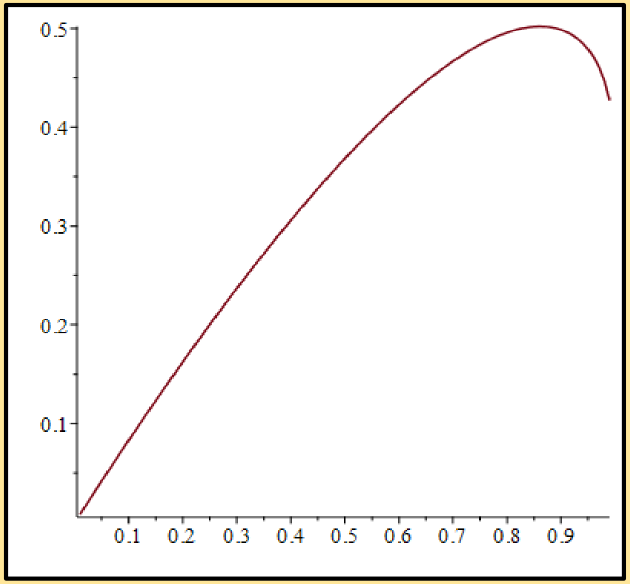
$$\sum_{i=n_1+1}^{N-1} \overline{A}_{i,n} p_i - B_n c = E_n, n = n_1, \ldots, N-1;$$

$$p_N = p_c, p_k = p_w, \mathrm{k} = 0..\mathrm{n}_1,$$

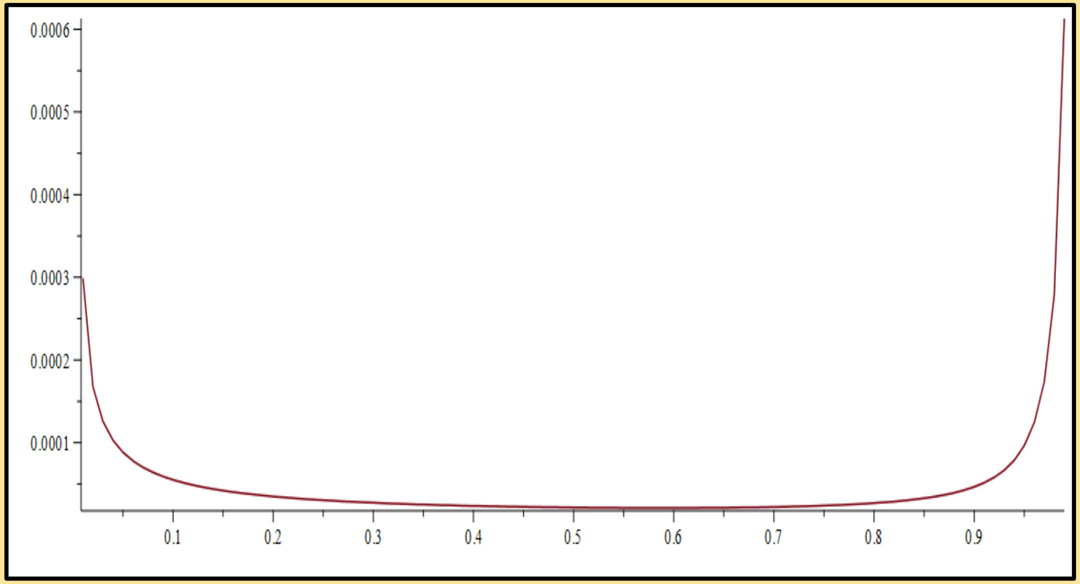
$$\overline{A}_{i,n} = \frac{2(\sqrt{x_{i-1}}A_{i-1,n} - \sqrt{x_i}A_{i,n})}{x_{i+1} - x_i}, i \ne 0, \overline{A}_{0,n} = -\frac{2\sqrt{x_0}A_{0,n}}{x_1 - x_0},$$

$$B_n = \frac{1}{\sqrt{x_n}}, n \ne 0, B_0 = 0,$$

$$E_n = -\sum_{i=0}^{n_1} \overline{A}_{i,n} p_{n_1} - \frac{2\sqrt{x_{N-1}}A_{N-1,n}p_N}{x_N - x_{N-1}}.$$

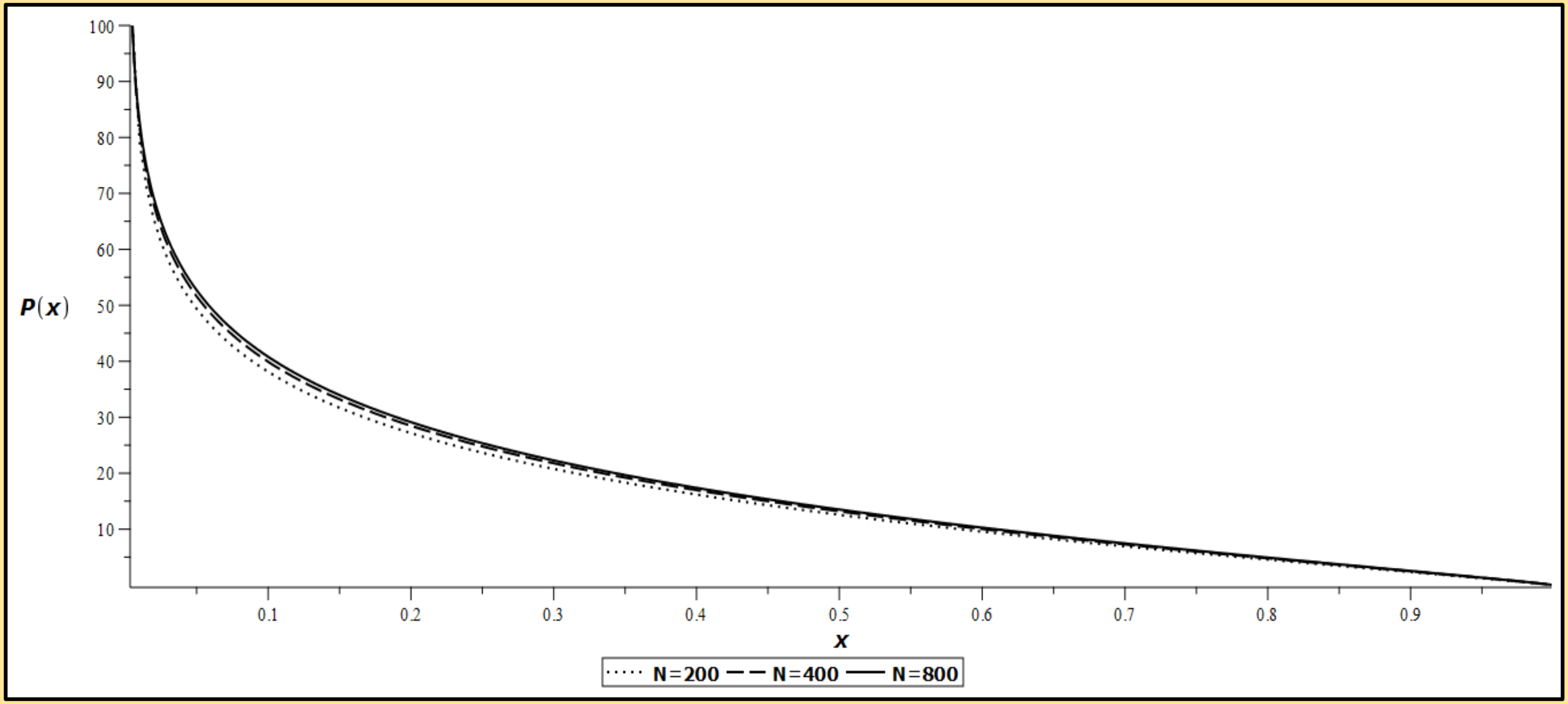



Exact value of $(K_x^\alpha f)(x)$




Absolute value of error $(K_x^\alpha f)(x)$

$\alpha = 0.5, \mathrm{f(x)} = \mathrm{x}, \mathrm{r}_0 = 0.001, R = 1, \mathrm{N} = 100.$
The calculation is maded on uniformly spaced grid.




Model parameteres: $\alpha = 0.25, r_0^2 = 0.005, R = 1, p_w = 100, p_c = 0.$
The calculations were made on a uniform grid.

The numerical scheme has a high convergence rate when grinding the mesh. Deviations of solutions increase in the region of high values of the pressure derivative modulus (Euler method error).

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Substituting $x = r^2$ in (1), we get

$$2\sqrt{x}\frac{d}{dx}\left(\sqrt{x}kK_x^\alpha\left[2\sqrt{x}\frac{\partial p}{\partial x}\right]\right) = \frac{\partial p}{\partial t}, \qquad p = p(x,t), \qquad r_0^2 < x < R^2, 0 < \mathrm{t} < \mathrm{T}$$

$$p(x,t) = p_w, 0 \le x \le r_0^2, p(R^2) = p_c, p(x,0) = p^0(x)$$

$$(K_x^\alpha f)(x) = 2^{-\alpha}({}_0I_x^{\alpha/2}[x^{-\alpha/2}({}_xI_{R^2}^{\alpha/2} f)(x)])(x).$$

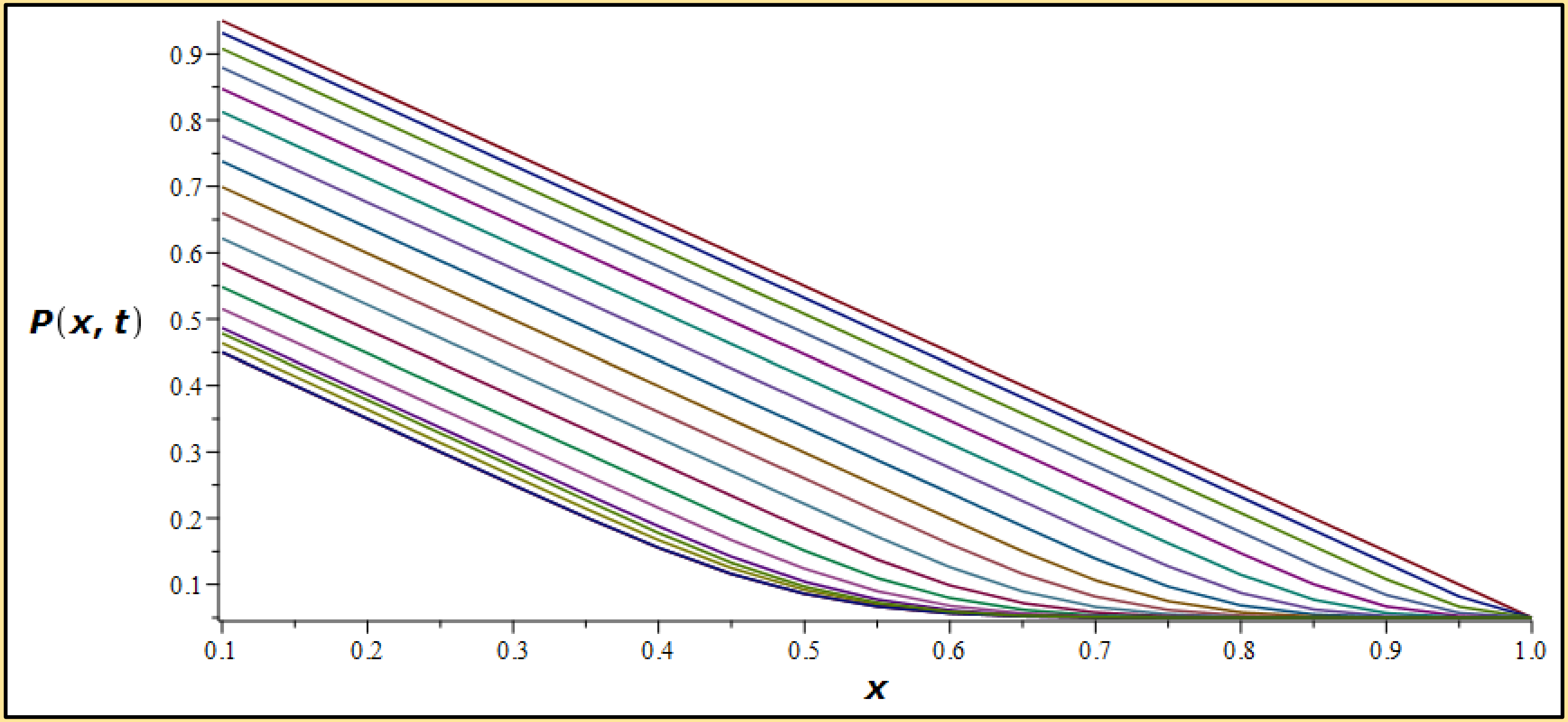
Explicit scheme

$$\frac{p_n^{i+1} - p_n^i}{\tau} = \frac{4\sqrt{x_n}}{h^2}\sum_{k=0}^{N} \left(\tilde{A}_{k,n+1} - \tilde{A}_{k,n}\right)\left(p_{n+1}^i - p_n^i\right), p_k^0 = p^0(x_n),$$

$$\mathrm{n} = n_1, \ldots, N_x - 1, \qquad \mathrm{i} = 1, \ldots, N_t;$$

$$p_N^i = p_c, p_j^i = p_w, \mathrm{j} = 0..\mathrm{n}_1,$$

$$\tilde{A}_{k,n} = \sqrt{x_n}\sqrt{x_k}A_{i,n}, \tau = \frac{\mathrm{T}}{N_t}, h = \frac{R^2}{N_x}.$$




Model parameteres: $\alpha = 0.25, r_0^2 = 0.05, R = 1, p(0,x) = 1 - \mathrm{x}, \mathrm{t} \in [0,0.5]$, $\tau = 0.025, \mathrm{h} = 0.05$.
The calculations were made on a uniform grid.

The explicit numerical scheme is conditional stable.

# Conclusion



**I.** Radial steady-state fluid flow model for porous media with the Riesz potential is considered. A numerical scheme for model with one-dimensional representation of the Riesz potential is presented.

**II.** Regularity and monotonicity of solution of the Dirichlet boundary value problem corresponding to the constant pressure in the bottomhole zone are discussed.

**III.** Radial non-steady-state fluid flow model for porous media with the Riesz potential is considered. Conditional stability of explicit numerical scheme for this model is showed.

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$$\frac{2}{(2-\alpha)\Gamma^2\left(\frac{\alpha}{2}\right)}\left(\frac{r_w^2}{s}\right)^{1-\frac{\alpha}{2}}\int_{r_w^2}^{R_c^2}\frac{u'(\xi)}{\xi^{\frac{1}{2}-\frac{\alpha}{2}}}F_1\left(1-\frac{\alpha}{2},1-\frac{\alpha}{2},1-\frac{\alpha}{2};2-\frac{\alpha}{2};\frac{r_w^2}{s},\frac{r_w^2}{\xi},\right)d\xi+$$
$$+{}_{r_w^2}I_s^{\frac{\alpha}{2}}\left[s^{-\frac{\alpha}{2}}\,{}_sI_{R_c^2}^{\frac{\alpha}{2}}\left(\sqrt{s}\frac{du}{ds}\right)\right]=\frac{C_1}{\sqrt{s}},\quad u(s)=p(\sqrt{s}),\quad s=r^2.\quad (3.1.14)$$