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AN INTRODUCTION TO GENERALIZED COMPLEX GEOMETRY

THE UNIVERSITY OF ZURICH

Contents

Introduction

These notes grew out of an introductory course on generalized complex geometry taught in the fall of 2022 at the University of Zurich. None of the content is original to the author of these notes. The author would like to thank

This course is far from a complete account of the subject, instead the aim is to get a quick overview and work towards essential milestones in the development of generalied geometry.

Generalized reduction. Review symplectic and holomorphic reduction.

1. stienonXu2007

2. zambon2008 generalization of coisotropic reduction

3. bursztynCavalcantiGualtieri2006

There are no real punch line examples. What about the GC structure on the instanton moduli space? from bursztynCavalcantiGualtieri2013?

Generalized Kähler structures? This really goes into sigma models and super symmetry as examples. I know a lot of about this topic, that would make it easier to prepare

Branes in prequantization and physics?

1. kapustinOrlov

GCG and supersymmetry? kapustinLi2005 for a discussion of sigma models, supersymmetry and generalized complex structures. This could also lead to T duality?

Geometric structures from differential forms

ZT1VUV

To revisit a few classical geometric structures and become accustomed with notation and conventions, we will dive into geometric structures provided by differentia forms and explore their integrability conditions.

[0] This contains Poisson geometry. This is a side note

Background

WE WILL ASSUME throughout this course that M is a smooth manifold. Whenever we use a category, it will be in a `monospaced` font, like so. [1] What about other equations?

[1] this is a side note about categories.

$$\sum_{i=0} i^2$$

This is supposed to be red.

This is a rather unconventional summation.

$$\begin{tikzcd} & (S,\omega) \arrow[dl, "\pi_M"'] \arrow[dr, "\pi_N"] & \\ (M,Q) & & (N,P) \end{tikzcd}$$

Figure 1: This is a Poisson span

Foliations

Reeb Foliation

Reduction of structure group

A very popular and powerful perspective on geometric structures is the reductio of structure groups.

compare from shaffhauser2009 Proposition 2.3

An oriented Riemannian n-manifold (M, g) corresponds to a reduction to $SO(n)$.

A complex structure is a reduction from GL_n to $U(n)$.

$V \oplus V^*$

We're using [2],

[2] Marco Gualtieri. *Generalized Complex Geometry*. PhD thesis, Oxford, 2004

Linear Algebra

Let V be a real vector space of dimension n. This chapter is devoted to the Linear algebra of the vector space $V \oplus V^*$.

marcutoooo

$V \oplus V^*$ always carries a bilinear, symmetric pairing

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X))$$

for $X, Y \in V$ and $\xi, \eta \in V^*$. Let $e_1, \ldots e_n$ be a basis of V and $e^1, \ldots, e^n$ its dual basis. We choose their union as a basis of $V \oplus V^*$ and compute

$$A_{\langle\ \rangle} = \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}$$

This shows that the canonical pairing $\langle\ \rangle$ is nondegenerate and has signature (n, n).

I could make this into an exercise.

Symmetries

As any metric space, we can consider symmetries

$$SO(V \oplus V^*) =$$

The special orthogonal Lie algebra $\mathfrak{so}(V \oplus V^*)$ is identified with $\Lambda^2(V \oplus V^*) = \Lambda^2 V \oplus V \otimes V^* \oplus \Lambda^2 V^*$. Elements are possibly of the form

$$\beta + A + B$$

For a metric vector space (W, g) we use here the correspondence between $\mathfrak{so}(W, g) \cong \Lambda^2 W$ that identifies $u \wedge v$ with $ug(v) - vg(u)$ for two elements $u, v \in W$.

We start by investigating $B \in \Lambda^2 V^*$. Write $B = B_{ij} e^i \wedge e^j$ in the same basis $e^1, \ldots, e^n$ as before. Pick an arbitrary element $X + \xi \in V \oplus V^*$

and compute

$$\begin{aligned} B\cdot(X+\xi) &= B_{ij}e^j\langle e^i, X+\xi\rangle - B_{ij}e^i\langle e^j, X+\xi\rangle \\ &= B_{ij}e^je^i(X) - B_{ij}e^ie^j(X) \\ &= \iota_X(B_{ij}e^i\wedge e^j) \\ &= \iota_X B \end{aligned}$$

In particular, we notice that $B^2=0$. This lets us compute its exponential $e^B \in SO(V\oplus V^*)$ as follows

$$e^B(X+\xi) = (1+B+\frac{1}{2}B^2+\cdots)(X+\xi) = X+\xi+\iota_X B$$

This symmetry has the form of a shear transformation and is commonly referred to as the B-field transform.

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

Isotropic subspaces

A subspace $U \subseteq V$ in a metric space (V,g) has a complement

$$U^\perp = \{x\in V \quad | \quad g(x,u)=0 \text{ for all } u\in U\}$$

A subspace $U\subseteq V$ is *isotropic* if $U\subseteq U^\perp$. In other words, $g\big|_U = 0$. We call a subspace U *coisotropic* if $U^\perp \subseteq U$.

Exercise: Shat that U^* is coisotropic if U is isotropic.

For our split signature metric $\langle\ \rangle$ an isotropic subspace $W\in V\oplus V^*$ can have at most dimension n. If it has dimension n, we call it *maximally isotropic* or *Lagrangian*. This is the case if W is both isotropic and coisotropic

This name comes from maximally isotropic subspaces of symplectic vector spaces (V,ω).

Exercise: Shat that $L \subseteq V\oplus V^*$ is Lagrangian iff it is isotropic and coisotropic.

Example 1. V and V^* are both maximally isotropic subspaces of $V\oplus V^*$.

Example 2. One way to construct a maximally isotropic subspace of $V\oplus V^*$ is to start with a subspace $U\subseteq V$ and consider

$$U\oplus Ann(U)\subseteq V\oplus V^*$$

This subspace is clearly isotropic and of maximal dimension because

$$\dim Ann(U) + dim(U) = \dim(V) = n$$

The Annihilator of a subspace is precisely the subspace $Ann(U)$ of V^* of covectors α which restrict to zero on U, that is, $\alpha\big|_U = 0$.

Lemma 3. Every maximal isotropic $L\in V\oplus V^*$ is of the form $L(E,\varepsilon)$

A B-field transform $B\in\Lambda^2V^*$ affects subspaces $L\subseteq V\oplus V^*$.

Spinors

Given a metric vector space (W, g), Spinors arise as a hidden representation of the Lie algebra of special orthogonal transformations $\mathfrak{so}(W, g)$. This does not correspond to a representation of $SO(W, g)$ when it is not simply connected. Instead, one constructs the universal cover, called the spin group $Spin(W, g)$ as a certain set contained in the Clifford algebra $Cl(W, g)$.

Reference needed, also suggested reading

More details can be found in wernli2019

The important application of spinors in our situation is as a means to encode maximally isotropic subspaces of $V \oplus V^*$.

Definition 4. The *Clifford algebra* of a metric vector space (W, g) is the quotient of the tensor algebra of W by the relation $v^2 = g(v, v)1$.

Why is this equivalent to the relation $vw + wv = 2g(v, w)$? wernli2019

General properties of the Clifford algebra.

Definition 5. The Spin group of (W, g) consists of the

The adjoint representation stabilizes $W \hookrightarrow Cl_W$. A vector $v \in W$ acts as a reflection across the g-orthogonal complement of v.

Exercise 6.

1. Describe the action of an element $uv \in Cl_W$ on $w \in W$.
2. How is the action of $-uv$ different?

The conclusion of this exercise should be that the standard representation of $Spin(W)$ on W is blind to certain subtle differences. We need spinors to detect these differences.

To construct spinors for $\mathbb{V} = V \oplus V^*$, we can take advantage of the split signature metric.

Now that we have spinors for $\mathbb{V}$, we can use the following.

Let $\varphi \in \Lambda^\bullet V^*$ be a spinor. Its *null space* L_φ is the subspace in $V \oplus V^*$

$$L_\varphi = \{v \in V \oplus V^* \quad | \quad v \cdot \varphi = 0\}$$

Null spaces of spinors are always isotropic. When L_φ is maximally isotropic, we call the spinor φ *pure*.

Exercises

Exercise 7. Show that $SO(n)$ is not simply connected for any $n \geq 2$.

taken from debray2016

Exact Courant algebroids

The reason we focussed intensively on the linear algebra of $V \oplus V^*$ is that it is precisely the generic fiber of the generalized tangent bundle $TM \oplus T^*M$ of an n-manifold M. This is the central object of interest in the study of generalized complex geometry as it is hosting generalized complex structures which we will define in the next chapter. For now, we will explore the geometric properties of $TM \oplus T^*M$ as a Courant algebroid. We will assume throughout this chapter that M is a smooth manifold.

Lie algebroids

In fact, we will start with yet another simplification to get acustomed with the world of *-oids.

Definition 8. A Lie algebroid $(A, [\,], \mathfrak{a})$ on M is a vector bundle $A \to M$ with a Lie bracket $[\,]$ on the space of sections $\Gamma(A)$ and a bundle map $A \xrightarrow{\mathfrak{a}} TM$ such that

1. conditions

■

A Poisson structure Q on M defines a bracket on 1-forms Ω^1_M.

$$[a, b]_Q = L_{Q(a)}b - L_{Q(b)}a - dQ(a, b)$$

This is the *Koszul bracket* which is the unique extension of the Poisson bracket on exact 1-forms that satisfies...

Example 9.

1. Poisson Lie algebroid
2. Foliations are Lie algebroids if they are integrable
3. Atiyah algebroid of a principal bundle

4. $T_{1,0}X$ of a complex manifold is Lie algebroid on its underlying smooth manifold.

■

Courant algebroids

Failure to be a Lie algebroid. Exactness, twists and Severa class. Symmetries of Courant algebroids.

Examples like

Other higher Courant algebroids like $T \oplus \mathbb{R} \oplus T^*$

Structures on Courant algebroids

Generalized complex structures

hu2006 for Hamiltonian symmetries of a GC structure
bursztyn2011 for Dirac structures.

Bibliography

[1] Marco Gualtieri. *Generalized Complex Geometry*. PhD thesis, Oxford, 2004.

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