Homework 1 - Problem 5B

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Problem Statement:
Consider the following sequence of numbers: $s_n = 1 + \prod_{i=0}^{n-1} s_i$, where $s_0 = 2$. Prove that any two numbers in this sequence are relatively prime.

Proof by Induction:

Basis:

$$s_0 = 2$$
$$s_1 = 1 + \prod_{i=0}^{0} s_i = 1 + s_0 = 1 + 2 = 3$$

Induction Hypothesis:
Suppose for all $n \geq 0$, s_n are pairwise relatively prime. In other words, $(\forall\, n \geq 1)\ \gcd(s_{n-1}, s_n) = 1$. Show that $\gcd(s_n, s_{n+1}) = 1$.

We know that the following property holds for gcd's,

$$\text{For some sequence of integers } a_i, \text{ if } gcd(a_i, a_{i+1}) = d \text{ and } gcd(a_{i+1}, a_{i+2}) = d, \text{ then } gcd(a_i, a_{i+2}) = d. \tag{1}$$

For the sequence of numbers s_i, if we can show that $\gcd(s_n, s_{n+1}) = 1$, then we'd have shown that all numbers in the sequence are relatively prime.

$$\begin{aligned}
s_{n+1} &= 1 + s_n \prod_{i=0}^{n-1} s_i \\
&= 1 + s_n (s_n - 1) && \text{, because } s_n - 1 = \prod_{i=0}^{n-1} s_i \text{ from } s_n = 1 + \prod_{i=0}^{n-1} s_i, \text{ by definition.} \\
1 &= s_{n+1} - (s_n - 1)\, s_n \\
&= a s_{n+1} - b s_n && \text{, where } a = 1 \text{ and } b = s_n - 1.
\end{aligned}$$

So we have shown that 1 is a linear combination of s_n and s_{n+1}, and there is no other positive integer smaller than 1 so we can conclude that $\gcd(s_n, s_{n+1}) = 1$. By the induction hypothesis and property 1, all numbers in the sequence s_i for $i = 0 \ldots n$ are pairwise relatively prime.

QED