EECE251

Circuit Analysis I

Set 3: Additional Analysis Techniques and Circuit Theorems

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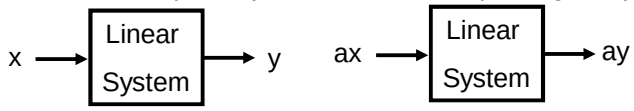
Reading Material

- Chapter 5 of the textbook
  - Section 5.1: Linearity
  - Section 5.2: Superposition
  - Section 5.3: Source Transformation, and Thévenin's and Norton's Theorems
  - Section 5.4: Maximum Power Transfer
- Reading assignment:
  - Review Section 5.5: Application Examples

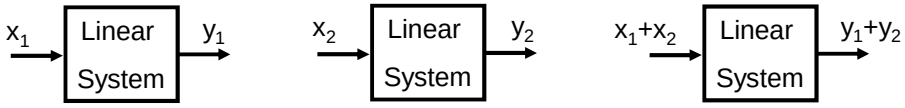
Linearity

- Simply put, a linear circuit (or in general a linear system) is a circuit (system) whose output is linearly related (or directly proportional) to its input.
- A system is linear if and only if it has these two properties:

1. If the input of the system is multiplied by a constant then the output is also multiplied by a same constant (homogeneity).

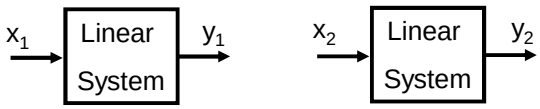


2. The output of the system to a sum of inputs is the sum of the outputs to each input applied separately (additivity).

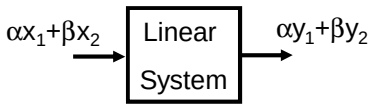


Linearity

- General definition:



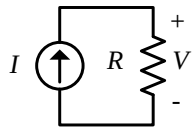
- For all α and β



Linearity

- In circuit theory, the inputs of a circuit are usually the independent voltage or current sources and the outputs are the voltages or currents of some elements of the circuit.

- **Example:** Let's consider a single resistor as a system. As shown below, the input of this system is an independent current source and the output is the voltage drop across the resistor. Is this system linear?



Linearity Theorem

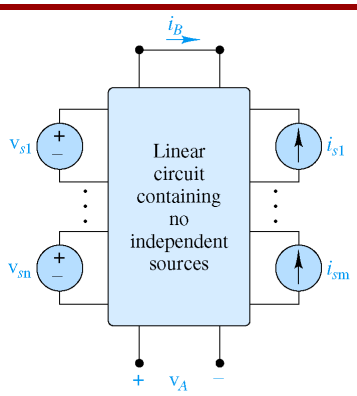
- In a more formal fashion, the following theorem is a general statement of linearity of a linear circuit:

- Linearity Theorem: For any linear circuit, any output voltage or current, denoted by variable y, is related linearly to the independent sources of the circuit, i.e.,

$$y = a_1 u_1 + a_2 u_2 + \cdots + a_m u_m$$

where u_1 through u_m are the voltage and current values of the independent sources in the circuit and a_1 through a_m are properly dimensioned constants.

A Graphical Illustration of the Linearity Theorem



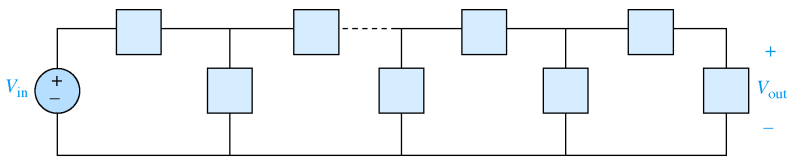
$$v_A = a_1 v_{s1} + \ldots + a_n v_{sn} + b_1 i_{s1} + \ldots + b_m i_{sm}$$

$$i_B = c_1 v_{s1} + \ldots + c_n v_{sn} + d_1 i_{s1} + \ldots + d_m i_{sm}$$

Proportionality Property

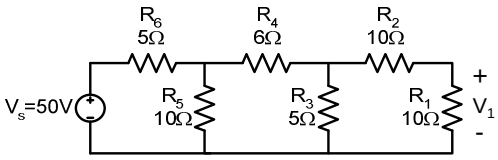
- In a linear circuit, when only one independent source is acting (alone), the output is proportional to that input (refer to linearity theorem). Since the system is linear if the input is multiplied by a constant, then the output is multiplied by the same constant.

- An interesting and illustrative example of usefulness of this proportionality property is the analysis of a ladder network (a circuit which has a patterned structure shown below where each box is a two-terminal element:



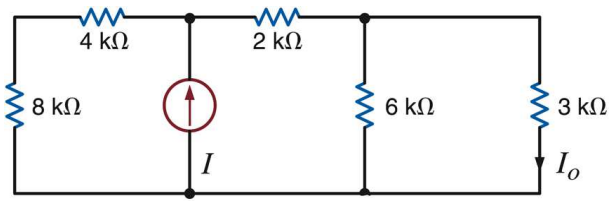
Resistive Ladder Network Example

- In the following circuit find the voltage across R_1.



Example

- Use linearity and the assumption that I_0=1mA to compute the correct value of I_0 in the following circuit if I=6mA.



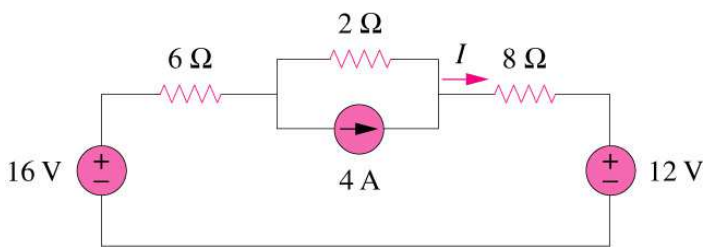
Superposition Property

- In a linear circuit containing more than one independent source, the voltage across or current through any element is the algebraic sum of the of the voltages across or currents through that element due to each independent source acting alone, with the remaining independent sources deactivated, i.e., their values set to zero.
- Steps for applying superposition principle:

1. Turn off all independent sources except one source. Find the desired voltage or current due to that source.
2. Repeat the previous step for each of the independent sources.
3. Find the total desired voltage or current by algebraically adding the contributions due to each of the independent sources

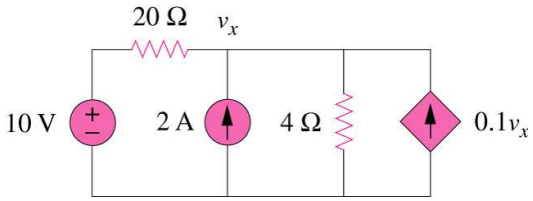
Example

- Using superposition principle find I in the following circuit:



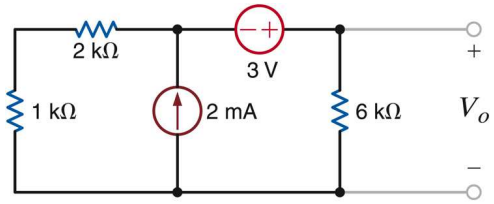
Example

- Using superposition find v_x in the following circuit.



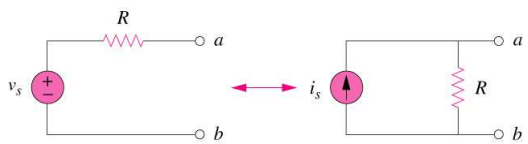
Example

- Use superposition to find V_0.

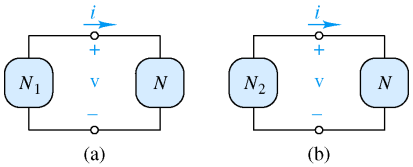


Source Transformation

- Source transformation refers to the process of replacing a voltage source in series with a resistor R with a current source in parallel with a resistor (with the same resistance value as R) and vice versa.

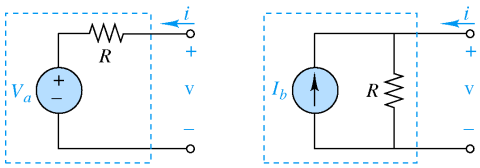


- In many circuits source transformation can simplify the analysis.
- Recall that a pair of two terminal networks (circuits) are equivalent if they have the same terminal v-i characteristics.



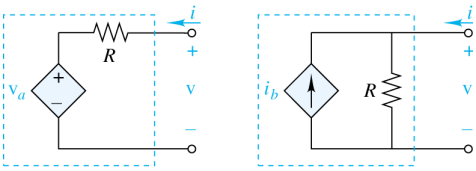
Source Transformation Theorem

- For independent sources (**Note** the direction of the current of the current source):



$V_a = RI_b$

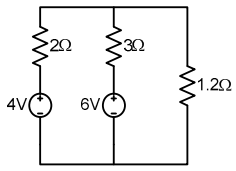
- For Dependent sources:



$v_a = Ri_b$

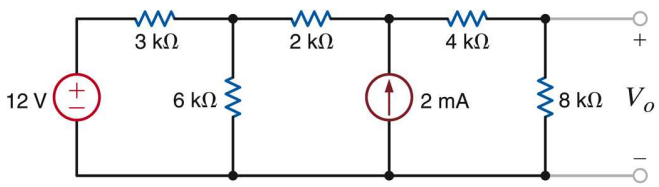
Example

- In the following circuit, find the current i in the 1.2Ω resistor.



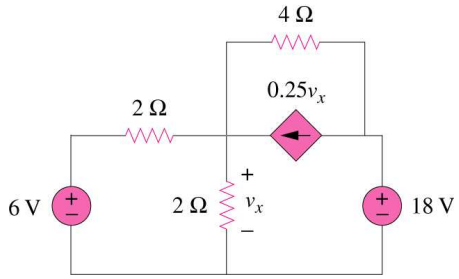
Example

- In the following circuit, use repeated application of source transformation to find V_0.



Example

- In the following circuit, use source transformation to find v_x.



Notes

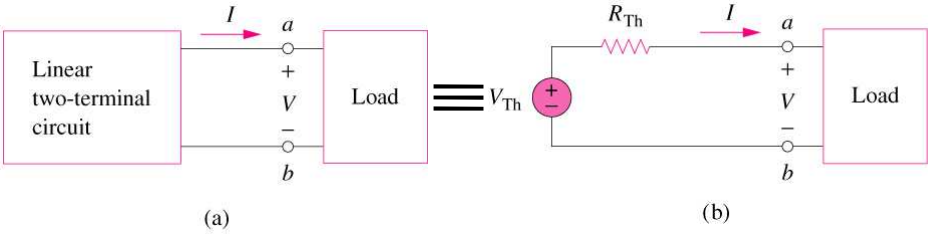
Thévenin's Theorem

- In practice, it often happens that a particular element (or block) in a circuit is variable (this variable block is usually called load) and the rest of the circuit elements are fixed.
- As an example think of different load speakers with different internal resistance that can be connected to an stereo amplifier.

- To avoid analyzing the whole circuit each time the variable element changes, in 1883, Leon Charles Thévenin (1857-1926), a French telegraph engineer came up with the interesting idea of replacing a complicated linear circuit by an equivalent circuit consisting of an independent voltage source in series with a resistor.

Thévenin's Theorem

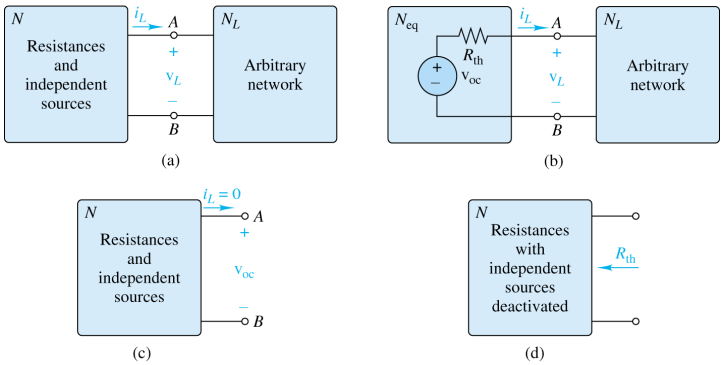
- Theorem:



- Formal proof of the Thévenin's theorem

Thévenin's Theorem

- The value of voltage source in the Thévenin equivalent circuit is the open-circuit voltage at the terminal of the circuit and R_{TH} is the equivalent resistance seen at the terminal when the independent sources inside the circuit are turned off.



Ways to Find Thévenin Equivalent of a circuit

- Approach discussed on previous page, that is $V_{th}=V_{oc}$ and to find R_{th} turn off the independent sources and find the equivalent resistance seen at the load terminal. This approach may be tricky when the circuit include dependent sources.

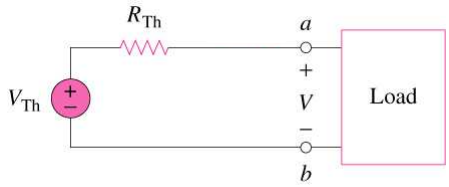
- Another approach is to short-circuit the load terminal. Find the short circuit current, I_{sc}, and then $V_{th}=V_{oc}$ and $R_{th}=V_{oc}/I_{sc}$.
  Can you prove that these equations are correct?

- UBC's 1A/2A approach! (I would like to thank Dr. Luis Linares for explaining this elegant technique to me.)

UBC's 1A/2A Method

- We have two unknowns V_{th} (or V_{Th}) and R_{th} (or R_{Th}). If we find two different equations for this circuit, we should be able to find V_{th} and R_{th}. Let's see if we can write two independent equations!

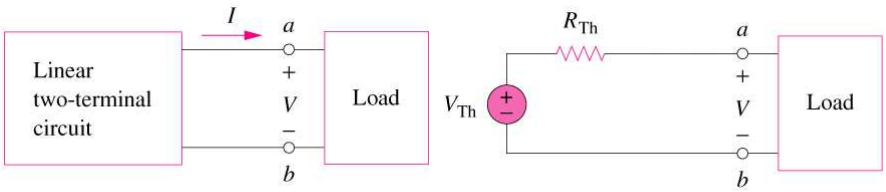
  Hint: What is the voltage V if the load is a current source of value "I" injecting current into node a.



(b)

- Do you think we can use two different values for current source I and thus write two equations with two unknowns. What values would you choose?

UBC's 1A/2A Method



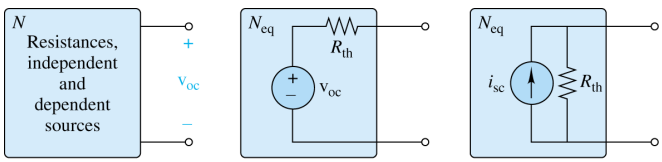
(a) (b)

- If the load is a current source injecting current I to node a, we have: $V_{th} + R_{th}\, I = V$
- If we use I=1A source and find $V=V_1$ and then use I=2A source and find $V=V_2$ then we have:
- $V_{th} + R_{th} = V_1$ and $V_{th} + 2\, R_{th} = V_2$
- Finding V_1 and V_2 from the original circuit, we can solve the above system of two equations two unknowns.

Notes

Norton's Theorem

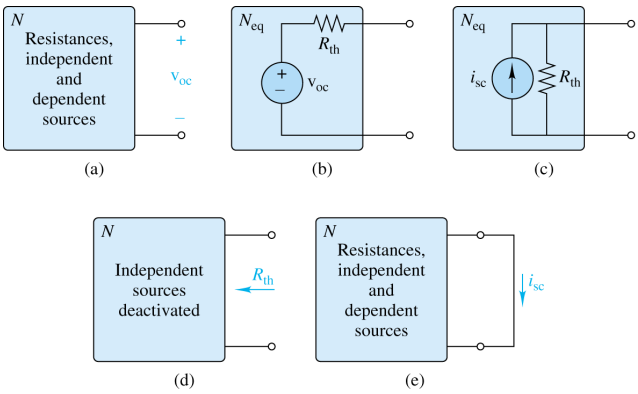
- In 1926 Edward Lawry Norton (1898-1983) a Bell Labs engineer came up with a similar theorem (think of source transformation):



(a) (b) (c)

Norton's Theorem

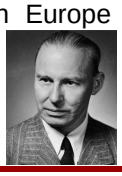
- The value of Norton's equivalent current source is the short circuit current through the terminal of the circuit and $R_N = R_{TH}$.



A Bit of History (According to Wikipedia!)

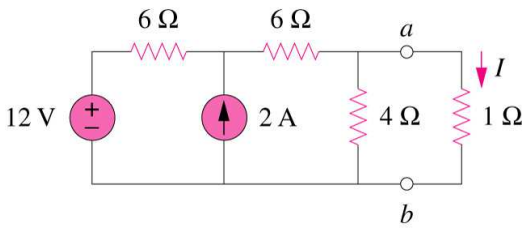
- Apparently, the so called Thévenin theorem was first discovered by German scientist Hermann von Helmholtz (1821–1894) in 1853, but was then rediscovered in 1883 by Thévenin.

- Interestingly, Norton's theorem was introduced in 1926 separately by two people: German mathematician and physicist Hans Ferdinand Mayer (1895–1980) and Bell Labs engineer Edward Lawry Norton. In Europe the theorem is known as Mayer-Norton theorem!



Example

- Use Thévenin's theorem to find the equivalent circuit to the left of the terminals a and b in the following circuit and then find I. Find Thévenin equivalent of the system using all three approaches that we have discussed.



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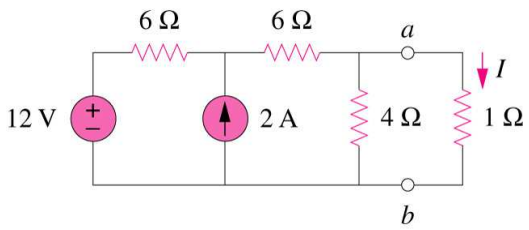

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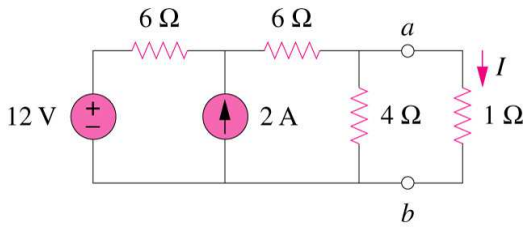
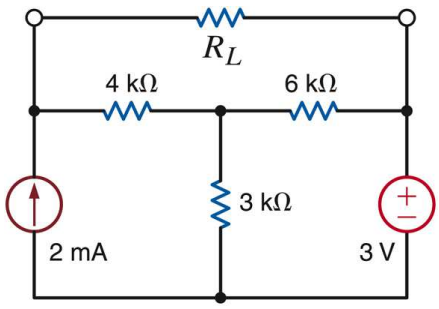


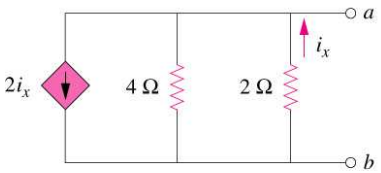
Example

- Find the Thevenin equivalent circuit seen by R_L.



Example

- Find the Thévenin equivalent of the following circuit:

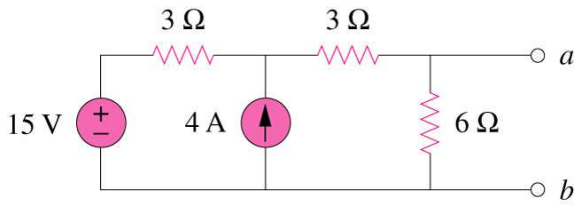


(a)

- Can you guess what is V_{th} before even picking up your pencil to solve this question?!

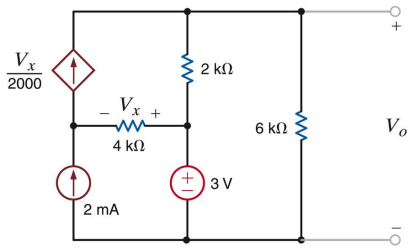
Example

- Find the Norton equivalent of the following circuit:



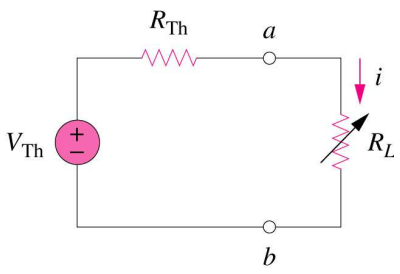
Example

- Find the Thévenin equivalent of the following circuit:



Maximum Power Transfer

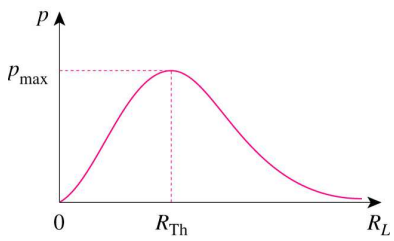
- In many practical circuits, it is desirable to provide maximum power to the load.



$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L$$

Maximum Power Transfer

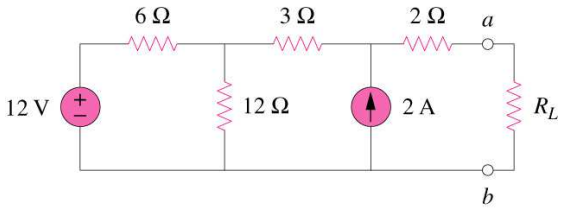
- Power delivered to the load as a function of R_L:



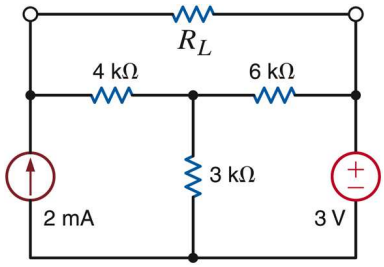
- Maximum power is transferred to the load when the load resistance equals the Thevenin resistance as seen from the load (that is, R_L=R_{Th})
- Formal proof:

Example

- Find the value of R_L for maximum power transfer in the following circuit. What is the maximum power?

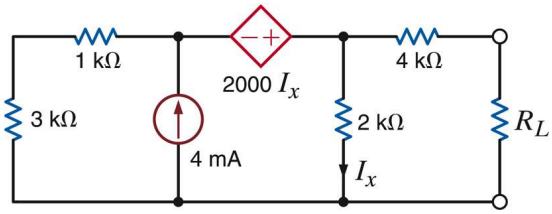


- Find the value of R_L to which the maximum power is transferred in the following circuit. What is the maximum power?



Example

- Find the value of R_L to which the maximum power is transferred in the following circuit. What is the maximum power?



Notes