Counting and Sampling Algorithms, Fall 2020/21
Homework 3, due Friday, October 2, 2020, 11:59pm

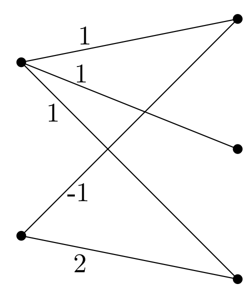
Problem 1

Read the proof of Lemma B in Section 2.2 of Prof. Jerrum's lecture notes (`http://www.maths.qmul.ac.uk/~mj/ETHbook/chapter2.pdf`). Note: In class we defined the problem #W-Bipartite-Match with different weights but that does not change the proof.

(a) Restate the proof (the reduction) as a pseudo code. In particular, give a pseudo code for the function `ComputeWeightedMatchingSum`$(G = (V = R \cup B, E, w))$ that takes a bipartite graph G with edge weights w and returns the sum of the weights of all matchings of G. This pseudo code uses `ComputeWeightedPerfectMatchingSum`(G) as a subroutine (you can think of this as a library call, an oracle, or a black box). Assuming each call to the subroutine takes a constant time, the running time of the function `ComputeWeightedMatchingSum`(G) is polynomial in the size of G.

(b) Give a rough big-Oh estimate of the running time of `ComputeWeightedMatchingSum`(G) based on your pseudo code. State the running time as a function of n, the number of vertices of G. Briefly reason the estimate.

(c) Demonstrate how the reduction works for the following graph G:



In particular:

(i) What is the range of k's in the reduction?

(ii) Draw each G_k. Do not forget to specify all the edge weights.

(iii) For each G_k, compute the sum of the weights of all perfect matchings in G_k by hand and state the resulting value (here you are simulating the subroutine call).

(iv) Show how to combine these values to get the return value of the reduction.

(v) Compute $p_{\mathrm{match}}(G)$ by hand and state the resulting value. How does it compare to the return value?

Problem 2

We are shuffling a deck of n distinct cards as follows. Suppose that the current deck is, from top to bottom, $a_1, a_2, \ldots, a_n$. With probability $1/2$, we swap the top two cards. Otherwise, we choose a random position in the deck $k \in \{1, \ldots, n-1\}$ and re-shuffle the deck by taking the top k cards and placing them at the bottom, i.e., we obtain $a_{k+1}, a_{k+2}, \ldots, a_n, a_1, a_2, \ldots, a_k$.

(a) For $n = 3$:

- (i) Describe the state space Ω of this Markov chain.
- (ii) Draw the transition graph of this Markov chain. Label each state. Do not forget to specify the probabilities on the arrows.
- (iii) Give the transition matrix P of this Markov chain.
- (iv) For every $x, y \in \Omega$, find the smallest t such that $P^t(x, y) > 0$.

(b) For general n:

- (i) What is $|\Omega|$?
- (ii) How many non-zero entries are there in each row of the transition matrix?
- (iii) For which $n \geq 2$ is the Markov chain aperiodic?
- (iv) For which $n \geq 2$ is the Markov chain irreducible?
- (v) Find a stationary distribution of this Markov chain. Is it unique?

Briefly reason all your answers in part (b).

Problem 3

[Levin-Peres-Wilmer, Exercise 2.3] Consider a random walk on the path $\{0, 1, \ldots, n\}$ in which the walk moves left or right with equal probability except when at n and 0. At n, it remains at n with probability $1/2$ and moves to $n - 1$ with probability $1/2$, and once the walk hits 0, it remains there forever.

(a) Is this Markov chain aperiodic? Is it irreducible?

(b) Find a stationary distribution of this MC. Is it unique?

(c) Compute the expected number of steps of the walk until it reaches 0, given that it starts at state n.

Briefly reason all your answers.

Note about all problems:

Make sure to label your answers to the individual parts of the problems with tags 1.(a), 2.(b)-(ii), etc.