Exercises

from Riehl, *Category Theory in Context*

I am self-studying this alongside Aluffi, *Algebra: Chapter 0*, so there is a bit of mixed notation.

Chapter 1

Categories, Functors, Natural Transformations

1.1 Abstract and concrete categories

Exercise 1.1.i.

(i) Consider a morphism $f : x \to y$. Show that if there exists a pair of morphisms $g, h : y \rightrightarrows x$ so that $gf = 1_x$ and $fh = 1_y$, then $g = h$ and f is an isomorphism.

(ii) Show that a morphism can have at most one inverse isomorphism.

(i) *Proof.* We can compose together $gfh = (gf)h = 1_X h = h$. But $gfh = g(fh) = g1_y = g$ by associativity. Therefore, $g = h$. Since $gf = 1_x$ and $fh = fg = 1_y$, f is an isomorphism. □

(ii) *Proof.* Suppose $f : x \to y$ has two inverses g and h. Then, as above, $g = g1_y = g(fh) = (gf)h = 1_x f = h$ by associativity. □

Exercise 1.1.ii. Let C be a category. Show that the collection of isomorphisms in C defines a subcategory, the maximal groupoid inside C.

Proof. Let X, Y, and Z be objects in C. We must show identity and composition, since we get associativity for free from C.

The identity morphisms 1_X are isomorphisms, since they are their own inverses. Therefore, they are present in the maximal groupoid.

Now, if $f : X \to Y$ and $g : Y \to Z$ are isomorphisms, then the composition $gf : X \to Z$ is an isomorphism with inverse $f^{-1}g^{-1}$ since $gff^{-1}g^{-1} = gg^{-1} = 1_Z$ and $f^{-1}g^{-1}gf = f^{-1}f = 1_X$.

Therefore, the maximal groupoid is in fact a category. □

Exercise 1.1.iii. For any category C and any object $c \in \mathsf{C}$, show that:

(i) There is a category c/C whose objects are morphisms $f : c \to x$ with domain c and in which a morphism from $f : c \to x$ to $g : c \to y$ is a map $h : x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"] & & y \end{tikzcd}$$

commutes, i.e., so that $g = hf$.

(ii) There is a category C/c whose objects are morphisms $f : x \to c$ with codomain c and in which a morphism from $f : x \to c$ to $g : y \to c$ is a map $h : x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

commutes, i.e., so that $f = gh$.

The categories c/C and C/c are called slice categories of C under and over c, respectively.

(i) See Exercise I.3.7, Aluffi.

(ii) See Example I.3.5, Aluffi.

1.2 Duality

Exercise 1.2.i. Defining C/c to be $(c/(\mathsf{C}^{\mathsf{op}}))^{\mathsf{op}}$, deduce Exercise 1.1.iii(ii) from Exercise 1.1.iii(i).

Proof. We must establish that C/c is in fact $(c/(\mathsf{C}^{\mathsf{op}}))^{\mathsf{op}}$. Then, everything (inverses, composition, associativity) follows immediately from duality.

First, notice that in $c/(\mathsf{C}^{\mathsf{op}})$, we have objects that are morphisms in C^{op}, i.e., the same that are in C but backwards:

$$\begin{tikzcd} & c \arrow[dl, "f^{\mathsf{op}}"'] \arrow[dr, "g^{\mathsf{op}}"] & \\ x \arrow[rr, "h^{\mathsf{op}}"] & & y \end{tikzcd}$$

To get to the desired commutative diagram, we have to apply the opposite operation once more. $\square$

Exercise 1.2.ii.

(i) Show that a morphism $f : x \to y$ is a split epimorphism in a category C if and only if for all $c \in \mathsf{C}$, the post-composition function $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ is surjective.

(ii) Argue by duality that f is a split monomorphism if and only if for all $c \in \mathsf{C}$, the pre-composition function $f^* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ is surjective.

(i) *Proof.* ($\Rightarrow$) Suppose f is a split epimorphism, i.e., it is a left inverse of some morphism $g : y \to x$. That is, $fg = 1_y$.

Let $h : c \to y$. We must show $f_*(j) = fj = h$ for some $j : c \to x$. Notice that $f(gh) = (fg)h = 1_y h = h$. Therefore, if we let $j := gh$, we are done.

($\Leftarrow$) Suppose f_* is surjective, i.e., for all $g : c \to y$ there exists an $h : c \to x$ such that $f_*(h) = fh = g$. In particular, for $g' = 1_y$, there exists $h' : y \to x$ such that $fh' = 1_y$. That is, f is a split epimorphism, the retraction of h'. □

(ii) *Proof.* Apply part (i) to the category C^{op}:

> $f^{\mathsf{op}} \in \mathsf{C}^{\mathsf{op}}(x, y)$ is a split epimorphism if and only if for all $c \in \mathsf{C}^{\mathsf{op}}$, the post-composition function $f^{\mathsf{op}}_* : \mathsf{C}^{\mathsf{op}}(c, x) \to \mathsf{C}^{\mathsf{op}}(c, y)$ is surjective.

But this is exactly the same as saying

> $f \in \mathsf{C}(y, x)$ is a split monomorphism if and only if for all $c \in \mathsf{C}$, the pre-composition function $f^* : \mathsf{C}^{\mathsf{op}}(x, c) \to \mathsf{C}^{\mathsf{op}}(y, c)$ is surjective.

because $\mathsf{C}^{\mathsf{op}}(x, y) = \mathsf{C}(y, x)$ and a split epimorphism $f^{\mathsf{op}} g^{\mathsf{op}} = 1_y$ becomes a split monomorphism $gf = 1_y$. □

Exercise 1.2.iii. Prove Lemma 1.2.11 by proving either (i) and (ii), then arguing by duality. Conclude that the monomorphisms in any category define a subcategory of that category and dually that the epimorphisms also define a subcategory.

Proof. (i) Let $f : x \rightarrowtail y$ and $g : y \rightarrowtail z$. We must show that gf is monic, i.e., $gfh = gfk \implies h = k$ for all $h, k : c \rightrightarrows x$. Suppose $gfh = gfk$. Since g is monic, we know that $g(fh) = g(fk) \implies fh = fk$. But then, since f is monic, $fh = fk \implies h = k$.

(ii) Let $f : x \to y$ and $g : y \to z$ such that $gf : x \rightarrowtail z$. We must show f is monic, i.e., $fh = fk \implies h = k$ for all $h, k : c \rightrightarrows x$. Suppose $fh = fk$. Then, by pre-composing g, $(gf)h = (gf)k$. Since gf is monic, $h = k$.

Now, we can show that the subcategory $\mathsf{C}_{\mathsf{Monic}}$ of the same objects as C and with only its monomorphisms is a category. First, since for all $h, k \in \mathsf{C}(c, x)$, $1_x h = 1_x k \implies h = k$, we have the identities in $\mathsf{C}_{\mathsf{Monic}}$. Then, due to (i), the compositions of all morphisms are in the subcategory. Since the objects are unchanged, we can conclude $\mathsf{C}_{\mathsf{Monic}}$ is a subcategory.

Dually, the opposite monomorphism f^{op} in C^{op} such that $f^{\mathsf{op}} h^{\mathsf{op}} = f^{\mathsf{op}} k^{\mathsf{op}} \implies h^{\mathsf{op}} = k^{\mathsf{op}}$ is the epimorphism f in C such that $hf = kf \implies h = k$. Therefore, the dual of (i) is (i') and the dual of (ii) is (ii'). Finally, the dual of the existence of the subcategory $\mathsf{C}_{\mathsf{Monic}}$ is a subcategory $\mathsf{C}_{\mathsf{Epic}}$ of epimorphisms. □

Exercise 1.2.vi. Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism. Argue by duality that a split monomorphism that is an epimorphism is also an isomorphism.

Proof. Suppose $f : x \rightarrowtail y$ is also a split epimorphism such that $fg = 1_y$ for some $g : y \to x$. Then, since f is monic, $f(gf) = (fg)f = (1_y)f = f = f(1_x) \implies gf = 1_x$. That is, f is an isomorphism.

By duality, a split epimorphism $f^{\mathsf{op}} g^{\mathsf{op}} = 1_y$ is a split homomorphism $gf = 1_y$. Therefore, a split homomorphism that is an epimorphism is an isomorphism too. $\square$

Exercise 1.2.vii. Regarding a poset $(\mathsf{P}, \leq)$ as a category, define the supremum of a subcollection of objects $a \in \mathsf{P}$ in such a way that the dual statement defines the infimum. Prove that the supremum of a subset of objects is unique, whenever it exists, in such a way that the dual proof demonstrates the uniqueness of the infimum.

Proof. Let a be the object such that $f_x : x \to a$ exists for all other x in the subcollection. That is, for every object x, $y \leq a$. Then, the dual definition is the object a^{op} in P^{op} such that $f_X^{\mathsf{op}} : x \to a$ always exists, i.e., the object a' in P such that $f_x : a \to x$ always exists.

Suppose there are two suprema a and a'. Then, by the definition for a, we just have a morphism $a \to a'$. But if a' is a supremum, there must also be a morphism $a' \to a$. Therefore, $a \leq a' \leq a$, which means $a = a'$ since P is a poset (and not a pre-ordered set). $\square$

1.3 Functoriality

Exercise 1.3.i. What is a functor between groups, regarded as one-object categories?

Solution. A functor $F : \mathsf{G} \to \mathsf{H}$ maps the single object $FG = H$ and each morphism (i.e., group element) such that for all $a, b : G \rightrightarrows G$, $F(ab) = (Fa)(Fb)$ and $Fe_G = e_H$.

That is, in the language of groups, a functor is a group homomorphism. $\square$

Exercise 1.3.ii. What is a functor between preorders, regarded as categories?

Solution. A functor $F : (\mathsf{P}, \leq) \to (\mathsf{Q}, \preccurlyeq)$ sends the objects of P to objects in Q such that if $a \leq b$, then $Fa \preccurlyeq Fb$. That is, we can regard F as a set function $P \to Q$ that is increasing with respect to the respective preorders. $\square$

Exercise 1.3.iii. Find an example to show that the objects and morphisms in the image of a functor $F : \mathsf{C} \to \mathsf{D}$ do not necessarily define a subcategory of D.

Solution. Let C be a groupoid with two groups generated by two elements $A = \langle a \rangle$ and $B = \langle b \rangle$:

$$\begin{tikzcd} A \arrow[loop below, "a"] & B \arrow[loop below, "b"] \end{tikzcd}$$

Let D be a groupoid with one group $G = \langle \alpha, \beta \rangle$. Now, let $F : \mathsf{C} \to \mathsf{D}$ with $FA = FB = G$, $Fa = \alpha$, and $Fb = \beta$.

The image of F is not a category. Both α and β are in the image, but their composition $\beta\alpha$ is not since there was no ba composition in C. □

> **Exercise 1.3.iv.** Verify that the constructions introduced in Definition 1.3.11 (functors $\mathsf{C}(c, -)$ and $\mathsf{C}(-, c)$ represented by c) are functorial.

Proof. Recall that $\mathsf{C}(c, -) : \mathsf{C} \to \mathsf{Set}$ sends x to the set $\mathsf{C}(c, x)$ and $f : x \to y$ to the post-composition function $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y) : h \mapsto fh$. Verify the functoriality axioms:

Let $f : x \to y$ and $g : y \to z$ be composable morphisms where $gf : x \to z$. Then, $(gf)_* : \mathsf{C}(c, x) \to \mathsf{C}(c, z)$. We can verify that $(gf)_*(h) = (gf)h = g(f(h)) = g(f_*h) = g_*f_*h$ for all $h : c \to z$. That is, $(gf)_* = g_*f_*$.

Consider an identity $1_x : x \to x$. Then, the post-composition function $(1_x)_* : \mathsf{C}(c, x) \to \mathsf{C}(c, x)$ sends morphisms to themselves because $h1_x = h$ and $1_x g = g$ by definition, i.e., it is $1_{\mathsf{C}(c,x)}$. □

> **Exercise 1.3.v.** What is the difference between a functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{D}$ and a functor $\mathsf{C} \to \mathsf{D}^{\mathsf{op}}$? What is the difference between a functor $\mathsf{C} \to \mathsf{D}$ and a functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{D}^{\mathsf{op}}$?

Solution. If a contravariant functor $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{D}$ sends $a \mapsto x$ and $f^{\mathsf{op}} : b \to a$ to $g : x \to y$, then the "same" functor $F' : \mathsf{C} \to \mathsf{D}^{\mathsf{op}}$ sends $a \mapsto x$ and $f : a \to b$ to $g^{\mathsf{op}} : y \to x$. For the functoriality axioms to hold, we must have that $F'f = Ff^{\mathsf{op}}$ for all morphisms. That is, we can consider F' as the "opposite functor" to F (in some sense, a composition of the opposite functor $\mathsf{D} \to \mathsf{D}^{\mathsf{op}}$ and F).

Likewise, any functor $G : \mathsf{C} \to \mathsf{D}$ must also act as a contravariant functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{D}^{\mathsf{op}}$. If $GfGg = G(fg)$, then can apply the "same" mapping between objects and morphisms to get $Gg^{\mathsf{op}}Gf^{\mathsf{op}} = G(g^{\mathsf{op}}f^{\mathsf{op}}) = G(gf)^{\mathsf{op}}$. □

> **Exercise 1.3.vi.** Given functors $F : \mathsf{D} \to \mathsf{C}$, and $G : \mathsf{E} \to \mathsf{C}$, show that there is a comma category $F\downarrow G$ which has
>
> - as objects, triples $(d \in \mathsf{D}, e \in \mathsf{E}, f : Fd \to Ge \in \mathsf{C})$, and
> - as morphisms $(d, e, f) \to (d', e', f')$, a pair of morphisms $(h : d \to d', k : e \to e')$
>
> so that
>
> $$\begin{tikzcd} Fd \arrow[r, "f"] \arrow[d, "Fh"'] & Ge \arrow[d, "Gk"] \\ Fd' \arrow[r, "f'"'] & Ge' \end{tikzcd}$$
>
> commutes in C, i.e., $f'(Fh) = (Gk)f$. Define a pair of projection functors $\mathrm{dom} : F\downarrow G \to \mathsf{D}$ and $\mathrm{cod} : F\downarrow G \to \mathsf{E}$.

Exercise 1.3.vii. Define functors to construct the slice categories c/C and C/c of exercise 1.1.iii as special cases of comma categories constructed in exercise 1.3.vi. What are the projection functors?

Solution. For c/C, let D be the category containing only c and its identity id_c. Then, let $F : \mathsf{D} \to \mathsf{C}$ be the "identity" such that $Fc = c$. Define $G : \mathsf{C} \to \mathsf{C}$ to also be the identity functor such that $Gx = x$ and $Gf = f$.

Then, objects in $F{\downarrow}G$ are triples $(c, x \in \mathsf{C}, f : c \to x \in \mathsf{C})$ and morphisms $(c, x, f) \to (c, y, g)$ are pairs $(\mathrm{id}_c : c \to c, h : x \to y)$ such that the diagram

$$\begin{tikzcd} & c \arrow[loop above, "\mathrm{id}_c"] \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"] & & y \end{tikzcd}$$

commutes. This is almost exactly the definition of c/C. The domain functor sends every object (c, x, f) to c and morphism to id_c. The codomain functor sends every object (c, x, f) to x and morphism to (id_c, h) to h, which recovers the exact definition of c/C.

For C/c, consider $G \downarrow F$, whose objects are triples $(x \in \mathsf{C}, c, f : x \to c \in \mathsf{C})$ and morphisms $(x, c, f) \to (y, c, g)$ are pairs $(h : x \to y, \mathrm{id}_c : c \to c)$ such that

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c \arrow[loop below, "\mathrm{id}_c"] & \end{tikzcd}$$

commutes. This is almost exactly the definition of C/c. This time, the domain functor sends $(x, c, f) \mapsto x$ and $(h, \mathrm{id}_c) \mapsto h$, which gives us exactly the definition of C/c. The codomain functor sends $(x, c, f) \mapsto c$ and $(h, \mathrm{id}_c) \to \mathrm{id}_c$. □

Exercise 1.3.viii. Lemma 1.3.8 shows that functors preserve isomorphisms. Find an example to demonstrate that functors need not reflect isomorphisms: that is, find a functor $F : \mathsf{C} \to \mathsf{D}$ and a morphism f in C so that Ff is an isomorphism in D but f is not an isomorphism in C.

Solution. Imagine the two categories C ($x \xrightarrow{f} y$) and D ($\begin{tikzcd} a \arrow[r, bend left, "g"] & b \arrow[l, bend left, "h"] \end{tikzcd}$) such that $g^{-1} = h$.

Define $F : \mathsf{C} \to \mathsf{D}$ such that $Fx = a$, $Fb = b$, and $Ff = g$.

Then, $Ff = g$ is an isomorphism but f is not. □

Exercise 1.3.ix. For any group G, we may define other groups:

- the center $Z(G) = \{h \in G \mid \forall g \in G, hg = gh\}$,
- the commutator subgroup $C(G)$, the subgroup generated by elements $ghg^{-1}h^{-1}$, and1
- the automorphism group $\mathrm{Aut}(G)$, the group of isomorphisms $\phi : G \to G$ in Group.

Trivially, all three constructions define a functor from the discrete category of groups (with only identity morphisms) to Group. Are these constructions functorial in

- the isomorphisms of groups? That is, do they extend to functors $\mathsf{Group}_{\mathsf{iso}} \to \mathsf{Group}$?
- the epimorphisms of groups? That is, do they extend to functors $\mathsf{Group}_{\mathsf{epi}} \to \mathsf{Group}$?
- all homomorphisms of groups? That is, do they extend to functors $\mathsf{Group} \to \mathsf{Group}$?

Exercise 1.3.x. Show that the construction of the set of conjugacy classes of elements of a group is functorial, defining a functor $\mathrm{Conj} : \mathsf{Group} \to \mathsf{Set}$. Conclude that any pair of groups whose sets of conjugacy classes of elements have differing cardinalities cannot be isomorphic.