MATH 135 Fall 2020: Extra Practice 3

Warm-Up Exercises

WE01. Prove the following two quantified statements.

(a) $\forall n \in \mathbb{N}, n+1 \geq 2$

Proof. Let $n \in \mathbb{N}$. Recall that 1 is the smallest natural. $n \geq 1 \iff n+1 \geq 2$. $\square$

(b) $\exists n \in \mathbb{Z}, \frac{5n-6}{3} \in \mathbb{Z}$

Proof. Select $n = 3$. Then, $\frac{5n-6}{3} = \frac{15-6}{3} = \frac{9}{3} = 3 \in \mathbb{Z}$. $\square$

WE02. Prove that for all $k \in \mathbb{Z}$, if k is odd, then $4k + 7$ is odd.

Proof. Let k be an odd integer. $4k + 7$ is odd if and only if it may be written as $2n + 1$ for some integer n. Let $n = 2k + 3$. Then, $2n + 1 = 2(2k + 3) + 1 = 4k + 6 + 1 = 4k + 7$, as required. $\square$

WE03. Consider the following proposition

For all $a, b \in \mathbb{Z}$, if $a^3 \mid b^3$, then $a \mid b$.

We now give three erroneous proofs of this proposition. Identify the major error in each proof, and explain why it is an error.

(a) *Consider $a = 2$, $b = 4$. Then $a^3 = 8$ and $b^3 = 64$. We see that $a^3 \mid b^3$ since $8 \mid 64$. Since $2 \mid 4$, we have $a \mid b$.*

This proof is erroneous as it only considers one specific case of a and b and not the general case of integer a and b.

(b) *Since $a \mid b$, there exists $k \in \mathbb{Z}$ such that $b = ka$. By cubing both sides, we get $b^3 = k^3a^3$. Since $k^3 \in \mathbb{Z}$, then $a^3 \mid b^3$.*

This proof supposes the conclusion instead of the hypothesis.

(c) *Since $a^3 \mid b^3$, there exists $k \in \mathbb{Z}$ such that $b^3 = ka^3$. Then $b = (ka^2/b^2)a$, hence $a \mid b$.*

The proof does not guarantee that $\frac{ka^2}{b^2}$ is an integer.

WE04. Let x be a real number. Prove that if $x^3 - 5x^2 + 3x \neq 15$, then $x \neq 5$.

Proof. Suppose for the contrapositive that $x = 5$. Then, $x^3 - 5x^2 + 3x = (5)^3 - 5(5)^2 + 3(5) = 15$, as required. Since the contrapositive is true, the original implication must be true. $\square$

WE05. Prove that there do not exist integers x and y such that $2x + 4y = 3$.

Proof. For the sake of contradiction, suppose the negation is true.

Consider the negation of the statement: there exist integers x and y such that $2x+4y=3$. Let x and y be such integers. Then, $x+2y$ is an integer. Therefore, $2x+4y=2(x+2y)$ is even. However, 3 is odd. An integer cannot be both even and odd, therefore, the negation is false, and the original statement is true. □

WE06. Prove that an integer is even if and only if its square is an even integer.

Proof. ($\Rightarrow$) Let n be an even integer. Then, $n=2k$ for some integer k. $n^2=(2k)^2=4k^2=2(2k^2)$. Since $2k^2$ is an integer, n^2 is even.

($\Leftarrow$) Let n be an even square integer. Then, $n=2k$ for some integer k and $n=x\cdot x$ for some integer x. Since $2k=x\cdot x$, and 2 is prime, 2 must divide x. Therefore, $x=2y$ for some integer y, which is the definition of being even.

Since the implication is true in both directions, the biconditional is true. □

Recommended Problems

RP01. Prove that $x^2+9\geq 6x$ for all real numbers x.

Proof. Let x be a real number. $x^2+9\geq 6x \iff x^2-6x+9\geq 0 \iff (x-3)^2\geq 0$. Since the square of a real is always non-negative, the statements are true. □

RP02. Prove that for all $r\in\mathbb{R}$ where $r\neq -1$ and $r\neq -2$,

$$\frac{2^{r+1}}{r+2}-\frac{2^r}{r+1}=\frac{r(2^r)}{(r+1)(r+2)}$$

Proof. Let r be a real number that is neither -1 nor -2. Then,

$$\begin{aligned}
LHS &= \frac{2^{r+1}}{r+2}-\frac{2^r}{r+1}\\
&= \frac{2^{r+1}(r+1)-2^r(r+2)}{(r+1)(r+2)}\\
&= \frac{r2^{r+1}+2^{r+1}-r2^r-2\cdot 2^r}{(r+1)(r+2)}\\
&= \frac{r2^{r+1}+\cancel{2^{r+1}}-r2^r-\cancel{2^{r+1}}}{(r+1)(r+2)}\\
&= \frac{r(2^{r+1}-2^r)}{(r+1)(r+2)}\\
&= \frac{r(2^r\cdot 2-2^r)}{(r+1)(r+2)}\\
&= \frac{r(2^r+\cancel{2^r}-\cancel{2^r})}{(r+1)(r+2)}\\
&= \frac{r(2^r)}{(r+1)(r+2)}\\
&= RHS
\end{aligned}$$

Since the left side equals the right side, the equality is true. ☐

RP03. Prove that there exists a real number x such that $x^2 - 6x + 11 \leq 2$.

Proof. Let $x = 3$. $x^2 - 6x + 11 = (3)^2 - 6(3) + 11 = 9 - 18 + 11 = 2 \leq 2$, as required. Since 3 is a real number, the statement is true. ☐

RP04. Prove or disprove each of the following statements.

(a) $\forall n \in \mathbb{Z}$, $\frac{5n-6}{3}$ is an integer.

Proof. Let $n = 1$ as a counter-example. Then, $\frac{5n-6}{3} = \frac{5-6}{3} = -\frac{1}{3}$, which is not an integer. Therefore, the statement is false. ☐

(b) $\forall a \in \mathbb{Z}$, $a^3 + a + 2$ is even.

Proof. Let a be an integer. Then, a is either even or odd. Suppose that a is even and can be written as $a = 2k$ for an integer k. Then, $a^3 + a + 2 = (2k)^3 + 2k + 2 = 8k^3 + 2k + 2 = 2(4k^3 + k + 1)$, an even number.

Suppose a is odd and can be written as $a = 2k+1$ for an integer k. Then, $a^3+a+2 = (2k+1)^3 + (2k+1) + 2 = 2k^3 + 12k^2 + 2k + 4 = 2(k^3 + 6k^2 + k + 2)$, an even number.

Therefore, the statement is true. ☐

(c) For every prime number p, $p + 7$ is composite.

Proof. Let p be a prime number.

If p is even, then $p = 2$, and $p + 7 = 9$ which is composite.

If p is odd, $p = 2k+1$ for some integer $k \geq 0$ (as there are no negative primes). Then, $p + 7 = 2k + 8 = 2(k + 4)$, which is even. The only even prime is 2, but $2k + 8 \geq 8$, so $p + 7$ is composite.

Therefore, since all primes are either even or odd, $p+7$ is composite for all primes. ☐

(d) For all $x \in \mathbb{R}$, $|x - 3| + |x - 7| \geq 10$.

Proof. Let $x = 3$ as a counter-example. Then, $|x-3|+|x-7| = |(3)-3|+|(3)-7| = 0 + 4 = 4 \ngeq 10$. Therefore, the statement is false. ☐

(e) There exists a natural number $m < 123456$ such that $123456m$ is a perfect square.

Proof. Since $123456 = 2^6 \cdot 3 \cdot 643$, let $m = 3 \cdot 643 = 1929$, which is less than 123456. Then, $123456m = 238146624 = 15432^2$. Since $123456m$ can be written as n^2 where $n = 15432 \in \mathbb{Z}$, it is a perfect square, and the statement is true. ☐

(f) $\exists k \in \mathbb{Z}, 8 \nmid (4k^2 + 12k + 8)$.

Proof. Consider the negation, $\forall k \in \mathbb{Z}, 8 \mid (4k^2 + 12k + 8)$. Notice that the open sentence is logically equivalent to $8 \mid (4k^2 + 12k)$. Let k be a natural number. Then, k is either even or odd.

Suppose that k is even and can be written as $k = 2n$. Then, $4k^2 = 16n^2 = 8(2n^2)$, so $8 \mid 4k^2$. Likewise, $12k = 24n = 8(3n)$, so $8 \mid 12k$. By DIC, $8 \mid (4k^2 + 12k)$.

Now, suppose that k is odd and can be written as $k = 2n + 1$. Then, $4k^2 + 12k = 4(4n^2 + 2n + 1) + 12(2n + 1) = 16n^2 + 40n + 16 = 8(2n^2 + 5n + 1)$, so $8 \mid (4k^2 + 12k)$.

Therefore, the negation is true, so the original statement is false. $\square$

RP05. Prove or disprove each of the following statements involving nested quantifiers.

(a) For all $n \in \mathbb{Z}$, there exists an integer $k > 2$ such that $k \mid (n^3 - n)$.

Proof. Let n be an integer. If $n = 0$ or $n = \pm 1$, $n^3 - n = 0$ and all integers (including any k) divide zero.

If $n > 1$, we select $k = n + 1 > 2$. Factor: $n^3 - n = n(n-1)(n+1)$. Then, $n^3 - n = [n(n-1)](n+1)$, so $k \mid (n^3 - n)$.

If $n < 1$, first let $m = -n$ so $n^3 - n = (-m)^3 + m = -(m^3 - m)$. Now, select $k = m + 1 > 2$. Then, $n^3 - n = -m(m-1)(m+1)$, so $k \mid (n^3 - n)$.

Therefore, the statement is true. $\square$

(b) For every positive integer a, there exists an integer b with $|b| < a$ such that b divides a.

Proof. We disprove by counter-example. Let $a = 1$. Then, $|b| < 1$, and the only such integer is 0. However, $0 \nmid 1$ since there is no integer k where $k \cdot 0 = 1$. Therefore, the statement is false. $\square$

(c) There exists an integer n such that $m(n-3) < 1$ for every integer m.

Proof. Consider the negation, that for all n, there is a m where $m(n-3) \geq 1$.

The inequality is equivalently written $mn \geq 3n + 1$. Take cases for the sign of n:

- If $n = 0$, then the inequality reads $0 \geq 0 + 1$, which is false for all m.
- If $n > 0$, then $m \geq \frac{3n+1}{n}$. Select $m = \lfloor \frac{3n+1}{n} \rfloor - 1$, and the inequality is false.
- If $n < 0$, then $m \leq \frac{3n+1}{n}$. Select $m = \lceil \frac{3n+1}{n} \rceil + 1$, and the inequality is false.

Therefore, since the negation is false, the original statement is true. $\square$

(d) $\exists n \in \mathbb{N}, \forall m \in \mathbb{Z}, -nm < 0$

Proof. Consider the negation $\forall n \in \mathbb{N}, \exists m \in \mathbb{Z}, -nm \geq 0$. Let n be a natural number.

We can choose an integer m, namely $m = -1$. Notice that because n is a natural number, $n > 0 \iff n(-1)(-1) > 0 \iff -nm > 0 \iff -nm \geq 0$.

Because the negation is true, the original statement is false. $\square$

RP06. Prove that for all integers a and b, if $a \mid (2b + 3)$ and $a \mid (3b + 5)$, then $a \mid 13$.

Proof. Let a and b be arbitrary integers, and assume that $a \mid (2b+3)$ and $a \mid (3b+5)$.

Recall the divisibility of integer combinations: since $2b+3$ and $3b+5$ are integers, a must divide $n(2b+3)+m(3b+5)$ for all integers n and m. Specifically, let $n=-39$ and $m=26$. Then, $n(2b+3)+m(3b+5) = -78b-117+78b+130 = 13$. Therefore, $a \mid 13$. □

RP07. Let a, b, c and d be positive integers. Prove that if $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Proof. Let a, b, c and d all be positive integers. Assume $\frac{a}{b} < \frac{c}{d}$, which is true when $ad < bc$, because b and d are positive. Now, adding ab and cd to both sides, respectively:

$$\begin{aligned} ad &< bc & \qquad ad &< bc \\ ad+ab &< bc+ab & \qquad ad+cd &< bc+cd \\ a(b+d) &< b(c+a) & \qquad d(a+c) &< c(b+d) \\ \frac{a}{b} &< \frac{a+c}{b+d} & \qquad \frac{a+c}{b+d} &< \frac{c}{d} \end{aligned}$$

Therefore, $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. □

RP08. Prove that for all integers n, if $1-n^2 > 0$, then $3n-2$ is an even integer.

Proof. Let n be an integer where $1-n^2>0$. Since squares of integers are positive, $1 > n^2$. This is only true when $|n| < 1$, but the only such integer is 0. $3(0)-2=-2$, which is even. □

RP09. Let a and b be integers. Prove each of the following implications.

(a) If $ab=4$, then $(a-b)^3 - 9(a-b) = 0$

Proof. Let a and b be integers with product 4.

Consider the possible values for a and b. 4's divisor pairs are $(\pm1, \pm4)$ and $(\pm2, \pm2)$. For all of these pairs, either $a=b$ or $a=b\pm3$. Specifically:

- If $b=\pm2$, then $a=b$
- If $b=1$, then $a=4=b+3$ (for $b=-1$, $a=-4=b-3$)
- If $b=4$, then $a=1=b-3$ (for $b=-4$, $a=-1=b+3$)

Notice that the conclusion factors to $(a-b)(a-b-3)(a-b+3)=0$. This is true when $a=b$ or $a=b\pm3$, which we just showed. □

(b) If a and b are positive, then $a^2(b+1)+b^2(a+1) \geq 4ab$

Proof. Let a and b be positive integers, i.e., at least 1.

If a and b are both at least 1, then $a+b\geq 2$, or $a+b-2>0$. Likewise, ab is a positive integer, so $ab(a+b-2)>0$.

$$\begin{aligned} ab(a+b-2) &> 0 \\ a^2b+b^2a-2ab &> 0 \end{aligned}$$

Recall that squares are non-negative:

$$\begin{aligned}(a-b)^2 + a^2b + b^2a - 2ab &> 0\\ a^2 - 2ab + b^2 + a^2b + b^2a - 2ab &> 0\\ a^2 + a^2b + b^2 + b^2a &> 4ab\\ a^2(b+1) + b^2(a+1) &> 4ab\end{aligned}$$

□

RP10. Let a, b, c and d be integers. Prove that if $a \mid b$ and $b \mid c$ and $c \mid d$, then $a \mid d$.

Proof. Let a, b, c, and d be integers where $a \mid b$, $b \mid c$, and $c \mid d$.

Recall the transitivity of divisibility: for integers x, y, and z, if $x \mid y$ and $y \mid z$, then $x \mid z$.

Then, $a \mid b$ and $b \mid c$ implies $a \mid c$. Likewise, $a \mid c$ and $c \mid d$ implies $a \mid d$. □

RP11. Prove that the product of any four consecutive integers is one less than a perfect square.

Proof. The statement is equivalently expressed that for any integer k, $k(k+1)(k+2)(k+3) = r^2 - 1$ for some positive integer r.

Let k be an integer. The product $k(k+1)(k+2)(k+3)$ expands to $k^4 + 6k^3 + 11k^2 + 6k$. As a fourth-degree polynomial, its square root would be a quadratic.

Expanding algebraically, the square of a quadratic in x, $ax^2 + bx + c$, is $a^2x^4 + 2abx^3 + (2ac + b^2)x^2 + 2bcx + c^2$.

Notice that when $a = c = 1$ and $b = 3$, this formula becomes $x^4 + 6x^3 + 11x^2 + 6x + 1$. The coefficients on x are precisely our original product (with a constant $+1$). Therefore, $x^4 + 6x^3 + 11x^2 + 6x = (x^2 + 3x + 1)^2 - 1$ for all real x.

We can now let $r = k^2 + 3k + 1$, which is a positive integer such that

$$\begin{aligned}r^2 - 1 &= (k^2 + 3k + 1)^2 - 1\\ &= k^4 + 6k^3 + 11k^2 + 6k + 1 - 1\\ &= k(k+1)(k+2)(k+3)\end{aligned}$$

and conclude that the statement is true. □

RP12. Let $x, y \in \mathbb{R}$. Prove that if $xy + 2x - 3y - 6 < 0$, then $x < 3$ or $y < -2$.

Proof. Let x and y be real solutions to $xy + 2x - 3y - 6 < 0$.

Notice that the inequality factors to $(x-3)(y+2) < 0$. This is true when x and y are non-zero and have opposite signs: either $x < 3$ and $y > -2$, or $x > 3$ and $y < -2$. Therefore, either $x < 3$ or $y < -2$. □

RP13. Is the following implication true for all integers a, b and c? Prove that your answer is correct.

$$a \mid b \textit{ if and only if } ac \mid bc$$

Yes, it is true.

Proof. Consider the two implications of the biconditional statement:

($\Rightarrow$) Let a, b, and c be integers where a divides b. Then, there exists an integer k where $b = ka$. Multiplying both sides by c, we have $bc = k(ac)$. However, this is exactly what is needed to show that $bc \mid ac$.

($\Leftarrow$) Let a, b, and c be integers where ac divides bc. Then, there exists an integer k where $bc = kac$. Dividing both sides by c, we have $b = ka$. However, this is exactly what is needed to show that $b \mid a$.

Therefore, since both expressions imply the other, $a \mid b$ if and only if $ac \mid bc$ for all integers a, b, and c. □

RP14. Let n be an integer. Prove that $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$.

Proof. Consider the two implications of the biconditional statement:

($\Rightarrow$) Let n be an integer where 2 divides $n^4 - 3$. This means there is an integer k where $n^4 - 3 = 2k$. Notice that this means $n^4 - 3$ is even, so $n^4 = 2(k + 1) + 1$ is odd. Even numbers raised to integer powers remain even, so n must be odd. Therefore, $n = 2m + 1$ for some integer m.

Now, expand $n^2 + 3$:

$$\begin{aligned} n^2 + 3 &= (2m + 1)^2 + 3 \\ &= 4m^2 + 4m + 1 + 3 \\ &= 4(m^2 + m + 1) \end{aligned}$$

Because $m^2 + m + 1$ is an integer, $4 \mid (n^2 + 3)$.

($\Leftarrow$) Let n be an integer where 4 divides $n^2 + 3$. This means there is an integer k where $n^2 + 3 = 4k$ or $n^2 = 4k - 3$, and

$$\begin{aligned} n^2 &= 4k - 3 \\ n^4 &= (4k - 3)^2 \\ n^4 &= 16k^2 - 24k + 9 \\ n^4 - 3 &= 16k^2 - 24k + 6 \\ &= 2(8k^2 - 12k + 3) \end{aligned}$$

Because $8k^2 - 12k + 3$ is an integer, $2 \mid (n^4 - 3)$.

Therefore, since both expressions imply the other, $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$. □

RP15. Let x and y be integers. Prove that if $xy = 0$ then $x = 0$ or $y = 0$.

Proof. Consider the contrapositive, $x \neq 0$ and $y \neq 0$ implies $xy \neq 0$.

Let x and y be non-zero integers. WLOG, take $x \leq y$.

Now, take cases of the signs of x and y:

- If $0 < x \leq y$, then $xy > 0$, since two positive numbers' product is a positive number.

- xy is also positive when $x \leq y < 0$, with two negative numbers.
- When $x < 0 < y$, i.e. the signs are opposite, $xy < 0$.

Since xy can never be 0 for any combination of non-zero integers, the contrapositive, and by extension, the original implication, is true. $\square$

RP16. Prove that $\forall a, b \in \mathbb{Z}, [(a \mid b \wedge b \mid a) \iff a = \pm b]$.

Proof. Let a and b be integers. Suppose a divides b and vice versa. Equivalently, integers p and q exist such that $a = pb$ and $b = qa$. Substituting, $a = pb = p(qa) \iff 1 = pq \iff p = \frac{1}{q}$.

The only integers of the form $\frac{1}{k}$ with integer k are 1 and -1. Therefore, $p = \frac{1}{q}$ if and only if $p = \pm 1$, i.e., $a = \pm b$. $\square$

RP17. Let a be an integer. Prove that $a^2 + 2a - 3$ is even if and only if a is odd.

Proof. Consider the two implications of the biconditional statement:

($\Rightarrow$) Let a be an odd integer, or, $a = 2k + 1$ for some integer k. Then,

$$\begin{aligned} 2a^2 + 2a - 3 & \\ &= (2k+1)^2 + 2(2k+1) - 3 \\ &= 4k^2 + 4k + 1 + 4k + 2 - 3 \\ &= 4k^2 + 8k - 2 \\ &= 2(2k^2 + 4k - 1) \end{aligned}$$

which is even, because $2k^2 + 4k - 1$ is an integer.

($\Leftarrow$) Consider the contrapositive, where even a implies odd $a^2 + 2a - 3$. Let a be an even integer, i.e., $a = 2k$ for some integer k. Then,

$$\begin{aligned} a^2 + 2a - 3 &= (2k)^2 + 2(2k) - 3 \\ &= 4k^2 - 4k - 3 \\ &= 2(2k^2 - 2k - 2) + 1 \end{aligned}$$

which is odd, because $2k^2 - 2k - 2$ is an integer. Since the contrapositive is true, the original implication is also true.

Therefore, since both implications hold, the statement is true. $\square$