MATH 135 Fall 2020: Extra Practice 4

Warm-Up Exercises

WE01. Evaluate $\sum_{i=3}^{8} 2^i$ and $\prod_{j=1}^{5} \frac{j}{3}$.

Solution. Simply expand along the sum/product:

$$\sum_{i=3}^{8} 2^i = 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 = 8 + 16 + 32 + 64 + 128 + 256 = 504$$

and

$$\prod_{j=1}^{5} \frac{j}{3} = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{3} \cdot \frac{4}{3} \cdot \frac{5}{3} = \frac{120}{243} = \frac{40}{81}$$

□

WE02. Let x be a real number. Using the Binomial Theorem, expand $\left(x - \frac{1}{x}\right)^7$.

Solution. Recall the Binomial Theorem, that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. Now, substitute $a = x$ and $b = -\frac{1}{x}$.

$$\begin{aligned}
\left(x - \frac{1}{x}\right)^7 &= \sum_{k=0}^{7} \binom{7}{k} x^{7-k} \left(-\frac{1}{x}\right)^k \\
&= \sum_{k=0}^{7} \binom{7}{k} x^{7-k} x^{-k} (-1)^k \\
&= \sum_{k=0}^{7} \binom{7}{k} (-1)^k x^{7-2k} \\
&= x^7 - 7x^{7-2} + 21x^{7-4} - 35x^{7-6} + 35x^{7-8} - 21x^{7-10} + 7x^{7-12} - x^{7-14} \\
&= x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}
\end{aligned}$$

□

Recommended Problems

RP01. Prove the following statements by induction.

(a) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} (2i - 1) = n^2$.

Proof. We will induct the statement $P(n) \equiv \sum_{i=1}^{n} (2i - 1) = n^2$ on n.

(Base Case) When $n = 1$, the left-hand side is

$$\begin{aligned}
\sum_{i=1}^{1} (2i - 1) &= 2(1) - 1 \\
&= 1 \\
&= 1^2
\end{aligned}$$

which is the right-hand side, so $P(1)$ holds.

(Inductive Step) Now, suppose that $P(k)$ holds for an arbitrary k. Then, we take the left-hand side of $P(k+1)$

$$\begin{aligned}\sum_{i=1}^{k+1}(2i-1) &= (2(k+1)-1)+\sum_{i=1}^{k}(2i-1) && \text{extracting } i=n+1\\ &= (2k+1)+k^2 && \text{by inductive hypothesis}\\ &= (k+1)^2 && \text{factoring}\end{aligned}$$

as desired to show that if $P(k)$ holds, then $P(k+1)$ holds.

Therefore, by induction, $P(n)$ holds for all n. □

(b) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ where r is any real number such that $r \neq 1$.

Proof. Let r be an arbitrary real other than 1. We will induct the statement $P(n) \equiv \sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ on n.

(Base Case) For $n = 1$, substitute into the LHS and expand the summation:

$$\sum_{i=1}^{1} r^i = r^0 + r^1 = 1 + r = (1+r)\frac{1-r}{1-r} = \frac{1-r^2}{1-r}$$

This is precisely the RHS of the equality, so $P(1)$ holds.

(Inductive Step) Now, suppose that $P(k)$ holds for an arbitrary k. Again, expand the summation but for the LHS of $P(k+1)$:

$$\begin{aligned}\sum_{i=0}^{k+1} r^i &= r^{k+1} + \sum_{i=0}^{k} r^i\\ &= r^{k+1} + \frac{1-r^{k+1}}{1-r} && \text{by inductive hypothesis}\\ &= \frac{(r^{k+1})(1-r)+1-r^{k+1}}{1-r}\\ &= \frac{r^{k+1}-r^{k+2}+1-r^{k+1}}{1-r}\\ &= \frac{1-r^{k+2}}{1-r}\end{aligned}$$

which is the other side of the equality. We have proved that if $P(n)$ holds, then $P(n+1)$ holds. Therefore, by induction, $P(n)$ holds for all natural n. □

(c) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}$.

Proof. We will induct the statement $P(n) \equiv \sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}$ on n.

First, verify the base case, $P(1)$. Then, we let $n = 1$ and have

$$\sum_{i=1}^{1} \frac{i}{(i+1)!} = 1 - \frac{1}{2!}$$

Expanding the summation, we can show that $P(1)$ holds:

$$\sum_{i=1}^{1} \frac{i}{(i+1)!} = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{2!}$$

Now, suppose $P(k)$ is true for some k, and consider $P(k+1)$:

$$\sum_{i=1}^{n+1} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+2)!}$$

Like above, we take out a term of the summation and simplify, so we have

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{i}{(i+1)!} &= \frac{k+1}{(k+2)!} + \sum_{i=1}^{k} \frac{i}{(i+1)!} \\
&= \frac{k+1}{(k+2)!} + 1 - \frac{1}{(k+1)!} && \text{by IH} \\
&= 1 + \frac{(k+1)-(k+2)}{(k+2)!} \\
&= 1 - \frac{1}{(k+2)!}
\end{aligned}$$

as required. We have proven $P(1)$ and that $P(k)$ implies $P(k+1)$, so, by induction, $P(n)$ is true for all natural n. □

(d) For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.

Proof. For induction on n, let $P(n) \equiv \sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.
Verify the base case $P(1)$:

$$\sum_{i=1}^{1} \frac{i}{2^i} = \frac{1}{2} = 2 - \frac{3}{2} = 2 - \frac{1+2}{2^1}$$

Suppose that $P(k)$ holds for some k, and consider $P(k+1)$. Now,

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{i}{2^i} &= \frac{k+1}{2^{k+1}} + \sum_{i=1}^{n} \frac{i}{2^i} \\
&= \frac{k+1}{2^{k+1}} + 2 - \frac{k+2}{2^k} && \text{by IH} \\
&= 2 + \frac{k+1-2(k+2)}{2^{k+1}} \\
&= 2 - \frac{k+3}{2^{k+1}}
\end{aligned}$$

as required. Because $P(1)$ holds and $P(k)$ implies $P(k+1)$, by induction, $P(n)$ holds for all n. □

(e) For all $n \in \mathbb{N}$, where $n \geq 4$, $n! > n^2$.

Proof. We will prove by induction on n. Let $P(n)$ be the statement $n! > n^2$.

To verify the base case $P(4)$, notice that $4! = 24$, that $4^2 = 16$, and that $24 > 16$.

Now, suppose that $P(k)$ is true for some $k \geq 4$. We must show that $P(k+1)$ holds, i.e., $(k+1)! > (k+1)^2$.

First, notice that $x^2 > x+1$ for all $x \geq 4$. Then, we can state the inductive hypothesis as $k! > k+1$. Multiplying both sides by $k+1$ gives $(k+1)! > (k+1)^2$, as desired.

Therefore, by induction, $n! > n^2$ for all $n \geq 4$. □

(f) For all $n \in \mathbb{N}$, $4^n - 1$ is divisible by 3.

Proof. Induct the statement "$4^n - 1$ is divisible by 3" on n.

For the base case, let $n = 1$ so $4^1 - 1 = 3$ and 3 is obviously divisible by 3.

Now, suppose that $4^k - 1$ is divisible by 3 for some natural number k. By definition, there exists an integer a where $4^k - 1 = 3a$.

Consider when $n = k+1$. Rearranging, $4^{k+1} - 1 = (4^{k+1} - 4) + 3 = 4(4^k - 1) + 3$. By our inductive hypothesis, this is equal to $4(3a) + 3 = 3(4a+1)$. Then, since $4^{k+1} - 1$ can be written as $3b$ for some integer b (namely, $b = 4a+1$), it is by definition divisible by 3.

Therefore, by induction, $4^n - 1$ is divisible by 3 for all $n \in \mathbb{N}$. □

RP02. Let x be a real number. Find the coefficient of x^{19} in the expansion of $(2x^3 - 3x)^9$.

Solution. Recall the Binomial Theorem, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. Let $a = 2x^3$, $b = -3x$, and $n = 9$. Then, we have $(2x^3 - 3x)^9 = \sum_{k=0}^{9} \binom{9}{k} 2^{9-k} (-3)^k x^{27-2k}$. We only care about when the exponent on x is 19, i.e., $27 - 2k = 19 \implies k = 4$. On this term of the summation, we have $\binom{9}{4} 2^5 (-3)^4 x^{19}$.

The coefficient is $\binom{9}{4} 2^5 (-3)^4 = 126 \cdot 32 \cdot 81 = 326592$. □

RP03. Let n be a non-negative integer. Prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Proof. We will induct the statement $P(n) \equiv \sum_{k=0}^{n} \binom{n}{k} = 2^n$ on $n \geq 0$.

For the base case, $P(0)$, we have

$$\sum_{k=0}^{0} \binom{0}{k} = \binom{0}{0} = 1 = 2^0.$$

Now, suppose $P(m)$ is true for some $m \geq 0$. Consider the summation in $P(m+1)$:

$$\begin{aligned}
\sum_{k=0}^{m+1} \binom{m+1}{k} &= \binom{m+1}{m+1} + \sum_{k=0}^{m} \binom{m+1}{k} \\
&= \binom{m+1}{m+1} + \sum_{k=0}^{m} \left(\binom{m}{k} + \binom{m}{k-1} \right) && \text{by PI} \\
&= \binom{m+1}{m+1} + \sum_{k=0}^{m} \binom{m}{k} + \sum_{k=0}^{m} \binom{m}{k-1} \\
&= 1 + 2^k + \sum_{k=0}^{m} \binom{m}{k-1} && \text{by IH}
\end{aligned}$$

Recall that negative binomial coefficients are undefined, so we can change the variable in the summation with $j = k+1$ and ignore the $k = 0$ term. Add and subtract a $\binom{m}{m}$ term to round out the summation and apply the IH once more:

$$\begin{aligned}
\sum_{k=0}^{m+1} \binom{m+1}{k} &= 1 + 2^k + \sum_{j=0}^{m-1} \binom{m}{j} \\
&= 1 + 2^k + \sum_{j=0}^{m-1} \binom{m}{j} + \binom{m}{m} - \binom{m}{m} \\
&= 1 + 2^k + \sum_{j=0}^{m} \binom{m}{j} - 1 \\
&= 1 + 2^k + 2^k - 1 && \text{by IH} \\
&= 2^{k+1}
\end{aligned}$$

which is what we wanted to show that $P(m+1)$ is true.

Therefore, by induction, $P(n)$ is true for all non-negative integer n. $\square$