MATH 245 Spring 2025:
Lecture Notes

Lecture notes taken, unless otherwise specified, by myself during the Spring 2025 offering of MATH 245, taught by Blake Madill.

Lectures

Chapter 1

Introduction

1.1 ?

Lecture 1
May 5

1.2 ?

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May 7

Recall: for $T : V \to V$ and an ideal $I = \{f(x) \in F[x] : f(T) = 0\}$.

Fact 1.2.1. If $\dim V < \infty$, then $I \neq \{0\}$.

Moreover, $I = \langle m(x) \rangle$ for a unique, monic, minimal polynomial $m(x)$.

Remark 1.2.2. $m(x)$ has minimal degree in I. If $f(x) \in I$, then $m(x) \mid f(x)$.

1.3 T-invariance

If $T : V \to V$ is linear and $W \leq V$ we want to consider $T : W \to W$ for inductive purposes.

Definition 1.3.1

Let $T : V \to V$, $W \leq V$. We say W is <u>T-invariant</u> if $T(W) \subseteq W$ (i.e., W is closed under T).

In this case, $T_W = T|_W : W \to W$ is well-defined.

Example 1.3.2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x, y) = (2x + 3y, 4x + 2y)$.

Is $W = \text{Span}\{(1, 1)\}$ T-invariant?

Solution. No. Notice that $T(1, 1) = (5, 6) \notin W$. $\square$

Example 1.3.3. Let $T : V \to V$ and $\lambda \in F$. The eigenspace $E_\lambda = \{v \in V : T(v) = \lambda v\}$ *is* T-invariant.

Proof. Let $v \in E_\lambda$ so that $T(v) = \lambda v$. We want to show $T(v)$ is also an eigenvector.

Then, $T(T(v)) = T(\lambda v) = \lambda T(v)$, which means $T(v) \in E_\lambda$. □

Proposition 1.3.4 (characteristic of the restriction divides characteristic of the operator)
Suppose $\dim V < \infty$ and $W \leq V$ is T-invariant where $T : V \to V$.

Let $f(x)$ and $g(x)$ be the characteristic polynomials for T and T_W.

Then, $g(x) \mid f(x)$.

Proof. Let $\beta = \{v_1, \dots, v_k\}$ be a basis for W. Extend it to a basis $\gamma = \{v_1, \dots, v_k, \dots, v_n\}$ for V.

Let $A = [T_W]_\beta$.[1] Then, $[T]_\gamma = \begin{bmatrix} [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \end{bmatrix}$.

But since $T(v_i) \in W$, $[T(v_i)]_\gamma = [T(v_i)]_\beta$.

Thus, $[T]_\gamma = \begin{bmatrix} A & \star \\ 0 & B \end{bmatrix}$ for some submatrices $\star$ and B.

Therefore, $f(x) = \det\big(xI - [T]_\gamma\big) = \det(xI - A)\det(xI - B)$. But $\det(xI - A)$ is just $g(x)$. □

Definition 1.3.5
Let $T : V \to V$ be a linear operator and $v \in V$. We call $W_{T,v} = \operatorname{Span}\{v, T(v), T^2(v), \dots\}$ the <u>T-cyclic subspace of V generated by v</u>.

Remark 1.3.6. $W_{T,v}$ is T-invariant, since $T(T^i(v)) = T^{i+1}(v) \in W_{T,v}$.

$W_{T,v}$ is the smallest T-invariant subspace of V which contains v.

Proposition 1.3.7
Let $T : V \to V$ and $v \neq 0$. Say $\dim W_{T,v} = k < \infty$.

Then, $\beta = \{v, T(v), \dots, T^{k-1}(v)\}$ is a basis.

Proof. Let $j \in \mathbb{N}$ be the maximal index such that $\{v, T(v), \dots, T^{j-1}(v)\}$ is linearly independent. This must exist since $v \neq 0$, so $\{v, T(v)\}$ is linearly independent.

Let $U = \operatorname{Span}\{v, T(v), \dots, T^{j-1}(v)\}$. We must show $U = W := W_{T,v}$. Clearly, $U \subseteq W$.

Consider $T^j(v)$. Since j is maximal, $T^j(v) = a_0 v + a_1 T(v) + \cdots + a_{j-1} T^{j-1}(v)$ where at least one $a_i \neq 0$ Hence, $T^j(v) \in U$. In fact, $T^i(v)$ for all $i \geq j$ is in U, so $W \subseteq U$. □

[1]The matrix of T_W with respect to the basis β.

Proposition 1.3.8

Let $T : V \to V$, $W = W_{T,v}$ for $v \neq 0$.

If $\dim W = k \leq \infty$ and $f(x) = x^k + a_{k-1}x^{k-1} + \cdots a_1 x + a_0 \in F[x]$ is such that $f(T)(v) = 0$, then $f(x)$ is the characteristic polynomial of T_W.

That is, if we find a k-degree polynomial that kills the generator, it must be the characteristic polynomial.

Proof. Let $\beta = \{T^i(v)\}$ be the basis from prop. 1.3.7. Then,

$$\begin{aligned} 0 = f(T)(v) &= T^k(v) + a_{k-1}T^{k-1}(v) + \cdots + a_1 T(v) + a_0 v \\ T^k(v) &= -a_0 - a_1 T(v) - \cdots - a_{k-1}T^{k-1}(v) \end{aligned}$$

so $[T_W]_\beta = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{bmatrix}$.

By assignment 1, $f(x)$ is the characteristic polynomial of $[T_W]_\beta$. □

1.4 The Cayley–Hamilton theorem

Theorem 1.4.1 (Cayley–Hamilton)

Let $\dim V < \infty$ and let $T : V \to V$ be linear.

If $f(x)$ is the characteristic polynomial for T, then $f(T) = 0$.

That is, a linear operator satisfies its own characteristic polynomial.

Why do we care?

Remark 1.4.2. In this case, if $f(x) \in I = \{g(x) : g(T) = 0\}$, then $I \neq \{0\}$. That means the minimal polynomial of T from fact 1.2.1 exists and $m(x) \mid f(x)$.

We can now prove the Cayley–Hamilton theorem: *Lecture 3 May 9*

Proof. If $v = 0$, then $f(T)(v) = 0$, so assume $v \neq 0$.

Let $W = W_{T,v}$ and consider $V_W : W \to W$.

Suppose the characteristic polynomial of T_W is $g(x)$ and let $\beta = \{v, T(v), \ldots, T^{k-1}(v)\}$ be a basis for W.

But since $T^k(v)$ is not in this basis, there exist $a_i \in F$, not all zero, such that

$$T^k(v) + a_{k-1}T^{k-1}(v) + \cdots + a_1 T(v) + a_0 v = 0$$

but by prop. 1.3.8, this is the characteristic polynomial $g(x)$. Therefore, $g(T)(v) = 0$.

Finally, since $g(x) \mid f(x)$, we also have $f(T)(v) = 0$. □

1.5 Properties of the minimal polynomial

Proposition 1.5.1

Let $T : V \to V$ and $\dim V < \infty$. Let $m(x)$ and $f(x)$ be the minimal and characteristic polynomials. Then, $m(x)$ and $f(x)$ have the same roots in F.

Example 1.5.2. Suppose $T : V \to V$ with $f(x) = (x-2)^2(x-3)$. Then, $m(x)$ must be $(x-2)(x-3)$ since it has the same roots and has minimal degree.

Proof. Since $m(x) \mid f(x)$, every root of $m(x)$ is a root of $f(x)$.

Let $\lambda \in F$ be a root of $f(x)$. Say $T(v) = \lambda v$ for $v \neq 0$. Then, consider $m(T)(v)$:

$$m(T)(v) = \sum a_i T^i(v) = \sum a_i(\lambda^i v) = m(\lambda)v$$

But we know $m(T) = 0$, so $m(\lambda)v = 0$. Finally, since $v \neq 0$, we must have $m(\lambda) = 0$. □

Definition 1.5.3

Let $A \in M_n(F)$. The minimal polynomial of A is the unique, monic $m(x) \in F[x]$ such that $\langle m(x) \rangle = \{f(x) : f(A) = 0\}$.

Exercise 1.5.4. Let $A \in M_n(F)$, $T_A : F^n \to F^n$, $T_A(x) = Ax$. Show that the minimal polynomial of A is the minimal polynomial of T_A.

Example 1.5.5. Find the minimal polynomial of $A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix}$.

Solution. Expand and find $f(x) = (x-2)^2(x-3)$.

Either $m(x) = f(x)$ or $m(x) = (x-2)(x-3)$. Since $(A-2I)(A-3I) = 0$, we have $m(x) = (x-2)(x-3)$. □

Remark 1.5.6. Jordan canonical form will provide a much better algorithm for computing these minimal polynomials.

The minimal polynomial is already helpful: it gives us eigenvalues. But we can do more.

Proposition 1.5.7

Let $T : V \to V$, $\dim V < \infty$. Let λ_i be the distinct eigenvalues of T in F. Then, T is diagonalizable if and only if the minimal polynomial of T is $m(x) = \prod(x - \lambda_i)$.

This is much faster than checking multiplicities of the characteristic polynomial against the geometric multiplicities from the operator's eigenspaces.

Proof. ($\Rightarrow$) Suppose T is diagonalizable and let $\beta = \{v_i\}$ be a basis for V consisting of eigenvectors for T. Let $p(x) = \prod(x - \lambda_i)$.

First, $p(x) \mid m(x)$ since each factor is certainly in $m(x)$.

Now, consider a $v_i \in \beta$ with $T(v_i) = \lambda_i v_i$. But we know that $p(x) = q_i(x)(x - \lambda_i)$ for some $q_i(x)$.

Then, $p(T)(v_i) = q_i(T)(v_i) \cdot (T - \lambda_i I)(v_i) = q_i(T)(v_i) \cdot (Tv_i - \lambda_i v_i) = 0$. Since $m(x)$ divides everything that makes T vanish, $m(x) \mid p(x)$.

Because $m(x)$ and $p(x)$ are monic, they must now be equal.

($\Leftarrow$) Assume the minimal polynomial of T is $m(x) = \prod(x - \lambda_i)$. Induct on the dimension.

If $\dim V = 1$, then we are done (1×1 matrices are diagonal).

Assume that if $\dim V < n$, T is diagonalizable. Take $\dim V = n$. Consider the subspace $W = \mathrm{im}_{T-\lambda_k I}(V)$. Since $T \circ (T - \lambda_k I) = (T - \lambda_k I) \circ T$, i.e., T(something in the range) is in the range, W is T-invariant. Therefore, $T_W : W \to W$ is linear.

Let $m_W(x)$ be the minimal polynomial of T_W. Then, $m(T_W) = 0$ and therefore $m_W(x) \mid m(x)$.

Since the kernel of $T - \lambda_k I$ is non-trivial (it contains the eigenvector), the nullity of $T - \lambda_k I$ is non-zero and the rank is less than n. That is, $\dim W < n$ and T_W is diagonalizable by the inductive hypothesis. $\square$