MATH 135 Fall 2020: Extra Practice 5

Warm-Up Exercises

WE01. What is the remainder when -98 is divided by 7?

$-98 \div 7 = -14$, so the remainder is 0.

WE02. Calculate $\gcd(10, -65)$.

We have $10 = 2 \cdot 5$ and $-65 = -1 \cdot 5 \cdot 13$, so the GCD is 5.

WE03. Let $a, b, c \in \mathbb{Z}$. Consider the implication S: If $\gcd(a, b) = 1$ and $c \mid (a + b)$, then $\gcd(b, c) = 1$. Fill in the blanks to complete a proof of S.

(a) Since $\gcd(a, b) = 1$, by $\boxed{\text{Bézout's Lemma}}$, there exist integers x and y such that $ax + by = 1$.

(b) Since $c \mid (a + b)$, by $\boxed{\text{definition}}$, there exists an integer k such that $a + b = ck$.

(c) Substituting $a = ck - b$ into the first equation, we get $1 = (ck - b)x + by = b(-x + y) + c(kx)$.

(d) Since 1 is a common divisor of b and c and $-x + y$ and kx are integers, $\gcd(b, c) = 1$ by $\boxed{\text{the GCD Characterization Theorem}}$.

WE04. Disprove: For all integers a, b, and c, if $a \mid (bc)$, then $a \mid b$ or $a \mid c$.

Proof. We prove the negation, there are integers a, b, and c where $a \mid bc$, $a \nmid b$, and $a \nmid c$.

Let $a = 15$, $b = 5$, and $c = 3$. Clearly, $a \nmid b$ and $a \nmid c$. However, $bc = 15$, and $15 \mid 15$. □

Recommended Problems

RP01.

(a) Use the Extended Euclidean Algorithm to find three integers x, y and $d = \gcd(1112, 768)$ such that $1112x + 768y = d$.

Solution. Apply the EEA with $x = 1112$ and $y = 768$.

x	y	r	q
1	0	1112	
0	1	768	
1	-1	344	1
-2	3	80	2
9	-13	24	4
-29	42	8	3
96	-139	0	3

Therefore, we have that $d = \gcd(1112, 768) = 8$, and that

$$1112(-29) + 768(42) = 8$$

That is, our solution is when $x = -29$ and $y = 42$. □

(b) Determine integers s and t such that $768s - 1112t = \gcd(768, -1112)$.

Solution. Since the GCD is invariant under sign changes, we immediately know that $\gcd(768, -1112) = 8$. We also have that $1112(-96) + 768(42) = 8$. But this is the same as saying $768(42) - 1112(96) = 8$, so $s = 42$ and $t = 96$. $\square$

RP02. Prove that for all $a \in \mathbb{Z}$, $\gcd(9a + 4, 2a + 1) = 1$.

Proof. Let a be an integer. We must show that $9a + 4$ and $2a + 1$ are coprime. Recall the Coprimeness Characterization Theorem: it suffices to find integers a and b such that $(9a + 4)a + (2a + 1)b = 1$.

Choose $a = -2$ and $b = 9$. Then,

$$\begin{aligned}(9a + 4)a + (2a + 1)b &= -2(9a + 4)a + 9(2a + 1) \\ &= -18a - 8 + 18a + 9 \\ &= 1\end{aligned}$$

as desired. Therefore, $\gcd(9a + 4, 2a + 1) = 1$. $\square$

RP03. Let $\gcd(x, y) = d$ for integers x and y. Express $\gcd(18x + 3y, 3x)$ in terms of d and prove that you are correct.

RP04. Let $a, b \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$, then $\gcd(2a + b, a + 2b) \in \{1, 3\}$.

RP05. Prove that for all integers a, b and k, if $b \neq 0$, then $\gcd(a, b) \leq \gcd(ak, b)$.

RP06. Prove that for all integers a, b and c: if $a \mid c$ and $b \mid c$ and $\gcd(a, b) = 1$, then $ab \mid c$.

RP07. Let $a, b, c \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$ and $c \mid a$, then $\gcd(b, c) = 1$.

RP08. Let a and b be integers. Prove that if $\gcd(a, b) = 1$, then $\gcd(a^m, b^n) = 1$ for all $m, n \in \mathbb{N}$. You may use the result which is proved in Example 14 in the notes.

RP09. Suppose a, b and n are integers. Prove that $n \mid \gcd(a, n) \cdot \gcd(b, n)$ if and only if $n \mid ab$.

RP10. How many positive divisors does 33480 have?

RP11. Prove that for all integers a and b, if $9a^2 = b^4$ where $a, b \in \mathbb{Z}$, then 3 is a common divisor of a and b.

RP12. Let $n \in \mathbb{N}$. Prove that if p is prime and $p \leq n$, then p does not divide $n! + 1$.

Challenges

C01. Prove that for any integer $a \neq 1$ and $n \in \mathbb{N}$, $\gcd\left(\frac{a^n - 1}{a - 1}, a - 1\right) = \gcd(n, a - 1)$.

C02. Let n be a positive integer for which $\gcd(n, n + 1) < \gcd(n, n + 2) < \cdots < \gcd(n, n + 20)$. Prove that $\gcd(n, n + 20) < \gcd(n, n + 21)$.

C03. Let a and b be nonnegative integers. Prove that $\gcd(2^a - 1, 2^b - 1) = 2^{\gcd(a,b)} - 1$.

C04. An integer n is *perfect* if the sum of all of its positive divisors (including 1 and itself) is $2n$.

(a) Is 6 a perfect number? Give reasons for your answer.

(b) Is 7 a perfect number? Give reasons for your answer.

(c) Prove the following statement: If k is a positive integer and $2^k - 1$ is prime, then $2^{k-1}(2^k - 1)$ is perfect.

C05. Let $a, b \in \mathbb{Z}$. Prove that $\gcd(a^n, b^n) = \gcd(a, b)^n$ for all $n \in \mathbb{N}$.