MATH 137 Fall 2020: Practice Midterm 1

Multiple Choice

MC01. For the sequence $\{a_n\}$ where $a_1 = 1$, $a_n = \sqrt{6 + a_{n-1}}$ for $n \geq 2$. The value of $\lim_{n\to\infty} a_n$ is

(a) -3
(b) -2
(c) 2
(d) $\boxed{\text{None of the above.}}$ $a_2 = \sqrt{7} > 2$ *and* $\{a_n\}$ *is increasing*

MC02. $\displaystyle\lim_{x\to 1} \frac{\sin(x-1)}{x^2-1}$

(a) $= 1$.
(b) $\boxed{= \frac{1}{2}.}$ *Translate to* $\displaystyle\lim_{y\to 0} \frac{\sin y}{y^2+2y} = \lim_{y\to 0} \frac{\sin y}{y} \cdot \frac{1}{y+2}$
(c) Does not exist.
(d) None of the above.

MC03. If f is not continuous at $x = 2$ then it must be the case that

(a) $f(2) \geq 0$.
(b) $f(2)$ is undefined.
(c) $f(2)$ is defined.
(d) $\boxed{\text{None of the above.}}$ *For examples,* $f(x) = \dfrac{1}{x-2}$ *and* $f(x) = \begin{cases} 0 & x \neq 2 \\ -1 & x = 2 \end{cases}$

MC04. The sequence defined by $a_n = \dfrac{n^2+1}{n+3}$

(a) converges.
(b) is non-increasing.
(c) $\boxed{\text{is bounded below.}}$ *Notice that* $a_n \to \infty$
(d) None of the above.

MC05. If $f(x) = 7$ for all $x \in \mathbb{R}$ then $f'(x)$

(a) $\boxed{\text{exists for all } x \in \mathbb{R}.}$ *And is equal to 0*
(b) is not continuous for all $x \in \mathbb{R}$.
(c) $= 1$.
(d) None of the above.

True/False

TF06. Three functions, f, g and h, are defined on an open interval I containing $x = a$. If for each $x \in I$, $g(x) < f(x) < h(x)$ and $\lim_{x\to a^+} g(x) = L = \lim_{x\to a^+} h(x)$, then $\lim_{x\to a} f(x) = L$.

False. If $\lim_{x\to a^-} g(x) \neq \lim_{x\to a^-} h(x)$, then $\lim_{x\to a^-} f(x)$ can be any value between (or undefined).

TF07. The Fundamental Trigonometric Limit tells us that if θ is small, then $\cos\theta \approx \theta$.

False. This is true for $\sin\theta$.

TF08. If f is continuous on $\mathbb{R}$ and $f(0) > 0$ then there exists $\delta > 0$ so that $f(x) > 0$ for all $x \in (0, \delta)$.

True. This follows from the ϵ–δ definition and that $\lim_{x\to 0} f(x) = f(0) > 0$.

TF09. $\lim_{x\to\infty} \frac{x^p}{e^x} = 0$ for all $p \in \mathbb{R}$.

True. For positive p, see course notes. For negative p, we have $\frac{1}{x^q e^x}$ for positive q, which converges to 0. For zero p, $\frac{1}{e^x}$ converges to 0.

TF10. If $f(a)$ exists, then $f'(a)$ exists too.

False. Consider for example $f(x) = \lfloor x \rfloor$, which is defined but is not continuous at $x = 2$.

Short Answer

SA01. For the function

$$f(x) = \begin{cases} 1 + \sin x & x < 0 \\ \cos x & 0 \leq x \leq \pi \\ \sin x & \pi < x \end{cases}$$

determine

(a) $\lim_{x\to 0} f(x)$, or write DNE if it does not exist.

Solution. From below, $\lim_{x\to 0^-}(1+\sin x) = 1+\sin 0 = 1$. From above, $\lim_{x\to 0^+} \cos x = \cos 0 = 1$. Since the one-sided limits agree, the limit exists and is $\boxed{1}$. □

(b) $\lim_{x\to \pi} f(x)$, or write DNE if it does not exist.

Solution. From below, $\lim_{x\to \pi^-} \cos x = \cos\pi = 1$. From above, $\lim_{x\to\pi^+} \sin x = \sin\pi = 0$. Since the one-sided limits do agree, the limit $\boxed{\text{DNE}}$. □

SA02. Write all solutions to $|x - 1| = |2x|$

Solution. For $x > 1$: $x - 1 = 2x \implies x = -1$.
For $0 < x < 1$: $-(x-1) = 2x \implies x = \frac{1}{3}$.
For $x < 0$: $-(x-1) = -(2x) \implies x = -1$.

Therefore, $\boxed{x \in \left\{-1, \frac{1}{3}\right\}}$. □

SA03. Give an example of a function such that the Extreme Value Theorem does not apply to it on the interval $[0, 5]$.

Solution. Let $\boxed{f(x) = \dfrac{1}{x-2}}$.

There is a vertical asymptote at $x = 2$, which breaks the Extreme Value Theorem. ☐

SA04. State the formal ϵ–δ definition of what it means for $\lim_{x \to a} f(x) = L$.

Solution. For all $\epsilon > 0$, there exists a $\delta > 0$, such that for every x in the domain of f, $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$. That is,

$$\boxed{\forall \epsilon > 0, \exists \delta > 0, \forall x, 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon}$$ ☐

Long Answer

LA01. Use the ϵ–N definition of the limit of a sequence to show $\lim_{n \to \infty} \dfrac{n+1}{3n+2} = \dfrac{1}{3}$.

Proof. Let $a_n = \frac{n+1}{3n+2}$ and $\epsilon > 0$. We must find N so $n \geq N$ implies $\left|a_n - \frac{1}{3}\right| < \epsilon$.

Because $a_n - \frac{1}{3} = \frac{3n+3-3n-2}{9n+6} = \frac{1}{9n+6}$, it suffices to show $\left|\frac{1}{9n+6}\right| < \epsilon$. Since $9n+6$ is positive for all positive n, we may drop the absolute value bars.

Let $N = \frac{1}{9\epsilon}$. Then,

$$\begin{aligned} n \geq N &\implies n \geq \frac{1}{9\epsilon} \\ &\implies 9n \geq \frac{1}{\epsilon} \\ &\implies 9n + 6 > \frac{1}{\epsilon} \\ &\implies \frac{1}{9n+6} < \epsilon \end{aligned}$$

exactly as desired.

Therefore, by the ϵ–N definition of a limit of a sequence, $\lim_{n \to \infty} a_n = \dfrac{1}{3}$. ☐

LA02. Compute the following sequence limits, or show that they do not exist.

(a) $\lim_{n \to \infty} \dfrac{\cos n}{n}$

Proof. We propose that the limit is 0 and prove it. Recall that $-1 \leq \cos n \leq 1$ for all n. Then, for positive n, $-\frac{1}{n} < \frac{\cos n}{n} < \frac{1}{n}$.

Trivially, $-\frac{1}{n} \to 0$ and $\frac{1}{n} \to 0$. The limits agree and $\frac{\cos n}{n}$ is bounded by them above and below.

Therefore, by the squeeze theorem, $\frac{\cos n}{n}$ also converges to 0. ☐

(b) $\lim_{n\to\infty} \dfrac{2n^2 - n - 1}{5n^2 + n - 3}$

Proof. Recall that for any rational function $\frac{f(x)}{g(x)}$, if $\deg f = \deg g$, the limit at infinity is the ratio of the leading coefficients.

Therefore, the limit is $\frac{2}{5}$. □

LA03. Consider the recursive sequence $a_1 = 5$ and $a_{n+1} = \dfrac{a_n + 1}{3}$ for $n \geq 1$.

(a) Prove that the sequence is decreasing and is bounded below by 0.

Proof. We prove by induction of the sentence $0 < a_{n+1} < a_n$ on n.

For the base case, notice that $a_1 = 5$ and $a_2 = \frac{5+1}{3} = 2$. We have $0 < a_2 < a_1$.

Now, suppose that $0 < a_{k+1} < a_k$ for some k. Then,

$$\begin{aligned} 1 < a_{k+1} + 1 &< a_k + 1 \\ \frac{1}{3} < \frac{a_{k+1}+1}{3} &< \frac{a_k+1}{3} \\ 0 < a_{k+2} &< a_{k+1} \end{aligned}$$

as desired. Therefore, by induction, $0 < a_{n+1} < a_n$ for all n, that is, a_n is decreasing and bounded below by 0. □

(b) Prove that the sequence converges and find its limit.

Proof. Because a_n is non-increasing and bounded below, the limit exists and is equal to L by the monotone convergence theorem.

Recall that if $a_n \to L$, then $a_{n+1} \to L$. Then,

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} a_{n+1} \\ L &= \lim_{n\to\infty} \frac{a_n + 1}{3} \\ L &= \frac{\lim_{n\to\infty} a_n + 1}{3} \\ L &= \frac{L+1}{3} \\ L &= \frac{1}{2} \end{aligned}$$

□

LA04. Use the ϵ–δ definition of the limit of a function to show that $\lim_{x\to 2} \left(x^2 + 2x - 3\right) = 5$.

Proof. Let $\epsilon > 0$. We must find δ so that $0 < |x - 2| < \delta$ implies $\left|(x^2 + 2x - 3) - 5\right| = |x^2 + 2x - 8| < \epsilon$.

We can limit δ by having it equal $\min(\{\frac{\epsilon}{7}, 1\})$. Then, when $|x-2| < \delta$ we have $|x+4| < 7$.

We now have $|x - 2| < \delta \leq \frac{\epsilon}{7}$ and $|x + 4| < 7$. Multiplying,

$$\begin{aligned} |x-2| \cdot |x+4| &< \frac{\epsilon}{7} \cdot 7 \\ |(x-2)(x+4)| &< \epsilon \\ |x^2 + 2x - 8| &< \epsilon \end{aligned}$$

Therefore, by the ϵ–δ definition of the limit of a function, $\lim_{x\to 2}\left(x^2+2x-3\right)=5$. $\square$

LA05. Compute the following function limits, if possible. If the limit does not exist, prove it.

(a) $\displaystyle\lim_{x\to 0}\frac{5x^2-3x}{2x^3-x^2}$

Solution. Recall the continuity of polynomials and quotients. It follows that all rational functions $\frac{p(x)}{q(x)}$ are continuous at any $x=a$ so long as $q(a)\neq 0$.

Let $f(x)=\frac{5x^2-3}{2x^3-x^2}$. At $x=0$, we have $2x^3-x^2=0$. Therefore, f is not continuous at $x=0$ and we analyze the one-sided limits to determine the type of discontinuity.

First, notice that we may factor as $f(x)=\frac{1}{x^2}\cdot\frac{5x-3}{2x-1}$.

Consider the sequence $a_n=\frac{1}{n}$, a sequence which converges to 0. We have

$$\lim_{n\to\infty}f(a_n)=\lim_{n\to\infty}n^2\left(\frac{\frac{5}{n}-3}{\frac{2}{n}-1}\right)=\infty$$

By the sequential characterization of limits, if the limit of $f(a_n)$ does not exist, so too does the limit f at $x=0$.

Therefore, $\displaystyle\lim_{x\to 0}\frac{5x^2-3x}{2x^3-x^2}$ does not exist. $\square$

(b) $\displaystyle\lim_{x\to 0}\frac{3x^2-1}{x^2-x+1}$

Proof. Recall again the continuity of rational functions.

Here, the denominator is $(0)^2-(0)+1=1\neq 0$, therefore we may simply evaluate the function at $x=0$. This is $\frac{0-1}{0-0+1}=-1$. $\square$

(c) $\displaystyle\lim_{x\to\infty}\frac{\ln x^2-\ln x}{x^2-x}$

Solution. Simplify:

$$\begin{aligned}
\lim_{x\to\infty}\frac{\ln x^2-\ln x}{x^2-x} &= \lim_{x\to\infty}\frac{2\ln x-\ln x}{x(x-1)} && \text{by logarithm laws}\\
&= \lim_{x\to\infty}\frac{\ln x}{x}\cdot\lim_{x\to\infty}\frac{1}{x-1} && \text{by limit laws}\\
&= 0\cdot 0 && \text{by FLL} \quad \square
\end{aligned}$$

LA06. Given the function

$$f(x)=\begin{cases}k^2+5 & x\leq -2\\ x^2+k & x>-2\end{cases}$$

If $f(x)$ is continuous at $x=-2$, determine all value(s) for k.

Solution. Recall that for f to be continuous at $x=-2$, $\lim_{x\to -2} f(x) = f(-2)$.

This limit exists if and only if the one-sided limits,

$$\lim_{x\to -2^-} f(x) = k^2+5 \qquad \text{and} \qquad \lim_{x\to -2^+} f(x) = 4+k$$

agree as x approaches -2 from above and below. That is,

$$\begin{aligned} k^2+5 &= 4+k \\ 0 &= k^2-k+1 \end{aligned}$$

which is a quadratic in k. However, the discriminant, $(-1)^2-4(1)(1)=-3$, is negative. This means there are no real solutions for k. $\square$

LA07. Determine, with justification, all vertical asymptotes of the function

$$f(x) = \frac{x+3}{|x^2-2x-15|}.$$

Solution. Recall the continuity of polynomials and quotients. It follows that all rational functions $\frac{p(x)}{q(x)}$ are continuous at any $x=a$ so long as $q(a)\neq 0$.

Factoring, $\left|x^2-2x-15\right| = |(x-5)(x+3)| = |x-5|\cdot|x+3|$. This is zero at $x=-3,5$. We consider these two options:

- Consider $x=-3$. The limit from below is:

  $$\lim_{x\to -3^-} \frac{x+3}{|x-5|\cdot|x+3|} = \lim_{x\to -3^-} \frac{x+3}{-(x-5)\cdot -(x+3)} = \lim_{x\to -3^-} \frac{1}{x-5} = \frac{1}{8}$$

  and the limit from above is:

  $$\lim_{x\to -3^+} \frac{x+3}{|x-5|\cdot|x+3|} = \lim_{x\to -3^+} \frac{x+3}{-(x-5)(x+3)} = \lim_{x\to -3^+} -\frac{1}{x-5} = -\frac{1}{8}$$

  These limits do not agree but they exist. Therefore, there is a jump discontinuity.

- Consider $x=5$. The limit from below is:

  $$\lim_{x\to 5^-} \frac{x+3}{|x-5|\cdot|x+3|} = \lim_{x\to 5^-} \frac{x+3}{-(x-5)(x+3)} = \lim_{x\to 5^-} -\frac{1}{x-5} = \lim_{x_0\to 0^-} -\frac{1}{x_0} = \infty$$

  This is enough to say that there exists a vertical asymptote at $x=5$.

Therefore, discontinuities exist only at $x=-3,5$, where $x=5$ is a vertical asymptote. $\square$

LA08. Suppose $A,B\in\mathbb{R}$, $A>0$, $B>0$ and $f:\mathbb{R}\to\mathbb{R}$ is a function such that if $|x-y|<A$ then $|f(x)-f(y)|<B|x-y|$ for all $x,y\in\mathbb{R}$.

Prove that f is continuous on $\mathbb{R}$.

Proof. Let A and B be positive reals, and f be a function on the reals such that $|x-y|<A$ implies $|f(x)-f(y)|<B|x-y|$ for any x and y.

We must show that $\lim_{n \to a} f(n) = f(a)$ for all a. That is, for any tolerance $\epsilon > 0$, we may find a δ such that $0 < |n - a| < \delta$ implies $|f(n) - f(a)| < \epsilon$.

Let $\epsilon > 0$ and a be a real. Select $\delta = \min(\{A, \frac{\epsilon}{B}\})$.

Suppose that $0 < |n - a| < \delta$. That is, $|n - a| < A$ and $|n - a| < \frac{\epsilon}{B}$.

It also follows that $|f(n) - f(a)| < B|n - a|$. But we supposed that $|n - a| < \frac{\epsilon}{B}$, so

$$|f(n) - f(a)| < B\frac{\epsilon}{B} = \epsilon$$

This is exactly what was needed to show that f is continuous for any a. □

LA09. Prove that $x^2 + x\cos x = 1$ has at least two real solutions.

Proof. Let $f(x) = x^2 + x\cos x$. Recall that polynomials and cosine are both continuous on $\mathbb{R}$. Therefore, their sum/product, f, is also continuous.

At $x = 0$, we have $f(x) = 0 + 0 = 0$.

At $x = -\pi$, we have $f(x) = \pi^2 - \pi(-1) = \pi^2 + \pi > 1$. We then have that $f(-\pi) < 1 < f(0)$. So, by the intermediate value theorem, there exists some $a \in (-\pi, 0)$ where $f(x) = 1$.

Likewise at $x = \pi$, we have $f(x) = \pi^2 + \pi(-1) = \pi^2 - \pi > 1$. We then have that $f(0) < 1 < f(\pi)$. So, by the intermediate value theorem, there exists some $b \in (0, \pi)$ where $f(x) = 1$.

Therefore, there must exist at least two real solutions, a and b, to $x^2 + x\cos x = 1$. □