
MATH 135 Fall 2020: Extra Practice 3
Warm-Up Exercises**WE01.** Prove the following two quantified statements.

(a) $\forall n \in \mathbb{N}, n + 1 \geq 2$

Proof. Let $n \in \mathbb{N}$. Recall that 1 is the smallest natural. $n \geq 1 \iff n + 1 \geq 2$. \square

(b) $\exists n \in \mathbb{Z}, \frac{5n-6}{3} \in \mathbb{Z}$

Proof. Select $n = 3$. Then, $\frac{5n-6}{3} = \frac{15-6}{3} = \frac{9}{3} = 3 \in \mathbb{Z}$. \square **WE02.** Prove that for all $k \in \mathbb{Z}$, if k is odd, then $4k + 7$ is odd.*Proof.* Let k be an odd integer. $4k + 7$ is odd if and only if it may be written as $2n + 1$ for some integer n . Let $n = 2k + 3$. Then, $2n + 1 = 2(2k + 3) + 1 = 4k + 6 + 1 = 4k + 7$, as required. \square **WE03.** Consider the following proposition*For all $a, b \in \mathbb{Z}$, if $a^3 \mid b^3$, then $a \mid b$.*

We now give three erroneous proofs of this proposition. Identify the major error in each proof, and explain why it is an error.

- (a)
- Consider $a = 2$, $b = 4$. Then $a^3 = 8$ and $b^3 = 64$. We see that $a^3 \mid b^3$ since $8 \mid 64$. Since $2 \mid 4$, we have $a \mid b$.*

This proof is erroneous as it only considers one specific case of a and b and not the general case of integer a and b .

- (b)
- Since $a \mid b$, there exists $k \in \mathbb{Z}$ such that $b = ka$. By cubing both sides, we get $b^3 = k^3 a^3$. Since $k^3 \in \mathbb{Z}$, then $a^3 \mid b^3$.*

This proof supposes the conclusion instead of the hypothesis.

- (c)
- Since $a^3 \mid b^3$, there exists $k \in \mathbb{Z}$ such that $b^3 = ka^3$. Then $b = (ka^2/b^2)a$, hence $a \mid b$.*

The proof does not guarantee that $\frac{ka^2}{b^2}$ is an integer.**WE04.** Let x be a real number. Prove that if $x^3 - 5x^2 + 3x \neq 15$, then $x \neq 5$.*Proof.* Suppose for the contrapositive that $x = 5$. Then, $x^3 - 5x^2 + 3x = (5)^3 - 5(5)^2 + 3(5) = 15$, as required. Since the contrapositive is true, the original implication must be true. \square **WE05.** Prove that there do not exist integers x and y such that $2x + 4y = 3$.

Proof. For the sake of contradiction, suppose the negation is true.

Consider the negation of the statement: there exist integers x and y such that $2x + 4y = 3$. Let x and y be such integers. Then, $x + 2y$ is an integer. Therefore, $2x + 4y = 2(x + 2y)$ is even. However, 3 is odd. An integer cannot be both even and odd, therefore, the negation is false, and the original statement is true. \square

WE06. Prove that an integer is even if and only if its square is an even integer.

Proof. (\Rightarrow) Let n be an even integer. Then, $n = 2k$ for some integer k . $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since $2k^2$ is an integer, n^2 is even.

(\Leftarrow) Let n be an even square integer. Then, $n = 2k$ for some integer k and $n = x \cdot x$ for some integer x . Since $2k = x \cdot x$, and 2 is prime, 2 must divide x . Therefore, $x = 2y$ for some integer y , which is the definition of being even.

Since the implication is true in both directions, the biconditional is true. \square

Recommended Problems

RP01. Prove that $x^2 + 9 \geq 6x$ for all real numbers x .

Proof. Let x be a real number. $x^2 + 9 \geq 6x \iff x^2 - 6x + 9 \geq 0 \iff (x - 3)^2 \geq 0$. Since the square of a real is always non-negative, the statements are true. \square

RP02. Prove that for all $r \in \mathbb{R}$ where $r \neq -1$ and $r \neq -2$,

$$\frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} = \frac{r(2^r)}{(r+1)(r+2)}$$

Proof. Let r be a real number that is neither -1 nor -2 . Then,

$$\begin{aligned} LHS &= \frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} \\ &= \frac{2^{r+1}(r+1) - 2^r(r+2)}{(r+1)(r+2)} \\ &= \frac{r2^{r+1} + 2^{r+1} - r2^r - 2 \cdot 2^r}{(r+1)(r+2)} \\ &= \frac{r2^{r+1} + \cancel{2^{r+1}} - r2^r - \cancel{2^{r+1}}}{(r+1)(r+2)} \\ &= \frac{r(2^{r+1} - 2^r)}{(r+1)(r+2)} \\ &= \frac{r(2^r \cdot 2 - 2^r)}{(r+1)(r+2)} \\ &= \frac{r(2^r + \cancel{2^r} - \cancel{2^r})}{(r+1)(r+2)} \\ &= \frac{r(2^r)}{(r+1)(r+2)} \\ &= RHS \end{aligned}$$

Since the left side equals the right side, the equality is true. \square

RP03. Prove that there exists a real number x such that $x^2 - 6x + 11 \leq 2$.

Proof. Let $x = 3$. $x^2 - 6x + 11 = (3)^2 - 6(3) + 11 = 9 - 18 + 11 = 2 \leq 2$, as required. Since 3 is a real number, the statement is true. \square

RP04. Prove or disprove each of the following statements.

(a) $\forall n \in \mathbb{Z}$, $\frac{5n-6}{3}$ is an integer.

Proof. Let $n = 1$ as a counter-example. Then, $\frac{5n-6}{3} = \frac{5-6}{3} = -\frac{1}{3}$, which is not an integer. Therefore, the statement is false. \square

(b) $\forall a \in \mathbb{Z}$, $a^3 + a + 2$ is even.

Proof. Let a be an integer. Then, a is either even or odd. Suppose that a is even and can be written as $a = 2k$ for an integer k . Then, $a^3 + a + 2 = (2k)^3 + 2k + 2 = 8k^3 + 2k + 2 = 2(4k^3 + k + 1)$, an even number.

Suppose a is odd and can be written as $a = 2k + 1$ for an integer k . Then, $a^3 + a + 2 = (2k + 1)^3 + (2k + 1) + 2 = 2k^3 + 12k^2 + 2k + 4 = 2(k^3 + 6k^2 + k + 2)$, an even number.

Therefore, the statement is true. \square

(c) For every prime number p , $p + 7$ is composite.

Proof. Let p be a prime number. \square

(d) For all $x \in \mathbb{R}$, $|x - 3| + |x - 7| \geq 10$.

Proof. Let $x = 3$ as a counter-example. Then, $|x - 3| + |x - 7| = |(3) - 3| + |(3) - 7| = 0 + 4 = 4 \not\geq 10$. Therefore, the statement is false. \square

(e) There exists a natural number $m < 123456$ such that $123456m$ is a perfect square.

Proof. Since $123456 = 2^6 \cdot 3 \cdot 643$, let $m = 3 \cdot 643 = 1929$, which is less than 123456. Then, $123456m = 238146624 = 15432^2$. Since $123456m$ can be written as n^2 where $n = 15432 \in \mathbb{Z}$, it is a perfect square, and the statement is true. \square

(f) $\exists k \in \mathbb{Z}$, $8 \nmid (4k^2 + 12k + 8)$.

Proof. Consider the negation, $\forall k \in \mathbb{Z}$, $8 \mid (4k^2 + 12k + 8)$. Notice that the open sentence is logically equivalent to $8 \mid (4k^2 + 12k)$. Let k be a natural number. Then, k is either even or odd.

Suppose that k is even and can be written as $k = 2n$. Then, $4k^2 = 16n^2 = 8(2n^2)$, so $8 \mid 4k^2$. Likewise, $12k = 24n = 8(3n)$, so $8 \mid 12k$. By DIC, $8 \mid (4k^2 + 12k)$.

Now, suppose that k is odd and can be written as $k = 2n + 1$. Then, $4k^2 + 12k = 4(4n^2 + 2n + 1) + 12(2n + 1) = 16n^2 + 40n + 16 = 8(2n^2 + 5n + 1)$, so $8 \mid (4k^2 + 12k)$.

Therefore, the negation is true, so the original statement is false. \square

RP05. Prove or disprove each of the following statements involving nested quantifiers.

- (a) For all $n \in \mathbb{Z}$, there exists an integer $k > 2$ such that $k \mid (n^3 - n)$.

Proof. Let n be an integer. If $n = 0$ or $n = \pm 1$, $n^3 - n = 0$ and all integers (including any k) divide zero.

If $n > 1$, we select $k = n + 1 > 2$. Factor: $n^3 - n = n(n - 1)(n + 1)$. Then, $n^3 - n = [n(n - 1)](n + 1)$, so $k \mid (n^3 - n)$.

If $n < 1$, first let $m = -n$ so $n^3 - n = (-m)^3 + m = -(m^3 - m)$. Now, select $k = m + 1 > 2$. Then, $n^3 - n = -m(m - 1)(m + 1)$, so $k \mid (n^3 - n)$.

Therefore, the statement is true. □