MATH 137 Fall 2020: Practice Assignment 1

Q01. Solve the following equations/inequalities:

1. $|x-2| = 5 \iff x-2 = \pm 5 \iff x = 2 \pm 5 \iff x = -3, 7$
2. $|x-4| = |3x+2|$: Take cases: $x-4 = 3x+2 \iff x = -1$ and $x-4 = -3x-2 \iff x = \frac{1}{2}$. Check: $|-1-4| \neq |-3+2|$ but $|\frac{1}{2}-4| = \frac{7}{2} = |\frac{3}{2}+2|$, so $x = \frac{1}{2}$.
3. $|x+1| > 1$ = "distance between x and -1 greater than 1" = $x \in \mathbb{R} \setminus [-2, 0] = (-\infty, -2) \cup (0, \infty)$
4. $|x+3| + |1-2x| \leq 5$. Take cases $x \in (-\infty, -3], [-3, \frac{1}{2}], [\frac{1}{2}, \infty)$:
   1. $-(x+3) + (1-2x) \leq 5 \iff x \geq -\frac{3}{7}$ but $x \in (-\infty, -3]$ so no solution
   2. $(x+3) + (1-2x) \leq 5 \iff x \geq -1$ but $x \in [-3, \frac{1}{2}]$ so $x \in [-1, \frac{1}{2}]$
   3. $(x+3) - (1-2x) \leq 5 \iff x \leq 1$ but $x \in [\frac{1}{2}, \infty)$ so $x \in [\frac{1}{2}, 1]$

   Therefore $x \in [-1, \frac{1}{2}] \cup [\frac{1}{2}, 1] = [-1, 1]$.
5. $|x-4||x+2| = |(x-4)(x+2)| = |x^2 - 2x - 8| > 7$. Take cases:
   1. $x^2 - 2x - 8 > 7 \iff x^2 - 2x - 15 > 0 \iff (x-5)(x+3) > 0 \iff x \in (-\infty, -3) \cup (5, \infty)$
   2. $x^2 - 2x - 8 < -7 \iff x^2 - 2x - 1 < 0 \iff (x-1)^2 < 0$ which is false for all real x.

   Therefore, $x \in (-\infty, -3) \cup (5-, \infty)$.

Q02. Prove that, for any real numbers a and b:

1. $|a| - |b| \leq |a+b|$

   Proof. Moving the $|b|$ term to the other side, $|a| \leq |a+b| + |b|$. Let $b' = -b$, so $|a| \leq |a + (-b')| + |-b'|$. Add some zeroes, $|a - 0| \leq |a - b'| + |b' - 0|$, which holds by the triangle inequality. $\square$

2. $|a| + |b| \leq |a+b|$

   Proof. Again move the $|b|$ term to the other side, $|a| \leq |a-b| + |b|$, and add some zeroes, $|a-0| \leq |a-b| + |b-0|$, which holds by the triangle inequality. $\square$

3. $|a-b| \leq |a+b|$ or $|a+b| \leq |a-b|$

   Proof. Consider for $a = b \neq 0$: the first is true as $0 \leq 2a$ and the second is false as $2a \not\leq 0$. Since the second is false for some case, it is false generally.

   For $a = -b \neq 0$: the first is now false as $2a \not\leq 0$.

   Therefore, both statements are false for some $x \in \mathbb{R}$, so they are not true. $\square$

Q03. Write the expression

$$\frac{1}{2}(a+b+|a-b|)$$

as a piecewise function. You should consider 2 cases. What is this expression calculating for any two real numbers a and b? What about the expression

$$\frac{1}{2}(a+b-|a-b|)?$$

Solution. Consider the cases:

$$\begin{aligned}\frac{1}{2}(a+b+|a-b|) &= \begin{cases}\frac{1}{2}(a+b+a-b) & a \geq b \\ \frac{1}{2}(a+b-a+b) & a < b\end{cases} \\ &= \begin{cases} a & a \geq b \\ b & a < b\end{cases}\end{aligned}$$

Which is just $\max\{a,b\}$.

Again, consider the cases:

$$\begin{aligned}\frac{1}{2}(a+b-|a-b|) &= \begin{cases}\frac{1}{2}(a+b-a+b) & a \geq b \\ \frac{1}{2}(a+b+a-b) & a < b\end{cases} \\ &= \begin{cases} b & a \geq b \\ a & a < b\end{cases}\end{aligned}$$

Which is just $\min\{a,b\}$. □