**MATH 137 Fall 2020: Practice Assignment 2**

**Q01.** Use the formal definition of limits to prove each statement below:

(a) $\lim_{n\to\infty} \frac{2n}{n+1} = 2$

*Proof.* Let $\epsilon > 0$. We have to find $N$ such that $n \geq N$ implies $\frac{2n}{n+1} \in (2-\epsilon, 2+\epsilon)$, or $\left|\frac{2n}{n+1} - 2\right| < \epsilon$. Simplifying:

$$\left|\frac{2n}{n+1} - 2\right| = \left|\frac{2n - 2(n+1)}{n+1}\right| = \left|\frac{2}{n+1}\right| = \frac{2}{n+1} < \epsilon$$

Now, take $N = \frac{2}{\epsilon}$. Then, $n \geq N \implies n \geq \frac{2}{\epsilon} \implies n+1 > \frac{2}{\epsilon} \implies \frac{2}{n+1} < \epsilon$ $\square$

(b) $\lim_{n\to\infty} \frac{6n - 3n^2 - 2}{(n-1)^2} = -3$

*Proof.* Let $\epsilon > 0$. We must find $N$ such that $n \geq N \implies \left|\frac{6n-3n^2-2}{(n-1)^2} - (-3)\right| < \epsilon$. Again, simplifying:

$$\left|\frac{-3n^2 + 6n - 2}{(n-1)^2} + 3\right| = \left|\frac{3n^2 - 6n + 2}{(n-1)^2} - 3\right| = \left|\frac{-1}{(n-1)^2}\right| = \frac{1}{(n-1)^2} < \epsilon$$

Let $N = \sqrt{\frac{1}{\epsilon}} + 2$. Then, $n \geq N$ implies that $n \geq \sqrt{\frac{1}{\epsilon}} + 2 \implies (n-1)^2 > \frac{1}{\epsilon} \implies \frac{1}{(n-1)^2} < \epsilon$ $\square$

(c) $\lim_{n\to\infty} 1 - 2^n = -\infty$

*Proof.* Let $M < 0$. We have to find $N$ such that $n \geq N$ implies $1 - 2^n < M$. Notice that since $2^n > 0$ for all $n$, this can be rewritten as $2^n > 1 - M$. Let $N = \log_2(1-M) + 1$. This is valid since $M$ is defined to be negative, so $1 - M$ is always positive. Now, $n \geq N \implies n > \log_2(1-M) \implies 2^n > 1 - M \implies 1 - 2^n < M$ $\square$

**Q02.** Determine if the following statements are true or false. If true, argue your case mathematically, if false, provide a counterexample.

(a) If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$, then $\lim_{n\to\infty}(a_n + b_n) = \infty$

*Proof.* Let $M > 0$. If $a_n \to \infty$, then there exists an $N_1$ such that $n \geq N_1$ implies $a_n > \frac{M}{2}$. Likewise, if $b_n \to \infty$, then there exists an $N_2$ such that $n \geq N_2$ implies $b_n > \frac{M}{2}$.

Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then $a_n + b_n > \frac{M}{2} + \frac{M}{2} = M$. Therefore, $a_n + b_n$ diverges to infinity. $\square$

(b) If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$, then $\lim_{n\to\infty}(a_n - b_n) = 0$

*Proof.* Let $a_n = n$ and $b_n = 2n$. Both diverge to positive infinity. However, $a_n - b_n = -n$, which diverges to negative infinity. Therefore, by counterexample, the statement is false. $\square$

(c) If $a_n \leq b_n \leq c_n$ for all $n$, $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = M$, then $\lim_{n\to\infty} b_n = K$ with $L \leq K \leq M$

*Proof.* Consider the similar proof presented below as Q05. Since $a_n \leq b_n \leq c_n \equiv a_n \leq b_n \wedge b_n \leq c_n$, we can apply that proof twice to show that $L \leq K$ and $K \leq M$, which is just $L \leq K \leq M$. $\square$

**Q03.** Consider the sequence

$$a_n = \begin{cases} 1 & \text{when } n \text{ is a perfect square} \\ \frac{1}{n} & \text{otherwise} \end{cases}$$

Use the definition of convergence to show this sequence does not have a limit of 0. Hint: consider building a contradiction.

*Proof.* Note that $a_n > 0$ for all $n$. If the statement is false and $a_n \to 0$, then, for any $\epsilon > 0$, we can find a $N > 0$ such that $n \geq N$ implies $a_n < \epsilon$. Alternatively stated, there is a tail of $\{a_n\}$ consisting only of numbers less than $\epsilon$. Select $\epsilon < 1$. Since $a_n = 1$ if $\sqrt{n} \in \mathbb{N}$, the corresponding tail given by $N$ must contain no square numbers. However, $N^2 \in \mathbb{N}$ and $N^2 > N$. Therefore, no such $N$ can exist, and the limit of $a_n$ cannot be zero. $\square$

**Q04.** Show that if a sequence $\{a_n\}$ converges to $L$ then there are infinitely many terms of the sequence that can be made arbitrarily close to one another. Specifically, show that eventually $|a_n - a_m|$ can be made arbitrarily small (i.e. for $n, m$ past a certain point) Note that $m$ and $n$ are not necessarily consecutive integers. Such sequences are called *Cauchy Sequences.*

*Proof.* For any $a_i$, by the definition of the limit, we can write it as $L + \epsilon_i$ for some $|\epsilon_i| > 0$. Now, rewrite $|a_n - a_m|$ as $|L + \epsilon_n - L - \epsilon_m| = |\epsilon_n - \epsilon_m|$. Since it is guaranteed by the definition of the limit that $\epsilon$ can be made arbitrarily small, the quantity $|\epsilon_n - \epsilon_m|$ can also be made arbitrarily small. $\square$

**Q05.** In this question we will prove the following:

If $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$ and $a_n \leq b_n$ for all $n$, then it must be the case that $L \leq M$

(a) Use the definition of limits to show that for all $\epsilon$, eventually

$$L - \epsilon < a_n \leq b_n < M + \epsilon$$

(the word "eventually" is meant to take the place of the statement "for all $n$ greater than some $N$")

*Proof.* Let $\epsilon > 0$. Then, by the definition of the limit, there is an $N_1$ such that for all $n \geq N_1$, $|a_n - L| < \epsilon \implies a_n > L - \epsilon$. Likewise, there is an $N_2$ such that for all $n \geq N_2$, $|b_n - M| < \epsilon \implies b_n < M + \epsilon$. Given that $a_n \leq b_n$ for all $n$, we can combine these inequalities by taking $n \geq \max\{N_1, N_2\} \implies L - \epsilon < a_n \leq b_n < M + \epsilon$ $\square$

(b) Since $\epsilon$ is arbitrary, the inequality above might help you "feel" that $L \leq M$. That is, we can make $\epsilon$ so small that "basically $L \leq M$". One way of showing this mathematically is to assume that $L > M$ and come up with a contradiction. That is, let $L = M + d$ for some positive number $d$. Use this to arrive at a contradiction and thus deduce $L \leq M$.

*Proof.* If $L > M$, we can let $L = M + d$ for some $d > 0$. Repeat the conclusion from part (a), $M + d - \epsilon < M + \epsilon$, and add $\epsilon$ to both sides: $M + d < M + 2\epsilon$. Since $d$ is defined, we can let $\epsilon < \frac{d}{2}$. Because $\frac{d}{2} > 0$, this is a valid choice of $\epsilon$. However, the inequality now reads $M + d < M + d$, which is clearly false. We can conclude that the inequality is false, so its negation $L \leq M$, is true. □

**Q06.** Prove (using the definition) that if $a_n > 0$ for all $n$ and $\lim\limits_{n\to\infty} a_n = L$, then $\lim\limits_{n\to\infty} \sqrt{a_n} = \sqrt{L}$. [Hint: Consider the cases $L = 0$ and $L \neq 0$ separately.]

*Proof.* Consider the case where $L = 0$. Let $\epsilon > 0$, which implies $\epsilon^2 > 0$. Given the known limit $\lim_{n\to\infty} a_n = 0$, we can find an $N$ such that $n \geq N$ implies $|a_n - 0| = a_n < \epsilon^2$. Taking the square roots of both sides, $\sqrt{a_n} < \epsilon$ as required.

Consider when $L \neq 0$. Because $\sqrt{L}$ exists, $L > 0$. Let $\epsilon > 0$. Given the known limit, we can find an $N$ such that $n \geq N$ implies $|a_n - L| < \epsilon$. Notice that we can use $a_n - L$ to create an expression for $\sqrt{a_n} - \sqrt{L}$:

$$\begin{aligned}
|a_n - L| &= \left|(\sqrt{a_n} - \sqrt{L})(\sqrt{a_n} + \sqrt{L})\right| \\
&= \left|\sqrt{a_n} - \sqrt{L}\right|\left|\sqrt{a_n} + \sqrt{L}\right| \\
\left|\sqrt{a_n} - \sqrt{L}\right| &= \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} < \frac{\epsilon}{\sqrt{a_n} + \sqrt{L}} < \epsilon
\end{aligned}$$

(since the denominator is a sum of two positive numbers) as required. □