# Appendix A

$$\frac{dT}{dt} = \kappa[M(t) - T(t)] + H(t) + Q(t)$$

1) The differential equation is a **linear, constant coefficient, nonhomogeneous** equation. If $Q(t) = tT$, the DE would be a **variable coefficient** equation.

2) For Picard's theorem to guarantee the existence of unique solutions, $M(t)$, $H(t)$ and $Q(t)$ must guarantee **continuity for the DE on the predetermined interval** (existence) and $f_T(t, T)$ **must be continuous on the interval** (uniqueness). This can be interpreted as the house having a **unique flow of temperature as time goes on.**

3)

$$\frac{dT}{dt} = \kappa[M(t) - T(t)] + H(t) + Q(t)$$

$$\frac{dT}{dt} - \kappa T(t) = \kappa M(t) + H(t) + Q(t)$$

**Solve for** $\mu(t)$**:**

We have $\quad p(t) = \kappa$

$$\int p(t) = \kappa t$$

Therefore $\quad \mu(t) = e^{\kappa t}$

**Multiply both sides by Integrating Factor**:

$$e^{\kappa t}[\frac{dT}{dt} - \kappa T(t)] = e^{\kappa t}[\kappa M(t) + H(t) + Q(t)]$$

$$\frac{d}{dt}[e^{\kappa t}T] = e^{\kappa t}[\kappa M(t) + H(t) + Q(t)]$$

$$e^{\kappa t}T = \int e^{\kappa t}[\kappa M(t) + H(t) + Q(t)]dt + C$$

$$T = \frac{\int e^{\kappa t}[\kappa M(t)+H(t)+Q(t)]dt + C}{e^{\kappa t}}$$

4)

a) "No people in it and the lights and machinery are off," so $H(t) = 0$
"No furnaces or air conditioners are running," so $Q(t) = 0$
"The outside temperature is constant with the value M," so $M(t) = M$

Our new DE is $\frac{dT}{dt} = \kappa[M - T(t)]$

b) The DE has equilibrium solutions if $\frac{dT}{dt} = 0$
$\kappa[M - T(t)] = 0$ if $T(t) = M$

c) If $T < M, \frac{dT}{dt} > 0$
If $T > M, \frac{dT}{dt} < 0$
$T$ goes towards $M$ as $t \to \infty$, so $T(t) = M$ is a stable equilibrium solution. This implies that **the temperature within the building would eventually be the same as that of the ambient temperature.**

**Using solution from part 3:**

$$T = \frac{\int e^{\kappa t}[\kappa M(t) + H(t) + Q(t)]dt + C}{e^{\kappa t}}$$

Apply the conditions from part 4a:

$$T = \frac{\kappa M \int e^{\kappa t} dt + C}{e^{\kappa t}}$$

$$T = M + Ce^{-\kappa t}$$

**Apply initial condition** $T(t_0) = T_0$

$$T_0 = M + Ce^{-\kappa t_0}$$

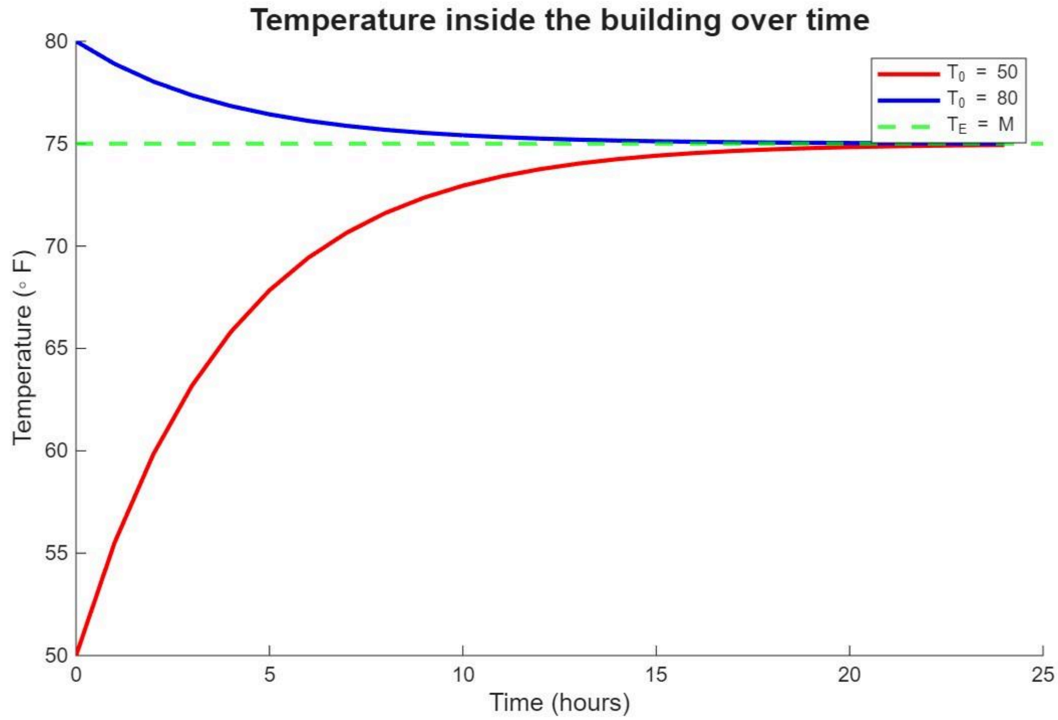
$$T_0 - M = Ce^{-\kappa t_0}$$

$$C = (T_0 - M)e^{\kappa t_0}$$

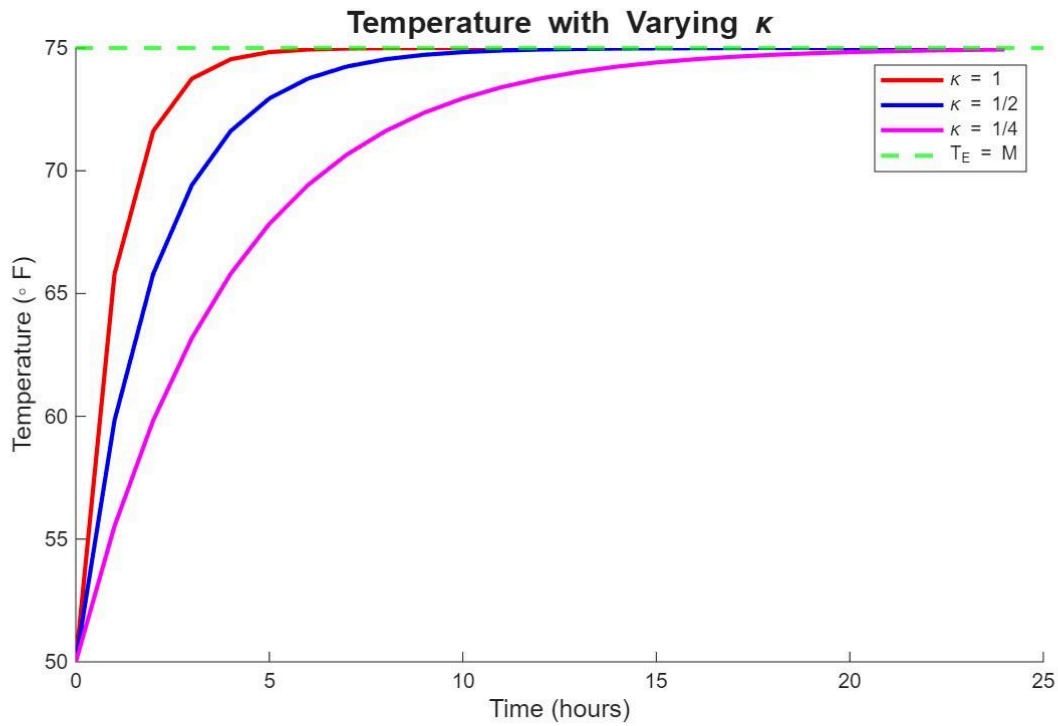
**General solution:**

$$T(t) = M + (T_0 - M)e^{\kappa t_0}e^{-\kappa t}$$

d)




These solutions confirm part c)'s answer: $T(t) \quad = \quad M$ is a stable equilibrium solution e)




From the graph, we can see that **while changing $\kappa$ does not affect the equilibrium solution, the rate of change of temperature is changed: a bigger $\kappa$ implies a greater rate of change, and vice versa.** The quality of the building's insulation could change $\kappa$, with quality insulation resulting in a slower heat loss (smaller $\kappa$), and poor insulation resulting in a sharper decrease in temperature (larger $\kappa$).

f)

$T(t) = M + (T_0 - M)e^{\kappa t_0}e^{-\kappa t}$

"The difference between the building's temperature and the outside temperature is $e^{-1}$ of the initial difference."

$T(t) - M_0 = e^{-1}(T_0 - M_0)$

Using the general solution:

$$(T_0 - M_0)e^{-1} = (T_0 - M)e^{\kappa t_0}e^{-\kappa t}$$

$$e^{-1} = e^{\kappa t_0}e^{-\kappa t}$$

$$e^{-1} = e^{\kappa(t_0 - t)}$$

$$-1 = \kappa(t_0 - t)$$

$$-1 = -\kappa(t - t_0)$$

$$-1 = -\kappa\Delta t$$

$$\Delta t = \frac{1}{\kappa}$$

Since the time constant resembles the change in time, **the units should be hours**

In part 4f), we saw that a smaller $\kappa$ implies smaller heat loss. Since $\Delta t$ is **inversely proportional** to $\kappa$, it can be inferred that **a larger** $\Delta t$ **would slow the building's response to outside temperature; therefore, we want a larger time constant**