Set Theory

Discrete Mathematic

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Set Theory

Set primitives
- Notation
- Ordered pairs and Cartesian product
- Relations and Functions

Venn diagrams and set operations

Subsets
- Set equality
- Empty set and Powersets
- Equality laws for sets
- Set properties

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Predefined sets

The sets of ...

- $\mathbb{R}$ all real numbers
- $\mathbb{R}^+$ all positive (not 0) real numbers
- $\mathbb{R}^-$ all negative (not 0) real numbers
- $\mathbb{R}^{nonneg}$ all nonnegative (positive or 0) real numbers
- $\mathbb{Z}$ all integers
- $\mathbb{N}$ or $\mathbb{Z}^+$ all positive integers
- $\mathbb{N}_0$ or $\mathbb{Z}^{nonneg}$ all nonnegative integers
- $\mathbb{Q}$ all rational numbers (quotients)

Set notation

Set-roster notation

$$\begin{aligned} A &= \{1, 2, 3\} \\ &= \{3, 1, 2\} \\ &= \{1, 2, 2, 3, 3, 3\} \end{aligned}$$

$$B = \{10, 11, 12, \ldots, 119\}$$

Set-builder notation

$$M = \{x \in S | P(x)\}$$

$$A = \{x \in \mathbb{Z} | -2 < x < 7\}$$
$$B = \{x \in \mathbb{N} | x^2 < 10\}$$
$$C = \{x \in \mathbb{R}^+ | log(x) < 10\}$$

What are the members of:

$$\{x \in \mathbb{N} | x^2 + 20 < 100\}$$

Ordered pairs

defined as sets

The ordered pair

$$(a, b) \neq (b, a) \neq (a, b, c)$$

Ordered pairs

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The ordered pair

$$(a, b) \neq (b, a) \neq (a, b, c)$$

can be defined as

$$\{\{a\}, \{a, b\}\}$$

Ordered pairs

defined as sets

The ordered pair

$$(a, b) \neq (b, a) \neq (a, b, c)$$

can be defined as

$$\{\{a\}, \{a, b\}\}$$

which is not the same as

$$\{\{b\}, \{b, a\}\}$$

Equality of ordered pairs

The pairs

$$(a, b)$$

and

$$(c, d)$$

are equal if, and only if

$$a = c \land b = d$$

Cartesian product of A and B is $A \times B$ read "A cross B"

$$A \times B = \{(a, b) | a \in A \land b \in B\}$$

Cartesian product of A and B is $A \times B$ read "A cross B"

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Example:

$$\{1, 2, 3\} \times \{3.4, 7.2\} =$$

$$\{(1, 3.4), (1, 7.2), (2, 3.4), (2, 7.2), (3, 3.4), (3, 7.2)\}$$

Cartesian product of A and B and C is $A \times B \times C$

$$(A \times B) \times C = \{((a, b), c) | (a, b) \in A \times B \wedge c \in C\}$$

$$(A \times B) \times C = \{((a, b), c) | (a \in A \wedge b \in B) \wedge c \in C\}$$

$$A \times B \times C = \{((a, b), c) | a \in A \wedge b \in B \wedge c \in C\}$$

Short hand for the pair-pair $((a, b), c)$ is called a tupple:

$$(a, b, c)$$

A relation R between two sets A and B can be defined as a subset of the cartesian product.

$$R \subseteq A \times B$$

- A is the **domain** of R
- B is the **co-domain** of R

$$x \; R \; y$$

tells that

$$(x, y) \in R$$

Relation example

Given a relation $Persons$, and the sets of IDs, $FirstNames$, $LastNames$

$$Person \subseteq IDs \times FirstNames \times LastNames$$

Where the subset is legal combinations of the three, normally defined by the tupples in a relational database.

Functions

- The function F from set A to set B
- is a relation with
  - domain A
  - and co-domain B
- $\forall x \in A, \exists y \in B, (x, y) \in F$
- $\forall x \in A \wedge y \in B \wedge z \in B, (x, y) \in F \wedge (x, z) \in F \rightarrow y = z$

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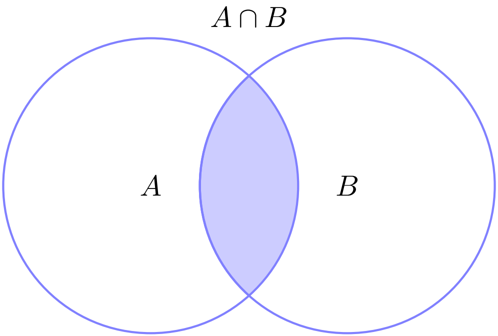
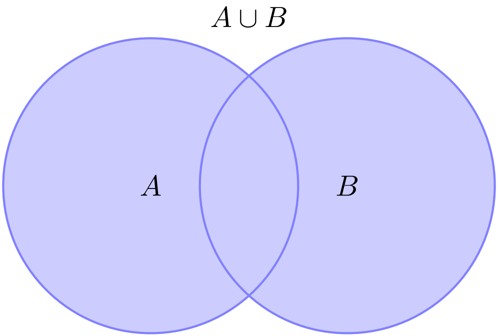
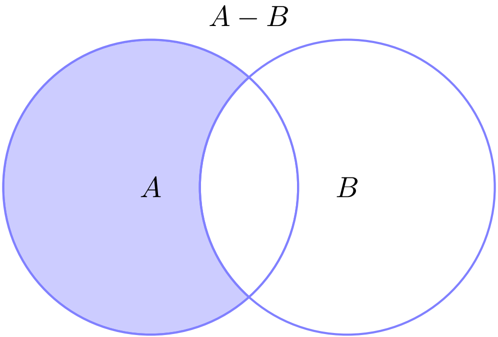
Intersection

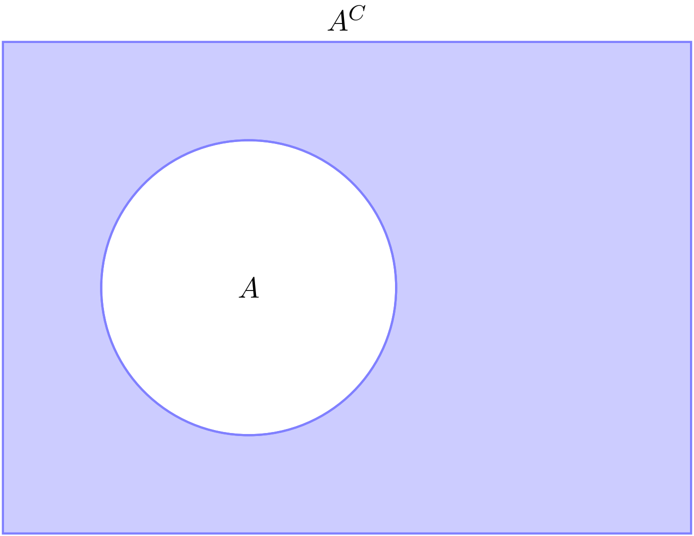


Union

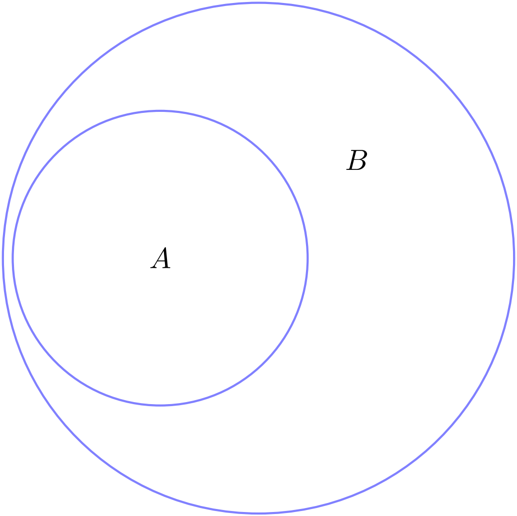



$A - B$
A
B

Complement



$A \subseteq B$



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$$A \subseteq B$$

$$A \subseteq B$$

$$\Updownarrow$$

$$\forall x, x \in A, x \in B$$

$$A \nsubseteq B$$

$$\Updownarrow$$

What should go here?

$$A \nsubseteq B$$

$$\Updownarrow$$

$$\exists x, x \in A, x \notin B$$

Proof outline

- suppose x is a particular, but arbitrarily chosen element of A
- prove that x is an element of B

Proper subset

$$A \subset B$$

Proper subset

$$A \subset B$$

$$\Updownarrow$$

$$A \subseteq B \land \exists x \in B, x \notin A$$

Definition of set equality

Definition of set equality

$$A = B$$

$$\Updownarrow$$

$$A \subseteq B \wedge B \subseteq A$$

Set equality

Proof outline

- prove that $A \subseteq B$
- prove that $B \subseteq A$

Create a program that

- Takes two sets A and B represented as **arrays**
- Evaluates wheter the sets are:
  - $A \subset B$ = -1
  - $A = B$ = 0
  - $A \supset B$ = 1
  - otherwise -2

```
fun compareSets(a: IntArray, b: IntArray): Int {
  if (a.size == 0) return if (b.size == 0) 0 else -1
  if (b.size == 0) return 1
  var result = 0
  var ia = 0
  var ib = 0
  while (ia < a.size && ib < b.size) {
    if (a[ia] == b[ib]) { ia++; ib++ }
    else if (a[ia] > b[ib]) {
      if (result == 1) return -2
      ib++; result = -1 // a is subset
      }
    else if (a[ia] < b[ib]) {
      if (result == -1) return -2
      ia++; result = 1 // a is superset
      }
    }
  if (ia == a.size && ib == b.size) return result;
  return -2 // sets
  }
```

The empty set

$$\emptyset = \{\}$$

There is only one empty set

The set with no elements

Disjoint sets

Partitions of sets

Sets A and B are disjoint if

$$A \cap B = \emptyset$$

Partions $\{A_1, A_2, A_3, \ldots, A_n\}$ of set A

- $A = \bigcup_{i=1}^{n} A_i$
- $\forall a, b \in \{1, 2, 3, \ldots, n\}, A_a \cap A_b = \emptyset$

$$\mathscr{P}(A)$$

The set of all subsets of A

$$\mathscr{P}(\{a,b,c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$$

De Morgan for sets

$$(A \cup B)^C = A^C \cap B^C$$

De Morgan for sets

$$(A \cup B)^C = A^C \cap B^C$$

$$(A \cap B)^C = A^C \cup B^C$$

Distributive law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Intersection and union with subsets

$$A \subseteq B \rightarrow A \cap B = A$$

Intersection and union with subsets

$$A \subseteq B \to A \cap B = A$$

$$A \subseteq B \to A \cup B = B$$

Set properties

Inclusion of intersection:

$$A \cap B \subseteq A$$

Inclusion in union:

$$A \subseteq A \cup B$$

Transitivity of subsets

$$A \subseteq B \land B \subseteq C \rightarrow A \subseteq C$$

Procedural definitions

of set operations

Intersection:

$$x \in X \cap Y \iff x \in X \land x \in Y$$

Union:

$$x \in X \cup Y \iff x \in X \lor x \in Y$$

Difference:

$$x \in X - Y \iff x \in X \land x \notin Y$$

Complement:

$$x \in X^C \iff x \notin X$$

Ordered Pair:

$$(x, y) \in X \times Y \iff x \in X \land y \in Y$$