Predicates and Quantified statements

Discrete Mathematic

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Predicates and Quantified Statements

Outline

Set notations

Predicates

Quantifiers

Programming in logic

Set notations

Predicates

Quantifiers

Programming in logic

Predefined sets

The sets of ...

- $\mathbb{R}$ all real numbers
- $\mathbb{R}^+$ all positive (not 0) real numbers
- $\mathbb{R}^-$ all negative (not 0) real numbers
- $\mathbb{R}^{nonneg}$ all nonnegative (positive or 0) real numbers
- $\mathbb{Z}$ all integers
- $\mathbb{N}$ or $\mathbb{Z}^+$ all positive integers (natural numbers)
- $\mathbb{N}_0$ or $\mathbb{Z}^{nonneg}$ all nonnegative integers
- $\mathbb{Q}$ all rational numbers (quotients)

Set notation

Set-roster notation

$$\begin{aligned} A &= \{1, 2, 3\} \\ &= \{3, 1, 2\} \\ &= \{1, 2, 2, 3, 3, 3\} \end{aligned}$$

$$B = \{10, 11, 12, \ldots, 119\}$$

Set notation

Set-builder notation

$$M = \{x \in S | P(x)\}$$

$$A = \{x \in \mathbb{Z} | -2 < x < 7\}$$
$$B = \{x \in \mathbb{N} | x^2 < 10\}$$
$$C = \{x \in \mathbb{R}^+ | log(x) < 10\}$$

What are the members of:

$$\{x \in \mathbb{N} | x^2 < 10\}$$

- A **statement** is a sentence that is either true or false, not both
- A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for all variables.
- The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable

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Let $P(x)$ be the predicate "$x^2 > x$". The domain of x is the real numbers: $x \in \mathbb{R}$

$$
\begin{aligned}
P(2) : 2^2 > 2 &\qquad \equiv 4 > 2 \equiv \text{true} \\
P\left(\frac{1}{2}\right) : \left(\frac{1}{2}\right)^2 > \frac{1}{2} &\qquad \equiv \frac{1}{4} > \frac{1}{2} \equiv \text{false} \\
P\left(-\frac{1}{2}\right) : \left(-\frac{1}{2}\right)^2 > -\frac{1}{2} &\qquad \equiv \frac{1}{4} > -\frac{1}{2} \equiv \text{true}
\end{aligned}
$$

Predicate truth set

- If $P(x)$ is a predicate
- And the domain of x is D
- Then the truth set of $P(x)$ is the set of all elements $e \in D$ where $P(e)$ is true.

$$\{x \in D | P(x)\}$$

Predicate truth set - example

Let $Q(n)$ be the predicate "n is a factor of 8". What is the truth set of $Q(n)$ when:

1. the domain of n is $\mathbb{N}$
2. the domain of n is $\mathbb{Z}$

Solutions:

1.
2.

Predicate truth set - example

Let $Q(n)$ be the predicate "n is a factor of 8". What is the truth set of $Q(n)$ when:

1. the domain of n is $\mathbb{N}$ (all natural numbers, positive integers)
2. the domain of n is $\mathbb{Z}$ (all integers)

Solutions:

1.
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Predicate truth set - example

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1. the domain of n is $\mathbb{N}$ (all natural numbers, positive integers)
2. the domain of n is $\mathbb{Z}$ (all integers)

Solutions:

1. $\{1, 2, 4, 8\}$
2.

Predicate truth set - example

Let $Q(n)$ be the predicate "n is a factor of 8". What is the truth set of $Q(n)$ when:

1. the domain of n is $\mathbb{N}$ (all natural numbers, positive integers)
2. the domain of n is $\mathbb{Z}$ (all integers)

Solutions:

1. $\{1, 2, 4, 8\}$
2. $\{-8, -4, -2, -1, 1, 2, 4, 8\}$

Exercise 2 - Truth sets

Let $P(x)$ be the predicate "$x^2 > x$", $x \in \mathbb{R}$

What is the truth set of $P(x)$?

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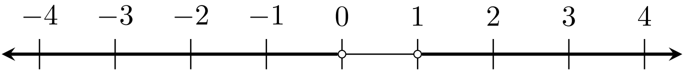
What is the truth set of $P(x)$?



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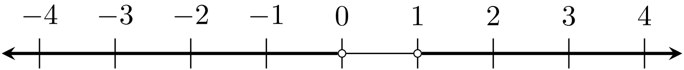

$$\{x \in \mathbb{R} \mid x < 0 \vee 1 < x\}$$

Set notations

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Programming in logic

$$\forall$$

All men are mortal.
Socrates is a man.
$\therefore$ Socrates i mortal

$\forall$ human beings x, x is mortal

$\forall x \in H, x$ is mortal

When Universal statements are true

With domains $D = \{1, 2, 3, 4, 5\}$ and $\mathbb{R}$ is all real numbers.

- "$\forall x \in D,\ x^2 \geq x$" is true
- "$\forall x \in \mathbb{R},\ x^2 \geq x$" is false

Solution:

- predicate must be proven **true for all** (and every) element of x

$$1 \geq 1,\ 4 \geq 2,\ 9 \geq 3,\ 16 \geq 4,\ 25 \geq 5$$

- predicate only has to be proven **false for a single** element of x

$$x = \frac{1}{3} : \left(\frac{1}{3}\right)^2 = \frac{1}{9} \ngeq \frac{1}{3}$$

$$\exists$$

There is a student in Discrete Math

$\exists$ a person p such that p is a Discrete Math student

$\exists p \in P,$ such that p is a Discrete Math student

When Existential statements are true

With domains $E = \{5, 6, 7, 8\}$ and $\mathbb{N}$ is all positive integers.

- "$\exists m \in \mathbb{N},\ m^2 = m$" is true
- "$\exists m \in E,\ m^2 = m$" is false

Solution:

- predicate must be proven **true for a single** element of m

$$1^2 = 1$$

- predicate only has to be proven **false all** elements of x

$$25 \neq 5,\ 36 \neq 6,\ 49 \neq 7,\ 64 \neq 8$$

Universal conditional statement

$$\forall x, \quad \text{if } P(x) \text{ then } Q(x)$$

$$\forall x,\ P(x) \to Q(x)$$

$$P(x) \implies Q(x)$$

Exercise 5 - Mapping quantifiers

Given the predicates $P(x)$ and $Q(x)$ and the domains $D = \{1, 2, 4, 25\}$ and $E = \{10, 20, 30, 35\}$ rewrite the following statements only using the operators "$\neg$", "$\wedge$", and "$\vee$"

- $\forall x \in D,\ P(x)$
- $\exists x \in E,\ Q(x)$

Exercise 5 - Mapping quantifiers

Given the predicates $P(x)$ and $Q(x)$ and the domains $D = \{1, 2, 4, 25\}$ and $E = \{10, 20, 30, 35\}$ rewrite the following statements only using the operators "$\neg$", "$\wedge$", and "$\vee$"

- $\forall x \in D,\ P(x)$: $P(1) \wedge P(2) \wedge P(4) \wedge P(25)$
- $\exists x \in E,\ Q(x)$

Exercise 5 - Mapping quantifiers

Given the predicates $P(x)$ and $Q(x)$ and the domains $D = \{1, 2, 4, 25\}$ and $E = \{10, 20, 30, 35\}$ rewrite the following statements only using the operators "$\neg$", "$\wedge$", and "$\vee$"

- $\forall x \in D,\ P(x)$: $P(1) \wedge P(2) \wedge P(4) \wedge P(25)$
- $\exists x \in E,\ Q(x)$: $Q(10) \vee Q(20) \vee Q(30) \vee Q(35)$

De Morgan for multiple operands

$$\neg(a \vee b \vee c)$$

$$\neg(a \vee b) \wedge \neg c$$

$$(\neg a \wedge \neg b) \wedge \neg c$$

$$\neg a \wedge \neg b \wedge \neg c$$

$$\neg(\forall x \in D,\ P(x))$$

$$\exists x \in D,\ \neg P(x)$$

Multi-quantified statements

$$\exists x \in D, \forall y \in E, P(x, y)$$

- There is a smallest positive integer!
- $\exists$ a positive integer m such that $\forall$ positive integers n, $m \leq n$
- $\exists m \in \mathbb{N}, \forall n \in \mathbb{N}, m \leq n$

$$\forall x \in D, \exists y \in E, P(x, y)$$

- There is no smallest positive real number!
- $\forall$ positive real numbers x, $\exists$ a positive real number y, such that $y < x$
- $\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+, y < x$

Remember "De Morgan" for qualified statements:

$$\neg(\forall x \in D,\ P(x)) \quad \equiv \quad \exists x \in D,\ \neg P(x)$$

$$\neg(\exists x \in D,\ P(x)) \quad \equiv \quad \forall x \in D,\ \neg P(x)$$

Use them in

$$\neg(\forall x \in D, \exists y \in E, P(x,y))$$

$$\exists x \in D, \neg(\exists y \in E, P(x,y))$$

$$\exists x \in D, \forall y \in E, \neg P(x,y)$$

Universal Modus Ponens

$$\begin{aligned} &\forall x, P(x) \to Q(x) \\ &P(a) \\ \therefore\ &Q(a) \end{aligned}$$

Universal Modus Tollens

$$\begin{aligned} &\forall x, P(x) \to Q(x) \\ &\neg Q(a) \\ \therefore\ &\neg P(a) \end{aligned}$$

Quantified converse error

$$\forall x, P(x) \rightarrow Q(x)$$
$$Q(a)$$
$$\therefore P(a) \quad \leftarrow \textbf{is not a valid conclusion}$$

Quantified inverse error

$$\forall x, P(x) \rightarrow Q(x)$$
$$\neg P(a)$$
$$\therefore \neg Q(a) \quad \leftarrow \textbf{is not a valid conclusion}$$

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Programming in logic

Prolog - PROgramming in LOgic

SWI Prolog

http://www.swi-prolog.org

On Mac: `brew install swi-prolog`

Prolog building blocks (clauses)

Fact	(only left side) is a **statement**
Rule	(both sides) is a **predicate**
Question	(right side)
Atom	numbers and lowercase
Variables	Upper case names