Sequences, Induction, and Recursion

Discrete Mathematics

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A **sequence** is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer

$$1 \leq n \leq 8 : n^2 = 1, 4, 9, \ldots, 64$$

$$1 \leq n : (-1)^n = -1, 1, -1, \ldots$$

$$1 \leq n : \frac{(-1)^{n+1}}{n^2} = 1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{25}, \ldots$$

Summations

$$\sum_{k=m}^{n} a_k$$

$$\sum_{k=m}^{n} a_k$$

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

Adding and removing a final term

$$\sum_{k=0}^{n} 2^k + 2^{n+1} = \sum_{k=0}^{n+1} 2^k$$

$$\sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=1}^{n-1} \frac{1}{k^2} + \frac{1}{n^2}$$

Telescopic sum

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}$$

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1) - k}{k(k+1)} = \frac{1}{k(k+1)}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

Telescopic sum

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(\frac{1}{1} - \frac{1}{1+1}\right) + \left(\frac{1}{2} - \frac{1}{2+1}\right) + \left(\frac{1}{3} - \frac{1}{3+1}\right) + \dots$$

$$\left(\frac{1}{n-1} - \frac{1}{n-1+1}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

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$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots$$

$$+ \left(\frac{1}{n-1} - \frac{1}{n} \right) + \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots$$

$$+ \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1}$$

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \frac{1}{1} + \left(-\frac{1}{2} + \frac{1}{2} \right) + \left(-\frac{1}{3} + \frac{1}{3} \right) + \left(-\frac{1}{4} + \dots \right.$$

$$\left. + \frac{1}{n-1} \right) + \left(-\frac{1}{n} + \frac{1}{n} \right) - \frac{1}{n+1}$$

Telescopic sum

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \frac{1}{1} - \frac{1}{n+1}$$

Products

$$\prod_{k=m}^{n} a_k$$

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$$\prod_{k=m}^{n} a_k$$

$$a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n$$

Adding and removing a final term

$$\prod_{k=0}^{n} 2^k \cdot 2^{n+1} = \prod_{k=0}^{n+1} 2^k$$

$$\prod_{k=1}^{n} k = \prod_{k=1}^{n-1} k \cdot n$$

Theorems

$$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$$

$$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$$

$$\prod_{k=m}^{n} a_k \cdot \prod_{k=m}^{n} b_k = \prod_{k=m}^{n} (a_k \cdot b_k)$$

$$k, m, n \in \mathbb{N}, m \leq n, a_k, b_k, c \in \mathbb{R}$$

Variables

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{j=1}^{7} \frac{1}{j} = \sum_{k=1}^{7} \frac{1}{k}$$

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$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{j=1}^{7} \frac{1}{j} = \sum_{k=1}^{7} \frac{1}{k}$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$$

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1 \end{cases}$$

In how many ways can one choose r elements from a total of n elements?

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Why is it that

$$n! \neq \prod_{k=1}^{n} k$$

Let $P(n)$ be a property (predicate) of integers.

- $a \in \mathbb{Z}$, $k \in \mathbb{Z}$
- If $P(a)$
- and $\forall k \geq a, P(k) \rightarrow P(k+1)$
- then $\forall n \geq a, P(n)$

Write a program that calculates the sum of all positive even integer numbers equal to or below a given input n

$$\sum_{k=1}^{n \div 2} 2 \cdot k$$

The $\div$ operator is integer division.
Try with $n = 100$, $n = 1.000$, and $n = 1.000.000$

Sum of first n integers

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$P(n) = \frac{n(n+1)}{2}$$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$P(n) = \frac{n(n+1)}{2}$$

$$P(1) = \frac{1(1+1)}{2} = 1$$

Inductive hypothesis

$$P(k) = \frac{k(k+1)}{2}$$

Inductive consequense

$$P(k+1) = P(k) + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

Inductive consequense

$$P(k+1) = P(k) + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

$$\frac{k(k+1)}{2} + \frac{2k+2}{2} = \frac{(k+1)(k+2)}{2}$$

Inductive consequense

$$P(k+1) = P(k) + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

$$\frac{k(k+1)}{2} + \frac{2k+2}{2} = \frac{(k+1)(k+2)}{2}$$

$$\frac{k(k+1)+2k+2}{2} = \frac{k^2+3k+2}{2}$$

Inductive consequense

$$\frac{k(k+1)}{2} + \frac{2k+2}{2} = \frac{(k+1)(k+2)}{2}$$

$$\frac{k(k+1)+2k+2}{2} = \frac{k^2+3k+2}{2}$$

$$\frac{k^2+k+2k+2}{2} = \frac{k^2+3k+2}{2}$$

Inductive consequense

$$\frac{k(k+1)+2k+2}{2} = \frac{k^2+3k+2}{2}$$

$$\frac{k^2+k+2k+2}{2} = \frac{k^2+3k+2}{2}$$

$$\frac{k^2+3k+2}{2} = \frac{k^2+3k+2}{2}$$

$$\sum_{k=1}^{100} = 1+2+3+\cdots+50+51+\cdots+98+99+100$$

$$= (1+100)+(2+99)+(3+98)+\cdots+(50+51)$$

$$= 101+101+101+\cdots+101 = 50\cdot 101 = 5050$$

$$\frac{100\cdot 101}{2} = 5050$$

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$$\sum_{k=1}^{99} = 1+2+3+\cdots+49+50+51+\cdots+98+99$$

$$= (1+99)+(2+98)+(3+97)+\cdots+(49+51)+50$$

$$= 100+100+100+\cdots+100+50 = 49 \cdot 100+50 = 4950$$

$$\frac{99 \cdot 100}{2} = 4950$$

Assignment revisited

$$\sum_{k=1}^{n \div 2} 2 \cdot k$$

$$= 2 \sum_{k=1}^{n \div 2} k = 2 \frac{(n \div 2)(n \div 2 + 1)}{2}$$

$$= (n \div 2)(n \div 2 + 1)$$

$$2.550, 250.500, 250.000.500.000$$

Sum of geometric sequence

$$r^0 + r^1 + r^2 + \cdots + r^n = \sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

$$P(n) = \frac{r^{n+1} - 1}{r - 1}$$

$$P(0) = \frac{r^{0+1} - 1}{r - 1} = \frac{r^1 - 1}{r - 1} = \frac{r - 1}{r - 1} = 1$$

$$r \in \mathbb{R}, r \neq 0, n \in \mathbb{N}_0$$

Inductive hypothesis

$$P(k) = \frac{r^{k+1} - 1}{r - 1}$$

Inductive consequense

$$P(k+1) = P(k) + r^{k+1} = \frac{r^{k+1+1} - 1}{r - 1}$$

Inductive consequense

$$P(k+1) = P(k) + r^{k+1} = \frac{r^{k+1+1} - 1}{r - 1}$$

$$\frac{r^{k+1} - 1}{r - 1} + r^{k+1} = \frac{r^{k+2} - 1}{r - 1}$$

Inductive consequense

$$P(k+1) = P(k) + r^{k+1} = \frac{r^{k+1+1} - 1}{r - 1}$$

$$\frac{r^{k+1} - 1}{r - 1} + r^{k+1} = \frac{r^{k+2} - 1}{r - 1}$$

$$\frac{r^{k+1} - 1}{r - 1} + \frac{r^{k+1}(r - 1)}{r - 1} = \frac{r^{k+2} - 1}{r - 1}$$

$$\frac{r^{k+1}-1}{r-1}+r^{k+1}=\frac{r^{k+2}-1}{r-1}$$

$$\frac{r^{k+1}-1}{r-1}+\frac{r^{k+1}(r-1)}{r-1}=\frac{r^{k+2}-1}{r-1}$$

$$\frac{(r^{k+1}-1)+r^{k+1}(r-1)}{r-1}=\frac{r^{k+2}-1}{r-1}$$

Inductive consequense

$$\frac{r^{k+1}-1}{r-1}+\frac{r^{k+1}(r-1)}{r-1}=\frac{r^{k+2}-1}{r-1}$$

$$\frac{(r^{k+1}-1)+r^{k+1}(r-1)}{r-1}=\frac{r^{k+2}-1}{r-1}$$

$$\frac{r^{k+1}-1+r^{k+1}r-r^{k+1}}{r-1}=\frac{r^{k+2}-1}{r-1}$$

Inductive consequense

$$\frac{(r^{k+1}-1)+r^{k+1}(r-1)}{r-1}=\frac{r^{k+2}-1}{r-1}$$

$$\frac{r^{k+1}-1+r^{k+1}r-r^{k+1}}{r-1}=\frac{r^{k+2}-1}{r-1}$$

$$\frac{r^{k+1}-1+r^{k+2}-r^{k+1}}{r-1}=\frac{r^{k+2}-1}{r-1}$$

Inductive consequense

$$\frac{r^{k+1} - 1 + r^{k+1}r - r^{k+1}}{r-1} = \frac{r^{k+2} - 1}{r-1}$$

$$\frac{r^{k+1} - 1 + r^{k+2} - r^{k+1}}{r-1} = \frac{r^{k+2} - 1}{r-1}$$

$$\frac{r^{k+2} - 1}{r-1} = \frac{r^{k+2} - 1}{r-1}$$

What is the sum of:

$$\sum_{k=3}^{n} (4r^k - 6k)$$

$$\sum_{k=3}^{n}(4r^k - 6k) = 4\sum_{k=3}^{n} r^k - 6\sum_{k=3}^{n} k$$

$$\sum_{k=3}^{n}(4r^k - 6k) = 4\sum_{k=3}^{n} r^k - 6\sum_{k=3}^{n} k$$

$$= 4\left(\sum_{k=0}^{n} r^k - \sum_{k=0}^{2} r^k\right) - 6\left(\sum_{k=1}^{n} k - \sum_{k=1}^{2} k\right)$$

$$\sum_{k=3}^{n}(4r^k - 6k) = 4\sum_{k=3}^{n} r^k - 6\sum_{k=3}^{n} k$$

$$= 4\left(\sum_{k=0}^{n} r^k - \sum_{k=0}^{2} r^k\right) - 6\left(\sum_{k=1}^{n} k - \sum_{k=1}^{2} k\right)$$

$$= 4\left(\frac{r^{n+1}-1}{r-1} - \frac{r^3-1}{r-1}\right) - 6\left(\frac{n(n+1)}{2} - \frac{2\cdot 3}{2}\right)$$

$$\sum_{k=3}^{n}(4r^k - 6k) = 4\sum_{k=3}^{n} r^k - 6\sum_{k=3}^{n} k$$

$$= 4\left(\sum_{k=0}^{n} r^k - \sum_{k=0}^{2} r^k\right) - 6\left(\sum_{k=1}^{n} k - \sum_{k=1}^{2} k\right)$$

$$= 4\left(\frac{r^{n+1}-1}{r-1} - \frac{r^3-1}{r-1}\right) - 6\left(\frac{n(n+1)}{2} - \frac{2\cdot 3}{2}\right)$$

$$= 4\left(\frac{r^{n+1}-r^3}{r-1}\right) - 3(n(n+1)-6)$$

Divisibility property

$$P(n) : (2^{2n} - 1) \mod 3 = 0$$

$$P(0) : (2^{2 \cdot 0} - 1) \mod 3 = 2^0 - 1 = 1 - 1 = 0$$

$$n \in \mathbb{N}_0$$

Inductive hypothesis

$$P(k) : (2^{2k} - 1) \mod 3 = 0$$

$$P(k+1) : (2^{2(k+1)} - 1) \mod 3 = 0$$

Inductive consequense

$$P(k+1) : (2^{2(k+1)} - 1) \mod 3 = 0$$
$$(2^{(2k+2)} - 1) \mod 3 = 0$$

$$P(k+1) : (2^{2(k+1)} - 1) \mod 3 = 0$$

$$(2^{(2k+2)} - 1) \mod 3 = 0$$

$$(2^{2k} \cdot 2^2 - 1) \mod 3 = 0$$

$$(2^{(2k+2)} - 1) \mod 3 = 0$$
$$(2^{2k} \cdot 2^2 - 1) \mod 3 = 0$$
$$(2^{2k} \cdot 4 - 1) \mod 3 = 0$$

$$(2^{2k} \cdot 2^2 - 1) \mod 3 = 0$$
$$(2^{2k} \cdot 4 - 1) \mod 3 = 0$$
$$(2^{2k} \cdot 4 - 1 - (2^{2k} - 1)) \mod 3 = 0$$

Inductive consequense

$$(2^{2k} \cdot 4 - 1) \mod 3 = 0$$

$$(2^{2k} \cdot 4 - 1 - (2^{2k} - 1)) \mod 3 = 0$$

$$(2^{2k} \cdot 4 - 2^{2k}) \mod 3 = 0$$

Inductive consequense

$$(2^{2k} \cdot 4 - 1 - (2^{2k} - 1)) \mod 3 = 0$$

$$(2^{2k} \cdot 4 - 2^{2k}) \mod 3 = 0$$

$$(2^{2k} \cdot 3) \mod 3 = 0$$

- A recurrence relation for a sequence $a_0, a_1, a_2, \ldots$ is a formula
- that relates a term a_k
- to one or more predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$
- where $k - i \geq 0$.
- Initial conditions specify the values of $a_0, a_1, a_2, \ldots, a_{i-1}$ if i is fixed
- or $a_0, a_1, a_2, \ldots, a_m$ where $m \geq 0$ if i depends on k

Fibonacci numbers

- Recurrence relation: $F_k = F_{k-1} + F_{k-2}$
- Initial conditions.
  - $F_0 = 1$
  - $F_1 = 1$

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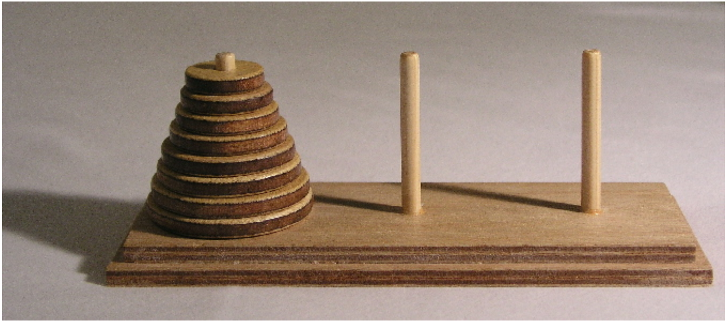
- Recurrence relation: $F_k = F_{k-1} + F_{k-2}$
- Initial conditions.
  - $F_0 = 1$
  - $F_1 = 1$
- Hereby follows:
  - $F_2 = F_1 + F_0 = 1 + 1 = 2$
  - $F_3 = F_2 + F_1 = 2 + 1 = 3$
  - $F_4 = F_3 + F_2 = 3 + 2 = 5$
  - $F_5 = F_4 + F_3 = 5 + 3 = 8$
  - $F_6 = F_5 + F_4 = 8 + 5 = 13$

Tower of Hanoi



- Recurrence relation: You can move a pile with k disks,
  - by moving the $k-1$ top disks to the pole wich is not the destination
  - moving k^{th} disk to the destination pole.
  - move the $k-1$ disks on top of the k^{th} disk on the destination.
- Initial conditions: A one disk pile can be moved to any bigger disk.

```java
class Hanoi {
  static int POLE_A = 0;
  static int POLE_B = 1;
  static int POLE_C = 2;
  static int[][] positions = new int[3][16];
  static int[] heights = new int[3];

  static void main(String... args) {
    for (int i = 0; i < 16; i++)
        positions[POLE_A][i] = 16 - i;
    heights[POLE_A] = 16;
    }

  static void move(int source, int dest, int start) {
    // ...
    }

  }
```

```
class Hanoi {
  ...
  static void move(int source, int target, int height) {
    if (height == 0) return;
    int temp = 2*(source + target)%3;
    move(source, temp, height - 1);
    positions[target][height] =
        positions[source][heights[source]];
    heights[target]++;
    heights[source]--;
    move(temp, target, height - 1);
    }

  }
```