Relations

Discrete Mathematics

Anders Kalhauge

Fall 2016

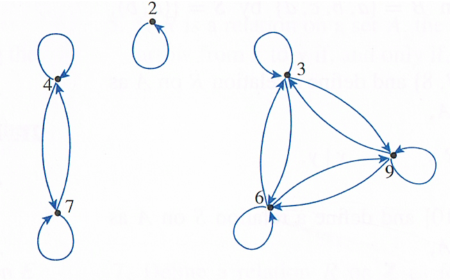
When R is a relation $x \, R \, y$

$$R \subseteq \{(y, x) \in B \times A\}$$

R^{-1} is the inverse of the relation R

$$R^{-1} = \{(y, x) \in B \times A | (x, y) \in R\}$$

$$A = \{2, 3, 4, 6, 7, 9\}$$



$$x, y \in A, x \, R \, y \iff 3 \mid (x - y)$$

Reflectivity

R is a relation on the set A

R is **reflective** if, and only if:

$$\forall x \in A, x \; R \; x$$

$$\forall x \in A, (x, x) \in R$$

R is a relation on the set A

R is **symmetric** if, and only if:

$$\forall x, y \in A, x \; R \; y \rightarrow y \; R \; x$$

$$\forall x, y \in A, (x, y) \in R \rightarrow (y, x) \in R$$

R is a relation on the set A

R is **transitive** if, and only if:

$$\forall x, y, z \in A, x \; R \; y \wedge y \; R \; z \rightarrow x \; R \; z$$

$$\forall x, y, z \in A, (x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R$$

$$x \, R \, y \iff x = y$$

- Reflexive $\forall x \in \mathbb{R}, x \, R \, x$:

$$x = x \text{ is true}$$

- Symetric $\forall x, y \in \mathbb{R}, x \, R \, y \rightarrow y \, R \, x$:

$$x = y \rightarrow y = x \text{ is true}$$

- Transitive $\forall x, y, z \in \mathbb{R}, x \, R \, y \land y \, R \, z \rightarrow x \, R \, z$

$$x = y \land y = z \rightarrow x = z \text{ is true}$$

$$x \, R \, y \iff x < y$$

- Reflexive $\forall x \in \mathbb{R}, x \, R \, x$:

$$x < x \text{ is false}$$

- Symetric $\forall x, y \in \mathbb{R}, x \, R \, y \to y \, R \, x$:

$$x < y \to y < x \text{ is false}$$

- Transitive $\forall x, y, z \in \mathbb{R}, x \, R \, y \land y \, R \, z \to x \, R \, z$

$$x < y \land y < z \to x < z \text{ is true}$$

$$x \, R \, y \iff 3 \mid (x - y)$$

- Reflexive $\forall x \in \mathbb{R}, x \, R \, x$:

$$3 \mid (x - x) \implies 3 \mid 0 \text{ is true}$$

- Symetric $\forall x, y \in \mathbb{R}, x \, R \, y \to y \, R \, x$:

$$3 \mid (x - y) \to 3 \mid (y - x) \text{ is true because } \exists k \in \mathbb{R} :$$

$$3 \mid (x - y) \iff x - y = 3k \iff y - x = -3k \iff 3 \mid (y - x)$$

- Transitive $\forall x, y, z \in \mathbb{R}, x \, R \, y \land y \, R \, z \to x \, R \, z$

$$3 \mid (x - y) \land 3 \mid (y - z) \to 3 \mid (x - z) \text{ is true}$$

R is a relation on the set A. The transitive closure of R is the relation R^t, also on A, that is:

- R^t is transitive
- $R \subseteq R^t$
- If S is any transitive relation that contains R, then $R^t \subseteq S$

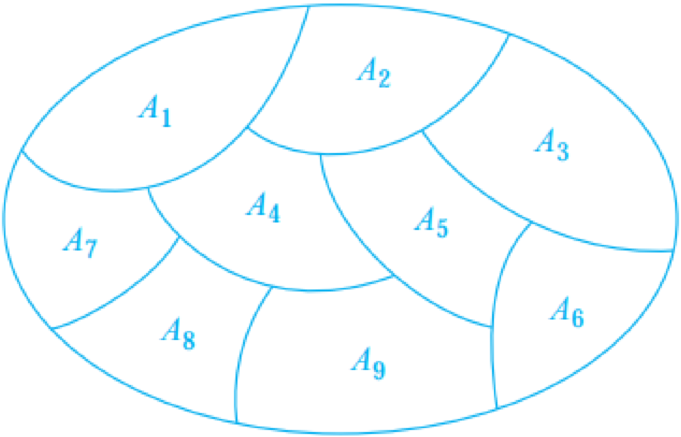
Define the term "reflective closure" of a relation R

Sets A and B are disjoint if

$$A \cap B = \emptyset$$

Partions $\{A_1, A_2, A_3, \ldots, A_n\}$ of set A

- $A = \bigcup_{i=1}^{n} A_i$
- $\forall a, b \in \{1, 2, 3, \ldots, n\}, a = b \vee A_a \cap A_b = \emptyset$



$$A = \bigcup_{i=1}^{9} A_i$$

Consider a set A partitioned in $A_1, A_2, A_3, \ldots, A_n$
R is the **relation induced by partition** of the set A

$$x \; R \; y \iff \exists A_i, x \in A_i \wedge y \in A_i$$

The set A is partitioned in $A_1, A_2, A_3, \ldots, A_n$ and $x, y, z \in A$

- Reflective $(x\,R\,x)$
- Symetric $(x\,R\,y \to y\,R\,x)$
- Transitive $(x\,R\,y \wedge y\,R\,z \to x\,R\,z)$

The set A is partitioned in $A_1, A_2, A_3, \dots, A_n$ and $x, y, z \in A$

- Reflective $(x \, R \, x)$

$$\exists A_i, x \in A_i \land x \in A_i \qquad \text{by definition of } x \, R \, x$$

- Symetric $(x \, R \, y \rightarrow y \, R \, x)$
- Transitive $(x \, R \, y \land y \, R \, z \rightarrow x \, R \, z)$

The set A is partitioned in $A_1, A_2, A_3, \ldots, A_n$ and $x, y, z \in A$

- Reflective $(x \, R \, x)$
- Symetric $(x \, R \, y \rightarrow y \, R \, x)$

$$\exists A_i, x \in A_i \land y \in A_i \quad \text{by definition of } x \, R \, y$$

- Transitive $(x \, R \, y \land y \, R \, z \rightarrow x \, R \, z)$

The set A is partitioned in $A_1, A_2, A_3, \ldots, A_n$ and $x, y, z \in A$

- Reflective $(x\,R\,x)$
- Symetric $(x\,R\,y \rightarrow y\,R\,x)$
- Transitive $(x\,R\,y \land y\,R\,z \rightarrow x\,R\,z)$

$$\exists A_i, x \in A_i \land y \in A_i \quad \text{by definition of } x\,R\,y$$

$$\exists A_j, y \in A_j \land z \in A_j \quad \text{by definition of } y\,R\,z$$

$$A_i \cap A_j = \emptyset \lor A_i = A_j$$

$$y \in A_i \land y \in A_j \iff y \in A_i \cap A_j \implies A_i \cap A_j \neq \emptyset$$

$$A_i = A_j \implies x \in A_i \land z \in A_i \quad \text{the definition of } x\,R\,z$$

We have looked at partitions of **finite** sets.

- Define a partition of an **infinite** set (e.g. $\mathbb{Z}$)
- Define an **infinite** partion of an infinite set

Exercise

We have looked at partitions of **finite** sets.

- Define a partition of an **infinite** set (e.g. $\mathbb{Z}$)
  - odd and even numbers
    $\{x \in \mathbb{Z} | x \mod 2 = 0\}, \{x \in \mathbb{Z} | x \mod 2 = 1\}$,
  - positive and negative numbers and zero
    $\mathbb{Z}^-, \{0\}, \mathbb{N}$
- Define an **infinite** partion of an infinite set

We have looked at partitions of **finite** sets.

- Define a partition of an **infinite** set (e.g. $\mathbb{Z}$)
  - odd and even numbers
    $\{x \in \mathbb{Z} | x \mod 2 = 0\}, \{x \in \mathbb{Z} | x \mod 2 = 1\}$,
  - positive and negative numbers and zero
    $\mathbb{Z}^-, \{0\}, \mathbb{N}$
- Define an **infinite** partion of an infinite set
  - $i \in \{0, 1, 2, \ldots\}, A_i = \{i, -i\}$

Equivalence Relation

cphbusiness
COPENHAGEN BUSINESS ACADEMY

A relation R on a set A is an **equivalence relation** if, and only if, R is

- Reflective $(x\, R\, x)$
- Symetric $(x\, R\, y \rightarrow y\, R\, x)$
- Transitive $(x\, R\, y \wedge y\, R\, z \rightarrow x\, R\, z)$

$x, y, z \in A$

Equvalence classes

$$[a] = \{x \in A | x \; R \; a\}$$

$$\forall x \in A, x \in [a] \iff x \; R \; a$$

A relation R on a set A is **antisymmetric** if, and only if,

- $\forall a, b \in A, a\,R\,b \land b\,R\,a \to a = b$

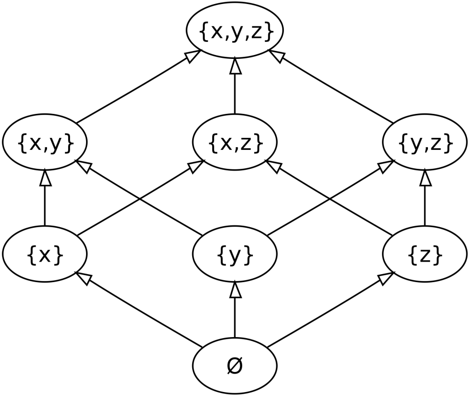
And it is **not** antisymmetric if, and only if

- $\exists a, b \in A, a\,R\,b \land b\,R\,a \land a \neq b$

A relation R on a set A is a **partial order relation** if, and only if, R is

- Reflective $(x\,R\,x)$
- Antisymmetric $(\forall a, b \in A, a\,R\,b \wedge b\,R\,a \rightarrow a = b)$
- Transitive $(x\,R\,y \wedge y\,R\,z \rightarrow x\,R\,z)$

$x, y, z \in A$



$$U, V \in \mathscr{P}(\{x, y, z\}), U \; R \; V \iff U \subseteq V$$

$\preceq$ is a partial order relation on a set A

- a and b of A are **comparable** if, and only if, either $a \preceq b$ or $b \preceq a$, same as: $(a,b) \in \preceq \lor (b,a) \in \preceq$
- a and b of A are **noncomparable** otherwise
  We will use $a \npreceq b$ and $b \npreceq a$

$$a, b \in A$$

Let A be the partially ordered set with respect to a relation $\preceq$

- $a \in A$ is the maximum element of A if, and only if,

$$\forall b \in A, b \preceq a \vee b \npreceq a$$

- $a \in A$ is the greatest element of A if ($\top$ of A), and only if,

$$\forall b \in A, b \preceq a$$

- $a \in A$ is the minimum element of A if, and only if,

$$\forall b \in A, a \preceq b \vee a \npreceq b$$

- $a \in A$ is the least element of A if ($\perp$ of A), and only if,

$$\forall b \in A, a \preceq b$$