Rumor Source Detection with Multiple Observations: Fundamental Limits and Algorithms

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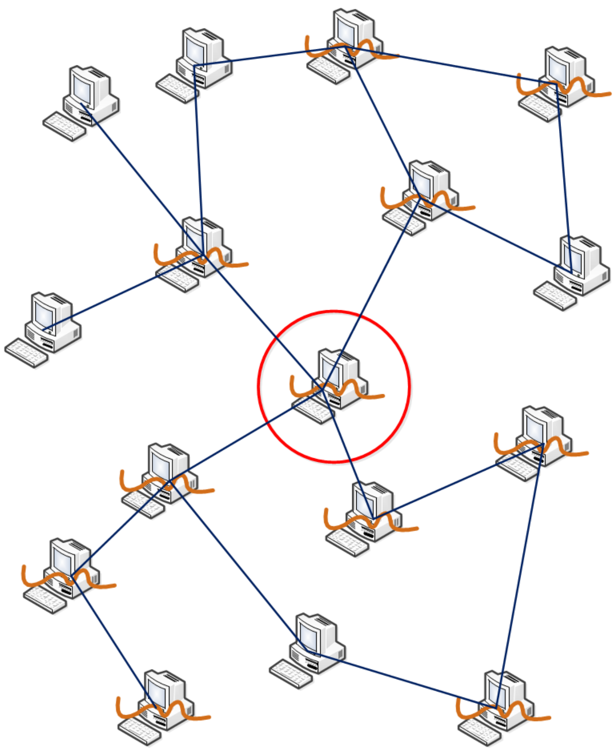
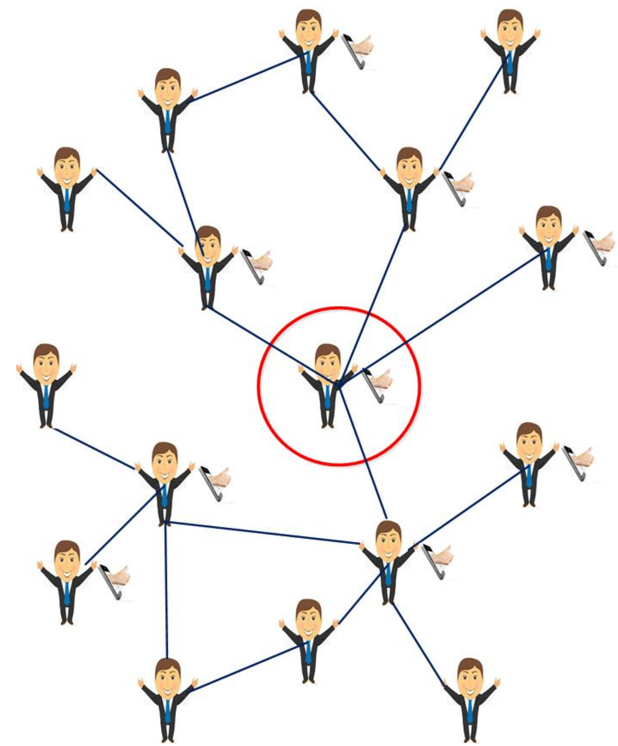
Background

- **How did virus epidemics begin?**



- **Who initiated a rumor in weibo/twitter?**



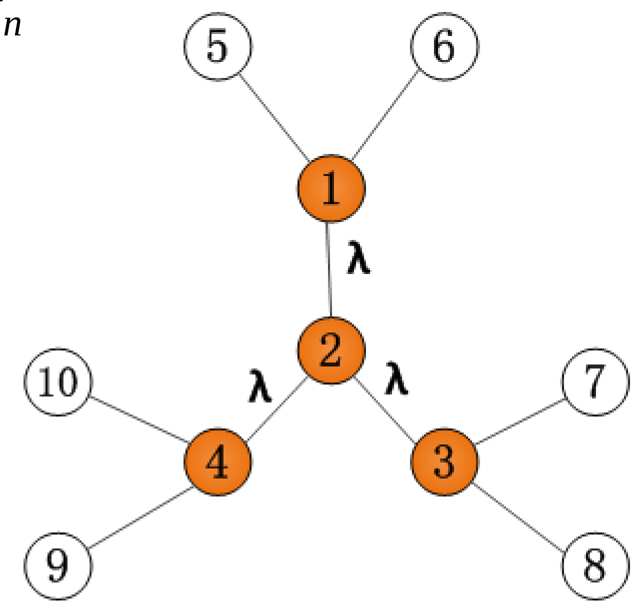
- A rumor, i.e., a “message”, has been passed around in a network.
- At some point we observe those who have possessed the message.
- How and how well can we figure out who initiated this spreading?

Outline

- SI model for rumor spreading
- Rumor center as maximum-likelihood (ML) detector
- Source detection with multiple instances
- Union rumor center
- Performance results
- Experiments
- Conclusions
- Literatures

SI Model for rumor spreading

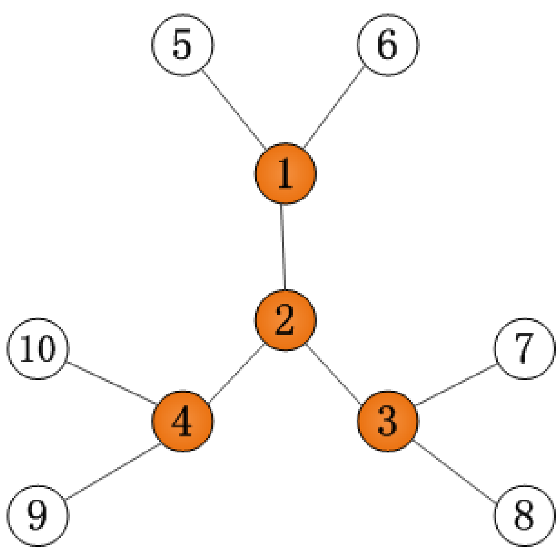
- **SI (susceptible-infectious) model as an undirected graph G =(V,E)**
  - An infected node keeps the rumor forever
- **Exponentially distributed infection time with parameter** λ
- **We observe the network G at some time and find n infected nodes denoted by** G_n



[ShaZamTIT11]

Rumor center
as maximum-likelihood (ML) detector

- **Rumor centrality R(v,G):** The total number of permitted permutations with source node v and infected nodes G.
- **Permitted permutation:** A possible order of infection starting from a postulated source node, obeying causality.
- **Rumor center (RC):** The node with the largest rumor centrality.



$v = 1$

$\{1,2,3,4\},\{1,2,4,3\}$ permitted;

$\{1,3,2,4\}$ not permitted.

$G = \{1,2,3,4\};$

$R(1,G) = 2, R(2,G) = 6,$

$R(3,G) = 2, R(4,G) = 2.$

Rumor center $= 2$

Key: **For the basic model, likelihood** $\propto R(v,G) \Rightarrow \mathrm{ML} = \mathrm{RC}.$

[ShaZamTIT11]

Performance results for the basic model

➢ For general graphs

- For node degree $\delta \geq 3$, "non-trivial" detection:

$$\lim_{n\to\infty} P_c(G_n) > 0$$

- For node degree $\delta = 2$, detection asymptotically impossible:

$$\lim_{n\to\infty} P_c(G_n) = 0$$

➢ For regular trees

- $$\lim_{n\to\infty} P_c(G_n) = \delta \cdot \mathrm{I}_{1/2}\left(\frac{1}{\delta-2}, \frac{\delta-1}{\delta-2}\right) - \delta + 1,$$

  where $\mathrm{I}_x(\alpha, \beta)$ is the incomplete beta function.

- $\lim_{n\to\infty} P_c(G_n) \nearrow 0.307$ as $\delta \to +\infty$.

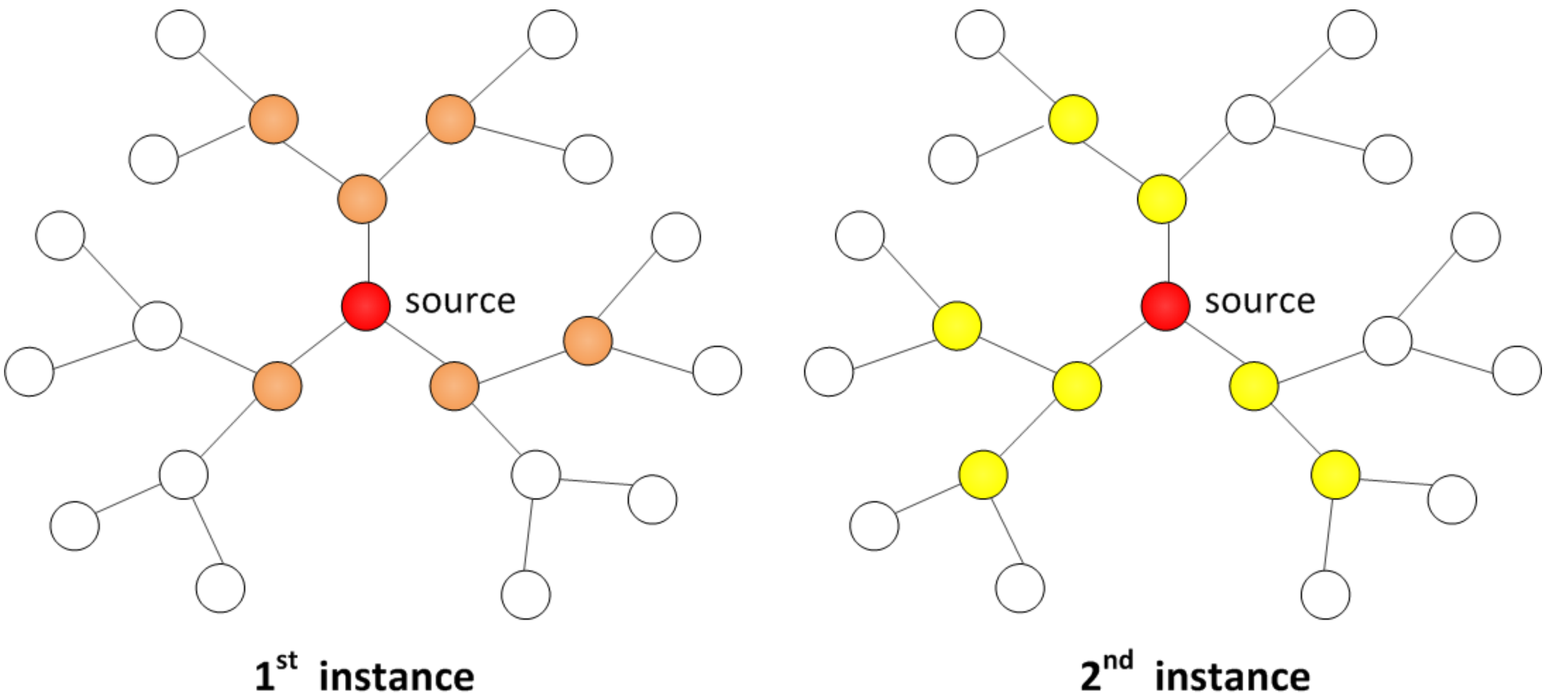
[ShaZamTIT11], [ShaZamSIGMETRICS12]

Source detection with multiple instances

- **A source may initiate multiple instances of spreading, rather than only once.**

  e.g., fraudulent email spams and recurring malcode, which are usually originated from a common culprit.

- **If multiple instances are available, how much can this diversity help?**



Union rumor center

➢ Union rumor center

Assume K independent instances of infected sets G_{n_j} , $j = 1, \ldots, K.$

- Union rumor centrality : $R_K(s^*, G_{n_1}, \ldots, G_{n_K}) = R(s^*, G_{n_1}) \cdots R(s^*, G_{n_K})$
- Union rumor center (URC): The node with the largest union rumor centrality.

➢ ML rumor source estimator

- For regular trees

$$\hat{s} \in \arg\max_{s^* \in \{\cap G_{1\to K}\}} \mathrm{P}_G(G_{n_1}, \ldots, G_{n_K} \mid s^*) = \arg\max_{s^* \in \{\cap G_{1\to K}\}} R_K(s^*, G_{n_1}, \ldots, G_{n_K}),$$

e.g. , **ML detector=URC.**

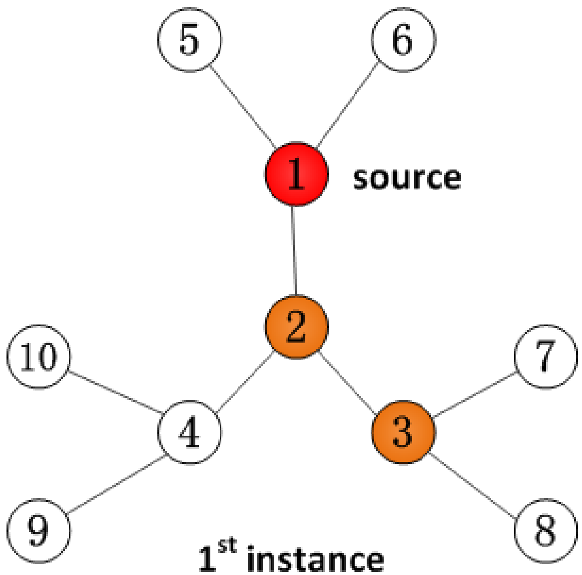
Tip: If s has the largest union rumor centrality among all its neighbors in $G_{n_1} \cap \ldots \cap G_{n_K}$ then s is a **URC**.

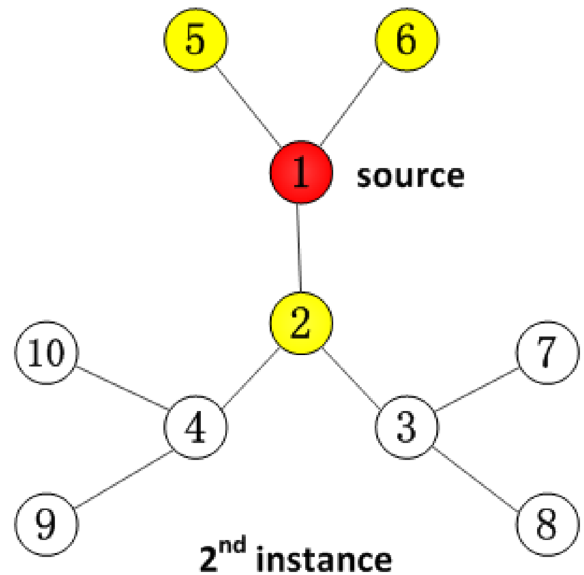
- For general trees

$$\hat{s} \in \arg\max_{s^* \in \{\cap G_{1\to K}\}} \prod_{j=1}^{K} \mathrm{P}(\sigma_s^* \mid s^*, G_{n_j}) \cdot R_K(s^*, G_{n_1}, \ldots, G_{n_K})$$

Union rumor center

- **For example**





$$G_1 = \{1,2,3\};$$
$$R(1,G) = 1,$$
$$R(2,G) = 2,$$
$$R(3,G) = 1.$$

$\Rightarrow$ $\mathrm{RC} = 2 \neq \mathrm{source}$

$$G_2 = \{1,2,5,6\};$$
$$R(1,G) = 6, R(2,G) = 2,$$
$$R(5,G) = 2, R(6,G) = 2.$$

$\Rightarrow$ $\mathrm{RC} = 1 = \mathrm{source}$

Union rumor center

$$G_1 \cap G_2 = \{1,2\};$$
$$R_2(1, G_1, G_2) = 1 \cdot 6 = 6,$$
$$R_2(2, G_1, G_2) = 2 \cdot 2 = 4.$$

$\Rightarrow$ $\mathrm{URC} = 1 = \mathrm{source}$

Performance results

- $\delta = 2, K = 2,$ Given G_{n_1}, G_{n_2}

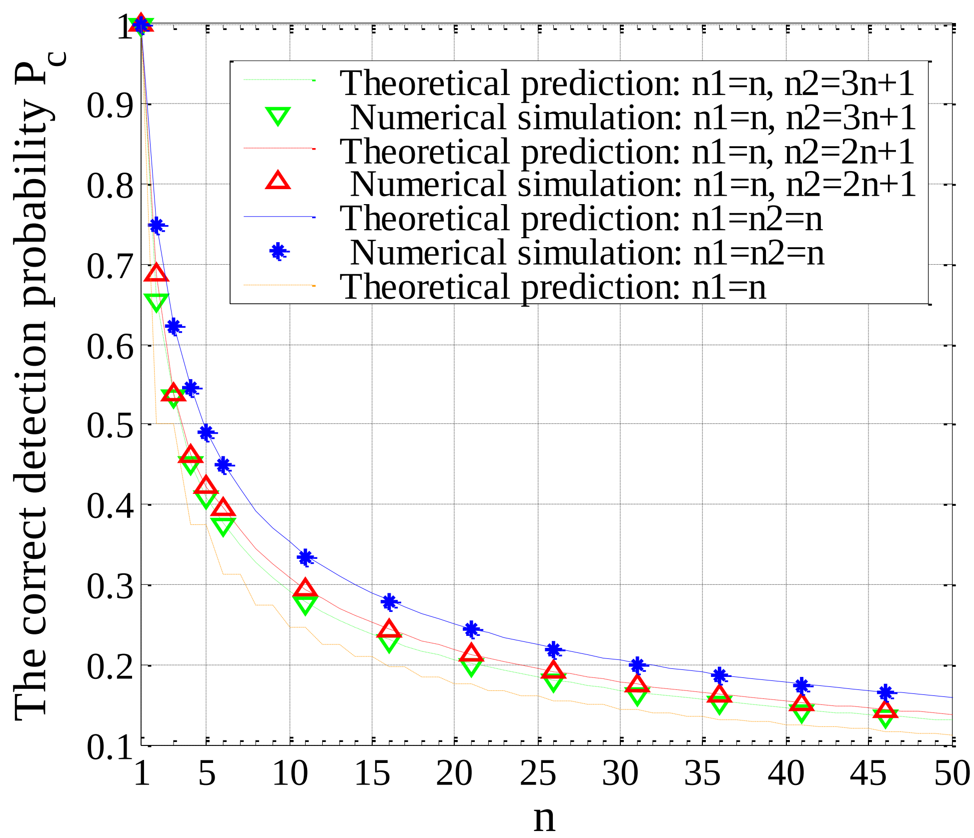
■ (1) $n_1 = n_2 = n \geq 1$

$$p_c = \binom{2n}{n} 2^{-2n+1}, \qquad n \geq 1.$$

■ (2) $n_1 = 2, n_2 > n_1$

$$p_c = \left\{ \sum_{m=0}^{\left\lfloor \frac{n_2}{n_1} \right\rfloor} \binom{n_2 - 1}{m} - \Pi\left(\frac{n_2}{n_1}\right) \binom{n_2 - 1}{\left\lfloor \frac{n_2}{n_1} \right\rfloor} \right\} \cdot 2^{-2n_2+1},$$

where $\Pi\left(\frac{n_2}{n_1}\right) = \begin{cases} 1 & \frac{n_2}{n_1} \in Z \\ 0 & \text{others.} \end{cases}$



$\delta = 2$

■ (3) $n_2 > n_1 \geq 3$

$$p_c = \left\{ \sum_{m=0}^{n_1 - 1} \left[\binom{n_2 - 1}{\left\lfloor m \frac{n_2}{n_1} \right\rfloor} \binom{n_1}{m} + S_{n_2 - 1}(m) \binom{n_1 - 1}{m} \right] - 2 \binom{n_2 - 1}{\left\lfloor \frac{n_2}{n_1} \right\rfloor} - \Pi\left(\frac{n_2}{n_1}\right) \binom{n_2 - 1}{\left\lfloor \frac{n_2}{n_1} \right\rfloor} \right\} \cdot 2^{-(n_1 + n_2)+2}, \quad \text{where } S_{n_2 - 1}(m) = \sum_{i = \left\lfloor m \frac{n_2}{n_1} \right\rfloor + 1}^{\left\lfloor (m+1) \frac{n_2}{n_1} \right\rfloor - 1} \binom{n_2 - 1}{i}.$$

Performance results

- $\delta = 3,\ K = 2,$ Given G_{n_1}, G_{n_2}

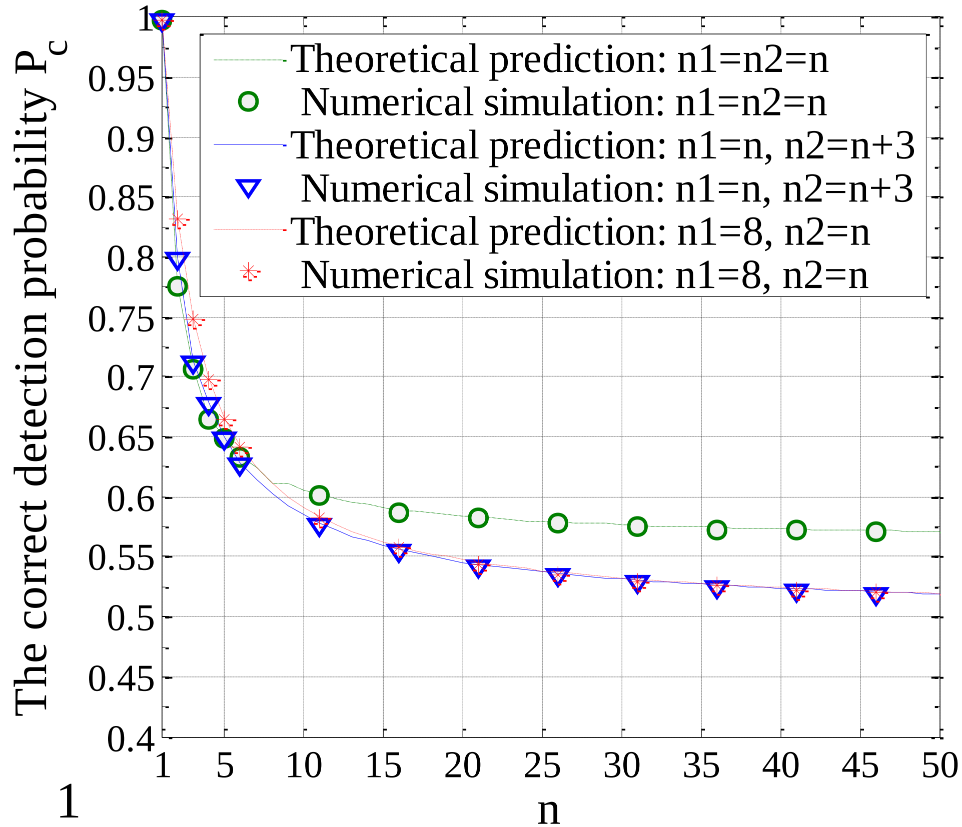
■ (1) $n_1 = n, n_2 = qn\ (q \in Z^+)$

$$\mathrm{P}_c = \frac{qn+q+2}{2(qn+1)}$$

■ (2) $n_1 = n, n_2 = qn+1\ (q \in Z^+)$

$$\mathrm{P}_c = \frac{qn+q+2}{2(qn+1)}$$

■ (3) $n_1 = n, n_2 = qn+t\ (q \in Z^+), t < n$



$\delta = 3$

$$\mathrm{P}_c = \frac{qn+q+2}{2(qn+1)} + \Delta\mathrm{P}_c \quad \text{with} \quad \Delta\mathrm{P}_c < \frac{1}{2(qn+1)}.$$

As $n \to +\infty$, $\lim\limits_{n\to+\infty} \mathrm{P}_c = \frac{1}{2}$

Performance results

➢ Monotonicity

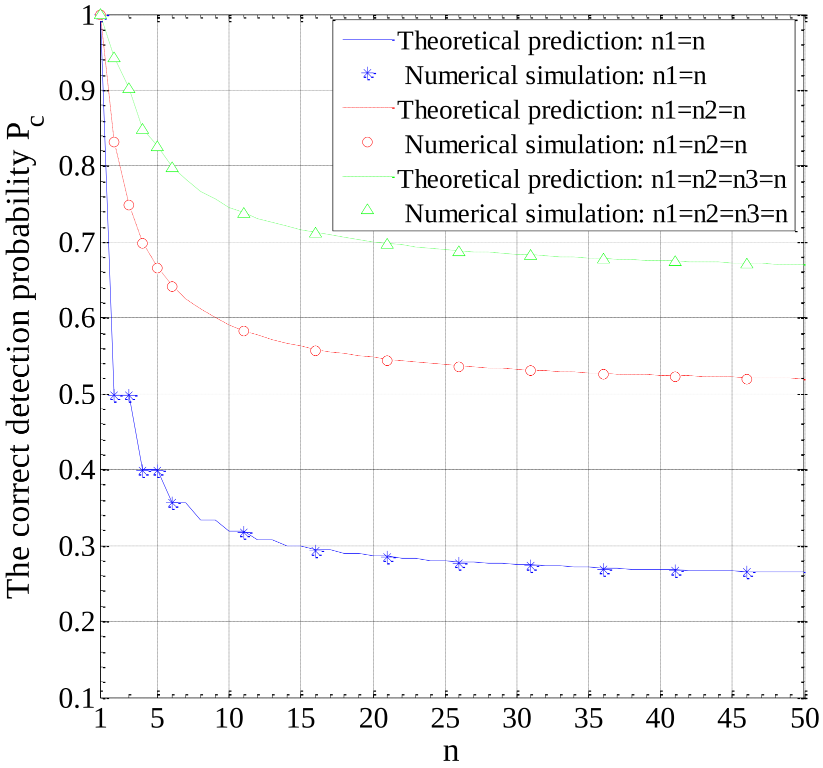
Given $G_{n_1}, G_{n_2}, \ldots, G_{n_K}$

- (1) P_c is increasing with δ
  - **Reliable detection with abundant connectivity**

    For any $K \geq 2$, as $\delta \to +\infty$, $\mathrm{P}_c \to 1$

  - **Reliable detection with abundant diversity**

    For any δ, as $K \to +\infty$, $\mathrm{P}_c \to 1$



$\mathrm{P}_c \ \ vs \ \ n, \delta = 3$

- (2) P_c is non-increasing with n_K, as fixed $n_1, \ldots, n_{K-1}$

Performance results

➢ Asymptotic regime

$$n_1, \ldots, n_K \to +\infty$$

■ (1) Any δ and any K

$$\lim_{n_1,\ldots,n_K \to +\infty} \mathrm{P}_c = \phi_K(\delta) := 1 - \delta\left(1 - \varphi_K\left(\frac{1}{\delta-2}, \frac{\delta-1}{\delta-2}\right)\right),$$

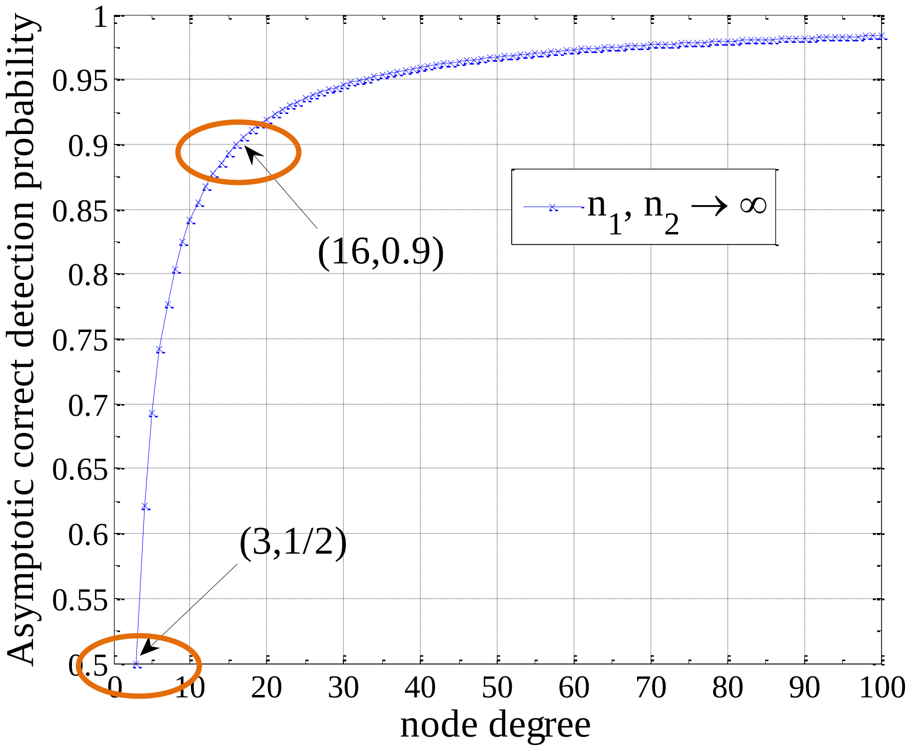
$$\text{where } \varphi_K(\alpha,\beta) = \int \cdots \int \frac{\Gamma(\alpha+\beta)^K}{\Gamma(\alpha)^K \Gamma(\beta)^K} \prod_{j=1}^{K} \left(x_j^{\alpha-1}(1-x_j)^{\beta-1}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_K,$$

$$\alpha = \frac{1}{\delta-2}, \beta = \frac{\delta-1}{\delta-2}.$$

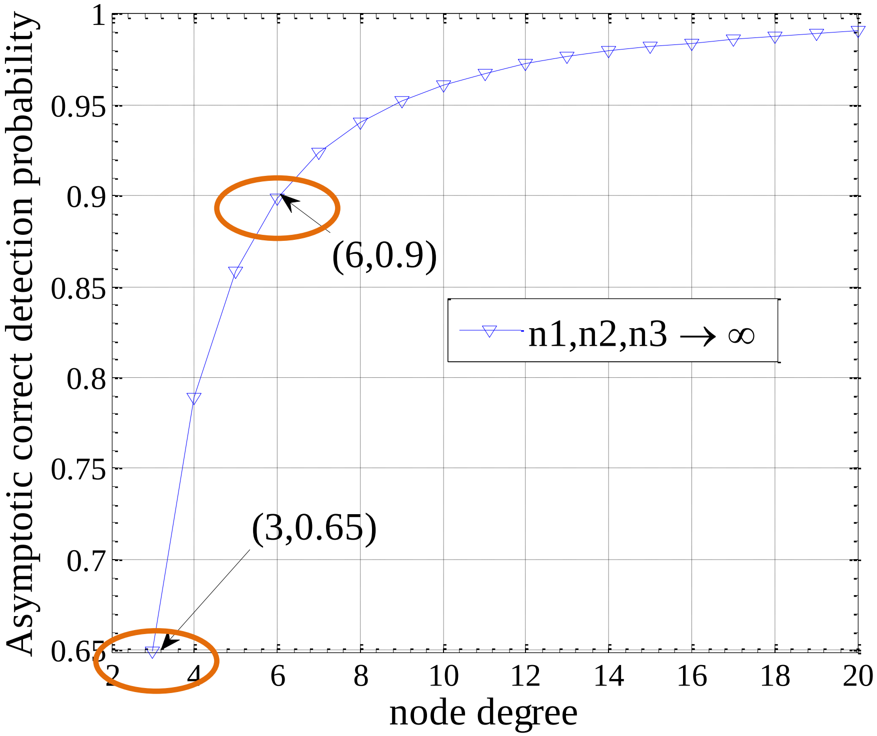
- As $\delta \to +\infty$, $\phi_K(\delta) \to 1$;
- Also, as $K \to +\infty$, $\phi_K(\delta) \to 1$.

Performance results

----Detection performance with multiple instances:



K=2



K=3

K=1 : $\lim_{n\to\infty} P_c(G_n) \nearrow$ **0.307**, $\delta \to +\infty.$

Performance results

➢ Asymptotic regime

■ (2) $\delta = 3$ and any K

$$\phi_K(3) = 1 - 3\cdot 2^{K-2}\int_0^1 \cdots \int_0^1 \frac{\prod_{j=1}^{K-1}\left(x_j(1-x_j)\right)}{\prod_{j=1}^{K-1} x_j + \prod_{j=1}^{K-1}\left(1-x_j\right)} \mathrm{d}x_1 \ldots \mathrm{d}x_{K-1},$$

which is increasing with K and is bounded as follows:

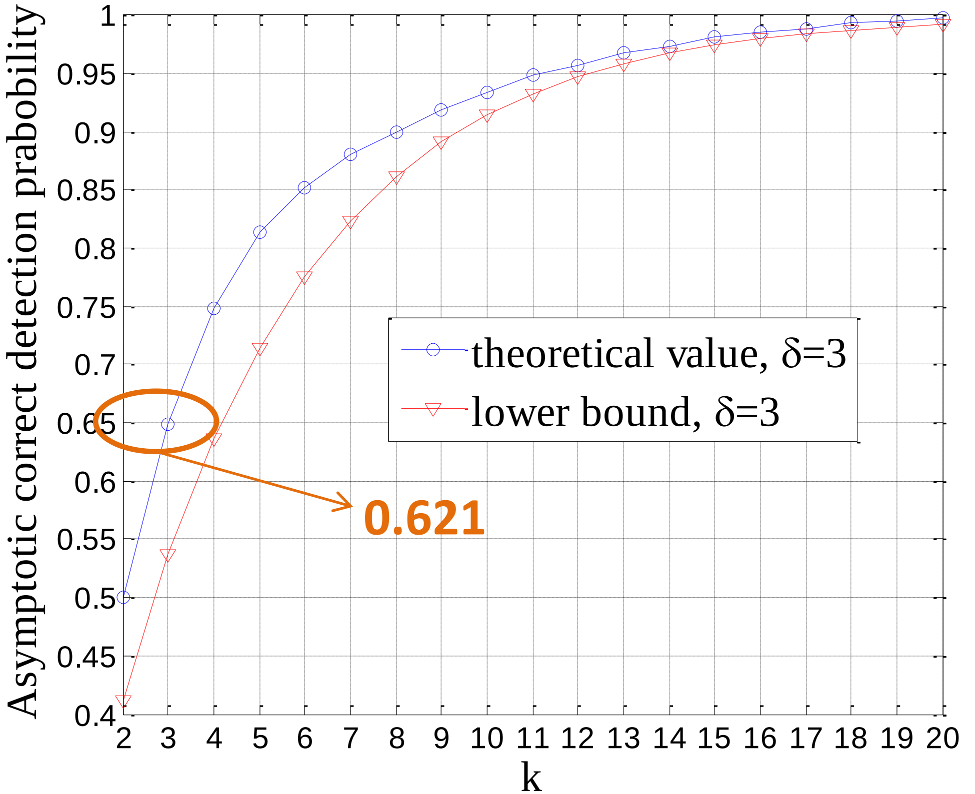
$$1 - \frac{3}{4}\left(\frac{\pi}{4}\right)^{K-1} < \lim_{n_1,\ldots n_K \to +\infty} \mathrm{P}_c < 1,\ K \in Z^+.$$

- **Exponential convergence with L.**
- As $K \to +\infty$, $\phi_K(3) \to 1$.

■ (3) $K = 2$

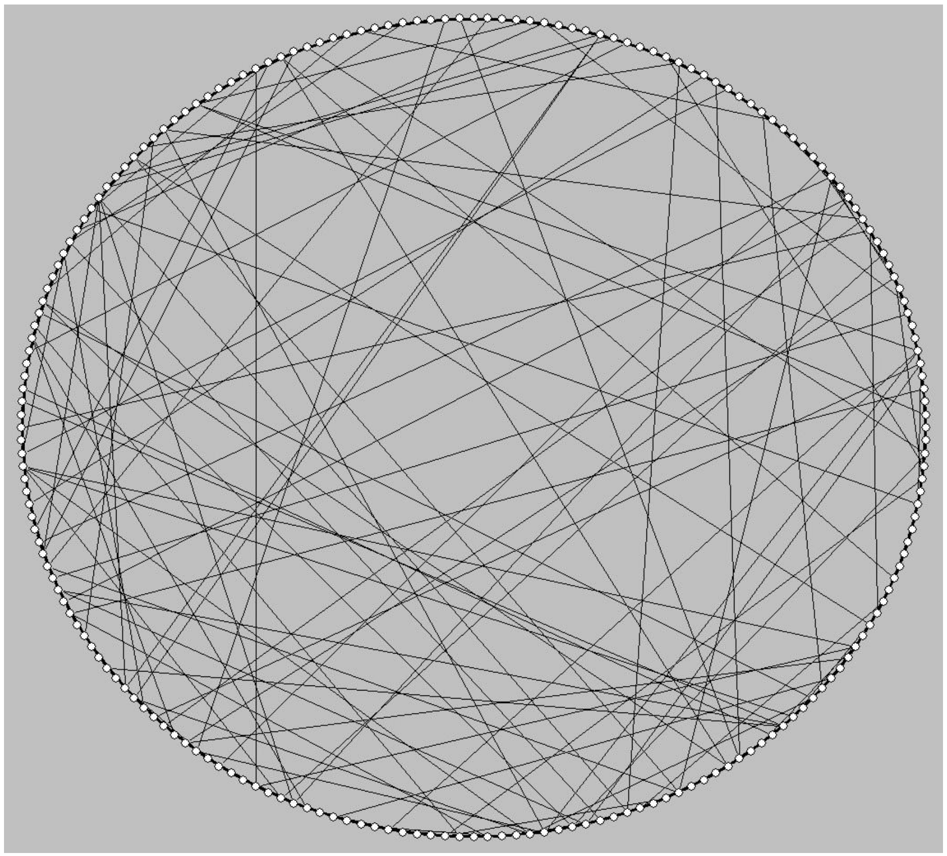
For $\delta = 2$, $\lim_{n_1,n_2 \to +\infty} \mathrm{P}_c = \frac{1}{2}$

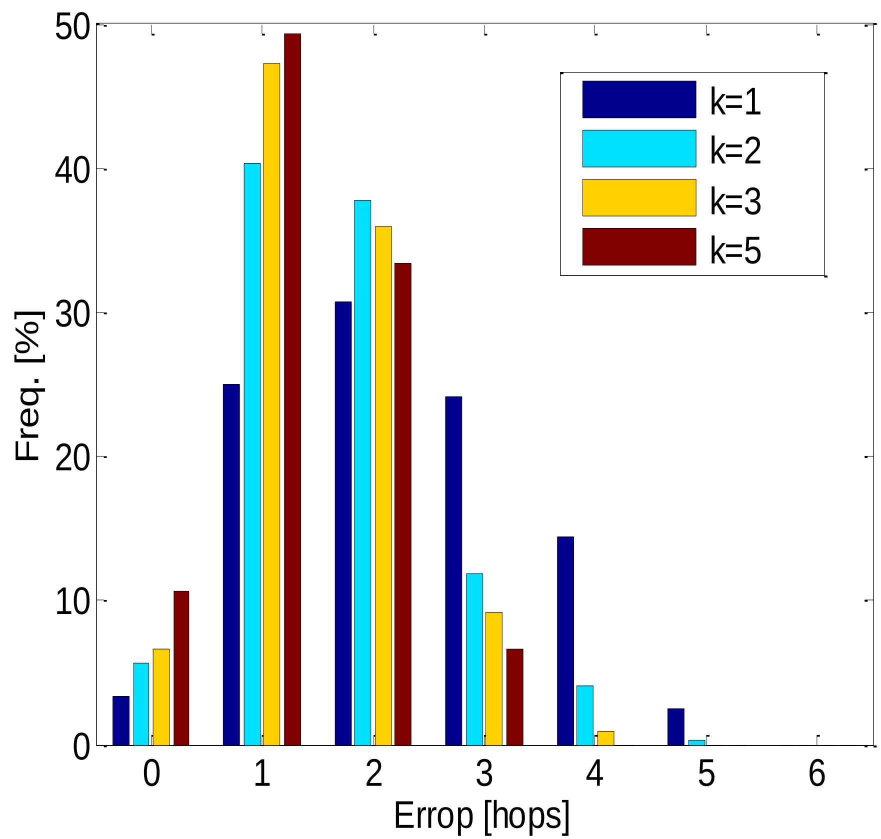
For $\delta = 3$, $\lim_{n_1,n_2 \to +\infty} \mathrm{P}_c = \frac{16}{\pi^2} - 1 \approx 0.621$



Experiments

Small-world networks (Watts-Strogatz model)

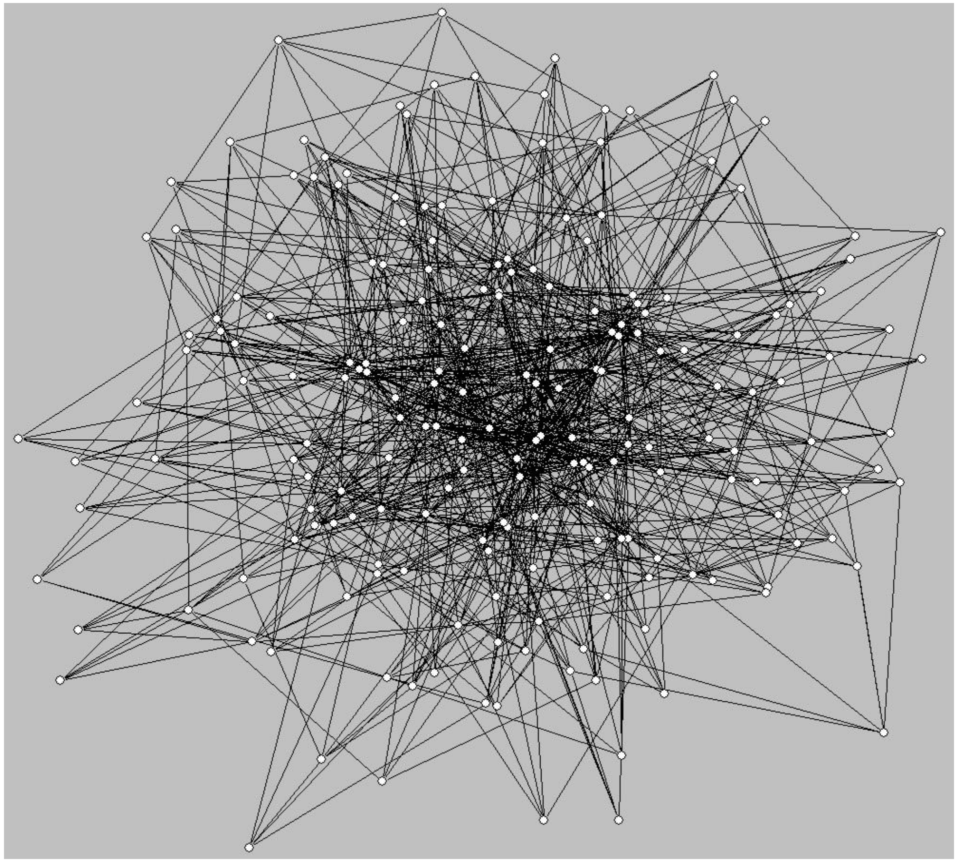


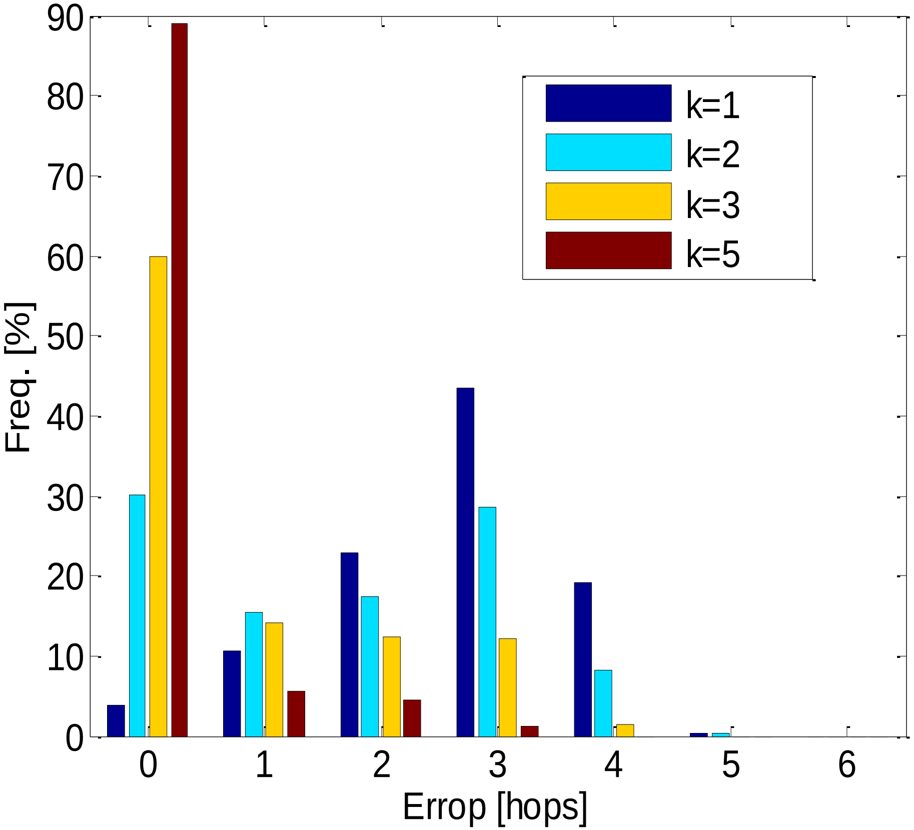


Network size=5000 , infected set size=400.

Experiments

Scale-free networks (Barabasi-Albert model)

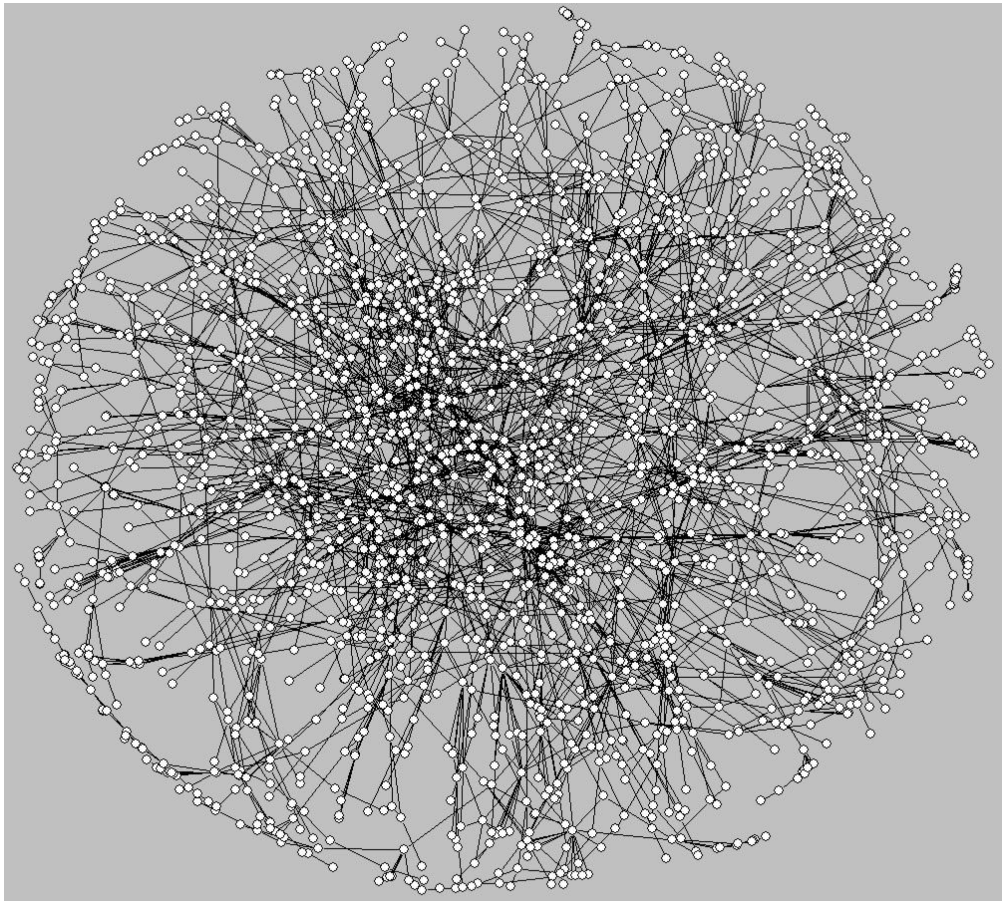


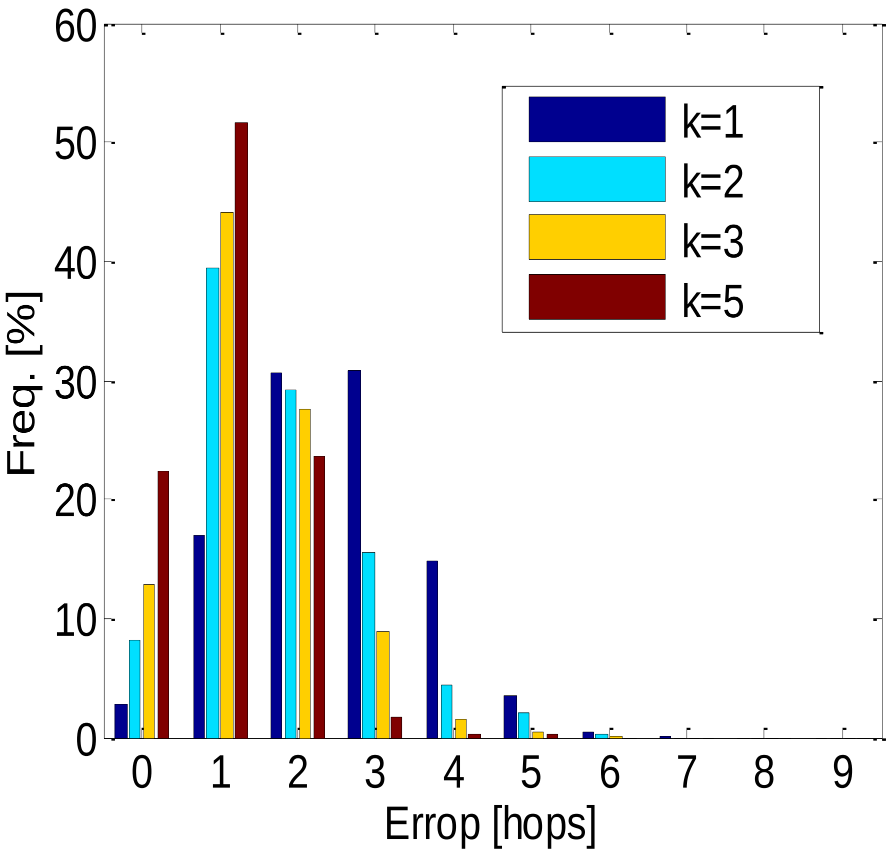


Network size=5000 , infected set size=400.

Experiments

Newman's scientific collaboration network





Source: http://konect.uni-koblenz.de/networks/opsahl-collaboration

Network size=13861 , infected set size=400.

Conclusions

- **Established a unified rumor source inference framework with multiple instances.**
- **Provided explicit analytical results for regular trees.**

— multiple independent observations dramatically enhances detectability.

— even two observations can more than double the detection probability of a single observation.

— P_c increases with δ as well as K, i.e., richer connectivity and diversity both enhance detection.

- **Leveraged the inference framework as an effective heuristic for general graphs.**
- **Still a long march towards a full-grown theory capable of handling all the factors in realistic scenarios.**

Literatures

----Estimation of rumor source

- Regular trees, general trees, general graphs:

breadth-first-search (BFS) heuristic + RC. [ShaZamTIT11]

- General infection time distribution, general random tree:

RC still achieves non-trivial detection (universal detector).

[ShaZamSIGMETRICS12]

- Limited (sparse) observations:

[PinThiVetPRL12], [KarFraISIT13], [LuoTayLenArxiv13]

- Multiple sources:

[LuoTayLenSP13]

- SIS or SIR infection processes:

[LuoTayICASSP13], [ZhuYinITA13]

- Other related models and algorithms:

[PraVreFalICDM12], [LokM'ezOhtZdeArxiv13], [AntLanSteSikSmuArxiv13]

A (partial) bibliography

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The End

Thank you!

Q&A